Session 4: Equivalent Martingale Measure

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Intuition behind Measure Change

Suppose we have a normally distributed random variable $X_t \sim N(-\kappa t, t)$, its terminal distribution is given by

$$f(x) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{(x+\kappa t)^2}{2t}\right].$$

For any bounded function g, we have the expectation relationship

$$\mathbb{E}[g(X_t)] = \int_{-\infty}^{\infty} g(x) f(x) dx.$$

Now, we introduce a new probability density function

$$f'(x) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{x^2}{2t}\right],$$

and write the expectation as

$$\mathbb{E}[g(X_t)] = \int_{-\infty}^{\infty} g(x) \frac{f(x)}{f'(x)} f'(x) dx.$$

Intuition behind Measure Change

After some simple algebra, we can show that:

$$\frac{f(x)}{f'(x)} = \exp\left(-\kappa x - \frac{1}{2}\kappa^2 t\right).$$

Let's introduce the notation $Z = \frac{f(x)}{f'(x)}$, we see that it satisfies the following properties

1. $Z > 0$, and $\mathbb{E}'[Z] = 1$
2. $\mathbb{E}[g(X)] = \mathbb{E}'[g(X)Z]$, where $\mathbb{E}$ and $\mathbb{E}'$ denotes integration with respect to f and f', respectively.
3. Note that the two probability measures f and f' are equivalent.

Z is known either as the likelihood ratio, or more commonly the Radon-Nikodym derivative. In most finance applications, we change the mean of the distribution so that the process is a martingale.

Girsanov Theorem

Building on this intuition, suppose we have a $\mathbb{P}$-Brownian motion W_t, so that

$$\mathbb{E}^{\mathbb{P}}[e^{W_t}] = e^{\frac{t}{2}}$$

What can we say about $\mathbb{E}^{\mathbb{Q}}[e^{W_t}]$, where $\mathbb{Q}$ is an equivalent measure of $\mathbb{P}$?
Define the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\kappa W_t - \frac{1}{2}\kappa^2 t\right).$$

We can evaluate $\mathbb{E}^{\mathbb{Q}}[e^{W_t}]$ using the Radon-Nikodym derivative as follows:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{W_t}\right] = \mathbb{E}^{\mathbb{P}}\left[e^{W_t} e^{-\kappa W_t - \frac{1}{2}\kappa^2 t}\right] = e^{-\kappa t + \frac{1}{2}t}$$

We can identify this as the moment generating function of a normal $N(-\kappa t, t)$ distribution.

$\Rightarrow$ So the marginal distribution of W_t under $\mathbb{Q}$ is also a normal with variance t, but with a non-zero mean of $-\kappa t$.

Girsanov Theorem

Using our definition of $\frac{d\mathbb{Q}}{d\mathbb{P}}$, we can show that if W_t is a Brownian motion under $\mathbb{P}$, then it becomes a Brownian motion with constant drift $-\kappa$ under $\mathbb{Q}$, i.e. $\tilde{W}_t = W_t + \kappa t$. In addition, $\tilde{W}_t$ follows the following $\mathbb{Q}$-Brownian motion properties:

1. $\mathbb{E}^{\mathbb{Q}}[\tilde{W}_t] = 0$
2. $\mathbb{E}^{\mathbb{Q}}[e^{\theta \tilde{W}_t}] = e^{\frac{1}{2}\theta^2 t}$
3. $\mathbb{E}^{\mathbb{Q}}[e^{\theta(\tilde{W}_{t+s} - \tilde{W}_s)}|s] = e^{\frac{1}{2}\theta^2 t}$

Girsanov Theorem

If W_t is a $\mathbb{P}$-Brownian motion and κ satisfies $\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \kappa_t^2 dt\right)\right] < \infty$, then there exists a measure $\mathbb{Q}$ such that

1. $\mathbb{Q}$ is equivalent to $\mathbb{P}$
2. $\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \kappa_t dW_t - \frac{1}{2}\int_0^T \kappa_t^2 dt\right)$
3. $\tilde{W}_t = W_t + \int_0^t \kappa_u du$ is a $\mathbb{Q}$-Brownian motion.

Girsanov Theorem – Example

Example Consider the process for X_t given by

$$dX_t = \mu dt + \sigma dW_t,$$

where W is a $\mathbb{P}$-Brownian motion. Determine if there's a measure $\mathbb{Q}$ such that the drift of the process X_t under $\mathbb{Q}$ is νdt instead of μdt.

Solution We begin by writing

$$dX_t = \nu dt + \sigma \left(dW_t + \left(\frac{\mu - \nu}{\sigma} \right) dt \right).$$

Next if we let $\kappa = \frac{\mu - \nu}{\sigma}$, then Girsanov theorem states the existence of a new measure $\mathbb{Q}$ such that

$$\tilde{W}_t = W_t + \int_0^t \frac{\mu - \nu}{\sigma} du$$

is a $\mathbb{Q}$-Brownian motion. So the stochastic differential equation for X_t under $\mathbb{Q}$ is given by

$$dX_t = \nu dt + \sigma d\tilde{W}_t. \quad \triangleleft$$

Girsanov Theorem – Example

Example Consider a stochastic process X_t satisfying the following SDE

$$dX_t = \mu dt + \sigma dW_t, \quad X_0 = 0,$$

where W is a $\mathbb{P}$-Brownian motion. Change the measure so that X_t is a driftless process, and then evaluate $\mathbb{E}[X_t^2]$ under the $\mathbb{P}$ and $\mathbb{Q}$.

Solution First we rewrite our SDE in the following format

$$dX_t = \sigma \left(dW_t + \frac{\mu}{\sigma} dt \right).$$

Then we let $\kappa_t = \frac{\mu}{\sigma}$, and Girsanov states the existence of an equivalent measure $\mathbb{Q}$ under which

$$\tilde{W}_t = W_t + \frac{\mu}{\sigma} t$$

and $\tilde{W}$ is a $\mathbb{Q}$-Brownian motion, hence

$$dX_t = \sigma d\tilde{W}_t.$$

Finally, we evaluate the expectations

$$\mathbb{E}^{\mathbb{P}}[X_t^2] = \mu^2 t^2 + \sigma^2 t, \quad \mathbb{E}^{\mathbb{Q}}[X_t^2] = \sigma^2 t. \quad \triangleleft$$

Girsanov Theorem – Example

Example Consider a stochastic process X_t satisfying the following SDE

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

where W is a $\mathbb{P}$-Brownian motion. Change the measure so that the drift of X_t is ν instead of μ.

Solution Again, rewriting our SDE in the following format

$$dX_t = \nu X_t dt + \sigma X_t \left(dW_t + \frac{\mu - \nu}{\sigma} dt \right),$$

we let $\kappa_t = \frac{\mu - \nu}{\sigma}$, and apply Girsanov to get an equivalent measure $\mathbb{Q}$ under which

$$\tilde{W}_t = W_t + \frac{\mu - \nu}{\sigma} t$$

is a $\mathbb{Q}$-Brownian motion. The process X_t satisfies the following SDE under this new measure

$$dX_t = \nu X_t dt + \sigma X_t d\tilde{W}_t,$$

where $\tilde{W}$ is a $\mathbb{Q}$-Brownian motion. $\triangleleft$

I didn't see how to prove the formula but I decided to go ahead and use it to invest, because there was in 1967-68 an abundance of vastly overpriced OTC options.

I used the formula to sell short (naked) the most extremely overpriced options.

As it happened, small company stocks were up, so naked shorts of options were a disaster. Amazingly, I ended up breaking even overall...

— Ed Thorp

Source: Option Theory — What I knew and When I Knew It

Before Black-Scholes:

Various people developed models of derivatives that are actuarial in that they define the value of an option as the expected discounted value of its payoffs.

This value does of course depend on the volatility of the stock. But they don't know what rate of return to use for growing the stock price into the future, and they don't know what rate to use for discounting the payoffs.

People who wanted to use this model had to forecast the return of the stock and figure out what discount rate to use as a consequence of its risk. It was personal.

—

Emanuel Derman

Source: A Stylized History of Quantitative Finance

Black-Scholes (1971–3)

> *Hedge to eliminate stock risk from option. Require that hedged portfolio, which is riskless, earns the known riskless rate. Then we get the same formula for the option value as the actuarial one, but where all growth and discount rates are riskless rates.*
>
> *The value of the option does not depend on the expected return of the stock, since that has been hedged away. Instead it depends on the riskless rate, which is known, and on the future volatility of the stock.*

—

Emanuel Derman

Source: A Stylized History of Quantitative Finance

Before Black-Scholes:

$$\text{Call} = S_0 \Phi\left(\frac{\log \frac{S_0}{K} + \left(\mu + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log \frac{S_0}{K} + \left(\mu - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}\right)$$

Black-Scholes:

$$\text{Call} = S_0 \Phi\left(\frac{\log \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log \frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}\right)$$

1997 Nobel Prize citation:

> *Robert C. Merton and Myron S. Scholes have, in collaboration with the late Fischer Black, developed a pioneering formula for the valuation of stock options.*
>
> *Their methodology has paved the way for economic valuations in many areas.*
>
> *It has also generated new types of financial instruments and facilitated more efficient risk management in society.*

— The Royal Swedish Academy of Sciences

Black-Scholes Assumptions

The Black-Scholes market model contains 2 SDEs

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t \\ dB_t = r B_t dt \end{cases}$$

The context is that the market contains

1. A risky asset S_t, typically a stock price process.
2. A non-risky asset B_t, typically a risk-free bond.

Assumptions made include:

1. Underlying is lognormal with constant mean and variance.
2. The risk-free rate r is a constant.
3. No dividend is paid during the life of the option.
4. Short selling is permitted.
5. No risk-free arbitrage opportunities.
6. Trading is possible in continuous time.
7. No transaction costs, no taxes and no trading limits.

Original Black-Scholes Derivation

In their original argument, Fisher Black and Myron Scholes consider a portfolio that consists of a short position of one call option and a long position that consists of a continuously changing quantity of stock. The size of the stock position is chosen so that the position tracks the option position as closely as possible, and, in our earlier notation, the value of this portfolio is given by

$$\Pi = \frac{\partial C}{\partial S} \cdot S - C,$$

where C denotes the unknown value of the call option. Over a small time interval, the change in the portfolio value is expressed as

$$\Delta\Pi = \frac{\partial C}{\partial S} \cdot \Delta S - \Delta C.$$

Itô's formula us allows us to write

$$\Delta C = \frac{\partial C}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \Delta t + \frac{\partial C}{\partial t}\Delta t.$$

Original Black-Scholes Derivation

Substituting, we obtain

$$\Delta\Pi = -\left(\frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 + \frac{\partial C}{\partial t}\right)\Delta t.$$

Since this portfolio is tradable, and since the return of this hedged portfolio is deterministic, arbitrage argument tells us that this return must equal the return from a risk-free investment. In other words, Π is expected to grow at the risk-free rate r. Hence

$$r\Pi\Delta t = -\left(\frac{1}{2}\frac{\partial^2 C}{\partial S}\sigma^2 S^2 + \frac{\partial C}{\partial t}\right)\Delta t.$$

Cancelling and rearranging yields

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 = rC.$$

Original Black-Scholes Derivation

In this derivation, Black-Scholes supply the key ingredient in their formulation — dynamic hedging. If we apply the Δ (difference) operator to both sides of

$$\Pi = \frac{\partial C}{\partial S} \cdot S - C,$$

why do we not apply the operator to $\frac{\partial C}{\partial S}$ when we write:

$$\Delta\Pi = \frac{\partial C}{\partial S} \cdot \Delta S - \Delta C \ ?$$

Consider how one would actually implement the management of a portfolio such as that given by Π. In practice, we will choose a discrete sequence of times $t_1, t_2, \ldots, t_n$, at which to do our hedging. If we take this point of view, and if we interpret the Δ as the change over the interval $[t_i, t_{i+1}]$, based on the investments that are held at time t_i, then the meaning of the step is simply

$$\Pi_{t_{i+1}} - \Pi_{t_i} = \frac{\partial C_{t_i}}{\partial S_{t_i}} \cdot \left(S_{t_{i+1}} - S_{t_i}\right) - \left(C_{t_{i+1}} - C_{t_i}\right).$$

This interpretation brings up the concept of dynamic hedging.

Numeraire

As we've shown in the previous discussions, we can write down the expectation of the stock price S as

$$\mathbb{E}_t[S_{t+\Delta t}] = S_t e^{r\Delta t}.$$

The expectation notation "$\mathbb{E}_t$" is commonly used to denote conditional expectation on information given till time step t. This relationship can be rearranged into

$$S_t e^{-rt} = \mathbb{E}_t[S_{t+\Delta t} e^{-r(t+\Delta t)}].$$

- In words, this means that the best estimate of the price ratio on the subsequent time step is just the price ratio on the current time step.
- The securities in the denominator of the price ratio expression is called the numeraire security.
- The only requirement for a particular security to qualify as a numeraire security is that it has to be strictly positive at all times.
- The risk-free money market account paying an interest of r is a popular choice of numeraire.

Equivalent Martingale Measure

- In a complete market, any derivative security is attainable. Since we can hedge a derivative product perfectly, the derivative security loses its randomness and behaves like a risk-less bond.
- So real world probabilities do not come into the picture in a risk-neutral valuation framework at all.
- If we hedge according to our risk-neutral valuation framework, then all risk is eliminated, and the hedged portfolio grows at a risk-free rate.
- Consequently, the hedged portfolio divided by the risk-free rate is a martingale.
- Two probabilities measures are equivalent if they agree on what is possible and what is impossible.

Equivalent Martingale Measure

- In other words, if one portfolio is an arbitrage in one measure, then it is an arbitrage in all other equivalent measures.
- If the option price we determined under the risk-neutral measure is arbitrage-free, then it is arbitrage-free in the real world.
- If we can express security price processes discounted by a numeraire security as a martingale, then there can be no arbitrage opportunities.
- Under the risk-neutral probabilities associated to this numeraire security, the option price is also a martingale, and we can therefore determine it's present value.
- The risk-free money market account B_t is a common choice for numeraire (used by Harrison and Kreps (1979)), but the choice is arbitrary.

Application of Measure Change – Black-Scholes

In a Black-Scholes economy, there are 2 marketed assets: a riskless money-market account compounding at the risk-free rate, and a stock. Let B_t denote the value of the money-market account with $B_0 = 1$, and let S_t denote the stock price process. The following equations described their dynamics:

$$\begin{aligned} dB_t &= rB_t dt \\ dS_t &= \mu S_t dt + \sigma S_t dW_t. \end{aligned}$$

Recall the martingale pricing theory we've outlined in earlier discussions. The value of B_t is strictly positive and can be used as a numeraire. Define the relative price process $X_t = \frac{S_t}{B_t}$, we can apply Itô's formula to obtain

$$dX_t = (\mu - r)X_t dt + \sigma X_t dW_t.$$

To identify the equivalent martingale measure we apply Girsanov's theorem with $\kappa = \frac{\mu - r}{\sigma}$ to obtain the new measure

$$dW_t^B = dW_t + \frac{\mu - r}{\sigma} dt,$$

where W_t^B is also a Brownian motion.

Application of Measure Change – Black-Scholes

We've used the notation to explicitly state the numeraire associated to our martingale measure. This allows us to write

$$\begin{aligned} dX_t &= (\mu - r)X_t dt + \sigma X_t \left(dW_t^B - \frac{\mu - r}{\sigma} dt \right) \\ &= \sigma X_t dW_t^B. \end{aligned}$$

This is the only measure which turns the relative price process into martingale. We can now determine what is the stock price process under this unique martingale measure $\mathbb{Q}^B$:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S \left(dW_t^B - \frac{\mu - r}{\sigma} dt \right) \\ &= r S_t dt + \sigma S_t dW_t^B. \end{aligned}$$

Under the equivalent martingale measure, the drift of the stock μ is irrelevant and is replaced by the risk-free interest rate r. The solution to this stochastic differential equation is

$$S_t = S_0 \exp \left[\left(r - \frac{1}{2}\sigma^2 \right) t + \sigma W_t^B \right].$$

Application of Measure Change – Black-Scholes

A European call option with strike K and maturing at time T where $V_T = (S_T - K)^+$ can be evaluated by martingale pricing theorem as follow

$$
\begin{aligned}
V_0 &= \mathbb{E}^B\left[\frac{(S_T - K)^+}{B_T}\right] \\
&= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}x} - K\right]^+ e^{-\frac{x^2}{2}}\, dx \\
&= S_0\Phi(d_1) - Ke^{-rT}\Phi(d_1 - \sigma\sqrt{T}), \quad d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.
\end{aligned}
$$

Binomial Tree and Martingale Measure

Example Consider a binomial tree model with $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$, $r = 25\%$. Interest rate is discretely compounded. Determine the risk-neutral probabilities under the risk-neutral measure with

1. the risk-free money market account as the numeraire security, and
2. the stock as the numeraire security.

Then determine the price of a European call option (V_0) struck at 10 on the 3^{rd} time step in both cases.

Ans:

1. $p^* = \frac{1}{2}$
2. $p^* = 0.8$

$V_0 = \frac{176}{125}$ in both cases.

Application of Measure Change – Black-Scholes

As we have seen earlier, the choice of numeraire is arbitrary, and that we don't always have to go for the obvious choice of picking the risk-free bond as numeraire. The stock price S_t is also strictly positive and can be used as a numeraire. Define the process $Y_t = \frac{B_t}{S_t}$, we apply Itô's formula to obtain

$$dY_t = (r - \mu + \sigma^2)Y_t dt - \sigma Y_t dW_t.$$

Next apply Girsanov's theorem with $\kappa = -\frac{(r-\mu)}{\sigma} - \sigma$, we have

$$dW_t^S = dW_t - \left(\frac{r-\mu}{\sigma} + \sigma\right) dt.$$

We have therefore identify the martingale measure for the process Y_t associated to the numeraire S_t. Itô's formula and Girsanov's theorem give us the inverted stock price process

$$\begin{aligned} d\frac{1}{S_t} &= (-\mu + \sigma^2)\frac{1}{S_t}dt - \sigma\frac{1}{S_t}dW_t \\ &= (-\mu + \sigma^2)\frac{1}{S_t}dt - \sigma\frac{1}{S_t}\left[dW_t^S + \left(\frac{r-\mu}{\sigma} + \sigma\right)dt\right] \\ &= -r\frac{1}{S_t}dt - \sigma\frac{1}{S_t}dW_t^S. \end{aligned}$$

Application of Measure Change – Black-Scholes

The solution to this stochastic differential equation is given by

$$\frac{1}{S_t} = \frac{1}{S_0} \exp\left[\left(-r - \frac{1}{2}\sigma^2\right)t - \sigma W_t^S\right].$$

Using our martingale pricing relationship, we have

$$\frac{V_0}{S_0} = \mathbb{E}^S\left[\frac{(S_T - K)^+}{S_T}\right]$$

$$\Rightarrow \quad V_0 = S_0 \mathbb{E}^S\left[\left(1 - K\frac{1}{S_T}\right)^+\right].$$