Session 6: Static Replication of European Payoff

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QF620 Stochastic Modelling in Finance

Implied Risk-Neutral Density

- Black-Scholes formula for European call and put options allow us to determine their prices by taking expectation of the option payoff on maturity, discount back to today.

- The formulation involves a lognormal process for the stock price

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

  where the volatility σ is a model parameter that we need to determine.

- Since the vanilla option market is very liquid, we do not need to rely on mathematical modelling to calculate the prices of options.

- Instead, the traded price of these options are published real-time by exchanges globally, and the process is now reversed - given that an option traded at a particular price, what is the implied volatility that we should plug into our Black-Scholes formula to give us this price, assuming that the underlying stock price is indeed following a lognormal process?

Implied Risk-Neutral Density

- One option price allows us to determine the implied volatility for a particular strike and maturity.
- The market is constantly providing information about option prices across a wide range of strikes for a given maturity.
- Given this information, we can now bring our analysis to the next level - instead of asking for just one single implied volatility to match one option price, we want to determine, for a given maturity, the implied distribution, that allows us to match the market volatility smile or skew.
- To this end, we need to apply Leibniz's rule:

$$I(x) = \int_{u(x)}^{v(x)} f(x,t)dt$$

$$\frac{dI(x)}{dx} = f(x,v(x))\frac{dv}{dx} - f(x,u(x))\frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial f(x,t)}{\partial x}dt$$

Implied Risk-Neutral Density

This allows us to extract risk-neutral probability density function from market-traded vanilla option prices. Let f denote the probability density, We have

$$C = \mathbb{E}[e^{-rT}(S_T - K)^+] = e^{-rT}\int_K^\infty (s-K)f(s)ds$$

$$P = \mathbb{E}[e^{-rT}(K - S_T)^+] = e^{-rT}\int_0^K (K-s)f(s)ds$$

These give us

$$\frac{\partial^2 C}{\partial K^2} = e^{-rT}f(K) \qquad \text{and} \qquad \frac{\partial^2 P}{\partial K^2} = e^{-rT}f(K).$$

To replicate any European payoff $h(S_T)$, we do

$$V_0 = e^{-rT}\mathbb{E}[h(S_T)] = e^{-rT}\int_0^\infty h(K)f(K)dK.$$

Breeden-Litzenberger Formula

Breeden-Litzenberger formula allows us to use the risk-neutral density backed-out from vanilla option prices to statically replicate any twice differentiable European payoff $h(S_T)$.

$\Rightarrow$ This is achieved using liquid vanilla European option prices observable in the market.

Let $F = S_0 e^{rT}$, we have

$$V_0 = e^{-rT}\mathbb{E}[h(S_T)] = \int_0^F h(K)\frac{\partial^2 P(K)}{\partial K^2}dK + \int_F^\infty h(K)\frac{\partial^2 C(K)}{\partial K^2}dK$$

Using integration-by-parts twice, we obtain

$$\begin{aligned} V_0 = e^{-rT} & h(F) + h'(F)(C(F) - P(F)) \\ & + \int_0^F h''(K)P(K)dK + \int_F^\infty h''(K)C(K)dK \end{aligned}$$

Breeden-Litzenberger Formula

- Note that we've use observed market prices of call and put options to form our static hedge.
- At no point do we apply the use of any option theory or stochastic differential equations, and definitely no Black-Scholes was used.
- It is important to note the difference between a static hedging and a dynamic hedging strategy.
- The implied risk-neutral distribution on maturity determined in this way is only applicable to static hedging of European options using European calls and puts.
- Dynamic hedging based on replication and trading strategy, on the other hand, is more general, and applies to all options.
- Black-Scholes formulation is based on dynamic hedging strategy, whereby we hedge against the changes caused by the stochastic process driving the stock price.
- The risk-neutral distribution at maturity provides no information whatsoever about the evolution of the stock price process. Hence, it is not useful in formulating dynamic hedging strategy.

Breeden-Litzenberger Formula

Example Suppose we want to derive the valuation formula for a log contract paying $\log \frac{S_T}{S_0}$ at maturity T, where S_t is the value of a stock.

1. Derive the valuation formula under Black-Scholes model.
2. Formulate the static replication portfolio using Breeden-Litzenberger formula.

Variance Swaps

Variance swaps are contracts which allow us to gain explicit volatility (and variance) exposure. This frees us from the need to worry about delta or gamma hedging if we were to use vanilla options to gain volatility exposure.

The payoff of a variance swap is given by

$$\text{Var Swap} \;=\; \text{Notional} \times \left(\sigma_R^2 - \sigma_K^2\right),$$

where σ_R^2 is the realised variance of the stock and σ_K^2 is the strike variance.

The realised variance σ_R^2 is quantified as

$$\sigma_R^2 = \frac{252}{N}\sum_{i=1}^{N}\left(\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right)^2,$$

where i labels the value of the stock on each day and N is the total number of days in the contract. Variance swaps capture the realised variance of the underlying asset. It is an intuitive contract based on the definition of historical variance. The contract is often described in terms of the fair strike σ_K^2.

Variance Swaps

To price a variance swap, we observe that the discrete sum over the log returns can be approximated by a continuous integral

$$\sum_{i=1}^{N}\left[\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right]^2 \approx \int_0^T \sigma_t^2 dt.$$

Itô's formula allows us to derive

$$\begin{aligned} d\log S_t &= \frac{dS_t}{S_t} - \frac{1}{2}\sigma_t^2 dt \\ \Rightarrow \quad \sigma_t^2 dt &= 2\left[\frac{dS_t}{S_t} - d\log S_t\right]. \end{aligned}$$

Integrating both sides and then take expectation, we obtain

$$\begin{aligned} \int_0^T \sigma_t^2 dt &= 2\int_0^T \frac{dS_t}{S_t} - 2\log\left(\frac{S_T}{S_0}\right) \\ \mathbb{E}\left[\int_0^T \sigma_t^2 dt\right] &= 2\mathbb{E}\left[\int_0^T \frac{dS_t}{S_t}\right] - 2\mathbb{E}\left[\log\left(\frac{S_T}{S_0}\right)\right]. \end{aligned}$$

Variance Swaps

This equation provides valuable hedging insights: the first term is a dynamic hedge consisting of holding $\frac{1}{S_t}$ of stock at each time step. The second term is a static hedge of holding a log contract to expiry. The dynamic hedge involves us borrowing \$1 to buy $\frac{1}{S_t}$ of stock and hold for a day. The expected return is

$$2\mathbb{E}\left[\int_0^T \frac{dS_t}{S_t}\right] = 2rT. \qquad \text{(assume no dividend)}$$

The term $\mathbb{E}\left[\log \frac{S_T}{S_0}\right]$ depends only on the initial stock price S_0 and the final stock price S_T. This is perfectly suited for the Breeden-Litzenberger formula:

$$\begin{aligned} e^{-rT}\mathbb{E}[h(S_T)] = e^{-rT}h(F) + h'(F)(C(F) - P(F)) \\ + \int_0^F P(K)h''(K)dK + \int_F^\infty C(K)h''(K)dK, \end{aligned}$$

where $F = e^{rT}S_0$.

Variance Swaps

Choose $h(S_T) = \log\left(\frac{S_T}{S_0}\right)$, then

$$\mathbb{E}\left[\log\left(\frac{S_T}{S_0}\right)\right] = \log\left(\frac{F}{S_0}\right) - e^{rT}\int_0^F \frac{P(K)}{K^2}dK - e^{rT}\int_F^\infty \frac{C(K)}{K^2}dK.$$

The log contract can be replicated using a portfolio of European put and call options. The weighting of the options is $\frac{1}{K^2}$. The portfolio contains all possible strikes. The portfolio has more weight for downside options than upside options - skew sensitive. The portfolio is asking us to trade a lot of low strike puts.

Finally, we obtain

$$\mathbb{E}\left[\int_0^T \sigma_t^2 dt\right] = 2e^{rT}\int_0^F \frac{P(K)}{K^2}dK + 2e^{rT}\int_F^\infty \frac{C(K)}{K^2}dK.$$

Static Replication of European Payoff

As discussed earlier, we can use static replication to value any twice differentiable payoff $h(S_T)$ with Breeden-Litzenberger formula:

$$\begin{aligned}V_0 = e^{-rT}h(F) &+ h'(F)(C(F) - P(F)) \\ &+ \int_0^F h''(K)P(K)dK + \int_F^\infty h''(K)C(K)dK\end{aligned}$$

⇒ Note that K in the integrals are a dummy variable — it doesn't imply that the payoff h has a strike.

⇒ However, we choose to use K to remind ourselves that the integrals are weighted across $P(K)$ and $C(K)$, i.e. put and call options across a wide range of strikes.

Since call and put options are worth the same with $K = F$, we can write

$$V_0 = e^{-rT}h(F) + \int_0^F h''(K)P(K)dK + \int_F^\infty h''(K)C(K)dK$$

Static Replication of European Payoff

Example A financial contract pays aS_T^b on maturity date T, where a, $b \in \mathbb{R}^+$ are positive real numbers. Use static replication method to replicate this payoff using vanilla European call and put options.

Solution With $h(S_T) = aS_T^b$, we have

$$h'(S_T) = abS_T^{b-1}, \quad h''(S_T) = ab(b-1)S_T^{b-2}.$$

Hence, the payoff, which is twice differentiable, can be static replicated with a portfolio of options as follow:

$$\begin{aligned} V_0 = e^{-rT} aF^b &+ \int_0^F ab(b-1)K^{b-2}P(K)\, dK \\ &+ \int_F^\infty ab(b-1)K^{b-2}C(K)\, dK. \end{aligned}$$

Static Replication of European Payoff

But what if the European payoff is not twice differentiable?

$\Rightarrow$ We can still use static replication – we just need to start from the risk-neutral density, rather than applying Breeden-Litzenberger formula directly.

To this end, we start with

$$
\begin{aligned}
V_0 = e^{-rT}\mathbb{E}[h(S_T)] &= e^{-rT}\int_0^\infty h(K)f(K)\,dK \\
&= \int_0^F h(K)\frac{\partial^2 P(K)}{\partial K}\,dK + \int_F^\infty h(K)\frac{\partial^2 C(K)}{\partial K^2}\,dK
\end{aligned}
$$

and perform integration-by-parts twice to obtain the replication formula.

Static Replication of European Payoff

Example Use static replication to value a cash-or-nothing digital call option with payoff:

$$h(S_T) = \mathbb{1}_{S_T \geq K_1} = \begin{cases} 1, & \text{if } S_T \geq K_1 \\ 0 & \text{otherwise} \end{cases}$$

where $K_1 \geq F = S_0 e^{rT}$.

Solution In this case, we start with

$$\int_{K_1}^{\infty} h(K) \frac{\partial^2 C(K)}{\partial K^2}\, dK.$$

We note that for $K \geq K_1$, $h(K) = 1$, $h'(K) = 0$, and $h''(K) = 0$. Using integration by parts, we obtain

$$\left[h(K)\frac{\partial C(K)}{\partial K}\right]_{K_1}^{\infty} - \left[h'(K)C(K)\right]_{K_1}^{\infty} + \int_{K_1}^{\infty} h''(K)C(K)\, dK = -\frac{\partial C(K_1)}{\partial K}$$

Static Replication of European Payoff

Example Call spread for $K_1 = 50$:

$$-\frac{\partial C(K_1)}{\partial K} \approx \frac{C(K_1 - \Delta K) - C(K_1)}{\Delta K}$$