# Expected Utility

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# Decision Theory

- Aim is to develop framework for theory of decision-making process, especially when outcome is random
- Interpret **lottery** as investment with two or more potential fixed outcomes, as represented by probability distribution
- Let $P$ and $P^*$ be any pair of lotteries, and let $\succeq$ be **binary relation** that indicates whether investor prefers $P$ or $P^*$
- Preference: $P^* \succeq P$ when $P^*$ is preferred to $P$
- Strict preference: $P^* \succ P$ when $P^* \succeq P$ and $P \not\succeq P^*$
- Indifference: $P^* \sim P$ when $P^* \succeq P$ and $P \succeq P^*$

## Preference Relation

- Completeness: $\succeq$ is **complete** when either $P^* \succeq P$ or $P \succeq P^*$ (or both) for all lotteries $P$ and $P^*$
- Transitivity: $\succeq$ is **transitive** when $P^{**} \succeq P^*$ and $P^* \succeq P$ imply that $P^{**} \succeq P$ for all lotteries $P$, $P^*$ and $P^{**}$
- If $\succeq$ is complete and transitive binary relation, then it represents **preference relation**
- Let $H$ be real-valued **utility function** that assigns value of $H(P)$ to any lottery $P$
- If $\succeq$ is preference relation, then it can represented by utility function: $P^* \succeq P$ iff $H(P^*) \geq H(P)$ for all $P$ and $P^*$

# Continuity

- So far, (simple) lotteries where outcomes are fixed
- But what if you flip a coin to decide between two lotteries?
- Represents **compound lottery** where each of two possible outcomes is (simple) lottery
- More generally, if $\lambda \in [0, 1]$, then $\lambda P^{**} + (1 - \lambda) P$ is compound lottery with probability $\lambda$ of receiving $P^{**}$ and probability $(1 - \lambda)$ of receiving $P$
- Continuity: $\succeq$ is **continuous** if when $P^{**} \succeq P^{*} \succeq P$, there exists some $\lambda \in [0, 1]$ such that $P^{*} \sim \lambda P^{**} + (1 - \lambda) P$

# Independence

- Independence: let $P$ and $P^*$ be any two lotteries with $P^* \succeq P$, then for all $\lambda \in (0, 1]$ and all other lotteries $P^{**}$:

$$\lambda P^* + (1 - \lambda) P^{**} \succeq \lambda P + (1 - \lambda) P^{**}$$

- Preference of $P^*$ over $P$ is independent of other potential outcome in compound lottery
- Also known as substitution axiom or "no regret" axiom
- In reality, experimental evidence of systematic violations of independence axiom

# Expected Utility

- Define **expected utility function** $V$:

$$V(P) = E[U(\tilde{x})]$$

- Here $E[\cdot]$ is expectation operator while $\tilde{x}$ is random variable that represents probability distribution for lottery $P$
- Then $U$ is **von Neumann–Morgenstern utility function**, which measures utility for individual outcomes
- If $\succeq$ is continuous preference relation that satisfies independence axiom, then it can be represented by $V$: $P^* \succeq P$ iff $V(P^*) \geq V(P)$ for all $P$ and $P^*$

# Risk Aversion

- Suppose that lottery $\tilde{\epsilon}$ has probability $p$ of gaining $\epsilon_+ > 0$ and probability $1 - p$ of losing $\epsilon_- < 0$
- Fair lottery must have expected gain of $p\epsilon_+ + (1 - p)\,\epsilon_- = 0$
- Suppose that investor is **risk averse**, in sense of being unwilling to accept fair lottery
- Then utility of investor's existing wealth $W$ must exceed expected utility of accepting fair lottery:

$$
\begin{aligned}
U(W) &= U(p\,(W + \epsilon_+) + (1 - p)\,(W + \epsilon_-)) \geq \\
E[U(W + \tilde{\epsilon})] &= pU(W + \epsilon_+) + (1 - p)\,U(W + \epsilon_-)
\end{aligned}
$$

- Hence risk aversion implies that utility function is concave

# Risk Premium

- Define (absolute) **risk premium** $\pi_a$ for fair lottery $\tilde{\epsilon}$ as

$$U(W - \pi_a) = E[U(W + \tilde{\epsilon})]$$

- Suppose that utility function is twice differentiable, and that $\tilde{\epsilon}$ is "small", so take Taylor series approximations:

$$\begin{aligned} U(W - \pi_a) &\approx U(W) - \pi_a U'(W) \\ E[U(W + \tilde{\epsilon})] &\approx E\left[U(W) + \tilde{\epsilon} U'(W) + \frac{1}{2}\tilde{\epsilon}^2 U''(W)\right] \\ &= U(W) + \frac{1}{2}\sigma_\epsilon^2 U''(W) \end{aligned}$$

# Absolute Risk Aversion

- Suppose that utility function is strictly increasing and **marginal utility** is strictly positive: $U'(W) > 0$, so investor will never be satiated
- Equate results to obtain expression for risk premium:

$$\pi_a = -\frac{1}{2}\sigma_\epsilon^2 \frac{U''(W)}{U'(W)} = \frac{1}{2}\sigma_\epsilon^2 R_a(W)$$

- Here $R_a(W) = -\frac{U''(W)}{U'(W)}$ is **(Arrow–Pratt) measure (or coefficient) of absolute risk aversion**
- If investor is risk averse, then utility function is concave, so $U''(W) \leq 0$ and $\pi_a \geq 0$

# Relative Risk Aversion

- Define **relative risk premium** $\pi_r$ for proportional lottery $\tilde{\epsilon}_r$:

$$U(W - \pi_r W) = E[U(W + \tilde{\epsilon}_r W)]$$

- Take Taylor series approximations and equate results:

$$\pi_r = -\frac{1}{2}\sigma^2_{\epsilon_r} W \frac{U''(W)}{U'(W)} = \frac{1}{2}\sigma^2_{\epsilon_r} R_r(W)$$

- Here $R_r(W) = -W\frac{U''(W)}{U'(W)} = WR_a(W)$ is **(Arrow–Pratt) measure (or coefficient) of relative risk aversion**

# Quadratic Utility

- Quadratic utility function:

$$U(W) = W - \frac{1}{2}bW^2, \quad b > 0$$

- Marginal utility is $U'(W) = 1 - bW$, so utility function is increasing for $W \leq 1/b$ and decreasing otherwise
- Absolute risk aversion is increasing:

$$R_a(W) = \frac{b}{1 - bW} \quad \Rightarrow \quad \frac{dR_a(W)}{dW} = \frac{b^2}{(1 - bW)^2} > 0$$

# Exponential Utility

- Exponential utility function:

$$U(W) = -e^{-bW}, \quad b > 0$$

- Utility function is strictly increasing, and marginal utility is strictly positive: $U'(W) = be^{-bW} > 0$
- Utility function is strictly concave, and marginal utility is strictly decreasing: $U''(W) = -b^2 e^{-bW} < 0$
- Absolute risk aversion is constant: $R_a(W) = b$
- Relative risk aversion is increasing: $R_r(W) = bW$

# Power Utility

- Power utility function:

$$U(W) = \frac{1}{1-\gamma} W^{1-\gamma}, \quad \gamma > 0$$

- Utility function is strictly increasing, and marginal utility is strictly positive: $U'(W) = W^{-\gamma} > 0$
- Utility function is strictly concave, and marginal utility is strictly decreasing: $U''(W) = -\gamma W^{-(\gamma+1)} < 0$
- Absolute risk aversion is decreasing: $R_a(W) = \frac{\gamma}{W}$
- Relative risk aversion is constant: $R_r(W) = \gamma$

# Preferences and Returns – Part 1

- Let $\tilde{R}_p$ be (one plus) random return on investor's portfolio, so that end-of-period wealth is given by $\tilde{W} = W_0 \tilde{R}_p$
- WLOG, set $W_0 = 1$, and take Taylor series expansion of utility function around mean return of $\mu_p$:

$$\begin{aligned} U\left(\tilde{R}_p\right) &= U(\mu_p) + U'(\mu_p)\left(\tilde{R}_p - \mu_p\right) + \\ &\quad \frac{1}{2}U''(\mu_p)\left(\tilde{R}_p - \mu_p\right)^2 + \cdots \\ \Rightarrow E\left[U\left(\tilde{R}_p\right)\right] &= U(\mu_p) + \frac{1}{2}\sigma_p^2 U''(\mu_p) + \cdots \end{aligned}$$

# Preferences and Returns – Part 2

- If utility function is quadratic, then expected utility only depends on mean and variance of return
- Implies satiation and increasing absolute risk aversion
- For other preferences, expected return will also depend on higher moments of return (such as skewness and kurtosis)
- Unless return distribution only depends on mean and variance, and is **stable** under addition
- Only stable distribution with finite variance is normal
- But normal distribution is unbounded from below, which violates **limited liability**

# Preferences and Returns – Part 3

- Let $\tilde{x} = \frac{\tilde{R}_p - \mu_p}{\sigma_p}$ be standard normal variable, so expected utility of end-of-period wealth:

$$E\left[U\left(\tilde{R}_p\right)\right] = \int_{-\infty}^{\infty} U(\mu_p + x\sigma_p)\,\phi(x)\,dx$$

- Here $\phi(\cdot)$ is standard normal probability density function
- If investor is not satiated, then increase in expected return will also increase expected utility:

$$\frac{\partial}{\partial \mu_p} E\left[U\left(\tilde{R}_p\right)\right] = \int_{-\infty}^{\infty} U'(\mu_p + x\sigma_p)\,\phi(x)\,dx > 0$$

# Preferences and Returns – Part 4

- If investor is risk averse, then marginal utility is decreasing:

$$U'(\mu_p + x\sigma_p) < U'(\mu_p - x\sigma_p) \quad \text{for} \quad x > 0$$

- Standard normal distribution is symmetric around zero, so increase in standard deviation of return will reduce expected utility for risk-averse investor:

$$\begin{aligned}\frac{\partial}{\partial \sigma_p} E\left[U\left(\tilde{R}_p\right)\right] &= \int_{-\infty}^{\infty} U'(\mu_p + x\sigma_p)\, x\phi(x)\, dx \\ &= \int_0^{\infty} \Big(U'(\mu_p + x\sigma_p) - U'(\mu_p - x\sigma_p)\Big)\, x\phi(x)\, dx < 0\end{aligned}$$

# Indifference Curve – Part 1

- What is shape of **indifference curve**, which represents combinations of $(\mu_p, \sigma_p)$ with same expected utility?
- Let $P_1$ and $P_2$ be two portfolios that lie on same indifference curve: $E\left[U\left(\tilde{R}_1\right)\right] = E\left[U\left(\tilde{R}_2\right)\right] = \overline{U}$
- Let $P_3$ be any convex combination of $P_1$ and $P_2$: $\tilde{R}_3 = w\tilde{R}_1 + (1-w)\,\tilde{R}_2$, where $w \in [0,1]$
- $P_3$ lies to left of line joining $P_1$ and $P_2$ in mean–std dev space:

$$
\begin{aligned}
& \mu_3 = w\mu_1 + (1-w)\,\mu_2 \\
& \sigma_3^2 = w^2\sigma_1^2 + 2w\,(1-w)\,\rho_{12}\sigma_1\sigma_2 + (1-w)^2\,\sigma_2^2 \\
\Rightarrow\ & \sigma_3 \leq w\sigma_1 + (1-w)\,\sigma_2
\end{aligned}
$$

# Indifference Curve – Part 2

- If investor is risk averse, then utility function is concave:

$$U\left(\tilde{R}_3\right) = U\left(w\tilde{R}_1 + (1-w)\,\tilde{R}_2\right) \geq wU\left(\tilde{R}_1\right) + (1-w)\,U\left(\tilde{R}_2\right)$$

- Hence $P_3$ offers higher expected utility than $P_1$ and $P_2$:

$$E\left[U\left(\tilde{R}_3\right)\right] \geq wE\left[U\left(\tilde{R}_1\right)\right] + (1-w)\,E\left[U\left(\tilde{R}_2\right)\right] = \overline{U}$$

- $P_3$ lies to northwest of indifference curve containing $P_1$ and $P_2$ in mean–standard deviation space, so risk-averse investor has convex indifference curves

# Indifference Curve – Part 3

- Expected utility is function of mean and standard deviation of return, so total derivative of equation for indifference curve:

$$dE\left[U\left(\tilde{R}_p\right)\right] = \frac{\partial}{\partial \mu_p} E\left[U\left(\tilde{R}_p\right)\right] d\mu_p + \frac{\partial}{\partial \sigma_p} E\left[U\left(\tilde{R}_p\right)\right] d\sigma_p$$
$$= 0$$

- Shows trade-off between risk and reward along indifference curve, and confirms that indifference curve has positive slope:

$$\frac{d\mu_p}{d\sigma_p} = -\frac{\partial}{\partial \sigma_p} E\left[U\left(\tilde{R}_p\right)\right] \Big/ \frac{\partial}{\partial \mu_p} E\left[U\left(\tilde{R}_p\right)\right] > 0$$