Session 7
Quantitative Analysis of Financial Markets
Volatility: GARCH

Christopher Ting

http://www.mysmu.edu/faculty/christophert/

Christopher Ting

✉: christopherting@smu.edu.sg

☎: 6828 0364

: LKCSB 5036

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Lesson Plan

Introduction

- The objective is to use historical data to produce estimates of the current and future levels of volatilities and correlations.
- You need to recognize that volatilities and correlations are not constant.
- Applications that motivate this module:
  - VaR
  - Valuation of derivatives
- Three important models:
  - EWMA: Exponentially Weighted Moving Average
  - ARCH: Autoregressive Conditional Heteroscedasticity
  - GARCH: Generalized Autoregressive Conditional Heteroscedasticity

Learning Outcomes of QA14

Chapter 10.

John C. Hull, Risk Management and Financial Institutions, 4th Edition (Hoboken, NJ: John Wiley & Sons, 2015).

- Define and distinguish between volatility, variance rate, and implied volatility.
- Describe the power law.
- Explain how various weighting schemes can be used in estimating volatility.
- Apply the exponentially weighted moving average (EWMA) model to estimate volatility.
- Describe the generalized autoregressive conditional heteroskedasticity (GARCH(p, q)) model for estimating volatility and its properties.

Learning Outcomes of QA14 (cont'd)

- Calculate volatility using the GARCH(1,1) model.
- Explain mean reversion and how it is captured in the GARCH(1,1) model.
- Explain the weights in the EWMA and GARCH(1,1) models.
- Explain how GARCH models perform in volatility forecasting.
- Describe the volatility term structure and the impact of volatility changes.

Lesson Plan

- Define value a risk (VaR).
- Describe the essential idea behind maximum likelihood estimation (MLE).
- Gain a deeper insights into GARCH processes.
- Apply GARCH processes to forecast VaR.

Risks and Risk Management

✓ Major Risks

- § Market risks
- § Credit risks
- § Operational risks
- § Liquidity risks
- § Legal risks
- § Political risks
- § Model risks

✓ Industry Practices

- § Regulatory capital adequacy
- § Bank's internal risk control
- § Corporations' investments
- § Firm's hedging of transactions
- § Exchanges' margining rules and practices

Introduction: Risk Measure

- Risk Management is a procedure for shaping a loss distribution.
- Despite some serious shortcomings, Value-at-Risk, or VaR, is the most popular portfolio risk measure used by risk management practitioners.
- VaR is a number constructed on day t such that the portfolio losses on day $t+1$ will only be larger than the VaR forecast with probability p, e.g. 5%.
- The main objective of this lesson is to see how GARCH model is applied in forecasting VaR.
  - ✓ Question: What risk does VaR address?

Value at Risk

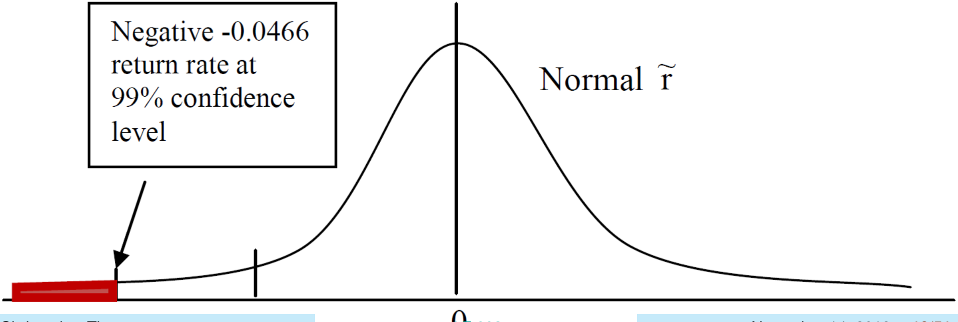
- VaR was popularized by J.P. Morgan in the 1990s. The executives at J.P. Morgan wanted their risk managers to generate one number at the end of each day to summarize the risk of the firm's entire portfolio.
- What they came up with was VaR.
- If the 95% VaR of a portfolio is \$400, then we expect the portfolio will lose \$400 or less in 95% of of the scenarios, and lose more than \$400 in 5% of the scenarios.
- We can define VaR for any confidence level, but 95% has become an extremely popular choice.
- VaR is a one-tailed confidence interval.

Definition of Value at Risk and Example

Definition of VaR

VaR is the maximum loss over a specified horizon at a given confidence level (e.g. 95%).

- Example: Suppose log return $\tilde{r} \overset{d}{\sim} N(\mu, \sigma^2)$. Suppose daily volatility $\sigma = 2\%$ and daily mean $\mu = 0$.



Example

- Suppose the portfolio value is $P_0 = \$100$ million.
- The daily log return is

$$\widetilde{r} = \ln\left(\frac{\widetilde{P}_1}{P_0}\right)$$

- Since $\widetilde{r} = -2.33 \times 0.02 = -0.0466$, we have

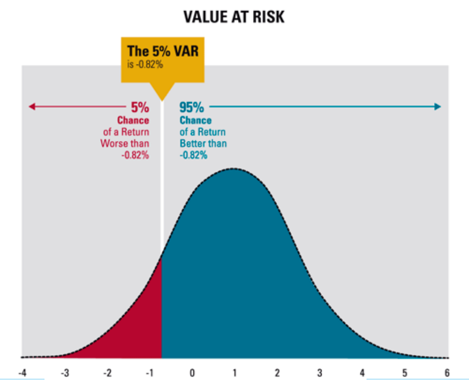
$$-0.0466 = \ln\left(\frac{\widetilde{P}_1}{\$100\text{m}}\right) \qquad \text{or} \qquad \widetilde{P}_1 = \$100\text{m} \times e^{-0.0466}$$

- The VaR is, at the 99% confidence level

$$\$100\text{m} \times \left(1 - e^{-0.0466}\right) = \$4.553\text{m}.$$

Value at Risk Time Horizon

- The time horizon needs to be specified for VaR.
- On trading desks, with liquid portfolios, it is common to measure the one-day 95% VaR.
- In other settings, in which less liquid assets may be involved, time frames of up to one year are not uncommon.



VaR Exceedance

- If an actual loss equals or exceeds the predicted VaR, that event is known as an **exceedance**.
- For a one-day 95% VaR, the probability of an exceedance event on any given day is 5%.
- Let the random variable L represent the loss to your portfolio. L is simply the negative of the return to your portfolio. If the return of your portfolio is -\$600, then the loss, L, is +\$600.
- For a given confidence level, α , then, we can define value at risk as

$$\mathbb{P}\left(L \geq \mathrm{VaR}_\alpha\right) = 1 - \alpha.$$

- If a risk manager says that the one-day 95% VaR of a portfolio is \$400, it means that there is a 5% probability that the portfolio will lose \$400 or more on any given day (that L will be more than \$400).

Remarks

- We can also define VaR directly in terms of returns. If we multiply both sides of the inequality in by -1, and replace $-L$ with R, we come up with

$$\mathbb{P}\left(R \leq -\mathrm{VaR}_\alpha\right) = 1 - \alpha.$$

- A loss of \$400 or more and a return of -\$400 or less are exactly the same.
- Notice that the definition does not invoke the assumption that the distribution is normal.

Absolute Versus Relative VaR

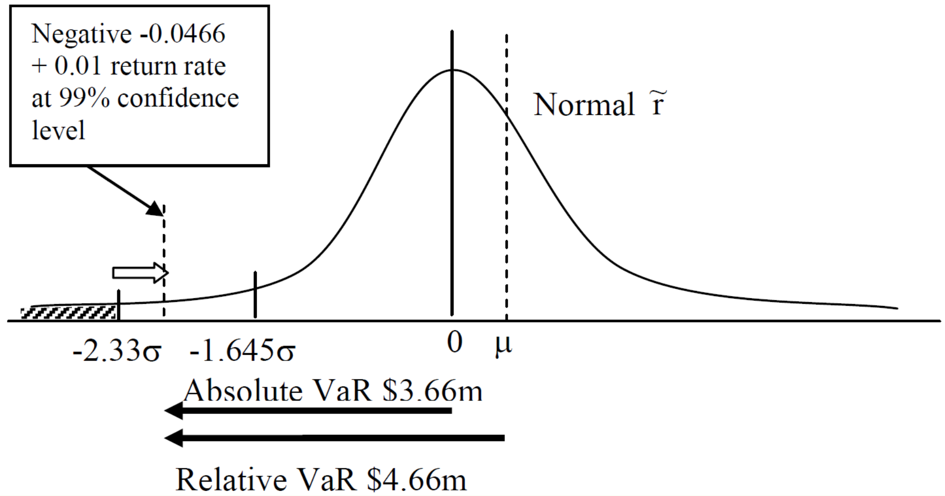
- Absolute VaR is measured with respect to the current marked-to-market portfolio value regardless of μ
- Relative VaR is computed taking into account the loss also of the expected profit μP_0.
- Suppose $\mu = 1\%$. At the 99% confidence level, the critical value is -2.33.

$$\mathbb{P}\big(\widetilde{r} < \mu - 2.33\sigma\big) = \mathbb{P}\big(\widetilde{r} < 0.01 - 2.33 \times 0.02\big) = \mathbb{P}\big(\widetilde{r} < -0.0366\big) = 1\%$$

- The loss is

$$P_0(1 - e^r) = \$100\text{m} \times \left(1 - e^{-0.0366}\right) = \$3.594\text{m}$$

Illustration



Definition of Volatility and Variance Rate

- Define σ_n as the volatility of a market variable on day n, as estimated at the end of day $n-1$.
- The square of the volatility, σ_n^2, on day n is the **variance rate**.
- Denote S_i as the value of a variable at the end of day i.
  - Define the log return as $u_i := \ln \dfrac{S_i}{S_{i-1}}$.
  - An unbiased estimate of the mean of u_i, using the most recent m observations with respect to "today" n is

$$\overline{u} = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}.$$

  - An unbiased estimate of the variance rate per day, σ_n^2, is

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i} - \overline{u})^2.$$

Really, what is volatility?

- Intuitively, volatility is a measure of a financial asset's readiness to move from one price to another price, in such a way that leaves the market participants unsure about the next quantum of price change.

- The more ready the asset's price is to move, the more uncertain its price in the future will be. The resulting price uncertainty makes the asset risky to the investors, as the price may move in the direction contrary to what they expect, and they become exposed to the consequent losses.

- Just like there are many different definitions of returns, volatility is defined differently for different purposes and from different sources.

Alternative Estimation

- For monitoring daily volatility, we define u_i as a percentage change in the market variable instead:

$$u_i = \frac{S_i - S_{i-1}}{S_{i-1}}.$$

- Assume $\overline{u}$ to be zero.
- Replace $m-1$ by m to obtain a maximum likelihood estimate:

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}^2.$$

Implied Volatilities

Definition

The implied volatility of an option is the volatility that gives the market price of the option when it is substituted into the pricing model.

- An example of implied volatility: **VIX Index**

Power Law

- An alternative to assuming normal distributions.
- For many variables that are encountered in practice, it is approximately true that the value of the variable, v, has the property that when x is large

$$\mathbb{P}\left(v > x\right) = Kx^{-\alpha},$$

where K and α are constants.

Weighting Scheme

- So far, we give an equal weight to $u_{n-1}^2, u_{n-2}^2, \ldots, u_{n-m}^2$.
- To estimate the *current* level of volatility, give more weight α_i to recent data.

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2. \tag{1}$$

- The weights α_i, $i = 1, 2, \ldots, m$ must satisfy the following:
  - positive: $\alpha_i > 0$.
  - lesser weight for older observations: $\alpha_i < \alpha_j$ when $i > j$.
  - summed to unity: $\sum_{i=1}^{m} \alpha_i = 1$.

Model for Estimating the Variance Rate: ARCH

- Let V_L be the long-run variance rate and γ be the weight assigned to V_L. The model for estimating the variance rate is

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}^2.$$

- The weights must sum to one.

$$\gamma + \sum_{i=1}^{m} \alpha_i = 1.$$

- Define $\omega := \gamma V_L$. The model is rewritten as

$$\sigma_n^2 = \omega + \sum_{i=1}^{m} \alpha_i u_{n-i}^2.$$

Model for Estimating the Variance Rate: ARCH

- The EMWA model is a special case of (1) where the weights α_i decrease exponentially as we move back through time.
- Specifically, with $0 < \lambda < 1$,

$$\alpha_{i+1} = \lambda \alpha_i,$$

the EMWA model is

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1 - \lambda) u_{n-1}^2.$$

- Only the current estimate of the variance rate and the most recent observation on the value of the market variable are needed.
- The RiskMetrics database uses the EWMA model with $\lambda = 0.94$ for updating daily volatility estimates.

Exponential Decline

- We know

$$\begin{aligned}\sigma_n^2 &= \lambda\sigma_n^2 + (1-\lambda)u_{n-1}^2 \\ &= \lambda\left(\lambda\sigma_{n-2}^2 + (1-\lambda)u_{n-2}^2\right) + (1-\lambda)u_{n-1}^2 \\ &= (1-\lambda)u_{n-1}^2 + \lambda(1-\lambda)u_{n-2}^2 + \lambda^2\sigma_{n-2}^2\end{aligned}$$

- Substituting for σ_{n-2}^2, then for σ_{n-3}^2, then for σ_{n-4}^2, and so on:

$$\begin{aligned}\sigma_n^2 =&(1-\lambda)u_{n-1}^2 + \lambda(1-\lambda)u_{n-2}^2 + \lambda^2(1-\lambda)u_{n-3}^2 + \cdots \\ &\cdots + \lambda^{m-1}(1-\lambda)u_{n-m}^2 + \lambda^m\sigma_{n-m}^2.\end{aligned}$$

- Weights start at $1-\lambda$ and decline at rate λ.

Attractions of EWMA

- Relatively little data needs to be stored
- We need only remember the current estimate of the variance rate and the most recent observation on the market variable
- Tracks volatility changes
- 0.94 is a popular choice for λ

ARCH

- A model for a process on the variance of u_t

$$\mathbb{V}(u_t) = \alpha_0 + \alpha_1 u_{t-1}^2$$

- More generally, with $e_t \overset{d}{\sim} N(0,1)$,

$$u_t = e_t \sqrt{\alpha_0 + \alpha_1 u_{t-1}^2}$$

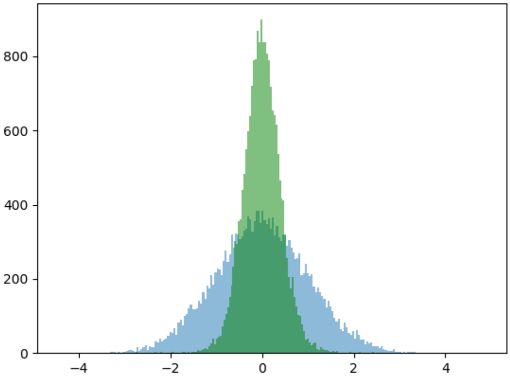
- The process u_t is unconditionally not a normal distribution. However, conditional on u_{t-1}, u_t is normally distributed.

ARCH Simulation Code

```
from __future__ import division, print_function
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import jarque_bera

alpha0, alpha1, n = 0.1, 0.4, 20000
np.random.seed(137)
e = np.random.normal(size=n)

plt.plot(e)
plt.show()

u = np.zeros(n)
u[0] = e[0]
for t in range(1,n):
   u[t] = e[t]*np.sqrt(alpha0 + alpha1 * u[t-1]**2)

plt.plot(u)
plt.show()

JBstat_e, JBstat_u = jarque_bera(e), jarque_bera(u)
print(JBstat_e)
print(JBstat_u)

he, hu = plt.hist(e, 200, alpha=0.5), plt.hist(u, 200, facecolor='g', alpha=0.5)
plt.savefig('ARCH_Simulate.png')
plt.show()
```

Histogram of Simulated ARCH(1) Process

- The Jarque-Bera test statistic of 1.74 does not reject the null hypothesis of normal distribution for the noise e_t in blue.
- But at 7,732, the Jarque-Bera test strongly rejects the null hypothesis for the ARCH(1) process u_t in green.



GARCH (1,1)

- Bollerslev in 1986 proposed the GARCH(1, 1) model.
- In GARCH(1,1), σ_n^2 is calculated from a **long-run average variance rate**, V_L, as well as from σ_{n-1}^2 and u_{n-1}^2.
- The GARCH(1,1) model is

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2.$$

- Since the weights must sum to unity, it follows that

$$\gamma + \alpha + \beta = 1.$$

- The EWMA model is a particular case of GARCH(1,1) where $\gamma = 0$, $\alpha = 1 - \lambda$ and $\beta = \lambda$.

Parameters of GARCH (1,1)

- Let $\omega := \gamma V_L$. The GARCH (1,1) is rewritten as

$$\sigma_n^2 = \omega + \alpha\, u_{n-1}^2 + \beta\, \sigma_{n-1}^2.$$

- Once the parameters ω, α, and β have been estimated, we can calculate γ as $1 - \alpha - \beta$.
- The **long-term variance** V_L can then be calculated as

$$V_L = \frac{\omega}{\gamma} = \frac{\omega}{1 - \alpha - \beta}.$$

- For a stable GARCH(1,1) process, we require $\alpha + \beta < 1$, and obviously $\omega > 0$.

Unconditional Stationarity

- While GARCH processes are conditionally nonstationary with changing variances, they are still unconditionally stationary processes.

$$
\begin{aligned}
\sigma_n^2 &= \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 \\
&= \omega + \alpha u_{n-1}^2 + \beta \Big(\omega + \alpha u_{n-2}^2 + \beta \sigma_{n-2}^2 \Big) \\
&= \omega \big(1 + \beta \big) + \alpha \big(u_{n-2}^2 + \beta u_{n-2} \big) + \beta^2 \Big(\omega + \alpha u_{n-3}^2 + \beta \sigma_{n-3}^2 \Big) \\
&= \omega \big(1 + \beta + \beta^2 + \cdots \big) + \alpha \big(u_{n-1}^2 + \beta u_{n-2}^2 + \beta^2 u_{n-3}^2 + \cdots \big)
\end{aligned}
$$

- Taking unconditional expectation on both sides, so that $\sigma^2 := \mathbb{E}\left(u_n^2\right) = \mathbb{E}\left(u_{n-1}^2\right) = \mathbb{E}\left(u_{n-2}^2\right) = \cdots$.

Unconditional Stationarity (cont'd)

- Then

$$\begin{aligned}\sigma^2 &= \frac{\omega}{1-\beta} + \alpha(\sigma^2 + \beta\sigma^2 + \beta^2\sigma^2 + \cdots) \\ &= \frac{\omega}{1-\beta} + \frac{\alpha\sigma^2}{1-\beta}\end{aligned}$$

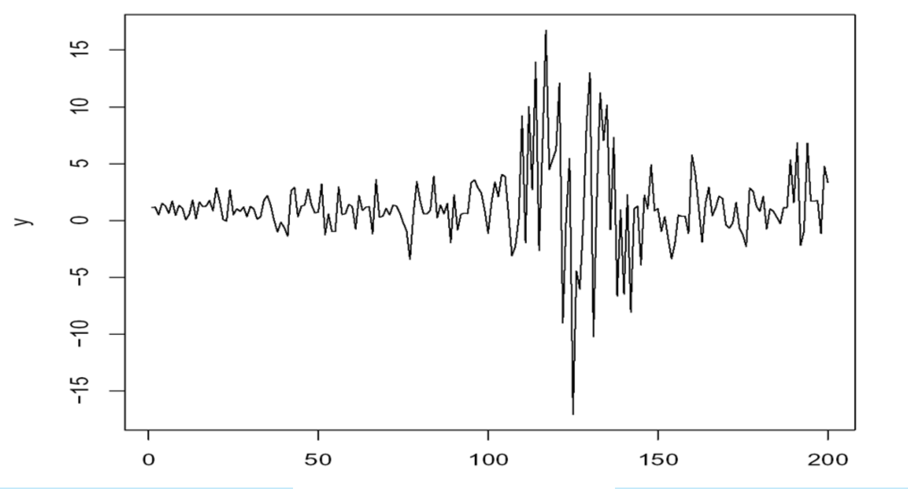
- The unconditional variance σ^2 is constant!

$$\sigma^2 = \frac{\omega}{1-\alpha-\beta} = V_L,$$

provided $\omega > 0$, and $|\alpha+\beta| < 1$.

Simulated GARCH Process

$$\sigma_n^2 = 0.50 + 0.25u_{n-1}^2 + 0.70\sigma_{n-1^2}$$



Example

- Suppose $\sigma_n^2 = 0.000002 + 0.13u_{n-1}^2 + 0.86\sigma_{n-1}^2$.
- The long-run variance rate is 0.0002 so that the long-run volatility per day is 1.41%.
- Suppose that the current estimate of the volatility is 1.6% per day and the most recent percentage change in the market variable is 1%.
- The new variance rate is

$$0.000002 + 0.13 \times 0.01^2 + 0.86 \times 0.016^2.$$

Consequently, the new volatility is 1.53% per day.

Mean Reversion

- The GARCH (1,1) model recognizes that over time the variance tends to get pulled back to a long-run average level of V_L.

- Stochastic differential equation for the variance V:

$$dV = a(V_L - V)dt + \xi V dz,$$

where time is measured in days, $a = 1 - \alpha - \beta$, and $\xi = \alpha\sqrt{2}$.

- Mean reversion: The variance has a drift that pulls it back to V_L at rate a. When $V > V_L$, the variance has a negative drift; when $V < V_L$, it has a positive drift.

Estimating a Constant Variance

- Maximum likelihood method involves choosing values for the parameters that maximize the chance (or likelihood) of observations occurring.
- Example 1: We observe that a certain event happens one time in 10 trials. What is our estimate of the proportion of the time, p, that it happens?
- The probability of the event happening on one particular trial and not on the others is $p(1-p)^9$.
- We maximize this probability to obtain a maximum likelihood estimate. Result: $p = 0.1$ (as expected)

Example 2

- Estimate the variance of observations from a normal distribution with mean zero

$$\prod_{i=1}^{m} \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{u_i^2}{2v}\right)$$

- Taking logarithm of the function is equivalent to maximizing

$$\sum_{i=1}^{m} \left(-\ln(v) - \frac{u_i^2}{v}\right)$$

- Result:

$$v = \frac{1}{m} \sum_{i=1}^{m} u_i^2 .$$

Application to GARCH

- We choose parameters that maximize

$$\prod_{i=1}^{m} \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{u_i^2}{2v}\right)$$

or

$$\sum_{i=1}^{m} \left(-\ln(v_i) - \frac{u_i^2}{v_i}\right)$$

S&P 500 Excel Application

- Start with trial values of ω, α, and β.
- Update variances.
- Calculate the likelihood as an objective function:

$$\sum_{i=1}^{m}\left(-\ln(v_i)-\frac{u_i^2}{v_i}\right)$$

- Use solver to search for values of ω, α, and β that maximize the likelihood.
- For efficient operation of Solver, set up spreadsheet so that three numbers that are the same order of magnitude are being searched for.

How Good Is the Model?

- If a GARCH model is working well, it should remove the autocorrelation in u_i^2/σ_i^2.
- We can test whether it has done so by considering the autocorrelation structure for the variables u_i^2 If these show very little autocorrelation, our model for σ_i^2 has succeeded in explaining the autocorrelations in u_i^2.
- Ljung-Box statistic for the u_i^2/σ_i^2 time series.
  - Chi-square

Variance Targeting

- One way of implementing GARCH(1,1) that increases stability is by using variance targeting.
- We set the long-run average volatility equal to the sample variance.
- Only two other parameters then have to be estimated.

Forecasting Future Volatility

- The variance rate estimated at the end of day $n-1$ for day n, when GARCH(1,1) is used, is

$$\sigma_n^2 = (1-\alpha-\beta)V_L + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2.$$

- If follows that

$$\sigma_n^2 - V_L = \alpha\big(u_{n-1}^2 - V_L\big) + \beta\big(\sigma_{n-1}^2 - V_L\big)$$

- On day $n+t$ in the future and with σ_n^2,

$$\sigma_{n+t}^2 - V_L = \alpha\big(u_{n+t-1}^2 - V_L\big) + \beta\big(\sigma_{n+t-1}^2 - V_L\big)$$

- The expected value of u_{n+t-1}^2 is σ_{n+t-1}^2. Hence

$$\mathbb{E}\left(\sigma_{n+t}^2 - V_L\right) = (\alpha+\beta)\,\mathbb{E}\left(\sigma_{n+t-1}^2 - V_L\right).$$

Forecasting Future Volatility (cont'd)

- Applying the above equation repeatedly yields

$$\mathbb{E}\left(\sigma_{n+t}^2 - V_L\right) = (\alpha+\beta)^t\left(\sigma_n^2 - V_L\right),$$

which is

$$\mathbb{E}\left(\sigma_{n+t}^2\right) = V_L + (\alpha+\beta)^t(\sigma_n^2 - V_L).$$

- This equation allows you to forecast the volatility on day $n+t$ using the information available at the end of day n.

- **Real-World Application**
The maintenance margin x is set as, given today's settlement price of F_n,

$$x \geqq 1.645\ \mathbb{E}\left(\sigma_{n+1}^2\right) F_n$$

Maximum Likelihood Estimators

- With $e_t \overset{d}{\sim} N(0,1)$, consider a model $Y_t = f\big(\boldsymbol{X};\boldsymbol{\theta}\big) + e_t$. Then $Y_t - f\big(\boldsymbol{X};\boldsymbol{\theta}\big)$ is distributed as i.i.d. $N\big(0,\sigma_e^2\big)$. Its probability density function is

$$\frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(-\frac{1}{2}\left(\frac{Y_t - f\big(\boldsymbol{X};\boldsymbol{\theta}\big)}{\sigma_e}\right)^2\right)$$

- Given a sample size N of $\{Y_1, Y_2, \ldots, Y_N,$ and $\boldsymbol{X}\}$, the likelihood function is

$$L = \left(\frac{1}{\sqrt{2\pi\sigma_e^2}}\right)^N \exp\left(-\frac{1}{2}\sum_{t=1}^{N}\left(\frac{Y_t - f\big(\boldsymbol{X};\boldsymbol{\theta}\big)}{\sigma_e}\right)^2\right)$$

- Suppose we find estimate values $\boldsymbol{\theta}$ and σ_e that maximizes this log-likelihood function $\log L$. Then the estimates are called Maximum Likelihood estimates.

Cramer-Rao Inequality

- The parameters to be estimated and data are assembled into vectors:

$$\boldsymbol{\Lambda} := \begin{pmatrix} \boldsymbol{\theta} \\ \sigma_e \end{pmatrix} \quad \text{and} \quad \boldsymbol{Z} := \begin{pmatrix} \boldsymbol{Y} & \boldsymbol{X} \end{pmatrix}$$

- Probabilities add up to 1:

$$\int_{-\infty}^{\infty} L(\boldsymbol{Z}\,;\,\boldsymbol{\Lambda})d\boldsymbol{Z} = 1; \qquad \int_{-\infty}^{\infty} \frac{\partial L(\boldsymbol{Z}\,;\,\boldsymbol{\Lambda})}{\partial \boldsymbol{\Lambda}} d\boldsymbol{Z} = 0$$

- **Fisher's information matrix**

$$\boldsymbol{R}(\boldsymbol{\Lambda}) := -\mathbb{E}\left(\frac{\partial^2 \log L}{\partial \boldsymbol{\Lambda} \partial \boldsymbol{\Lambda}^\top}\right)$$

- Let $h(\boldsymbol{Z})$ be an unbiased estimator of $\boldsymbol{\Lambda}$. Then

$$\mathbb{C}\big(h(\boldsymbol{Z})\big) \geq \boldsymbol{R}^{-1}$$

- Question: What's the interpretation of Cramer-Rao Inequality?

Volatility Term Structures

- Suppose it is day n. Define

$$V(t) := \mathbb{E}\left(\sigma_{n+t}^2\right), \qquad \text{and} \qquad a := \ln \frac{1}{\alpha + \beta}.$$

- The predictive equation becomes

$$V(t) = V_L + e^{-at}\big(V(0) - V_L\big).$$

Here, $V(t)$ is an estimate of the instantaneous variance rate in t days.

Volatility Term Structures (cont'd)

- The average variance rate per day between today and time T is given by

$$\frac{1}{T}\int_0^T V(t)\,dt = V_L + \frac{1-e^{-aT}}{aT}\big(V(0)-V_L\big).$$

- Then the volatility per annum for an option lasting T days is

$$\sigma(T) = \sqrt{252\left(V_L + \frac{1-e^{-aT}}{aT}(V(0)-V_L)\right)}.$$

- So from GARCH(1,1), we have a volatility term structure, which is the relationship between the forward-looking volatilities and the maturities.

S&P Volatility Term Structure Predicted from GARCH(1,1)

- Note that a is positive since $\alpha + \beta < 1$.
- $\omega = 0.0000013465$, $\alpha = 0.083394$, and $\beta = 0.910116$.

$$a = \ln\left(\frac{1}{0.083394 + 0.910116}\right) = 0.006511$$

Option Life (days)	10	30	50	100	500
Volatility (% per annum)	27.4	27.1	26.9	26.4	24.3

Impact of Volatility Changes

- We note that $V(0) = \sigma(0)^2/252$.

- When instantaneous volatility $\sigma(0)$ changes by $\Delta\sigma(0)$, volatility for T-day maturity changes by approximately

$$\Delta\sigma(T) \approx \frac{1 - e^{-aT}}{aT}\frac{\sigma(0)}{\sigma(T)}\Delta\sigma(0).$$

- Impact of 1% change in the instantaneous volatility predicted from GARCH (1,1)

Option Life (days)	10	30	50	100	500
Volatility increase (%)	0.97	0.92	0.87	0.77	0.33

Summary

- In the EWMA and the GARCH(1,1) models, the weights assigned to observations decrease exponentially as the observations become older.
- The GARCH(1,1) model differs from the EWMA model in that some weight is also assigned to the long-run average variance rate. It has a structure that enables forecasts of the future level of variance rate to be produced relatively easily.
- Maximum likelihood methods are usually used to estimate parameters from historical data in the EWMA, GARCH(1,1), and similar models.
- Once its parameters have been determined, a GARCH(1,1) model can be judged by how well it removes autocorrelation from the u_i^2.