QF620 Additional Examples
Session 6: Static Replication of European Payoff

1 Examples

1. An exotic European option pays S_T^2 if $K_1 < S_T < K_2$ on the maturity date of the option (where $K_1, K_2 \in \mathbb{R}^+,\ K_1 < K_2$). Let S_t follows the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where W_t is a Brownian motion under the real world probability measure. In the market there is also a risk-free bond following the differential equation $dB_t = rB_t dt$. Derive a risk-neutral valuation formula for this option.

2. Consider the same option payoff as the previous question (i.e. pays S_T^2 if $K_1 < S_T < K_2$). You have access to the vanilla European call option market, i.e. $C(K)$ is observable for all K. Defining the payoff function h as

$$h(S_T) = S_T^2 \mathbb{1}_{K_1 < S_T < K_2},$$

and starting from

$$\int_0^\infty h(K) \frac{\partial^2 C(K)}{\partial K^2} dK,$$

determine how you would go about applying the Breeden-Litzenberger approach to form a static replication portfolio of this exotic option using vanilla European call options.

2 Suggested Solutions

1. First we move from the real world probability measure to the risk-neutral measure associated to the risk-free bond (B_t) numeraire:

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$

Solve to obtain

$$S_T = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^*}$$
$$S_T^2 = S_0^2 e^{\left(2r - \sigma^2\right)T + 2\sigma W_T^*}$$

The exercise region is determined by inequalities

$$K_1 < S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x} < K_2$$

$$\Rightarrow \quad x_l^* = \frac{\log \frac{K_1}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} < x < \frac{\log \frac{K_2}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x_h^*$$

So the option value is given by

$$\begin{aligned}
e^{-rT}\mathbb{E}^*[S_T^2 \mathbb{1}_{K_1 < S_T < K_2}] &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S_T^2 \mathbb{1}_{K_1 < S_T < K_2} e^{-\frac{x^2}{2}} dx \\
&= \frac{e^{-rT}}{\sqrt{2\pi}} S_0^2 e^{(2r - \sigma^2)T} \int_{x_h^*}^{x_h^*} e^{2\sigma\sqrt{T}x - \frac{x^2}{2}} dx \\
&= S_0^2 e^{(r + \sigma^2)T}[\Phi(x_h^* - 2\sigma\sqrt{T}) - \Phi(x_l^* - 2\sigma\sqrt{T})]. \quad \triangleleft
\end{aligned}$$

2.

$$\begin{aligned}
\int_0^{\infty} K^2 \mathbb{1}_{K_1 < S_T < K_2} \frac{\partial^2 C(K)}{\partial K^2} dK &= \int_{K_1}^{K_2} K^2 \frac{\partial^2 C(K)}{\partial K^2} dK \\
&= \left[K^2 \frac{\partial C(K)}{\partial K}\right]_{K_1}^{K_2} - 2\int_{K_1}^{K_2} K \frac{\partial C(K)}{\partial K} dK \\
&= \left[K_2^2 \frac{\partial C(K_2)}{\partial K} - K_1^2 \frac{\partial C(K_1)}{\partial K}\right] \\
&\quad - 2[K_2 C(K_2) - K_1 C(K_1)] + 2\int_{K_1}^{K_2} C(K) dK. \quad \triangleleft
\end{aligned}$$