SINGAPORE MANAGEMENT UNIVERSITY

Session 5: Option Valuation & Stochastic Volatility Models

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QF620 Stochastic Modelling in Finance

Radon-Nikodym Derivative as Numeraire Ratio

Let V_t denote a derivative price. If we value it under the risk-neutral measure $\mathbb{Q}^N$ associated to the numeraire N_t, we have

$$\frac{V_t}{N_t} = \mathbb{E}^{\mathbb{Q}^N}\left[\frac{V_T}{N_T}\right].$$

Alternatively, we could value it under the risk-neutral measure $\mathbb{Q}^M$ associated to the numeraire M_t, and we will have

$$\frac{V_t}{M_t} = \mathbb{E}^{\mathbb{Q}^M}\left[\frac{V_T}{M_T}\right].$$

In an arbitrage-free market, derivative price is unique. Therefore, we have

$$\begin{aligned} N_t\mathbb{E}^{\mathbb{Q}^N}\left[\frac{V_T}{N_T}\right] &= M_t\mathbb{E}^{\mathbb{Q}^M}\left[\frac{V_T}{M_T}\right] \\ \Rightarrow \mathbb{E}^{\mathbb{Q}^N}[G_T] &= \mathbb{E}^{\mathbb{Q}^M}\left[G_T\frac{N_T/N_t}{M_T/M_t}\right]. \end{aligned}$$

where $G_T = \frac{V_T}{N_T}$.

Radon-Nikodym Derivative as Numeraire Ratio

- Since V, N, M are general, this result holds for all random variables G and all numeraires N and M.
- We have now a way to express the expectation of G_T under the measure $\mathbb{Q}^N$ in terms of an expectation under the measure $\mathbb{Q}^M$.

Comparing this relationship with our Radon-Nikodym derivative

$$\mathbb{E}^{\mathbb{Q}^N}[G_T] = \mathbb{E}^{\mathbb{Q}^M}\left[G_T \frac{d\mathbb{Q}^N}{d\mathbb{Q}^M}\right],$$

we conclude that

$$\frac{d\mathbb{Q}^N}{d\mathbb{Q}^M} = \frac{N_T/N_t}{M_T/M_t}.$$

Digital (Binary) Options

Digital (or binary) options are characterized by the binary state of their payoff. Two most common types are:

1. Cash-or-nothing: you either get a unit cash amount or nothing
2. Asset-or-nothing: you either get the underlying asset or nothing

A cash-or-nothing digital option pays 1 is $S_T > K$ on maturity and 0 otherwise. We can write the payoff as

$$V_{\text{Cash Digital}}(T) = \mathbb{1}_{S_T > K}.$$

An asset-or-nothing digital option pays S_T is $S_T > K$ on maturity and 0 otherwise. We can write the payoff as

$$V_{\text{Asset Digital}}(T) = S_T \mathbb{1}_{S_T > K}.$$

Digital Option – Cash-or-nothing Digital

Under Black-Scholes model, the stock price follows the lognormal process

$$dS_t = rS_t dt + \sigma S_t dW_t^*$$

We can price the cash digital call option as

$$\begin{aligned}
V_{\text{Cash Digital}}(0) &= e^{-rT}\mathbb{E}^{\mathbb{Q}^*}\left[\mathbb{1}_{S_T>K}\right] \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbb{1}_{S_T>K}e^{-\frac{x^2}{2}}dx \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}e^{-\frac{x^2}{2}}dx \\
&= e^{-rT}\Phi\left(\frac{\log\left(\frac{S_0}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right).
\end{aligned}$$

Digital Option – Asset-or-nothing Digital

Again based on Black-Scholes model:

$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$

the asset digital call option can be priced as

$$\begin{aligned}
V_{\text{Asset Digital}}(0) &= e^{-rT}\mathbb{E}^{\mathbb{Q}^*}\left[S_T \mathbb{1}_{S_T>K}\right] \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} S_T \mathbb{1}_{S_T>K} e^{-\frac{x^2}{2}}\,dx \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty} S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}x} e^{-\frac{x^2}{2}}\,dx \\
&= S_0 \Phi\left(\frac{\log\left(\frac{S_0}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right).
\end{aligned}$$

Solving the integral involves the process of completing the square for x.

$\Rightarrow$ We can in fact avoid this step by a change of measure.

Digital Option – Asset-or-nothing Digital

As we have seen, it is not necessary to choose the risk-free bond as the numeraire.

$\Rightarrow$ We can also use the stock as the numeraire and price the option under the risk-neutral measure associated to the stock $\mathbb{Q}^S$.

In this case, our asset-or-nothing digital formula becomes

$$V_{\text{Asset Digital}}(0) = e^{-rT}\mathbb{E}^{\mathbb{Q}^*}\left[S_T \mathbb{1}_{S_T>K}\right] = S_0\mathbb{E}^{\mathbb{Q}^S}\left[\mathbb{1}_{S_T>K}\right],$$

where we've used the Radon-Nikodym derivative to change the measure from $\mathbb{Q}^*$ to $\mathbb{Q}^S$

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}^*} = \frac{S_T/S_0}{e^{rT}/1} = e^{-\frac{1}{2}\sigma^2 T+\sigma W_T^*}.$$

Comparing this to the Radon-Nikodym derivative, we note that $\mathbb{Q}^S$ is equivalent to $\mathbb{Q}^*$, and $W_t^S = W_t^* + \kappa t$ is a $\mathbb{Q}^S$-Brownian motion (where $\kappa = -\sigma$).

Digital Option – Asset-or-nothing Digital

Under this measure change, we have a new Brownian motion W_t^S where

$$dW_t^S = dW^* - \sigma dt.$$

Now our lognormal SDE for our stock price process becomes

$$dS_t = (r + \sigma^2)S_t dt + \sigma S_t dW_t^S,$$

with a solution of

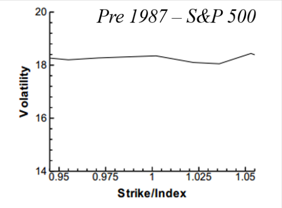
$$S_T = S_0 \exp\left[\left(r + \frac{1}{2}\sigma^2\right)T + \sigma W_T^S\right].$$

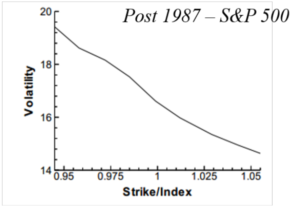
And so our asset-or-nothing digital option formula becomes

$$\begin{aligned} V_{\text{Asset Digital}}(0) &= S_0 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{1}_{S_T > K} e^{-\frac{x^2}{2}} dx \\ &= S_0 \int_{x^*}^{\infty} e^{-\frac{x^2}{2}} dx \\ &= S_0 \Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right). \end{aligned}$$

Volatility Smile

Black-Scholes assumes that the volatility of stock returns is constant through time and strikes. Is this true?





A distinct volatility smile/skew manifested after the 1987 crash across a wide range of market – in anticipation of extreme market moves.

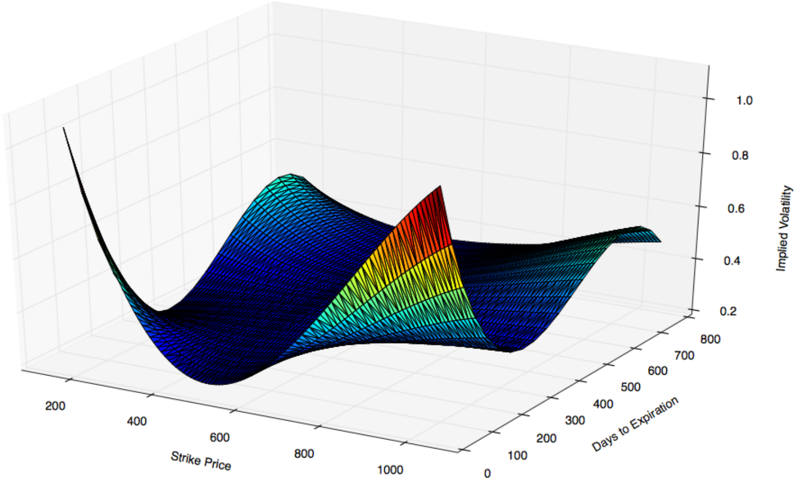
If the Black-Scholes assumptions are correct, then the implied volatilities of options should fall on a horizontal line when plotted against strike price.

Volatility Smile

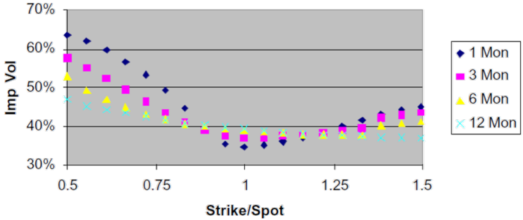
Background

- According to classical formulation, the Black-Scholes implied volatility of an option should be independent of its strike and expiration.
- Prior to the stock market crash of October 1987, the volatility surface of index options was indeed fairly flat.
- The Black-Scholes model assumes that a stock's return volatility is a constant, independent of strike and time to expiration.
- The smile's appearance after the 1987 crash was clearly connected in some way with the shock of discovering, for the first time since 1929, that a huge market could drop by 20% or more in a day or two.
- In a *liquid* option market, option prices are determined by supply and demand, not by a valuation formula.

Volatility Smile



Volatility Smile



Implied volatilities across strike is steepest for short expiries, and is flatter for longer expiries. Higher implied volatilities translate to higher option prices.

The return of the underlying asset is typically negatively correlated with volatility — rising volatility is commonly interpreted as early warning of financial crisis.

In a liquid option market, the volatility we substitute into our Black-Scholes formula is implied from traded option prices, i.e.

$$\text{Mkt Option Price} = \text{Black-Scholes}(S_0, K, r, \sigma_{\text{imp}}, T)$$

Extension to Black-Scholes Model

European options are often priced and hedged using the Black-Scholes model.

⇒ There is a one-to-one relation between the price of a European option and the volatility σ.

Consequently, option prices are often quoted by stating the implied volatility, the unique value of the volatility which yields the options price when used in the Black-Scholes formula.

In theory, the volatility σ should be a constant (according to Black-Scholes). In practice however, options with different strikes require different volatilities to match their market prices.

Handling these market skews and smiles correctly is critical to the trading desks, since they usually have large exposures across a wide range of strikes.

The inherent contradiction of using different volatilities for different options makes it difficult to successfully manage these risks using Black-Scholes model.

Black76 Lognormal Model

As we have seen earlier (Session 3), given

$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$

we can use Itô's formula to derive the SDE for $F_t = e^{r(T-t)}S_t$, and this is given by

$$dF_t = \sigma F_t dW_t^*.$$

On the maturity of a call option the payoff is $(S_T - K)^+ = (F_T - K)^+$, and we have the Black76 formula

$$\text{Black76LognormalCall} = e^{-rT}\left[F_0\Phi\left(\frac{\log\left(\frac{F_0}{K}\right)+\frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log\left(\frac{F_0}{K}\right)-\frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}\right)\right]$$

This model is identical to the Black-Scholes formula when interest rate r is deterministic.

Black76 Normal Model

An immediate and straightforward extension is the Black76 Normal model, where we model the forward price process as

$$dF_t = \sigma F_0 dW_t^*.$$

In this case F_t follows an arithmetic Brownian motion.

Solving the SDE, we have $F_T = F_0 + \sigma F_0 W_T^*$. The call option is in-the-money when

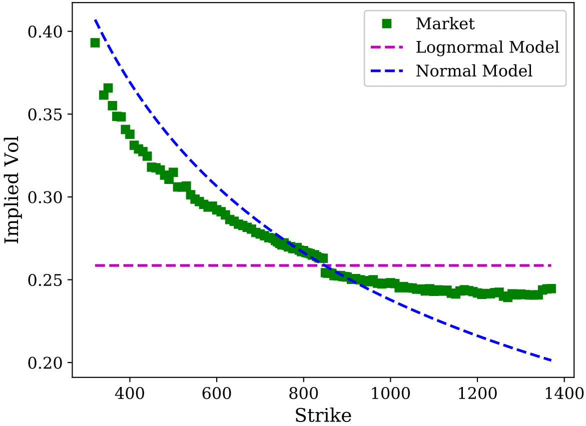
$$F_T > K \quad \Rightarrow \quad x > \frac{K - F_0}{\sigma F_0 \sqrt{T}} = x^*$$

hence a call option can be valued as

$$\text{Black76NormalCall} = e^{-rT}\left[(F_0 - K)\Phi(-x^*) + \sigma\sqrt{T}F_0\phi(x^*)\right]$$

Fitting Market Implied Volatilities

Take Google's call and put options on 2013-08-30 as an example. We look at options expiring on 2015-01-17, the spot stock price is 846.9, and the at-the-money volatility is ≈ 0.26.



Black76 Displaced-Diffusion Model

From the graph, it appears that the implied volatility smile we observed in the market is between normal and lognormal.

We have seen earlier that the displaced-diffusion (shifted lognormal) model can be thought of as a weighted average between a normal and a lognormal model. Under a Black76 displaced-diffusion model, we have:

$$dF_t = \sigma[\beta F_t + (1-\beta)F_0]dW_t^*$$

Recall that the solution is given by

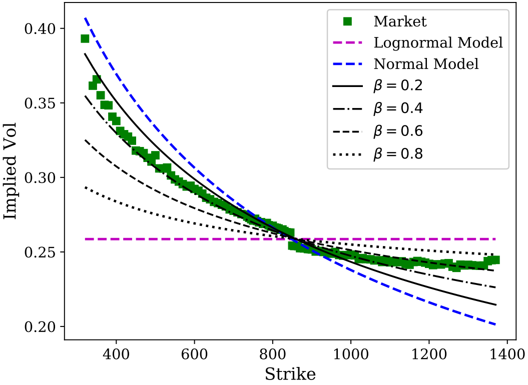
$$F_T = \frac{F_0}{\beta} e^{-\frac{\beta^2\sigma^2 T}{2} + \beta\sigma W_T^*} - \frac{1-\beta}{\beta} F_0$$

The option price under the displaced-diffusion model is

$$\text{Black76Disp.Diff.} = \text{Black76Lognormal}\left(\frac{F_0}{\beta},\ K + \frac{1-\beta}{\beta}F_0,\ \sigma\beta,\ T\right)$$

Fitting Market Implied Volatilities

Observe that we are able to obtain a closer fit to the market using Black76 displaced diffusion model by choosing the right β parameter.



However, the fit is still not sufficiently accurate.

Stochastic Volatility

Direct observation of the daily log-return of any underlying should convince us that volatility is stochastic instead of deterministic.

In other words, instead of treating it as a constant, it should also be described by a stochastic differential equation. As the simplest extension, we let it follow a driftless lognormal process

$$d\sigma_t = \nu \sigma_t dW_t^\sigma,$$

where ν is the volatility of volatility. This SDE is solved by

$$\begin{aligned}\sigma_T &= \sigma_0 \exp\left[-\frac{1}{2}\nu^2 T + \nu W_T^\sigma\right] \\ &= \sigma_0 \exp\left[-\frac{1}{2}\nu^2 T + \nu\sqrt{T} N(0,1)\right].\end{aligned}$$

In other words, instead of letting σ be a constant, it is now evolving according to its own SDE, and is a random variable.

Heston Model

The Heston model is formulated by Steven Heston (1993) and is given by the stochastic differential equations:

$$\begin{cases} dS_t = rS_t dt + \sqrt{V_t} S_t dW_t^S \\ dV_t = \kappa(\theta - V_t)dt + \nu\sqrt{V_t} dW_t^V \end{cases}$$

where $dW_t^S dW_t^V = \rho dt$.

Heston models the variance as a stochastic process, following a mean-reverting square-root diffusion process.

The value of vanilla European options are determined by a 1-d integral which has to be evaluated numerically.

Heston model is popular among the equity desks.

SABR Model

The SABR model (stochastic alpha-beta-rho) is pioneered by Patrick Hagan (2002) and is characterised by the SDEs

$$\begin{cases} dF_t = \alpha_t F_t^{\beta} dW_t^F \\ d\alpha_t = \nu \alpha_t dW_t^{\alpha} \end{cases}$$

where $dW_t^F dW_t^{\alpha} = \rho dt$.

The volatility is stochastic and follows a zero-drift lognormal dynamics. Hagan derived the formula for implied volatility σ_{SABR} as an analytical function of the model parameters.

To value vanilla European options, we just need to calculate σ_{SABR} and substitute this implied volatility into the Black's formula to convert to price.

This is much quicker than the Heston model. SABR model is widely used across a range of asset classes.

SABR Model

$$
\begin{aligned}
&\sigma_{\text{SABR}}(F_0, K, \alpha, \beta, \rho, \nu) \\
&= \frac{\alpha}{(F_0K)^{(1-\beta)/2}\left\{1 + \frac{(1-\beta)^2}{24}\log^2\left(\frac{F_0}{K}\right) + \frac{(1-\beta)^4}{1920}\log^4\left(\frac{F_0}{K}\right) + \cdots\right\}} \\
&\quad \times \frac{z}{x(z)} \times \left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(F_0K)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{(F_0K)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right]T + \cdots\right.
\end{aligned}
$$

where

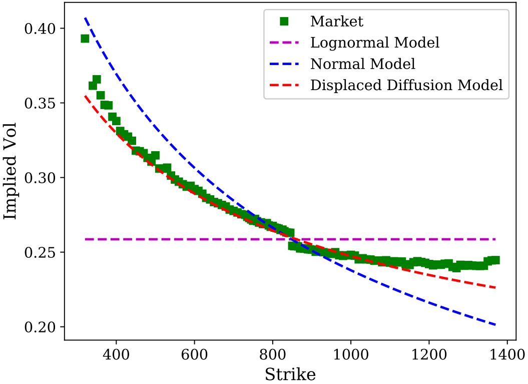
$$
z = \frac{\nu}{\alpha}(F_0K)^{(1-\beta)/2}\log\left(\frac{F_0}{K}\right),
$$

and

$$
x(z) = \log\left[\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right].
$$

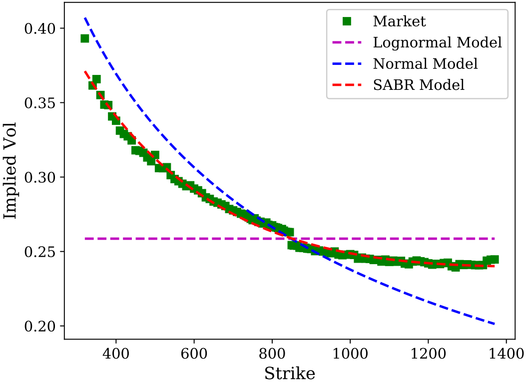
Program provided in `sabr.py`

SABR Model



Displaced-diffusion model can only fit to implied volatility skew – there will be mismatch if the implied volatility surface also exhibit "smile" characteristic.

SABR Model



SABR model is able to fit both skew and smile in the implied volatility surface – it is a popular model due to the ease of calculation.

Behavior of Model Parameters – ρ

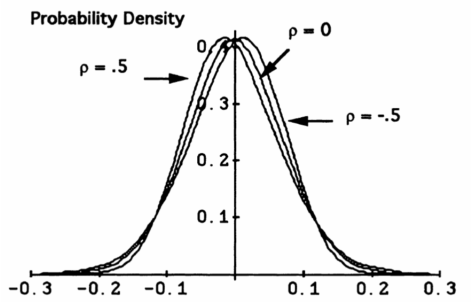
Implication on Distribution

- The correlation parameter ρ is proportional to the skewness of stock returns.
- Intuitively, a negative correlation results in high volatility when the stock price drops, and this spreads the left tail of the probability density. The right tail is associated with low volatility and is not spread out.
- A negative correlation creates a fat left tail and a thin right tail in the stock return distribution.

Implication on Pricing

- This increases the prices of out-of-the-money puts and decreases the prices of out-of-the-money calls relative to the Black-Scholes model price.
- Intuitively, out-of-the-money put options benefit substantially from a fat left tail.
- A positive correlation will have completely opposite effects — it creates a fat right tail and a thin left tail.

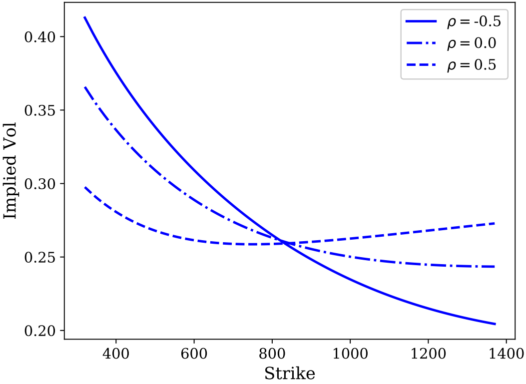
Behavior of Model Parameters – ρ



⇒ Positive correlation between stock and volatility is associated with positive skew in return distribution.

⇒ Negative correlation between stock and volatility is associated with negative skew in return distribution.

Behavior of Model Parameters – ρ



Negative correlation increases the price of out-of-the-money put options and decreases the price of out-of-the-money call options.

Behavior of Model Parameters – ν

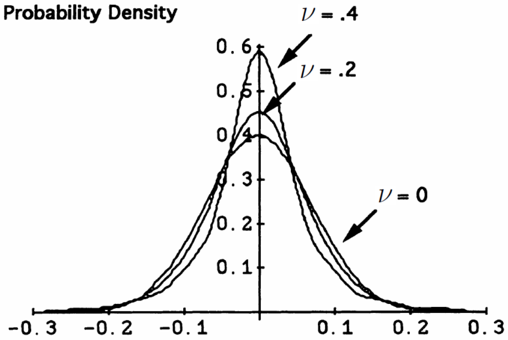
Implication on Distribution

- When the volatility of volatility parameter is 0, we are back to a normal log-return distribution (if $\beta = 0$).
- Otherwise, it increases the kurtosis of stock returns, creating two fat tails in both ends of the distribution.
- This has the effect of raising out-of-the-money puts and out-of-the-money call prices.

Implication on Pricing

- If volatility is uncorrelated with stock return, then increasing the volatility of volatility only increases the kurtosis of spot return.
- In this case, random volatility is associated with increases in the prices of far-from-the-money options relative to near-the-money options.
- In contrast, the correlation of volatility with the spot return produces skewness.

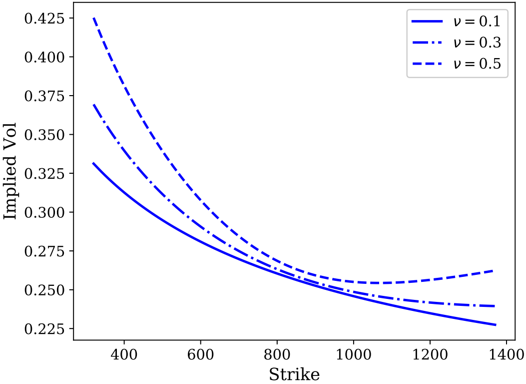
Behavior of Model Parameters – ν



⇒ Increasing volatility-of-volatility has the effect of increasing the kurtosis of return.

⇒ When the volatility-of-volatility parameter is 0, volatility will be deterministic.

Behavior of Model Parameters – ν



Larger volatility-of-volatility ν increases the price of out-of-the-money call and put options.