Session 7: Greeks and Dynamic Hedging

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QF620 Stochastic Modelling in Finance

Greeks

Black-Scholes formulation allows us to derive the valuation formulas for European call and put options in closed-form:

$$\text{European Call} = C(S_0, K, r, \sigma, T) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$
$$\text{European Put} = P(S_0, K, r, \sigma, T) = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$$

where

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T} = \frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}.$$

$\Rightarrow$ We use Greeks to measure the sensitivity of an option with respect to changes in the input parameters.

$\Rightarrow$ Under the Black-Scholes framework, we can express these sensitivities in closed-form by taking the partial derivatives of the formulas with respect to the input parameters.

Greeks

Before we proceed, we shall establish a few useful relationships to aid our derivations. First, we note that

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$\sigma\sqrt{T}d_1 - \left(r + \frac{\sigma^2}{2}\right)T = \log\frac{S_0}{K}$$

$$e^{\sigma\sqrt{T}d_1 - \left(r + \frac{\sigma^2}{2}\right)T} = \frac{S_0}{K}$$

$$Ke^{-rT} = S_0 e^{\frac{\sigma^2 T}{2} - \sigma\sqrt{T}d_1}.$$

Given the definition of the standard normal cumulative distribution function

$$\Phi(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} e^{-\frac{x^2}{2}}\,dx,$$

we can use Leibniz's rule to show that

$$\frac{\partial\Phi(d_1)}{\partial d_1} = \phi(d_1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}, \qquad \frac{\partial\Phi(d_2)}{\partial d_2} = \phi(d_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}}.$$

Greeks

Also, we note the following derivatives by chain rule:

$$\frac{\partial \Phi(d_1)}{\partial S_0} = \frac{\partial \Phi(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S_0}, \qquad \frac{\partial \Phi(d_2)}{\partial S_0} = \frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S_0}.$$

where

$$\frac{\partial d_1}{\partial S_0} = \frac{1}{\sigma S_0\sqrt{T}} = \frac{\partial d_2}{\partial S_0}.$$

Finally, we note the following derivatives

$$\begin{aligned}\frac{\partial d_1}{\partial \sigma} &= \frac{\sigma\sqrt{T}\sigma T - \sqrt{T}\left(\log\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T\right)}{\sigma^2 T}\\ &= \sqrt{T} - \frac{d_1}{\sigma}\\ \frac{\partial d_2}{\partial \sigma} &= -\sqrt{T} - \frac{d_2}{\sigma}\end{aligned}$$

Greeks — Delta (Δ)

Delta measures the sensitivity of the option to the stock price. For a European call option, we have

$$
\begin{aligned}
\frac{\partial C}{\partial S_0} &= \Phi(d_1) + S_0 \frac{\partial \Phi(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S_0} - Ke^{-rT}\frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S_0} \\
&= \Phi(d_1) + S_0 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \frac{1}{\sigma S_0 \sqrt{T}} - Ke^{-rT} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \cdot \frac{1}{\sigma S_0 \sqrt{T}} \\
&= \Phi(d_1) + S_0 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \frac{1}{\sigma S_0 \sqrt{T}} - S_0 e^{\frac{\sigma^2 T}{2} - \sigma\sqrt{T} d_1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \cdot \frac{1}{\sigma S_0 \sqrt{T}} \\
&= \Phi(d_1) + S_0 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \frac{1}{\sigma S_0 \sqrt{T}} - S_0 e^{\frac{\sigma^2 T}{2} - \sigma\sqrt{T} d_1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(d_1 - \sigma\sqrt{T})^2}{2}} \cdot \frac{1}{\sigma S_0 \sqrt{T}} \\
&= \Phi(d_1) + \underbrace{S_0 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \frac{1}{\sigma S_0 \sqrt{T}} - S_0 e^{\frac{\sigma^2 T}{2} - \sigma\sqrt{T} d_1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2} + \sigma\sqrt{T} d_1 - \frac{\sigma^2 T}{2}} \cdot \frac{1}{\sigma S_0 \sqrt{T}}}_{0} \\
&= \Phi(d_1)
\end{aligned}
$$

Greeks — Delta (Δ)

So the delta of a call option is given by

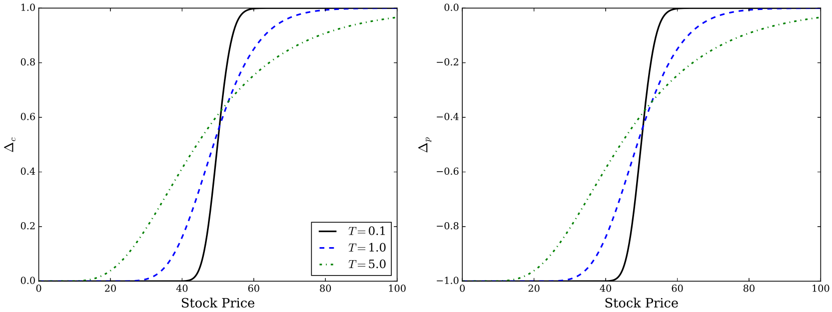
$$
\begin{aligned}
\frac{\partial C}{\partial S_0} &= \Phi(d_1) \\
&= \Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)
\end{aligned}
$$

Having obtain the delta of a European call option, we can make use put-call parity to derive the expression for the delta of a European put option:

$$
\begin{aligned}
C(S_0, K, r, \sigma, T) - P(S_0, K, r, \sigma, T) &= S_0 - Ke^{-rT} \\
\frac{\partial C}{\partial S_0} - \frac{\partial P}{\partial S_0} &= 1 \\
\Rightarrow \quad \frac{\partial P}{\partial S_0} &= \Phi(d_1) - 1.
\end{aligned}
$$

Greeks — Delta (Δ)

Example: $S_0 = 50$, $\sigma = 0.2$, $r = 0.5\%$, $K = 50$.



Greeks — Vega

Vega measures an option's sensitivity to volatility. For a European call option, we have

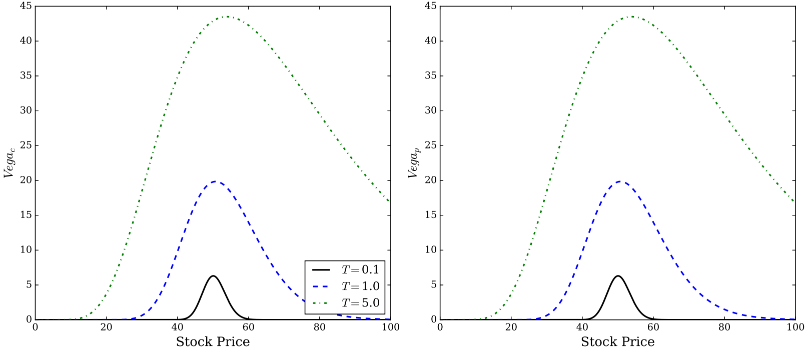
$$\begin{aligned}
\frac{\partial C}{\partial \sigma} &= S_0 \frac{\partial \Phi(d_1)}{\partial d_1} \frac{\partial d_1}{\partial \sigma} - Ke^{-rT} \frac{\partial \Phi(d_2)}{\partial d_2} \frac{\partial d_2}{\partial \sigma} \\
&= S_0 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \left(\sqrt{T} - \frac{d_1}{\sigma}\right) - Ke^{-rT} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \cdot \left(-\sqrt{T} - \frac{d_2}{\sigma}\right) \\
&= S_0 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \left(\sqrt{T} - \frac{d_1}{\sigma}\right) - S_0 e^{\frac{\sigma^2 T}{2} - \sigma\sqrt{T} d_1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2} + \sigma\sqrt{T} d_1 - \frac{\sigma^2 T}{2}} \cdot \left(-\sqrt{T} - \frac{d_2}{\sigma}\right) \\
&= \frac{S_0}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \left[\sqrt{T} - \frac{d_1}{\sigma} + \sqrt{T} + \frac{d_2}{\sigma}\right] \\
&= \frac{S_0 \sqrt{T}}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}.
\end{aligned}$$

Similarly, we can use put-call parity to show that

$$\frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} = \frac{S_0\sqrt{T}}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}.$$

Greeks — Vega

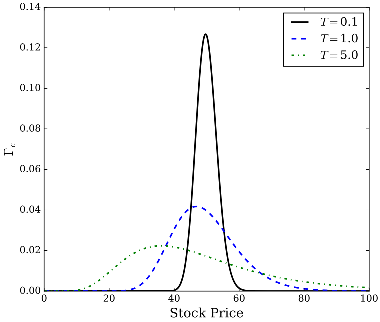
Example: $S_0 = 50$, $\sigma = 0.2$, $r = 0.5\%$, $K = 50$.

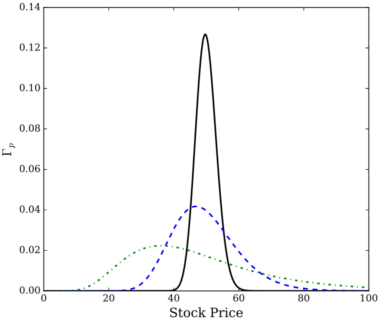


Greeks — Gamma (Γ)

Gamma measures the sensitivity of delta to the stock. Taking partial derivatives of the closed-form expressions for delta, it is straightforward to show that

$$\frac{\partial^2 C}{\partial S_0^2} = \frac{\partial^2 P}{\partial S_0^2} = \frac{e^{-\frac{d_1^2}{2}}}{\sigma S_0 \sqrt{2\pi T}}.$$





Black-Scholes' Dynamic Hedging Strategy

A portfolio is a pair of processes ϕ_t and ψ_t which describe the units of security and bond which we hold at time t, respectively.

⇒ The process can take positive or negative values (short sale is allowed).

The pair (ϕ_t, ψ_t) is a dynamic trading strategy detailing the amount of each component to be held at each instant.

Of particular interest to us is the set of trading strategies that are financially self-contained, or equivalently, self-financing.

⇒ A portfolio is self-financing if and only if the change in its value only depends on the change of the asset prices.

Let S_t, B_t and V_t denote the value of the stock price, the bond price and the value of the portfolio (ϕ_t, ψ_t), respectively.

At time t, the worth of the portfolio is given by

$$V_t = \phi_t S_t + \psi_t B_t.$$

Black-Scholes' Dynamic Hedging Strategy

At the next instant, the portfolio's value will change due to changes in S_t and B_t.

- ⇒ Consequently, the portfolio needs to be readjusted in accordance to the trading strategy (ϕ_t, ψ_t).
- ⇒ If the cost of the adjustment is perfectly matched by the profits or losses made by the portfolio then no extra money is required from outside — the portfolio is self-financing.

Consider a trading strategy (ϕ_t, ψ_t), where ϕ_t and ψ_t are both previsible functions of stock, time and/or bond, giving rise to a portfolio $V_t = \phi_t S_t + \psi_t B_t$. The trading strategy is said to be self-financing if

$$dV_t = \phi_t dS_t + \psi_t dB_t.$$

If we apply Itô's Formula and product rule of stochastic calculus to V_t, we obtain

$$dV_t = \underbrace{\phi_t dS_t + S_t d\phi_t + d\phi_t dS_t}_{d(\phi_t S_t)} + \underbrace{\psi_t dB_t + B_t d\psi_t + d\psi_t dB_t}_{d(\psi_t B_t)}.$$

Black-Scholes' Dynamic Hedging Strategy

Since B_t is a deterministic process, we note that $d\psi_t dB_t = 0$ (why?).

Collecting terms and comparing against the self-financing portfolio requirement, we note that the self-financing property requires that

$$S_t d\phi_t + d\phi_t dS_t + B_t d\psi_t = 0.$$

It should be clear that the self-financing property is not an automatic property of a portfolio. We need to apply the Itô's formula check to the SDEs to verify that no capital injection or withdrawal is required.

If a replicating strategy (ϕ_t, ψ_t) exists, then the price of X at time t must be

$$V_t = \phi_t S_t + \psi_t B_t.$$

Any price that deviates from this value will be open to arbitrage.

Black-Scholes' Dynamic Hedging Strategy

⇒ If we hold ϕ and ψ units of S and B respectively at time t then we have a portfolio whose time t value is $\phi S_t + \psi B_t$.

⇒ The assumptions of Black-Scholes are that we have a frictionless market, meaning that S and B can be traded in arbitrary amounts with no transaction costs, and short positions are allowed.

⇒ In particular this means we can invest in, or borrow from, the risk-free account at the same rate of interest r.

Suppose we form a portfolio in which we are long one unit of the call option and short a self-financing portfolio (ϕ_t, ψ_t). The value of this portfolio at time t is

$$X_t = C_t - \phi_t S_t - \psi_t B_t, \quad X_0 = 0.$$

By Itô's formula and the self-financing property,

$$dX_t = \frac{\partial C}{\partial S} dS_t + \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} \right) dt - \phi_t dS_t - \psi_t dB_t.$$

Black-Scholes' Dynamic Hedging Strategy

If we choose $\phi_t = \frac{\partial C}{\partial S}(t, S_t)$ and use the fact that $dB_t = rB_t dt$, we see that

$$dX_t = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} - \psi_t r B_t \right) dt.$$

If we choose

$$\psi_t = \frac{\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2}}{rB_t},$$

then $X_t = 0$, and

$$C_t = \phi_t S_t + \psi_t B_t = S_t \frac{\partial C}{\partial S} + \frac{\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2}}{r},$$

and C_t will satisfy the PDE (with boundary condition $C(T, S_T) = (S_T - K)^+$)

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC.$$

Interpretation of Black-Scholes Formula

The Black-Scholes formula is given by

$$C = S\Phi\left(\frac{\log\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\left(\frac{S}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right).$$

Recall that we need to hold ϕ_t amount of stock in the replicating portfolio, this is given by

$$\phi_t = \frac{\partial C}{\partial S} = \Phi\left(\frac{\log\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right).$$

Since the CDF is always positive, we know that the replication portfolio will always consist of a long position in the stock, and since CDF is never greater than one, we know that we'll never have to hold more than one stock. As for the bond holding, we know that we need ψ_t. Looking at the Black-Scholes formula, the amount of bond we're holding is obviously

$$\psi_t B_t = -Ke^{-rT}\Phi\left(\frac{\log\left(\frac{S}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right).$$

Interpretation of Black-Scholes Formula

The negative sign for $\psi_t B_t$ tells us that we're always short bond in the replication portfolio. To recap, Black-Scholes provided us with a statement about dynamic hedging/trading strategies:

$$C(S,K,\sigma,r,T) = \underbrace{S\Phi\left(\frac{\log\frac{S}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)}_{\text{stock holding}} - \underbrace{Ke^{-rT}\Phi\left(\frac{\log\frac{S}{K}+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)}_{\text{bond holding}}$$

If we're unable to purchase a call option, we can synthesize one using the following trading strategy (provided the underlying stock market and the risk-free bond market are both liquid):

$$(\phi_t,\psi_t) :\longrightarrow \begin{cases} \phi_t = \Delta_t = \dfrac{\partial C}{\partial S} = \Phi\left(\dfrac{\log\frac{S}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) \\ \\ \psi_t B_t = -Ke^{-rT}\Phi\left(\dfrac{\log\frac{S}{K}+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) \end{cases}$$

Interpretation of Black-Scholes Formula

- The idea is to show that the option price is a solution of a partial differential equation.
- To this end, they assume that they can replicate the option price with a self-financing and previsible trading strategy in stock and cash.
- Self-financing means that there's no further capital injection or withdrawal required after trade initiation.
- Previsible means that investment decisions are based on information available at any given day, without any forecasting of future performance required.
- The key insight is that the price of an option is exactly equal to the cost of setting up the replicating portfolio, i.e. the hedge.

Interpretation of Black-Scholes Formula

There are five important observations in this results:

1. The option price does not depend on the historical drift μ! μ can be interpreted as the market trend, which is surprising, since the main use of derivatives is to hedge against the market moves.
2. The dynamic hedging strategy enables the seller to hedge against unfavourable market moves by cancelling the risk arising from the market trend. Whether the market is bullish or bearish, the option price (option premium) is the same.
3. From a statistical point of view, we have one parameter less to identify. This is very important as the historical trend is very difficult to estimate.
4. However, the risk due to stock price fluctuations is still there and has a huge impact on the price via the volatility parameter σ.
5. Provided we are able to solve the partial differential equation, we know how much stock we must hold as $\Delta(t, S_t) = \frac{\partial V}{\partial S_t}(t, S_t)$, and therefore also how much cash we must hold in our replicating portfolio.

Interpreting Black-Scholes: Digital Options

In earlier sections, we have discussed the valuation of cash-or-nothing digital (call) option

$$e^{-rT}\mathbb{E}^*\left[\mathbb{1}_{S_T>K}\right] = e^{-rT}\Phi\left(\frac{\log\frac{S}{K}+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

and the valuation of asset-or-nothing digital (call) option as

$$e^{-rT}\mathbb{E}^*\left[S_T\mathbb{1}_{S_T>K}\right] = S\Phi\left(\frac{\log\frac{S}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

Hence we can interpret Black-Scholes (call) option formula as

$$\text{Vanilla Call} = \text{Asset Digital Call} - K \times \text{Cash Digital Call}$$

Interpreting Black-Scholes: Probability Measures

We can also interpret Black-Scholes formula from a probabilistic point of view. Note that

$$e^{-rT}\mathbb{E}^*\left[\mathbb{1}_{S_T>K}\right] = e^{-rT}\mathbb{P}^*(S_T > K)$$

We have also shown earlier that, using Girsanov's theorem and the change of measure, we can obtain

$$e^{-rT}\mathbb{E}^*\left[S_T\mathbb{1}_{S_T>K}\right] = S\mathbb{E}^S\left[\mathbb{1}_{S_T>K}\right] = S\mathbb{P}^S(S_T > K)$$

Hence, Black-Scholes (call) option formula can be interpreted as

$$\text{Vanilla Call} = S \times \mathbb{P}^S(S_T > K) - Ke^{-rT} \times \mathbb{P}^*(S_T > K)$$