Session 3

ℚuantitative Analysis of Financial Markets

A Derivation of the Capital Asset Pricing Model

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LKCSB 5036

October 20, 2018

Broad Lesson Plan

Background

- Markowitz (1959) model suggests that investors choose a portfolio that will minimize the variance of portfolio return, given a specific level of expected return, or maximize expected return, given a specific level of variance.

- Sharpe (1964) and Lintner (1965) introduce two (hypothetical) constructs:
  - **risk-less instrument**
  - **risky market portfolio**

New result of CAPM

Expected rate of excess return is proportional to the market risk premium, without having to do mean-variance optimization.

Assumptions

- No taxes
- No transaction costs
- All investors are risk-averse.
- All investors know their utility function of terminal wealth.
- All investors can maximize their utility function.
- All investors can choose among portfolios solely on the basis of mean and variance.
- all investors have homogeneous views regarding the parameters of the joint probability distribution of all security returns.
- All investors can borrow and lend at a given risk-less rate of interest.

Portfolio Construction, Expected Return

- Consider a portfolio with w portion invested in an asset i of expected return $r_i := \mathbb{E}(r_{i,t})$ and $1-w$ portion invested in the market portfolio of expected return $r_m := \mathbb{E}\left(r_{m,t}\right)$.

- The return of this portfolio, denoted by $r_{w,t}$, is a weighted average of $r_{i,t}$ and $r_{m,t}$.

$$r_{w,t} = w r_{i,t} + (1-w) r_{m,t}. \tag{1}$$

- By the linear property of the expectation operator $\mathbb{E}\left(\ \cdot\ \right)$, the expected return of this portfolio is

$$r_w = w r_i + (1-w) r_m. \tag{2}$$

Lemma: Variance of $aX + bY$

- Let $\mu_X = \mathbb{E}(X)$, $\mu_Y = \mathbb{E}(Y)$, and $\mu_Z = \mathbb{E}(Z)$.
- Let $Z := aX + bY$.
- $\mu_Z = \mathbb{E}(Z) = a\,\mathbb{E}(X) + b\,\mathbb{E}(Y) = a\mu_X + b\mu_Y$
- The definition of variance of is

$$
\begin{aligned}
\mathbb{V}\left(Z\right) &= \mathbb{E}\left((Z-\mu_Z)^2\right) = \mathbb{E}\left((aX - a\mu_X + bY - b\mu_Y)^2\right) \\
&= \mathbb{E}\left((aX-a\mu_X)^2 + (bY-b\mu_Y)^2 + 2(aX-a\mu_X)(bY-b\mu_Y)\right) \\
&= a^2\,\mathbb{E}\left((X-\mu_X)^2\right) + b^2\,\mathbb{E}\left((Y-\mu_Y)^2\right) \\
&\qquad + 2ab\,\mathbb{E}\left((X-\mu_X)(Y-\mu_Y)\right) \\
&= a^2\,\mathbb{V}(X) + b^2\,\mathbb{V}(Y) + 2ab\,\mathbb{C}(X,Y).
\end{aligned}
$$

Variance of Portfolio's Return

- To (1), apply the variance operator $\mathbb{V}\left(\ \cdot\ \right)$. Using the lemma, we get

$$\mathbb{V}\left(r_{w,t}\right) = w^2\,\mathbb{V}\left(r_{i,t}\right) + (1-w)^2\,\mathbb{V}\left(r_{m,t}\right) + 2w(1-w)\,\mathbb{C}\left(r_{i,t}, r_{m,t}\right).$$

- For convenience, we denote
  - § $\sigma_w^2 := \mathbb{V}\left(r_{w,t}\right)$, $\quad\sigma_i^2 := \mathbb{V}\left(r_{i,t}\right)$ and $\sigma_m^2 = \mathbb{V}\left(r_{m,t}\right)$
  - § The covariance $\sigma_{im} := \mathbb{C}\left(r_{i,t}, r_{m,t}\right)$.

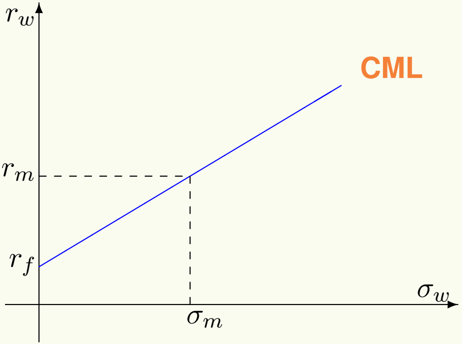
- With these notations, the variance $\mathbb{V}\left(r_{w,t}\right)$ simplifies to

$$\sigma_w^2 = w^2\sigma_i^2 + 2w(1-w)\sigma_{im} + (1-w)^2\sigma_m^2. \tag{3}$$

Slope of Capital Market Line

- The slope of the CML is the Sharpe ratio. At $w = 0$, or for σ_m, we have

$$\frac{r_m - r_f}{\sigma_m} = \left.\frac{dr_w}{d\sigma_w}\right|_{w=0}.$$



Derivation of Slope

- It is tedious to compute $\dfrac{dr_w}{d\sigma_w}$ directly.
- Instead, we have, by chain rule,

$$\frac{dr_w}{d\sigma_w} = \frac{\dfrac{dr_w}{dw}}{\dfrac{d\sigma_w}{dw}}.$$

- From (2), we obtain $\dfrac{dr_w}{dw} = r_i - r_m$.
- From (3), we obtain

$$2\sigma_w \frac{d\sigma_w}{dw} = 2w\sigma_i^2 + 2(1-2w)\sigma_{im}, -2(1-w)\sigma_m^2,$$

equivalently,

$$\frac{d\sigma_w}{dw} = \frac{w\sigma_i^2 + (1-2w)\sigma_{im} - (1-w)\sigma_m^2}{\sigma_w}.$$

Slope at $w = 0$

- Putting everything together,

$$\frac{dr_w}{d\sigma_w} = \frac{\dfrac{dr_w}{dw}}{\dfrac{d\sigma_w}{dw}} = \frac{r_i - r_m}{\dfrac{w\sigma_i^2 + (1-2w)\sigma_{im} - (1-w)\sigma_m^2}{\sigma_w}}.$$

- At $w = 0$, $\sigma_w = \sigma_m$. Moreover, given that the slope is the Sharpe ratio, we have

$$\frac{r_m - r_f}{\sigma_m} = \frac{r_i - r_m}{\left(\dfrac{\sigma_{im} - \sigma_m^2}{\sigma_m}\right)}$$

$$r_m - r_f = \frac{r_i - r_m}{\left(\dfrac{\sigma_{im} - \sigma_m^2}{\sigma_m^2}\right)} = \frac{r_i - r_m}{\left(\dfrac{\sigma_{im}}{\sigma_m^2} - 1\right)}.$$

Slope at $w = 0$ (Cont'd)

- For any asset i that is not a market portfolio, $\frac{\sigma_{im}}{\sigma_m^2} - 1 \neq 0$. So we multiple it to both sides to obtain

$$\begin{aligned} (r_m - r_f)\left(\frac{\sigma_{im}}{\sigma_m^2} - 1\right) &= r_i - r_m, \\ \Longrightarrow \qquad \frac{\sigma_{im}}{\sigma_m^2}(r_m - r_f) - (r_m - r_f) &= r_i - r_m. \end{aligned}$$

- Knowing that $\frac{\sigma_{im}}{\sigma_m^2} = \beta_i$, we write,

$$\beta_i(r_m - r_f) = (r_m - r_f) + r_i - r_m = r_i - r_f.$$

- Hence CAPM ensues:

$$r_i - r_f = \beta_i(r_m - r_f). \tag{4}$$

Security Market Line

- From (4), we obtain **the security market line**:

$$r_i = r_f + (r_m - r_f)\beta_i. \tag{5}$$

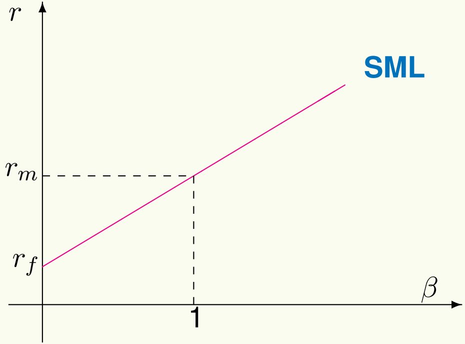
- It shows you the relationship between β_i and the required return r_i.

- It is a useful tool in determining if an asset being considered for a portfolio offers a reasonable expected return for risk.

- If the security's (β_i, r_i) is plotted above (below) the SML, it is considered undervalued (overvalued), and the security gives a greater (smaller) return against its inherent risk.

Undervalued versus Overvalued

- From (5), we see that the market portfolio has $\beta_m = 1$.
- You can easily draw an SML with two points $(0, r_f)$ and $(1, r_m)$.
- Now you know why the risk-less rate is so important.



Treynor Ratio and Alpha

- All of the portfolios on the SML have the same Treynor ratio as does the market portfolio, i.e.

$$\frac{r_i - r_f}{\beta_i} = \frac{r_m - r_f}{1} = \text{slope of the SML.}$$

- A **stock picking** rule of thumb for assets with positive beta is to buy if the Treynor ratio will be above the SML and sell if it will be below.

- The abnormal extra return above the market's return at a given level of risk is what is called the **alpha**.

Takeaways

1. CAPM says that expected excess return $r_i - r_f$ on the portfolio is proportional to the expected excess return $r_m - r_f$ on the market portfolio.

2. The beta β_i of portfolio i is essentially the covariance between the portfolio and the market, and normalized by the variance of the market portfolio.

3. Expected excess return $r_m - r_f$ is also known as the market risk premium.

4. Duality of risk and expected return

RISK FACTOR $\longleftrightarrow$ RISK PREMIUM