Assignment 4

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1. a) $Y_t = 2 + 0.5 Y_{t-1} + 0.4 Y_{t-2} + u_t$

$= 2 + 0.5 B \cdot Y_t + 0.4 B^2 Y_t + u_t$ where B is backward shift operator

$\Rightarrow Y_t - 0.5 B Y_t - 0.4 B^2 Y_t = 2 + u_t$

$(1 - 0.5B - 0.4B^2) Y_t = 2 + u_t$

↳ characteristic equation.

$1 - 0.5B - 0.4B^2 = 0$

Using quadratic formula,

$$B = \frac{-5 \pm \sqrt{185}}{8}$$

since $\left(\frac{5}{8}\right)^2 + \left(\frac{\sqrt{185}}{8}\right)^2 > 1$, the roots of the characteristic equation lie outside of the unit circle.

$\therefore Y_t$ is stationary.

b) When Y_t is stationary, $E(Y_t)$ is identical for all values of t.

$\therefore E(Y_t) = E(2) + 0.5E(Y_{t-1}) + 0.4E(Y_{t-2}) + E(u_t)$

$\Rightarrow \mu = 2 + 0.5\mu + 0.4\mu + 0$

$\mu = 20$

c) $Y_t = 2 + 0.5 Y_{t-1} + 0.4 Y_{t-2} + u_t$

$\times Y_{t-k}$, $Y_{t-k} Y_t = 2Y_{t-k} + 0.5 Y_{t-k} Y_{t-1} + 0.4 Y_{t-k} Y_{t-1} + Y_{t-k} u_t$

If we take unconditional expectation on both sides noting that $E(Y_{t-k} Y_t) = \gamma(k) + \mu^2$ for any k, then

$\gamma(k) + \mu^2 = 2\mu + 0.5(\gamma(k-1) + \mu^2) + 0.4(\gamma(k-2) + \mu^2)$

In the previous question, $\mu = 2 + 0.5\mu + 0.4\mu$. So

$\mu^2 = 2\mu + 0.5\mu^2 + 0.4\mu^2$

Therefore, $\gamma(k) = 0.5\,\gamma(k-1) + 0.4\,\gamma(k-2)$

Dividing both sides by $\gamma(0)$ will give us correlation.

$$\rho(k) = 0.5\rho(k-1) + 0.4\rho(k-2)$$

Here, $\rho(0) = 1$ and $\rho(j) = \rho(-j)$.

$$\rho(1) = 0.5\rho(0) + 0.4\rho(1).$$

$$\therefore \rho(1) = \frac{5}{6}$$

$$\rho(2) = 0.5\rho(1) + 0.4\rho(0)$$

$$\therefore \rho(2) = \frac{49}{60}$$

As shown in the equation above, autocorrelations of higher lags are determined by summation of two previous autocorrelations multiplied by a constant (0.5 and 0.4 in this question) respectively.

For example, $\rho(5) = 0.5\rho(4) + 0.4\rho(3)$.

Here, $\rho(4)$ and $\rho(3)$ can be expressed using $\rho(3), \rho(2)$ and $\rho(2), \rho(1)$ respectively. In this way, the autocorrelation of any higher lag can be calculated by summing up $\rho(1)$ and $\rho(2)$ after multiplying a constant to each ρ.

Therefore, autocorrelations of higher lags have linear relationship with the first two autocorrelations.

a) By adding the covariance of AR(1) and MA(1), we can get the covariance of ARMA(1,1).

$$C(Y_t, Y_{t-1}) = \lambda V(Y_{t-k}) + \alpha \sigma_u^2$$

$$C(Y_t, Y_{t-k}) = \lambda^k V(Y_{t-k}) \quad \text{where } k > 1$$

If $|\lambda| < 1$, ARMA(1,1) process is covariance-stationary with constant variance $\sigma_u^2\left(1 + \frac{(\lambda+\alpha)^2}{1-\lambda^2}\right)$

Therefore, ACF of ARMA(1,1) when $k = 1$ is

$$\Rightarrow \frac{\lambda \times \sigma_u^2\left(1 + \frac{(\lambda+\alpha)^2}{1-\lambda^2}\right) + \alpha\sigma_u^2}{\sigma_u^2\left(1 + \frac{(\lambda+\alpha)^2}{1-\lambda^2}\right)}$$

$$= \lambda + \frac{(1-\lambda^2)\alpha}{1 + 2\lambda\alpha + \alpha^2}$$

$$= \frac{(1+\alpha\lambda)(\lambda+\alpha)}{1 + 2\lambda\alpha + \alpha^2}.$$

When $k > 1$, ACF of ARMA(1,1) is

$$\frac{\lambda^k \sigma_u^2\left(1 + \frac{(\lambda+\alpha)^2}{1-\lambda^2}\right)}{\sigma_u^2\left(1 + \frac{(\lambda+\alpha)^2}{1-\lambda^2}\right)} = \lambda^k.$$

b) ACF of ARMA(1,1) when $k=1$

$$\Rightarrow \frac{(1+\alpha\lambda)(\lambda+\alpha)}{1+2\lambda\alpha+\alpha^2}$$

when $\lambda = -\alpha$ ACF becomes 0.