Linear Factor Models

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Efficient Frontier with Riskless Asset

- $n \geq 2$ risky assets and riskless asset with return R_f
- Let $\mathbf{R}$ be $n \times 1$ vector of expected returns for risky assets
- Let $\mathbf{V}$ be $n \times n$ is covariance matrix of returns for risky assets
- Let $\mathbf{w}$ be $n \times 1$ vector of portfolio weights for risky assets
- Weights for frontier portfolio with expected return R_p:

$$\mathbf{w} = \lambda \mathbf{V}^{-1} \left(\mathbf{R} - R_f \mathbf{e} \right); \qquad \lambda = \frac{R_p - R_f}{\zeta - 2\alpha R_f + \delta R_f^2};$$

$$\alpha = \mathbf{R}' \mathbf{V}^{-1} \mathbf{e}; \qquad \zeta = \mathbf{R}' \mathbf{V}^{-1} \mathbf{R}; \qquad \delta = \mathbf{e}' \mathbf{V}^{-1} \mathbf{e}$$

Tangency Portfolio

- Efficient frontier is also called **capital market line (CML)**:

$$R_p = R_f + \left(\zeta - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}} \sigma_p$$

- Let $\mathbf{w}_m$ be portfolio weights for frontier portfolio with zero weight in riskless asset, so $\mathbf{e}'\mathbf{w}_m = 1$:

$$\mathbf{w}_m = \lambda_m \mathbf{V}^{-1} \left(\mathbf{R} - R_f \mathbf{e}\right); \qquad \lambda_m = \frac{1}{\alpha - \delta R_f}$$

- "Tangency" portfolio is point where efficient frontier is tangent to risky-asset-only frontier when $R_f < R_{mv} = \alpha/\delta$

Capital Asset Pricing Model

- Assume that all investors hold mean-variance efficient portfolios, agree on **R** and **V**, and can borrow and lend at R_f
- Then tangency portfolio represents aggregate portfolio of risky assets, or **market portfolio**
- Previously, assumed that all investors are "price takers", in sense that allocation choices have no effect on asset prices
- Now assume that asset prices adjust to produce **market equilibrium**, where supply of risky assets equals demand
- This **capital asset pricing model (CAPM)** is example of **endowment economy**, in which asset supply is fixed and asset prices (and returns) adjust to produce market equilibrium

Security Market Line

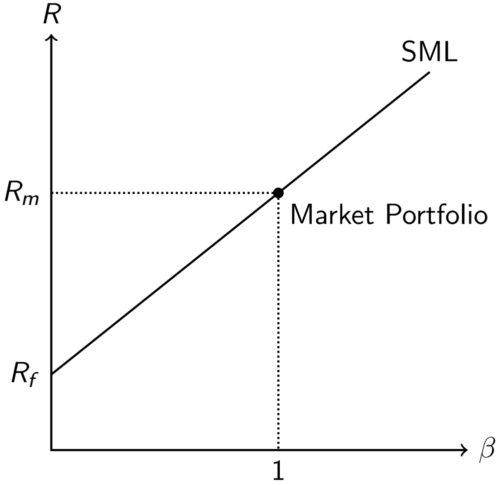
- Let R_m be expected market return, and let $\vec{\sigma}_m$ be $n \times 1$ vector of covariances between asset returns and market return:

$$\vec{\sigma}_m = \mathbf{V}\mathbf{w}_m = \lambda_m (\mathbf{R} - R_f\mathbf{e})\,;$$
$$\sigma_m^2 = \mathbf{w}_m'\mathbf{V}\mathbf{w}_m = \lambda_m (R_m - R_f)$$

- Divide and rearrange to get **security market line (SML)**:

$$\mathbf{R} - R_f\mathbf{e} = \frac{\vec{\sigma}_m}{\sigma_m^2}(R_m - R_f) = \vec{\beta}(R_m - R_f)$$

- Here $\vec{\beta}$ is $n \times 1$ vector such that $\beta_i \equiv \sigma_{im}/\sigma_m^2$, where σ_{im} is covariance of return between i'th asset and market portfolio


R
SML
R_m
Market Portfolio
R_f
β
1

Realised Returns – Part 1

- Let $\tilde{\nu}_i$ and $\tilde{\nu}_m$ be unexpected components of i'th asset return and market return respectively, and use result for CAPM:

$$\begin{aligned}\tilde{R}_i &= R_i + \tilde{\nu}_i \\ &= R_f + \beta_i \left(R_m - R_f\right) + \tilde{\nu}_i \\ &= R_f + \beta_i \left(\tilde{R}_m - \tilde{\nu}_m - R_f\right) + \tilde{\nu}_i\end{aligned}$$

- Define $\tilde{\epsilon}_i = \tilde{\nu}_i - \beta_i \tilde{\nu}_m$ to obtain relation between realised asset return and contemporaneous realised market return:

$$\tilde{R}_i - R_f = \beta_i \left(\tilde{R}_m - R_f\right) + \tilde{\epsilon}_i$$

Realised Returns – Part 2

- Here $\tilde{R}_m$ represents **risk factor** for (systematic) market risk, which captures effect of market risk on realised asset return
- Hence β_i represents degree of asset's exposure to market risk
- Notice that $\tilde{\epsilon}_i$ is uncorrelated with realised market return:

$$\begin{aligned}\mathrm{Cov}\left(\tilde{R}_m, \tilde{\epsilon}_i\right) &= \mathrm{Cov}\left(\tilde{R}_m, \tilde{\nu}_i\right) - \beta_i \mathrm{Cov}\left(\tilde{R}_m, \tilde{\nu}_m\right) \\ &= \mathrm{Cov}\left(\tilde{R}_m, \tilde{R}_i\right) - \beta_i \mathrm{Cov}\left(\tilde{R}_m, \tilde{R}_m\right) \\ &= 0\end{aligned}$$

- Hence $\tilde{\epsilon}_i$ represents risk factor for idiosyncratic risk

Market Model

- Allow for non-zero intercept to obtain **market model**:

$$\tilde{R}_i - R_f = \alpha_i + \beta_i \left(\tilde{R}_m - R_f \right) + \tilde{\epsilon}_i$$

- Notice that market model represents linear regression model, with (excess) asset return as dependent variable and (excess) market return as explanatory variable
- Slope coefficient from market model regression provides convenient estimate of asset beta, while intercept coefficient provides empirical test of CAPM

Idiosyncratic Risk

- Decompose variance of asset return into systematic and idiosyncratic portions:

$$\mathrm{Var}\left(\tilde{R}_i\right) = \mathrm{Var}\left(\beta_i \tilde{R}_m + \tilde{\epsilon}_i\right) = \beta_i^2 \sigma_m^2 + \sigma_{\epsilon_i}^2$$

- Idiosyncratic risk can be almost eliminated by combining individual assets into well-diversified portfolio
- Optimal for investors to hold combination of market portfolio and riskless asset, which avoids exposure to idiosyncratic risk
- Hence idiosyncratic risk will not be "priced", in sense that investors will not be compensated for bearing idiosyncratic risk

Market Price of Risk – Part 1

- Let $\rho_{im} = \sigma_{im}/\sigma_i\sigma_m$ be correlation of return between asset i and market portfolio, so that $\beta_i = \rho_{im}\sigma_i/\sigma_m$:

$$R_i - R_f = \rho_{im}\sigma_i \left(\frac{R_m - R_f}{\sigma_m} \right) = \rho_{im}\sigma_i S_m$$

- Here $S_m = (R_m - R_f)/\sigma_m$ is Sharpe ratio of market portfolio, which represents market price of systematic risk
- Let w_{mi} be i'th element of $\mathbf{w}_m$ and let $\mathbf{v}_i$ be i'th row of $\mathbf{V}$:

$$\frac{\partial \sigma_m^2}{\partial w_{mi}} = \frac{\partial \mathbf{w}'_m \mathbf{V} \mathbf{w}_m}{\partial w_{mi}} = 2\mathbf{v}_i \mathbf{w}_m = 2 \sum_{j=1}^{n} w_{mj}\sigma_{ij}$$

Market Price of Risk – Part 2

- Use $\tilde{R}_m = \sum_{j=1}^{n} w_{mj} \tilde{R}_j$ to determine covariance of returns:

$$\sigma_{im} = \mathrm{Cov}\left(\tilde{R}_i, \tilde{R}_m\right) = \mathrm{Cov}\left(\tilde{R}_i, \sum_{j=1}^{n} w_{mj} \tilde{R}_j\right) = \sum_{j=1}^{n} w_{mj} \sigma_{ij}$$

- Hence $\rho_{im}\sigma_i$ represents marginal increase in (systematic) market risk from marginal increase in weight on asset i:

$$\frac{\partial \sigma_m}{\partial w_{mi}} = \frac{1}{2\sigma_m} \frac{\partial \sigma_m^2}{\partial w_{mi}} = \frac{1}{\sigma_m} \sum_{j=1}^{n} w_{mj} \sigma_{ij} = \frac{\sigma_{im}}{\sigma_m} = \rho_{im} \sigma_i$$

CAPM without Riskless Asset – Part 1

- If all investors hold mean-variance efficient portfolios, then market portfolio will also be mean-variance efficient:

$$\mathbf{w}_m = \frac{\zeta \mathbf{V}^{-1}\mathbf{e} - \alpha \mathbf{V}^{-1}\mathbf{R}}{\zeta\delta - \alpha^2} + \left(\frac{\delta \mathbf{V}^{-1}\mathbf{R} - \alpha \mathbf{V}^{-1}\mathbf{e}}{\zeta\delta - \alpha^2} \right) R_m$$

- Let $\tilde{R}_p$ be return for *any* portfolio of risky assets:

$$\begin{aligned} \mathrm{Cov}\left(\tilde{R}_p, \tilde{R}_m\right) = \mathbf{w}_p' \mathbf{V} \mathbf{w}_m &= \frac{\zeta - \alpha R_p}{\zeta\delta - \alpha^2} + \left(\frac{\delta R_p - \alpha}{\zeta\delta - \alpha^2} \right) R_m \\ &= \frac{\zeta - \alpha R_m}{\zeta\delta - \alpha^2} + \left(\frac{\delta R_m - \alpha}{\zeta\delta - \alpha^2} \right) R_p \end{aligned}$$

CAPM without Riskless Asset – Part 2

- Rearrange to get equation for portfolio expected return:

$$
\begin{aligned}
R_p &= \frac{\alpha R_m - \zeta}{\delta R_m - \alpha} + \mathrm{Cov}\Big(\tilde{R}_p, \tilde{R}_m\Big) \frac{\zeta\delta - \alpha^2}{\delta R_m - \alpha} \\
&= \frac{\alpha R_m - \zeta}{\delta R_m - \alpha} + \beta_p \left(\frac{\zeta\delta - \alpha^2}{\delta R_m - \alpha}\right) \sigma_m^2
\end{aligned}
$$

- Let R_{zm} be expected return for frontier portfolio that is orthogonal to market portfolio, and use $R_{mv} = \alpha/\delta$:

$$
R_{zm} = R_{mv} - \frac{\zeta\delta - \alpha^2}{\delta^2 \left(R_m - R_{mv}\right)} = \frac{\alpha R_m - \zeta}{\delta R_m - \alpha}
$$

CAPM without Riskless Asset – Part 3

- Use equation for variance of return for market portfolio:

$$\left(\frac{\zeta\delta - \alpha^2}{\delta R_m - \alpha}\right)\sigma_m^2 = \frac{\zeta\delta - \alpha^2}{\delta\left(R_m - R_{mv}\right)}\left[\frac{1}{\delta} + \frac{\delta\left(R_m - R_{mv}\right)^2}{\zeta\delta - \alpha^2}\right]$$
$$= R_m - R_{mv} + \frac{\zeta\delta - \alpha^2}{\delta^2\left(R_m - R_{mv}\right)}$$
$$= R_m - R_{zm}$$

- Substitute to get result of Black's **zero-beta CAPM**:

$$R_p = R_{zm} + \beta_p\left(R_m - R_{zm}\right)$$

Arbitrage Opportunity

- **Arbitrage portfolio** is portfolio that requires zero initial investment (i.e., long positions are financed by short positions)
- Arbitrage opportunity exists when payoff of arbitrage portfolio is (non-strictly) positive in all states, and strictly positive in at least one state
- If arbitrage portfolio provides riskless payoff, then arbitrage opportunity exists unless payoff is zero
- Equivalently, arbitrage opportunity exists unless riskless return on regular portfolio is equal to risk-free rate
- Absence of arbitrage implies **law of one price**: different assets with same payoffs must have same price

Arbitrage Pricing Theory

- **Arbitrage pricing theory (APT)** is single-period asset pricing model developed by Stephen Ross in 1976
- CAPM uses market equilibrium to derive derive asset-pricing relation, while APT uses absence of (asymptotic) arbitrage and law of one price
- Unlike CAPM, APT does not impose restrictions on investor preferences or return distribution
- Unlike CAPM, APT allows for multiple systematic risk factors (but also does not specify nature of systematic risk factors)
- Suppose there exist $k \geq 1$ systematic risk factors, which drive realised returns of $n > k$ linearly independent risky assets

Linear Factor Model

- Assume that realised asset returns follow linear factor model:

$$\tilde{R}_i = a_i + \sum_{z=1}^{k} b_{iz}\tilde{f}_z + \tilde{\epsilon}_i$$

- Here $\tilde{f}_z$ is mean-zero random realisation of z'th systematic risk factor, while $\tilde{\epsilon}_i$ is mean-zero random realisation of idiosyncratic risk factor for i'th asset
- Then a_i represents expected return for i'th asset, while b_{iz} represents sensitivity to (or "loading on") z'th risk factor

Risk Factors

- Assume that systematic risk factors have unit variance and are uncorrelated with one another:

$$E\left[\tilde{f}_z^2\right] = 1; \qquad E\left[\tilde{f}_x \tilde{f}_z\right] = 0 \quad \forall \quad x \neq z$$

- Idiosyncratic risk factors are uncorrelated with systematic risk factors, and also uncorrelated across assets:

$$E\left[\tilde{\epsilon}_i \tilde{f}_z\right] = 0; \qquad E[\tilde{\epsilon}_i \tilde{\epsilon}_j] = 0 \quad \forall \quad i \neq j$$

- Assume that idiosyncratic risk is finite: $E\left[\tilde{\epsilon}_i^2\right] = s_i^2 < S^2 \ \forall \ i$

Asymptotic Arbitrage – Part 1

- Suppose that economy contains n risky assets
- Let a_i be expected return on i'th asset
- Let σ_{ij} be covariance of return between i'th and j'th assets
- Construct arbitrage portfolio with amount W_i^n invested in each asset, such that $\sum_{i=1}^n W_i^n = 0$
- Consider sequence of arbitrage portfolios for $n = 2, 3, \ldots$
- Asymptotic arbitrage opportunity exists when:
  - Portfolio payoff becomes certain as $n \to \infty$, and
  - Portfolio expected payoff is always bounded above zero

Asymptotic Arbitrage – Part 2

- Portfolio payoff becomes certain when variance of portfolio payoff disappears as n becomes large:

$$\lim_{n\to\infty}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} W_i^n W_j^n \sigma_{ij}\right\} = 0$$

- Portfolio expected payoff is always bounded above zero:

$$\sum_{i=1}^{n} W_i^n a_i > 0 \qquad \forall \qquad n \geq 2$$

APT Pricing Rule – Part 1

- If no asymptotic arbitrage opportunities exist, then expected asset returns must be given by:

$$a_i = \lambda_0 + \sum_{z=1}^{k} b_{iz}\lambda_z + \nu_i$$

- Here λ_z represents risk premium for z'th risk factor, while ν_i represents "pricing error" for i'th asset
- If riskless asset exists, then λ_0 is approximately equal to risk-free rate: $R_f = \lambda_0 + \nu_f$

APT Pricing Rule – Part 2

- Moreover, pricing errors must sum to zero, and be orthogonal to factor sensitivities:

$$\sum_{i=1}^{n} \nu_i = 0; \qquad \sum_{i=1}^{n} b_{iz}\nu_i = 0 \quad \forall \quad z = 1, \ldots, k$$

- Finally, mean squared pricing errors must disappear as n becomes large:

$$\lim_{n\to\infty} \left\{ \frac{1}{n} \sum_{i=1}^{n} \nu_i^2 \right\} = 0$$

Proof of APT Pricing Rule – Part 1

- Consider regression of $\mathbf{a} = (a_1, \ldots, a_n)'$ on set of explanatory variables given by $\mathbf{b}_z = (b_{1z}, \ldots, b_{nz})'$ for $z = 1, \ldots, k$:

$$\mathbf{a} = \lambda_0 + \sum_{z=1}^{k} \lambda_z \mathbf{b}_z + \vec{\nu}$$

- Here λ_0 is intercept coefficient while λ_z's are slope coefficients
- Then $\vec{\nu} = (\nu_1, \ldots, \nu_n)'$ is vector of regression residuals:

$$\sum_{i=1}^{n} \nu_i = 0; \qquad \sum_{i=1}^{n} b_{iz}\nu_i = 0 \quad \forall \quad z = 1, \ldots, k$$

Proof of APT Pricing Rule – Part 2

- Consider arbitrage portfolio where amount of investment is proportional to relative pricing error:

$$W_i^n = \frac{\nu_i}{\sqrt{n \sum_{i=1}^n \nu_i^2}}$$

- Use $\sum_{i=1}^n b_{iz} \nu_i = 0$ to get payoff for arbitrage portfolio:

$$\tilde{R}_p = \sum_{i=1}^n W_i^n \tilde{R}_i = \frac{1}{\sqrt{n \sum_{i=1}^n \nu_i^2}} \left[\sum_{i=1}^n \nu_i \left(a_i + \tilde{\epsilon}_i \right) \right]$$

Proof of APT Pricing Rule – Part 3

- Use $E[\tilde{\epsilon}_i] = 0$ to get expected payoff for arbitrage portfolio:

$$E\left[\tilde{R}_p\right] = \frac{1}{\sqrt{n\sum_{i=1}^{n}\nu_i^2}}\left[\sum_{i=1}^{n}\nu_i a_i\right]$$

- Substitute for a_i and use $\sum_{i=1}^{n}\nu_i = \sum_{i=1}^{n} b_{iz}\nu_i = 0$:

$$E\left[\tilde{R}_p\right] = \frac{1}{\sqrt{n\sum_{i=1}^{n}\nu_i^2}}\left[\sum_{i=1}^{n}\nu_i^2\right] = \left[\frac{1}{n}\sum_{i=1}^{n}\nu_i^2\right]^{\frac{1}{2}} > 0$$

Proof of APT Pricing Rule – Part 4

- Deviation from mean for payoff of arbitrage portfolio:

$$\tilde{R}_p - E\left[\tilde{R}_p\right] = \frac{1}{\sqrt{n\sum_{i=1}^{n}\nu_i^2}}\left[\sum_{i=1}^{n}\nu_i\tilde{\epsilon}_i\right]$$

- Use $E[\tilde{\epsilon}_i\tilde{\epsilon}_j] = 0$ and $E\left[\tilde{\epsilon}_i^2\right] = s_i^2$ to get variance of deviation from mean for payoff of arbitrage portfolio:

$$E\left[\left(\tilde{R}_p - E\left[\tilde{R}_p\right]\right)^2\right] = \frac{\sum_{i=1}^{n}\nu_i^2 s_i^2}{n\sum_{i=1}^{n}\nu_i^2} < \frac{S^2}{n}$$

Proof of APT Pricing Rule – Part 5

- Variance of deviation from mean disappears as n becomes large, so payoff of arbitrage portfolio becomes riskless:

$$\lim_{n\to\infty} \tilde{R}_p = E\left[\tilde{R}_p\right] = \left[\frac{1}{n}\sum_{i=1}^{n} \nu_i^2\right]^{\frac{1}{2}}$$

- Absence of asymptotic arbitrage requires that payoff be zero:

$$\lim_{n\to\infty} \left\{\frac{1}{n}\sum_{i=1}^{n} \nu_i^2\right\} = 0$$