Multi-Factor Models 3.0

November 13, 2018

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1 Part 1: Performance Measurement

1.1 import data

In [1]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import statsmodels.api as sm
import scipy.optimize as optimize
plt.rcParams['font.family'] = 'serif'
plt.rcParams['figure.facecolor'] = '1.'
```

In [2]:
```
IndPtf = pd.read_excel('Industry_Portfolios.xlsx')
IndPtf['Date'] = pd.to_datetime(IndPtf['Date'], format = '%Y%m')
IndPtf = IndPtf.set_index('Date')

RiskFactor = pd.read_excel('Risk_factors.xlsx')
RiskFactor['Date'] = pd.to_datetime(RiskFactor['Date'], format = '%Y%m')
RiskFactor = RiskFactor.set_index('Date')
```

In [3]:
```
print(RiskFactor.shape)
RiskFactor.head()
```

```
(120, 4)
```

Out[3]:

Date	Rf	Rm-Rf	SMB	HML
2004-01-01	0.07	2.15	2.67	1.55
2004-02-01	0.06	1.40	-1.17	0.45
2004-03-01	0.09	-1.32	1.84	0.07
2004-04-01	0.08	-1.83	-2.59	-1.67
2004-05-01	0.06	1.17	-0.10	-0.33

In [4]:
```
print(IndPtf.shape)
IndPtf.head()
```

```
(120, 10)
```

```
Out[4]:             NoDur  Durbl  Manuf  Enrgy  HiTec  Telcm  Shops  Hlth  Utils  \
        Date
        2004-01-01   0.06  -1.07  -0.62   0.44   4.53   1.41   0.45  3.09   1.92
        2004-02-01   4.25  -0.07   1.95   4.69  -2.92  -0.52   6.09  0.89   2.07
        2004-03-01  -0.09  -1.15  -0.27  -0.13  -2.55  -2.07   0.29 -3.96   1.13
        2004-04-01   1.42   2.30  -0.17   2.52  -4.91  -0.48  -2.70  3.54  -3.55
        2004-05-01  -1.89  -1.64   1.61   0.39   4.85  -2.95   0.30 -0.42   1.28

                    Other
        Date
        2004-01-01   2.88
        2004-02-01   2.16
        2004-03-01  -0.63
        2004-04-01  -3.76
        2004-05-01   1.86
```

```
In [5]: ExcRet = IndPtf.sub(RiskFactor['Rf'], axis =0)

        ExcRet.head()
```

```
Out[5]:             NoDur  Durbl  Manuf  Enrgy  HiTec  Telcm  Shops  Hlth  Utils  \
        Date
        2004-01-01  -0.01  -1.14  -0.69   0.37   4.46   1.34   0.38  3.02   1.85
        2004-02-01   4.19  -0.13   1.89   4.63  -2.98  -0.58   6.03  0.83   2.01
        2004-03-01  -0.18  -1.24  -0.36  -0.22  -2.64  -2.16   0.20 -4.05   1.04
        2004-04-01   1.34   2.22  -0.25   2.44  -4.99  -0.56  -2.78  3.46  -3.63
        2004-05-01  -1.95  -1.70   1.55   0.33   4.79  -3.01   0.24 -0.48   1.22

                    Other
        Date
        2004-01-01   2.81
        2004-02-01   2.10
        2004-03-01  -0.72
        2004-04-01  -3.84
        2004-05-01   1.80
```

1.2 Sharpe Ratio

```
In [6]: Data = pd.DataFrame()
        Data['Sharpe Ratio'] = ExcRet.apply(lambda x: ( x.mean() )/( x.std() ))
```

1.3 Sortino Ratio

```
In [7]: SV= ExcRet.apply(lambda x: np.square( x.apply(lambda y: min(y, 0)) ).mean() )
        Data['Sortino Ratio'] = ExcRet.apply(lambda x: x.mean() ) / np.sqrt(SV)
```

1.4 Jensen's Alpha

In [8]:
```
def calculate_CAPM(Rm_Rf, Ri_Rf):
    ''' calculate Alpha and Beta for CAPM
    input:
    --- Rm_Rf: series of market portfolio excess return
                pandas.core.series.Series type

    --- Ri_Rf: series of i_th portfolio excess return
                 pandas.core.series.Series type

    output:
    --- result.params: OLS parameters
                             pandas.core.series.Series type

    '''
    X = sm.add_constant(Rm_Rf)
    y = Ri_Rf
    results = sm.OLS(y, X).fit()

    return results.params

CAPM = pd.concat([calculate_CAPM(RiskFactor['Rm-Rf'], ExcRet[name])
                  for name in ExcRet.columns], axis = 1)

CAPM
```

Out[8]:
```
              0         1         2         3         4         5         6  \
const  0.369717 -0.417903  0.160494  0.504485 -0.064024  0.194348  0.274093
Rm-Rf  0.653744  1.649374  1.167929  0.965527  1.132387  0.901721  0.829515

              7         8         9
const  0.236968  0.446523 -0.387508
Rm-Rf  0.675890  0.537009  1.206992
```

In [9]:
```
Data['Jensen\'s Alpha'] = CAPM.loc['const', :].values
```

1.5 3-factor alpha

In [10]:
```
def calculate_3_FACTOR(Rx_Rf, Ri_Rf):
    ''' calculate Alpha and Beta for three-factor model
    input:
    --- Rx_Rf: series of explanatory factors
                pandas.core.series.Series type

    --- Ri_Rf: series of i_th portfolio excess return
                pandas.core.series.Series type

    output:
```

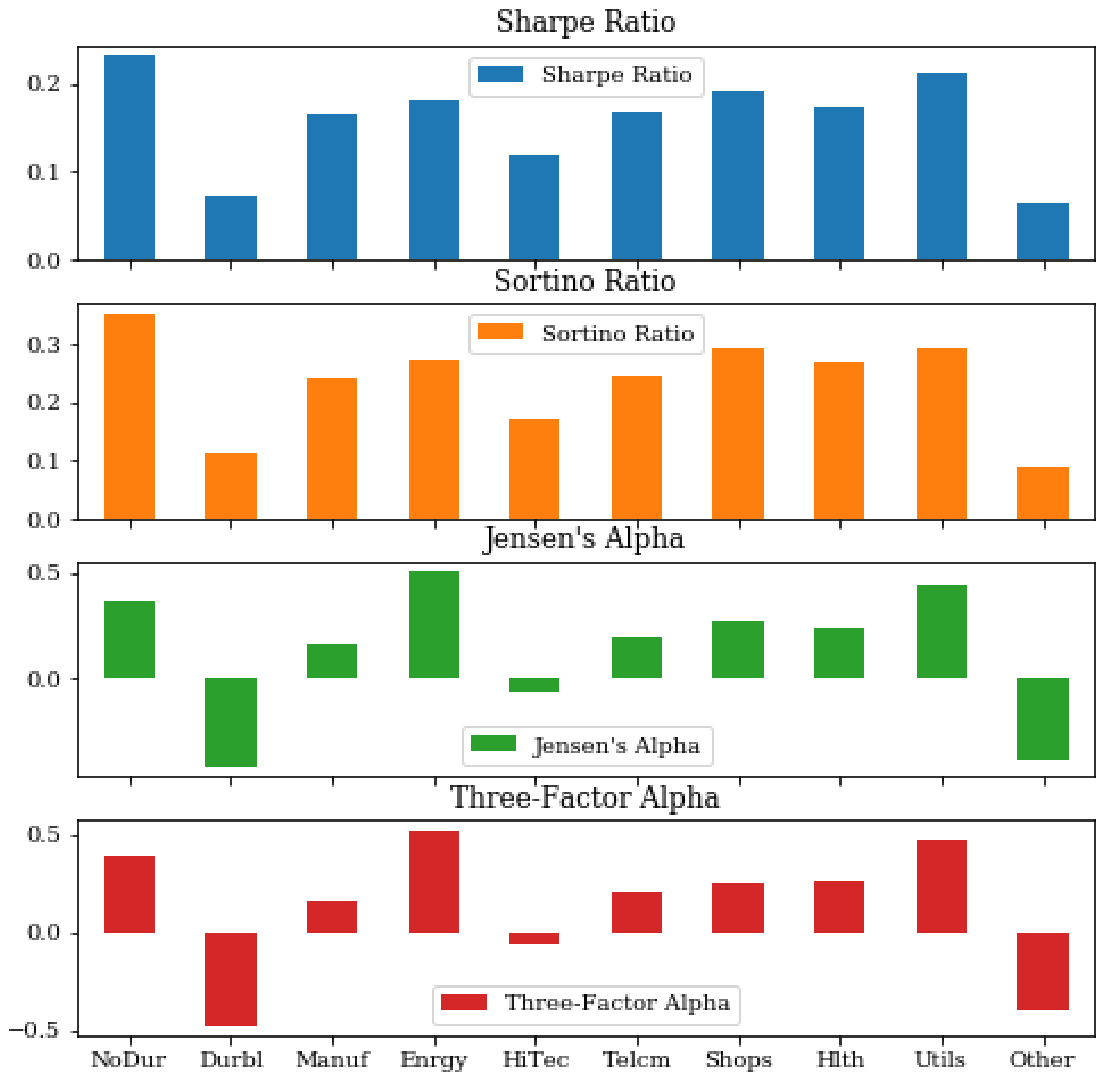
```python
    --- result.params: OLS parameters
                                pandas.core.series.Series type

    '''
    X = sm.add_constant(Rx_Rf)
    y = Ri_Rf
    results = sm.OLS(y, X).fit()

    return results.params

THREE_FACTOR = pd.concat([calculate_3_FACTOR(RiskFactor.iloc[:, 1:], ExcRet[name])
                          for name in ExcRet.columns], axis = 1)

THREE_FACTOR
```

Out[10]:

```
              0         1         2         3         4         5         6  \
const  0.386704 -0.474342  0.153285  0.523007 -0.065979  0.200724  0.255941
Rm-Rf  0.712134  1.447452  1.142282  1.028354  1.152803  0.924137  0.770227
SMB   -0.229102  0.670878  0.087388 -0.259360  0.335674 -0.080299  0.280191
HML   -0.023342  0.240949  0.027727 -0.008158 -0.556947 -0.019063 -0.039080

              7         8         9
const  0.257472  0.474411 -0.404412
Rm-Rf  0.751976  0.631827  1.123473
SMB   -0.212655 -0.387961 -0.061676
HML   -0.143765 -0.016881  0.547325
```

In [11]:

```python
Data['Three-Factor Alpha'] = THREE_FACTOR.loc['const', :].values

Data.plot.bar(figsize = (8, 2*Data.shape[1]), rot=0, subplots=True)
plt.show()
```



In [12]: Data.T

Out[12]:

```
                      NoDur     Durbl     Manuf     Enrgy     HiTec  \
Sharpe Ratio       0.231099  0.072356  0.166616  0.181708  0.118552
Sortino Ratio      0.350804  0.111967  0.241260  0.273612  0.170620
Jensen's Alpha     0.369717 -0.417903  0.160494  0.504485 -0.064024
Three-Factor Alpha 0.386704 -0.474342  0.153285  0.523007 -0.065979

                      Telcm     Shops      Hlth     Utils     Other
Sharpe Ratio       0.169064  0.191753  0.172529  0.210948  0.064693
Sortino Ratio      0.244940  0.293032  0.270294  0.290044  0.087351
Jensen's Alpha     0.194348  0.274093  0.236968  0.446523 -0.387508
Three-Factor Alpha 0.200724  0.255941  0.257472  0.474411 -0.404412
```

Sharpe ratio measures the risk premium per unit of total risk;

Sortino Ratio measures the risk premium per unit of downside risk;
Jensen's Alpha measures the pricing error of CAPM;
Three-factor Alpha measures the pricing error of three-factor model.

2 Part 2: Minimum-Variance Frontier Revisited

2.1 Calculate efficient frontier via scipy.optimize.minimize

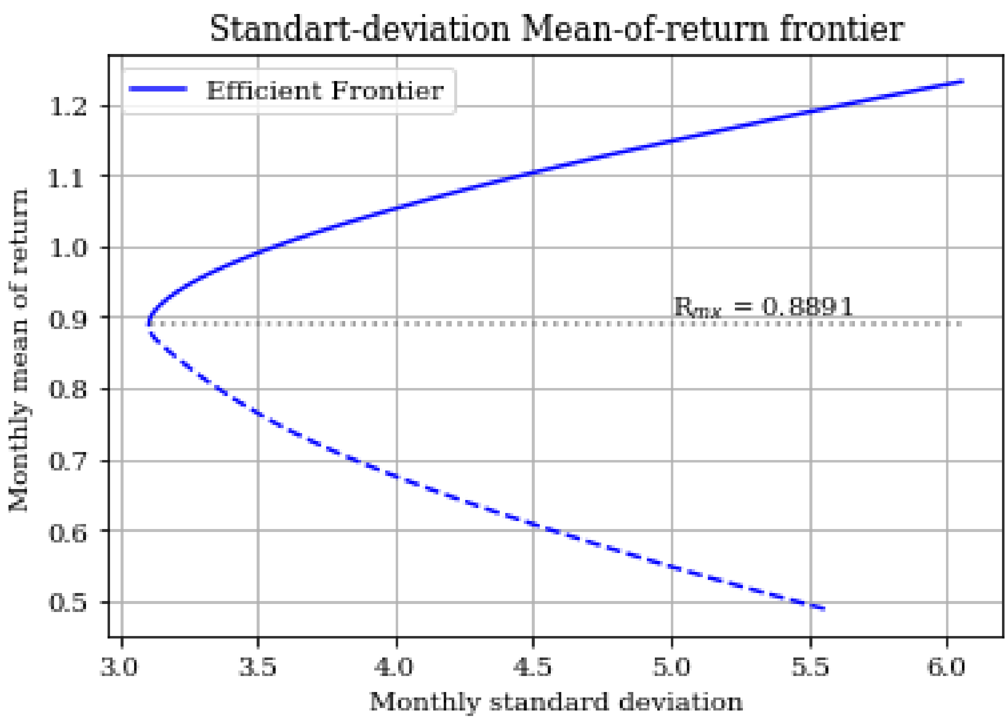
Before we apply Monte Carlo method, we firstly use scipy.optimize.minimize to find out the frontier portfolio weight.

In [13]:
```
# because this is a long-only portfolio, so possible return should be
# no less than the smallest single asset return and no more than
# largest single asset return.
mu_lst = np.arange( IndPtf.mean().min(), IndPtf.mean().max(), 0.001)
w_lst = []
for mu in mu_lst:
    # aiming function that we're going to minimize
    f = lambda w: np.dot(IndPtf.values, w.T).std()

    # boundary of weight
    bnds = ((0,1),)*10

    # constaints of mean of return and sum of portfolio weight
    cons = ({'type': 'eq', 'fun': lambda w:  np.dot(IndPtf.values, w.T).mean() - mu},
            {'type': 'eq', 'fun': lambda w:  sum(w) - 1})

    # initialization
    w0 = ([0.1]*10)

    # minimize standard deviation
    w_lst.append(optimize.minimize(f, w0, method='SLSQP',
                                   bounds=bnds, constraints=cons,
                                  options={'maxiter': 10000000, 'ftol': 1e-06}).x)
```

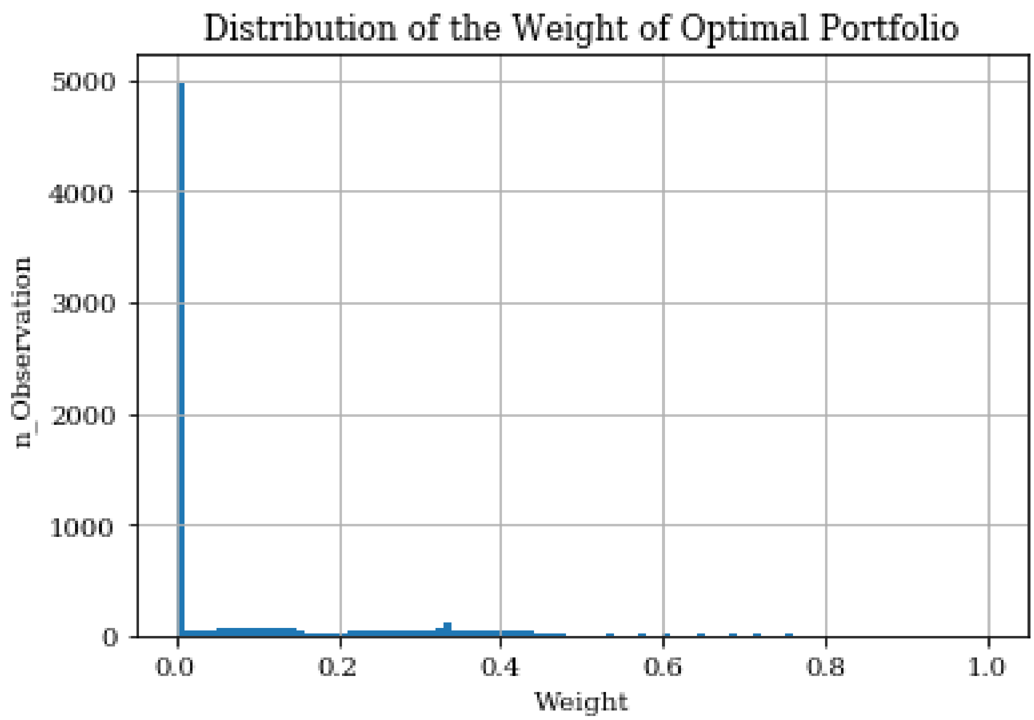
In [14]:
```
# calculate the (annual) sigma for frontier portfolio
FrontierSigma = [np.dot(IndPtf.values, w.T).std() for w in w_lst]

# calculate the (annual) mean of return for frontier portfolio
FrontierMeanReturn = [np.dot(IndPtf.values, w.T).mean() for w in w_lst]
```

In [15]:
```
ind = FrontierSigma.index(min(FrontierSigma))

plt.plot(FrontierSigma[ind:], FrontierMeanReturn[ind:],
         '-', color = 'blue', label = 'Efficient Frontier')
plt.plot(FrontierSigma[:ind], FrontierMeanReturn[:ind],
         '--', color = 'blue')
```

```
plt.plot( [FrontierSigma[ind], max(FrontierSigma)], [FrontierMeanReturn[ind]]*2,
         ':', color = 'gray')
plt.text(5, FrontierMeanReturn[ind], 'R$%s$ = '%'_{mx}'\
         +str(round(FrontierMeanReturn[ind], 4)), verticalalignment='bottom')

plt.xlabel('Monthly standard deviation')
plt.ylabel('Monthly mean of return')
plt.title('Standart-deviation Mean-of-return frontier')
plt.legend(loc = 'best')
plt.grid(True)
plt.show()
```




```
In [16]: w_ = np.array(w_lst)
         w_ = w_.reshape(-1, 1)

In [17]: plt.hist(w_, bins = 100)
         plt.title('Distribution of the Weight of Optimal Portfolio')
         plt.ylabel('n_Observation')
         plt.xlabel('Weight')
         plt.grid(True)
         plt.show()
```



We can find that weight of optimal portfolio countains a great many extremely small value.

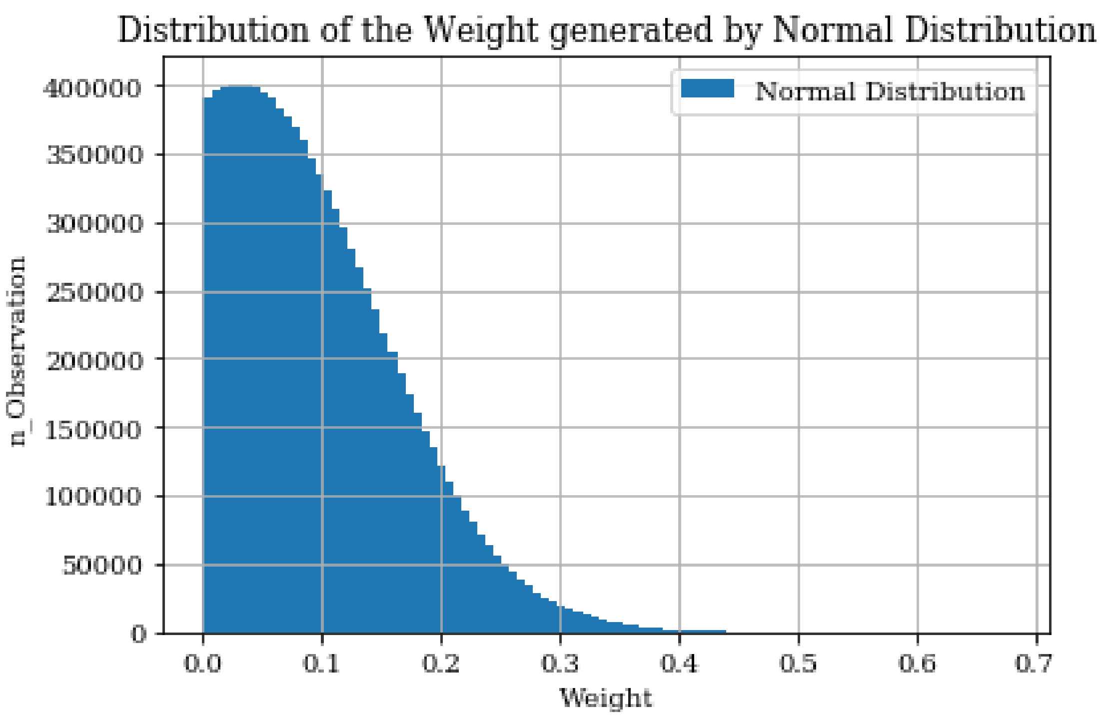
2.2 Simulate efficient frontier via random weight generated by normal distribution

In [18]:
```
# Monte Carlo method

# random seed, keep the random number still
np.random.seed(101)

# numbers of random portfolio
n_portfolio =  int(1e6)

# generate random weight from normal distribution
w_normal = abs(np.random.randn(n_portfolio, 10))

# uncomment this line and comment the previous line if you want uniform distribution
#w_normal = np.random.rand(n_portfolio).reshape(-1, 10)

w_normal /= w_normal.sum(axis = 1, keepdims = True)
w_normal.shape
```

Out[18]:
```
(1000000, 10)
```

In [19]:
```
w_normal_ = w_normal.reshape(-1, 1)
```

```
plt.hist(w_normal_, bins = 100, label ='Normal Distribution')

plt.title('Distribution of the Weight generated by Normal Distribution')
plt.legend(loc = 'best')
plt.ylabel('n_Observation')
plt.xlabel('Weight')
plt.grid(True)
plt.show()
```



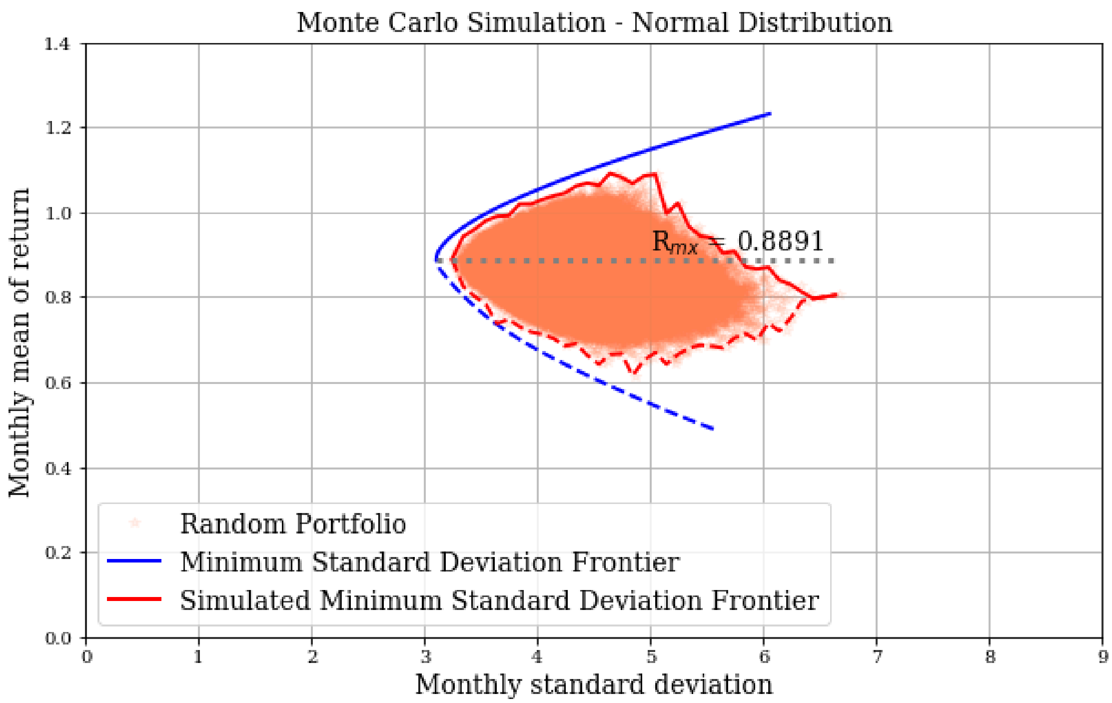
Weight generated from normal distribution are not very likely to located on the optimal portfolio, because it doesn't countain enough extremely weight.

```
In [20]: # calculate sigma and mu for random portfolio
         Portfolios = np.dot(IndPtf.values, w_normal.T)
         Mean_normal = Portfolios.mean(axis = 0)
         Std_normal = Portfolios.std(axis = 0)
```

```
In [21]: # filter away points inside of the frontier
         step = 0.1

         ind_ = np.where(Std_normal == min(Std_normal) )
         sigma = Std_normal[ind_][0]
         sigma_normal = [sigma]
         normal_line_upper = [ Mean_normal[ind_][0] ]
         normal_line_lower = [ Mean_normal[ind_][0] ]
```

```
    while sigma < max(Std_normal):
        sigma += step
        ind = ((Std_normal.reshape(-1,) >= sigma)
                    & (Std_normal.reshape(-1,) < sigma + step))

        while not ind.any() == True and  sigma < max(Std_normal):
            sigma += 0.1
            ind = ((Std_normal.reshape(-1,) >= sigma)
                    & (Std_normal.reshape(-1,) < sigma + step))

        if  ind.any() == True:
            sigma_normal.append(sigma)
            normal_line_upper.append(np.max(Mean_normal[ ind ]))
            normal_line_lower.append(np.min(Mean_normal[ ind ]))
```

In [22]:
```
z = 2
fig = plt.figure(figsize = (z*5, z*3))

ind = FrontierSigma.index(min(FrontierSigma))
plt.plot(Std_normal, Mean_normal,
         '*', color = 'coral', alpha = 0.1, label = 'Random Portfolio')

plt.plot(FrontierSigma[ind:], FrontierMeanReturn[ind:],
         '-', lw = z, color = 'blue', label = 'Minimum Standard Deviation Frontier')
plt.plot(FrontierSigma[:ind], FrontierMeanReturn[:ind],
         '--', lw = z, color = 'blue')

plt.plot(sigma_normal, normal_line_upper,
         '-', lw = z, color = 'red', label = 'Simulated Minimum Standard Deviation Fro
plt.plot(sigma_normal, normal_line_lower,
         '--', lw = z, color = 'red')

plt.plot( [FrontierSigma[ind], max(Std_normal)], [FrontierMeanReturn[ind]]*2,
         ':', lw = 1.5*z, color = 'gray')
plt.text(5, FrontierMeanReturn[ind], 'R$%s$ = '%'_{mx}'\
         +str(round(FrontierMeanReturn[ind], 4)), verticalalignment='bottom', fontsize

plt.xlim(0, 9)
plt.ylim(0, 1.4)
plt.xlabel('Monthly standard deviation', fontsize = 7*z)
plt.ylabel('Monthly mean of return', fontsize = 7*z)
plt.title('Monte Carlo Simulation - Normal Distribution', fontsize = 7*z)
plt.grid(True)
plt.legend(loc = 'best', fontsize = 7*z)
plt.show()
```



No surprise, the simulation fails to be perfectly located on the frontier.

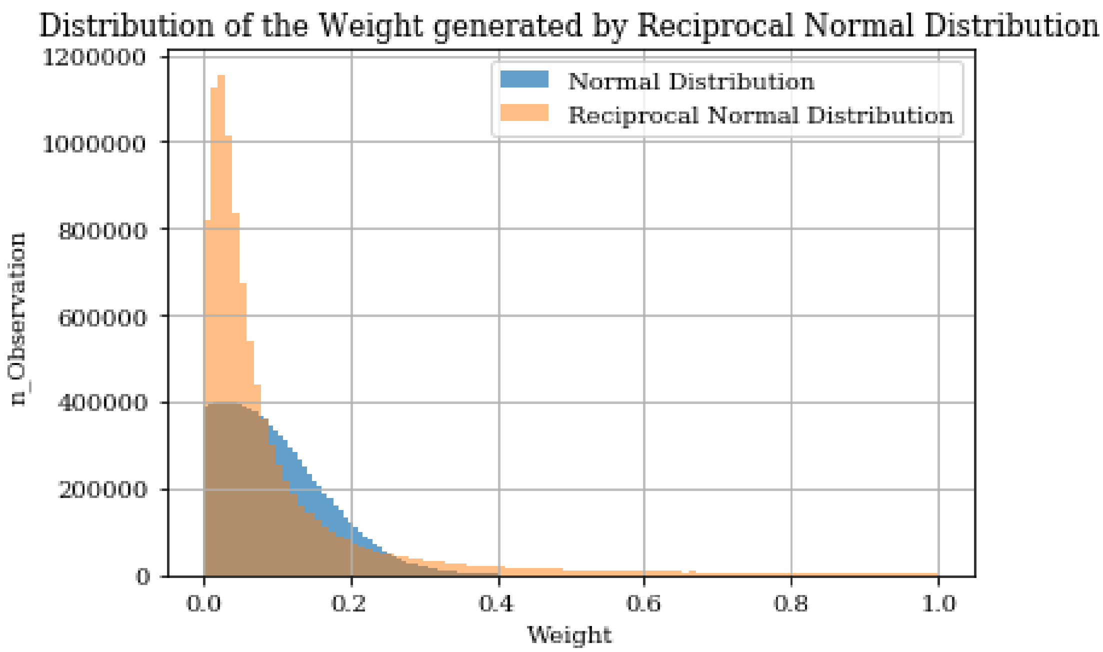
2.3 Simulate efficient frontier via random generated by weightreciprocal normal distribution

In [23]:
```
# Monte Carlo method

# random seed, keep the random number still
np.random.seed(101)

# numbers of random portfolio
n_portfolio =  int(1e6)

# generate random weight from a reciprocal normal distribution
w_reciprocal = abs(1/np.random.randn(n_portfolio, 10))

w_reciprocal /= w_reciprocal.sum(axis = 1, keepdims = True)
w_reciprocal.shape
```

Out[23]:
```
(1000000, 10)
```

In [24]:
```
w_reciprocal_ = w_reciprocal.reshape(-1, 1)

plt.hist(w_normal_, bins = 100,
         alpha = 0.7, label ='Normal Distribution')
plt.hist(w_reciprocal_, bins = 100,
```

```
                alpha = 0.5, label ='Reciprocal Normal Distribution')

plt.title('Distribution of the Weight generated by Reciprocal Normal Distribution')
plt.legend(loc = 'best')
plt.ylabel('n_Observation')
plt.xlabel('Weight')
plt.grid(True)
plt.show()
```



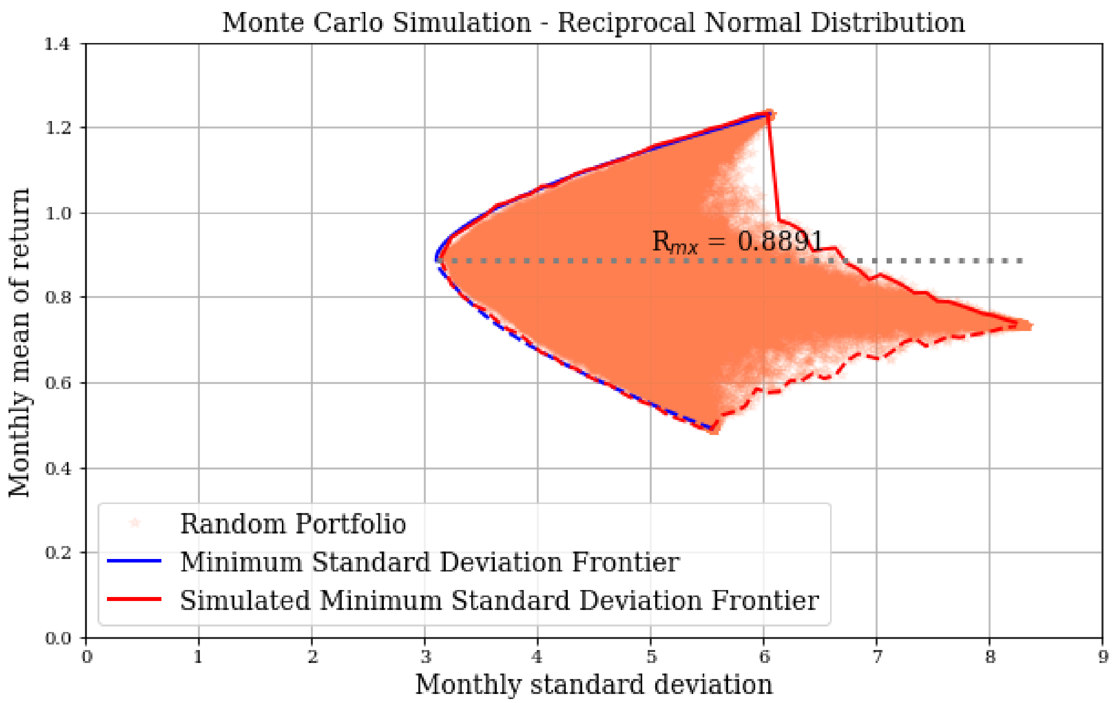
A reciprocal of normal distribution can generate more extreme values than the original one.

In [25]:
```
# calculate sigma and mu for reciprocal random portfolio
Portfolios = np.dot(IndPtf.values, w_reciprocal.T)
Mean_reciprocal = Portfolios.mean(axis = 0)
Std_reciprocal = Portfolios.std(axis = 0)
```

In [26]:
```
# filter away points inside of the frontier
step = 0.1

ind_ = np.where(Std_reciprocal == min(Std_reciprocal) )
sigma = Std_reciprocal[ind_][0]
sigma_reciprocal = [sigma]
reciprocal_line_upper = [ Mean_reciprocal[ind_][0] ]
reciprocal_line_lower = [ Mean_reciprocal[ind_][0] ]

while sigma < max(Std_reciprocal):
    sigma += step
```

```
        ind = ((Std_reciprocal.reshape(-1,) >= sigma)
                & (Std_reciprocal.reshape(-1,) < sigma + step))

        while not ind.any() == True and  sigma < max(Std_reciprocal):
            sigma += 0.1
            ind = ((Std_reciprocal.reshape(-1,) >= sigma)
                & (Std_reciprocal.reshape(-1,) < sigma + step))

        if  ind.any() == True:
            sigma_reciprocal.append(sigma)
            reciprocal_line_upper.append(np.max(Mean_reciprocal[ ind ]))
            reciprocal_line_lower.append(np.min(Mean_reciprocal[ ind ]))
```

```
In [27]: z = 2
         fig = plt.figure(figsize = (z*5, z*3))

         ind = FrontierSigma.index(min(FrontierSigma))
         plt.plot(Std_reciprocal, Mean_reciprocal,
                  '*', color = 'coral', alpha = 0.1, label = 'Random Portfolio')

         plt.plot(FrontierSigma[ind:], FrontierMeanReturn[ind:],
                  '-', lw = z, color = 'blue', label = 'Minimum Standard Deviation Frontier')
         plt.plot(FrontierSigma[:ind], FrontierMeanReturn[:ind],
                  '--', lw = z, color = 'blue')

         plt.plot(sigma_reciprocal, reciprocal_line_upper,
                  '-', lw = z, color = 'red', label = 'Simulated Minimum Standard Deviation Fro
         plt.plot(sigma_reciprocal, reciprocal_line_lower,
                  '--', lw = z, color = 'red')

         plt.plot( [FrontierSigma[ind], max(Std_reciprocal)], [FrontierMeanReturn[ind]]*2,
                  ':', lw = 1.5*z, color = 'gray')
         plt.text(5, FrontierMeanReturn[ind], 'R$%s$ = '%'_{mx}'\
                  +str(round(FrontierMeanReturn[ind], 4)), verticalalignment='bottom', fontsiz

         plt.xlim(0, 9)
         plt.ylim(0, 1.4)
         plt.xlabel('Monthly standard deviation', fontsize = 7*z)
         plt.ylabel('Monthly mean of return', fontsize = 7*z)
         plt.title('Monte Carlo Simulation - Reciprocal Normal Distribution', fontsize = 7*z)
         plt.grid(True)
         plt.legend(loc = 'best', fontsize = 7*z)
         plt.show()
```



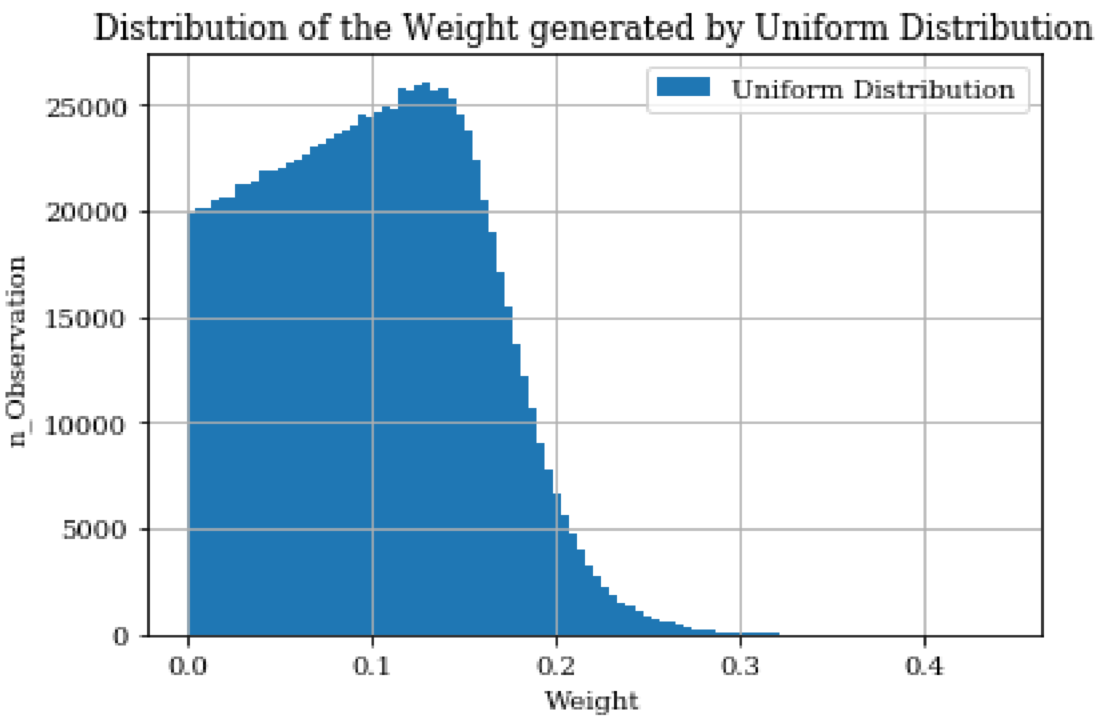
2.4 Complementary Work : Uniform Distribution

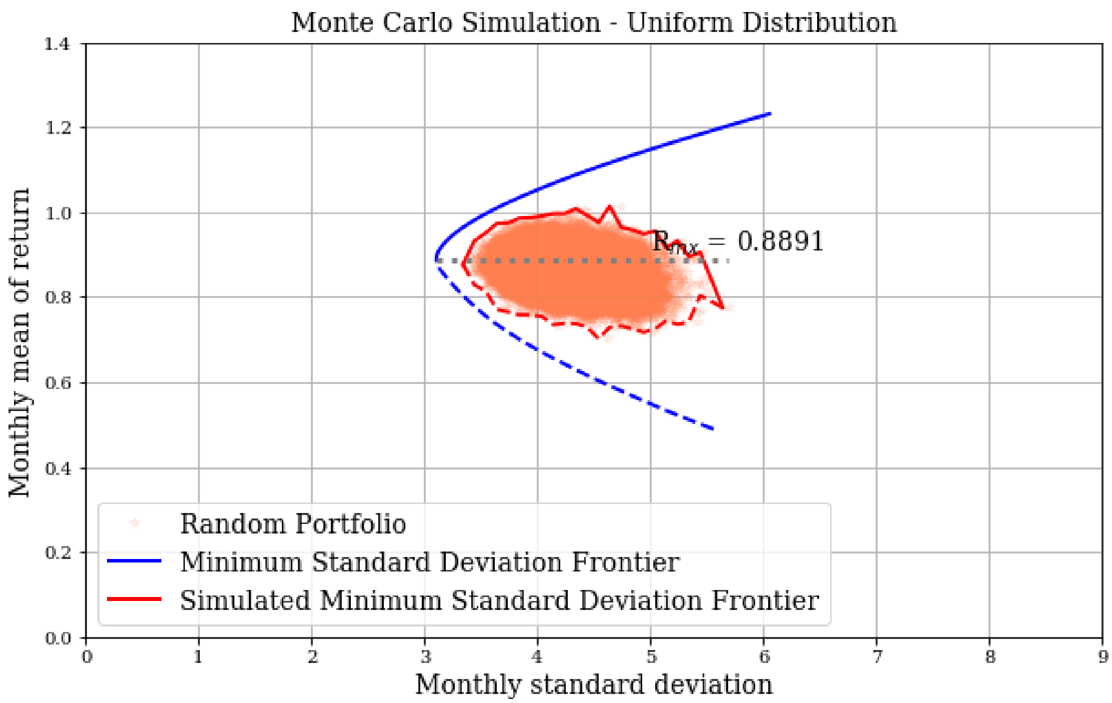
Uniform distribution fails to outperform nornal distribution according to my experiment.

In [28]:
```
# Monte Carlo method
#############################################################
# random seed, keep the random number still
np.random.seed(101)

# numbers of random portfolio
n_portfolio =  int(1e6)

# generate random weight from uniform distribution
w_uniform = np.random.rand(n_portfolio).reshape(-1, 10)

w_uniform /= w_uniform.sum(axis = 1, keepdims = True)

w_uniform_ = w_uniform.reshape(-1, 1)

plt.hist(w_uniform_, bins = 100, label ='Uniform Distribution')

plt.title('Distribution of the Weight generated by Uniform Distribution')
plt.legend(loc = 'best')
plt.ylabel('n_Observation')
```

```python
plt.xlabel('Weight')
plt.grid(True)
plt.show()

############################################################
# calculate sigma and mu for random portfolio
Portfolios = np.dot(IndPtf.values, w_uniform.T)
Mean_uniform = Portfolios.mean(axis = 0)
Std_uniform = Portfolios.std(axis = 0)

############################################################
# filter away points inside of the frontier
step = 0.1

ind_ = np.where(Std_uniform == min(Std_uniform) )
sigma = Std_uniform[ind_][0]
sigma_uniform = [sigma]
uniform_line_upper = [ Mean_uniform[ind_][0] ]
uniform_line_lower = [ Mean_uniform[ind_][0] ]

while sigma < max(Std_uniform):
    sigma += step
    ind = ((Std_uniform.reshape(-1,) >= sigma)
           & (Std_uniform.reshape(-1,) < sigma + step))

    while not ind.any() == True and  sigma < max(Std_uniform):
        sigma += 0.1
        ind = ((Std_uniform.reshape(-1,) >= sigma)
           & (Std_uniform.reshape(-1,) < sigma + step))

    if  ind.any() == True:
        sigma_uniform.append(sigma)
        uniform_line_upper.append(np.max(Mean_uniform[ ind ]))
        uniform_line_lower.append(np.min(Mean_uniform[ ind ]))

############################################################
z = 2
fig = plt.figure(figsize = (z*5, z*3))

ind = FrontierSigma.index(min(FrontierSigma))
plt.plot(Std_uniform, Mean_uniform,
         '*', color = 'coral', alpha = 0.1, label = 'Random Portfolio')

plt.plot(FrontierSigma[ind:], FrontierMeanReturn[ind:],
         '-', lw = z, color = 'blue', label = 'Minimum Standard Deviation Frontier')
plt.plot(FrontierSigma[:ind], FrontierMeanReturn[:ind],
         '--', lw = z, color = 'blue')
```

```
plt.plot(sigma_uniform, uniform_line_upper,
         '-', lw = z, color = 'red', label = 'Simulated Minimum Standard Deviation Fro
plt.plot(sigma_uniform, uniform_line_lower,
         '--', lw = z, color = 'red')

plt.plot( [FrontierSigma[ind], max(Std_uniform)], [FrontierMeanReturn[ind]]*2,
         ':', lw = 1.5*z, color = 'gray')
plt.text(5, FrontierMeanReturn[ind], 'R$%s$ = '%'_{mx}'\
         +str(round(FrontierMeanReturn[ind], 4)), verticalalignment='bottom', fontsize

plt.xlim(0,9)
plt.ylim(0,1.4)
plt.xlabel('Monthly standard deviation', fontsize = 7*z)
plt.ylabel('Monthly mean of return', fontsize = 7*z)
plt.title('Monte Carlo Simulation - Uniform Distribution', fontsize = 7*z)
plt.grid(True)
plt.legend(loc = 'best', fontsize = 7*z)
plt.show()
```





In [29]:
```
# Monte Carlo method
##############################################################
# random seed, keep the random number still
np.random.seed(101)

# numbers of random portfolio
n_portfolio =  int(1e6)

# generate random weight from uniform distribution
w_uniform_r = 1/np.random.rand(n_portfolio).reshape(-1, 10)

w_uniform_r /= w_uniform_r.sum(axis = 1, keepdims = True)

w_uniform_r_ = w_uniform_r.reshape(-1, 1)

plt.hist(w_uniform_, bins = 100,
         alpha = 0.7, label ='Uniform Distribution')
plt.hist(w_uniform_r_, bins = 100,
         alpha = 0.5, label ='Reciprocal Uniform Distribution')

plt.title('Distribution of the Weight generated by Reciprocal Uniform Distribution')
plt.legend(loc = 'best')
plt.ylabel('n_Observation')
plt.xlabel('Weight')
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plt.grid(True)
plt.show()
#############################################################
# calculate sigma and mu for random portfolio
Portfolios = np.dot(IndPtf.values, w_uniform_r.T)
Mean_uniform_r = Portfolios.mean(axis = 0)
Std_uniform_r = Portfolios.std(axis = 0)

#############################################################
# filter away points inside of the frontier
step = 0.1

ind_ = np.where(Std_uniform_r == min(Std_uniform_r) )
sigma = Std_uniform_r[ind_][0]
sigma_uniform_r = [sigma]
uniform_r_line_upper = [ Mean_uniform_r[ind_][0] ]
uniform_r_line_lower = [ Mean_uniform_r[ind_][0] ]

while sigma < max(Std_uniform_r):
    sigma += step
    ind = ((Std_uniform_r.reshape(-1,) >= sigma)
           & (Std_uniform_r.reshape(-1,) < sigma + step))

    while not ind.any() == True and  sigma < max(Std_uniform_r):
        sigma += 0.1
        ind = ((Std_uniform_r.reshape(-1,) >= sigma)
               & (Std_uniform_r.reshape(-1,) < sigma + step))

    if  ind.any() == True:
        sigma_uniform_r.append(sigma)
        uniform_r_line_upper.append(np.max(Mean_uniform_r[ ind ]))
        uniform_r_line_lower.append(np.min(Mean_uniform_r[ ind ]))

#############################################################
z = 2
fig = plt.figure(figsize = (z*5, z*3))

ind = FrontierSigma.index(min(FrontierSigma))
plt.plot(Std_uniform_r, Mean_uniform_r,
         '*', color = 'coral', alpha = 0.1, label = 'Random Portfolio')

plt.plot(FrontierSigma[ind:], FrontierMeanReturn[ind:],
         '-', lw = z, color = 'blue', label = 'Minimum Standard Deviation Frontier')
plt.plot(FrontierSigma[:ind], FrontierMeanReturn[:ind],
         '--', lw = z, color = 'blue')

plt.plot(sigma_uniform_r, uniform_r_line_upper,
         '-', lw = z, color = 'red', label = 'Simulated Minimum Standard Deviation Fro
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plt.plot(sigma_uniform_r, uniform_r_line_lower,
         '--', lw = z, color = 'red')

plt.plot( [FrontierSigma[ind], max(Std_uniform_r)], [FrontierMeanReturn[ind]]*2,
         ':', lw = 1.5*z, color = 'gray')
plt.text(5, FrontierMeanReturn[ind], 'R$%s$ = '%'_{mx}'\
         +str(round(FrontierMeanReturn[ind], 4)), verticalalignment='bottom', fontsize

plt.xlim(0,9)
plt.ylim(0,1.4)
plt.xlabel('Monthly standard deviation', fontsize = 7*z)
plt.ylabel('Monthly mean of return', fontsize = 7*z)
plt.title('Monte Carlo Simulation - Reciprocal Uniform Distribution', fontsize = 7*z)
plt.grid(True)
plt.legend(loc = 'best', fontsize = 7*z)
plt.show()
```

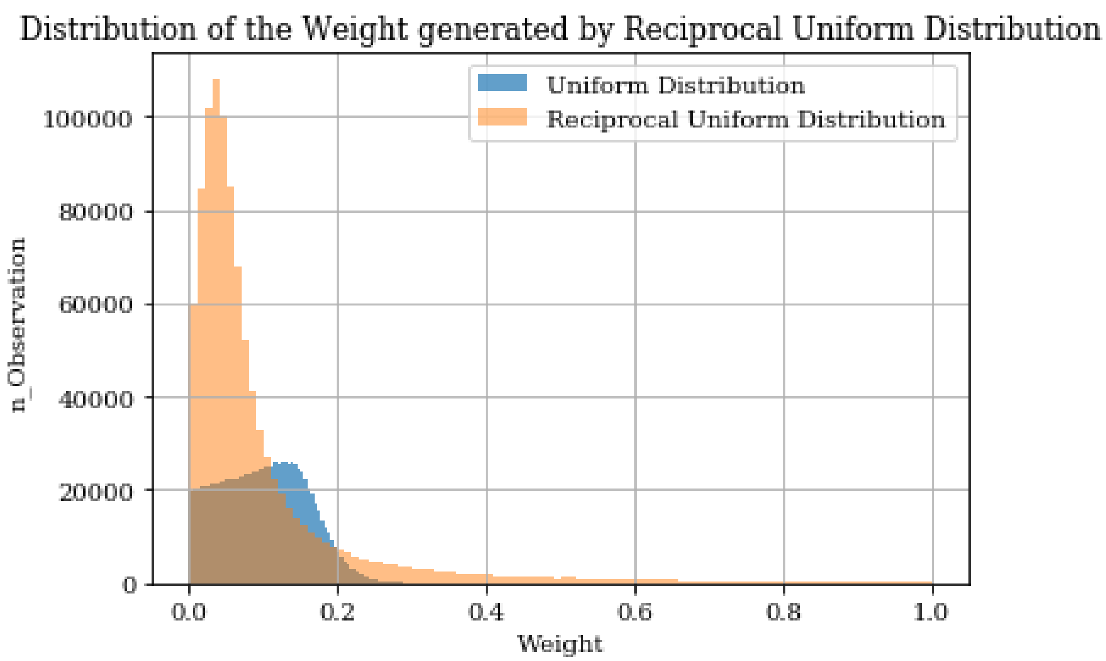

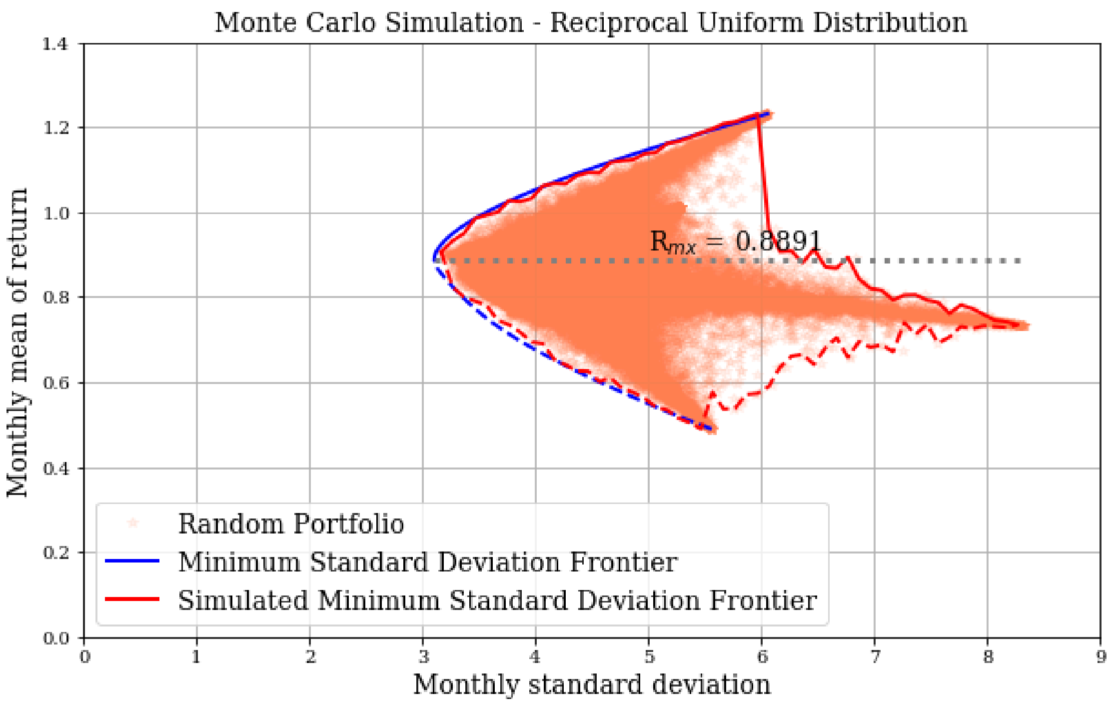
Monte Carlo Simulation - Reciprocal Uniform Distribution
R_{mx} = 0.8891
Random Portfolio
Minimum Standard Deviation Frontier
Simulated Minimum Standard Deviation Frontier
Monthly mean of return
Monthly standard deviation
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5
6
7
8
9