SINGAPORE MANAGEMENT UNIVERSITY

Session 2: Stochastic Integrals & Itô Formula

Tee Chyng Wen

QF620 Stochastic Modelling in Finance

From Brownian Motions to Stochastic Processes

Brownian motion is the natural candidate to be used to model the evolution of the stock price process S_t. We can write the future stock price as its present price plus a deterministic and a stochastic components:

$$\underbrace{S_{t+\Delta t}}_{\text{future price}} = \underbrace{S_t}_{\text{present price}} + \underbrace{\mu(t, S_t)\Delta t}_{\text{deterministic}} + \underbrace{\sigma(t, S_t)(W_{t+\Delta t} - W_t)}_{\text{stochastic}}$$

By using Brownian motion increment to form the stochastic component of our model, we're effectively using independent normally distributed increment to drive our stock price process. Now we take the limit of $\Delta t \rightarrow 0^+$, and obtain the stochastic differential equation (sde)

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t.$$

To solve this sde, we wish we could write it in the following format:

$$\frac{dS_t}{dt} = \mu(t, S_t) + \sigma(t, S_t)\frac{dW_t}{dt}.$$

However, this is not feasible given that one of Brownian motion's properties is that it is nowhere differentiable.

Stochastic Integrals

Since the differentiation formulation doesn't work, let's try the integration formulation by expressing the stock price process as follow:

$$S_T = S_0 + \underbrace{\int_0^T \mu(u, S_u)\, du}_{\text{Riemann integral}} + \underbrace{\int_0^T \sigma(u, S_u)\, dW_u}_{\text{stochastic integral}} .$$

Note that on the right hand side, the first integral is a classic Riemann integral, and we know how to manage it. Recall the definition of a Riemann integral

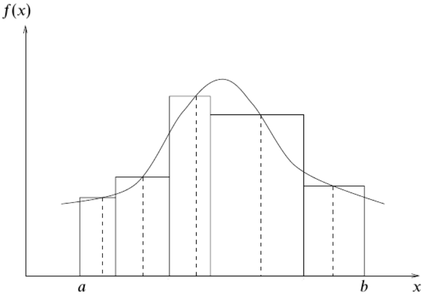
Riemann Integration

Let f be a regular function and P_N be a partition of the interval $[0, T]$, given by $\{t_0 = 0, t_1, t_2, \ldots, t_N = T\}$, then f is Riemann integrable if the following limit converges

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_i) \times (t_{i+1} - t_i), \quad x \in [t_i, t_{i+1}].$$

Definition of Riemann Integrals

We can define and visualise an integral as the area under the curve. Consider a function $f : \mathbb{R} \to \mathbb{R}$ which we would like to integral over the interval $[a, b]$.



Partitioning the interval into

$$\{x_0 = a,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5 = b\}$$

We can approximate the area as

$$S = \sum_{k=1}^{5} f(\bar{x}_k)(x_k - x_{k-1}).$$

The integral is therefore defined as $n \to \infty$.

Definition of Stochastic Integrals

What about the second integral? Let's define it using the same approach as the Riemann integral before. With the same notations, we define the stochastic integral as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i) \times (W_{t_{i+1}} - W_{t_i}).$$

This limit exists in an appropriate sense. With this definition for stochastic integral, we note that

- the result of a stochastic integral is a random variable, as opposed to what we get from Riemann integral.
- in Riemann integral of a function $f(x)$, x_i can be any point in the interval $[t_i, t_{i+1}]$, whereas in stochastic integral, x_i must be taken at the left side of each interval (t_i). This is due to the "previsibility" requirement, that $f(x_i)$ is only known at the beginning of the interval. Taking $x_i = t_{i+1}$ for instance will lead to a different result.

Integrating Stochastic Processes

Example Evaluate the stochastic integral

$$\int_0^T W_t \, dW_t.$$

Solution We partition T with $\{t_0 = 0, t_1, t_2, \ldots, t_n\}$. The stochastic integral is then given by

$$\begin{aligned}
\int_0^T W_t \, dW_t &= \lim_{n\to\infty} \sum_{i=1}^{n} W_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) \\
&= \frac{1}{2} \lim_{n\to\infty} \sum_{i=1}^{n} (W_{t_i}^2 - W_{t_i}^2 + 2W_{t_{i-1}}W_{t_i} - W_{t_{i-1}}^2 - W_{t_{i-1}}^2) \\
&= \frac{1}{2} \lim_{n\to\infty} \sum_{i=1}^{n} (W_{t_i}^2 - W_{t_{i-1}}^2 - (W_{t_i} - W_{t_{i-1}})^2) \\
&= \frac{1}{2} W_T^2 - \frac{1}{2} W_0^2 - \lim_{n\to\infty} \frac{1}{2} \sum_{i=1}^{n} (W_{t_i} - W_{t_{i-1}})^2 = \frac{1}{2} W_T^2 - \frac{T}{2}. \quad \triangleleft
\end{aligned}$$

Integrating Stochastic Processes

Example Evaluate the following integral

$$\int_0^T W_t \, dt.$$

Solution Similar to the previous question, we partition T into $\{t_0 = 0, t_1, t_2, \ldots, t_n\}$ of uniform size Δt. We have

$$\int_0^T W_t \, dt = \lim_{n\to\infty} \sum_{i=1}^{n} w_{t_{i-1}}(t_i - t_{i-1}).$$

$$\begin{aligned}
&\sum_{i=1}^{n} w_{t_{i-1}}(t_i - t_{i-1}) \\
&\quad = -w_0 t_0 + (w_0 - w_1)t_1 + \cdots + (w_{t_{n-2}} - w_{t_{n-1}})t_{n-1} + w_{t_{n-1}} t_n \\
&\quad = -w_0 t_0 + \sum_{i=1}^{n-1} (w_{t_{i-1}} - w_{t_i})t_i + (w_{t_{n-1}} - w_{t_{n-2}} + w_{t_{n-2}} - \cdots + w_0 - w_0)t_n \\
&\quad = -w_0 t_0 + \sum_{i=1}^{n-1} (w_{t_{i-1}} - w_{t_i})t_i + t_n \sum_{i=1}^{n-1} (w_{t_i} - w_{t_{i-1}}) + w_0 t_n \\
&\quad = \sum_{i=1}^{n-1} (w_{t_i} - w_{t_{i-1}})(t_n - t_i).
\end{aligned}$$

Integrating Stochastic Processes

This is just a weighted sum of Brownian motion increments. We conclude from the definition of Brownian motion that the integral is also a normal distribution, with mean 0 and a variance of

$$\begin{aligned}
V[I] = \mathbb{E}[I_T^2] &= \mathbb{E}\left[\int_0^T w_t \, dt \int_0^T w_s \, ds\right] \\
&= \int_0^T \int_0^T \mathbb{E}[w_t w_s] \, dt \, ds \\
&= \int_0^T \int_0^T \min(t, s) \, dt \, ds \\
&= \int_0^T \left(\int_0^s t \, dt + \int_s^T s dt\right) \, ds \\
&= \int_0^T \left(\frac{s^2}{2} + s(T - s)\right) ds = \frac{T^3}{3}.
\end{aligned}$$

So we have $I_T \sim N\left(0, \frac{T^3}{3}\right)$. $\triangleleft$

Properties of Stochastic Integrals

Consider the random variable I defined as the stochastic integral

$$I = \int_0^T f(u)\, dW_u, \quad f : \mathbb{R} \to \mathbb{R},\ f(x).$$

A few key properties I satisfies are as follow:

1. $\mathbb{E}[I] = 0$
2. $\mathbb{E}[I^2] = \mathbb{E}\left[\left(\int_0^T f(u)\, dW_u\right)^2\right] = \mathbb{E}\left[\int_0^T f(u)^2\, du\right]$
3. $\mathbb{E}\left[\int_0^T f(u)\, dW_u \times \int_0^T g(u)\, dW_u\right] = \mathbb{E}\left[\int_0^T f(u)g(u)\, du\right]$
4. If f is a deterministic function, then

$$I \sim N\left(0, \int_0^T f(u)^2\, du\right)$$

5. Itô's Isometry theorem states that

$$\mathbb{E}\left[\left(\int_0^T X_t\, dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T X_t^2\, dt\right].$$

Brown, a botanist, discovered the motion of pollen particles in water in 1827.

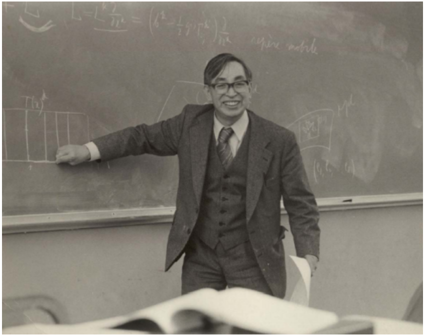
At the beginning of the 20^{th} *century, Brownian motion was studied by Einstein, Perrin and other physicists.*

In 1923, against this scientific background, Wiener defined probability measures in path spaces, and used the concept of Lebesgue integrals to lay the mathematical foundations of stochastic analysis.

In 1942, Dr. Itô began to reconstruct from scratch the concept of stochastic integrals, and its associated theory of analysis. He created the theory of stochastic differential equations, which describe motion due to random events.

Kiyosi Itô (1915–2008)

> *...I finally devised stochastic differential equations, after painstaking solitary endeavours.*
>
> — Kiyosi Itô (1915–2008)



Why Itô's Formula?

From a theoretical viewpoint:

- Now that we have defined the stochastic integral, we want to be able to manipulate it without coming back to the definition.
- General rules similar to ordinary calculus chain rules, product rules etc. will be very handy.

From a practical viewpoint:

- We have now defined the stock price process, and knowing that an option price is a function of that random process,
- We want to study and understand the infinitesimal evolution of the option price process as well.

The purpose of computing is insight, not numbers.

— Richard Hamming (1915–1998)

There is no doubt that the field of quantitative finance has been thoroughly transformed by the basic insights provided by Ito's calculus, both on a conceptual and on a computational level.

— Hans Föllmer (b. 1941)

Formulation of Itô's Formula

Suppose the stock price discrete dynamics is given by

$$\Delta S_t = \mu(t, S_t)\Delta t + \sigma(t, S_t)\Delta W_t,$$

using Taylor expansion from ordinary calculus for a smooth function $f : \mathbb{R} \to \mathbb{R}$, we have

$$f(S_t + \Delta S_t) = f(S_t) + f'(S_t)\Delta S_t + \frac{1}{2!} f''(S_t)(\Delta S_t)^2 + \mathcal{O}\big((\Delta S_t)^3\big).$$

The next step is obviously to keep Δt, take the limit of it tending towards 0, and neglect all the higher order terms (i.e. from Δt^2 onwards). To that end, we must first analyse the expression

$$(\Delta S_t)^2 = \mu(t, S_t)^2(\Delta t)^2 + 2\mu(t, S_t)\sigma(t, S_t)(\Delta t)(\Delta W_t) + \sigma(t, S_t)^2(\Delta W_t)^2.$$

Under the limit $\Delta t \to 0^+$, we obtain

$$df(S_t) = f'(S_t)dS_t + \frac{1}{2} f''(S_t)\sigma(t, S_t)^2 dt$$

Note that there's an extra term compared to differentiation in ordinary calculus.

Itô's Formula

This leads us to the famous Itô's formula (sometimes known as Itô's lemma):

Itô's Formula (Function of a Stochastic Process)

If X_t is a stochastic process satisfying

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

and the function $f : \mathbb{R} \to \mathbb{R}$, f, f', f'' are continuous, then $Y_t = f(X_t)$ is also a stochastic process and is given by

$$dY_t = \left(\mu_t f'(X_t) + \frac{1}{2}\sigma_t^2 f''(X_t)\right) dt + \sigma_t f'(X_t) dW_t.$$

Example Suppose $dX_t = \mu dt + \sigma dW_t$, and $Y_t = X_t^2$. Derive the stochastic differential equation for dY_t.

Itô's Formula

More generally, Itô's Formula also allows us to write down the stochastic differential equation of a function of stochastic processes and time:

Itô's Formula (Function of a Stochastic Process & Time)

If X_t is a stochastic process satisfying

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

and the function $g : \mathbb{R}^2 \to \mathbb{R},\ g(t,x),\ g_t,\ g_x,\ g_{xx}$ are continuous, then $Y_t = g(t, X_t)$ is also a stochastic process and is given by

$$dY_t = \left[g_t(t,X_t) + \mu_t g_x(t,X_t) + \frac{1}{2}\sigma_t^2 g_{xx}(t,X_t) \right] dt + \sigma_t g_x(t,X_t) dW_t.$$

Example Suppose $dX_t = \mu dt + \sigma dW_t$, and $Y_t = e^{X_t + t}$. Derive the stochastic differential equation for dY_t.

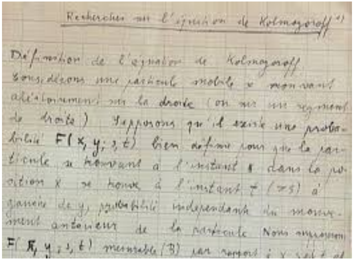
Wolfgang Doeblin (1915–1940)

Jewish-German who escaped from Nazi Germany to France and became French citizen.

Exceptional mathematician, who kept on working on Markov processes as a soldier during WWII.

Sent his final work to the French Academy of Sciences in a sealed envelope before committing suicide to avoid capture by German troops.

In honour of his contribution, Itô's formula is occasionally referred to as Itô-Doeblin Theorem.


Recherches sur l'équation de Kolmogoroff [1]

Définition de l'équation de Kolmogoroff
Considérons une particule mobile x se mouvant aléatoirement sur la droite (ou sur un segment de droite). Supposons qu'il existe une probabilité F(x, y; s, t) bien définie pour que la particule se trouvant à l'instant s dans la position x se trouve à l'instant t (> s) à gauche de y, probabilité indépendante du mouvement antérieur de la particule. Nous supposons F(x, y; s, t) mesurable (B) par rapport à x, s, t [illegible]

Wolfgang Doeblin (1915–1940)

Solving Stochastic Integrals

Example Show that

$$\int_0^T W_t \, dW_t = \frac{W_T^2}{2} - \frac{T}{2}$$

by applying Itô formula to $X_t = f(W_t) = W_t^2$.

Example Apply Itô formula to the function $X_t = f(t, W_t) = tW_t$, and show that

$$\int_0^T W_t \, dt = TW_T - \int_0^T t \, dW_t = \int_0^T (T - t) \, dW_t.$$

Use this to show that

$$V\left[\int_0^T W_t \, dt\right] = \frac{T^3}{3}.$$

Mackerel Three Ways

It's a beautiful fish to work with. It has delicate flavour, yet can take strong accompaniments.

— Michel Roux Jr.

$$\mathbb{E}[e^{\theta W_t}]$$

Three Ways

It's a beautiful equation to work with. It is a simple expectation, yet illustrates so many different concepts.

— Master Quant

Evaluating $\mathbb{E}[e^{\theta W_t}]$

Let W_t denote a standard Brownian motion. What is the expectation $\mathbb{E}[e^{\theta W_t}]$? There are several ways we can obtain this result. For a start, we know that $W_t \sim N(0,t)$, so we can proceed by evaluating the expectation:

$$\mathbb{E}\left[e^{\theta W_t}\right] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\theta x} e^{-\frac{x^2}{2t}}\ dx,$$

By completing the square, we obtain

$$\mathbb{E}[e^{\theta W_t}] = e^{\frac{\theta^2}{2}t}.$$

Alternatively, we note that $W_t \sim \sqrt{t}N(0,1)$. Let $X \sim N(0,1)$, we have

$$\mathbb{E}\left[e^{\theta\sqrt{t}X}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\theta\sqrt{t}x} e^{-\frac{x^2}{2}}\ dx.$$

This will lead us to the same result. We can also obtain the result using Moment Generating Function (MGF) of normally distributed random variables.

Evaluating $\mathbb{E}[e^{\theta W_t}]$

Now we shall introduce yet another method using Itô's formula. Consider the function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=e^{\theta x}$. Define a new stochastic variable $X_t=f(W_t)=e^{\theta W_t}$. By Itô's formula,

$$\begin{aligned} dX_t &= \theta X_t dW_t + \frac{1}{2}\theta^2 X_t dt \\ \Rightarrow \quad X_t &= X_0 + \theta\int_0^t X_u dW_u + \frac{\theta^2}{2}\int_0^t X_u du. \end{aligned}$$

Taking expectation on both sides, we have

$$\mathbb{E}[X_t] = \mathbb{E}[X_0] + \frac{\theta^2}{2}\int_0^t \mathbb{E}[X_u]du.$$

Now differentiate both sides with respect to t, we have

$$\frac{d\mathbb{E}[X_t]}{dt} = \frac{\theta^2}{2}\mathbb{E}[X_t],$$

this is now an ordinary differential equation, and we can solve it by

$$\mathbb{E}[X_t] = \mathbb{E}[e^{\theta W_t}] = e^{\frac{\theta^2}{2}t}.$$

Stochastic Process from SDE

- Itô's formula allows us to generate SDEs from stochastic processes and vice versa.
- Some of the stochastic differential equations can be solved explicitly, so that we can write it down as a stochastic process.

Itô's formula allows us to determine such a solution. Consider the following stochastic differential equation

$$dX_t = \sigma X_t dW_t, \quad X_0 = 1.$$

We define the function $f : \mathbb{R} \to \mathbb{R}, f(x) = \log(x)$, and the stochastic process $Y_t = \log(X_t)$. Using Itô's formula, we obtain

$$dY_t = -\frac{1}{2}\sigma^2 dt + \sigma W_t \quad \Rightarrow \quad X_t = \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_t\right).$$

Chain Rule for Stochastic Calculus

Itô's formula allows us to derive the stochastic differential equations for a given functional expression of a stochastic process. Consider the exponential Brownian motion

$$X_t = \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_t\right).$$

To determine the stochastic differential equation satisfied by X_t, we can apply Itô's formula to the function $f:\mathbb{R}^2 \to \mathbb{R}, f(t,x) = e^{-\frac{1}{2}\sigma^2 t + \sigma x}$ to show that

$$dX_t = \sigma X_t dW_t.$$

Alternatively, chain rule applies as well, and we note that if we let

$$Y_t = -\frac{1}{2}\sigma^2 t + \sigma W_t \quad \Rightarrow \quad dY_t = -\frac{1}{2}\sigma^2 dt + \sigma dW_t,$$

then we can express X_t as $X_t = \exp(Y_t)$. We use Itô's formula with the function $f:\mathbb{R}\to\mathbb{R}, f(x) = e^x$ to derive the same result.

Product Rule for Stochastic Calculus

- In ordinary calculus, we have the product rule $(uv)' = u'v + uv'$.
- We can derive the corresponding product rule for stochastic processes.
- Let X_t and Y_t be stochastic processes with known SDEs, what is the SDEs satisfied by the stochastic process $Z_t = X_t Y_t$?

Consider the smooth function $f : \mathbb{R}^2 \to \mathbb{R}, f(x,y) = xy$. The derivatives are given by

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial^2 f}{\partial x \partial y} = 1, \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0.$$

It follows from Itô's formula that

$$dZ_t = d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t.$$

Product Rule — Different Brownian Motions

Suppose two stochastic processes X_t and Y_t are adapted to the same Brownian motion such that

$$dX_t = \mu_X dt + \sigma_X dW_t, \qquad dY_t = \mu_Y dt + \sigma_Y dW_t,$$

then by applying Itô's formula to $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = xy$, $Z_t = f(X_t, Y_t)$, we arrive at

$$dZ_t = X_t dY_t + Y_t dX_t + \sigma_X \sigma_Y dt.$$

On the other hand, if X_t and Y_t are adapted to two different and independent Brownian motions such that

$$dX_t = \mu_X dt + \sigma_X dW_t, \qquad dY_t = \mu_Y dt + \sigma_Y d\tilde{W}_t,$$

then Itô's formula yields a result identical to the case in ordinary calculus

$$dZ_t = d(X_t Y_t) = X_t dY_t + Y_t dX_t.$$

Question What if they are adapted to two Brownian motions with a correlation of ρ?

Box Calculus Rules

Notice that in all our derivation for stochastic calculus so far, we have used the rule that if

$$dY_t = \mu dt + \sigma dW_t,$$

then

$$(dY_t)^2 = \sigma^2 dt.$$

This rule is based on the formalism of box calculus, which states that

$$\begin{aligned}(dt)^2 &= 0\\(dt dW_t) &= (dW_t dt) = 0\\(dW_t)^2 &= dt.\end{aligned}$$

This allows us to handle and evaluate the product of terms as follow:

$$(a dt + b dW_t)(\alpha dt + \beta dW_t) = b\beta dt.$$