State Prices

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Economic Environment

- Assume k distinct states of nature and n distinct assets
- Let $\pi_s > 0$ be probability of state s, so that $\sum_{s=1}^{k} \pi_s = 1$
- Let X_{si} be payoff for one share of asset i in state s
- Let $\mathbf{X}_i = [X_{1i}, \ldots, X_{ki}]'$ be $k \times 1$ vector of payoffs for asset i
- Complete set of payoffs for all assets in all states, where i'th column of $\mathbf{X}$ represents payoffs for i'th asset:

$$\mathbf{X} = \begin{bmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{k1} & \cdots & X_{kn} \end{bmatrix}$$

Complete Market

- Market is **complete** if $n \geq k$ and $\mathbf{X}$ has k linearly independent columns and rows (i.e., $\mathbf{X}$ has rank k)
- WLOG, assume that $n = k$ and $\mathbf{X}$ is invertible, since can form k portfolios with linearly independent payoffs when $n > k$
- Let $\mathbf{Y} = [Y_1, \ldots, Y_k]'$ be any $k \times 1$ vector of payoffs
- To get portfolio with payoffs given by $\mathbf{Y}$, let $\mathbf{N} = [N_1, \ldots, N_k]'$ be $k \times 1$ vector of number of shares to invest for each asset:

$$\mathbf{Y} = \mathbf{XN} \quad \Rightarrow \quad \mathbf{N} = \mathbf{X}^{-1}\mathbf{Y}$$

- Hence $\mathbf{X}_i$'s **span** space consisting of all payoff vectors, so can form portfolio to provide any desired set of payoffs

State Prices

- Let $\mathbf{P} = [P_1, \ldots, P_k]'$ be $k \times 1$ vector of initial prices for one share of each asset
- Assuming no arbitrage, portfolio with payoffs given by $\mathbf{Y}$ must have initial price of $P_Y = \mathbf{P}'\mathbf{N} = \mathbf{P}'\mathbf{X}^{-1}\mathbf{Y}$
- Let $\mathbf{e}_s = [0, \ldots, 1, \ldots, 0]'$ be **elementary security** with unit payoff in state s and zero payoff in all other states
- Also known as **primitive security** or **Arrow–Debreu security**
- Initial prices of elementary securities:

$$p_s = \mathbf{P}'\mathbf{X}^{-1}\mathbf{e}_s \quad \forall \quad s = 1, \ldots, k$$

Pricing Kernel

- There exists unique set of state prices in complete market
- Investors who are nonsatiated will always be willing to pay for more consumption, so state prices must be strictly positive
- Assuming no arbitrage, initial price of portfolio with payoffs given by **Y**:

$$P_Y = \sum_{s=1}^{k} p_s Y_s = \sum_{s=1}^{k} \pi_s \left(\frac{p_s}{\pi_s} \right) Y_s = \sum_{s=1}^{k} \pi_s M_s Y_s = E\left[\tilde{M} \tilde{Y} \right]$$

- Hence there exists unique pricing kernel in complete market, with elements given by $M_s = p_s / \pi_s > 0$

Risk-Neutral Probabilities – Part 1

- Consider riskless asset with unit payoff in every state:

$$P_f = \sum_{s=1}^{k} p_s = \sum_{s=1}^{k} \pi_s M_s = E\left[\tilde{M}\right] = \frac{1}{R_f}$$

- Define $\widehat{\pi}_s = R_f p_s > 0$ for all $s = 1, \ldots, k$, which resembles set of probabilities since $\sum_{s=1}^{k} \widehat{\pi}_s = 1$
- Then initial price of portfolio with payoffs given by $\mathbf{Y}$:

$$P_Y = \sum_{s=1}^{k} p_s Y_s = \frac{1}{R_f} \sum_{s=1}^{k} \widehat{\pi}_s Y_s = \frac{1}{R_f} \widehat{E}\left[\tilde{Y}\right]$$

Risk-Neutral Probabilities – Part 2

- Here $\widehat{E}[\cdot]$ denotes expectation under the "pseudo" probability distribution represented by $\widehat{\pi}$
- Hence $\widehat{E}\left[\tilde{Y}\right]$ represents certainty equivalent of payoffs Y, which is discounted by risk-free rate
- Notice that $\widehat{\pi}$ puts more weight on states where pricing kernel is larger than average (and vice versa):

$$\widehat{\pi}_s = R_f p_s = R_f M_s \pi_s = \left(\frac{M_s}{E\left[\tilde{M}\right]} \right) \pi_s$$

- Hence $\widehat{\pi}$ puts more weight on "bad" states where consumption is low (and marginal utility is high), and vice versa

Risk-Neutral Probabilities – Part 3

- Consider special case where all investors are risk neutral, so pricing kernel is same in every state: $M_s = P_f = R_f^{-1} \ \forall \ s$
- Then pseudo probability $\widehat{\pi}_s$ matches true probability π_s:

$$\widehat{\pi}_s = R_f p_s = R_f M_s \pi_s = \pi_s$$

- Hence $\widehat{\pi}_s$ is known as **risk-neutral probability** while π_s is known as **physical probability**
- Interpret $\widehat{\pi}_s$ as risk-adjusted probability for state s, so can price assets as if all investors are risk neutral

Multiple Time Periods

- Suppose that trading and decision-making occurs over multiple time periods: $t = 1, \ldots, T$
- Suppose that k distinct states of nature in each time period
- Complete market requires that number assets (with linearly independent payoffs) be $n = k^T$
- However, market will be **dynamically complete** when number of assets (with linearly independent payoffs) is $n = k$
- Can form portfolio to provide any desired set of payoffs, but may need to rebalance portfolio in every time period