SINGAPORE MANAGEMENT UNIVERSITY

Session 8: Barrier Options

Tee Chyng Wen

QF620 Stochastic Modelling in Finance

Barrier Options are Path-dependent

Path-dependent options

Options having payoffs which depend on the path taken by the underlying asset.

$\Rightarrow$ Instead of just the asset's value on maturity.

Certain aspects of a barrier option's contract are triggered if the asset price becomes too low or too high, making them path-dependent.

Rationale: if you have a very specific view on the direction of the market, then you would want to capitalize on your view with as cheap an option as possible.

If an investor wants only part of the payoff of a call option, one can avoid having to pay for all the upside potential by truncating away payoff.

$\Rightarrow$ This option is cheaper than a vanilla call, and the investor gets rewarded if the view is realized.

Trading Barrier Options

Why would anyone buy a barrier option?

⇒ For the purpose of hedging they are not particularly useful.

⇒ However, they are cheaper than vanilla European options.

⇒ If an investor has a very strong view on how the asset price will move, barrier options allow the view to be specified more precisely.

An important issue with barrier options is the question of how to agree what it means for asset price to cross the barrier.

- The asset price is only observable when a trade is made, and there is the technical issue of checking all transactions to see whether an asset price actually crossed the barrier.
- For this reason, truly continuous barrier options are rare.
- Instead, the option price is generally sampled on a daily basis.
- The option is therefore really a discrete barrier option.
- However, the continuous barrier is a very good approximation to the daily sampled barrier.

Barrier Options Terminology

Out Barrier

Only pays if the barrier level B is not breached. If the barrier is hit, then the option is said to have knocked out.

In Barrier

Only pays if the barrier level B is breached. If the barrier is hit, then the option is said to have knocked in.

Direction of barrier B with respect to spot asset price S_0:

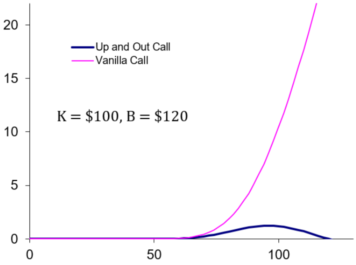
- If the barrier is above the asset price today, we call it up.
- If the barrier is below the asset price today, we call it down.

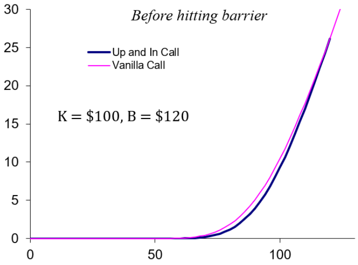
Variants: up-and-out, up-and-in, down-and-out, down-and-in.

Barrier Options *vs* Vanilla European Options

Example: $K = 100,\ B = 120,\ S_0 < B.$

Up-and-out & up-and-in call *vs* vanilla call

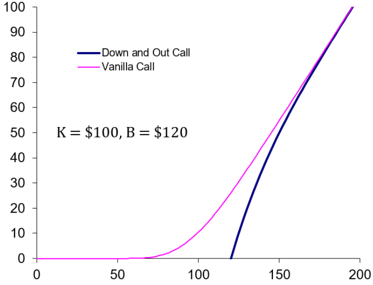


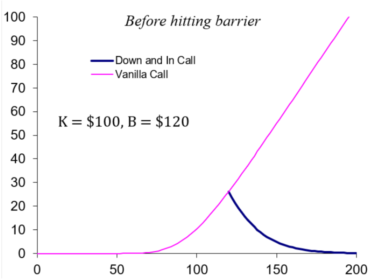


Barrier Options *vs* Vanilla European Options

Example: $K = 100,\ B = 120,\ S_0 > B.$

Down-and-out & down-and-in call *vs* vanilla call

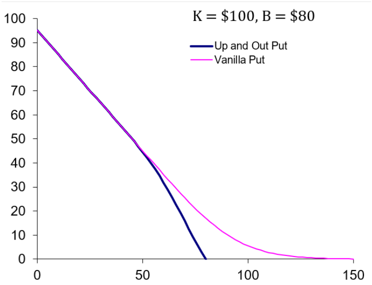


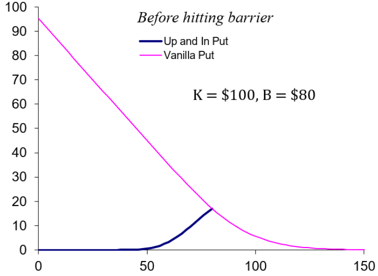


Barrier Options *vs* Vanilla European Options

Example: $K = 100,\ B = 120,\ S_0 < B$.

Up-and-out & up-and-in put *vs* vanilla put

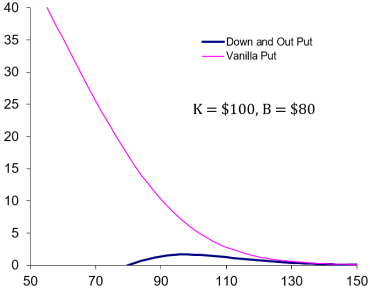


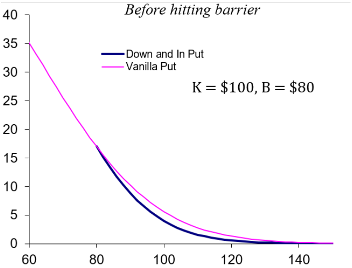


Barrier Options *vs* Vanilla European Options

Example: $K = 100,\ B = 120,\ S_0 > B.$

Down-and-out & down-and-in put *vs* vanilla put





Relationship with Vanilla Options

Clearly, if we own both the “out” and “in” version of the same option, one of it will pay the European option payout while the other one will be inactive/knockout and pay nothing.

Therefore, barrier options can be related to vanilla options by the simple relationship

$$\text{Out Barrier} + \text{In Barrier} = \text{Vanilla European}$$

Given this relationship, we only need to know how to value one type of option (either “in” or “out”), and we can then proceed to derive the value of the other barrier.

Since the “out” barrier is generally more liquid, we tend to focus more on deriving the valuation formulae for “out” barrier options.

Valuing Down-and-out Call Option

Consider a down-and-out call barrier option. Let m_T denote the minimum value of the stock price over the period $[0, T]$, the payoff of this option is given by

$$(S_T - K)\mathbb{1}_{S_T \geq K, m_T \geq B}$$

To evaluate the expectation of this payoff under the risk-neutral measure, we need to know

1. the probability that the barrier will be breached
2. the distribution of the final value of the asset given that the barrier has not been breached.

To compute these, what we really need is the joint distribution of the minimum and the terminal value for a Brownian motion with drift.

Valuing Down-and-out Call Option

We can use reflection principle to derive the expression for the joint law of minimum and terminal value of a driftless Brownian motion.

However, reflection principle isn't sufficient, as we are modelling stock drifting at the risk-free rate in the risk-neutral measure.

$\Rightarrow$ Note that working with discounted prices will not help here, as the barrier will depend on the actual price, not the discounted price.

In order to handle drift, we need to use Girsanov's theorem as well.

We start with an extension to the reflection principle to incorporate Girsanov's theorem, in order to derive joint distributions of minima and terminal values.

Next, we will return to evaluate the expectation of the barrier option payoff above.

Reflection Principle Revisited

Let W_t be a Brownian motion, and m_T be the minimum value of W_t over the interval $[0, T]$. We want to compute the probability of the event defined by

$$m_T \leq y, W_T \geq x$$

for $x \geq y$ and $y < 0$.

The Brownian motion needs to descends at least as far as y and then comes back up to level x.

Let τ denote the time the Brownian motion hit the level y. Suppose instead of continuing the Brownian motion after time τ, we restart it and replace it by its value reflected at the level y.

In other words, we define the following Brownian motion (based on Reflection Principle)

$$\tilde{W}_t = \begin{cases} W_t, & t < \tau \\ 2y - W_t, & t \geq \tau \end{cases}$$

Reflection Principle Revisited

The event $W_T \geq x$ and $m_T \leq y$ becomes $\tilde{W}_T \leq 2y - x$. Note that the event $\tilde{W}_T \leq 2y - x$ can only occur if $m_T \leq y$ also occurs.

Thus the event is now much simpler

$$\tilde{W}_T \leq 2y - x.$$

Reflection Principle states that $\tilde{W}_t$ is also a Brownian motion.

A stopping time is a random time such that the event before a time τ is determined by information available at time τ. In conclusion

$$\mathbb{P}(W_T \geq x, m_T \leq y) = \mathbb{P}(W_T \leq 2y - x)$$

The property that Brownian motion starts anew at stopping times is sometimes called the strong Markov property.

Girsanov's Theorem Revisited

Let A be an event, and $\mathbb{1}_A$ be the indicator which is 1 if A occurs and 0 otherwise. We can define a measure via

$$\tilde{\mathbb{P}}(A) = \mathbb{E}\left[\mathbb{1}_A \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}\right]$$

Obviously, probabilities should be positive, so $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}$ should also be positive. The probability of the global event (the whole sample space) is 1, therefore

$$\mathbb{E}\left[\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}\right] = 1.$$

Another property of $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}$ is that it ought to be a martingale. Let $X = \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}$, we define it as

$$dX_t = -\kappa X_t dW_t, \quad \Rightarrow \quad X_T = X_0 e^{-\frac{1}{2}\kappa^2 t - \kappa W_T}, \quad X_0 = 1.$$

Girsanov's Theorem Revisited

Hence, we have

$$\tilde{\mathbb{P}}(W_t < x) = \mathbb{E}\left[\mathbb{1}_{W_t < x} X_t\right] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{x} e^{-\frac{u^2}{2t}} e^{-\frac{\kappa^2}{2}t - \kappa u} du.$$

Completing the squares, we have

$$\tilde{\mathbb{P}}(W_t < x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{x} e^{-\frac{(u+\kappa t)^2}{2t}} du.$$

A simple change of variables yields

$$\mathbb{P}(W_t < x + \kappa t).$$

We have shown that the probability of $W_t < x$ in the new measure is equal to the probability that $W_t - \kappa t$ is less than x in the old measure.

$\Rightarrow$ In other words, the process W_t is distributed as a Brownian motion with drift $-\kappa t$ in the new measure.

Joint Distribution of Minimum and Terminal Value

In the previous pages, we have derived the joint law of the minimum and terminal value for a Brownian motion without drift.

In this section, we combine that result with our results on Girsanov's theorem to derive the joint law for a Brownian motion with drift.

Let W_t be a Brownian motion. Let $Y_t = \sigma W_t$, and m_{Y_t} be the minimum of Y_t up to time t. We have for $y < 0$ and $x \geq y$ that

$$\mathbb{P}(m_{Y_t} \leq y, Y_t \geq x) = \mathbb{P}(Y_t \leq 2y - x).$$

This follows from the result for the Brownian motion. We wish to prove a similar result for a generalized Brownian motion with drift. Let

$$Z_t = \mu t + \sigma W_t,$$

and m_{Z_t} denotes its minimum up to time t.

Joint Distribution of Minimum and Terminal Value

Theorem

For $y < 0$ and $x \geq y$,

$$\begin{aligned}\mathbb{P}(m_{Z_t} \leq y, Z_t \geq x) &= e^{2\mu y \sigma^{-2}} \mathbb{P}(Z_t \leq 2y - x + 2\mu t) \\ &= e^{2\mu y \sigma^{-2}} \Phi\left(\frac{2y - x + \mu t}{\sigma\sqrt{t}}\right)\end{aligned}$$

In this week's additional example, we work through the proof of this theorem step-by-step.

Given this theorem, we can now deduce the law of the minimum of a Brownian motion with drift. We have

$$\begin{aligned}\mathbb{P}(m_{Z_t} \leq y) &= \mathbb{P}(m_{Z_t} \leq y, Z_t \leq y) + \mathbb{P}(m_{Z_t} \leq y, Z_t \geq y) \\ &= \mathbb{P}(Z_t \leq y) + \mathbb{P}(m_{Z_t} \leq y, Z_t \geq y) \\ &= \Phi\left(\frac{y - \mu t}{\sigma\sqrt{t}}\right) + e^{2\mu y \sigma^{-2}} \Phi\left(\frac{y + \mu t}{\sigma\sqrt{t}}\right).\end{aligned}$$

Joint Distribution of Minimum and Terminal Value

Given that

$$\mathbb{P}(m_{Z_t} \leq y) = \mathbb{P}(m_{Z_t} \leq y, Z_t \geq x) + \mathbb{P}(m_{Z_t} \leq y, Z_t \leq x),$$

we have

$$\begin{aligned}\mathbb{P}(m_{Z_t} \leq y, Z_t \leq x) = \Phi\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) &+ e^{2\mu y\sigma^{-2}}\Phi\left(\frac{y+\mu t}{\sigma\sqrt{t}}\right) \\ &- e^{2\mu y\sigma^{-2}}\Phi\left(\frac{2y-x+\mu t}{\sigma\sqrt{t}}\right).\end{aligned}$$

Going through a similar procedure, we can show that

$$\mathbb{P}(m_{Z_t} \geq y, Z_t \geq x) = \Phi\left(\frac{\mu t-x}{\sigma\sqrt{t}}\right) - e^{2\mu y\sigma^{-2}}\Phi\left(\frac{2y-x+\mu t}{\sigma\sqrt{t}}\right).$$

Payoffs of Barrier Options

A down-and-out barrier call pays

$$(S_T - K)^+ \mathbb{1}_{m_T > B},$$

where K is the strike, B is the barrier level and $m_T = \min_{t\in[0,T]} S_T$ is the minimum value taken by S_T throughout the period $t \in [0, T]$.

Similarly, a up-and-out barrier call pays

$$(S_T - K)^+ \mathbb{1}_{M_T \leq B},$$

where $M_T = \max_{t\in[0,T]} S_T$ is the maximum value taken by S_T throughout the period $t \in [0, T]$.

The "in" barrier is the direct opposite — a down-and-in barrier call pays

$$(S_T - K)^+ \mathbb{1}_{m_T \leq B},$$

while a up-and-in barrier call pays

$$(S_T - K)^+ \mathbb{1}_{M_T > B}.$$

Expectation Pricing of Barrier Options

We are now in place to derive the valuation formula for a down-and-out barrier call.

Let B denote the barrier where $B < K$, the payoff of the barrier option is

$$(S_T - K)\mathbb{1}_{m_T \geq B, S_T \geq K}$$

Taking expectation, the first term evaluates to

$$\begin{aligned}\mathbb{E}\left[S_T \mathbb{1}_{m_T \geq B, S_T \geq K}\right] &= S_0 \mathbb{E}\left[\mathbb{1}_{m_T \geq B, S_T \geq K}\right] \\ &= S_0 \mathbb{P}(m_T \geq B, S_T \geq K) \\ &= S_0 \mathbb{P}(m'_T \geq \log B, \log S_T \geq \log K).\end{aligned}$$

Expectation Pricing of Barrier Options

Under the risk-neutral measure associated with the stock numeraire $\mathbb{Q}^S$, $\log \frac{S_T}{S_0}$ is a Brownian motion with drift $r + \frac{\sigma^2}{2}$. Let

$$
\begin{aligned}
x &= \log K - \log S_0 = -\log \frac{S_0}{K} \\
y &= \log B - \log S_0 = \log \frac{B}{S_0} \\
\mu &= r + \frac{\sigma^2}{2},
\end{aligned}
$$

and obtain

$$
S_0 \Phi \left(\frac{\log \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} \right) - S_0 \left(\frac{B}{S_0} \right)^{1+2r\sigma^{-2}} \Phi \left(\frac{\log \frac{B^2}{S_0 K} + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} \right).
$$

Expectation Pricing of Barrier Options

The second term, under the risk-neutral measure associated with the risk-free bond numeraire $\mathbb{Q}^*$ (i.e. with $\mu = r - \frac{\sigma^2}{2}$), evaluates to

$$-K\mathbb{E}\left[\mathbb{1}_{m_T \geq B, S_T \geq K}\right] = -Ke^{-rT}\Phi\left(\frac{\log\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) + Ke^{-rT}\left(\frac{B}{S_0}\right)^{-1+2r\sigma^{-2}}\Phi\left(\frac{\log\frac{B^2}{S_0K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).$$