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1 Analytical Option Formulae

1.1 Black-Scholes Vanilla Call Option

The Black-Scholes model for the stock price process is defined as

$$dS_t = rS_t dt + \sigma S_t dW_t$$

By solving the stochastic differential equation using Itô's formula, we can get

$$S_t = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_t}$$

Vanilla European call option price is derived as below.

$$V_o^c = e^{-rT}E[(S_t-K)^+] = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}-K)^+ e^{-\frac{x^2}{2}}dx$$

Here,

$$S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x} > K$$

$$x > \frac{\log\frac{K}{S_0}-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x^*$$

Then,

$$V_o^c = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}-K)e^{-\frac{x^2}{2}}dx$$

$$= e^{-rT}\frac{1}{\sqrt{2\pi}}\left[S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T}\int_{x^*}^{\infty}e^{\sigma\sqrt{T}x}e^{-\frac{x^2}{2}}dx - K\int_{x^*}^{\infty}e^{-\frac{x^2}{2}}dx\right]$$

$$= e^{-rT}\left[S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}e^{-\frac{x^2-2\sigma\sqrt{T}x+\sigma^2T-\sigma^2T}{2}}dx - K\Phi(-x^*)\right]$$

$$= S_0\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}e^{-\frac{(x-\sigma\sqrt{T})^2}{2}}dx - Ke^{-rT}\Phi(-x^*)$$

Let

$$y = x-\sigma\sqrt{T} \Rightarrow dy = dx$$

$$x = x^*, y = x^* - \sigma\sqrt{T}$$

Then,

$$V_o^c = S_0 \frac{1}{\sqrt{2\pi}} \int_{x^*-\sigma\sqrt{T}}^{\infty} e^{-\frac{y^2}{2}} dy - Ke^{-rT}\Phi(-x^*)$$

$$= S_0 \Big[\Phi(\infty) - \Phi(x^* - \sigma\sqrt{T})\Big] - Ke^{-rT}\Phi(-x^*)$$

$$= S_0\Phi(-x^* + \sigma\sqrt{T}) - Ke^{-rT}\Phi(-x^*)$$

$$V_o^c = S_0\Phi\left(\frac{\log\frac{S_0}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\frac{S_0}{K} + \big(r - \frac{\sigma^2}{2}\big)T}{\sigma\sqrt{T}}\right)$$

1.2 Black-Scholes Vanilla Put Option

$$V_o^p = e^{-rT}E[(K - S_t)^+] = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(K - S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x})^+ e^{-\frac{x^2}{2}} dx$$

Here,

$$S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x} < K$$

$$x < \frac{\log\frac{K}{S_0} - \big(r - \frac{\sigma^2}{2}\big)T}{\sigma\sqrt{T}} = x^*$$

Then,

$$V_o^p = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x^*}\Big(K - S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}\Big)e^{-\frac{x^2}{2}} dx$$

$$= e^{-rT}K\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x^*} e^{-\frac{x^2}{2}} dx - e^{-rT}S_0\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x^*} e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}e^{-\frac{x^2}{2}} dx$$

$$= e^{-rT}K\Phi(x^*) - S_0 e^{-\frac{1}{2}\sigma^2 T}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x^*} e^{\sigma\sqrt{T}x}e^{-\frac{x^2}{2}} dx$$

$$= e^{-rT}K\Phi(x^*) - S_0\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x^*} e^{-\frac{(x-\sigma\sqrt{T})^2}{2}} dx$$

Let

$$y = x - \sigma\sqrt{T} \Rightarrow dy = dx$$

$$x = x^*, y = x^* - \sigma\sqrt{T}$$

Then,

$$V_o^p = e^{-rT}K\Phi(x^*) - S_0\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x^*-\sigma\sqrt{T}} e^{-\frac{y^2}{2}} dy$$

$$= e^{-rT}K\Phi(x^*) - S_0[\Phi(x^* - \sigma\sqrt{T})]$$

$$V_o^p = e^{-rT}K\Phi\left(\frac{\log\frac{K}{S_0} - \big(r - \frac{\sigma^2}{2}\big)T}{\sigma\sqrt{T}}\right) - S_0\Phi\left(\frac{\log\frac{K}{S_0} - (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

1.3 Black-Scholes Digital Cash-or-Nothing Call Option

$$V^c_{CashDigital}(0) = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{1}_{S_T>K}e^{-\frac{x^2}{2}}\,dx$$
$$= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}e^{-\frac{x^2}{2}}\,dx$$

where,

$$x^* = \frac{\log\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Therefore,

$$V^c_{CashDigital}(0) = e^{-rT}\Phi\left(\frac{\log\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

1.4 Black-Scholes Digital Cash-or-Nothing Put Option

$$V^p_{CashDigital}(0) = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{1}_{K>S_T}e^{-\frac{x^2}{2}}\,dx$$
$$= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x^*}e^{-\frac{x^2}{2}}\,dx$$

where,

$$x^* = \frac{\log\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Therefore,

$$V^p_{CashDigital}(0) = e^{-rT}\Phi\left(\frac{\log\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

1.5 Black-Scholes Digital Asset-or-Nothing Call Option

$$V^c_{AssetDigital}(0) = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}S_T\mathbf{1}_{S_T>K}e^{-\frac{x^2}{2}}\,dx$$
$$= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}S_Te^{-\frac{x^2}{2}}\,dx$$

where,

$$x^* = \frac{\log\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Therefore,

$$V^c_{AssetDigital}(0) = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}S_0e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}e^{-\frac{x^2}{2}}\,dx$$

From the same process as Black-Scholes vanilla call option,

$$V^c_{AssetDigital}(0) = S_0\Phi\left(\frac{\log\frac{S_0}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

1.6 Black-Scholes Digital Asset-or-Nothing Put Option

$$V^p_{AssetDigital}(0) = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} S_T \mathbf{1}_{K>S_T} e^{-\frac{x^2}{2}} dx$$

$$= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x^*} S_T e^{-\frac{x^2}{2}} dx$$

where,

$$x^* = \frac{\log\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Therefore,

$$V^p_{AssetDigital}(0) = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x^*} S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x} e^{-\frac{x^2}{2}} dx$$

From the same process as Black-Scholes vanilla put option,

$$V^p_{AssetDigital}(0) = S_0\Phi\left(\frac{\log\frac{K}{S_0} - (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

1.7 Bachelier Vanilla Call Option

The Bachelier model for the stock price process is defined as

$$dS_t = \sigma S_0 dW_t$$

$$S_T = S_0(1 + \sigma W_T)$$

Vanilla European call option price is derived as below.

$$V^c_o = E[(S_t - K)^+]$$

$$= E[(S_0(1 + \sigma W_T) - K)^+]$$

$$= E[(S_0(1 + \sigma\sqrt{T}x) - K)^+], \qquad X \sim N(0,1)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(S_0 + \sigma S_0\sqrt{T}x - K)^+ e^{-\frac{x^2}{2}} dx$$

Here,

$$S_0 + \sigma S_0\sqrt{T}x - K > 0$$

$$x > \frac{K - S_0}{\sigma S_0\sqrt{T}} = x^*$$

Then,

$$V^c_o = \frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}(S_0 + \sigma S_0\sqrt{T}x - K) e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}}\left(\int_{x^*}^{\infty}(S_0-K)e^{-\frac{x^2}{2}}\,dx + \int_{x^*}^{\infty}\sigma S_0\sqrt{T}xe^{-\frac{x^2}{2}}\,dx\right)$$

$$= (S_0-K)\Big[\Phi(\infty)-\Phi(x^*)\Big] + \frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}\sigma S_0\sqrt{T}xe^{-\frac{x^2}{2}}\,dx$$

Let $u=-\frac{x^2}{2}$, $du=-xdx$

$$V_o^c = (S_0-K)\Phi(-x^*) - \sigma S_0\sqrt{T}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}e^u du$$

$$= (S_0-K)\Phi(-x^*) - \sigma S_0\sqrt{T}\frac{1}{\sqrt{2\pi}}\big[e^u\big]_{-\frac{x^{*2}}{2}}^{\infty}$$

$$= (S_0-K)\Phi(-x^*) - \sigma S_0\sqrt{T}\frac{1}{\sqrt{2\pi}}\big[e^{-\frac{x^2}{2}}\big]_{x^*}^{\infty}$$

$$= (S_0-K)\Phi(-x^*) - \sigma S_0\sqrt{T}\frac{1}{\sqrt{2\pi}}\Big[0-e^{-\frac{x^{*2}}{2}}\Big]$$

$$= (S_0-K)\Phi(-x^*) + \sigma S_0\sqrt{T}\phi(-x^*)$$

$$= (S_0-K)\Phi\left(\frac{S_0-K}{\sigma S_0\sqrt{T}}\right) + \sigma S_0\sqrt{T}\phi\left(\frac{S_0-K}{\sigma S_0\sqrt{T}}\right)$$

1.8 Bachelier Vanilla Put Option

$$V_o^p = E[(K-S_t)^+]$$

$$= E[(K-S_0(1+\sigma W_T))^+]$$

$$= E[(K-S_0(1+\sigma\sqrt{T}x))^+], \qquad X\sim N(0,1)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(K-S_0-\sigma S_0\sqrt{T}x)^+e^{-\frac{x^2}{2}}\,dx$$

Here,

$$K-S_0-\sigma S_0\sqrt{T}x>0$$

$$x<\frac{K-S_0}{\sigma S_0\sqrt{T}}=x^*$$

Then,

$$V_o^p = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x^*}(K-S_0-\sigma S_0\sqrt{T}x)e^{-\frac{x^2}{2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{x^*}(K-S_0)e^{-\frac{x^2}{2}}\,dx - \int_{-\infty}^{x^*}\sigma S_0\sqrt{T}xe^{-\frac{x^2}{2}}\,dx\right)$$

$$= (K-S_0)\Phi(x^*) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x^*}\sigma S_0\sqrt{T}xe^{-\frac{x^2}{2}}\,dx$$

Let $u = -\frac{x^2}{2}$, $du = -xdx$

$$\begin{aligned}
V_o^p &= (K - S_0)\Phi(x^*) - \sigma S_0\sqrt{T}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x^*} e^u du \\
&= (K - S_0)\Phi(x^*) - \sigma S_0\sqrt{T}\frac{1}{\sqrt{2\pi}}\left[e^u\right]_{-\infty}^{-\frac{{x^*}^2}{2}} \\
&= (K - S_0)\Phi(x^*) - \sigma S_0\sqrt{T}\frac{1}{\sqrt{2\pi}}\left[e^{-\frac{x^2}{2}}\right]_{-\infty}^{x^*} \\
&= (K - S_0)\Phi(x^*) - \sigma S_0\sqrt{T}\phi(x^*) \\
&= (K - S_0)\Phi\left(\frac{K - S_0}{\sigma S_0\sqrt{T}}\right) - \sigma S_0\sqrt{T}\phi\left(\frac{K - S_0}{\sigma S_0\sqrt{T}}\right)
\end{aligned}$$

1.9 Bachelier Digital Cash-or-Nothing Call Option

$$\begin{aligned}
V_{CashDigital}^c(0) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathbf{1}_{S_T>K} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty} e^{-\frac{x^2}{2}} dx
\end{aligned}$$

where,

$$x^* = \frac{K - S_0}{\sigma S_0\sqrt{T}}$$

Therefore,

$$V_{CashDigital}^c(0) = \Phi\left(\frac{S_0 - K}{\sigma S_0\sqrt{T}}\right)$$

1.10 Bachelier Digital Cash-or-Nothing Put Option

$$\begin{aligned}
V_{CashDigital}^p(0) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathbf{1}_{K>S_T} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{x^*} e^{-\frac{x^2}{2}} dx
\end{aligned}$$

where,

$$x^* = \frac{K - S_0}{\sigma S_0\sqrt{T}}$$

Therefore,

$$V_{CashDigital}^p(0) = \Phi\left(\frac{K - S_0}{\sigma S_0\sqrt{T}}\right)$$

1.11 Bachelier Digital Asset-or-Nothing Call Option

$$V^c_{AssetDigital}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} S_T \mathbf{1}_{S_T>K} e^{-\frac{x^2}{2}} dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty} S_T e^{-\frac{x^2}{2}} dx$$

where,

$$x^* = \frac{K - S_0}{\sigma S_0\sqrt{T}}$$

Therefore,

$$V^c_{AssetDigital}(0) = \frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty} S_0(1+\sigma\sqrt{T}x) e^{-\frac{x^2}{2}} dx$$

From the same process as Bachelier vanilla call option,

$$V^c_{AssetDigital}(0) = S_0\Phi\left(\frac{S_0 - K}{\sigma S_0\sqrt{T}}\right) + \sigma S_0\sqrt{T}\phi\left(\frac{S_0 - K}{\sigma S_0\sqrt{T}}\right)$$

1.12 Bachelier Digital Asset-or-Nothing Put Option

$$V^p_{AssetDigital}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} S_T \mathbf{1}_{K>S_T} e^{-\frac{x^2}{2}} dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{x^*} S_T e^{-\frac{x^2}{2}} dx$$

where,

$$x^* = \frac{K - S_0}{\sigma S_0\sqrt{T}}$$

Therefore,

$$V^p_{AssetDigital}(0) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{x^*} S_0(1+\sigma\sqrt{T}x) e^{-\frac{x^2}{2}} dx$$

From the same process as Bachelier vanilla put option,

$$V^p_{AssetDigital}(0) = S_0\Phi\left(\frac{K - S_0}{\sigma S_0\sqrt{T}}\right) + \sigma S_0\sqrt{T}\phi\left(\frac{K - S_0}{\sigma S_0\sqrt{T}}\right)$$

1.13 Black76 Vanilla Call Option

$$dF_t = \sigma F_t W_t$$
$$F_t = F_0 e^{-\frac{\sigma^2 T}{2} + \sigma W_t}$$
$$F_0 = S_0 e^{rT}$$

Vanilla European call option price is derived as below.

$$V_o^c = e^{-rT}E[(F_t - K)^+]$$
$$= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(F_0 e^{-\frac{\sigma^2 T}{2}+\sigma\sqrt{T}x} - K)^+ e^{-\frac{x^2}{2}}dx$$
$$= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x} - K)^+ e^{-\frac{x^2}{2}}dx$$

The result above is exactly same as Black-Scholes model. Therefore, when we rearrange Black-Scholes model for forward price, we can get Black76 model.

$$\text{Black-Scholes Model : } V_o^c = S_0\Phi\left(\frac{\log\frac{S_0}{K} + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\frac{S_0}{K} + \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$\text{Black76 Model : } V_o^c = e^{-rT}\left[F_0\Phi\left(\frac{\log\frac{F_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{F_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right]$$

1.14 Black76 Other options

As shown in Black76 Vanilla call option, when we rearrange Black-Scholes models for forward price, we can get Black76 model.

Black76 Vanilla Put Option :

$$V_o^p = e^{-rT}K\Phi\left(\frac{\log\frac{K}{S_0} - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - S_0\Phi\left(\frac{\log\frac{K}{S_0} - (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$
$$= e^{-rT}\left[K\Phi\left(\frac{\log\frac{K}{F_0} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - F_0\Phi\left(\frac{\log\frac{K}{F_0} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right]$$

Black76 Digital Cash-or-Nothing Call Option :

$$V_{CashDigital}^c(0) = e^{-rT}\Phi\left(\frac{\log\frac{S_0}{K} + \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$
$$= e^{-rT}\Phi\left(\frac{\log\frac{F_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)$$

Black76 Digital Cash-or-Nothing Put Option :

$$V_{CashDigital}^p(0) = e^{-rT}\Phi\left(\frac{\log\frac{K}{S_0} - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$
$$= e^{-rT}\Phi\left(\frac{\log\frac{K}{F_0} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)$$

Black76 Digital Asset-or-Nothing Call Option :

$$V^c_{AssetDigital}(0) = S_0\Phi\left(\frac{\log\frac{S_0}{K} + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

$$= e^{-rT}F_0\Phi\left(\frac{\log\frac{F_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)$$

Black76 Digital Asset-or-Nothing Put Option :

$$V^p_{AssetDigital}(0) = S_0\Phi\left(\frac{\log\frac{K}{S_0} - (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

$$= e^{-rT}F_0\Phi\left(\frac{\log\frac{K}{F_0} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)$$

1.15 Displaced-diffusion Model

The Displaced-diffusion model for the forward price process is defined as

$$dF_t = \sigma[\beta F_t + (1-\beta)F_0]dW_t$$

Let

$$X_t = \log[\beta F_t + (1-\beta)F_0] = f(F_t)$$

$$f'(F_t) = \frac{\beta}{\beta F_t + (1-\beta)F_0}, \quad f''(F_t) = -\frac{\beta^2}{(\beta F_t + (1-\beta)F_0)^2}$$

Applying Itô's formula,

$$dX_t = f'(F_t)dF_t + \frac{1}{2}f''(F_t)(dF_t)^2$$

$$= \beta\sigma W_t - \frac{\beta^2\sigma^2}{2}dt$$

$$\int_0^T dX_t = \int_0^T \beta\sigma dW_t - \int_0^T \frac{\beta^2\sigma^2}{2}dt$$

$$X_T - X_0 = \beta\sigma W_T - \frac{\beta^2\sigma^2}{2}T$$

$$\log[\beta F_t + (1-\beta)F_0] - log[\beta F_0 + (1-\beta)F_0] = \beta\sigma W_T - \frac{\beta^2\sigma^2}{2}$$

$$\log\left[\frac{\beta F_T + (1-\beta)F_0}{F_0}\right] = \beta\sigma W_T - \frac{\beta^2\sigma^2}{2}$$

$$\frac{\beta F_T + (1-\beta)F_0}{F_0} = e^{-\frac{\beta^2\sigma^2 T}{2} + \beta\sigma W_T}$$

$$\beta F_T + (1-\beta)F_0 = F_0 e^{-\frac{\beta^2\sigma^2 T}{2}+\beta\sigma W_T}$$
$$F_T = \frac{F_0}{\beta} e^{-\frac{\beta^2\sigma^2 T}{2}+\beta\sigma W_T} - \frac{1-\beta}{\beta}F_0$$
$$\text{Forward price of Black76 : } F_t = F_0 e^{-\frac{\sigma^2 T}{2}+\sigma W_t}$$

Comparing this equation with Black76's F_t, we know that DisplacedDiffusion(F_0, k, σ, β, T) is equal to Black76($\frac{F_0}{\beta}$, $K + \frac{1-\beta}{\beta}F_0$, $\sigma\beta$, T). When we substitute Black76's parameters with those of Displaced-diffusion model, we derive Displaced-diffusion model.

Vanilla Call Option :

$$V_0^c = e^{-rT}\left[\frac{F_0}{\beta}\Phi\left(\frac{\log\frac{F_0}{\beta K+(1-\beta)F_0} + \frac{1}{2}\sigma^2\beta^2 T}{\sigma\beta\sqrt{T}}\right) - \left(K + \frac{1-\beta}{\beta}F_0\right)\Phi\left(\frac{\log\frac{F_0}{\beta K+(1-\beta)F_0} - \frac{1}{2}\sigma^2\beta^2 T}{\sigma\beta\sqrt{T}}\right)\right]$$

Vanilla Put Option :

$$= e^{-rT}\left[\left(K + \frac{1-\beta}{\beta}F_0\right)\Phi\left(\frac{\log\frac{\beta K+(1-\beta)F_0}{F_0} + \frac{1}{2}\sigma^2\beta^2 T}{\sigma\beta\sqrt{T}}\right) - \frac{F_0}{\beta}\Phi\left(\frac{\log\frac{\beta K+(1-\beta)F_0}{F_0} - \frac{1}{2}\sigma^2\beta^2 T}{\sigma\beta\sqrt{T}}\right)\right]$$

Digital Cash-or-Nothing Call Option :

$$= e^{-rT}\Phi\left(\frac{\log\frac{F_0}{\beta K+(1-\beta)F_0} - \frac{1}{2}\sigma^2\beta^2 T}{\sigma\beta\sqrt{T}}\right)$$

Digital Cash-or-Nothing Put Option :

$$= e^{-rT}\Phi\left(\frac{\log\frac{\beta K+(1-\beta)F_0}{F_0} + \frac{1}{2}\sigma^2\beta^2 T}{\sigma\beta\sqrt{T}}\right)$$

Digital Asset-or-Nothing Call Option :

$$= e^{-rT}\left[\frac{F_0}{\beta}\Phi\left(\frac{\log\frac{F_0}{\beta K+(1-\beta)F_0} + \frac{1}{2}\sigma^2\beta^2 T}{\sigma\beta\sqrt{T}}\right) - \frac{1-\beta}{\beta}F_0\Phi\left(\frac{\log\frac{F_0}{\beta K+(1-\beta)F_0} - \frac{1}{2}\sigma^2\beta^2 T}{\sigma\beta\sqrt{T}}\right)\right]$$

Digital Asset-or-Nothing Put Option :

$$= e^{-rT}\left[\frac{1-\beta}{\beta}F_0\Phi\left(\frac{\log\frac{\beta K+(1-\beta)F_0}{F_0} + \frac{1}{2}\sigma^2\beta^2 T}{\sigma\beta\sqrt{T}}\right) - \frac{F_0}{\beta}\Phi\left(\frac{\log\frac{\beta K+(1-\beta)F_0}{F_0} - \frac{1}{2}\sigma^2\beta^2 T}{\sigma\beta\sqrt{T}}\right)\right]$$