

QF620 Additional Examples

Session 7: Greeks and Dynamic Hedging

1 Examples

1. Suppose the stock price S_t follows the Black-Scholes process:

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$

Show that the value of a European contract paying $\log \frac{S_T}{K}$ on maturity T can be written as:

$$V_0 = e^{-rT} \left[\log S_0 + \left(r - \frac{\sigma^2}{2} \right) T \right].$$

Use this formula to derive the delta (Δ) and gamma (Γ) of this contract.

2. Consider the Black-Scholes process

$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$

where W_t^* is a \mathbb{Q}^* -Brownian motion. The call option formula is given by

$$C(S_0, K, r, \sigma, T) = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2),$$

where

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T} = \frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}.$$

Find $\frac{\partial C}{\partial K}$.

3. In the real world probability measure \mathbb{P} , the stock price process follows the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where W_t is a \mathbb{P} -Brownian motion. In the market there is also a risk-free bond following the deterministic process $dB_t = rB_t dt$. We form a portfolio $V_t = \phi_t S_t + \psi_t B_t$ following the trading strategy (ϕ_t, ψ_t) . If $\forall t : \phi_t = 1$, find an appropriate choice of ψ_t so that V_t is a self-financing portfolio.

4. **Discussions** Suppose we have access to the following tradables in the market

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t \\ dB_t = r B_t dt \end{cases}$$

We form a portfolio $V_t = \phi_t S_t + \psi_t B_t$ following the trading strategy (ϕ_t, ψ_t) . If the strategy is self-financing, then we have

$$\begin{aligned} dV_t &= \phi_t dS_t + \psi_t dB_t \\ &= \phi_t(\mu S_t dt + \sigma S_t dW_t) + \psi_t(r B_t dt) \\ &= (\phi_t \mu S_t + \psi_t r B_t) dt + \phi_t \sigma S_t dW_t. \end{aligned}$$

The self-financing portfolio V_t can also be thought of as a function of time and stock, such that $V_t = f(t, S_t)$, where $f(t, x)$ is a bivariate function. Applying Itô's Formula, we obtain

$$\begin{aligned} dV_t &= \frac{\partial f}{\partial t}(t, S_t) dt + \frac{\partial f}{\partial x}(t, S_t) dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, S_t) (dS_t)^2 \\ &= \left(\frac{\partial f}{\partial t}(t, S_t) + \mu S_t \frac{\partial f}{\partial x}(t, S_t) + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t) \right) dt + \sigma S_t \frac{\partial f}{\partial x}(t, S_t) dW_t. \end{aligned}$$

Comparing the dW_t term, we get $\phi_t = \frac{\partial f}{\partial x}(t, S_t)$. Next we compare the dt term and obtain

$$\begin{aligned} \frac{\partial f}{\partial x}(t, S_t) \mu S_t + \psi_t r B_t &= \frac{\partial f}{\partial t}(t, S_t) + \mu S_t \frac{\partial f}{\partial x}(t, S_t) + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t) \\ \psi_t &= \frac{\frac{\partial f}{\partial t}(t, S_t) + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t)}{r B_t}. \quad \triangleleft \end{aligned}$$

2 Suggested Solutions

1. Taking the partial derivatives of V_0 with respect to S_0 , we obtain

$$\Delta = \frac{e^{-rT}}{S_0}, \quad \Gamma = -\frac{e^{-rT}}{S_0^2}. \quad \triangleleft$$

2. Noting that

$$Ke^{-rT} = S_0 e^{\frac{\sigma^2 T}{2} - \sigma \sqrt{T} d_1} \quad \text{and} \quad \frac{\partial d_1}{\partial K} = \frac{1}{S_0/K} \cdot \left(-\frac{S_0}{K^2} \right) \cdot \frac{1}{\sigma \sqrt{T}} = -\frac{1}{K \sigma \sqrt{T}} = \frac{\partial d_2}{\partial K},$$

we can proceed with the partial derivative with respect to K as follow:

$$\begin{aligned} \frac{\partial C}{\partial K} &= S_0 \frac{\partial \Phi(d_1)}{\partial d_1} \frac{\partial d_1}{\partial K} - Ke^{-rT} \frac{\partial \Phi(d_2)}{\partial d_2} \frac{\partial d_2}{\partial K} - e^{-rT} \Phi(d_2) \\ &= S_0 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \left(-\frac{1}{K \sigma \sqrt{T}} \right) - S_0 e^{\frac{\sigma^2 T}{2} - \sigma \sqrt{T} d_1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \cdot \left(-\frac{1}{K \sigma \sqrt{T}} \right) - e^{-rT} \Phi(d_2) \\ &= S_0 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot \left(-\frac{1}{K \sigma \sqrt{T}} \right) - S_0 e^{\frac{\sigma^2 T}{2} - \sigma \sqrt{T} d_1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2} + \sigma \sqrt{T} d_1 - \frac{\sigma^2 T}{2}} \cdot \left(-\frac{1}{K \sigma \sqrt{T}} \right) - e^{-rT} \Phi(d_2) \\ &\quad \underbrace{\hspace{15em}}_0 \\ &= -e^{-rT} \Phi(d_2). \end{aligned}$$

This is what we would expect to get — recall that the valuation formula for a cash-or-nothing digital option is given by

$$V_0 = e^{-rT} \Phi(d_2),$$

and the way to hedge this digital option is to form a call spread with strikes very close to each other, giving us the following relationship:

$$\lim_{\Delta K \rightarrow 0} \frac{C(K) - C(K + \Delta K)}{\Delta K} = - \lim_{\Delta K \rightarrow 0} \frac{C(K + \Delta K) - C(K)}{\Delta K} = -\frac{\partial C}{\partial K}. \quad \triangleleft$$

3. For V_t to be a self-financing portfolio, we need to have

$$dV_t = \phi_t dS_t + \psi_t dB_t.$$

Given that $\phi_t = 1$, we have $d\phi_t = 0$. Applying Itô's Formula and stochastic calculus chain rule to V_t , we obtain

$$dV_t = dS_t + \psi_t dB_t + B_t d\psi_t + \cancel{\psi_t dB_t} \xrightarrow{0}$$

Comparing against the self-financing requirement, we note that V_t is self-financing if $d\psi_t = 0$. In other words, the self-financing requirement stipulates that we need $\psi_t = \text{const.}$ \triangleleft

Note that although this portfolio is self-financing, it is not a dynamic hedging strategy for a call option.