Session 1
ℚuantitative Analysis of Financial Markets
Probabilities

Christopher Ting

http://www.mysmu.edu/faculty/christophert/

Christopher Ting

✉: christophert@smu.edu.sg

☎: 6828 0364

: LKCSB 5036

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Notations

- Scalars: letters that are not bold, e.g., x
- Column vectors: lower case bold letters, e.g., $\boldsymbol{x}$
- Row vectors: transposes of column vectors, e.g., $\boldsymbol{x}^\top$
- Matrices: upper case bold letters, e.g., $\boldsymbol{X}$
- The i-th element of a vector $\boldsymbol{b}$: b_i.
- The (i, j) element of the matrix $\boldsymbol{A}$: a_{ij}.
- Red symbols highlight that they denote **random variables**.

Introduction

- The concept of probability is very important in $\mathbb{Q}$uantitative Finance.

- Probability can be quite tricky yet fun at times. A thorough grasp of the concepts of unconditional probability, conditional probability, joint probability, and the relationship among them is necessary.

Learning Outcomes of QA01

Chapter 2.

Michael Miller, Mathematics and Statistics for Financial Risk Management, 2nd Edition (Hoboken, NJ: John Wiley & Sons, 2013).

- Describe and distinguish between continuous and discrete random variables.
- Define and distinguish between the probability density function, the cumulative distribution function, and the inverse cumulative distribution function.
- Calculate the probability of an event given a discrete probability function.
- Distinguish between independent and mutually exclusive events.
- Define joint probability, describe a probability matrix, and calculate joint probabilities using probability matrices.
- Define and calculate a conditional probability, and distinguish between conditional and unconditional probabilities.

Events, Random Variables, and Probability Distribution

- An event in probability theory refers to something of interest for the analyst, and its occurrence is uncertain.
- An outcome is a realization of one possible result out of many possibilities.
- Mutually exclusive events are events that cannot happen at the same time.
- Exhaustive events are event that include all possible outcomes.
- In quantitative analysis, a random variable is a mapping from the space of events to a number.
- A probability distribution describes the probabilities of all the possible outcomes for a random variable.

Relevant Examples

- Intra-day trading
  - Outcome 1: The signal/forecast generated by an algorithmic system is correct.
  - Outcome 2: The signal/forecast generated by an algorithmic system is incorrect.
- Trinomial tree
- Moody's long-term credit rating
  - 21 outcomes: Aaa, Aa1, Aa2, Aa3, A1, A2, A3, Baa1, Baa2, Baa3, Ba1, Ba2, Ba3, B1, B2, B3, Caa1, Caa2, Caa3, Ca, C.
- Corporate Earnings
  - Three outcomes: better than, equal to, or worst than consensus estimate
- Daily return r given daily volatility σ
  - Outcomes: $-1 < r \leq -3\sigma,\ -3\sigma < r \leq -2\sigma,\ -2\sigma < r \leq -\sigma,$ $-\sigma < r \leq 0,\ 0 < r \leq \sigma,\ \sigma < r \leq 2\sigma,\ 2\sigma < r \leq 3\sigma,\ r > 3\sigma$

Discrete vs. Continuous Random Variables

- A discrete random variable X can take on only a countable number of values.

$$\mathbb{P}(X = x_i) = p_i, \qquad i = 1, 2, \ldots, n.$$

- A continuous random variable X can take on any value within a given range.

$$\mathbb{P}(r_1 < X < r_2) = p.$$

The probability of X being between r_1 and r_2 is equal to p.

Probability Density Functions

- For a continuous random variable we can define a probability density function (PDF), which tells us the likelihood of outcomes occurring between any two points.
- For the probability p of X lying between r_1 and r_2, we define the density function $f(x)$ as follows:

$$\int_{r_1}^{r_2} f(x)\, dx = p.$$

Sample Problem

✂ Suppose the probability density function for the price of a zero coupon bond in percentage point x of the par or face value is,

$$f(x) = \frac{8}{9}x,$$

for $0 < x \leq 3/2$.

1. Do the probabilities sum to 1?

2. What is the probability that the price of the bond is between 90% and 100%?

Cumulative Distribution Functions (CDF)

- A cumulative distribution function tells us the probability of a random variable being less than a certain value.
- By integrating the probability density function from its lower bound to the certain value a, the CDF is obtained.

$$F(a) = \int_{-\infty}^{a} f(x)\,dx = \mathbb{P}\left(X \le a\right).$$

- We emphasize that $0 \le F(x) \le 1$, and $F(x)$ is non-decreasing.
- By the fundamental theorem of calculus, $f(x) = \dfrac{dF(x)}{dx}$.
- Moreover,

$$\mathbb{P}\left(a < X \le b\right) = \int_{a}^{b} f(x)\,dx = F(b) - F(a).$$

- The probability that a random variable is greater than a certain value a is

$$\mathbb{P}\left(X > a\right) = 1 - F(a).$$

Inverse Cumulative Distribution Functions

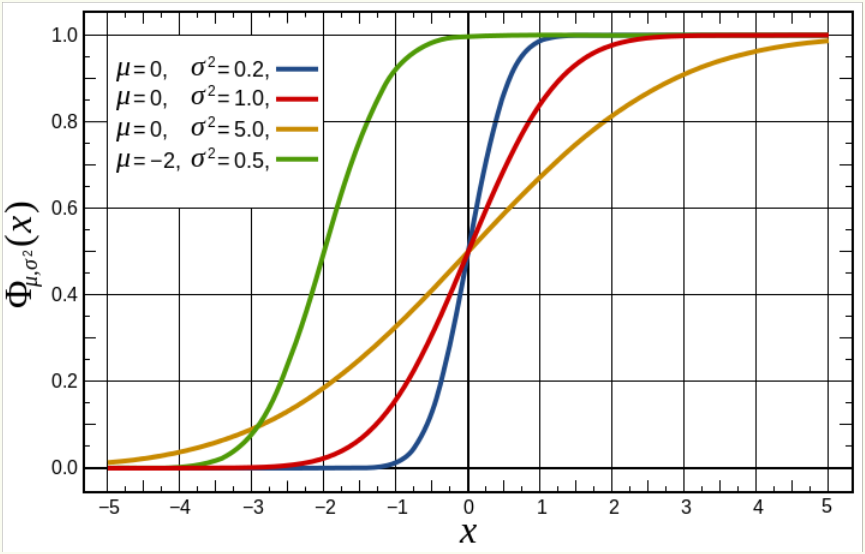
- Let $F(a)$ be the cumulative distribution function. We define the inverse function $F^1(p)$, the inverse cumulative distribution, as follows:

$$F(a) = p \Longleftrightarrow F^{-1}(p) = a.$$

- The inverse distribution function is also called the quantile function.
- The 95-th percentile is $F^{-1}(0.95)$.
- Properties of $F^{-1}(p)$

  1. $F^{-1}(p)$ is non-decreasing
  2. $F^{-1}(y) \leq x$ if and only if $y \leq F(x)$.
  3. If Y has a uniform distribution in the interval $[0, 1]$, then $F^{-1}(Y)$ is a random variable with distribution F.

Normal Distribution CDF



wikipedia.org/wiki/File:Normal_Dis

Sample Problem

✂ Consider the cumulative distribution function for $0 \leq a \leq 10$,

$$F(a) = \frac{a^2}{100}.$$

1. Derive the inverse cumulative distribution function.
2. Find the value of a such that 25% of the distribution is less than or equal to a.

Wikipedia Example

✂ The cumulative distribution function of Exponential(λ) (i.e. intensity λ and expected value (mean) $1/\lambda$) is

$$F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

1. Find the probability density function.
2. Find the quantile function for Exponential(λ).
3. Suppose $\lambda = \ln(2)$. Find the median.

Mutually Exclusive Events

- For a given random variable, the probability of any of two mutually exclusive events A and B occurring is just the sum of their individual probabilities.

$$\mathbb{P}\left(A \cup B\right) = \mathbb{P}(A) + \mathbb{P}(B).$$

- It is the probability of either A or B occurring, which is true only for mutually exclusive events.

- Question: Calculate the probability that a stock return is either below -10% or above 10%, given that

$$\mathbb{P}\left(R < -10\%\right) = 14\%, \qquad \mathbb{P}\left(R > 10\%\right) = 17\%,$$

Independent Events and Joint Probability

- What happens when we have more than one random variable?
- Event: It rains tomorrow and the return on stock SIA is greater than 0.5%?
- If the outcome of one random variable is not influenced by the outcome of the other random variable, then we say those variables are independent. The joint probability of W and R is such that

$$\begin{aligned}\mathbb{P}\left(W = \text{rain and } R > 0.5\%\right) &= \mathbb{P}\left(\text{rain } \cap R > 0.5\%\right)\\ &= \mathbb{P}\left(\mathrm{rain}\right) \times \mathbb{P}\left(R > 0.5\%\right).\end{aligned}$$

- Question: According to the most recent weather forecast, there is a 20% chance of rain tomorrow. The probability that SIA returns more than 0.5% on any given day is 40%. The two events are independent. What is the probability that it rains and SIA returns more than 0.5% tomorrow?

Probability Matrices

- When dealing with the joint probabilities of two variables, it is often convenient to summarize the various probabilities in a probability matrix or probability table.

- Example: Stock Grading by Equity Analyst and Credit Rating Agency

		Stock		
		Outperform	Underperform	Total %
Bonds	Upgrade	15%	5%	20%
	No Change	30%	25%	55%
	Downgrade	5%	20%	25%
	Total %	50%	50%	100%

Sample Problem

- Bonds versus Stock Matrix

		Stock		
		Outperform	Underperform	Total %
Bonds	Upgrade	5%	0%	5%
	No Change	40%	Y%	Z%
	Downgrade	X%	30%	35%
	Total %	50%	50%	100%

- What are the values of X, Y, and Z?

Conditional Probability

- What is the probability that the stock market is up **given that** it is raining?

$$\mathbb{P}\left(M = \text{up} \middle| W = \text{rain}\right)$$

- Using the conditional probability, we can calculate the probability that it will rain and that the market will be up.

$$\mathbb{P}\left(M = \text{up and } W = \text{rain}\right) = \mathbb{P}\left(M = \text{up} \middle| W = \text{rain}\right) \mathbb{P}\left(W = \text{rain}\right).$$

$$\Updownarrow$$

$$\mathbb{P}\left(M = \text{up and } W = \text{rain}\right) = \mathbb{P}\left(W = \text{rain} \middle| M = \text{up}\right) \mathbb{P}\left(M = \text{up}\right).$$

Independence

- Another way to define the concept of independence: If $\mathbb{P}\left(M = \text{up} \middle| W = \text{rain}\right) = \mathbb{P}\left(M = \text{up}\right)$, the two random variables, M and W, are independent.

- Show that if M and W are independent, then

$$\mathbb{P}\left(M = \text{up and } W = \text{rain}\right) = \mathbb{P}\left(M = \text{up}\right)\mathbb{P}\left(W = \text{rain}\right).$$

Using Conditional Probabilities

- We can also use conditional probabilities to calculate unconditional probabilities. On any given day, either it rains or it does not rain. The probability that the market will be up, then, is simply the probability of the market being up when it is raining plus the probability of the market being up when it is not raining.

$$\begin{aligned}\mathbb{P}\left(M = \text{up}\right) &= \mathbb{P}\left(M = \text{up and } W = \text{rain}\right) + \mathbb{P}\left(M = \text{up and } W = \overline{\text{rain}}\right) \\ &= \mathbb{P}\left(M = \text{up} \middle| W = \text{rain}\right)\mathbb{P}\left(W = \text{rain}\right) \\ &\quad + \mathbb{P}\left(M = \text{up} \middle| W = \overline{\text{rain}}\right)\mathbb{P}\left(W = \overline{\text{rain}}\right).\end{aligned}$$

- In general, if a random variable X has n possible values, $x_1, x_2, \ldots, x_n$, we have the law of total probability:

$$\mathbb{P}\left(Y\right) = \sum_{i=1}^{n} \mathbb{P}\left(Y \middle| X = x_i\right)\mathbb{P}\left(X = x_i\right).$$

Important Concepts

- Event, Outcome, Random Variable
- Probability Mass Function vs. Probability Density Function
- Cumulative Distribution Function
- Inverse Cumulative Distribution Function = ____________ Function
- Conditional, Joint, and Marginal Probabilities Connected in Bayes' Theorem

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

- Independent Events
- Mutually exclusive Events
- Law of total probability