Moment Generating Function
for Standard Normal Random Variables

We often call the mean/expectation of a random variable X its first moment (i.e. $\mathbb{E}[X]$), as it is the sum or integral of the random variable itself weighted by the probability mass/density function.

We call $\mathbb{E}[X^2]$ the second moment, and it should be obvious that this is related to the variance of X. In general, we call $\mathbb{E}[X^k]$ the k^{th} moment of X. We often need to evaluate higher-order moments of random variable X in a wide range of applications.

In practice, to calculate the moments of a random variable, it is often simpler to use the moment generating function. This is particularly true for higher-order moments, when direct evaluation of the sum or integral becomes tedious.

For a random variable X with probability density/mass function f, and a real number t, the moment generating function (MGF) is defined as

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx}\ f(x)\ dx.$$

Given this definition, once we have obtained an expression for $M_X(t)$, the n^{th}-moment of a distribution can be deduced from the MGF as follow

$$\mathbb{E}[X^n] = \left.\frac{d^n M_X(t)}{dt^n}\right|_{t=0}$$

In words, the n^{th} moment is obtained by differentiating $M_X(t)$ for n times and then setting t to 0.

To see why this is the case, we expand the exponential function with Taylor series

$$\begin{aligned} M_X(t) = \mathbb{E}[e^{tX}] &= \mathbb{E}\left[1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots\right] \\ &= 1 + t\mathbb{E}[X] + \frac{t^2}{2!}\mathbb{E}[X^2] + \frac{t^3}{3!}\mathbb{E}[X^3] + \cdots \end{aligned}$$

Differentiating with respect to t once, we obtain

$$\begin{aligned} \frac{dM_X(t)}{dt} &= \mathbb{E}[X] + \frac{t}{1}\mathbb{E}[X^2] + \frac{t^2}{2!}\mathbb{E}[X^3] + \cdots \\ \Rightarrow \quad \frac{dM_X(0)}{dt} &= \mathbb{E}[X]. \end{aligned}$$

Differentiating for a second time, we obtain

$$\begin{aligned} \frac{d^2M_X(t)}{dt^2} &= \mathbb{E}[X^2] + \frac{t}{1}\mathbb{E}[X^3] + \cdots \\ \Rightarrow \quad \frac{d^2M_X(0)}{dt^2} &= \mathbb{E}[X^2]. \end{aligned}$$

For instance, the mean and variance of X can be obtained from the first 2 moments as follows:

$$\mathbb{E}[X] = \frac{dM_X(0)}{dt}$$
$$V[X] = \frac{d^2 M_X(0)}{dt^2} - \left[\frac{dM_X(0)}{dt}\right]^2$$

Let $X \sim N(\mu, \sigma^2)$ be normally distributed. The MGF is given by

$$M_X(t) = \mathbb{E}\left[e^{tX}\right] = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right).$$

To show this, we need to complete the square in the exponential function as follows

$$\begin{aligned}
M_X(t) = \mathbb{E}\left[e^{tX}\right] &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{tx} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{tx-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{tx-\frac{x^2-2\mu x+\mu^2}{2\sigma^2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{-2\sigma^2 tx+x^2-2\mu x+\mu^2}{2\sigma^2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{x^2-2(\mu+\sigma^2 t)x+\mu^2}{2\sigma^2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{x^2-2(\mu+\sigma^2 t)x+(\mu^2+2\mu\sigma^2 t+\sigma^4 t^2)-2\mu\sigma^2 t+\sigma^4 t^2}{2\sigma^2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{x^2-2(\mu+\sigma^2 t)x+(\mu+\sigma^2 t)^2}{2\sigma^2}+\mu t+\frac{\sigma^2 t^2}{2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}+\mu t+\frac{\sigma^2 t^2}{2}}\, dx \\
&= e^{\mu t+\frac{\sigma^2 t^2}{2}} \underbrace{\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}}\, dx}_{1} = e^{\mu t+\frac{\sigma^2 t^2}{2}}
\end{aligned}$$

In the last step, the exponential function $\exp(\mu t + \sigma^2 t^2)$ is not a function of x, and can be extracted from the integral, leaving the term which integrates to 1.

As an example, consider $Z \sim N(0,1)$, i.e. Z follows a standard normal distribution. In this case, the moment generating function is given by setting $\mu = 0$ and $\sigma^2 = 1$. We have

$$\begin{aligned}
M_Z(t) &= e^{\frac{t^2}{2}} \\
M'_Z(t) &= te^{\frac{t^2}{2}} \\
M''_Z(t) &= e^{\frac{t^2}{2}} + t^2 e^{\frac{t^2}{2}} \\
M'''_Z(t) &= 3te^{\frac{t^2}{2}} + t^3 e^{\frac{t^2}{2}} \\
M''''_Z(t) &= 3e^{\frac{t^2}{2}} + 6t^2 e^{\frac{t^2}{2}} + t^4 e^{\frac{t^2}{2}}
\end{aligned}$$

These give

$$
\begin{aligned}
\mathbb{E}[Z] &= M_Z'(0) = 0 \\
\mathbb{E}[Z^2] &= M_Z''(0) = 1 \\
\mathbb{E}[Z^3] &= M_Z'''(0) = 0 \\
\mathbb{E}[Z^4] &= M_Z''''(0) = 3
\end{aligned}
$$

Next, consider a Brownian motion $W_t \sim N(0,t) \sim \sqrt{t}N(0,1)$. Let $Z \sim N(0,1)$, we have

$$
\begin{aligned}
\mathbb{E}[W_t^2] &= \mathbb{E}[tZ^2] = t\mathbb{E}[Z^2] = t \\
\mathbb{E}[W_t^4] &= \mathbb{E}[t^2 Z^4] = t^2\mathbb{E}[Z^4] = 3t
\end{aligned}
$$