QF620 Additional Examples
Session 1: Brownian Motion and Martingale

1 Examples

1. Let $S_n = \sum_{i=1}^{n} X_i$ denote an n-step random walk, where X_i is independent and identically distributed with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = -1) = 1 - p$. Evaluate

   (a) $\mathbb{E}[S_n]$.

   (b) $V[S_n]$.

   (c) $\mathbb{E}_m[S_n]$ (conditional on S_m, where $m < n$)

   (d) $V_m[S_n]$ (conditional on S_m, where $m < n$)

2. (a) If W_t is a Brownian motion, show that W_t^3 isn't a martingale.

   (b) If $W_t^3 + f(W_t, t)$ is a martingale, determine $f(W_t, t)$.

3. If W_t and W_s are Brownian processes, and $t > s$, determine

$$\mathbb{E}[(W_t - W_s)^4].$$

4. If W_t is a Brownian motion, show that $e^{\theta W_t - \frac{\theta^2 t}{2}}$ is a martingale.

5. Let W_t denote a Brownian motion. Which of the following are also Brownian motions?

   (a) $-W_t$

   (b) cW_{t/c^2}, where $c \in \mathbb{R}$

   (c) $\sqrt{t}W_t$

   (d)
$$W_t :\to \begin{cases} 0, & t = 0 \\ tW_{1/t}, & t \in \mathbb{R}^+ \end{cases}$$

6. Let X_i denote a sequence of random variables taking the values of either $+1$ or -1 with equal probability $\frac{1}{2}$, and let $S_n = \sum_i^n X_i$ where $n \in \mathbb{N}$. If $m \in \mathbb{N},\ m < n$, show that $\mathbb{E}[S_n - S_m] = 0$ and $\text{Cov}(S_n - S_m, S_m) = 0$.

7. Let W_t denote a Brownian motion, write down the probability density function of W_t. Let $0 < s < t$, write down the probability density function of $W_t - W_s$.

8. Let W_t denote a Brownian motion. Evaluate the expectation $\mathbb{E}[W_t]$, $\mathbb{E}[W_t^2]$, and $\mathbb{E}[W_t^4]$.

9. Let W_t denote a Brownian motion. Evaluate the expectation $\mathbb{E}[W_t]$ *in full by making use of its probability density function.*

10. Let W_t denote a Brownian motion. Find $\mathbb{E}[W_t^{125}]$.

11. Let W_t denote a Brownian motion. If $W_1 > 0$, what is the probability $\mathbb{P}(W_2 > 0 | W_1 > 0)$?

12. Let W_t denote a Brownian motion. What is the probability $\mathbb{P}(W_1 \times W_2 > 0)$?

13. Let X be a standard normally distributed random variable, i.e. $X \sim N(0,1)$, show that its mean, mode and median is given by

$$\text{Mean} = 0, \quad \text{Mode} = 0, \quad \text{Median} = 0.$$

14. Let X be a normally distributed random variable, i.e. $X \sim N(\mu, \sigma^2)$, show that its mean, mode and median is given by

$$\text{Mean} = \mu, \quad \text{Mode} = \mu, \quad \text{Median} = \mu.$$

15. Show that

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \mu.$$

16. Show that

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \sigma^2.$$

17. Consider a normally distributed random variable $X \sim N(\mu, \sigma^2)$. Define $Y = e^X$, show that

$$\mathbb{E}[Y] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = e^{\mu + \frac{1}{2}\sigma^2}.$$

18. Consider a normally distributed random variable $X \sim N(\mu, \sigma^2)$. Evaluate $\mathbb{E}[e^{\theta X}]$, where $\theta \in \mathbb{R}$ is a constant, using the following method:

    (a) completing the square.

    (b) moment generating function.

19. If $X \sim N(0,1)$. Let $Y_t = \sqrt{t}X$, show that $V[Y_t] = t$.

20. Let $f(t,x) = tx^2$. Work out the Taylor expansion up to the 2^{nd} order.

2 Suggested Solutions

1. First, given the probability distribution of X_i, we note that

$$\begin{aligned}\mathbb{E}[X_i] &= p \times (1) + (1-p) \times (-1) \\ &= 2p - 1\end{aligned}$$

and

$$\begin{aligned}V[X_i] &= \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 \\ &= \left[p \times (1)^2 + (1-p) \times (-1)^2\right] - (2p-1)^2 \\ &= 4p(1-p)\end{aligned}$$

(a) The unconditional expectation of the random walk is given by

$$\begin{aligned}\mathbb{E}[S_n] &= \mathbb{E}[X_1 + X_2 + \cdots + X_n] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n] \\ &= \underbrace{(2p-1) + (2p-1) + \cdots + (2p-1)}_{\text{n terms}} = n(2p-1)\end{aligned}$$

(b) The unconditional variance of the random walk is given by

$$\begin{aligned}V[S_n] &= V[X_1] + V[X_2] + \cdots + V[X_n] \\ &= \underbrace{4p(1-p) + 4p(1-p) + \cdots + 4p(1-p)}_{\text{n terms}} = 4np(1-p)\end{aligned}$$

(c) The conditional expectation of the random walk is given by

$$\begin{aligned}\mathbb{E}_m[S_n] &= \mathbb{E}_m\left[S_m + \sum_{i=m+1}^{n} X_i\right] \\ &= S_m + \mathbb{E}_m[X_{m+1}] + \mathbb{E}_m[X_{m+2}] + \cdots + \mathbb{E}_m[X_n] \\ &= S_m + (n-m)(2p-1)\end{aligned}$$

(d) The conditional variance of the random walk is given by

$$\begin{aligned}V_m[S_n] &= V_m\left[S_m + \sum_{i=m+1}^{n} X_i\right] \\ &= V_m\left[\sum_{i=m+1}^{n} X_i\right] \\ &= V_m[X_{m+1} + X_{m+2} + \cdots + X_n] \\ &= (n-m)4p(1-p)\end{aligned}$$

2. (a) To show this, we check against the 2 basic definitions of a Brownian motion:

$$\begin{aligned}\mathbb{E}[W_t^3] &= 0 \\ \mathbb{E}[W_t^3|s] &= \mathbb{E}[(W_t - W_s + W_s)^3|s] \\ &= \mathbb{E}[(W_t - W_s)^3 + 3(W_t - W_s)^2 W_s + 3(W_t - W_s)W_s^2 + W_s^3|s] \\ &= 3(t-s)W_s + W_s^3.\end{aligned}$$

It is obvious that the second criterion (Brownian increment) is not satisfied, hence W_t^3 doesn't satisfy the definition of a standard Brownian motion.

(b) Taking inspiration from the previous question, we postulate that $W_t^3 - 3tW_t$ might be a Brownian motion. To verify this, we check that

$$\begin{aligned}
\mathbb{E}[W_t^3 - 3tW_t] &= \mathbb{E}[W_t^3] - 3t\mathbb{E}[W_t] = 0 \\
\mathbb{E}[W_t^3 - 3tW_t|s] &= \mathbb{E}[W_t^3|s] - 3t\mathbb{E}[W_t|s] \\
&= 3(t-s)W_s + W_s^3 - 3tW_s \\
&= W_s^3 - 3sW_s.
\end{aligned}$$

3. First we note that if $X \sim N(0,1)$, then we have the following

$$\mathbb{E}[X] = 0, \quad \mathbb{E}[X^2] = 1, \quad \mathbb{E}[X^3] = 0, \quad \mathbb{E}[X^4] = 3.$$

Next, note that

$$(W_t - W_s)^4 \sim N(0,(t-s))^4 = (t-s)^2 N(0,1)^4 = (t-s)^2 X^4.$$

Hence,

$$\mathbb{E}[(W_t - W_s)^4] = \mathbb{E}[(t-s)^2 X^4] = 3(t-s)^2.$$

4. We can show that

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right)|s\right] &= \mathbb{E}\left[\exp\left(\sigma(W_t - W_s)\right)\exp\left(\sigma W_s - \frac{1}{2}\sigma^2 t\right)|s\right] \\
&= \exp\left(\sigma W_s - \frac{1}{2}\sigma^2 t\right)\mathbb{E}[\exp(\sigma(W_t - W_s))|s] \\
&= \exp\left(\sigma W_s - \frac{1}{2}\sigma^2 t\right)\exp\left[\frac{1}{2}\sigma^2(t-s)\right] \\
&= \exp\left(\sigma W_s - \frac{1}{2}\sigma^2 s\right).
\end{aligned}$$

5. Recall the definition of Brownian motions:

- $W_0 = 0$
- $W_t \sim N(0,t)$
- Given $0 < s < t$, then $W_t - W_s \sim N(0, t-s)$, where $W_t - W_s$ is independent of $W_s - W_0$.

To check whether a process is a Brownian motion, it suffice to check whether these criteria are satisfied.

(a) Yes. Because

- $-W_0 = 0$
- $-W_t \sim N(0,t)$
- Given $0 < s < t$, $(-W_t) - (-W_s) \sim N(0, t-s)$, and we have independent increment since

$$\mathrm{Cov}(W_t - W_s, W_s - W_0) = \mathbb{E}[(W_t - W_s)W_s] = \mathbb{E}[W_t W_s] - \mathbb{E}[W_s^2] = s - s = 0.$$

(b) Yes. Because

- $cW_{0/c^2} = cW_0 = 0$
- $cW_{t/c^2} \sim cN\left(0, \frac{t}{c^2}\right) \sim N(0,t)$
- Given $0 < s < t$, we have

$$cW_{t/c^2} - cW_{s/c^2} \sim N(0,t) - N(0,s) \sim N(0, t-s),$$

and we also have independent increment since

$$\begin{aligned}
\operatorname{Cov}(cW_{t/c^2} - cW_{s/c^2}, cW_{s/c^2} - cW_{0/c^2}) &= \mathbb{E}[(cW_{t/c^2} - cW_{s/c^2})cW_{s/c^2}] \\
&= \mathbb{E}[cW_{t/c^2} \times cW_{s/c^2}] - \mathbb{E}[c^2 W_{s/c^2}^2] \\
&= c^2 \cdot \frac{s}{c^2} - c^2 \cdot \frac{s}{c^2} = 0.
\end{aligned}$$

(c) No. Because although $\sqrt{0}W_0 = 0$, the 2^{nd} condition is already violated

$$\sqrt{t}W_t \sim \sqrt{t}N(0,t) \sim N(0,t^2) \neq N(0,t).$$

(d) We can show that the process starts at 0 and is normally distributed with mean 0 and variance t. To show independent increment, we have

$$\begin{aligned}
\operatorname{Cov}\left[tW_{1/t}, (t+s)W_{1/(t+s)}\right] &= t(t+s)\operatorname{Cov}\left[W_{1/t}, W_{1/(t+s)}\right] \\
&= t(t+s) \times \frac{1}{t+s} = t.
\end{aligned}$$

6. Since S_n is made up of a sequence of identical and independently distributed Bernoulli trial with mean 0 and variance 1, we have

$$\forall n \in \mathbb{N}: \ \mathbb{E}[S_n] = 0.$$

And so

$$\mathbb{E}[S_n - S_m] = \mathbb{E}[S_n] - \mathbb{E}[S_m] = 0.$$

Next, we note that

$$\begin{aligned}
\operatorname{Cov}(S_n - S_m, S_m) &= \mathbb{E}[(S_n - S_m)S_m] - \mathbb{E}[S_n - S_m]\mathbb{E}[S_m] \\
&= \mathbb{E}[(S_n - S_m)S_m] - 0 \cdot 0 \\
&= \mathbb{E}[S_n S_m] - \mathbb{E}[S_m^2] \\
&= \mathbb{E}[\mathbb{E}[S_n S_m | m]]] - m \\
&= \mathbb{E}[S_m \mathbb{E}[S_n | m]] - m \\
&= \mathbb{E}[S_m^2] - m \\
&= m - m = 0. \quad \triangleleft
\end{aligned}$$

7. We know that $W_t \sim N(0,t)$, i.e. it is normally distributed with 0 mean and a variance of t, which measures the time elapsed. A normal probability density function $N(\mu, \sigma^2)$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

So the probability density function for W_t is

$$f(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

By the definition of Brownian motion, $W_t - W_s \sim N(0, t-s)$. So the probability density function of $W_t - W_s$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x^2}{2(t-s)}}. \quad \lhd$$

8. Since $W_t \sim N(0, t)$, we can already conclude that

$$\mathbb{E}[W_t] = 0 \quad \because \text{ mean is } 0$$

and

$$\begin{aligned} V[W_t] = t &= \mathbb{E}[W_t^2] - \mathbb{E}[W_t]^2 \quad \because \text{ variance is } t \\ &= \mathbb{E}[W_t^2] - 0 \\ \Rightarrow \quad t &= \mathbb{E}[W_t^2]. \quad \lhd \end{aligned}$$

Finally, we have

$$\begin{aligned} \mathbb{E}[W_t^4] &= \mathbb{E}[t^2 X^4] \quad \text{where } X \sim N(0,1) \\ &= 3t^2. \quad \lhd \end{aligned}$$

9. We need to evaluate

$$\mathbb{E}[W_t] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} w e^{\frac{-w^2}{2t}} dw.$$

Let

$$u = \frac{w^2}{2t} \quad \Rightarrow \quad du = \frac{w}{t} dw.$$

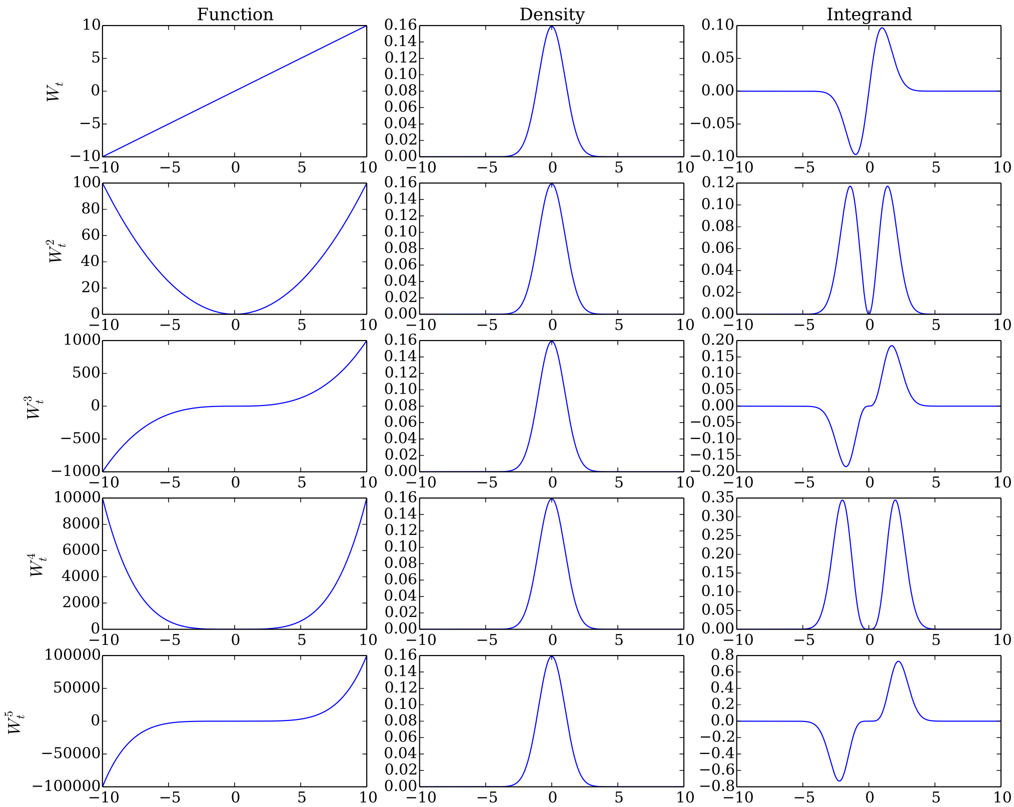
We have

$$\frac{1}{\sqrt{2\pi t}} \int w e^{-\frac{w^2}{2t}} dw = \frac{\sqrt{t}}{\sqrt{2\pi}} \int e^{-\frac{w^2}{2t}} \frac{w}{t} dw = \frac{\sqrt{t}}{\sqrt{2\pi}} \int e^{-u} du = \frac{\sqrt{t}}{\sqrt{2\pi}} [-e^{-u} + C] = \frac{\sqrt{t}}{\sqrt{2\pi}} \left[-e^{-\frac{w^2}{2t}} + C \right].$$

So back to our definite integral

$$\begin{aligned} \mathbb{E}[W_t] &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} w e^{-\frac{w^2}{2t}} dw \\ &= \frac{\sqrt{t}}{\sqrt{2\pi}} \left[-e^{-\frac{w^2}{2t}} \right]_{-\infty}^{\infty} \\ &= 0. \quad \lhd \end{aligned}$$

10. You should have observed a pattern in the previous question – when taking the expectation of W_t, all odd-powered expectations evaluate to 0. This is due to the fact that W_t has symmetrical probability density function:



Since taking expectation involves integrating the random variable weighted by the density function (which is symmetrical across the y-axis), the integrand, given by

$$x^n \times e^{-\frac{x^2}{2t}}, \quad n = 1, 2, 3, \cdots$$

when n is odd will always remain 0, as we would end up integrating equal area above and below the x-axis. So $\mathbb{E}[W_t^{125}] = 0$. $\triangleleft$

11. Given that $W_1 > 0$, two cases will yield the required event: $\{W_2 > 0 | W_1 > 0\}$. The first is that W_2 is an upward step, which occurs with probability $\frac{1}{2}$. The second case is when W_2 steps down, but the step size is not as large as W_1, so that W_2 is still above the x-axis. The event

$$|W_2 - W_1| < |W_1 - W_0|$$

occurs with probability $\frac{1}{2}$. So

$$\begin{aligned}
\mathbb{P}(W_2 > 0 | W_1 > 0) &= \mathbb{P}(W_2 > W_1) + \mathbb{P}(\{W_2 < W_1\} \cap \{|W_2 - W_1| < |W_1 - W_0|\}) \\
&= \frac{1}{2} + \mathbb{P}(\{|W_2 - W_1| < |W_1 - W_0|\} | \{W_2 < W_1\}) \mathbb{P}\Big(\{W_2 < W_1\}\Big) \\
&= \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}.
\end{aligned} \quad \triangleleft$$

12. In order to have the event $\{W_1 \times W_2 > 0\}$, W_1 and W_2 would both need to be simultaneously positive or negative. This probability can be calculated as follow:

$$\begin{aligned}
\mathbb{P}(W_1 \times W_2 > 0) &= \mathbb{P}\Big(\{W_2 > 0\} \cap \{W_1 > 0\}\Big) + \mathbb{P}\Big(\{W_2 < 0\} \cap \{W_1 < 0\}\Big) \\
&= \mathbb{P}\Big(\{W_2 > 0\} | \{W_1 > 0\}\Big)\mathbb{P}\Big(W_1 > 0\Big) + \mathbb{P}\Big(\{W_2 < 0\} | \{W_1 < 0\}\Big)\mathbb{P}\Big(W_1 < 0\Big) \\
&= \frac{3}{4} \times \frac{1}{2} + \frac{3}{4} \times \frac{1}{2} = \frac{3}{4}. \quad \triangleleft
\end{aligned}$$

■ Without thinking it through, it might be tempting to guess that $\mathbb{P}(W_1 \times W_2 > 0) = \mathbb{P}(W_1 \times W_2 < 0) = \frac{1}{2}$, given the independent increment property. Why is this "intuition" wrong?

13. If $X \sim N(0, 1)$, then the mean is given by $\mathbb{E}[X] = 0$. The mode is determined as follow:

$$\begin{aligned}
\phi(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\
\phi'(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times (-x) = 0 \qquad \Rightarrow \qquad x = 0.
\end{aligned}$$

The median is given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du = 0.5.$$

Since the probability density function of the standard normal distribution is symmetric across the y-axis, we can infer that $\Phi(0) = 0.5$. $\triangleleft$

14. For $X \sim N(\mu, \sigma^2)$, mean is given by $\mathbb{E}[X] = \mu$. The mode is given by

$$f'(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \times \left[-\frac{(x-\mu)}{\sigma^2}\right] = 0 \qquad \Rightarrow \qquad x = \mu.$$

Again due to the symmetric property of normal distribution, we know that

$$F(m) = \int_{-\infty}^{m} f(x)dx = 0.5 \qquad \Rightarrow \qquad m = F^{-1}(0.5) = \mu. \quad \triangleleft$$

15. This can be worked out using either moment generating function (MGF) or just basic integration. Under the MGF approach, we see that

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \mathbb{E}[X], \quad X \sim N(\mu, \sigma^2).$$

The MGF for X, a normally distributed random variable, is given by

$$M_X(t) = \mathbb{E}[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Taking the first derivative, we obtain

$$\frac{dM_X(t)}{dt} = (\mu + \sigma^2 t) e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

We can therefore conclude that

$$\mathbb{E}[X] = \frac{dM_X(0)}{dt} = \mu.$$

Alternatively, using basic integration approach, we can write

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \underbrace{\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx}_{\text{first integral}} + \underbrace{\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\mu e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx}_{\text{second integral}}$$

The second integral evaluates to μ. The first integral evaluates to 0. To see this, let $u = \frac{x-\mu}{\sqrt{2}\sigma}$, we have $du = \frac{dx}{\sqrt{2}\sigma}$, and hence the first integral becomes

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} u e^{-u^2}\,du,$$

which is 0. $\triangleleft$

16. Similar to the previous question, this can be worked out using either MGF or just basic integration technique. To use the MGF approach, first expand the integral, and we obtain

$$\begin{aligned}\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x^2-2x\mu+\mu^2)e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx\\ &= \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mu^2\\ &= \mathbb{E}[X^2] - \mu^2\end{aligned}$$

Using the MGF approach, taking the 2^{nd} derivative of the normal random variable's moment generating function with respect to t, we obtain

$$\begin{aligned}\frac{dM_X(t)}{dt} &= (\mu + \sigma^2 t)e^{\mu t + \frac{1}{2}\sigma^2 t^2}\\ \frac{d^2M_X(t)}{dt^2} &= (\sigma^2 + (\mu+\sigma^2 t)^2)e^{\mu t + \frac{1}{2}\sigma^2 t^2}\end{aligned}$$

Hence

$$\mathbb{E}[X^2] = \frac{d^2M_X(0)}{dt^2} = \sigma^2 + \mu^2$$

And so we've shown that the integral evaluates to σ^2. Alternatively, we can just use integration by parts as follow:

$$\int u\,dv = uv - \int v\,du,$$

where

$$\begin{aligned}u = x-\mu &\quad\Rightarrow\quad du = dx,\\ v = -\sigma^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} &\quad\Rightarrow\quad dv = (x-\mu)e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} u\,dv\\ &= -\frac{1}{\sqrt{2\pi}\sigma}\,(x-\mu)\sigma^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}\Big|_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\sigma^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx\\ &= 0 + \sigma^2\times 1 = \sigma^2. \quad\triangleleft\end{aligned}$$

17. The MGF of a normally distributed random variable $X \sim N(\mu, \sigma^2)$ is given by

$$\mathbb{E}[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Hence we have

$$\mathbb{E}[Y] = \mathbb{E}[e^X] = \mathbb{E}[e^{tX}]\big|_{t=1} = e^{\mu + \frac{1}{2}\sigma^2}$$

Alternatively, we can solve the question by completing the square (see the next question). $\triangleleft$

18. (a) By completing the square, we have

$$\begin{aligned}
\mathbb{E}[e^{\theta X}] &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{\theta x} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{\theta x - \frac{x^2 - 2\mu x + \mu^2}{2\sigma^2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{x^2 - 2\mu x + \mu^2 - 2\sigma^2\theta x}{2\sigma^2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{x^2 - 2(\mu+\sigma^2\theta)x + \mu^2}{2\sigma^2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{x^2 - 2(\mu+\sigma^2\theta)x + (\mu+\sigma^2\theta)^2 - (\mu+\sigma^2\theta)^2 + \mu^2}{2\sigma^2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2\theta))^2 - (\mu+\sigma^2\theta)^2 + \mu^2}{2\sigma^2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2\theta))^2 - \mu^2 - 2\mu\sigma^2\theta - \sigma^4\theta^2 + \mu^2}{2\sigma^2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2\theta))^2}{2\sigma^2}} e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2}\,dx \\
&= e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2} \quad \triangleleft
\end{aligned}$$

(b) By the MGF approach, we have

$$\mathbb{E}[e^{\theta X}] = \mathbb{E}[e^{tX}]\big|_{t=\theta} = e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2} \quad \triangleleft$$

19. Since $X \sim N(0,1)$, we know that $V[X] = 1$, as the variance of X is already given to be equal to 1. Therefore,

$$V[Y_t] = V[\sqrt{t}X] = t \cdot V[X] = t.$$

20. First work out the partial derivatives:

$$\frac{\partial f}{\partial t} = x^2, \quad \frac{\partial f}{\partial x} = 2tx, \quad \frac{\partial^2 f}{\partial t^2} = 0, \quad \frac{\partial^2 f}{\partial x^2} = 2t, \quad \frac{\partial^2 f}{\partial t \partial x} = 2x.$$

Expanding around (t_0, x_0), we obtain

$$\begin{aligned}
f(t,x)|_{(t_0,x_0)} = f(t_0,x_0) &+ \frac{\partial f}{\partial t}(t-t_0) + \frac{\partial f}{\partial x}(x-x_0) \\
&+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial t^2}(t-t_0)^2 + 2\frac{\partial^2 f}{\partial t \partial x}(t-t_0)(x-x_0) + \frac{\partial^2 f}{\partial x^2}(x-x_0)^2\right] \\
&+ \cdots
\end{aligned}$$

Writing

$$\begin{aligned}\Delta t &= t - t_0,\\ \Delta x &= x - x_0,\\ \Delta f &= f(t,x) - f(t_0,x_0),\end{aligned}$$

we have

$$\Delta f \approx x_0^2 \Delta t + 2t_0 x_0 \Delta x + 2x_0 \Delta t \Delta x + t_0 \Delta x^2. \quad \triangleleft$$