SMU
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Session 3: Stochastic Differential Equations

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QF620 Stochastic Modelling in Finance

Bachelier Model (1900)

Louis Bachelier was the first mathematician to use random walk to analyse stock prices (in 1900).

He modelled stock price as normally distributed (instead of Black-Scholes' lognormal model):

$$S_T = S_0\left(1 + \sigma W_T\right), \quad W_T \sim N(0, T).$$

Given this definition, we can then proceed to derive valuation formulas for vanilla European options.

However, a shortcoming of this model is that the lack of a lower bound at 0.

In other words, while this is a good model for interest rates, it leads to non-zero probability for negative stock prices.

Bachelier Model – Arithmetic Brownian Process

The Bachelier model for the stock price process is defined as

$$dS_t = \sigma S_0 dW_t.$$

Bachelier's model has the following properties:

1. Terminal stock price is normally distributed as

$$S_T \sim N(S_0, \sigma^2 S_0^2 T)$$

2. If we assume interest rate is 0, a vanilla European call option price is given by

$$V_0^c = (S_0 - K)\Phi\left(\frac{S_0 - K}{S_0\sigma\sqrt{T}}\right) + S_0\sigma\sqrt{T}\phi\left(\frac{S_0 - K}{S_0\sigma\sqrt{T}}\right)$$

3. When the option is at the money ($S_0 = K$), the price is exactly

$$V_0^c = \sigma S_0\sqrt{\frac{T}{2\pi}}$$

Black-Scholes Model (1976)

Black-Scholes states that

$$S_T = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}W_T\right], \quad W_T \sim N(0, T).$$

Given this definition, we can readily verify that

$$\mathbb{E}[S_T] = S_0 e^{rT}.$$

Alternatively, we can also write S_T as

$$\log S_T = \log S_0 + \left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}N(0,1).$$

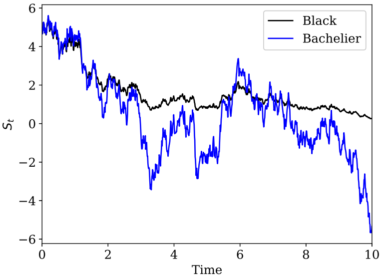
With this, we see that the log of the terminal stock price, $\log(S_T)$, follows a normal distribution.

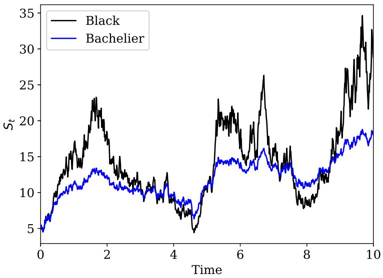
Black-Scholes *vs.* Bachelier

Below are 2 sample paths (same Brownian motion) from the 2 models:

$$\text{Black-Scholes:} \quad S_{t+\Delta t} = S_t e^{\left(r-\frac{\sigma^2}{2}\right)\Delta T+\sigma(W_{t+\Delta t}-W_t)}$$

$$\text{Bachelier:} \quad S_{t+\Delta t} = S_t + \sigma S_0(W_{t+\Delta t} - W_t)$$



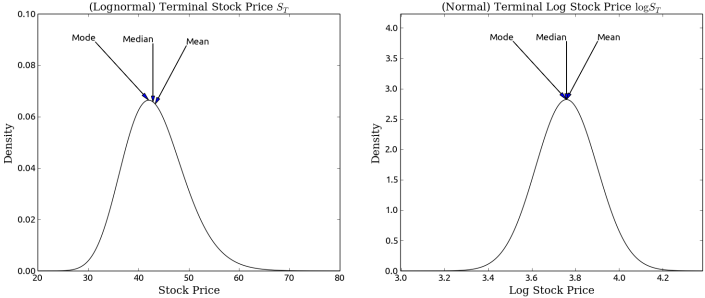


Question How do the two models compare?

Distribution under Black-Scholes Model

What is the probability distribution of the terminal stock price S_T?

$\Rightarrow$ If $\log(S_T)$ is normal, it follows that the price S_T is lognormal.



Example: $S_0 = \$40$, $r = 0.16$, $\sigma = 0.4$, $T = 0.5$.

The ordering of the mean, median and mode remains unchange. In particular, the median of a lognormal distribution is always below its mean.

Distribution under Black-Scholes Model

Since the median is always lying below the mean, it follows that, more often than not, the realised value of a lognormal random variable falls below its expected value.

Thus, more often than not, the terminal stock price S_T falls below its expected value.

	Normal $\log(S_T)$	Lognormal S_T
Mean	$\log(S_0) + \left(r - \frac{\sigma^2}{2}\right) T$	$S_0 e^{rT}$
Median	$\log(S_0) + \left(r - \frac{\sigma^2}{2}\right) T$	$S_0 e^{(r - \frac{\sigma^2}{2})T}$
Mode	$\log(S_0) + \left(r - \frac{\sigma^2}{2}\right) T$	$S_0 e^{(r - \frac{3\sigma^2}{2})T}$
Variance	$\sigma^2 T$	$S_0^2 e^{2rT} \left(e^{\sigma^2 T} - 1\right)$
Range	$-\infty < \log(S_T) < +\infty$	$0 \leq S_T < +\infty$

Black-Scholes assume that the return $\log(S_T)$ is normally distributed, and thus the price ratio S_T is lognormally distributed.

Black-Scholes Model – Geometric Brownian Process

The Black-Scholes model for the stock price process is defined as

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Black-Scholes model has the following properties:

1. The stochastic differential equation can be solved by applying Itô's formula to the function $X_t = f(S_t)$, where $f(x) = \log(x)$.
2. Vanilla European call option price is given by

$$V_0^c = S_0 \Phi\left(\frac{\log\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

3. For at-the-money option ($S_0 = K$), we can approximate the price as (assuming $r \approx 0$)

$$V_0^c \approx \sigma S_0 \sqrt{\frac{T}{2\pi}}$$

Black-Scholes Model – Geometric Brownian Process

Questions

1. The price for an at-the-money $1y$ call option is \$100. What is the price for an at-the-money $2y$ call option?

2. Give a quick approximation for the price of at-the-money call option on a stock ($S_0 = \$50$) with an annual volatility of 10%.

Black-Scholes Model – Geometric Brownian Process

Questions (Skew) Put-call parity states that

$$V_0^c - V_0^p = S_0 - Ke^{-rT}.$$

1. At what strike level $\bar{K}$ does call and put options have equal price?

2. Which of the following 2 options is more expensive according to Black-Scholes model:
   - A call option strike at $\bar{K} + \$10$
   - A put option strike at $\bar{K} - \$10$

3. A call option gives you an unlimited potential upside, but a put option gives you a limited potential upside equal to the strike price. Why do they still worth the same at $\bar{K}$?

Black Model (1976) – Forward Price Process

So far we've regarded the underlying as the stochastic variables and derived stochastic differential equations to describe their price dynamic.

Black proposed using the forward price instead of the spot price. Given the definition of the forward price

$$F_t = e^{r(T-t)}S_t$$

and the underlying price process of

$$dS_t = rS_t dt + \sigma S_t dW_t$$

Applying Itô's formula to the function $f(t,x) = e^{r(T-t)}x$ allows us to write down the stochastic differential equation for the forward price

$$dF_t = \sigma F_t dW_t,$$

which is a more compact notation, as it is driftless and is therefore a martingale.

Black Model (1976) – Forward Price Process

The Black76 model is defined on the forward price and is given by

$$dF_t = \sigma F_t dW_t.$$

As this is also a geometric process, we can solve this stochastic differential equation by applying Itô's formula to $X_t = f(F_t)$ where $f(x) = \log(x)$.

The solution is given by:

$$F_T = F_0 e^{-\frac{\sigma^2 T}{2} + \sigma W_T}$$

Let $D(0,T) = e^{-rT}$ denote the "discount factor", under this model the price of a vanilla European call option is given by

$$V_0^c = D(0,T)\left[F\Phi\left(\frac{\log\frac{F}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{F}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right].$$

Displaced-Diffusion Model – Shifted Lognormal

The displaced-diffusion model was formulated by Rubinstein in 1983. Consider the following forward price process:

$$dF_t = \sigma F_t dW_t$$

We say that F_T follows a lognormal distribution. Based on this definition, we call the following a displaced-diffusion, or shifted lognormal, process:

$$d(F_t + \alpha) = \sigma(F_t + \alpha)dW_t, \qquad \alpha \in \mathbb{R}.$$

Since α is a constant, the process can be written as

$$d(F_t + \alpha) = dF_t = \sigma(F_t + \alpha)dW_t$$

Let $X_t = F_t + \alpha$, we can readily see that:

$$dX_t = \sigma X_t dW_t, \qquad X_T = F_T + \alpha.$$

Displaced-Diffusion Model – Pricing Options

The following stochastic differential equation is the most commonly used form for displaced-diffusion process

$$dF_t = \sigma[\beta F_t + (1-\beta)F_0]dW_t, \qquad \beta \in [0,1].$$

To solve this, we apply Itô formula to the function f

$$X_t = f(F_t), \qquad \text{where } f(x) = \log[\beta x + (1-\beta)F_0]$$

to obtain

$$F_T = \frac{F_0}{\beta}\exp\left[-\frac{\beta^2\sigma^2 T}{2} + \beta\sigma_T W_T\right] - \frac{1-\beta}{\beta}F_0.$$

Question Suppose we define

$$\text{Black76Call}(F_0, K, \sigma, T) = D(0,T)\left[F_0\Phi\left(\frac{\log\frac{F_0}{K}+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{F_0}{K}-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right]$$

Can we express a vanilla displaced-diffusion call option price using Black76Call?

Displaced-Diffusion Model – Negative Lower Bound

The displaced diffusion process can also be written as

$$d\left(F_t + \frac{1-\beta}{\beta}F_0\right) = \sigma\beta\left(F_t + \frac{1-\beta}{\beta}F_0\right)dW_t.$$

Written in this way, it should be clear that with $\left(F_t + \frac{1-\beta}{\beta}F_0\right)$ modeled as a geometric Brownian process, it is strictly positive.

As long as the β parameter is positive, the forward rate process F_t can now take on negative values, since the process is well-defined as long as $F_t + \frac{1-\beta}{\beta}F_0 > 0$.

In other words, the forward rate process is allowed to be negative, so long as $F_t > \frac{\beta-1}{\beta}F_0$.

When $F_0 > 0$, any choice of $0 < \beta < 1$ will provide a negative value as the lowerbound to the forward rate process.

Mean-reverting Process – Vasicek Model

The Ornstein-Uhlenbeck process is used in solid-state physics to model gas molecules under the influence of pressure and temperature.

In 1977, Vasicek adapted this model to model interest rate as a mean reverting stochastic process, given by

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t.$$

Applying Itô formula to $X_t = e^{\kappa t} r_t = f(t, r_t)$, we obtain

$$\begin{aligned} d(e^{\kappa t} r_t) &= \kappa e^{\kappa t} r_t dt + e^{\kappa t} dr_t \\ &= \kappa\theta e^{\kappa t} dt + \sigma e^{\kappa t} dW_t. \end{aligned}$$

Integrating both sides from 0 to t, we can obtain a solution to the stochastic differential equation

$$\int_0^t d(e^{\kappa u} r_u) = \int_0^t \kappa\theta e^{\kappa u} du + \int_0^t \sigma e^{\kappa u} dW_u$$

$$r_t = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma \int_0^t e^{\kappa(u-t)} dW_u.$$

Mean-reverting Process – Vasicek Model

Taking expectation on both sides gives us the mean

$$\mathbb{E}[r_t] = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}).$$

Recall Itô's Isometry theorem states that

$$\mathbb{E}\left[\left(\int_0^T X_t dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T X_t^2 dt\right].$$

Applying it to our case,

$$\begin{aligned} V[r_t] &= \mathbb{E}\left[\left(\sigma \int_0^t e^{\kappa(u-t)} dW_u\right)^2\right] \\ &= \mathbb{E}\left[\sigma^2 \int_0^t e^{2\kappa(u-t)} du\right] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right). \end{aligned}$$

The distribution of r_t is therefore given by

$$r_t \sim N\left(r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}), \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})\right).$$

Cox-Ingersoll-Ross Process

The interest rate dynamic proposed by Cox-Ingersoll-Ross (CIR) is described by the following stochastic differential equation

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t,$$

with the corresponding expression in integral form

$$r_t = r_0 + \kappa\int_0^t (\theta - r_u)du + \sigma\int_0^t \sqrt{r_u}dW_u.$$

This cannot be solved explicitly, but that shouldn't stop us from determining it's mean and variance. The mean can be obtained by taking the expectation on both sides of the integral expression, giving

$$\mathbb{E}[r_t] = r_0 + \kappa\int_0^t \left(\theta - \mathbb{E}[r_u]\right) du.$$

Differentiating both sides with respect to t yields

$$\frac{d}{dt}\mathbb{E}[r_t] = \kappa\theta - \kappa\mathbb{E}[r_t].$$

Cox-Ingersoll-Ross Process

Using the total derivative relationship, we see that

$$\frac{d}{dt}\left[e^{\kappa t}\mathbb{E}[r_t]\right] = e^{\kappa t}\frac{d}{dt}\mathbb{E}[r_t] + \kappa e^{\kappa t}\mathbb{E}[r_t] = \kappa\theta e^{\kappa t}.$$

Integrating from 0 to t, we obtain an expression for the mean

$$\begin{aligned}\int_0^t d\left[e^{\kappa t}\mathbb{E}[r_u]\right] &= \kappa\theta\int_0^t e^{\kappa u}dt \\ e^{\kappa t}\mathbb{E}[r_t] - r_0 &= \theta(e^{\kappa t}-1) \\ \Rightarrow \mathbb{E}[r_t] &= \theta + e^{-\kappa t}(r_0-\theta).\end{aligned}$$

When t tends towards infinity, we have $\lim_{t\to\infty}\mathbb{E}[r_t] = \theta$. Next we proceed to determine the variance. First we derive the stochastic differential equation satisfied by the squared interest rate process. Define $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2$ and $X_t = f(r_t)$, by Itô's formula

$$\begin{aligned}dX_t &= f'(r_t)dr_t + \frac{1}{2}f''(r_t)(dr_t)^2 \\ &= [2r_t\kappa(\theta-r_t)+\sigma^2 r_t]dt + 2\sigma r_t^{3/2}dW_t.\end{aligned}$$

Cox-Ingersoll-Ross Process

The corresponding integral form is given by

$$X_t = r_t^2 = r_0^2 + \int_0^t [2r_u\kappa(\theta - r_u) + \sigma^2 r_u]du + \int_0^t 2\sigma r_u^{3/2} dW_u.$$

Taking expectation on both sides

$$\frac{d}{dt}\mathbb{E}[r_t^2] = r_0^2 + \int_0^t \left[(2\kappa\theta + \sigma^2)\mathbb{E}[r_u] - 2\kappa\mathbb{E}[r_u^2]\right] du.$$

Differentiating both sides with respect to t

$$\frac{d}{dt}\mathbb{E}[r_t^2] = (2\kappa\theta + \sigma^2)\mathbb{E}[r_t] - 2\kappa\mathbb{E}[r_t^2].$$

Again using the total derivative relationship, and substituting for the expression for $\mathbb{E}[r_t]$ determined earlier, we see that

$$\begin{aligned}\frac{d}{dt}\left[e^{2\kappa t}\mathbb{E}[r_t^2]\right] &= e^{2\kappa t}\frac{d}{dt}\mathbb{E}[r_t^2] + 2\kappa e^{2\kappa t}\mathbb{E}[r_t^2] \\ &= e^{2\kappa t}(2\kappa\theta + \sigma^2)\mathbb{E}[r_t] \\ &= e^{2\kappa t}(2\kappa\theta + \sigma^2)[\theta + e^{-\kappa t}(r_0 - \theta)] \\ &= \theta e^{2\kappa t}(2\kappa\theta + \sigma^2) + e^{\kappa t}(2\kappa\theta + \sigma^2)(r_0 - \theta).\end{aligned}$$

Cox-Ingersoll-Ross Process

Integrating both sides from 0 to t and rearranging, we have

$$\int_0^t d\left[e^{2\kappa t}\mathbb{E}[r_u^2]\right] = \int_0^t \theta e^{2\kappa u}(2\kappa\theta+\sigma^2) + e^{\kappa u}(2\kappa\theta+\sigma^2)(r_0-\theta)dt$$

$$e^{2\kappa t}\mathbb{E}[r_t^2] - r_0^2 = \frac{\theta}{2\kappa}(e^{2\kappa t}-1)(2\kappa\theta+\sigma^2) + \frac{1}{\kappa}(e^{\kappa t}-1)(2\kappa\theta+\sigma^2)(r_0-\theta)$$

$$\Rightarrow \mathbb{E}[r_t^2] = e^{-2\kappa t}r_0^2 + \frac{\theta}{2\kappa}(1-e^{-2\kappa t})(2\kappa\theta+\sigma^2) + \frac{1}{\kappa}(e^{-\kappa t}-e^{-2\kappa t})(2\kappa\theta+\sigma^2)(r_0-\theta).$$

We're now in place to determine the variance of the CIR process

$$\begin{aligned} V[r_t] = \mathbb{E}[r_t^2] - \mathbb{E}[r_t]^2 &= r_0\frac{\sigma^2}{\kappa}(e^{-\kappa t}-e^{-2\kappa t}) + \frac{\theta\sigma^2}{2\kappa}(1-e^{-\kappa t})^2 \\ &= \frac{\theta\sigma^2}{2\kappa} + e^{-\kappa t}\frac{\sigma^2}{\kappa}(r_0-\theta) + e^{-2\kappa t}\frac{\sigma^2}{\kappa}\left(\frac{\theta}{2}-r_0\right). \end{aligned}$$

Cox-Ingersoll-Ross Process

The distribution of r_t under the CIR process is therefore given by

$$\mathbb{E}[r_t|r_0] = \theta + e^{-\kappa t}(r_0 - \theta)$$

$$V[r_t|r_0] = \frac{\theta\sigma^2}{2\kappa} + e^{-\kappa t}\frac{\sigma^2}{\kappa}(r_0 - \theta) + e^{-2\kappa t}\frac{\sigma^2}{\kappa}\left(\frac{\theta}{2} - r_0\right)$$

and with the following long term distribution

$$\lim_{t\to\infty} \mathbb{E}[r_t|r_0] = \theta$$

$$\lim_{t\to\infty} V[r_t|r_0] = \frac{\theta\sigma^2}{2\kappa}.$$

SDEs and Martingale

Example Use Itô's formula to derive the stochastic differential equations of the following processes, and determine which of them are martingales:

1. $X_t = W_t^2$ (no)
2. $X_t = 2 + t + e^{W_t}$ (no)
3. $X_t = W_t^2 + \tilde{W}_t^2$, where W_t and $\tilde{W}_t$ are independent Brownian. (no)
4. $X_t = W_t^2 - t$ (yes)
5. $X_t = W_t^3$ (no)
6. $X_t = e^{\theta W_t - \frac{\theta^2 t}{2}}$ (yes)