pg 8.

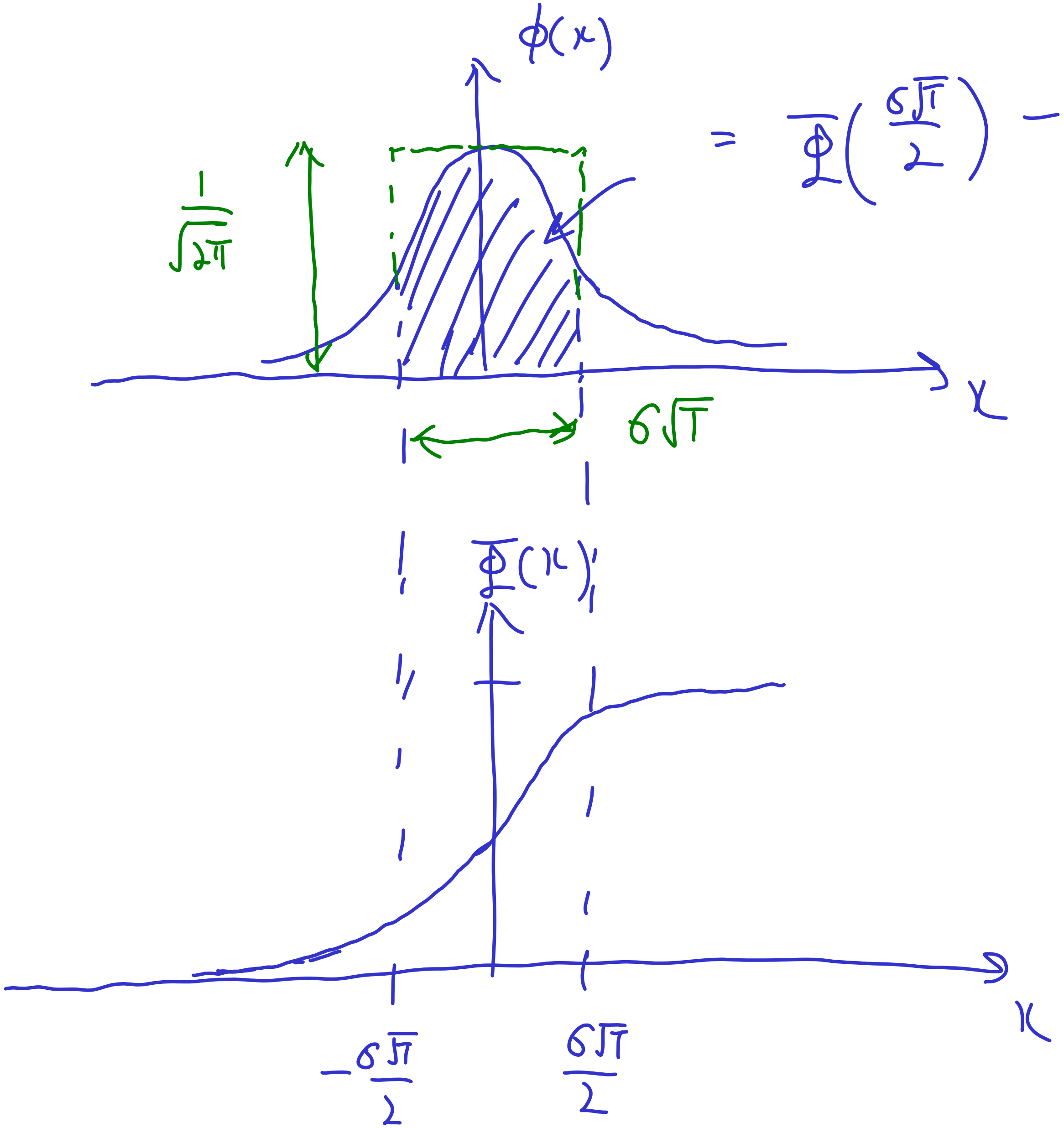
$$V_0^c = S_0\, \Phi\left(\frac{\log \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) - Ke^{-rT}\, \Phi\left(\frac{\log \frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right)$$

$r \simeq 0, \quad S_0 = K$:

$$V_0^c \simeq S_0\, \Phi\left(\frac{\sigma\sqrt{T}}{2}\right) - S_0\, e^{-0}\, \Phi\left(-\frac{\sigma\sqrt{T}}{2}\right)$$

$$= S_0 \left[\Phi\left(\frac{\sigma\sqrt{T}}{2}\right) - \Phi\left(-\frac{\sigma\sqrt{T}}{2}\right) \right]$$

$$\simeq S_0 \cdot \frac{\sigma\sqrt{T}}{\sqrt{2\pi}}$$



$$= \Phi\left(\frac{\sigma\sqrt{\pi}}{2}\right) - \Phi\left(-\frac{\sigma\sqrt{\pi}}{2}\right) \simeq \frac{\sigma\sqrt{\pi}}{\sqrt{2\pi}}$$

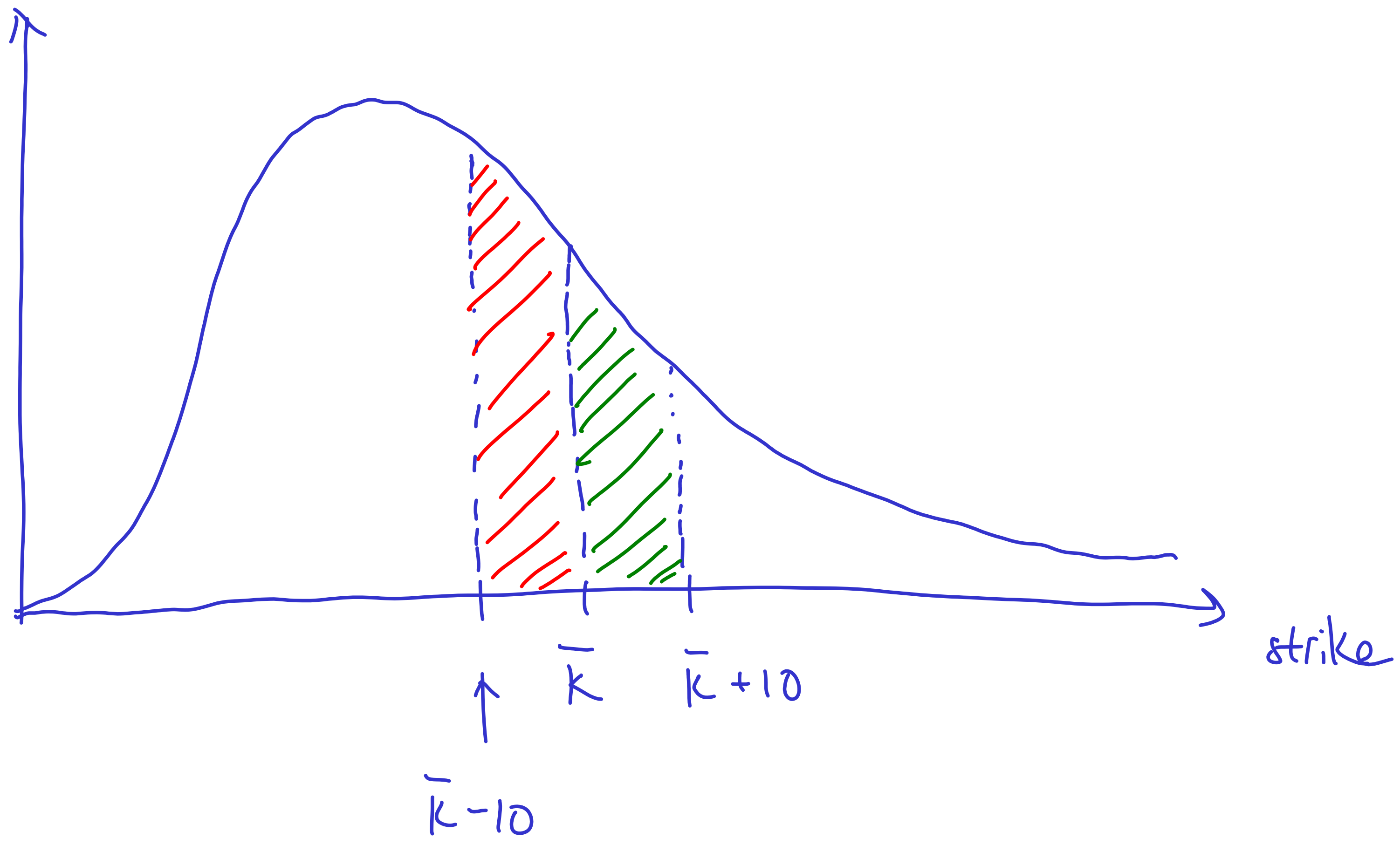
pg10.

$$V^c - V^p = S - Ke^{-rT} = 0$$

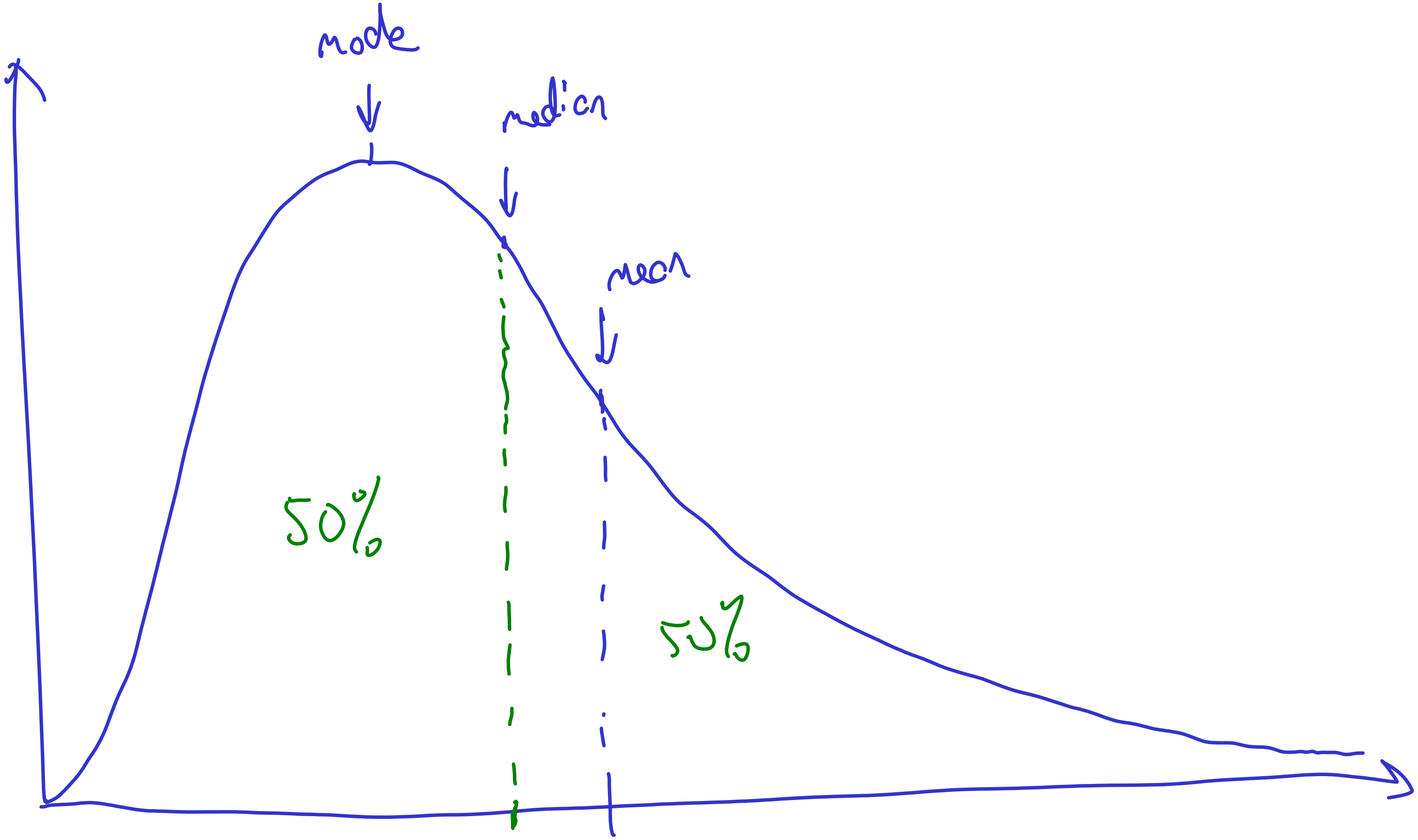
①

$$\Rightarrow \quad K = Se^{rT} = \bar{K}$$

②



3

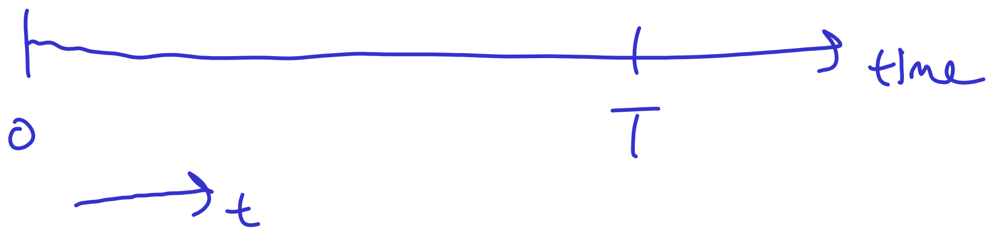


$$\mathbb{E}[S_T] = \bar{K}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{>50\%}\quad\underbrace{\qquad\qquad\qquad\qquad}_{<50\%}$$

pg 11.

$$F_0 = S_0 e^{rT} \quad , \quad F_t = S_t e^{r(T-t)} = f(t, S_t)$$



$$\frac{\partial f}{\partial t} = -x r e^{r(T-t)} \quad , \quad \frac{\partial f}{\partial x} = e^{r(T-t)} \quad , \quad \frac{\partial^2 f}{\partial x^2} = 0$$

Itô's formula: $$dF_t = \frac{\partial f(t, S_t)}{\partial t} dt + \frac{\partial f(t, S_t)}{\partial x} dS_t + \frac{1}{2} \frac{\partial^2 f(t, S_t)}{\partial x^2} (dS_t)^2$$

$$= \cancel{-r S_t e^{r(T-t)} dt} + e^{r(T-t)} \left[\cancel{r S_t dt} + \sigma S_t dW_t \right] + 0$$

$$dF_t = \sigma S_t e^{r(T-t)} dW_t = \sigma F_t dW_t$$

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t \longrightarrow S_T = S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma W_T}$$

$$dF_t = \sigma F_t\,dW_t \longrightarrow F_T = F_0 e^{-\frac{\sigma^2 T}{2} + \sigma W_T}$$

$$\text{BlackScholes}(S_0, K, r, \sigma, T)$$

$$\text{Black76}(F_0, K, \sigma, T)$$

pg14.

$$dF_t = \sigma[\beta F_t + (1-\beta) F_0] dW_t$$

if $\beta = 1$: $dF_t = \sigma F_t dW_t$

if $\beta = 0$: $dF_t = \sigma F_0 dW_t$

$$0 < \beta < 1$$

$$\Rightarrow \quad dF_t = \underbrace{\sigma \beta F_t dW_t}_{\text{lognormal}} + \underbrace{\sigma(1-\beta) F_0 dW_t}_{\text{normal}}$$

$$dF_t = \sigma\left[\beta F_t + (1-\beta)F_0\right]dW_t$$

$$X_t = \log\left[\beta F_t + (1-\beta)F_0\right] = f(F_t)$$

$$f'(F_t) = \frac{\beta}{\beta F_t + (1-\beta)F_0}, \quad f''(F_t) = -\frac{\beta^2}{\left[\beta F_t + (1-\beta)F_0\right]^2}$$

Itô's formula:

$$dX_t = f'(F_t)dF_t + \frac{1}{2}f''(F_t)\left(dF_t\right)^2 \qquad dt$$

$$= \frac{\beta}{\beta F_t + (1-\beta)F_0}\cdot\sigma\left[\beta F_t + (1-\beta)F_0\right]dW_t - \frac{1}{2}\frac{\beta^2}{\left[\beta F_t + (1-\beta)F_0\right]^2}\sigma^2\left[\beta F_t + (1-\beta)F_0\right]^2$$

$$= \beta\sigma\, dW_t - \frac{\beta^2\sigma^2}{2}dt$$

$$dX_t = \beta\sigma\, dW_t - \frac{\beta^2\sigma^2}{2} dt$$

$$\int_0^T dX_t = \int_0^T \beta\sigma\, dW_t - \int_0^T \frac{\beta^2\sigma^2}{2} dt$$

$$X_T - X_0 = \beta\sigma W_T - \frac{\beta^2\sigma^2}{2} T$$

$$\log\left[\beta F_T + (1-\beta)F_0\right] - \log\left[\beta F_0 + (1-\beta)F_0\right] = \beta\sigma W_T - \frac{\beta^2\sigma^2 T}{2}$$

$$\log\left[\frac{\beta F_T + (1-\beta)F_0}{F_0}\right] = \beta\sigma W_T - \frac{\beta^2\sigma^2 T}{2}$$

$$\frac{\beta F_T + (1-\beta)F_0}{F_0} = e^{-\frac{\beta^2\sigma^2 T}{2} + \beta\sigma W_T}$$

$$\beta F_T + (1-\beta) F_0 = F_0 e^{-\frac{\beta^2 \sigma^2 T}{2} + \beta \sigma W_T}$$

$$F_T = \frac{F_0}{\beta} e^{-\frac{\beta^2 \sigma^2 T}{2} + \beta \sigma W_T} - \left(\frac{1-\beta}{\beta}\right) F_0$$

Compared with : $F_T = F_0 e^{-\frac{\sigma^2 T}{2} + \sigma W_T}$

$$\text{DisplacedDiffusion}(F_0, K, \sigma, \beta, T)$$

$$= \text{Black76}\left(\frac{F_0}{\beta}, K + \left(\frac{1-\beta}{\beta}\right) F_0, \sigma\beta, T\right)$$

BlackScholes : $$S_0\,\Phi\left(\frac{\log\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)-Ke^{-rT}\,\Phi\left(\frac{\log\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

Black76: $$e^{-rT}F_0\,\Phi\left(\frac{\log\frac{F_0}{K}+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)-e^{-rT}K\,\Phi\left(\frac{\log\frac{F_0}{K}-\frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}\right)$$

$$F_0=S_0e^{rT}$$

$$\Rightarrow\quad S_0\,\Phi\left(\frac{\log\frac{S_0e^{rT}}{K}+\frac{\sigma^2T}{2}}{\sigma\sqrt{T}}\right)-Ke^{-rT}\,\Phi\left(\frac{\log\frac{S_0e^{rT}}{K}-\frac{\sigma^2T}{2}}{\sigma\sqrt{T}}\right)$$

$$\Rightarrow \quad S_0 \Phi\left(\frac{\log\frac{S_0}{K} + rT + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\frac{S_0}{K} + rT - \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}\right)$$

$$\log\left(\frac{S_0}{K} \times e^{rT}\right) = \log\frac{S_0}{K} + \log e^{rT}$$

pg16

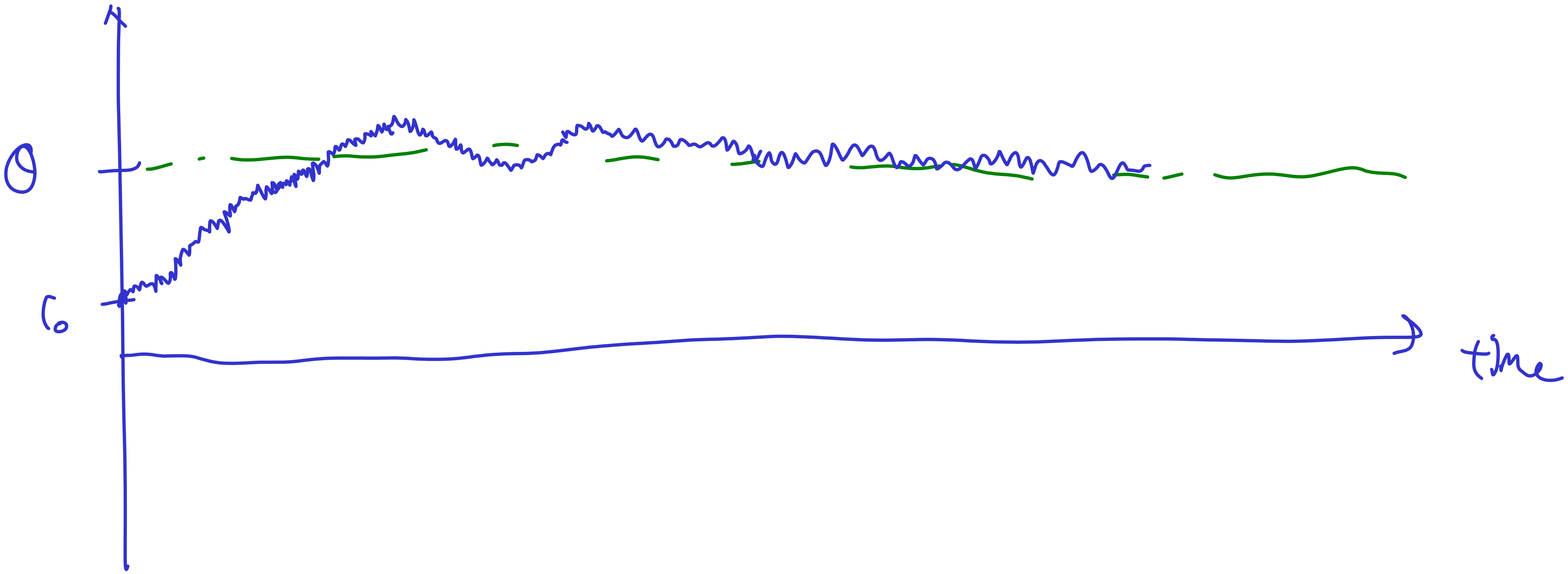
$$dr_t = \kappa(\theta - r_t)\,dt + \underbrace{\sigma\, dW_t}_{\text{arithmetic Brownian motion}}$$

long-run average

mean reversion speed

$$\mathbb{E}[dr_t] = \kappa(\theta - \mathbb{E}[r_t])\,dt + 0$$

$$\begin{cases} r_t = \theta & : \Rightarrow \mathbb{E}[dr_t] = 0 \\ r_t < \theta & : \Rightarrow \mathbb{E}[dr_t] > 0 \\ r_t > \theta & : \Rightarrow \mathbb{E}[dr_t] < 0 \end{cases}$$



θ
r_0
time

pg16.

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t$$

$$f(t,x) = xe^{\kappa t}$$

$$\frac{\partial f}{\partial t} = \kappa x e^{\kappa t}, \quad \frac{\partial f}{\partial x} = e^{\kappa t}, \quad \frac{\partial^2 f}{\partial x^2} = 0$$

$$X_t = e^{\kappa t} r_t = f(t, r_t)$$

Itô's formula:

$$dX_t = \frac{\partial f(t,r_t)}{\partial t} dt + \frac{\partial f(t,r_t)}{\partial x} dr_t + \frac{1}{2} \cancel{\frac{\partial^2 f(t,r_t)}{\partial x^2}}^{0} (dr_t)^2$$

$$= \cancel{\kappa r_t e^{\kappa t} dt} + e^{\kappa t}\left[\kappa(\theta - \cancel{r_t})dt + \sigma dW_t\right] + 0$$

$$dX_t = \theta \kappa e^{\kappa t} dt + \sigma e^{\kappa t} dW_t$$

$$\int_0^t dX_u = \kappa\theta \int_0^t e^{\kappa u} du + \sigma \int_0^t e^{\kappa u} dW_u$$

$$X_t - X_0 = \theta \cdot \left[e^{ku} \right]_0^t + \sigma \int_0^t e^{ku} \, dW_u$$

$$r_t e^{kt} - r_0 e^0 = \theta \left(e^{kt} - e^0 \right) + \sigma \int_0^t e^{ku} \, dW_u$$

$$r_t = r_0 e^{-kt} + \theta \left(1 - e^{-kt} \right) + \sigma \int_0^t e^{k(u-t)} \, dW_u$$

pg3.

$$\frac{f(x)}{f'(x)} = \frac{\cancel{\frac{1}{\sqrt{2\pi t}}}\, e^{-\frac{(x+kt)^2}{2t}}}{\cancel{\frac{1}{\sqrt{2\pi t}}}\, e^{-\frac{x^2}{2t}}}$$

$$= \exp\left[-\frac{(x+kt)^2}{2t} + \frac{x^2}{2t}\right]$$

$$= \exp\left[-\frac{x^2+2xkt+k^2t^2}{2t} + \frac{x^2}{2t}\right]$$

$$= \exp\left[-kx - \frac{k^2t}{2}\right]$$

$$\mathbb{E}\left[g(x)\right] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx = \int_{-\infty}^{\infty} g(x) \frac{f(x)}{f'(x)} f'(x)\, dx$$

$$= \mathbb{E}'\left[g(x) \cdot Z \right]$$

<u>pg 4.</u>

$W_t \rightarrow \mathbb{P}$ - Brownian

$$\mathbb{E}^{Q}\left[e^{W_t}\right] = \mathbb{E}^{P}\left[e^{W_t} \cdot \frac{dQ}{dP}\right]$$

$$= \mathbb{E}^{P}\left[e^{W_t} \cdot e^{-kW_t - \frac{1}{2}k^2 t}\right]$$

$$= \mathbb{E}^{P}\left[e^{(1-k)W_t}\right] e^{-\frac{1}{2}k^2 t}$$

$$= e^{(1-k)^2 \cdot \frac{t}{2}} \quad e^{-\frac{1}{2}k^2 t}$$

$$= e^{\frac{t}{2} - kt + \frac{k^2 t}{2} - \frac{k^2 t}{2}} = e^{\frac{t}{2} - kt}$$

Pg 5.

$W_t \rightarrow \mathbb{P}$ -Brownian

$\tilde{W}_t \rightarrow \mathbb{Q}$ -Browian

$$\mathbb{E}^{P}[W_t] = 0 \quad , \quad \mathbb{E}^{Q}[\tilde{W}_t] = 0$$

$$\mathbb{E}^{Q}[W_t] = \mathbb{E}^{Q}[\tilde{W}_t - kt] = -kt$$

$$\mathbb{E}^{P}[\tilde{W}_t] = \mathbb{E}^{P}[W_t + kt] = kt$$

pg6.

$$dX_t = \mu\, dt + \sigma\, dW_t \quad , \quad W_t \to \mathbb{P}\text{-Brownian}$$

$$dX_t = \nu\, dt - \nu\, dt + \mu\, dt + \sigma\, dW_t$$

$$= \nu\, dt + (\mu - \nu)\, dt + \sigma\, dW_t$$

$$= \nu\, dt + \sigma \left(dW_t + \frac{\mu - \nu}{\sigma}\, dt \right)$$

$$= \nu\, dt + \sigma\, d\tilde{W}_t$$

Pg 7.

$$dX_t = \mu\, dt + \sigma\, dW_t, \qquad W_t \rightarrow \mathbb{P}\text{-Brownian}$$

$$= \sigma\left(dW_t + \frac{\mu}{\sigma}\, dt\right)$$

$$= \sigma\, d\tilde{W}_t, \qquad \tilde{W}_t \rightarrow \mathbb{Q}\text{-Brownian}$$