Session 7
Quantitative Analysis of Financial Markets
Stationary Processes

Christopher Ting

http://www.mysmu.edu/faculty/christophert/

Christopher Ting

: christophert@smu.edu.sg

: 6828 0364

: LKCSB 5036

February 18, 2018

Broad Lesson Plan

Learning Outcomes

✂ Understand thoroughly the nature of white noise.

✂ Describe AR, MA, and ARMA time series using white noise as the building block.

✂ Compute the unconditional and conditional means, as well as variances of these stationary processes.

✂ Describe the autocorrelation function and how the function is estimated and tested for statistical significance (Box and Pierce Q-statistic, Ljung and Box statistic).

✂ Define back-shift operator B^n and describe the relationship between MA and AR(∞) processes.

✂ Thoroughly understand the Yule-Walker equations and their applications to obtain partial autocorrelation function.

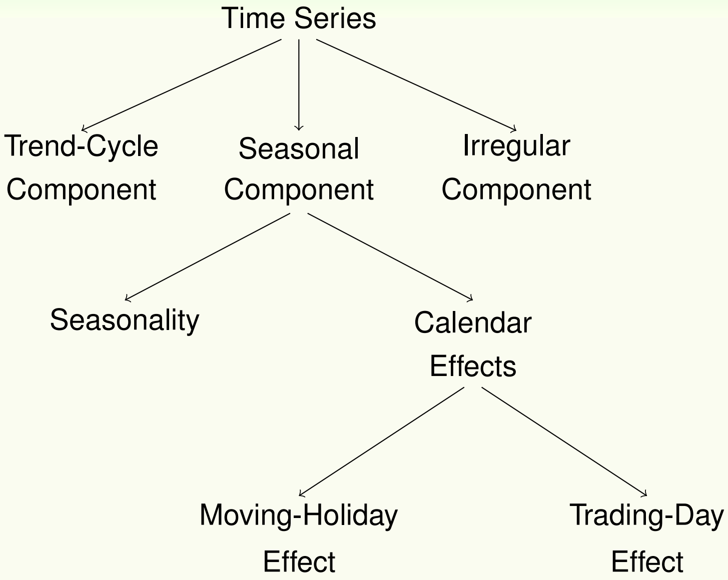
Components of a Time Series

☛ In general, a time series Y_t has the following components:

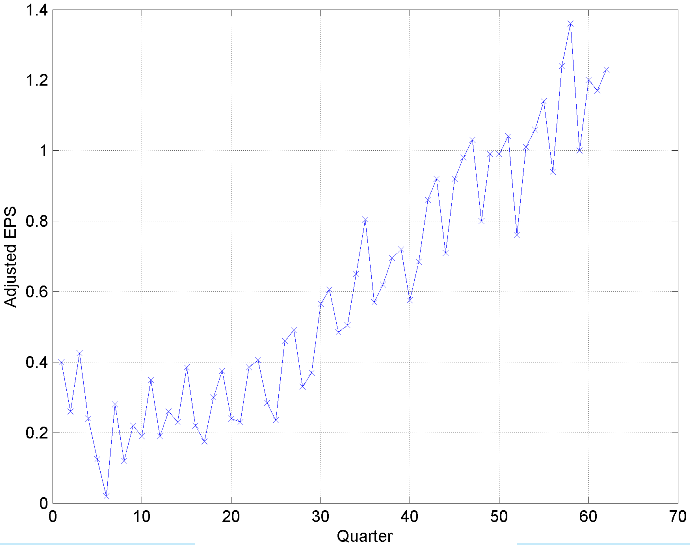
(a) **trend-cycle (TC) component**: the combined long-term and growth cycle movement of the time series

(b) **seasonal (S) component**: the systematic variations of the time series

(c) **irregular (I) component**: the random fluctuations of short-term movements of the time series.


Time Series
Trend-Cycle Component
Seasonal Component
Irregular Component
Seasonality
Calendar Effects
Moving-Holiday Effect
Trading-Day Effect

Example: Nike's EPS



Basic Building Block: White Noise $\{u_t\}$

♪ Properties of white noise

- Identically distributed, stationary process
- $\mathbb{E}(u_t) = 0$
- $\mathbb{V}(u_t) = \sigma_u^2$
- $\mathbb{C}(u_t\,,\, u_{t+k}) = 0$ for any $k \neq 0$
- Stronger definition: u_t is independent of u_{t+k} for all $k \neq 0$

♪ How does white noise sound like?

- Generate white noise with Matlab: `u = randn(8192*10, 1);`
- Create an audioplayer object: `pu = audioplayer(u, 8192);`
- Listen to white noise: `play(pu);`

AR, MA, and ARMA

With basic white noise process $\{u_t\}$, $t = -\infty, \ldots, \infty$, the following processes are generated:

- Autoregressive order one: AR(1) process

$$Y_t = \theta + \lambda\, Y_{t-1} + u_t$$

- Moving Average order one: MA(1) process

$$Y_t = \theta + u_t + \alpha\, u_{t-1}$$

- Autoregressive Moving Average order one: ARMA(1, 1) process

$$Y_t = \theta + \lambda\, Y_{t-1} + u_t + \alpha\, u_{t-1}$$

Autoregressive Process AR(1)

⊥ Repeated substitution for Y_t in the AR(1) process

$$Y_t = \theta + \lambda(\theta + \lambda Y_{t-2} + u_{t-1}) + u_t$$

leads to

$$Y_t = (1 + \lambda + \lambda^2 + \cdots)\theta + (u_t + \lambda u_{t-1} + \lambda^2 u_{t-2} + \cdots).$$

⊥ For each t, provided $|\lambda| < 1$,

$$\begin{aligned}
\mathbb{E}(Y_t) &= (1 + \lambda + \lambda^2 + \cdots)\theta = \frac{\theta}{1-\lambda}; \\
\mathbb{V}(Y_t) &= \mathbb{V}(u_t + \lambda u_{t-1} + \lambda^2 u_{t-2} + \cdots) = \sigma_u^2(1 + \lambda^2 + \lambda^4 + \cdots) \\
&= \frac{\sigma_u^2}{1-\lambda^2}; \\
\mathbb{C}(Y_t\,,\,Y_{t-1}) &= \mathbb{C}\big(\theta + \lambda Y_{t-1} + u_t\,,\,Y_{t-1}\big) = \lambda\frac{\sigma_u^2}{1-\lambda^2}.
\end{aligned}$$

Autocorrelation Coefficients of AR(1)

⟶ Autocovariance at lag k is a function of k only

$$\mathbb{C}(Y_t\,,\,Y_{t-k}) = \mathbb{C}\left(\theta\sum_{j=0}^{k-1}\lambda^j + \lambda^k Y_{t-k} + \sum_{j=0}^{k-1}\lambda^j u_{t-j}\,,\,Y_{t-k}\right) = \lambda^k\,\mathbb{V}(Y_t) =: \gamma(k)$$

⟶ Accordingly, the serial correlation is

$$\mathrm{Corr}(Y_t\,,\,Y_{t-k}) = \frac{\mathbb{C}(Y_t\,,\,Y_{t-k})}{\mathbb{V}(Y_t)} = \lambda^k$$

⟶ For a symmetrical time shift $t \longrightarrow t+k$,

$$\mathrm{Corr}(Y_{t+k}\,,\,Y_t) = \frac{\mathbb{C}(Y_{t+k}\,,\,Y_t)}{\mathbb{V}(Y_{t+k})} = \lambda^k$$

⟶ Hence

$$\rho(k) := \mathrm{Corr}\big(Y_t\,,\,Y_{t\pm k}\big) = \lambda^{|k|}$$

Illustrative Example of an AR(1) Process

Suppose

$$Y_t = 2.5 + 0.5Y_{t-1} + u_t\,, \qquad \text{where } \mathbb{E}(u_t) = 0, \mathbb{V}(u_t) = 3$$

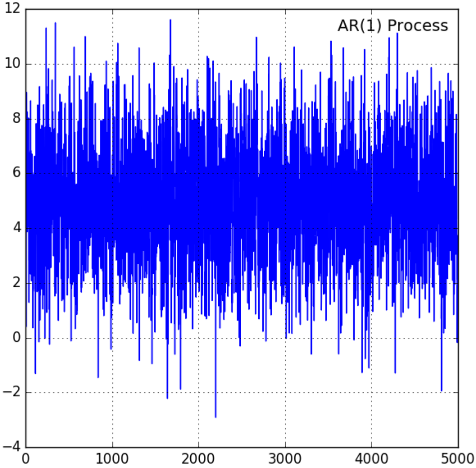
1. What is the mean of Y_t?
2. What is the variance of Y_t?
3. What is the first-order autocovariance of this AR(1) process?
4. What is the first-order autocorrelation of this AR(1) process?
5. What is the third-order autocorrelation of this AR(1) process?

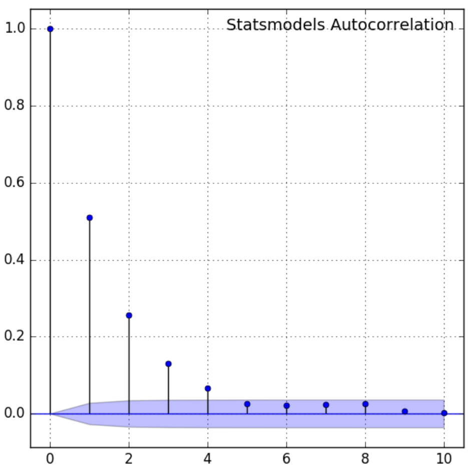
Simulating an AR(1) Process

- Simulation parameters:
  - number of samples 5,000
  - $\sigma_u^2 = 3$
  - $\theta = 2.5, \qquad \lambda_1 = 0.5$

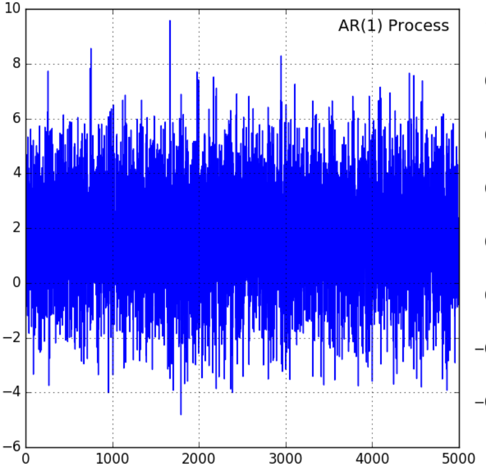
```
n = 5000
su2 = 3
theta, lambda1 = 2.5, 0.5
np.random.seed(20180219)
u = np.random.normal(0, 1, n)
mu, su = np.mean(u), np.std(u, ddof=1)
u -= mu      # ensure that mean of u is really 0
u /= su;     # ensure that variance of u is really 1
u *= np.sqrt(su2)
Y = np.zeros(n, dtype=float)
Y[0] = theta + u[0]
for t in range(1, n):
     Y[t] = theta + lambda1*Y[t-1] + u[t]
```

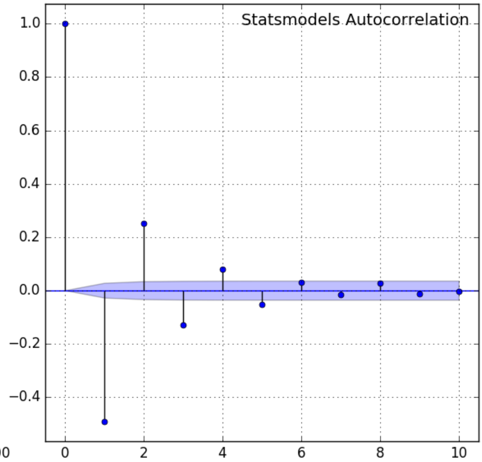
Sample Autocorrelation Function of AR(1)





Another Example of AR(1), $\lambda = -0.5$





Moving Average Process MA(1)

$\Rightarrow$ For MA(1) process $Y_t = \theta + u_t + \alpha\, u_{t-1}$,

$$\begin{aligned}
\mathbb{E}\big(Y_t\big) &= \theta \\
\mathbb{V}\big(Y_t\big) &= (1+\alpha^2)\sigma_u^2 \\
\mathbb{C}\big(Y_t\,,\,Y_{t-1}\big) &= \mathbb{C}\big(\theta + u_t + \alpha u_{t-1}\,,\,\theta + u_{t-1} + \alpha u_{t-2}\big) = \alpha\,\sigma_u^2 \\
\mathrm{Corr}\big(Y_t\,,\,Y_{t-1}\big) &= \frac{\alpha}{1+\alpha^2} \\
\mathrm{Corr}\big(Y_t\,,\,Y_{t-k}\big) &= 0 \quad \text{for } k > 1
\end{aligned}$$

$\Rightarrow$ Hence, MA(1) process is covariance-stationary with constant mean θ, constant variance $\sigma_u^2(1+\alpha^2)$ and autocorrelation function of $k=1$ only nonzero.

Illustrative Example of an MA(1) Process

Suppose

$$Y_t = 2.5 + u_t + 0.5u_{t-1}\,, \qquad \text{where } \mathbb{E}(u_t) = 0, \mathbb{V}(u_t) = 3$$

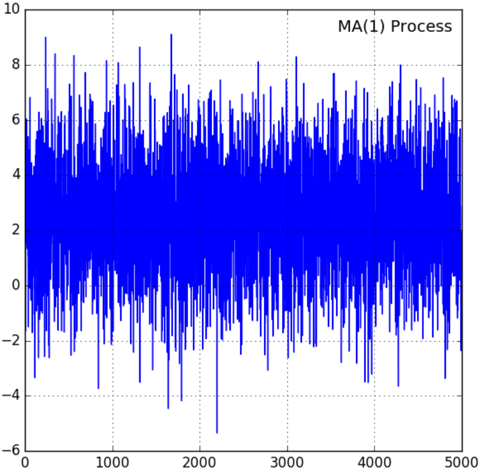
1. What is the mean of Y_t?
2. What is the variance of Y_t?
3. What is the first-order autocovariance of this MA(1) process?
4. What is the first-order autocorrelation of this MA(1) process?
5. What is the third-order autocorrelation of this MA(1) process?

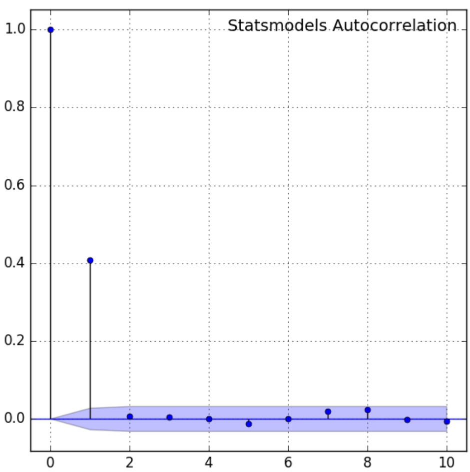
Simulating an MA(1) Process

- Simulation parameters:
  - number of samples 5,000
  - $\sigma_u^2 = 3$
  - $\theta = 2.5, \qquad \alpha = 0.5$

```
n = 5000
su2 = 3
theta, alpha = 2.5, 0.5
np.random.seed(20180219)
u = np.random.normal(0, 1, n)
mu, su = np.mean(u), np.std(u, ddof=1)
u -= mu      # ensure that mean of u is really 0
u /= su;     # ensure that variance of u is really 1
u *= np.sqrt(su2)
Y = np.zeros(n, dtype=float)
Y[0] = theta + u[0]
for t in range(1, n):
     Y[t] = theta + alpha*u[t-1] + u[t]
```

Sample ACF of MA(1) Process





ARMA(1,1)

⇉ Repeated substitution of $Y_t = \theta + \lambda Y_{t-1} + u_t + \alpha\, u_{t-1}$ leads to

$$Y_t = \theta \sum_{i=0}^{\infty} \lambda^i + u_t + (\lambda + \alpha) \sum_{i=0}^{\infty} \lambda^i u_{t-i}.$$

⇉ For each t, provided $|\lambda| < 1$

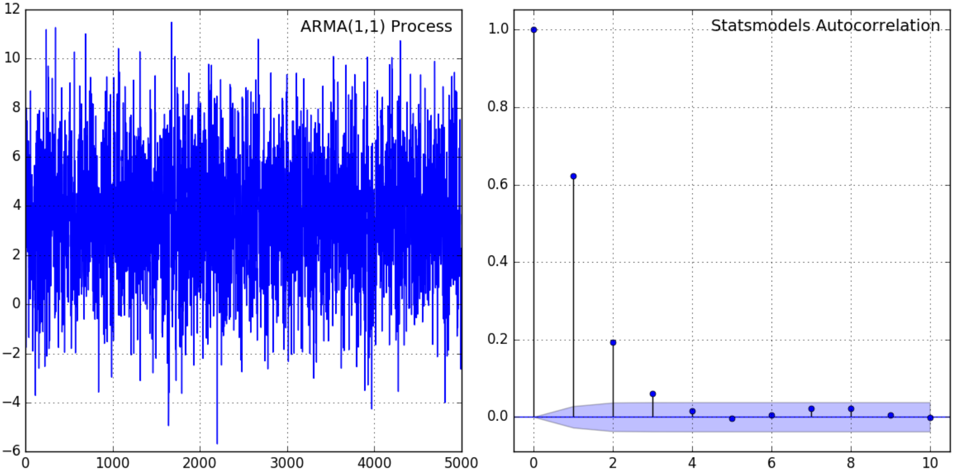
$$\mathbb{E}(Y_t) = \frac{\theta}{1-\lambda}, \quad \mathbb{V}(Y_t) = \sigma_u^2 \left(1 + (\lambda+\alpha)^2 \sum_{i=0}^{\infty} \lambda^{2i} \right) = \sigma_u^2 \left(1 + \frac{(\lambda+\alpha)^2}{1-\lambda^2} \right)$$

$$\mathbb{C}(Y_t\,,\, Y_{t-1}) = \lambda\, \mathbb{V}(Y_{t-k}) + \alpha\, \sigma_u^2$$

$$\mathbb{C}(Y_t\,,\, Y_{t-k}) = \lambda^k\, \mathbb{V}(Y_{t-k})\,, \quad \text{for } k > 1$$

⇉ Hence, ARMA(1,1) process is covariance-stationary with constant mean $\dfrac{\theta}{1-\lambda}$, constant variance $\sigma_u^2 \left(1 + \dfrac{(\lambda+\alpha)^2}{1-\lambda^2} \right)$, and autocovariance of a function of k only.

Sample ACF of ARMA(1,1) Process



Autocorrelation Function of ARMA(1,1)

Question 1: Derive the autocorrelation function of ARMA(1,1) process

Question 2: If $\lambda = -\alpha$, what is the ACF of ARMA(1,1) at lag 1?

Changing Conditional Means

AR(1)

ℵ At $t+1$, information about Y_t is already known, so for AR(1) process, conditional mean is a random variable

$$\mathbb{E}\big(Y_{t+1}\big|Y_t\big) = \theta + \lambda Y_t + \mathbb{E}\big(u_{t+1}\big|Y_t\big) = \theta + \lambda Y_t \neq \frac{\theta}{1-\lambda}$$

ℵ Conditional variance is constant

$$\mathbb{V}\big(Y_{t+1}\big|Y_t\big) = \mathbb{V}\big(u_{t+1}\big|Y_t\big) = \sigma_u^2 < \frac{\sigma_u^2}{1-\lambda^2}$$

MA(1)

ℵ Conditional mean is stochastic but conditional variance is constant:

$$\begin{aligned}\mathbb{E}\big(Y_{t+1}\big|Y_t\big) &= \theta + \alpha u_t \\ \mathbb{V}\big(Y_{t+1}\big|Y_t\big) &= \sigma_u^2 < \sigma_u^2(1+\alpha^2)\end{aligned}$$

Sample Autocorrelation Function

◇ Sample autocovariance lag k, for $k = 0, 1, 2, \ldots, p$, with $p < T/4$.

$$c(k) = \frac{1}{T} \sum_{t=1}^{T-k} \big(Y_t - \overline{Y}\big)\big(Y_{t+k} - \overline{Y}\big)$$

◇ The estimator $c(k)$ is consistent, as $\lim_{T\uparrow\infty} c(k) = \mathbb{C}\big(Y_k\,,\, Y_{t-k}\big)$.

◇ Sample autocorrelation at lag k is then

$$r(k) = \frac{c(k)}{c(0)} = \frac{\sum_{t=1}^{T-k} \big(Y_t - \overline{Y}\big)\big(Y_{t+k} - \overline{Y}\big)}{\sum_{t=1}^{T} \big(Y_t - \overline{Y}\big)^2},$$

which is also consistent, as $\lim_{T\uparrow\infty} r(k) = \rho(k)$.

◇ Variance of $r(k)$ for AR(1) process

$$\mathbb{V}\big(r(k)\big) \approx \frac{1}{T} \left(\frac{(1+\lambda^2)(1-\lambda^{2k})}{1-\lambda^2} - 2k\lambda^{2k} \right)$$

Test for $\rho = 0$, AR(1)

♡ Test hypothesis $H_0 : \rho(j) = 0$ for all $j > 0$ or $H_0 = \lambda = 0$. Then,

$$\mathbb{V}\big(r(k)\big) \approx \frac{1}{T}.$$

♡ To test that the j^{th} autocorrelation is zero, the test statistic is

$$z_j = \frac{r(j) - 0}{\sqrt{1/T}} \overset{d}{\sim} N(0, 1).$$

♡ Reject $H_0 : \rho(j) = 0$ at 5% significance level if $|z| > 1.96$.

Test for $\rho = 0$, MA(q)

- For MA(q) processes, the variance of $r(k)$ is

$$\mathbb{V}\big(r(k)\big) \approx \frac{1}{T}\left(1 + 2\sum_{i=1}^{q} \rho(i)^2\right).$$

- Thus for MA(1) process, $\mathbb{V}\big(r(k)\big)$ for $k > 1$ is estimated by

$$\frac{1}{T}\big(1 + 2r(1)^2\big) > \frac{1}{T},$$

and rejection of the null hypothesis allows the identification of MA(1).

Joint Test Statistic

♯ To test if m autocorrelations are jointly zero, the null hypothesis is

$$H_0 : \rho(1) = \rho(2) = \cdots = \rho(m) = 0.$$

♯ The Box and Pierce Q-statistic for asymptotic test is

$$Q_m = T \sum_{k=1}^{m} r(k)^2 = \sum_{k=1}^{m} \left(\sqrt{T}\, r(k)\right)^2 = \sum_{k=1}^{m} z_k^2 \overset{d}{\sim} \chi_m^2.$$

♯ The Ljung and Box test statistic provides an approximate correction for finite sample:

$$Q_m = T(T+2) \sum_{k=1}^{m} \frac{r(k)^2}{T-k} \overset{d}{\sim} \chi_m^2.$$

Backward Shift Operator & Characteristic Equation

- Backward shift operator B:

$$BY_t := Y_{t-1}$$

- Applying the backward shift operator k times,

$$B^k Y_t = Y_{t-k}$$

- AR(p) process

$$Y_t = \theta + \lambda_1 Y_{t-1} + \lambda_2 Y_{t-2} + \cdots + \lambda_p Y_{t-p} + u_t = \theta + \sum_{i=1}^{p} \lambda_i B^i Y_t + u_t$$

$$\phi(B) Y_t := \left(1 - \sum_{i=1}^{p} \lambda_i B^i \right) Y_t = \theta + u_t$$

- The equation $\phi(B) = 0$ is called the characteristic equation of the AR(p) process. For AR(p) to be stationary, the roots of the characteristic equation must lie **outside** the unit circle.

Representation of MA(1)

‡ MA(1) process is representable as AR(∞) process

$$\begin{aligned} Y_t &= \theta + u_t + \alpha\, u_{t-1} \\ &= \theta + (1 + \alpha\, B) u_t \\ (1+\alpha B)^{-1} Y_t &= (1+\alpha B)^{-1}\theta + u_t \\ \big(1 - \alpha B + \alpha^2 B^2 - \alpha^3 B^3 + \cdots\big) Y_t &= c + u_t \end{aligned}$$

where $c := (1+\alpha B)^{-1}\theta$. Hence

$$Y_t = c + \alpha Y_{t-1} - \alpha^2 Y_{t-2} + \alpha^3 Y_{t-3} - \alpha^4 Y_{t-4} + \cdots + u_t$$

‡ Invertibility requires $(1+\alpha B) = 0$ lie outside the unit circle, which is satisfied if $|\alpha| < 1$.

‡ A stationary MA(q) process is said to be invertible if it can be represented as a stationary AR(∞) process.

‡ What's the point of inverting an MA process?

Yule-Walker Equations (1)

- Multiply both sides of AR(p) process by Y_{t-k},

$$Y_{t-k}Y_t = Y_{t-k}\theta + \lambda_1 Y_{t-k}Y_{t-1} + \lambda_2 Y_{t-k}Y_{t-2} + \cdots + \lambda_p Y_{t-k}Y_{t-p} + Y_{t-k}u_t$$

- Taking unconditional expectations on both sides and noting that $\mathbb{E}\big(Y_{t-k}Y_t\big) = \gamma(k) + \mu^2$ for any k, then

$$\gamma(k)+\mu^2 = \mu\theta+\lambda_1\big(\gamma(k-1)+\mu^2\big)+\lambda_2\big(\gamma(k-2)+\mu^2\big)+\cdots+\lambda_p\big(\gamma(k-p)+\mu^2\big)$$

- The unconditional means of an AR(p) process is $\mu = \theta + \lambda_1\mu + \cdots + \lambda_p\mu$. So $\mu^2 = \mu\theta + \lambda_1\mu^2 + \cdots + \lambda_p\mu^2$. Hence for $k > 1$,

$$\gamma(k) = \lambda_1\gamma(k-1) + \lambda_2\gamma(k-2) + \cdots + \lambda_p\gamma(k-p)$$

- Dividing both sides by $\gamma(0)$ to obtain correlations:

$$\rho(k) = \lambda_1\rho(k-1) + \lambda_2\rho(k-2) + \cdots + \lambda_p\rho(k-p)$$

Yule-Walker Equations (2)

- Note that $\rho(0) = 1$ and $\rho(j) = \rho(-j)$
- For each k, there is a corresponding equation with p parameters of $\lambda_1, \lambda_2, \ldots, \lambda_p$, resulting in the Yule-Walker equations:

$$
\begin{aligned}
\rho(1) &= \lambda_1 + \lambda_2\rho(1) + \cdots + \lambda_p\rho(p-1) \\
\rho(2) &= \lambda_1\rho(1) + \lambda_2 + \cdots + \lambda_p\rho(p-2) \\
\rho(3) &= \lambda_1\rho(2) + \lambda_2\rho(1) + \cdots + \lambda_p\rho(p-3) \\
\vdots &= \vdots + \vdots + \cdots + \vdots \\
\rho(p) &= \lambda_1\rho(p-1) + \lambda_2\rho(p-2) + \cdots + \lambda_p
\end{aligned}
$$

Yule-Walker Equations in Matrix Form (1)

✡ Replace the ACF $\rho(p)$ by sample ACF $r(k)$, the p linear equations can be solved as

$$R := \begin{pmatrix} r(1) \\ r(2) \\ \vdots \\ r(p) \end{pmatrix}, \quad \Phi := \begin{pmatrix} 1 & r(1) & r(2) & \cdots & r(p-1) \\ r(1) & 1 & r(1) & \cdots & r(p-2) \\ r(2) & r(1) & 1 & \cdots & r(p-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r(p-1) & r(p-2) & r(p-3) & \cdots & 1 \end{pmatrix}$$

$$\Lambda := \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_p \end{pmatrix}$$

Yule-Walker Equations in Matrix Form (2)

- Write compactly as $R = \Phi\Lambda$, so the parameters can be estimated as

$$\widehat{\Lambda} = \Phi^{-1} R\,,$$

along with $\widehat{\mu} = \dfrac{1}{T}\sum_{t=1}^{T} Y_t$.

- Also,

$$\widehat{\theta} = \widehat{\mu}\Big(1 - \widehat{\lambda}_1 - \widehat{\lambda}_2 - \cdots - \widehat{\lambda}_p\Big).$$

- The estimate for the variance of u_t is

$$\widehat{\sigma}_u^2 = \left(\frac{1}{T}\sum_{t=1}^{T}\big(Y_t - \widehat{\mu}\big)^2\right)\Big(1 - \widehat{\lambda}_1 r(1) - \widehat{\lambda}_2 r(2) - \cdots - \widehat{\lambda}_p r(p)\Big)\,.$$

Partial Autocorrelation Function

- The sample ACF allows the identification of either an AR or an MA process depending on whether the sample $r(k)$ decays slowly or are clearly zero after some lag k.

- However, even if an AR(p) is identified, it is still difficult to identify the order p, since all AR(p) processes show similar decay patterns of ACF.

- Is it possible to identify the order of an AR process?

- Yes, use the Yule-Walker equations to generate a sequence of $\widehat{\lambda}_{kk}, k = 1, 2, \ldots, p$ which is the PACF.

Sample PACF Calculation (1)

☺ Suppose the true process is AR(p). Take $k = 1$ and based on AR(1)

$$Y_t = \theta + \lambda_{11} Y_{t-1} + u_t$$

The Yule-Walker equation for $k = 1$ gives

$$\widehat{\lambda}_{11} = r(1)$$

☺ Next take $k = 2$, i.e., assume that it is AR(2). The Yule-Walker system is

$$\begin{pmatrix} \widehat{\lambda}_{21} \\ \widehat{\lambda}_{22} \end{pmatrix} = \begin{pmatrix} 1 & r(1) \\ r(1) & 1 \end{pmatrix}^{-1} \begin{pmatrix} r(1) \\ r(2) \end{pmatrix}$$

Solution for $\widehat{\lambda}_{22}$ is

$$\widehat{\lambda}_{22} = \frac{r(2) - r(1)^2}{1 - r(1)^2}$$

Sample PACF Calculation (2)

- In general, to calculate the p-th value, take $k = p$ and the AR(p) is

$$Y_t = \theta + \lambda_{p1} Y_{t-1} + \lambda_{p2} Y_{t-2} + \cdots + \lambda_{kp} Y_{t-p} + u_t$$

- Solve the Yule-Walker equations

$$\widehat{\Lambda} = \Phi^{-1} R \,,$$

- If the true order is p, theoretically,

$$\lambda_{11} \neq 0, \lambda_{22} \neq 0, \ldots, \lambda_{pp} \neq 0$$
$$\lambda_{p+1p+1} = \lambda_{p+2p+2} = \cdots = 0$$

Duality between AR and MA, ACF and PACF

- While an AR(p) process has a decaying ACF infinite in extent, the PACF cuts off after lag p.

- Recall that an MA(1) process is invertible into AR(∞). In general, this property holds for MA(q) processes.

- So while the ACF of an MA(q) process cuts off after lag q, the PACF is infinite in extent.

- ARMA(p, q)'s ACF follows the same pattern as that of an AR(p) process after $q - p + 1$ initial values $\rho_0, \rho_1, \cdots, \rho_{q-p}$ (if $q - p < 0$, no initial values), while its PACF (for lag $k > p - q$) behaves like that of an MA(q) process.

Takeaways

- White noise is the ultimate randomness.
- AR, MA, and ARMA are covariance stationary processes for modeling time series.
- Autocovariance and autocorrelation functions are dependent on the duration and not the absolute clock time.
- Simulation of AR, MA, and ARMA processes
- Ljung and Box test statistic for testing autocorrelations
- Characteristic equation from backward shift operator for checking stationarity
- Invertibility condition for MA processes
- Yule-Walker analysis of AR(p) processes
- PACF and ACF are needed in stationary time series analysis.