pg 28.

$$\sum_{i=1}^{n} \left(W_{t_i} - W_{t_{i-1}} \right)^2 = \left(W_{t_1} - W_{t_0} \right)^2 + \left(W_{t_2} - W_{t_1} \right)^2 + \dots$$

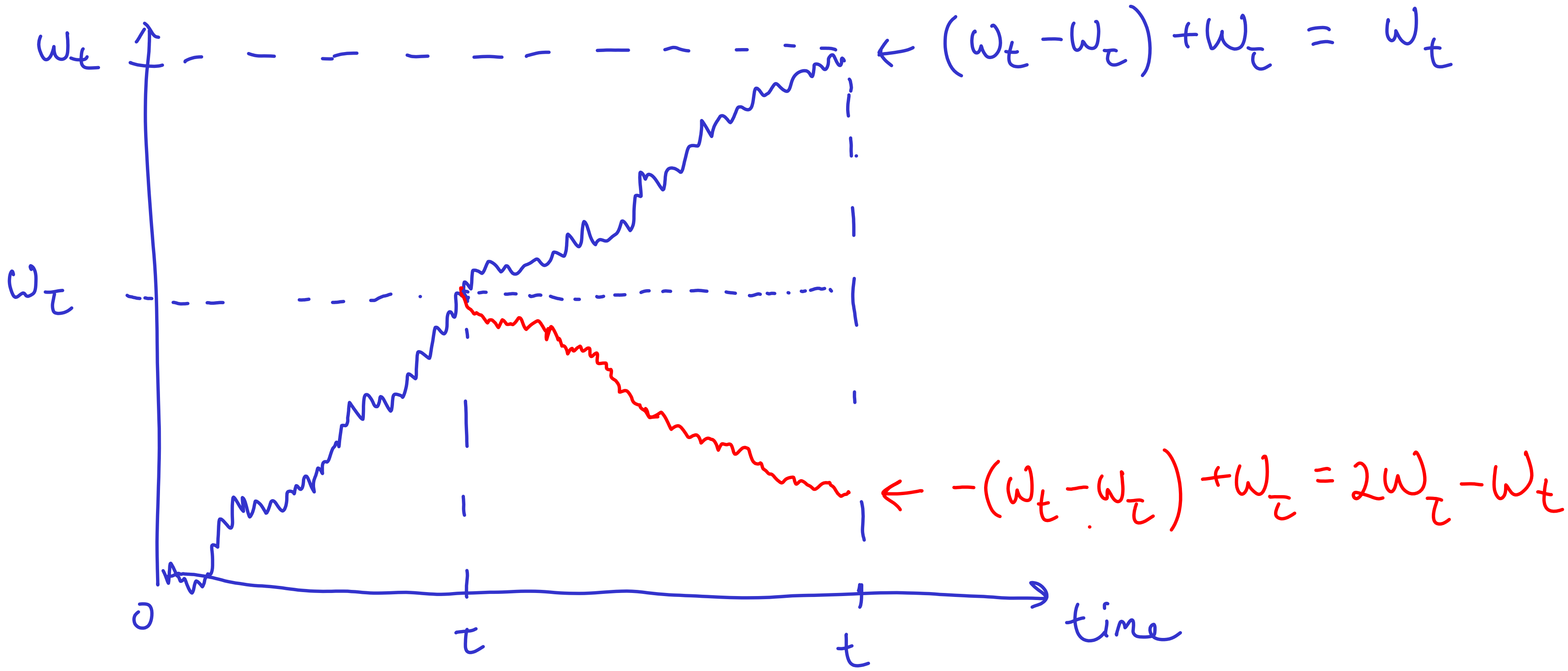
$$+ \left(W_{t_n} - W_{t_{n-1}} \right)^2$$

$$\mathbb{E}\left[\sum_{i=1}^{n} \left(W_{t_i} - W_{t_{i-1}} \right)^2 \right] = (t_1 - t_0) + (t_2 - t_1) + \dots + (t_n - t_{n-1})$$

$$= t_n - t_0 = t_n$$

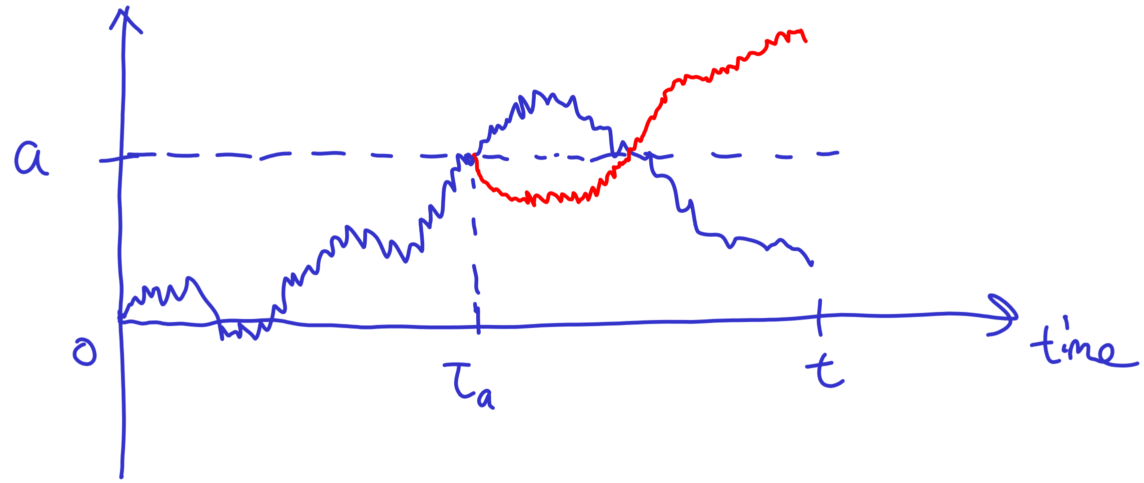
$$\lim_{n \to \infty} \sum_{i=1}^{n} \left(W_{t_i} - W_{t_{i-1}} \right)^2 = t_n \quad \text{with } \mathbb{P} = 1$$

pg30.



W_t
W_τ
0
τ
t
time
$\leftarrow (W_t - W_\tau) + W_\tau = W_t$
$\leftarrow -(W_t - W_\tau) + W_\tau = 2W_\tau - W_t$

pg 31.



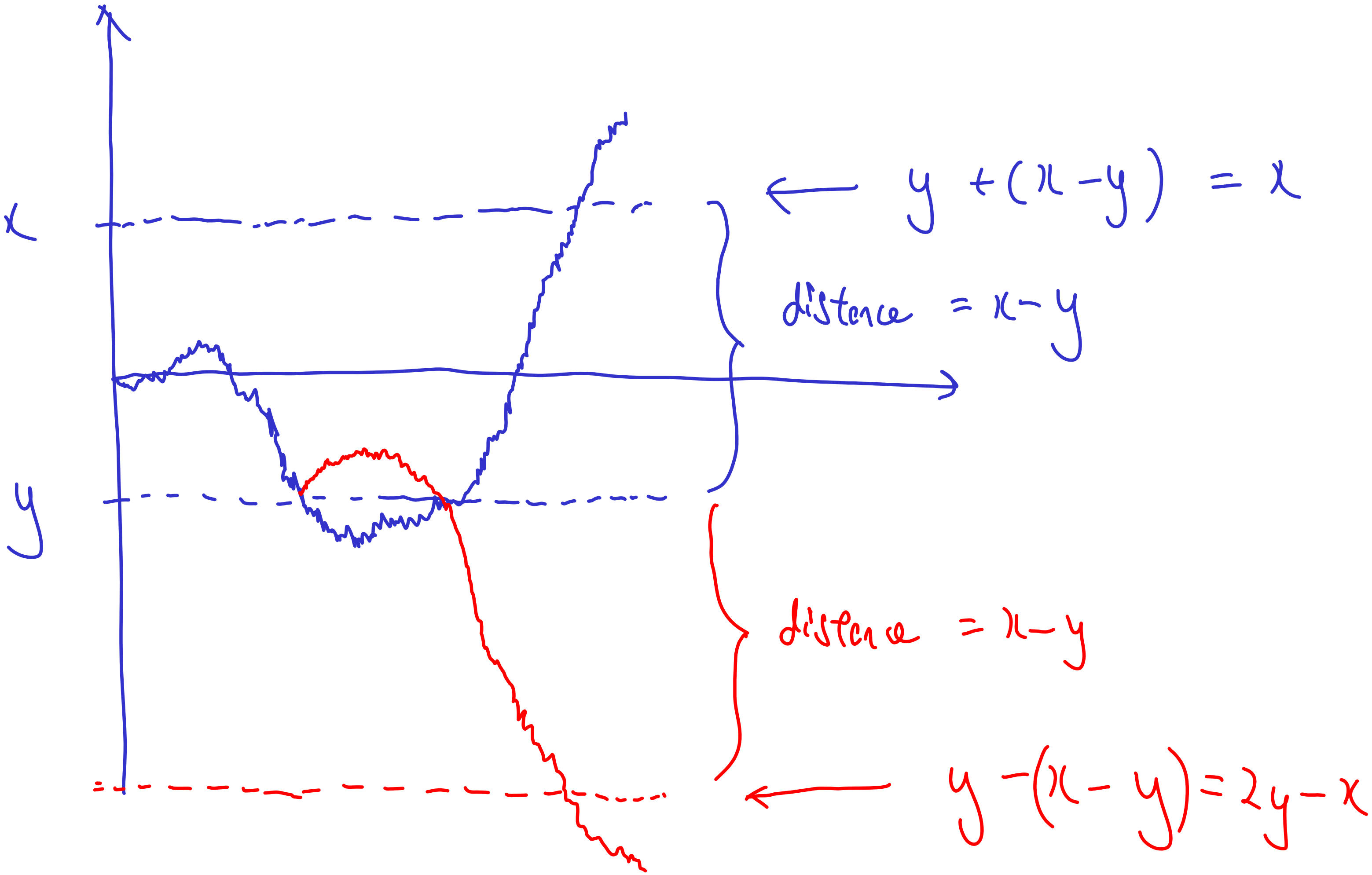
$$\mathbb{P}(\tau_a < t) = \mathbb{P}(\text{touch } a \text{ before } t)$$

$$= \mathbb{P}(\text{touch } a \text{ before } t \cap \text{ends up above } a)$$

$$+ \mathbb{P}(\text{touch } a \text{ before } t \cap \text{ends up below } a)$$

$$= 2 \times \mathbb{P}(\text{touch } a \text{ before } t \cap \text{ends up above } a) = 2\,\mathbb{P}(W_t > a)$$

pg 32.



pg 35.

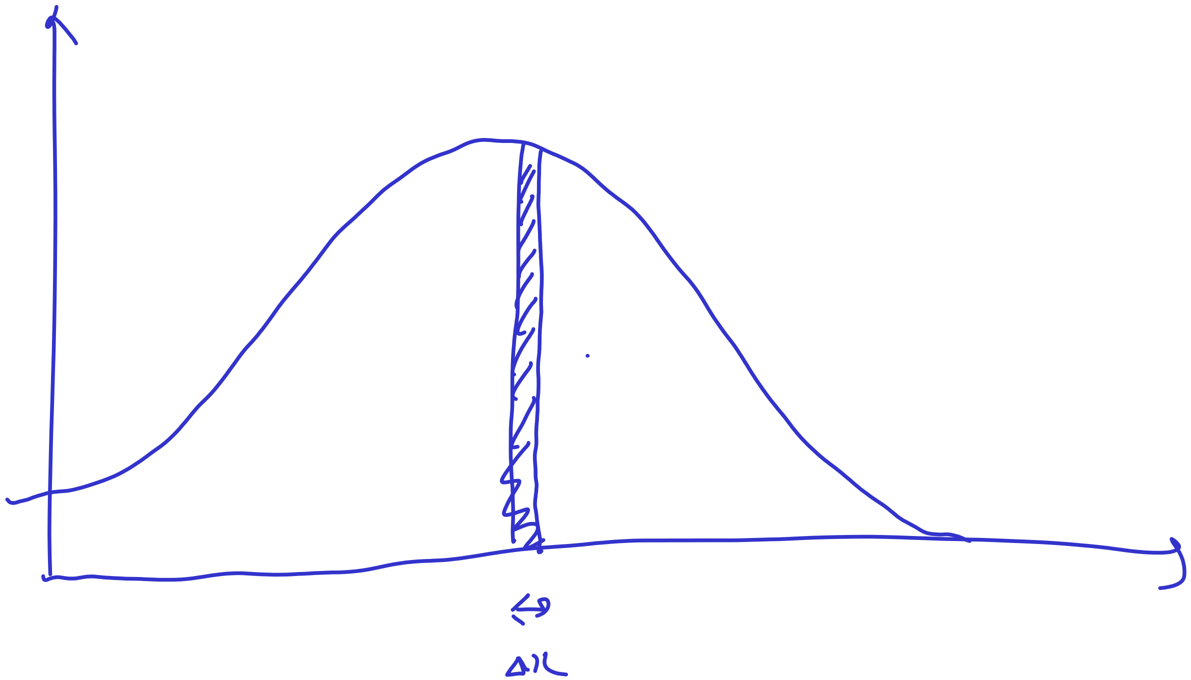
$$dS_t = \mu \, dt + \sigma \, dW_t$$

$$\int_0^t dS_u = \int_0^t \mu \, du + \int_0^t \sigma \, dW_u$$

$$S_t - S_0 = \mu \times (t-0) + \sigma \times (W_t - W_0)$$

$$\Rightarrow \quad S_t = S_0 + \mu t + \sigma W_t$$

pg 3.



pg6.

$$\int_0^T W_t \, dW_t = \frac{W_T^2}{2} - \frac{T}{2}$$

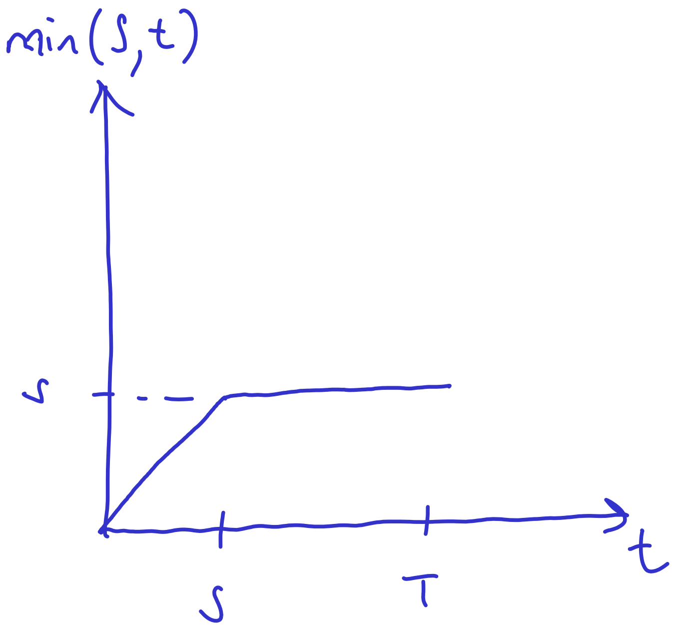
$$\int x \, dx = \frac{x^2}{2}$$

pg 8.

inner integral $= \int_0^T \min(t,s)\,dt$

$$= \int_0^s t\,dt + \int_s^T s\,dt$$

$$= \left[\frac{t^2}{2}\right]_0^s + s(T-s)$$



pg9.

$\int t \, dt$ trivial

$\int W_t \, dt$

$\int_0^T t \, dW_t$

$$V\left[\int_0^T W_t \, dW_t\right] = \mathbb{E}\left[\left(\int_0^T W_t \, dW_t\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^T W_t^2 \, dt\right]$$

$$= \int_0^T \mathbb{E}\left[W_t^2\right] dt = \int_0^T t \, dt$$

$$\int_0^T t \, dW_t \sim N\left(0, \frac{T^3}{3}\right)$$

$$\mathbb{E}\left[\int_0^T t \, dW_t\right] = 0$$

$$V\left[\int_0^T t \, dW_t\right] = \mathbb{E}\left[\left(\int_0^T t \, dW_t\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^T t^2 \, dt\right]$$

$$= \frac{t^3}{3}\Bigg|_0^T = \frac{T^3}{3}$$

pg14.

$$f(x+\Delta x) = f(x) + f'(x)\cdot\Delta x + \frac{1}{2!}\cdot f''(x)\cdot(\Delta x)^2 + \dots$$

$$f(x+\Delta x) - f(x) = f'(x)\cdot\Delta x + \frac{1}{2!} f''(x)(\Delta x)^2 + \dots$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) + \frac{1}{2!} f''(x)(\Delta x) + \dots$$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$$

$$\frac{df(x)}{dx} = f'(x)$$

pg 14.

$$f(S_t + \Delta S_t) - f(S_t) = f'(S_t) \cdot \Delta S_t + \frac{1}{2!} f''(S_t) (\Delta S_t)^2 + \dots$$

$$\Delta t \to 0 \quad : \quad \Delta t \to dt, \quad \Delta S_t \to dS_t, \quad (\Delta W_t)^2 = dt$$

$$df = f'(S_t)\, dS_t + \frac{1}{2} f''(S_t)\, \sigma(t, S_t)^2\, dt$$

$\sigma(t, S_t)^2$

$$= f'(S_t) \Big(\mu(t, S_t)\, dt + \sigma(t, S_t)\, dW_t \Big) + \frac{1}{2} f''(S_t)\, dt$$

✓

$$= \Big[f'(S_t)\, \mu(t, S_t) + \frac{1}{2} f''(S_t) \Big] dt + \sigma(t, S_t)\, f'(S_t)\, dW_t$$

$\sigma(t, S_t)^2$

$$\Delta S_t = \mu(t,S_t)\,\Delta t + \sigma(t,S_t)\,\Delta W_t$$

$$(\Delta S_t)^2 = \underbrace{\mu(t,S_t)^2\,\Delta t^2}_{\Delta t^2} + 2\mu(t,S_t)\,\sigma(t,S_t)\,\underbrace{\Delta t\,\Delta W_t}_{\Delta t^{3/2}}$$

$$+ \sigma(t,S_t)^2\,\underbrace{\Delta W_t^2}_{\Delta t}$$

$\Delta t \to 0$: $(\Delta t)^2 = 0$, $\Delta t\,\Delta W_t = 0$,

$\Delta W_t = dW_t$, $(\Delta W_t)^2 = dt$, $\Delta t = dt$

pg 15.

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t$$

$$Y_t = f(X_t)$$

$$\boxed{dY_t = f'(X_t)\, dX_t + \frac{1}{2!} \cdot f''(X_t)\,(dX_t)^2}$$

$$= f'(X_t)\left(\mu_t \, dt + \sigma_t \, dW_t\right) + \frac{1}{2} \cdot f''(X_t)\left(\sigma_t^2 \, dt\right)$$

$$= \left[\mu_t f'(X_t) + \frac{1}{2}\sigma_t^2 f''(X_t)\right] dt + \sigma_t f'(X_t)\, dW_t$$

pg 15.

$$dX_t = \mu\, dt + \sigma\, dW_t$$

$$Y_t = f(X_t) = X_t^2$$

$f(x) = x^2$

$f'(x) = 2x$, $f'(X_t) = 2X_t$

$f''(x) = 2$, $f''(X_t) = 2$

By Itô's formula,

$$dY_t = f'(X_t)\, dX_t + \frac{1}{2} \cdot f''(X_t)\, (dX_t)^2$$

$$= 2X_t \left(\mu\, dt + \sigma\, dW_t \right) + \frac{1}{2} \times 2 \times \sigma^2\, dt$$

$$dY_t = \left(2\mu X_t + \sigma^2 \right) dt + 2\sigma X_t\, dW_t$$

$$f(X_t) = X_t^2$$

$$\frac{df}{dX_t} = 2X_t$$ ← undefined

$$\frac{d^2 f}{dX_t^2} = 2$$

pg 16. ordinary bivariate functions:

$$g(x+\Delta x, y+\Delta y) = g(x,y) + \frac{\partial g}{\partial x}\cdot\Delta x + \frac{\partial g}{\partial y}\cdot\Delta y$$

$$+ \frac{1}{2!}\left[\frac{\partial^2 g}{\partial x^2}(\Delta x)^2 + 2\cdot\frac{\partial^2 g}{\partial x \partial y}\Delta x\,\Delta y + \frac{\partial^2 g}{\partial y^2}\cdot(\Delta y)^2\right]$$

$$+ \cdots\cdots$$

Stochastic:

$g(t,x)$

$$g(t+\Delta t,\ W_t+\Delta W_t) = g(t,W_t) + \frac{\partial g}{\partial t}\cdot \Delta t + \frac{\partial g}{\partial x}\Delta W_t$$

$$+\frac{1}{2!}\left[\cancel{\frac{\partial^2 g}{\partial t^2}(\Delta t)^2} + \cancel{2\frac{\partial^2 g}{\partial t\,\partial x}\Delta t\,\Delta W_t} + \frac{\partial^2 g}{\partial x^2}(\Delta W_t)^2\right]$$

$+ \dots$

$\lim \Delta t \to 0$: $\Delta t = dt$, $(\Delta W_t)^2 = dt$, $\Delta W_t = dW_t$, $(\Delta t)^2 = 0$, $\Delta t\,\Delta W_t = 0$

$$dg = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dW_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}dt$$

$$dX_t = \mu_t dt + \sigma_t dW_t$$

$$Y_t = g(t, X_t)$$

By Itô formula:

$$dY_t = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} (dX_t)^2$$

$$= \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} \left(\mu_t dt + \sigma_t dW_t \right) + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} \left(\sigma_t^2 dt \right)$$

$$= \left[\frac{\partial g}{\partial t} + \mu_t \frac{\partial g}{\partial x} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 g}{\partial x^2} \right] dt + \sigma_t \frac{\partial g}{\partial x} dW_t$$