Date No.

Assignment 2

CHANJUNG KIM

1

a) $dS_t = rS_t dt + \sigma S_t dW_t$

$X_t = \log(S_t) = f(S_t) \quad f(x) = \log x,\ f'(x) = \frac{1}{x},\ f''(x) = -\frac{1}{x^2}$

Itô's formula :

$$dX_t = f'(S_t)\, dS_t + \frac{1}{2} f''(S_t)(dS_t)^2$$

$$= \frac{1}{S_t}(rS_t dt + \sigma S_t dW_t) - \frac{1}{2}\frac{1}{S_t^2}\sigma^2 S_t^2 dt$$

$$= rdt + \sigma dW_t - \frac{1}{2}\sigma^2 dt$$

$$dX_t = \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t$$

$\therefore$ If $Y_t = \log(S_t^2)$, $dY_t = (2r - \sigma^2)dt + 2\sigma dW_t$

b) $$\int_0^t dX_u = \int_0^t \left(r - \frac{1}{2}\sigma^2\right)du + \int_0^t \sigma dW_u$$

$$X_t - X_0 = \left(r - \frac{1}{2}\sigma^2\right)t + \sigma dW_t$$

$$\log\left(\frac{S_t}{S_0}\right) = \left(r - \frac{1}{2}\sigma^2\right)t + \sigma dW_t$$

$$S_t = S_0 \cdot e^{(r - \frac{1}{2}\sigma^2)t + \sigma dW_t}$$

$$S_t^2 = S_0^2\, e^{2(r - \frac{1}{2}\sigma^2)t + 2\sigma dW_t}$$

c) $$E[S_t] = E\left[S_0 \cdot e^{(r - \frac{1}{2}\sigma^2)t + \sigma dW_t}\right] \qquad W_t \sim \sqrt{t}\, N(0,1)$$

$$= S_0 \cdot e^{(r - \frac{1}{2}\sigma^2)t}\, E\left[e^{\sigma dW_t}\right]$$

From MGF of normal distribution, $E[e^{\sigma dW_t}] = e^{\frac{1}{2}\sigma^2 t}$

$$\therefore S_0 \cdot e^{(r - \frac{1}{2}\sigma^2)t}\, E\left[e^{\sigma dW_t}\right] = S_0 e^{rt}$$

$$E[S_t] = S e^{rt}$$

A'ZONE

$$E[S_t^2] = E[S_0^2 \, e^{2(r-\frac{1}{2}\sigma^2)t + 2\sigma dW_t}]$$

$$= S_0^2 \, e^{2(r-\frac{1}{2}\sigma^2)t} \, E[e^{2\sigma dW_t}]$$

here, $E[e^{2\sigma dW_t}] = e^{\frac{1}{2}4\sigma^2 t}$.

$$\therefore S_0^2 \cdot e^{2(r-\frac{1}{2}\sigma^2)t} \, E[e^{2\sigma dW_t}] = S_0^2 \cdot e^{2rt + \sigma^2 t}$$

$$E[S_t^2] = S^2 e^{(2r+\sigma^2)t}$$

2. $dS_t = rS_t dt + \sigma S_t dW_t$

Since risk-free rate is used for the growth rate of S_t, this SDE represents Q-measure world, and it's numeraire is the value of money-market account. B_t ($dB_t = rB_t dt$)

Under p-measure,

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$dB_t = rB_t dt$$

$$f(s,b) = \frac{s}{b}$$

$$\frac{\partial f}{\partial b} = -\frac{s}{b^2}, \quad \frac{\partial f}{\partial s} = \frac{1}{b}, \quad \frac{\partial^2 f}{\partial s^2} = 0$$

$$X_t = f(S_t, B_t) = \frac{S_t}{B_t}$$

Applying Itô's formula

$$dX_t = \frac{\partial f(S_t, B_t)}{\partial b} dB_t + \frac{\partial f(S_t, B_t)}{\partial S} dS_t + \frac{1}{2}\frac{\partial^2 f(S_t, B_t)}{\partial S^2}(dS_t)^2$$

$$= -\frac{S_t}{B_t^2} rB_t dt + \frac{1}{B_t}(\mu S_t dt + \sigma S_t dW_t)$$

$$dX_t = -rX_t dt + \mu X_t dt + \sigma X_t dW_t$$

$$= (\mu - r)X_t dt + \sigma X_t dW_t$$

$$= \sigma X_t \left(dW_t + \frac{\mu - r}{\sigma} dt \right)$$

$$= \sigma X_t dW_t^B$$

$\therefore \; dW_t^B = dW_t + \frac{\mu - r}{\sigma} dt$

$dW_t = dW_t^B - \frac{\mu - r}{\sigma} dt$

If we substitute dW_t of p-measure with $dW_t^B - \frac{\mu - r}{\sigma} dt$

In below equation, we will get Q-measure equation.

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$dS_t = \mu S_t dt + \sigma S_t \left(dW_t^B - \frac{\mu - r}{\sigma} dt \right)$$

$$= r S_t dt + \sigma S_t dW_t^B$$

3. Let $X_t = f\left(\frac{1}{S_t}\right) = \log \frac{1}{S_t}$ $\quad f'(x) = \frac{1}{x} \quad f''(x) = -\frac{1}{x^2}$

$$dX_t = S_t \, d\frac{1}{S_t} - \frac{1}{2} S_t^2 \left(d\frac{1}{S_t}\right)^2$$

$$= S_t \left(-r \cdot \frac{1}{S_t} dt - \sigma \frac{1}{S_t} dW_t^S \right) - \frac{1}{2} S_t^2 \sigma^2 \cdot \frac{1}{S_t^2} dt$$

$$dX_t = -r dt - \sigma dW_t^S - \frac{1}{2}\sigma^2 dt$$

$$= \left(-r - \frac{1}{2}\sigma^2\right) dt - \sigma dW_t^S$$

$$\int_0^T dX_t = \int_0^T \left(-r - \frac{1}{2}\sigma^2\right) dt - \int_0^T \sigma dW_t^S$$

$$X_T - X_0 = \left(-r - \frac{1}{2}\sigma^2\right) t - \sigma W_t^S$$

$$\log\left(\frac{\frac{1}{S_t}}{\frac{1}{S_0}}\right) = \left(-r - \frac{1}{2}\sigma^2\right) t - \sigma W_t^S$$

$$\frac{1}{S_t} = \frac{1}{S_0} \exp\left[\left(-r - \frac{1}{2}\sigma^2\right) t - \sigma W_t^S \right]$$

$$V_0 = S_0 \, \tilde{E}^S \left[\left(1 - k \frac{1}{S_T}\right)^+ \right]$$

$$= S_0 \, \tilde{E}^S \left[\left(1 - k \cdot \frac{1}{S_0} e^{\left(-r - \frac{1}{2}\sigma^2\right) t - \sigma W_t^S}\right)^+ \right]$$

$$= S_0 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(1 - \frac{k}{S_0} e^{(-r - \frac{1}{2}\sigma^2)t - \sigma\sqrt{T}x}\right)^+ e^{-\frac{x^2}{2}} dx \quad \because x \sim N(0,1)$$

here, $1 - \frac{k}{S_0} e^{(-r-\frac{1}{2}\sigma^2)t - \sigma\sqrt{T}x} > 0$

$$1 > \frac{k}{S_0} e^{(-r-\frac{1}{2}\sigma^2)t - \sigma\sqrt{T}x}$$

$$\frac{S_0}{k} > e^{(-r-\frac{1}{2}\sigma^2)t - \sigma\sqrt{T}x}$$

$$\log\left(\frac{S_0}{k}\right) > (-r - \frac{1}{2}\sigma^2)t - \sigma\sqrt{T}x$$

$$\frac{\log\left(\frac{S_0}{k}\right) + \left(r + \frac{1}{2}\sigma^2\right)t}{-\sigma\sqrt{T}} < x.$$

let $\frac{\log\left(\frac{S_0}{k}\right) + \left(r + \frac{1}{2}\sigma^2\right)t}{-\sigma\sqrt{T}} = x^*$

$$V_0 = S_0 \cdot \frac{1}{\sqrt{2\pi}} \int_{x^*}^{\infty} \left(1 - \frac{k}{S_0} e^{(-r-\frac{1}{2}\sigma^2)t - \sigma\sqrt{T}x}\right) e^{-\frac{x^2}{2}} dx$$

$$= S_0 \cdot \frac{1}{\sqrt{2\pi}} \left[\int_{x^*}^{\infty} e^{-\frac{x^2}{2}} dx - \int_{x^*}^{\infty} \frac{k}{S_0} e^{(-r-\frac{1}{2}\sigma^2)t - \sigma\sqrt{T}x} e^{-\frac{x^2}{2}} dx\right]$$

$$= S_0 \Phi(-x^*) - \frac{S_0}{\sqrt{2\pi}} \int_{x^*}^{\infty} \frac{k}{S_0} e^{(-r-\frac{1}{2}\sigma^2)t - \sigma\sqrt{T}x} e^{-\frac{x^2}{2}} dx$$

$$= S_0 \Phi(-x^*) - \frac{k}{\sqrt{2\pi}} e^{(-r-\frac{1}{2}\sigma^2)t} \int_{x^*}^{\infty} e^{-\sigma\sqrt{T}x - \frac{x^2}{2}} dx.$$

$$= S_0 \Phi(-x^*) - \frac{k}{\sqrt{2\pi}} e^{-rt} \int_{x^*}^{\infty} e^{-\frac{(x+\sigma\sqrt{T})^2}{2}} dx$$

substitute $y = x + \sigma\sqrt{T}$, $dy = dx$

$x = x^*$, $y = x^* + \sigma\sqrt{T}$.

$$V_0 = S_0 \Phi(-x^*) - k \cdot e^{-rt} \frac{1}{\sqrt{2\pi}} \int_{x^* + \sigma\sqrt{T}}^{\infty} e^{-\frac{y^2}{2}} dy$$

$$V_0 = S_0 \Phi(-x^*) - k \cdot e^{-rt} \Phi(-x^* - \sigma\sqrt{T})$$

$$V_0 = S_0 \Phi\left(\frac{\log\left(\frac{S_0}{k}\right) + \left(r + \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{T}}\right) - k e^{-rt} \Phi\left(\frac{\log\left(\frac{S_0}{k}\right) + \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{T}}\right)$$