pg3.

$X_i \sim \text{iid}$

$$\mathbb{E}[X_i] = \frac{1}{2} \times (+1) + \frac{1}{2} \times (-1) = 0$$

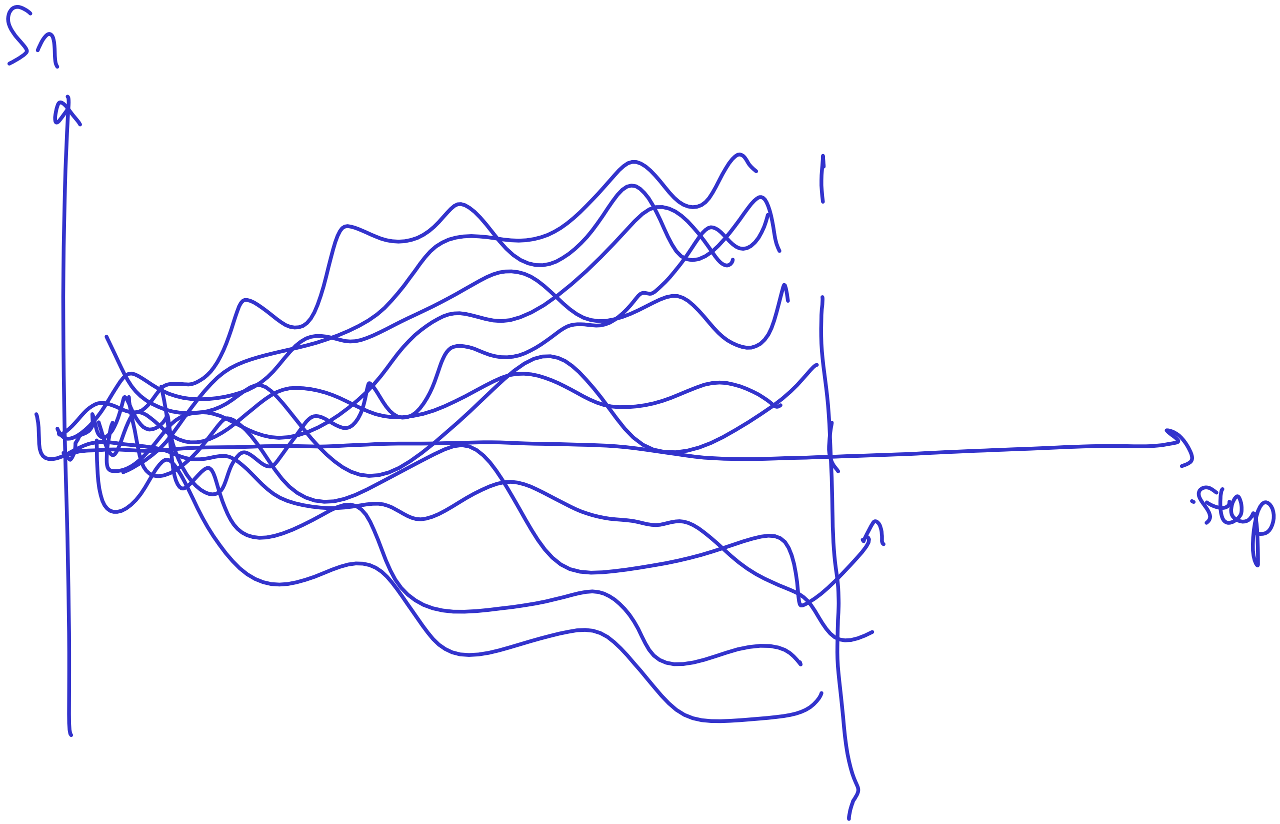
$$V[X_i] = \frac{1}{2} \times (+1)^2 + \frac{1}{2} \times (-1)^2 - 0^2 = 1$$

$$V[S_n] = V[X_1 + X_2 + X_3 + \dots + X_n]$$

$$= V[X_1] + V[X_2] + \dots + V[X_n]$$

$$= 1 + 1 + \dots + 1$$

$$= n$$



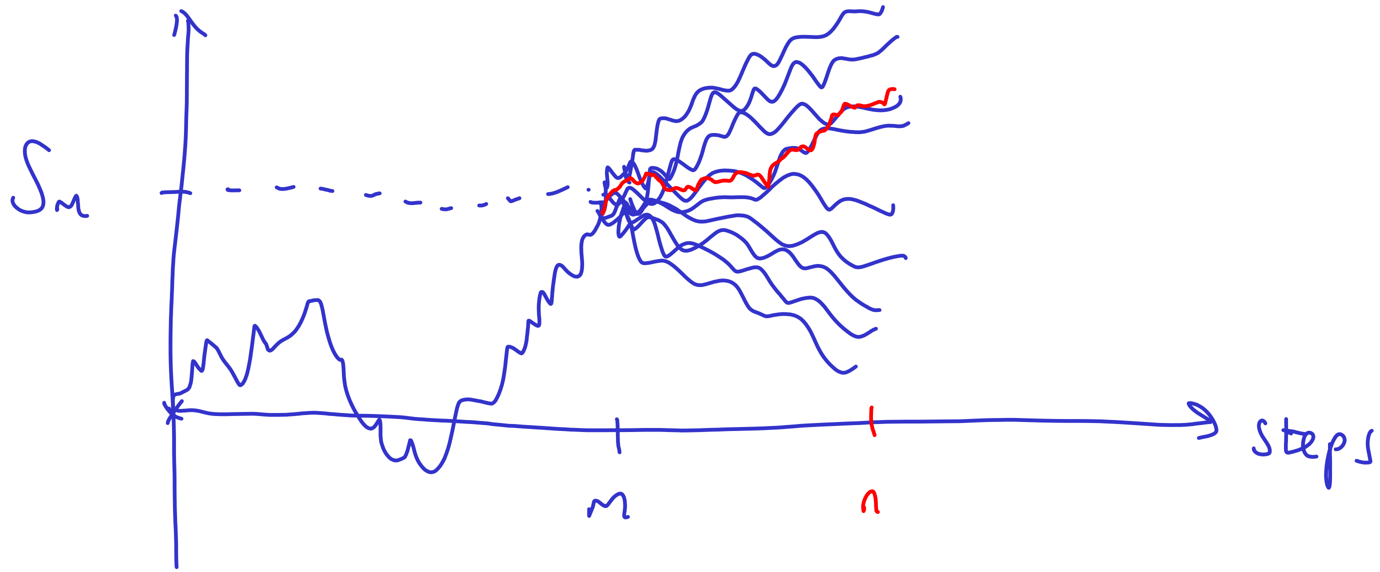
$$\mathbb{E}[S_n] = 0$$

$$V[S_n] = n$$

$$\mathbb{E}_m[S_n] = \mathbb{E}_m\left[\sum_{i=1}^{n} X_i\right]$$

$$= \mathbb{E}_m\left[S_m + \sum_{i=m+1}^{n} X_i\right]$$

$$= S_m + \mathbb{E}_m\left[\sum_{i=m+1}^{n} X_i\right] = S_m + 0$$



$$\mathrm{Cov}(S_n, S_m) = \mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{m} X_i\right)$$

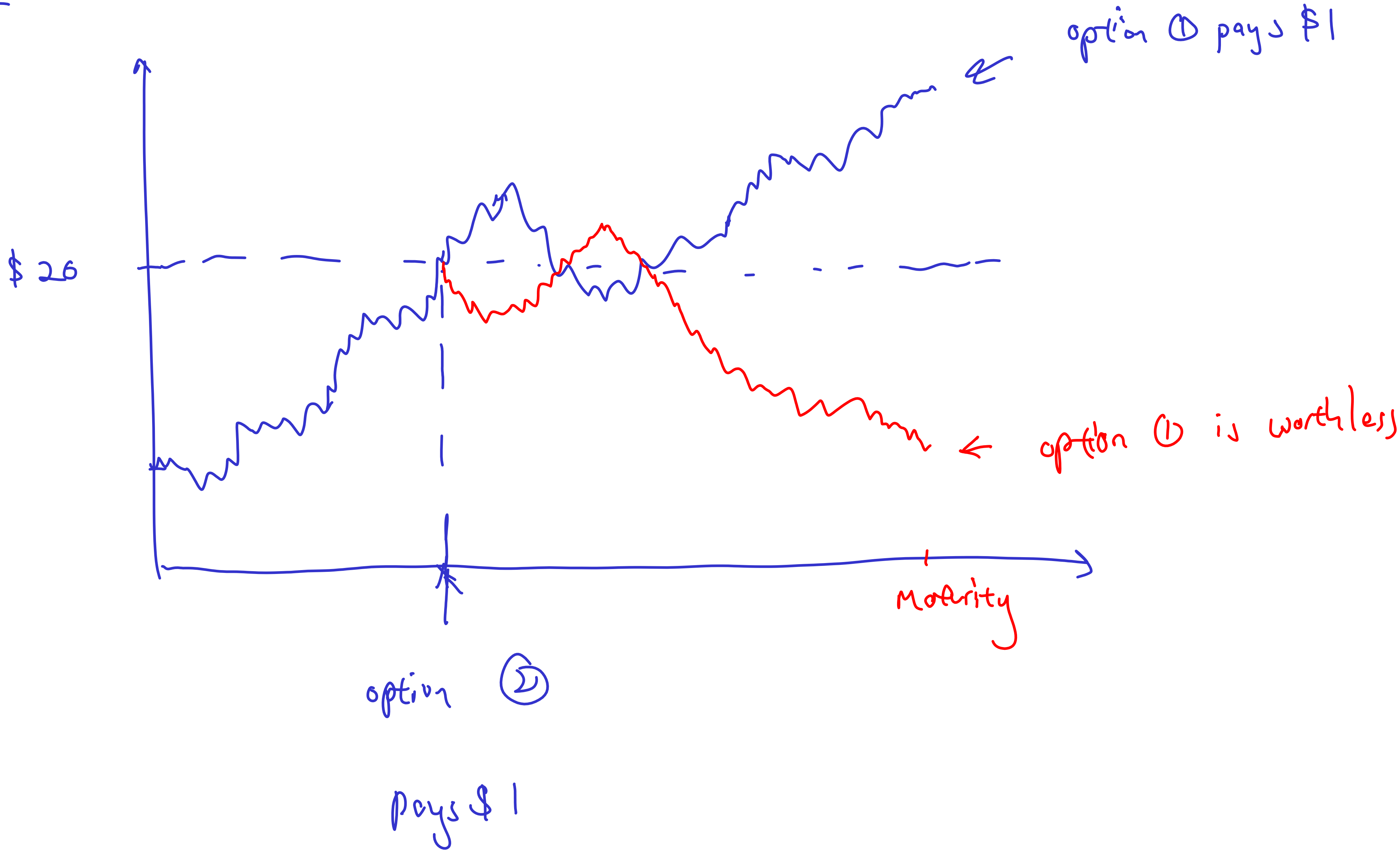
$$= \mathrm{Cov}\left(\sum_{i=1}^{m} X_i + \sum_{i=m+1}^{n} X_i, \sum_{i=1}^{m} X_i\right)$$

$$= \mathrm{Cov}\left(\sum_{i=1}^{m} X_i, \sum_{i=1}^{m} X_i\right) + \mathrm{Cov}\left(\sum_{i=m+1}^{n} X_i, \sum_{i=1}^{m} X_i\right)$$

$$= V[S_m] + 0$$

$$= m$$

pg6.


option ① pays $1
$26
option ① is worthless
maturity
option ② pays $1

pg 12.

$$W_n(0) = 0$$

$$W_n\left(\frac{1}{n}\right) = W_n(0) + \frac{X_1}{\sqrt{n}}$$

$$W_n\left(\frac{2}{n}\right) = W_n\left(\frac{1}{n}\right) + \frac{X_2}{\sqrt{n}} = \frac{\sum_{i=1}^{2} X_i}{\sqrt{n}}$$

$$W_n\left(\frac{3}{n}\right) = W_n\left(\frac{2}{n}\right) + \frac{X_3}{\sqrt{n}} = \frac{\sum_{i=1}^{3} X_i}{\sqrt{n}}$$

$$\vdots$$

let $t = \frac{i}{n}$, $\Rightarrow W_n(t) = \dfrac{\sum_{i=1}^{nt} X_i}{\sqrt{n}}$

$$W_n(t) = \sqrt{t} \times \frac{\sum_{i=1}^{nt} X_i}{\sqrt{nt}} \xrightarrow[n\to\infty]{CLT} \sqrt{t} \cdot N(0,1)$$

$$\Downarrow$$

$$N(0,t)$$

$$\mathbb{E}\left[\frac{\sum_{i=1}^{nt} X_i}{\sqrt{nt}}\right] = 0$$

$$V\left[\frac{\sum_{i=1}^{nt} X_i}{\sqrt{nt}}\right] = \frac{1}{nt} V\left[\sum_{i=1}^{nt} X_i\right] = 1$$

pg15.

$$Z \sim N(0,1)$$

$$X_t = \sqrt{t} \cdot Z$$

① $X_0 = \sqrt{0} \times Z = 0$ ✓

② $X_t = \sqrt{t}\, Z \sim \sqrt{t}\, N(0,1) \sim N(0,t)$ ✓

③ $X_{s+t} - X_s = \sqrt{s+t}\, N(0,1) - \sqrt{s} \cdot N(0,1)$

$$\sim \left(\sqrt{s+t} - \sqrt{s}\right) \cdot N(0,1)$$

$$\sim N\left(0, \left(\sqrt{s+t} - \sqrt{s}\right)^2\right) \sim N\left(0, (s+t) - 2\sqrt{s+t}\sqrt{s} + s\right)$$

Alternatively:

3

$$\mathrm{Cov}(X_{t+s}, X_s) = \mathrm{Cov}(\sqrt{t+s} \cdot Z, \sqrt{s} \cdot Z)$$

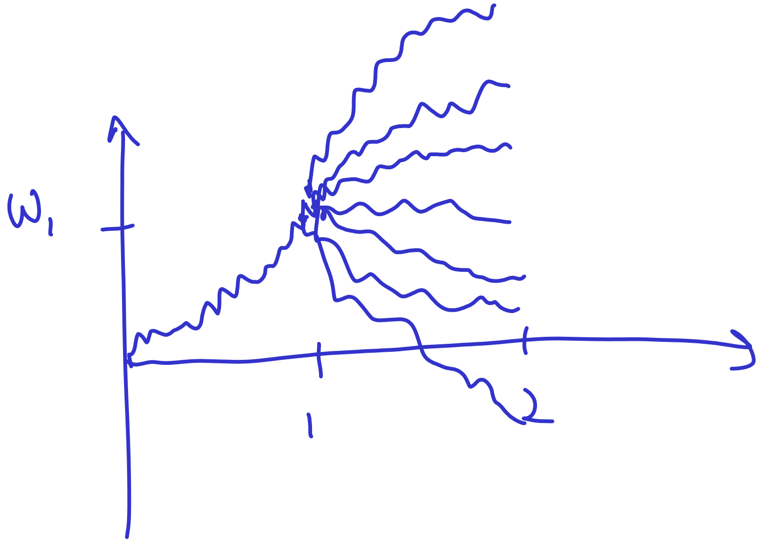
$$= \mathbb{E}\left[\sqrt{t+s}\sqrt{s} \cdot Z^2\right] - 0 \times 0$$

$$= \sqrt{(t+s)s} \; \mathbb{E}[Z^2]$$

$$= \sqrt{(t+s)s} \neq s$$

pg19.

$$\mathbb{P}\left(W_2 < 0 \,\middle|\, W_1 > 0\right) = \mathbb{P}\left(W_2 \text{ moves down}\right) \times \mathbb{P}\left(|W_2 - W_1| > |W_1 - W_0|\right)$$



$$= \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{4}.$$

$$\therefore \quad \mathbb{P}\left(W_2 > 0 \,\middle|\, W_1 > 0\right) = \frac{3}{4}$$

Q19.

$$\mathbb{P}(W_1 \times W_2 > 0) = \mathbb{P}(W_1 > 0, W_2 > 0) + \mathbb{P}(W_1 < 0, W_2 < 0)$$

$$\mathbb{P}(W_1 > 0, W_2 > 0) = \mathbb{P}(W_2 > 0 \mid W_1 > 0)\,\mathbb{P}(W_1 > 0)$$

$$= \frac{3}{4} \times \frac{1}{2}$$

$$= \frac{3}{8}$$

$$\mathbb{P}(W_1 < 0, W_2 < 0) = \frac{3}{8}$$

$$\therefore \mathbb{P}(W_1 \times W_2 > 0) = \frac{3}{8} + \frac{3}{8} = \frac{3}{4}$$

pg 21.

$$\mathbb{E}_s\left[W_t^2 - t\right] = \mathbb{E}_s\left[W_t^2\right] - t$$

$$= \mathbb{E}_s\left[\left(\underbrace{W_t - W_s} + W_s\right)^2\right] - t$$

$$= \mathbb{E}_s\left[\left(W_t - W_s\right)^2 + 2\left(W_t - W_s\right)W_s + W_s^2\right] - t$$

$$= (t-s) + 2 \times 0 \times W_s + W_s^2 - t$$

$$= W_s^2 - s$$

pg23.

$$X \sim N(\mu, \sigma^2)$$

$$\mathbb{E}\left[e^{\theta X}\right] = e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2}$$

$$W_t \sim N(0, t)$$

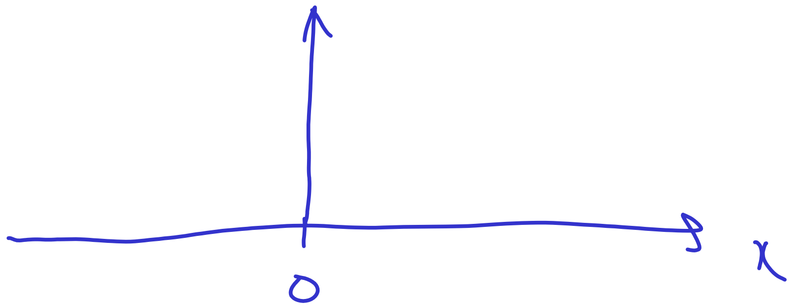
$$\mathbb{E}\left[e^{W_t}\right] = e^{0 + \frac{1}{2}\cdot t} = e^{\frac{t}{2}}$$

$$\mathbb{E}\left[e^{\theta W_t}\right] = e^{0 + \frac{1}{2}\cdot t\cdot\theta^2} = e^{\frac{t\theta^2}{2}}$$

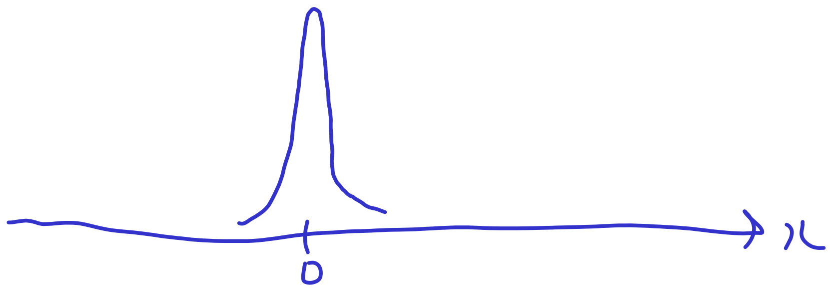
valuation

Martin Gale's theorem

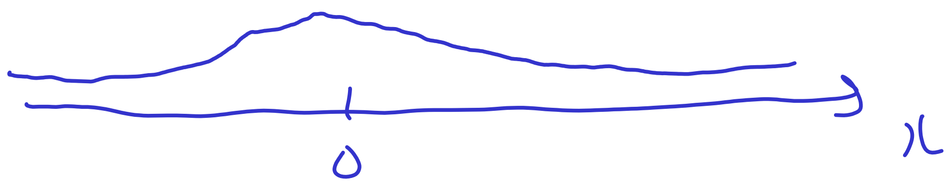
$t = 0$

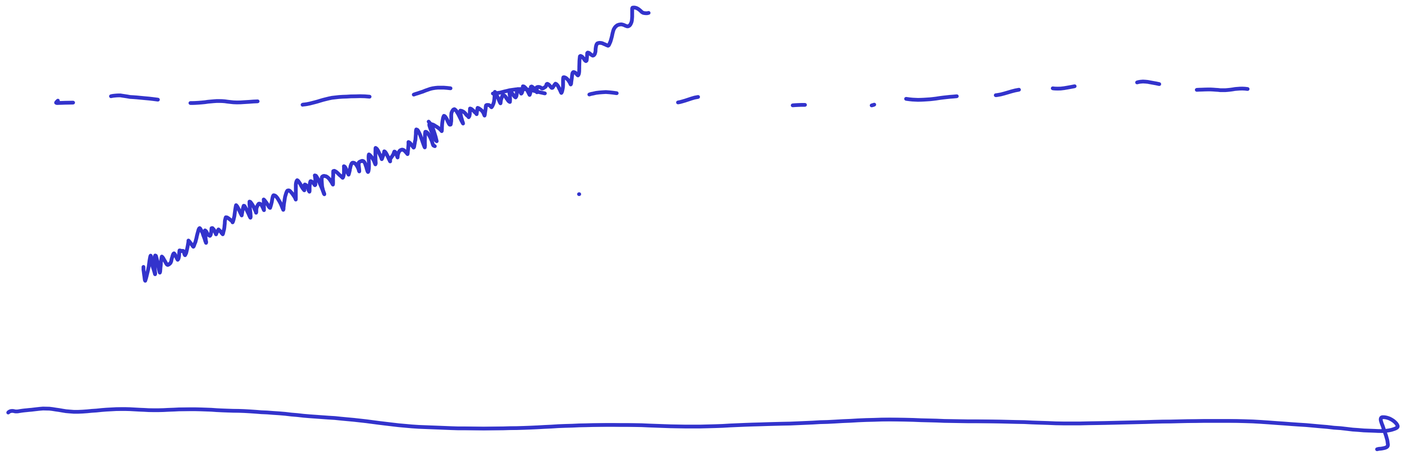


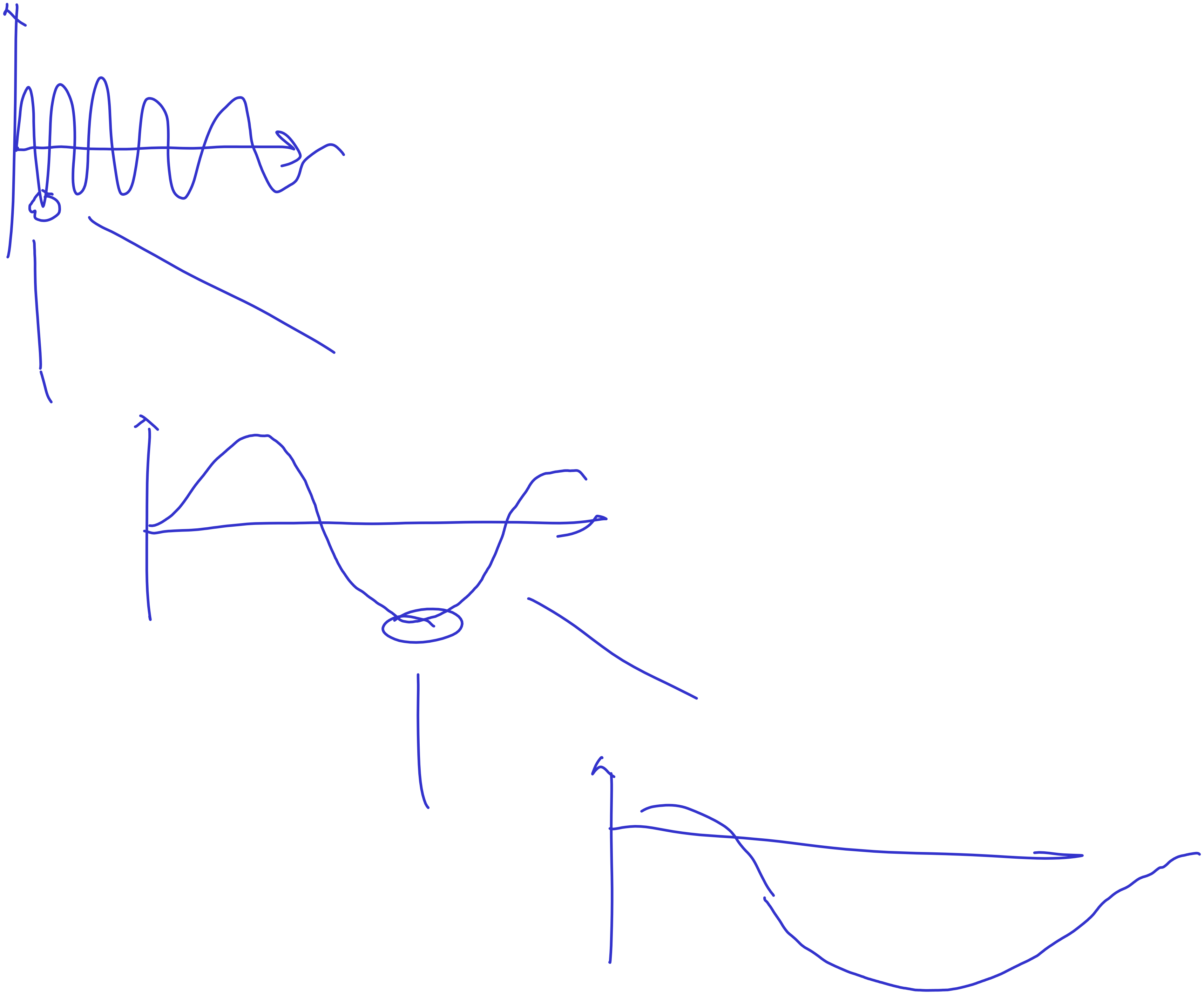
$t = 1$

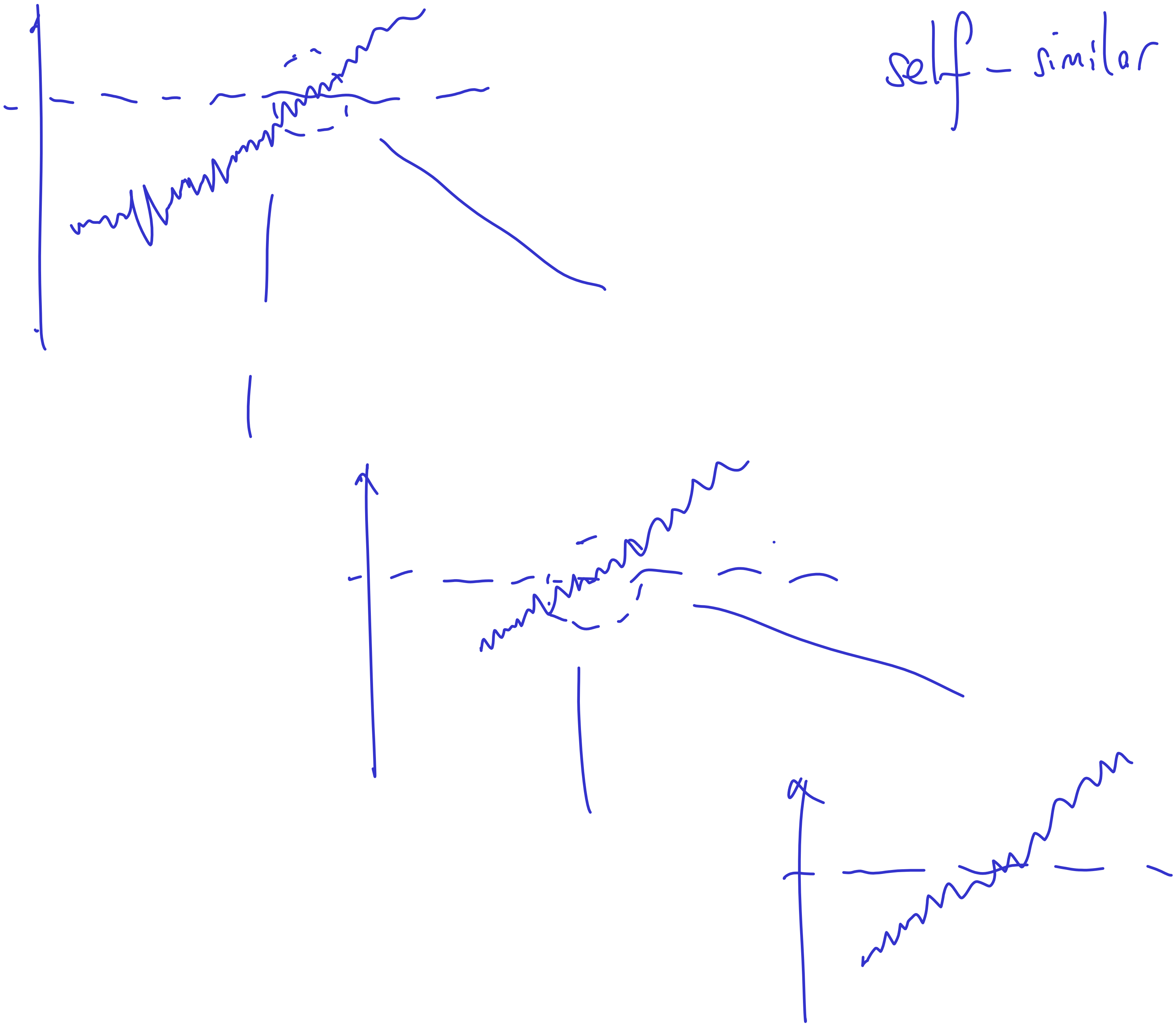


$t = 1000$









self-similar

<u>pg 26.</u>

$$S_{t+\Delta t} = S_t + \mu(t, S_t)\cdot\Delta t + \sigma(t, S_t)\left(W_{t+\Delta t} - W_t\right)$$

$$S_{t+\Delta t} - S_t = \mu(t, S_t)\cdot\Delta t + \sigma(t, S_t)\left(W_{t+\Delta t} - W_t\right)$$

$$\Delta S_t = \mu(t, S_t)\cdot\Delta t + \sigma(t, S_t)\,\Delta W_t$$

$$\lim \Delta t \to 0^+ : \Rightarrow dS_t = \mu(t, S_t)\,dt + \sigma(t, S_t)\,dW_t$$

pg 27. ② $W_t : \to \alpha W_{t/\alpha^2}$

1. $\alpha W_{0/\alpha^2} = 0$ ✓

2. $\alpha W_{t/\alpha^2} \sim \alpha N\left(0, t/\alpha^2\right) \sim N(0,t)$ ✓

3. $\mathrm{Cov}\left(\alpha W_{t/\alpha^2}, \alpha W_{s/\alpha^2}\right) = s \qquad (s<t)$

$$= \mathbb{E}\left[\alpha^2 W_{t/\alpha^2} W_{s/\alpha^2}\right] - 0$$

$$= \alpha^2\, \mathbb{E}\left[\left(W_{t/\alpha^2} - W_{s/\alpha^2} + W_{s/\alpha^2}\right) W_{s/\alpha^2}\right]$$

$$= \alpha^2 \left(\mathbb{E}\left[\left(W_{t/\alpha^2} - W_{s/\alpha^2}\right) W_{s/\alpha^2} \right] + \mathbb{E}\left[W^2_{s/\alpha^2} \right] \right)$$

$$= \alpha^2 \left(0 + s/\alpha^2 \right) = s$$

Using variance:

$$V\left[\alpha W_{t/\alpha^2} - \alpha W_{s/\alpha^2}\right] = V\left[\alpha W_{t/\alpha^2}\right] + V\left[\alpha W_{s/\alpha^2}\right] - 2\,\mathrm{Cov}\left(\alpha W_{t/\alpha^2}, \alpha W_{s/\alpha^2}\right)$$

$$= \alpha^2 \times \frac{t}{\alpha^2} + \alpha^2 \times \frac{s}{\alpha^2} - 2 \times \alpha^2 \times \frac{s}{\alpha^2}$$

$$= t + s - 2s$$

$$= t - s$$