Stochastic Modelling Assignment 1

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1.

a) $\mathbb{P}(W_2 < 0 \mid W_1 > 0) = \mathbb{P}(W_2 \text{ moves down}) \times \mathbb{P}(|W_2 - W_1| > |W_1 - W_0|)$

$= \frac{1}{2} \times \frac{1}{2} = \underline{\underline{\frac{1}{4}}}$

b) $P(W_1 \times W_2 < 0) = \underbrace{\mathbb{P}(W_2 < 0 \cap W_1 > 0)} + \mathbb{P}(W_2 > 0 \cap W_1 < 0)$

$\hookrightarrow \mathbb{P}(W_2 < 0 \mid W_1 > 0) \cdot P(W_1 > 0) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}.$

$\frac{1}{8} \times 2 = \underline{\underline{\frac{1}{4}}}$

c) $\mathbb{P}(W_1 < 0 \cap W_2 < 0) = \mathbb{P}(W_2 < 0 \mid W_1 < 0) \times \mathbb{P}(W_1 < 0)$

$= \frac{3}{4} \times \frac{1}{2} = \underline{\underline{\frac{3}{8}}}$

2. $E[|W_{t+\Delta t} - W_t|] = \underbrace{E[W_{t+\Delta t} - W_t]}_{\text{when } W_{t+\Delta t} > W_t} + \underbrace{E[-W_{t+\Delta t} + W_t]}_{\text{when } W_{t+\Delta t} < W_t}$

$= 2E[W_{t+\Delta t} - W_t] \quad (W_{t+\Delta t} > W_t)$

here, $W_{t+\Delta t} - W_t = X \sim N(0, \Delta t)$

$\therefore 2E[W_{t+\Delta t} - W_t] = 2 \times \int_0^\infty x \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{x^2}{2\Delta t}} dx$

$= 2 \times \frac{1}{2} \frac{\sqrt{2\Delta t}}{\sqrt{\pi}}.$

$= \frac{\sqrt{2\Delta t}}{\sqrt{\pi}}.$

3. $V\left[(W_t - W_s)^2\right] = E\left[(W_t - W_s)^4\right] - E\left[(W_t - W_s)^2\right]^2$

$$= E\left[(t-s)^2 X^4\right] - (t-s)^2 \quad \left(X \sim N(0,1)\right)$$

$$= 3(t-s)^2 - (t-s)^2$$

$$= 2(t-s)^2$$

4.

a) $X_t = W_t^2 = f(x)$

$$dX_t = f'(x) \cdot dW_t + \frac{1}{2} f''(x)(dW_t)^2$$

$$= 2 \cdot W_t \cdot dW_t + \frac{1}{2} \cdot 2 \cdot dt$$

$$dX_t = 2W_t dW_t + dt$$

b) $X_t = t + e^{W_t} = f(t, x)$

$$dX_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dW_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dW_t)^2$$

$$= dt + e^{W_t} dW_t + \frac{1}{2} e^{W_t} dt$$

$$dX_t = \left(1 + \frac{1}{2} e^{W_t}\right) dt + e^{W_t} dW_t$$

c) $X_t = W_t^3 - 3tW_t = f(t, x)$

$$dX_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dW_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dW_t)^2$$

$$= -3W_t dt + (3W_t^2 - 3t) dW_t + \frac{1}{2} 6W_t dt$$

$$dX_t = (3W_t^2 - 3t) dW_t$$

d) $X_t = e^{t+W_t}$

$dX_t = e^{t+W_t}dt + e^{t+W_t}dW_t + \frac{1}{2}e^{t+W_t}dt$

$dX_t = \frac{3}{2}e^{t+W_t}dt + e^{t+W_t}dW_t = \frac{3}{2}X_t dt + X_t dW_t$

e) $X_t = e^{\frac{t}{2}}\sin(W_t)$

$dX_t = \frac{1}{2}\cdot e^{\frac{t}{2}}\cdot\sin(W_t)\,dt + e^{\frac{t}{2}}\cdot\cos(W_t)\cdot dW_t - \frac{1}{2}e^{\frac{t}{2}}\cdot\sin(W_t)\cdot dt$

$dX_t = e^{\frac{t}{2}}\cos(W_t)\cdot dW_t$

f) $X_t = e^{W_t - \frac{t}{2}}$

$dX_t = -\frac{1}{2}e^{W_t-\frac{t}{2}}dt + e^{W_t-\frac{t}{2}}dW_t + \frac{1}{2}e^{W_t-\frac{t}{2}}\cdot dt$

$dX_t = e^{W_t-\frac{t}{2}}dW_t = X_t dW_t$

5. $X_t = Y_t Z_t$

$dX_t = Y_t dZ_t + Z_t dY_t + dY_t dZ_t$

$= Y_t A(t)dt + Y_t B(t)dW_t + Z_t b(t) Y_t dW_t + b(t) Y_t B(t) dt$

$= \Big(A(t) + b(t)B(t)\Big)Y_t dt + \Big(B(t) + Z_t b(t)\Big)Y_t dW_t$

6. $Y_t = \frac{W_t}{\tilde{W}_t} = f(W_t, \tilde{W}_t)$.

Let W_t and $\tilde{W}_t$, x and y respectively

$$Y_t = \frac{\partial f}{\partial x} dW_t + \frac{\partial f}{\partial y} d\tilde{W}_t + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(d\tilde{W}_t)^2 + \cancel{\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dW_t)^2} \to 0$$

$$= \frac{1}{\tilde{W}_t} dW_t - \frac{W_t}{\tilde{W}_t^2} d\tilde{W}_t + \frac{1}{2} \times 2 \times \frac{W_t}{\tilde{W}_t^3} dt$$

$$= \frac{1}{\tilde{W}_t} dW_t - \frac{W_t}{\tilde{W}_t^2} d\tilde{W}_t + \frac{W_t}{\tilde{W}_t^3} dt$$

7.

a) $dr_t = \theta dt + \sigma dW_t$

$$\int_0^T dr_t = \int_0^T \theta dt + \int_0^T \sigma dW_t.$$

$$r_T - r_0 = \theta T + \sigma W_T$$

$$E\left(\int_0^T r_t dt\right) = E\left(\int_0^T (r_0 + \theta t + \sigma W_t) dt\right)$$

$$= r_0 T + \theta \cdot \frac{1}{2} T^2 + \cancel{\sigma \int_0^T E(W_t) dt} \to 0$$

$$= r_0 T + \frac{\theta}{2} T^2$$

b) $$V\left[\int_0^T r_t dt\right] = E\left[\left(\int_0^T r_t dt\right)^2\right] - T^2\left(r_0 + \frac{\theta T}{2}\right)^2$$

$$= E\left[\left(\int_0^T (r_0 + \theta t + \sigma W_t) dt\right)^2\right] - T^2\left(r_0 + \frac{\theta T}{2}\right)^2$$

$$= E\left[\left(T\left(r_0 + \frac{\theta T}{2}\right) + \int_0^T \sigma W_t dt\right)^2\right] - T^2\left(r_0 + \frac{\theta T}{2}\right)^2$$

$$E\left[\cancel{T^2\left(r_0+\frac{\theta T}{2}\right)^2}+\left(\int_0^T \sigma W_t\,dt\right)^2+\cancelto{0}{2T\left(r_0+\theta\right)\int_0^T \sigma W_t\,dt}\right]-\cancel{T^2\left(r_0+\frac{\theta T}{2}\right)^2}$$

$$= E\left[\left(\int_0^T \sigma W_t\,dt\right)^2\right] \quad = \quad \sigma^2 E\left[\left(\int_0^T W_t\,dt\right)^2\right]$$

$$= \sigma^2 \cdot \frac{T^3}{3}$$