pg25.

$$\begin{cases} dF_t = \alpha_t F_t^{\beta} dW_t^F \\ d\alpha_t = \nu \alpha_t dW_t^{\alpha} \end{cases}, \quad dW_t^F \, dW_t^{\alpha} = \rho \, dt$$

high F_t, high α_t

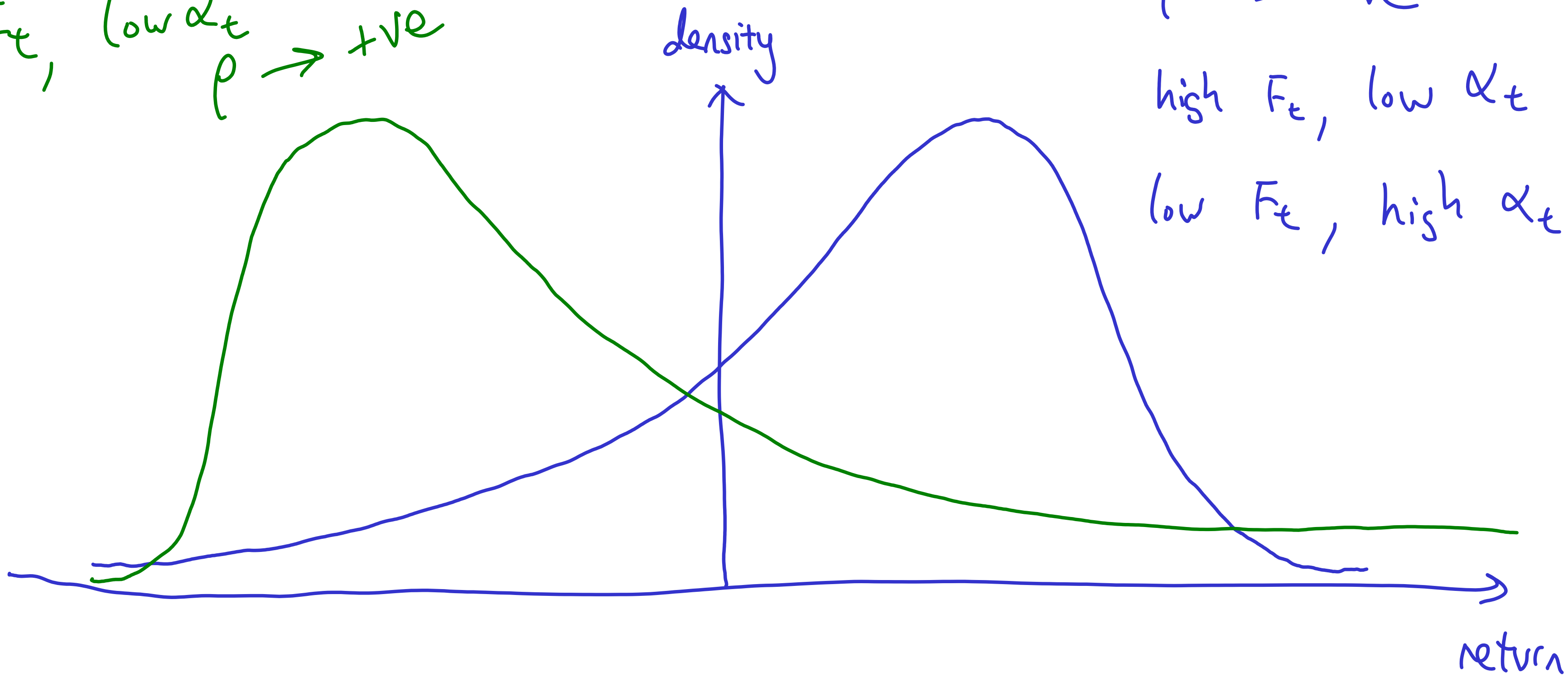
low F_t, low α_t

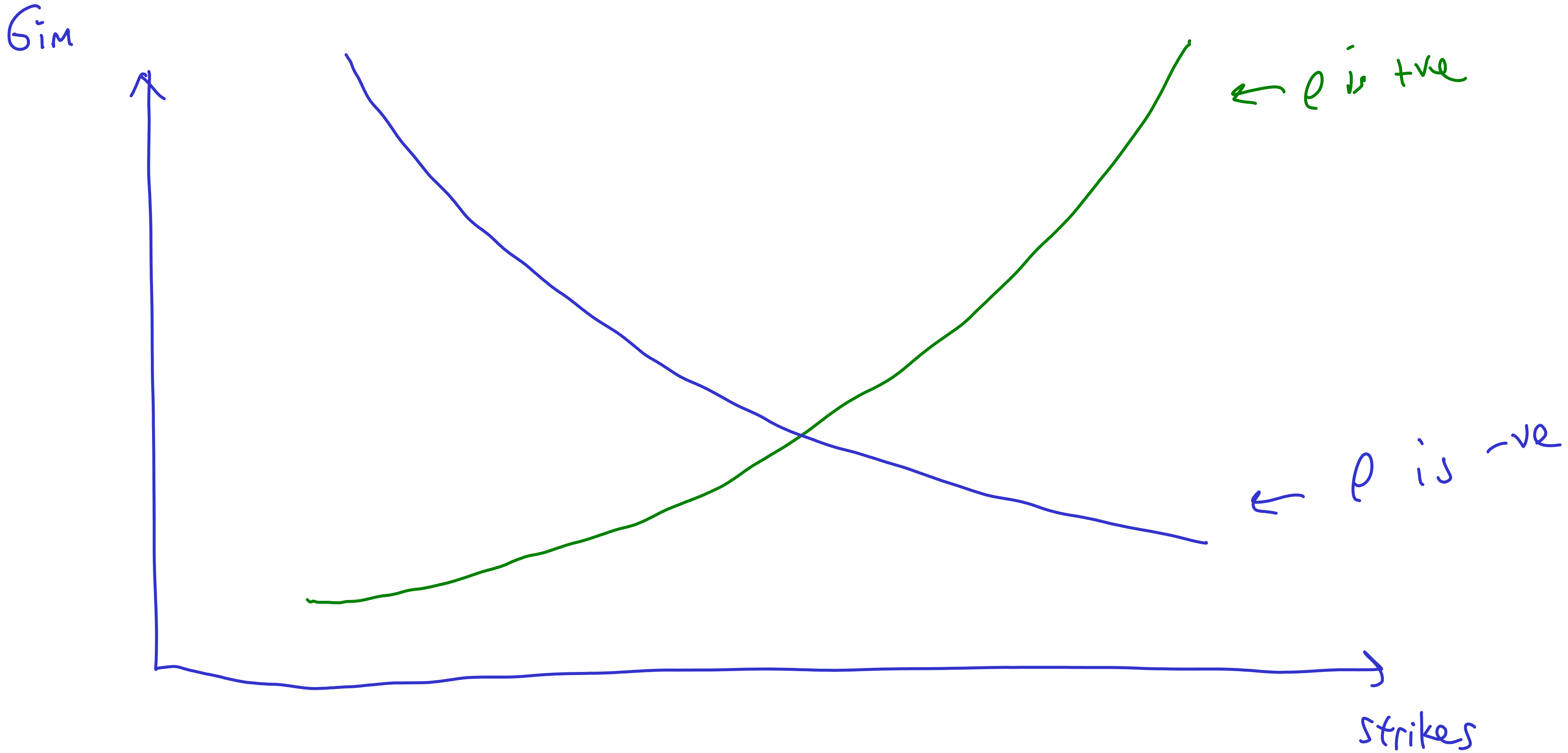
$\rho \rightarrow$ +ve

$\rho \rightarrow$ -ve

high F_t, low α_t

low F_t, high α_t

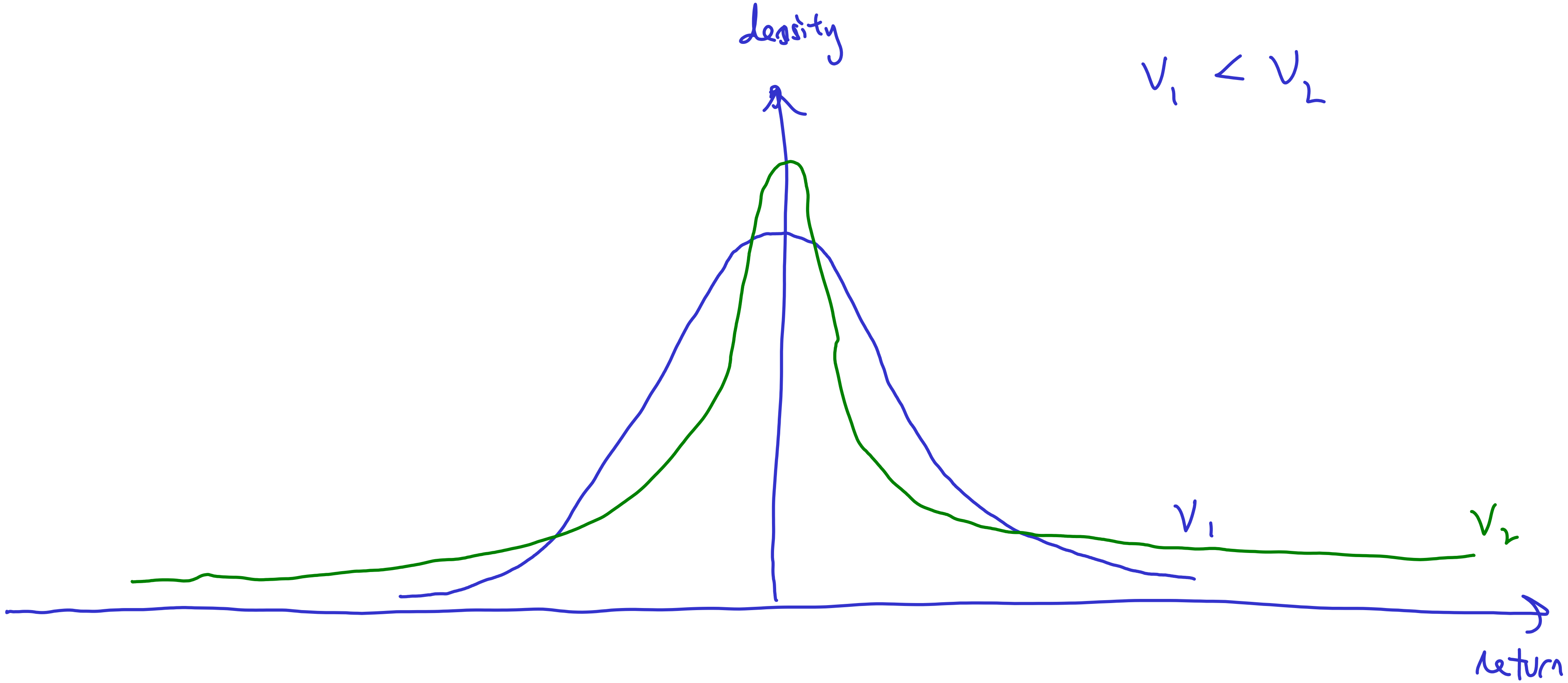


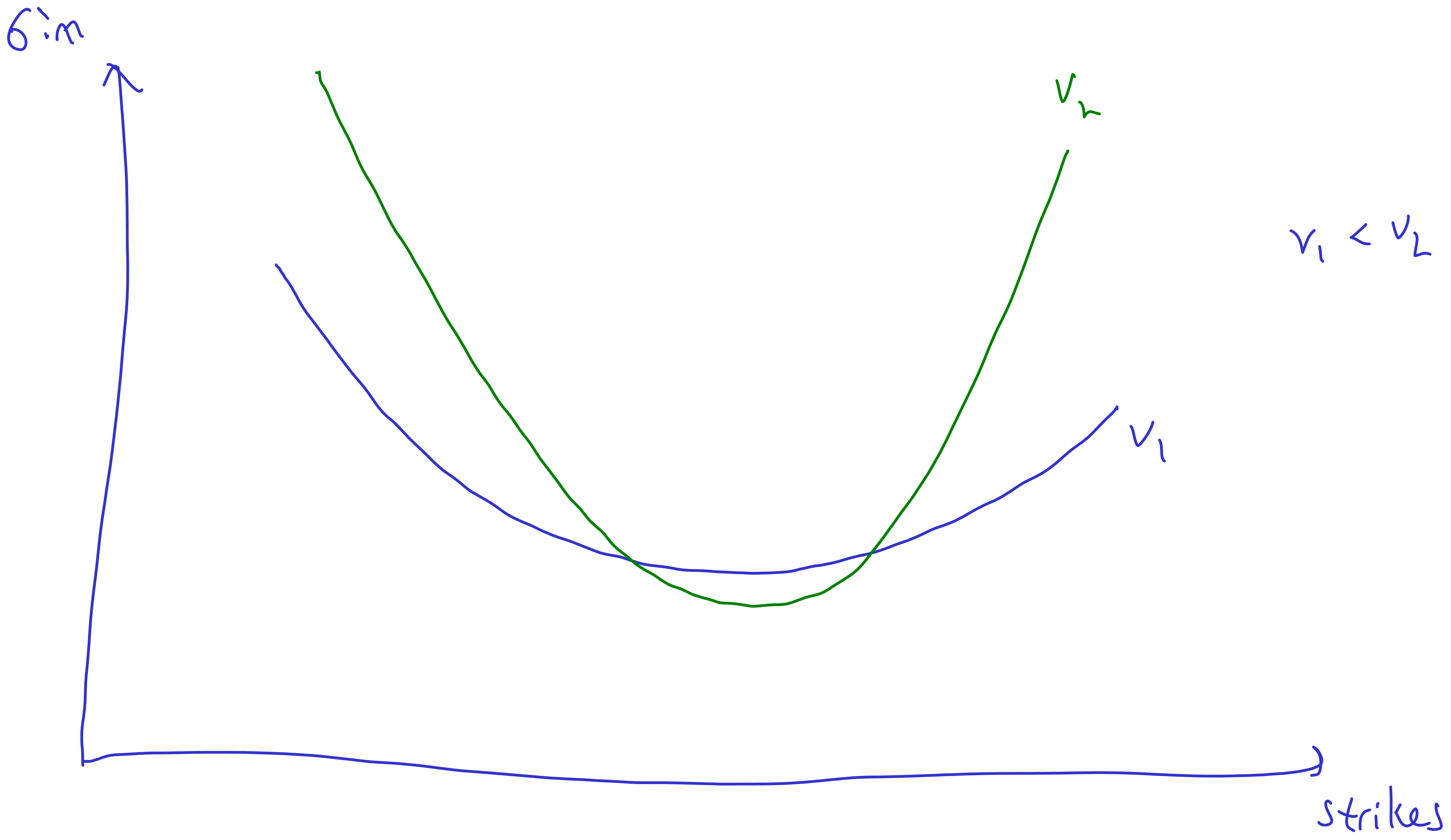


Gim
ρ is +ve
ρ is -ve
strikes

Pg28.

$$d\alpha_t = \nu\,\alpha_t\,dW_t^{\alpha}$$





$v_1 < v_2$

$$dF_t = \sigma \left[\beta F_t + (1-\beta) F_0 \right] dW_t$$

$$dF_t = \sigma F_t^{\beta} \, dW_t$$

pg 4.

"Model-free"

$S \to$ stock

$f(s) \to$ risk-neutral density of S

$$\text{Call} = e^{-rT}\,\mathbb{E}\left[(S_T - K)^+\right] = e^{-rT}\int_K^{\infty}(s-K)\,f(s)\,ds$$

---

e.g. Black-Scholes model

$$\text{Call} = \frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}\left(S_0\,e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}\,x} - K\right)e^{-\frac{x^2}{2}}\,dx$$

$$C(K_1) = e^{-rT} \int_{K_1}^{\infty} (S - K_1) f(S) dS$$

$$C(K_2) = e^{-rT} \int_{K_2}^{\infty} (S - K_2) f(S) dS$$

$$C(K_3) = e^{-rT} \int_{K_3}^{\infty} (S - K_3) f(S) dS$$

$$\vdots \qquad \qquad \vdots$$

$$C(K) = e^{-rT} \int_K^\infty (s-K)\, f(s)\, ds$$

$$\frac{\partial C(K)}{\partial K} = \underbrace{\cancel{e^{-rT} \cdot (\infty - K)\, f(\infty) \cdot \frac{d}{dK}(\infty)}}_{\to 0} - \underbrace{\cancel{e^{-rT} \cdot (K-K)\, f(K) \cdot \frac{d}{dK}(K)}}_{\to 0}$$

$$- e^{-rT} \int_K^\infty f(s)\, ds$$

$$\frac{\partial C(K)}{\partial K} = - e^{-rT} \int_K^\infty f(s)\, ds$$

$$\frac{\partial^2 C(K)}{\partial K^2} = - \underbrace{\cancel{e^{-rT} \cdot f(\infty) \cdot \frac{d\infty}{dK}}}_{\to 0} + e^{-rT} f(K) \cdot \frac{dK}{dK} - 0$$

$$\frac{\partial^2 C(K)}{\partial K^2} = e^{-rT} f(K)$$

$$f(K) = \underbrace{e^{rT} \cdot \frac{\partial^2 C(K)}{\partial K^2}}_{\text{when } K > F} \quad \text{or} \quad \underbrace{e^{rT} \frac{\partial^2 P(K)}{\partial K^2}}_{\text{when } K < F}$$

$$\frac{\partial C(K)}{\partial K} \simeq \frac{C(K+\Delta K) - C(K)}{\Delta K}$$

finite difference approximation

$$\frac{\partial^2 C(K)}{\partial K^2} \simeq \frac{C(K+\Delta K) - 2C(K) + C(K-\Delta K)}{(\Delta K)^2}$$

$\rightarrow$ noisy

$$V_0 = e^{-rT} \int_0^F h(K)\, f(K)\, dK + e^{-rT} \int_F^\infty h(K)\, f(K)\, dK$$

$$= e^{-rT} \int_0^F h(K) \cdot e^{rT} \cdot \frac{\partial^2 P(K)}{\partial K^2}\, dK + e^{-rT} \int_F^\infty h(K)\, e^{rT} \frac{\partial^2 C(K)}{\partial K^2}\, dK$$

$$= \underbrace{\int_0^F h(K) \frac{\partial^2 P(K)}{\partial K^2}\, dK}_{1} + \underbrace{\int_F^\infty h(K) \frac{\partial^2 C(K)}{\partial K^2}\, dK}_{2}$$

(1)

$$\int_0^F h(K)\,\frac{\partial^2 P(K)}{\partial K^2}\,dK \qquad\qquad \int uv' = uv - \int u'v$$

$$= \left[h(K)\,\frac{\partial P(K)}{\partial K}\right]_0^F - \int_0^F h'(K)\cdot\frac{\partial P(K)}{\partial K}\,dK$$

$$= h(F)\,\frac{\partial P(F)}{\partial K} - \cancelto{0}{h(0)\,\frac{\partial P(0)}{\partial K}} - \left\{\left[h'(K)\,P(K)\right]_0^F - \int_0^F h''(K)\,P(K)\,dK\right\}$$

$$= h(F)\,\frac{\partial P(F)}{\partial K} - h'(F)\,P(F) + \cancelto{0}{h'(0)\,P(0)} + \int_0^F h''(K)\,P(K)\,dK$$

$$= h(F)\,\frac{\partial P(F)}{\partial K} - h'(F)\,P(F) + \int_0^F h''(K)\,P(K)\,dK$$

② $$\int_F^\infty h(K)\,\frac{\partial^2 C(K)}{\partial K^2}\,dK$$

$$= \left[h(K)\,\frac{\partial C(K)}{\partial K} \right]_F^\infty - \int_F^\infty h'(K)\,\frac{\partial C(K)}{\partial K}\,dK$$

$$= h(\infty)\,\cancelto{0}{\frac{\partial C(\infty)}{\partial K}} - h(F)\,\frac{\partial C(F)}{\partial K} - \left\{ \left[h'(K)\,C(K) \right]_F^\infty - \int_F^\infty h''(K)\,C(K)\,dK \right\}$$

$$= -h(F)\,\frac{\partial C(F)}{\partial K} - h'(\infty)\,\cancelto{0}{C(\infty)} + h'(F)\,C(F) + \int_F^\infty h''(K)\,C(K)\,dK$$

$$= -h(F)\,\frac{\partial C(F)}{\partial K} + h'(F)\,C(F) + \int_F^\infty h''(K)\,C(K)\,dK$$

$$① = h(F)\frac{\partial P(F)}{\partial K} - h'(F)P(F) + \int_0^F h''(K)P(K)\,dK$$

$$② = -h(F)\frac{\partial C(F)}{\partial K} + h'(F)C(F) + \int_F^\infty h''(K)C(K)\,dK$$

$$C(K) - P(K) = S - Ke^{-rT}$$

$$\frac{\partial C}{\partial K} - \frac{\partial P}{\partial K} = 0 - e^{-rT}$$

$$\therefore \frac{\partial P}{\partial K} - \frac{\partial C}{\partial K} = e^{-rT}$$

$$① + ② =$$

pg 7.

pays $\log\left(\frac{S_T}{S_0}\right)$

① Black-Scholes model: $dS_t = rS_t dt + \sigma S_t dW_t^*$

$$\Rightarrow S_T = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T + \sigma W_T^*}$$

$$\log(S_T) = \log(S_0) + \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^*$$

$$\log\left(\frac{S_T}{S_0}\right) = \left(r-\frac{\sigma^2}{2}\right)T + \sigma W_T^*$$

$$V_0^{BS} = e^{-rT}\,\mathbb{E}^*\left[\log\frac{S_T}{S_0}\right] = e^{-rT} \times \left(r - \frac{\sigma^2}{2}\right)T$$

pg8.

② Static replication : $h(S_T) = \log \frac{S_T}{S_0}$

$$h'(S_T) = \frac{1}{S_T/S_0} \cdot \frac{1}{S_0} = \frac{1}{S_T}$$

$$h''(S_T) = -\frac{1}{S_T^2}$$

$$V_0 = e^{-rT} \cdot \log \frac{F}{S_0} + \cancel{h'(F)\big(C(F) - P(F)\big)} \rightarrow 0$$

$$- \int_0^F \frac{1}{K^2} \cdot P(K) \cdot dK - \int_F^\infty \frac{1}{K^2} \cdot C(K) \cdot dK$$

$$= e^{-rT} \cdot \log\left(\frac{S_0 e^{rT}}{S_0}\right) - \int_0^F \frac{P(K)}{K^2} dK - \int_F^\infty \frac{C(K)}{K^2} dK$$

$$= e^{-rT}\left(rT\right) - \int_0^F \frac{P(K)}{K^2}\,dK - \int_F^\infty \frac{C(K)}{K^2}\,dK$$

Black-Scholes

$$e^{-rT}\cdot\frac{\sigma^2 T}{2}$$

Static replication

$$\int_0^F \frac{P(K)}{K^2}\,dK + \int_F^\infty \frac{C(K)}{K^2}\,dK$$