<u>pg 16.</u>

$$dX_t = \mu\, dt + \sigma\, dW_t \,, \quad Y_t = e^{t+X_t} = g(t, X_t)$$

$$g(t,x) = e^{t+x} \,, \quad \frac{\partial g}{\partial t} = e^{t+x} \,, \quad \frac{\partial g}{\partial x} = e^{t+x} \,, \quad \frac{\partial^2 g}{\partial x^2} = e^{t+x}$$

Itô's formula:

$$dY_t = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} (dX_t)^2$$

$$= Y_t\, dt + Y_t \left(\mu\, dt + \sigma\, dW_t \right) + \frac{1}{2} \cdot Y_t \cdot \sigma^2\, dt$$

$$= \left(1 + \frac{\sigma^2}{2} + \mu \right) Y_t\, dt + \sigma\, Y_t\, dW_t$$

pg 19.

$$X_t = f(W_t) = W_t^2$$

$$f(x) = x^2 , \quad f'(x) = 2x , \quad f''(x) = 2$$

Itô formula:

$$dX_t = f'(W_t)\, dW_t + \frac{1}{2} f''(W_t) \left(dW_t \right)^2$$

$$= 2W_t\, dW_t + \frac{1}{2} \times 2 \times dt$$

$$dX_t = 2W_t\, dW_t + dt$$

$$\int_0^T dX_t = 2\int_0^T W_t\, dW_t + \int_0^T dt$$

$$\Rightarrow \int_0^T W_t\, dW_t = \frac{1}{2}\left[\int_0^T dX_t - \int_0^T dt \right]$$

$$= \frac{1}{2}\left[\left(X_T - X_0 \right) - (T-0) \right]$$

$$= \frac{1}{2}\left[\left(W_T^2 - 0^2 \right) - T \right]$$

Itô : $f(W_t) = W_t^3$

$$\Rightarrow \int_0^T W_t^2 \, dW_t = \underline{\qquad\qquad}$$

Pg 19.

$$X_t = tW_t = f(t, W_t)$$

$$f(t,x) = tx \ , \quad \frac{\partial f}{\partial t} = x \ , \quad \frac{\partial f}{\partial x} = t \ , \quad \frac{\partial^2 f}{\partial x^2} = 0$$

Itô's formula:

$$dX_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dW_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dW_t)^2$$

$$dX_t = W_t \, dt + t \, dW_t + 0$$

$$\int_0^T dX_t = \int_0^T W_t \, dt + \int_0^T t \, dW_t$$

$$X_T - X_0 = \int_0^T W_t \, dt + \int_0^T t \, dW_t$$

$$TW_T - 0 \cdot W_0 = \int_0^T W_t \, dt + \int_0^T t \, dW_t$$

$$\int_0^T W_t \, dt = T W_T - \int_0^T t \, dW_t$$

$$= T \int_0^T dW_t - \int_0^T t \, dW_t$$

$$= \int_0^T (T - t) \, dW_t \leftarrow \text{normal dist.}$$

$$\mathbb{E}\left[\int_0^T W_t \, dt \right] = 0 \quad \because \text{stochastic integral 0 mean}$$

$$V\left[\int_0^T W_t \, dt \right] = V\left[\int_0^T (T-t) \, dW_t \right] \quad \because \text{Itô Isometry}$$

$$= \mathbb{E}\left[\left(\int_0^T (T-t) \, dW_t \right)^2 \right] = \mathbb{E}\left[\int_0^T (T-t)^2 \, dt \right]$$

pg23.

$$X_t = f(W_t) = e^{\theta W_t}$$

$$f(x) = e^{\theta x}, \quad f'(x) = \theta e^{\theta x}, \quad f''(x) = \theta^2 e^{\theta x}$$

Itô's formula:

$$dX_t = f'(W_t)\,dW_t + \frac{1}{2} f''(W_t)\,(dW_t)^2$$

$$= \theta X_t\,dW_t + \frac{1}{2}\theta^2 X_t\,dt$$

$$\int_0^t dX_u = \theta \int_0^t X_u\,dW_u + \frac{1}{2}\theta^2 \int_0^t X_u\,du$$

$$X_t - X_0 = \theta \int_0^t X_u\,dW_u + \frac{1}{2}\theta^2 \int_0^t X_u\,du$$

pg23.

Leibniz's rule

$$I(t) = \int_{u(t)}^{v(t)} f(t,x)\,dx$$

$$\frac{dI(t)}{dt} = f(t, v(t)) \cdot \frac{dv}{dt} - f(t, u(t)) \cdot \frac{du}{dt} + \int_{u(t)}^{v(t)} \frac{\partial f}{\partial t}(t,x)\,dx .$$

$$I(t) = \frac{\theta^2}{2} \int_0^t \mathbb{E}[X_u]\,du$$

$$\frac{dI(t)}{dt} = \frac{\theta^2}{2}\,\mathbb{E}[X_t] \cdot \cancelto{1}{\frac{dt}{dt}} - \frac{\theta^2}{2}\,\mathbb{E}[X_0] \cdot \cancelto{0}{\frac{d(0)}{dt}} + \frac{\theta^2}{2} \int_0^t \cancelto{0}{\frac{\partial}{\partial t}\mathbb{E}[X_u]}\,du$$

Pg 23.

$$\frac{d\,\mathbb{E}[X_t]}{dt} = \frac{\theta^2}{2}\,\mathbb{E}[X_t]$$

$$\frac{d\,\mathbb{E}[X_t]}{\mathbb{E}[X_t]} = \frac{\theta^2}{2}\,dt$$

$$\int_0^t \frac{d\,\mathbb{E}[X_u]}{\mathbb{E}[X_u]} = \int_0^t \frac{\theta^2}{2}\,du$$

$$\Big[\log \mathbb{E}[X_u]\Big]_0^t = \frac{\theta^2}{2}t$$

$$\log \mathbb{E}[X_t] - \log \mathbb{E}[X_0] = \frac{\theta^2}{2}t$$

$$\log \mathbb{E}[X_t] = \frac{\theta^2 t}{2}$$

$$\mathbb{E}[X_t] = e^{\frac{\theta^2 t}{2}}$$

pg24.

$$dX_t = \sigma X_t \, dW_t \quad , \quad X_0 = 1$$

$$Y_t = \log(X_t) = f(X_t)$$

$$f(x) = \log(x), \quad f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}$$

Itô formula:

$$dY_t = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)(dX_t)^2$$

$$= \frac{1}{X_t} \cdot \sigma X_t \, dW_t - \frac{1}{2} \cdot \frac{1}{X_t^2} \cdot \sigma^2 X_t^2 \, dt$$

$$= \sigma \, dW_t - \frac{1}{2} \sigma^2 dt$$

$$\int_0^t dY_u = \int_0^t \sigma \, dW_u - \frac{1}{2} \sigma^2 \int_0^t du$$

$$Y_t - Y_0 = \sigma W_t - \frac{1}{2}\sigma^2 t$$

$$\log(X_t) - \log(X_0) = -\frac{1}{2}\sigma^2 t + \sigma W_t$$

$$X_t = e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}$$

Normal process

$$dX_t = \mu\, dt + \sigma\, dW_t$$

$$\int_0^t dX_u = \mu t + \sigma W_t$$

$$X_t = X_0 + \mu t + \sigma W_t$$

$$dX_t = \sigma X_t \, dW_t$$

$$\frac{dX_t}{X_t} = \sigma \, dW_t$$

$$\int_0^t \frac{dX_u}{X_u} = \int_0^t \sigma \, dW_u$$

$$\log X_t - \cancelto{0}{\log X_0} = \sigma W_t$$

$$\log X_t = \sigma W_t \quad \longleftarrow \text{missing } dt \text{ terms}$$

pg 24.

$$X_t = e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}$$

$$\mathbb{E}[X_t] = e^{-\frac{1}{2}\sigma^2 t}\,\mathbb{E}\left[e^{\sigma W_t}\right] = e^{-\frac{1}{2}\sigma^2 t}\; e^{\frac{1}{2}\sigma^2 t}$$

$$= e^{0} = 1$$

pg25

$$X_t = e^{-\frac{\sigma^2 t}{2} + \sigma W_t} = f(t, W_t)$$

$$f(t,x) = e^{-\frac{\sigma^2 t}{2} + \sigma x}$$

$$dX_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dW_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dW_t)^2$$

$$X_t = e^{Y_t} = f(Y_t), \quad Y_t = -\frac{\sigma^2 t}{2} + \sigma W_t$$

$$dY_t = -\frac{\sigma^2}{2} dt + \sigma dW_t$$

$$dX_t = f'(Y_t) dY_t + \frac{1}{2} f''(Y_t) (dY_t)^2$$

pg28.

	dt	dW_t
dt	0	0
dW_t	0	dt

Pg 7.

$$dS_t = \sigma S_0 \, dW_t$$

$$\int_0^T dS_t = \int_0^T \sigma S_0 \, dW_t$$

$$S_T - S_0 = \sigma S_0 (W_T - \cancelto{0}{W_0})$$

$$S_T = S_0 + \sigma S_0 W_T = S_0(1 + \sigma W_T)$$

$$S_T = f(W_T) = S_0(1 + \sigma W_T)$$

$$f'(W_T) = \sigma S_0 , \quad f''(W_t) = 0$$

$$dS_t = f'(W_t)\, dW_t + \frac{1}{2} \cancelto{0}{f''(W_t)} (dW_t)^2$$

$$= \sigma S_0 \, dW_t$$

pg3.

$$\text{Call} = \mathbb{E}\left[(S_T - K)^+\right]$$

$$= \mathbb{E}\left[\left(S_0(1+\sigma W_T) - K\right)^+\right]$$

$$= \mathbb{E}\left[\left(S_0(1+\sigma\sqrt{T}X) - K\right)^+\right], \quad X \sim N(0,1)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(S_0 + \sigma S_0\sqrt{T}x - K\right)^+ e^{-\frac{x^2}{2}}\,dx$$

$$\left(S_0 + S_0 \sigma \sqrt{T}\, x - K \right) > 0$$

$$x > \frac{K - S_0}{S_0 \sigma \sqrt{T}} = x^*$$

continue

$$\frac{1}{\sqrt{2\pi}} \int_{x^*}^{\infty} \left(S_0 + S_0 \sigma \sqrt{T}\, x - K \right) e^{-\frac{x^2}{2}} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{x^*}^{\infty} (S_0 - K) e^{-\frac{x^2}{2}} \, dx + \int_{x^*}^{\infty} S_0 \sigma \sqrt{T}\, x\, e^{-\frac{x^2}{2}} \, dx \right)$$

$$= (S_0 - K) \left[\Phi(\infty) - \Phi(x^*) \right] + \frac{1}{\sqrt{2\pi}} \int_{x^*}^{\infty} S_0 \sigma \sqrt{T}\, x\, e^{-\frac{x^2}{2}} \, dx$$

$$= (S_0 - K)\,\Phi(-x^*) + \frac{1}{\sqrt{2\pi}}\,\sigma S_0 \sqrt{T} \int_{x^*}^{\infty} e^{-u}\,du$$

$$\boxed{\begin{aligned} &\text{let} \quad u = \frac{x^2}{2} \\ &\qquad du = x\,dx \end{aligned}}$$

$$= (S_0 - K)\,\Phi(-x^*) + \frac{1}{\sqrt{2\pi}}\,\sigma S_0 \sqrt{T} \left[-e^{-u} \right]_{x^*}^{\infty}$$

$$= (S_0 - k)\,\Phi(-x^*) + \frac{1}{\sqrt{2\pi}}\,\sigma S_0 \sqrt{T} \left[-e^{-\frac{x^2}{2}} \right]_{x^*}^{\infty}$$

$$= (S_0 - k)\,\Phi(-x^*) + \frac{1}{\sqrt{2\pi}}\,\sigma S_0 \sqrt{T} \left[-0 + e^{-\frac{(x^*)^2}{2}} \right]$$

$$= (S_0 - K)\,\Phi(-x^*) + \sigma S_0 \sqrt{T}\,\phi(-x^*)$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \phi(x) = \phi(-x)$$

ATM: $S_0 = K$.

$$V^c = 0 + \sigma S_0 \sqrt{T} \cdot \frac{1}{\sqrt{2\pi}} e^{-0} = \sigma S_0 \sqrt{\frac{T}{2\pi}}$$

$$\frac{1}{\sqrt{2\pi}} \simeq 0.4 \qquad \simeq \sigma S_0 \sqrt{T} \times 0.4$$

Pg 4.

$$S_T = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T}$$

$$\mathbb{E}[S_T] = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T} \ \mathbb{E}\left[e^{\sigma W_T}\right]$$

$$= S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T} \cdot e^{\frac{\sigma^2 T}{2}}$$

$$= S_0 e^{rT}$$

Pg 8.

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t$$

$$X_t = \log(S_t) = f(S_t), \qquad f(x) = \log(x)$$

$$f'(x) = \frac{1}{x}, \qquad f''(x) = -\frac{1}{x^2}$$

Itô's formula:

$$dX_t = f'(S_t)\,dS_t + \frac{1}{2} f''(S_t)(dS_t)^2$$

$$= \frac{1}{S_t}\left(rS_t\,dt + \sigma S_t\,dW_t\right) - \frac{1}{2}\frac{1}{S_t^2}\cdot \sigma^2 S_t^2\,dt$$

$$= r\,dt + \sigma\,dW_t - \frac{\sigma^2}{2}\,dt$$

$$dX_t = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma\,dW_t$$

$$\int_0^T dX_t = \int_0^T \left(r - \frac{\sigma^2}{2}\right)dt + \int_0^T \sigma\,dW_t$$

$$X_T - X_0 = \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T$$

$$\log(S_T) - \log(S_0) = \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T$$

$$\log\left(\frac{S_T}{S_0}\right) = \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T$$

$$\frac{S_T}{S_0} = e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T}$$

$$S_T = S_0 \, e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T}$$

pg 8-

$$e^{-rT}\,\mathbb{E}\left[\left(S_T-K\right)^+\right]$$

$$= e^{-rT}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x}-K\right)^+ e^{-\frac{x^2}{2}}\,dx$$

$$S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x} > K$$

$$x > \frac{\log\frac{K}{S_0}-(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = x^*$$

continue:

$$= e^{-rT}\cdot\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}\left(S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x}-K\right)e^{-\frac{x^2}{2}}\,dx$$

$$= e^{-rT} \frac{1}{\sqrt{2\pi}} \left[S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T} \int_{x^*}^{\infty} e^{\sigma\sqrt{T}x} e^{-\frac{x^2}{2}} dx - K \int_{x^*}^{\infty} e^{-\frac{x^2}{2}} dx \right]$$

$$= e^{-rT} \cdot \left[\frac{1}{\sqrt{2\pi}} S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T} \int_{x^*}^{\infty} e^{-\frac{\overbrace{x^2-2\sigma\sqrt{T}x+\sigma^2T}-\sigma^2T}{2}} dx - K\,\Phi(-x^*) \right]$$

$$= \frac{1}{\sqrt{2\pi}} S_0 e^{-\frac{\sigma^2}{2}T} \int_{x^*}^{\infty} e^{-\frac{(x-\sigma\sqrt{T})^2}{2}} e^{\frac{\sigma^2T}{2}} dx - K e^{-rT} \Phi(-x^*)$$

$$= \frac{1}{\sqrt{2\pi}} S_0 \int_{x^*}^{\infty} e^{-\frac{(x-\sigma\sqrt{T})^2}{2}} dx - K e^{-rT} \Phi(-x^*)$$

let $y = x - \sigma\sqrt{T} \Rightarrow dy = dx$

$x = x^*,\ y = x^* - \sigma\sqrt{T}$; $x \to \infty,\ y \to \infty$

$$= \frac{1}{\sqrt{2\pi}} S_0 \int_{x^* - \sigma\sqrt{T}}^{\infty} e^{-\frac{y^2}{2}} dy \quad - K e^{-rT} \Phi(-x^*)$$

$$= S_0 \left[\Phi(\infty) - \Phi\left(x^* - \sigma\sqrt{T}\right) \right] - K e^{-rT} \Phi(-x^*)$$

$$= S_0 \, \Phi\left(-x^* + \sigma\sqrt{T}\right) - K e^{-rT} \Phi(-x^*)$$