Stochastic Discount Factor

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Investor Preferences

- Consider investor with **time-separable** utility of consumption:

$$V\left(C_0, \tilde{C}_1\right) = U(C_0) + \delta E\left[U\left(\tilde{C}_1\right)\right]$$

- Here $\delta \in (0, 1)$ is **subjective discount factor** that reflects investor's rate of time preference, while U is (strictly increasing and concave) utility function of consumption
- Investor can trade in n risky assets with initial price of P_i, final payoff of $\tilde{X}_i$, and return of $\tilde{R}_i = \tilde{X}_i / P_i$, for $i = 1, \ldots, n$
- Investor starts with wealth W_0 and invests proportion w_i of $(W_0 - C_0)$ in asset i, subject to constraint: $\sum_{i=1}^{n} w_i = 1$

Consumption and Portfolio Choice Problem

- Investor's intertemporal budget constraint:

$$\tilde{C}_1 = (W_0 - C_0) \sum_{i=1}^{n} w_i \tilde{R}_i$$

- Positive correlation between returns and final consumption
- Investor's consumption and portfolio choice problem, subject to budget constraint and portfolio weight constraint:

$$\max_{C_0, \{w_i\}} \left\{ U(C_0) + \delta E\left[U\left(\tilde{C}_1 \right) \right] \right\}$$

Optimality Conditions

- First-order condition for optimal consumption:

$$U'(C_0) = \delta E\left[U'\left(\tilde{C}_1\right)\sum_{i=1}^{n} w_i \tilde{R}_i\right]$$

- First-order conditions for optimal asset allocation:

$$\delta E\left[U'\left(\tilde{C}_1\right)\tilde{R}_i\right] = \frac{\lambda}{W_0 - C_0} \quad \forall \quad i = 1, \ldots, n$$

- Here λ is Lagrange multiplier for portfolio weight constraint

Asset Allocation

- All assets in optimal portfolio must provide same expected marginal-utility-weighted return:

$$E\left[U'\left(\tilde{C}_1\right)\tilde{R}_i\right] = E\left[U'\left(\tilde{C}_1\right)\tilde{R}_j\right]$$

- Suppose that i'th asset provides higher expected marginal-utility-weighted return than other assets
- Investor will shift investment into i'th asset, which increases correlation between i'th asset return and final consumption
- But marginal utility is decreasing function of consumption, so less weight on larger realisations of i'th asset return, and hence decrease in expected marginal-utility-weighted return

Intertemporal Allocation

- Rearrange first-order condition for optimal consumption and use equality of expected marginal-utility-weighted returns:

$$U'(C_0) = \sum_{i=1}^{n} w_i \left(\delta E \left[U'\left(\tilde{C}_1 \right) \tilde{R}_i \right] \right) = \delta E \left[U'\left(\tilde{C}_1 \right) \tilde{R}_i \right]$$

- Investor shifts consumption until marginal utility from one unit of initial consumption is same as discounted expected marginal utility from $\tilde{R}_i$ units of final consumption
- Applies to all assets (and also optimal portfolio) given equality of expected marginal-utility-weighted returns

Asset Pricing Equation

- Rearrange to get asset pricing equation:

$$E\left[\delta \frac{U'\left(\tilde{C}_1\right)}{U'(C_0)} \tilde{R}_i\right] = 1 \quad \Leftrightarrow \quad P_i = E\left[\delta \frac{U'\left(\tilde{C}_1\right)}{U'(C_0)} \tilde{X}_i\right]$$

- Define $\tilde{M} = \delta U'\left(\tilde{C}_1\right) / U'(C_0) > 0$, which represents **intertemporal marginal rate of substitution (IMRS)**:

$$E\left[\tilde{M}\tilde{R}_i\right] = 1 \quad \Leftrightarrow \quad P_i = E\left[\tilde{M}\tilde{X}_i\right]$$

- Interpret $\tilde{M}$ as **pricing kernel** or **stochastic discount factor**

Riskless Asset

- Intertemporal allocation for riskless asset with return R_f:

$$U'(C_0) = \delta E\left[U'\left(\tilde{C}_1\right)\right] R_f$$

- Suppose that riskless asset is only tradeable asset:

$$C_1 = (W_0 - C_0)\, R_f$$

- Consider investor with power utility: $U(C) = (1-\gamma)^{-1}\, C^{1-\gamma}$:

$$C_0^{-\gamma} = \delta C_1^{-\gamma} R_f \quad \Rightarrow \quad R_f = \frac{1}{\delta}\left(\frac{C_1}{C_0}\right)^{\gamma}$$

Elasticity of Intertemporal Substitution

- Take natural logarithm on both sides of equation:

$$\ln R_f = \gamma \ln\left(\frac{C_1}{C_0}\right) - \ln\delta$$

- Define **elasticity of intertemporal substitution (EIS)**:

$$\epsilon = \frac{\partial \ln(C_1/C_0)}{\partial \ln R_f} = \frac{1}{\gamma}$$

- Hence curvature of utility function determines (relative) risk aversion as well as elasticity of intertemporal substitution

Optimal Initial Consumption

- Solve for optimal initial consumption:

$$C_0^{-\gamma} = \delta \left(W_0 - C_0\right)^{-\gamma} R_f^{1-\gamma} \quad \Rightarrow \quad C_0 = \frac{R_f^{1-\epsilon}}{\delta^\epsilon + R_f^{1-\epsilon}} W_0$$

- Optimal initial consumption is independent of risk-free rate when $\epsilon = \gamma = 1$, which corresponds to logarithmic utility
- Otherwise, effect of change in risk-free rate depends on EIS:

$$\frac{dC_0}{dR_f} = (1-\epsilon) \frac{\delta^\epsilon R_f^{-\epsilon}}{\left(\delta^\epsilon + R_f^{1-\epsilon}\right)^2} W_0 \lesseqgtr 0 \quad \Leftrightarrow \quad \epsilon \gtreqless 1$$

Substitution Effect vs Income Effect

- Rise in interest rate produces **substitution effect**: investor increases saving and transfers consumption from initial time period to final time period
- Rise in interest rate also produces **income effect**: investor increases consumption in both time periods, in response to expectation of increased future wealth
- Hence substitution and income effects have opposite effects on consumption in initial time period
- Substitution effect exactly offsets income effect when $\epsilon = \gamma = 1$; otherwise, substitution effect outweighs income effect when $\epsilon > 1$ and $\gamma < 1$ (and vice versa)

Consumption CAPM – Part 1

- Expand expectation of product in asset pricing equation:

$$
\begin{aligned}
1 &= E\left[\tilde{M}\right] E\left[\tilde{R}_i\right] + \mathrm{Cov}\left[\tilde{M}, \tilde{R}_i\right] \\
&= E\left[\tilde{M}\right] \left(E\left[\tilde{R}_i\right] + \frac{\mathrm{Cov}\left[\tilde{M}, \tilde{R}_i\right]}{E\left[\tilde{M}\right]} \right)
\end{aligned}
$$

- Rearrange and use result that $E\left[\tilde{M}\right] = R_f^{-1}$:

$$
E\left[\tilde{R}_i\right] - R_f = -\frac{\mathrm{Cov}\left[\tilde{M}, \tilde{R}_i\right]}{E\left[\tilde{M}\right]} = -\frac{\mathrm{Cov}\left[U'\left(\tilde{C}_1\right), \tilde{R}_i\right]}{E\left[U'\left(\tilde{C}_1\right)\right]}
$$

Consumption CAPM – Part 2

- Suppose return on i'th asset has negative correlation with marginal utility of final consumption
- Implies that asset return tends to be high when marginal utility is low, and vice versa
- Hence investor is likely to receive more consumption when consumption is less valuable, and vice versa
- Asset has undesirable payoff characteristics, so investor will demand large risk premium for holding asset
- Conversely, asset with positive covariance provides hedge against low consumption, so investor will be willing to pay

Relationship to CAPM – Part 1

- Suppose that market return has perfect negative correlation with marginal utility of final consumption:

$$U'\left(\tilde{C}_1\right) = -\kappa \tilde{R}_m, \qquad \kappa > 0$$

- All investors must hold combination of market portfolio and riskless asset, so that final consumption has perfect positive correlation with market return
- Risk premium for market portfolio:

$$E\left[\tilde{R}_m\right] - R_f = -\frac{\text{Cov}\left[U'\left(\tilde{C}_1\right), \tilde{R}_m\right]}{E\left[U'\left(\tilde{C}_1\right)\right]} = \frac{\kappa \text{Var}\left[\tilde{R}_m\right]}{E\left[U'\left(\tilde{C}_1\right)\right]}$$

Relationship to CAPM – Part 2

- Substitute for $E\left[U'\left(\tilde{C}_1\right)\right]$ in risk premium for i'th asset:

$$E\left[\tilde{R}_i\right] - R_f = -\frac{\text{Cov}\left[U'\left(\tilde{C}_1\right), \tilde{R}_i\right]}{\kappa \text{Var}\left[\tilde{R}_m\right]} \left(E\left[\tilde{R}_m\right] - R_f\right)$$

- Use result that $\text{Cov}\left[U'\left(\tilde{C}_1\right), \tilde{R}_i\right] = -\kappa \text{Cov}\left[\tilde{R}_i, \tilde{R}_m\right]$:

$$\begin{aligned} E\left[\tilde{R}_i\right] - R_f &= \frac{\text{Cov}\left[\tilde{R}_i, \tilde{R}_m\right]}{\text{Var}\left[\tilde{R}_m\right]} \left(E\left[\tilde{R}_m\right] - R_f\right) \\ &= \beta_i \left(E\left[\tilde{R}_m\right] - R_f\right) \end{aligned}$$

Volatility Bound

- Use $\text{Cov}\left[\tilde{M}, \tilde{R}_i\right] = \rho_{Mi}\sigma_M\sigma_i$ in pricing equation for Consumption CAPM, and rearrange to get Sharpe ratio:

$$\frac{E\left[\tilde{R}_i\right] - R_f}{\sigma_i} = -\rho_{Mi}\frac{\sigma_M}{E\left[\tilde{M}\right]}$$

- Apply $\rho_{Mi} \in [-1, 1]$ to get **Hansen–Jagannathan bound**:

$$\frac{\sigma_M}{E\left[\tilde{M}\right]} \geq \left|\frac{E\left[\tilde{R}_i\right] - R_f}{\sigma_i}\right|$$

Equity Premium Puzzle – Part 1

- Use result for variance of pricing kernel:

$$\text{Var}\left[\tilde{M}\right] = E\left[\tilde{M}^2\right] - E\left[\tilde{M}\right]^2$$
$$\Rightarrow \frac{\sigma_M}{E\left[\tilde{M}\right]} = \left(\frac{E\left[\tilde{M}^2\right]}{E\left[\tilde{M}\right]^2} - 1\right)^{\frac{1}{2}}$$

- Consider investor with power utility: $U(C) = (1-\gamma)^{-1} C^{1-\gamma}$:

$$\tilde{M} = \delta \left(\frac{\tilde{C}_1}{C_0}\right)^{-\gamma} = \delta \exp\left[-\gamma \ln\left(\frac{\tilde{C}_1}{C_0}\right)\right]$$

Equity Premium Puzzle – Part 2

- Suppose that consumption growth has lognormal distribution with mean μ_c and variance σ_c^2:

$$\ln\left(\frac{\tilde{C}_1}{C_0}\right) = \mu_c + \sigma_c \tilde{z}, \qquad \tilde{z} \sim N(0,1)$$

- Use result for lognormal random variable:

$$E\left[\tilde{M}\right] = \delta E\left[e^{-\gamma(\mu_c + \sigma_c \tilde{z})}\right] = \delta e^{-\gamma\mu_c + \frac{1}{2}\gamma^2\sigma_c^2};$$
$$E\left[\tilde{M}^2\right] = \delta E\left[e^{-2\gamma(\mu_c + \sigma_c \tilde{z})}\right] = \delta e^{-2\gamma\mu_c + 2\gamma^2\sigma_c^2}$$

Equity Premium Puzzle – Part 3

- Substitute for $E\left[\tilde{M}\right]$ and $E\left[\tilde{M}^2\right]$ in previous equation:

$$\frac{\sigma_M}{E\left[\tilde{M}\right]} = \left(\frac{e^{-2\gamma\mu_c+2\gamma^2\sigma_c^2}}{e^{-2\gamma\mu_c+\gamma^2\sigma_c^2}} - 1\right)^{\frac{1}{2}} = \left(e^{\gamma^2\sigma_c^2} - 1\right)^{\frac{1}{2}}$$

- Apply first-order Taylor series approximation and H–J bound:

$$\frac{\sigma_M}{E\left[\tilde{M}\right]} \approx \gamma\sigma_c \geq \left|\frac{E\left[\tilde{R}_i\right] - R_f}{\sigma_i}\right|$$

Equity Premium Puzzle – Part 4

- Risk premium is around 7% per year for U.S. stock market
- Standard deviation of market return is around 17% per year
- Hence Sharpe ratio of market portfolio is around 0.41
- Based on per capita aggregate consumption, $\sigma_c \approx 2\%$ per year
- Hence investor with power utility who consumes per capita aggregate consumption must have $\gamma \gtrsim 20$
- Represents **equity premium puzzle** since implied level of relative risk aversion is implausibly high