QF620 Additional Examples
Session 8: Barrier Options

1 Examples

1. Let W_t denote a Brownian motion, and $m_t = \min_{[0,t]} W_t$ denote the minimum value taken by W_t in the period $[0, t]$. Show that

   (a)
$$\mathbb{P}(W_t > x) = \Phi\left(-\frac{x}{\sqrt{t}}\right)$$

   (b)
$$\mathbb{P}(m_t \leq y, W_t > x) = \Phi\left(\frac{2y - x}{\sqrt{t}}\right)$$

   (c)
$$\mathbb{P}(m_t > y, W_t > x) = \Phi\left(-\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{2y - x}{\sqrt{t}}\right)$$

2. Let W_t denote a Brownian motion, and $M_t = \max_{[0,t]} W_t$ denote the maximum value taken by W_t in the period $[0, t]$. Show that

   (a)
$$\mathbb{P}(W_t \leq x) = \Phi\left(\frac{x}{\sqrt{t}}\right)$$

   (b)
$$\mathbb{P}(M_t > y, W_t \leq x) = \Phi\left(\frac{x - 2y}{\sqrt{t}}\right)$$

   (c)
$$\mathbb{P}(M_t \leq y, W_t \leq x) = \Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{x - 2y}{\sqrt{t}}\right)$$

3. Consider the following SDE
$$dZ_t = \mu dt + \sigma dW_t.$$

   Let $\mathbb{Q}$ denote the measure under which W_t is a Brownian motion. Let $\tilde{\mathbb{Q}}$ denote the probability measure where Z_t is driftless. We have

$$\begin{aligned} dZ_t &= \sigma\left(dW_t + \frac{\mu}{\sigma}dt\right) \\ &= \sigma d\tilde{W}_t, \end{aligned}$$

   with the Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \exp\left(-\frac{1}{2}\kappa^2 T - \kappa W_T\right), \quad \kappa = \frac{\mu}{\sigma}, \quad d\tilde{W}_t = dW_t + \kappa dt.$$

(a) Consider the expectation

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{m_t>y, Z_t>x}\right] &= \tilde{\mathbb{E}}\left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{Q}}}\mathbb{1}_{m_t>y, Z_t>x}\right] \\ &= \frac{1}{\sqrt{2\pi t}\sigma}\int_x^\infty \left(e^{-\frac{x^2}{2\sigma^2 t}} - e^{-\frac{(2y-x)^2}{2\sigma^2 t}}\right) e^{-\frac{1}{2}\kappa^2 t + \kappa\frac{x}{\sigma}}\, dx\end{aligned}$$

By completing the square, show that

$$\mathbb{E}\left[\mathbb{1}_{m_t>y, Z_t>x}\right] = \Phi\left(\frac{\mu t - x}{\sigma\sqrt{t}}\right) - e^{2\mu y\sigma^{-2}}\Phi\left(\frac{2y - x + \mu t}{\sigma\sqrt{t}}\right).$$

(b) Consider the expectation

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{m_t\leq y, Z_t\leq x}\right] &= \tilde{\mathbb{E}}\left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{Q}}}\mathbb{1}_{m_t\leq y, Z_t\leq x}\right] \\ &= \frac{1}{\sqrt{2\pi t}\sigma}\int_{-\infty}^x \left(e^{-\frac{x^2}{2\sigma^2 t}} - e^{-\frac{(x-2y)^2}{2\sigma^2 t}}\right) e^{-\frac{1}{2}\kappa^2 t + \kappa\frac{x}{\sigma}}\, dx\end{aligned}$$

By completing the square, show that

$$\mathbb{E}\left[\mathbb{1}_{m_t\leq y, Z_t\leq x}\right] = \Phi\left(\frac{x - \mu t}{\sigma\sqrt{t}}\right) - e^{2\mu y\sigma^{-2}}\Phi\left(\frac{x - 2y - \mu t}{\sigma\sqrt{t}}\right).$$

2 Suggested Solutions

1. (a)

$$\begin{aligned}\mathbb{P}(W_t > x) &= 1 - \mathbb{P}(W_t \le x)\\ &= 1 - \mathbb{P}\left(N(0,1) \le \frac{x}{\sqrt{t}}\right)\\ &= \Phi\left(-\frac{x}{\sqrt{t}}\right)\end{aligned}$$

(b)

$$\begin{aligned}\mathbb{P}(m_t \le y, W_t > x) &= \mathbb{P}(W_t \le 2y - x)\\ &= \mathbb{P}\left(N(0,1) \le \frac{2y-x}{\sqrt{t}}\right)\\ &= \Phi\left(\frac{2y-x}{\sqrt{t}}\right)\end{aligned}$$

(c)

$$\begin{aligned}\mathbb{P}(W_t > x) &= \mathbb{P}(m_t \le y, W_t > x) + \mathbb{P}(m_t > y, W_t > x)\\ \Rightarrow \quad \mathbb{P}(m_t > y, W_t > x) &= \Phi\left(-\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{2y-x}{\sqrt{t}}\right)\end{aligned}$$

2. (a)

$$\mathbb{P}(W_t \le x) = \Phi\left(\frac{x}{\sqrt{t}}\right)$$

(b)

$$\begin{aligned}\mathbb{P}(M_t > y, W_t \le x) &= \mathbb{P}(W_t > 2y - x)\\ &= \Phi\left(\frac{x-2y}{\sqrt{t}}\right)\end{aligned}$$

(c)

$$\begin{aligned}\mathbb{P}(W_t \le x) &= \mathbb{P}(M_t > y, W_t \le x) + \mathbb{P}(M_t \le y, W_t \le x)\\ \Rightarrow \quad \mathbb{P}(M_t \le y, W_t \le x) &= \Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{x-2y}{\sqrt{t}}\right)\end{aligned}$$

3. Here we cover the general derivation (without normalising σ to 1). First note that

$$\begin{aligned}\mathbb{P}(N(0,1) \le y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}}\, dx\\ \mathbb{P}(N(0,t) \le y) &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{y} e^{-\frac{x^2}{2t}}\, dx\\ \mathbb{P}(N(0,\sigma^2 t) \le y) &= \frac{1}{\sqrt{2\pi t}\sigma} \int_{-\infty}^{y} e^{-\frac{x^2}{2\sigma^2 t}}\, dx.\end{aligned}$$

(a)

$$\begin{aligned}\tilde{\mathbb{E}}\left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{Q}}} \mathbb{1}_{m_t > y, Z_t > x}\right] &= \frac{1}{\sqrt{2\pi t}\sigma} \int_{x}^{\infty} e^{-\frac{1}{2}\kappa^2 t + \kappa \frac{x}{\sigma}} \left(e^{-\frac{x^2}{2\sigma^2 t}} - e^{-\frac{(2y-x)^2}{2\sigma^2 t}}\right) dx\\ &= \frac{1}{\sqrt{2\pi t}\sigma}\left[\int_{x}^{\infty}\left(e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}} dx - e^{2\mu y \sigma^{-2}} \int_{x}^{\infty} e^{-\frac{(x-2y-\mu t)^2}{2\sigma^2 t}}\right) dx\right]\\ &= \Phi\left(\frac{\mu t - x}{\sigma\sqrt{t}}\right) - e^{2\mu y \sigma^{-2}} \Phi\left(\frac{2y - x + \mu t}{\sigma\sqrt{t}}\right).\end{aligned}$$

(b)

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{Q}}}\mathbb{1}_{m_t\le y, Z_t\le x}\right] &= \frac{1}{\sqrt{2\pi t}\sigma}\int_{-\infty}^{x} e^{-\frac{1}{2}\kappa^2 t+\kappa\frac{x}{\sigma}}\left(e^{-\frac{x^2}{2\sigma^2 t}} - e^{-\frac{(x-2y)^2}{2\sigma^2 t}}\right)dx \\
&= \frac{1}{\sqrt{2\pi t}\sigma}\left[\int_{-\infty}^{x}\left(e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}dx - e^{2\mu y\sigma^{-2}}\int_{-\infty}^{x} e^{-\frac{(x-2y-\mu t)^2}{2\sigma^2 t}}\right)dx\right] \\
&= \Phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right) - e^{2\mu y\sigma^{-2}}\Phi\left(\frac{x-2y-\mu t}{\sigma\sqrt{t}}\right).
\end{aligned}$$

In both cases, we've used the complete square method as follow

$$\begin{aligned}
-\frac{x^2}{2\sigma^2 t} - \frac{\kappa^2 t}{2} + \frac{\kappa x}{\sigma} &= -\frac{x^2+\kappa^2\sigma^2 t^2 - 2\kappa x\sigma t}{2\sigma^2 t} = -\frac{(x-\kappa\sigma t)^2}{2\sigma^2 t} \\
-\frac{(2y-x)^2}{2\sigma^2 t} - \frac{\kappa^2 t}{2} + \frac{\kappa x}{\sigma} &= -\frac{4y^2-4xy+x^2+\kappa^2\sigma^2 t^2 - 2\sigma t\kappa x}{2\sigma^2 t} \\
&= -\frac{x^2-2x(2y+\sigma t\kappa)+4y^2+\kappa^2\sigma^2 t^2+4y\kappa\sigma t-4y\kappa\sigma t}{2\sigma^2 t} \\
&= -\frac{x^2-2x(2y+\sigma t\kappa)+(2y+\sigma t\kappa)^2-4y\kappa\sigma t}{2\sigma^2 t} \\
&= -\frac{(x-2y-\sigma t x)^2}{2\sigma^2 t} + \frac{2y\kappa}{\sigma}.
\end{aligned}$$