Assignment 4.

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1. a) $dS_t = rS_t dt + \sigma S_t dW_t^*$

$$S_t = S_0 \cdot e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t^*}$$

$$V_t = \log\left(\frac{S_0 \cdot e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t^*}}{k}\right).$$

Therefore, the value of the European contract is

$$e^{rt} E\left[\log\left(\frac{S_0 \cdot e^{(r-\frac{1}{2}\sigma^2)t + \sigma\sqrt{t}\lambda}}{k}\right)\right] \quad \text{where } \lambda \sim N(0, 1)$$

$$= e^{-rt} E\left[\log\left(\frac{S_0}{k}\right) + \left(r - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}\lambda\right]$$

$$= e^{-rt}\left(\log\left(\frac{S_0}{k}\right) + \left(r - \frac{1}{2}\sigma^2\right)t\right)$$

b) delta

If we take partial derivative of above equation with respect to S_0, we obtain

$$e^{-rt} \times \frac{1}{S_0}$$

gamma.

Taking partial derivative of delta with respect to S_0, we obtain

$$-e^{rt} \times \frac{1}{S_0^2}$$

2. $$V^c_{\text{Cash Digital}} = e^{-rt}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathbb{1}_{S_T > K}\, e^{-\frac{x^2}{2}}\,dx.$$

$$= e^{-rt}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty} e^{-\frac{x^2}{2}}\,dx.$$

$$x^* = \frac{\log\frac{K}{S_0} - (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

Therefore, $V^c_{\text{Cash Digital}} = e^{-rt}\,\Phi\left(\frac{\log\frac{S_0}{K} + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$

a) Delta

Let $\frac{\log\frac{S_0}{K} + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_2$.

$$\frac{\partial V^c}{\partial S_0} = e^{-rt}\cdot\frac{\partial\Phi(d_2)}{\partial d_2}\cdot\frac{\partial d_2}{\partial S_0}$$

$$= e^{-rt}\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{d_2^2}{2}}\cdot\frac{1}{\sigma S_0\sqrt{T}}$$

$$= e^{-rt}\cdot\frac{1}{\sigma S_0\sqrt{T}}\cdot\frac{1}{\sqrt{2\pi}}\, e^{-\frac{d_2^2}{2}}$$

b). Vega.

Let $\frac{\log\frac{S_0}{K} + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_2$.

$$\frac{\partial V}{\partial \sigma} = e^{-rt}\frac{\Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial \sigma}$$

$$= e^{-rt}\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{d_2^2}{2}}\cdot\left(-\sqrt{T} - \frac{d_2}{\sigma}\right)$$

3. a) $V_t = f(t, S_t)$

$$dV_t = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS_t)^2$$

$$= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S}(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2 dt.$$

$$= \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2\right) dt + \frac{\partial V}{\partial S}\sigma S_t dW_t$$

b) $\Pi_t = V_t + \Delta \times S_t$

$d\Pi_t = dV_t + \Delta \times dS_t.$

$$= \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2\right) dt + \frac{\partial V}{\partial S}\sigma S_t dW_t$$

$$+ \Delta \mu S_t dt + \Delta \sigma S_t dW_t.$$

For the portfolio to be a risk-free portfolio, dW_t term should disappear from the equation.

Therefore, $\frac{\partial V}{\partial S}\sigma S_t dW_t + \Delta \sigma S_t dW_t = 0$

$$\Delta = -\frac{\partial V}{\partial S}$$

c) From 3-(b), we know that $\Delta = -\frac{\partial V}{\partial S}$.

Then we obtain

$$d\pi_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2\right)dt.$$

Since the return of this portfolio is deterministic, the return from this portfolio must be equal to risk-free rate r. to get rid of arbitrage

$$\therefore\ d\pi_t = r\pi_t dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2\right)dt.$$

Substituting π_t with $V_t - \frac{\partial V}{\partial S}S_t$, (where $\Delta = -\frac{\partial V}{\partial S}$) we obtain

$$rV_t - r\frac{\partial V}{\partial S}S_t = \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2$$

$$rV_t = \frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2$$

Therefore, V_t satisfies the Black-Scholes PDE.