QF620 Stochastic Modelling
End-term Revision Pack

1 Practice Questions

1. Suppose the stock price follows the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t^*$$

where W_t^* is a standard Brownian motion under the risk-neutral measure associated with the risk-free bond as numeraire. Derive the valuation formula for an option paying

$$\left(\sqrt{S_T} - K\right)^+$$

on the maturity date T.

2. Suppose the stock price follows the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t^*$$

where W_t^* is a standard Brownian motion under the risk-neutral measure associated with the risk-free bond as numeraire. We want to derive a valuation formula for an option paying

$$S_T\left(S_T - K\right)^+$$

on the maturity date T.

   (a) Derive the formula by taking the expectation of the payoff directly in the $\mathbb{Q}^*$ measure.

   (b) Derive the formula by first changing to the risk-neutral measure associated to the stock as numeraire $\mathbb{Q}^S$ using Radon-Nikodym derivative.

3. (a) Consider the stochastic process $Y_t = \exp(\nu t + \sigma W_t)$. What is the necessary relationship between ν and σ for Y_t to be a martingale?

   (b) Consider the stock price process

$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$

   where W_t^* is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}^*$ associated with the risk-free bond numeraire $dB_t = rB_t dt$. What is the Radon-Nikodym derivative $\frac{d\mathbb{Q}^S}{d\mathbb{Q}^*}$ that change the risk-neutral measure from the risk-free bond numeraire to a stock numeraire?

4. Suppose the stock price follows the process

$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$

where W_t^* is a $\mathbb{Q}^*$-Brownian motion.

(a) Derive a valuation formula for a *forward contract* paying

$$S^{\frac{1}{n}} - K^{\frac{1}{m}},$$

where $n,\ m \in \mathbb{N}$.

(b) Derive a valuation formula for an *option* paying

$$\left(S^{\frac{1}{n}} - K^{\frac{1}{m}}\right)^+,$$

where $n,\ m \in \mathbb{N}$.

5. An asset-or-nothing digital call option pays (on maturity T)

$$V_T = S_T \mathbb{1}_{S_T \geq K^*}$$

Suppose $K^* > F$, starting with

$$\int_F^\infty h(K) \frac{\partial^2 C(K)}{\partial K^2} dK,$$

use static-replication approach to value the contract.

6. A contract pays (on maturity date T)

$$V_T = \sqrt{S_T}.$$

Derive a valuation formula for the contract at time 0 using

(a) Black-Scholes model

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$

(b) Static replication approach.

$$\int_0^F h(K) \frac{\partial^2 P(K)}{\partial K^2} dK + \int_F^\infty h(K) \frac{\partial^2 C(K)}{\partial K^2} dK$$

7. Consider the Black-Scholes process

$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$

where W_t^* is a $\mathbb{Q}^*$-Brownian motion. The call option formula is given by

$$C(S_0, K, r, \sigma, T) = S_0 \Phi(d_1) - Ke^{-rT} \Phi(d_2),$$

where

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T} = \frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}.$$

Find $\frac{\partial C}{\partial K}$.

8. Volga (or vomma) measures the vega's sensitivity to volatility. Let C denote the value of a call option, volga is defined as

$$\frac{\partial^2 C}{\partial \sigma^2}.$$

Under Black-Scholes model, the value of a European call option is expressed as

$$C(S_0, K, r, \sigma, T) = S_0 \Phi\left(d_1\right) - K e^{-rT} \Phi\left(d_2\right),$$

where

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}, \qquad d_2 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}.$$

Derive the expression for volga under Black-Scholes model.

2 Suggested Solutions

1. First, solve the sde to obtain

$$
\begin{aligned}
S_T &= S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T^*} \\
\Rightarrow \quad \sqrt{S_T} &= \sqrt{S_0} e^{\frac{1}{2}\left(r-\frac{\sigma^2}{2}\right)T+\frac{\sigma}{2} W_T^*}
\end{aligned}
$$

The option is in-the-money when

$$
\begin{aligned}
\sqrt{S_T} &> K \\
\sqrt{S_0} e^{\frac{1}{2}\left(r-\frac{\sigma^2}{2}\right)T+\frac{\sigma}{2} W_T^*} &> K \\
x &> \frac{2\log\frac{K}{\sqrt{S_0}} - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x^*
\end{aligned}
$$

Now we can proceed to evaluate the expectation

$$
\begin{aligned}
V_0 &= e^{-rT}\mathbb{E}^*\left[\left(\sqrt{S_T}-K\right)^+\right] \\
&= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}\left(\sqrt{S_T}-K\right)\, e^{-\frac{x^2}{2}}\, dx \\
&= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}\sqrt{S_0}e^{\frac{1}{2}\left(r-\frac{\sigma^2}{2}\right)T+\frac{\sigma}{2}\sqrt{T}x}\, e^{-\frac{x^2}{2}}\, dx - Ke^{-rT}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty} e^{-\frac{x^2}{2}}\, dx \\
&= e^{-\frac{rT}{2}}\frac{1}{\sqrt{2\pi}}\sqrt{S_0}e^{-\frac{\sigma^2 T}{4}}\int_{x^*}^{\infty} e^{-\frac{x^2-\sigma\sqrt{T}x+\frac{\sigma^2 T}{4}-\frac{\sigma^2 T}{4}}{2}}\, dx - Ke^{-rT}\Phi(-x^*) \\
&= e^{-\frac{rT}{2}}\frac{1}{\sqrt{2\pi}}\sqrt{S_0}e^{\frac{\sigma^2 T}{8}}\int_{x^*}^{\infty} e^{-\frac{\left(x-\frac{\sigma\sqrt{T}}{2}\right)^2}{2}}\, dx - Ke^{-rT}\Phi(-x^*) \\
&= e^{-\frac{rT}{2}}\sqrt{S_0}e^{\frac{\sigma^2 T}{8}}\Phi\left(-x^*+\frac{\sigma\sqrt{T}}{2}\right) - Ke^{-rT}\Phi(-x^*) \quad \triangleleft
\end{aligned}
$$

2. (a) Under the $\mathbb{Q}^*$ measure, we have

$$S_T = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T^*}.$$

The option is in-the-money when

$$x > \frac{\log\frac{K}{S_0} - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x^*$$

Taking the expectation, we obtain

$$\begin{aligned}
V_0 &= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}\left(S_T^2 - S_T K\right)\ e^{-\frac{x^2}{2}}\ dx \\
&= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty} S_0^2 e^{2\left(r-\frac{\sigma^2}{2}\right)T+2\sigma\sqrt{T}x}\ e^{-\frac{x^2}{2}}\ dx - e^{-rT}\frac{K}{\sqrt{2\pi}}\int_{x^*}^{\infty} S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}\ e^{-\frac{x^2}{2}}\ dx \\
&= e^{rT}\frac{S_0^2}{\sqrt{2\pi}}e^{-\sigma^2 T}\int_{x^*}^{\infty} e^{-\frac{x^2-4\sigma\sqrt{T}x+4\sigma^2T-4\sigma^2T}{2}}\ dx - KS_0\Phi\left(-x^*+\sigma\sqrt{T}\right) \\
&= e^{rT}\frac{S_0^2}{\sqrt{2\pi}}e^{\sigma^2 T}\int_{x^*}^{\infty} e^{-\frac{\left(x-2\sigma\sqrt{T}\right)^2}{2}}\ dx - KS_0\Phi\left(-x^*+\sigma\sqrt{T}\right) \\
&= e^{rT}S_0^2 e^{\sigma^2 T}\Phi\left(-x^*+2\sigma\sqrt{T}\right) - KS_0\Phi\left(-x^*+\sigma\sqrt{T}\right) \\
&= S_0^2 e^{(r+\sigma^2)T}\Phi\left(\frac{\log\frac{S_0}{K}+\left(r+\frac{3\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - S_0 K\Phi\left(\frac{\log\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) \quad \triangleleft
\end{aligned}$$

(b) Under the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}^*} = \frac{S_T/S_0}{e^{rT}/1} = e^{-\frac{\sigma^2 T}{2}+\sigma W_T^*},$$

we note that

$$dW_T^S = dW_T^* - \sigma dt,$$

where W_t^S is a standard Brownian motion under measure $\mathbb{Q}^S$, associated with the stock as numeraire. Therefore, we have

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t^* \\ &= (r+\sigma^2)S_t dt + \sigma S_t dW_t^S \end{aligned}$$

Solving this sde yields

$$S_T = S_0 e^{\left(r+\frac{\sigma^2}{2}\right)T+\sigma W_T^S}.$$

Note that $S_T > K$ when

$$\begin{aligned} S_0 e^{\left(r+\frac{\sigma^2}{2}\right)T+\sigma W_T^S} &> K \\ x &> \frac{\log\frac{K}{S_0} - \left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x^S \end{aligned}$$

To evaluate the expectation, we start by changing the measure as follows:

$$\begin{aligned} V_0 &= e^{-rT}\mathbb{E}^*\left[S_T(S_T-K)^+\right] \\ &= S_0\mathbb{E}^*\left[\frac{S_T/S_0}{e^{rT}/1}(S_T-K)^+\right] \\ &= S_0\mathbb{E}^*\left[\frac{d\mathbb{Q}^S}{d\mathbb{Q}^*}(S_T-K)^+\right] \\ &= S_0\mathbb{E}^S[(S_T-K)^+], \end{aligned}$$

proceeding to evaluate the expectation, we obtain

$$\begin{aligned} V_0 &= S_0\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty} S_0 e^{\left(r+\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}\, e^{-\frac{x^2}{2}}\, dx - S_0 K\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty} e^{-\frac{x^2}{2}}\, dx \\ &= S_0^2 e^{(r+\sigma^2)T}\Phi\left(-x^s+\sigma\sqrt{T}\right) - S_0 K\Phi\left(-x^s\right) \\ &= S_0^2 e^{(r+\sigma^2)T}\Phi\left(\frac{\log\frac{S_0}{K}+\left(r+\frac{3\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - S_0 K\Phi\left(\frac{\log\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right). \quad \triangleleft \end{aligned}$$

3. (a) By Itô's formula,

$$dY_t = \left(\nu + \frac{1}{2}\sigma^2\right) Y_t dt + \sigma^2 Y_t dW_t.$$

So the necessary relationship is

$$\nu + \frac{1}{2}\sigma^2 = 0. \quad \triangleleft$$

(b) The solutions to the differential equations are

$$S_T = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T}, \quad B_T = B_0 e^{rT}.$$

So the Radon-Nikodym derivative is given by

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}^*} = \frac{S_T/S_0}{B_T/B_0} = e^{-\frac{\sigma^2}{2}T + \sigma W_T}. \quad \triangleleft$$

4. (a) The solution to the stochastic differential equation is given by

$$S_T = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^*} \qquad \Rightarrow \qquad S_T^{1/n} = S_0^{1/n} e^{\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n} + \frac{\sigma}{n} W_T^*}.$$

The forward contract is valued as

$$\begin{aligned} f &= e^{-rT}\mathbb{E}^*\left[S_T^{1/n} - K^{1/m}\right] \\ &= e^{-rT}\mathbb{E}^*\left[S_0^{1/n} e^{\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n} + \frac{\sigma}{n} W_T^*} - K^{1/m}\right] \\ &= e^{-rT}\left[S_0^{1/n} e^{\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n} + \frac{\sigma^2 T}{2n^2}} - K^{1/m}\right]. \quad \lhd \end{aligned}$$

(b) First we solve for the integration region

$$\begin{aligned} S_T^{1/n} - K^{1/m} &> 0 \\ x^* &> \frac{n \log \frac{K^{1/m}}{S_0^{1/n}} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x^*. \end{aligned}$$

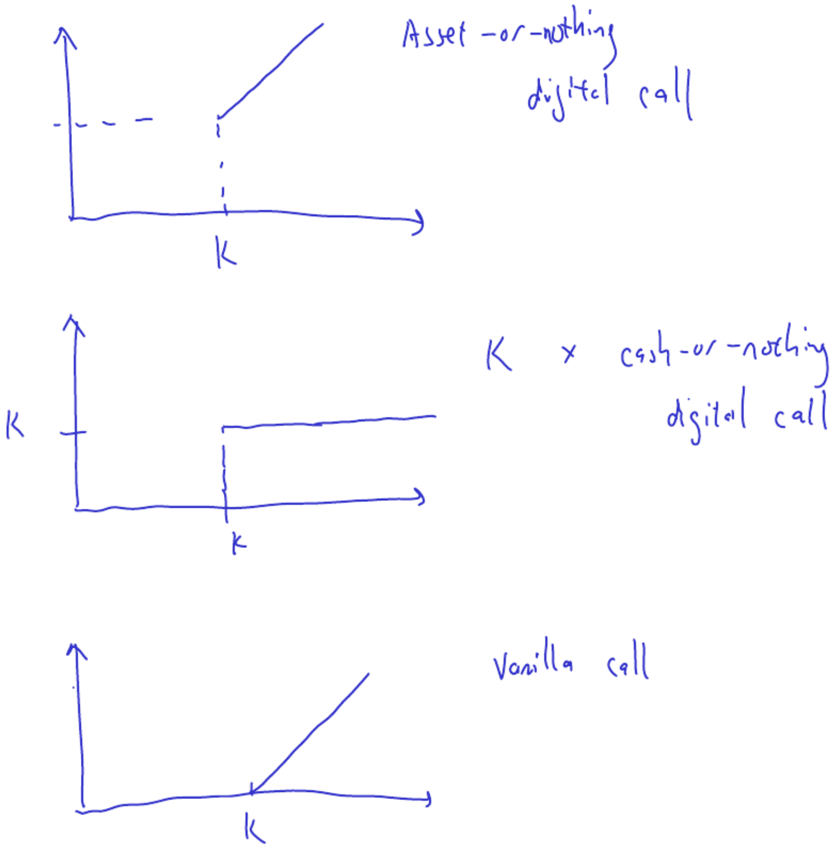
The option can be valued as

$$\begin{aligned} V_0 &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} \left(S_0^{1/n} e^{\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n} + \frac{\sigma\sqrt{T}}{n} x} - K^{1/m}\right) e^{-\frac{x^2}{2}}\, dx \\ &= \frac{e^{-rT}}{\sqrt{2\pi}} S_0^{1/n} e^{\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n}} \int_{x^*}^{\infty} e^{-\frac{\left(x - \frac{\sigma\sqrt{T}}{n}\right)^2}{2} + \frac{\sigma^2 T}{2n^2}}\, dx - \frac{e^{-rT}}{\sqrt{2\pi}} K^{1/m} \int_{x^*}^{\infty} e^{-\frac{x^2}{2}}\, dx \\ &= S_0^{1/n} e^{-rT + \frac{rT}{n} - \frac{\sigma^2 T}{2n} + \frac{\sigma^2 T}{2n^2}} \Phi\left(-x^* + \frac{\sigma\sqrt{T}}{n}\right) - e^{-rT} K^{1/m} \Phi(-x^*). \quad \lhd \end{aligned}$$

5. We have

$$\begin{aligned}
\int_F^\infty h(K)\frac{\partial^2 C(K)}{\partial K^2}dK &= \int_{K^*}^\infty K\frac{\partial^2 C(K)}{\partial K^2}dK \\
&= \left[K\frac{\partial C(K)}{\partial K}\right]_{K^*}^\infty - \int_{K^*}^\infty 1\cdot\frac{\partial C(K)}{\partial K}dK \\
&= -K^*\frac{\partial C(K^*)}{\partial K} - [C(K)]_{K^*}^\infty + \int_{K^*}^\infty 0\cdot C(K)dK \\
&= -K^*\frac{\partial C(K^*)}{\partial K} + C(K^*) \quad \triangleleft
\end{aligned}$$

This expression carries important insights about replication — it is a statement that an asset-or-nothing digital call option can be replicated with K^* amount of cash-or-nothing digital call option and a vanilla call, as shown in the figure below:



6. (a) The solution to the Black-Scholes sde is given by

$$S_T = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T^*}$$
$$\Rightarrow \quad \sqrt{S_T} = \sqrt{S_0} e^{\frac{1}{2}\left(r-\frac{\sigma^2}{2}\right)T+\frac{\sigma}{2}W_T^*}$$

Under martingale valuation framework, we obtain

$$\begin{aligned} V_0 &= e^{-rT}\mathbb{E}^*\left[\sqrt{S_0}e^{\frac{1}{2}\left(r-\frac{\sigma^2}{2}\right)T+\frac{\sigma}{2}W_T^*}\right] \\ &= e^{-rT}\sqrt{S_0}e^{\frac{1}{2}\left(r-\frac{\sigma^2}{2}\right)T+\frac{\sigma^2 T}{8}} \\ &= \sqrt{S_0}e^{-\frac{rT}{2}}e^{-\frac{\sigma^2 T}{8}} \quad \triangleleft \end{aligned}$$

(b) The payoff is twice differentiable:

$$h(S_T) = \sqrt{S_T}, \qquad h'(S_T) = \frac{1}{2S_T^{1/2}}, \qquad h''(S_T) = -\frac{1}{4S_T^{3/2}}$$

Using Breeden-Litzenberger formula, we have

$$\begin{aligned} V_0 &= e^{-rT}h(F) + \int_0^F h''(K)P(K)dK + \int_F^\infty h''(K)C(K)dK \\ &= \sqrt{S_0}e^{-\frac{rT}{2}} - \int_0^F \frac{1}{4K^{3/2}}P(K)dK - \int_F^\infty \frac{1}{4K^{3/2}}C(K)dK \quad \triangleleft \end{aligned}$$

7. Noting that

$$Ke^{-rT} = S_0 e^{\frac{\sigma^2 T}{2} - \sigma\sqrt{T} d_1} \quad \text{and} \quad \frac{\partial d_1}{\partial K} = \frac{1}{S_0/K} \cdot \left(-\frac{S_0}{K^2}\right) \cdot \frac{1}{\sigma\sqrt{T}} = -\frac{1}{K\sigma\sqrt{T}} = \frac{\partial d_2}{\partial K},$$

we can proceed with the partial derivative with respect to K as follow:

$$\begin{aligned}
\frac{\partial C}{\partial K} &= S_0 \frac{\partial \Phi(d_1)}{\partial d_1} \frac{\partial d_1}{\partial K} - Ke^{-rT} \frac{\partial \Phi(d_2)}{\partial d_2} \frac{\partial d_2}{\partial K} - e^{-rT}\Phi(d_2) \\
&= S_0 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \left(-\frac{1}{K\sigma\sqrt{T}}\right) - S_0 e^{\frac{\sigma^2 T}{2} - \sigma\sqrt{T} d_1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \cdot \left(-\frac{1}{K\sigma\sqrt{T}}\right) - e^{-rT}\Phi(d_2) \\
&= \underbrace{S_0 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \left(-\frac{1}{K\sigma\sqrt{T}}\right) - S_0 e^{\frac{\sigma^2 T}{2} - \sigma\sqrt{T} d_1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2} + \sigma\sqrt{T} d_1 - \frac{\sigma^2 T}{2}} \cdot \left(-\frac{1}{K\sigma\sqrt{T}}\right)}_{0} - e^{-rT}\Phi(d_2) \\
&= -e^{-rT}\Phi(d_2).
\end{aligned}$$

This is what we would expect to get — recall that the valuation formula for a cash-or-nothing digital option is given by

$$V_0 = e^{-rT}\Phi(d_2),$$

and the way to hedge this digital option is to form a call spread with strikes very close to each other, giving us the following relationship:

$$\lim_{\Delta K \to 0} \frac{C(K) - C(K + \Delta K)}{\Delta K} = -\lim_{\Delta K \to 0} \frac{C(K + \Delta K) - C(K)}{\Delta K} = -\frac{\partial C}{\partial K}. \quad \triangleleft$$

8. First derive the expression for vega

$$
\begin{aligned}
C(S,K,r,\sigma,T) &= S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)\\
\frac{\partial C}{\partial \sigma} &= S_0\frac{\partial \Phi(d_1)}{\partial d_1}\frac{\partial d_1}{\partial \sigma} - Ke^{-rT}\frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial \sigma}\\
&= S_0 \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \cdot \left(\sqrt{T} - \frac{d_1}{\sigma}\right) - Ke^{-rT} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}} \cdot \left(-\sqrt{T} - \frac{d_2}{\sigma}\right)\\
&= S_0 \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \cdot \left(\sqrt{T} - \frac{d_1}{\sigma}\right) - S_0 e^{\frac{\sigma^2 T}{2} - \sigma\sqrt{T}d_1} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2} + \sigma\sqrt{T}d_1 - \frac{\sigma^2 T}{2}} \cdot \left(-\sqrt{T} - \frac{d_2}{\sigma}\right)\\
&= \frac{S_0}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\left[\sqrt{T} - \frac{d_1}{\sigma} + \sqrt{T} + \frac{d_2}{\sigma}\right]\\
&= \frac{S_0\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}.
\end{aligned}
$$

Differentiating one more time with respect to σ, we obtain

$$
\begin{aligned}
\frac{\partial^2 C}{\partial \sigma^2} &= \frac{S_0\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \cdot (-d_1) \cdot \frac{\partial d_1}{\partial \sigma}\\
&= \frac{S_0\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \cdot (-d_1) \cdot \left(\sqrt{T} - \frac{d_1}{\sigma}\right)\\
&= \frac{S_0\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \cdot (-d_1) \cdot \left(\sqrt{T} - \frac{d_2 + \sigma\sqrt{T}}{\sigma}\right)\\
&= \frac{S_0\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\frac{d_1 d_2}{\sigma} \quad \triangleleft
\end{aligned}
$$