Mean-Variance Analysis

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Investment Environment

- Frictionless market: no taxes, transactions costs, etc.
- $n \geq 2$ risky assets with no restrictions on short selling
- Let $\mathbf{R} = (R_1, \ldots, R_n)'$ be $n \times 1$ vector of expected returns
- Let $\mathbf{V}$ be $n \times n$ covariance matrix of returns, which must be **symmetric**: $\mathbf{V}' = \mathbf{V}$ and **positive definite**:

$$\mathbf{z}'\mathbf{V}\mathbf{z} > 0 \quad \text{for any} \quad \mathbf{z} \neq 0$$

- If no redundant assets, then returns must be **linearly independent** and covariance matrix must be **invertible**:

$$\exists \quad \mathbf{V}^{-1} \quad \text{such that} \quad \mathbf{V}^{-1}\mathbf{V} = \mathbf{I}$$

Optimal Asset Allocation – Part 1

- Let $\mathbf{w} = (w_1, \ldots, w_n)'$ be $n \times 1$ vector of portfolio weights, which must sum to one: $\mathbf{w}'\mathbf{e} = 1$, where $\mathbf{e} = (1, \ldots, 1)'$
- Then $\mathbf{w}'\mathbf{R}$ is expected return for portfolio, and $\mathbf{w}'\mathbf{V}\mathbf{w} > 0$ is variance of return for portfolio
- Suppose that investor wants to create portfolio with expected return of R_p and lowest possible variance of return
- Use Lagrangian to represent investor's asset allocation (or portfolio choice) problem:

$$\min_{\{\mathbf{w},\lambda,\gamma\}} \mathcal{L} = \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w} + \lambda\left(R_p - \mathbf{w}'\mathbf{R}\right) + \gamma\left(1 - \mathbf{w}'\mathbf{e}\right)$$

Optimal Asset Allocation – Part 2

- Use first-order condition to solve for optimal portfolio weights:

$$\mathbf{V}\mathbf{w} - \lambda\mathbf{R} - \gamma\mathbf{e} = 0 \quad \Rightarrow \quad \mathbf{w}^* = \lambda\mathbf{V}^{-1}\mathbf{R} + \gamma\mathbf{V}^{-1}\mathbf{e}$$

- Pre-multiply both sides by $\mathbf{R}'$:

$$\mathbf{R}'\mathbf{w}^* = \lambda\mathbf{R}'\mathbf{V}^{-1}\mathbf{R} + \gamma\mathbf{R}'\mathbf{V}^{-1}\mathbf{e} = R_p$$

- Pre-multiply both sides by $\mathbf{e}'$:

$$\mathbf{e}'\mathbf{w}^* = \lambda\mathbf{e}'\mathbf{V}^{-1}\mathbf{R} + \gamma\mathbf{e}'\mathbf{V}^{-1}\mathbf{e} = 1$$

Optimal Asset Allocation – Part 3

- Solve simultaneous equations for Lagrange multipliers:

$$\lambda = \frac{\delta R_p - \alpha}{\zeta\delta - \alpha^2}; \qquad \gamma = \frac{\zeta - \alpha R_p}{\zeta\delta - \alpha^2}$$

- Here α is scalar, while ζ and δ are positive scalars:

$$\alpha = \mathbf{R}'\mathbf{V}^{-1}\mathbf{e}; \qquad \zeta = \mathbf{R}'\mathbf{V}^{-1}\mathbf{R}; \qquad \delta = \mathbf{e}'\mathbf{V}^{-1}\mathbf{e}$$

- Denominator of Lagrange multipliers is strictly positive:

$$(\alpha\mathbf{R} - \zeta\mathbf{e})'\,\mathbf{V}^{-1}\,(\alpha\mathbf{R} - \zeta\mathbf{e}) = \zeta\left(\zeta\delta - \alpha^2\right) > 0$$

Optimal Asset Allocation – Part 4

- Necessary and sufficient condition for $\mathbf{w}^*$ to be portfolio weights for **frontier portfolio** with expected return of R_p:

$$\mathbf{w}^* = \left(\frac{\delta R_p - \alpha}{\zeta\delta - \alpha^2}\right)\mathbf{V}^{-1}\mathbf{R} + \left(\frac{\zeta - \alpha R_p}{\zeta\delta - \alpha^2}\right)\mathbf{V}^{-1}\mathbf{e}$$

- Simplify to $\mathbf{w}^* = \mathbf{a} + \mathbf{b}R_p$ where:

$$\mathbf{a} = \frac{\zeta\mathbf{V}^{-1}\mathbf{e} - \alpha\mathbf{V}^{-1}\mathbf{R}}{\zeta\delta - \alpha^2}; \qquad \mathbf{b} = \frac{\delta\mathbf{V}^{-1}\mathbf{R} - \alpha\mathbf{V}^{-1}\mathbf{e}}{\zeta\delta - \alpha^2}$$

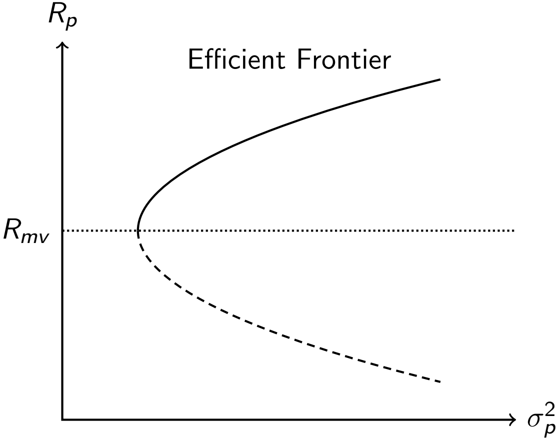
- Forms minimum-variance frontier, where portfolios have lowest level of risk for given expected return

Portfolio Frontier – Part 1

- Calculate variance of return for frontier portfolio, to show that minimum-variance frontier forms parabola in $\left(\sigma_p^2, R_p\right)$ space:

$$
\begin{aligned}
\sigma_p^2 = \mathbf{w}'\mathbf{V}\mathbf{w} &= \left(\mathbf{a} + \mathbf{b}R_p\right)' \mathbf{V} \left(\mathbf{a} + \mathbf{b}R_p\right) \\
&= \frac{\delta R_p^2 - 2\alpha R_p + \zeta}{\zeta\delta - \alpha^2} \\
&= \frac{1}{\delta} + \frac{\delta}{\zeta\delta - \alpha^2}\left(R_p - \frac{\alpha}{\delta}\right)^2
\end{aligned}
$$

- $R_{mv} = \frac{\alpha}{\delta}$ is expected return for minimum variance portfolio
- Top half (where $R_p \geq R_{mv}$) is **efficient fronter**, where portfolios have highest expected return for given level of risk


R_p
Efficient Frontier
R_{mv}
σ_p^2

Portfolio Frontier – Part 2

- Rewrite variance of return for frontier portfolio as:

$$\frac{\sigma_p^2}{\eta_1} - \frac{(R_p - R_{mv})^2}{\eta_2} = 1; \qquad \eta_1 = \frac{1}{\delta}; \qquad \eta_2 = \frac{\zeta\delta - \alpha^2}{\delta^2}$$

- Represents hyperbola with center at $(0, R_{mv})$ and asymptotes:

$$R_p = R_{mv} \pm \left(\frac{\zeta\delta - \alpha^2}{\delta}\right)^{\frac{1}{2}} \sigma_p$$

- Frontier forms hyperbola in mean–standard deviation space

Portfolio Separation

- Let p_1 and p_2 be two distinct frontier portfolios, and let q be any frontier portfolio
- There always exists θ such that $R_q = \theta R_{p_1} + (1-\theta) R_{p_2}$, so can invest θ in p_1 and $1-\theta$ in p_2 to replicate q:

$$
\begin{aligned}
\theta \mathbf{w_{p_1}} + (1-\theta)\mathbf{w_{p_2}} &= \theta\left(\mathbf{a} + \mathbf{b}R_{p_1}\right) + (1-\theta)\left(\mathbf{a} + \mathbf{b}R_{p_2}\right) \\
&= \mathbf{a} + \mathbf{b}\left(\theta R_{p_1} + (1-\theta) R_{p_2}\right) \\
&= \mathbf{a} + \mathbf{b}R_q \\
&= \mathbf{w_q}
\end{aligned}
$$

- Can generate entire frontier using (affine) combinations of any two distinct frontier portfolios

Orthogonal Frontier Portfolios – Part 1

- Covariance of return between any two frontier portfolios:

$$\begin{aligned}\mathbf{w}'_{\mathbf{p_1}}\mathbf{V}\mathbf{w}_{\mathbf{p_2}} &= \left(\mathbf{a}+\mathbf{b}R_{p_1}\right)'\mathbf{V}\left(\mathbf{a}+\mathbf{b}R_{p_2}\right) \\ &= \frac{1}{\delta}+\frac{\delta}{\zeta\delta-\alpha^2}\left(R_{p_1}-R_{mv}\right)\left(R_{p_2}-R_{mv}\right)\end{aligned}$$

- Set covariance to zero and solve for R_{p_2}:

$$R_{p_2} = R_{mv} - \frac{\zeta\delta-\alpha^2}{\delta^2\left(R_{p_1}-R_{mv}\right)}$$

- If p_1 is efficient, then p_2 is inefficient (and vice versa)

Orthogonal Frontier Portfolios – Part 2

- Slope of frontier at any point in mean–std dev space:

$$\frac{\partial R_p}{\partial \sigma_p} = \frac{\zeta\delta - \alpha^2}{\delta\left(R_p - R_{mv}\right)}\sigma_p$$

- Evaluate at (σ_{p_1}, R_{p_1}) to get slope of frontier at point corresponding to p_1
- Equation for line tangent to frontier at (σ_{p_1}, R_{p_1}):

$$R_p = R_0 + \left[\frac{\zeta\delta - \alpha^2}{\delta\left(R_{p_1} - R_{mv}\right)}\sigma_{p_1}\right]\sigma_p$$

Orthogonal Frontier Portfolios – Part 2

- Evaluate at (σ_{p_1}, R_{p_1}) and solve for intercept:

$$
\begin{aligned}
R_0 &= R_{p_1} - \frac{\zeta\delta - \alpha^2}{\delta\left(R_{p_1} - R_{mv}\right)}\sigma_{p_1}^2 \\
&= R_{p_1} - \frac{\zeta\delta - \alpha^2}{\delta\left(R_{p_1} - R_{mv}\right)}\left[\frac{1}{\delta} + \frac{\delta}{\zeta\delta - \alpha^2}\left(R_{p_1} - R_{mv}\right)^2\right] \\
&= R_{mv} - \frac{\zeta\delta - \alpha^2}{\delta^2\left(R_{p_1} - R_{mv}\right)} \\
&= R_{p_2}
\end{aligned}
$$

- Hence orthogonal portfolio is on same level as intercept

Asset Allocation with Riskless Asset – Part 1

- $n \geq 2$ risky assets and riskless asset with return of R_f
- Let $\mathbf{w}$ be vector of portfolio weights for risky assets, then $1 - \mathbf{w}'\mathbf{e}$ is proportion of wealth invested in riskless asset
- Here $\mathbf{w}'\mathbf{e} < 1$ represents lending while $\mathbf{w}'\mathbf{e} > 1$ represents borrowing (both at risk-free rate)
- Consider portfolio with expected return of R_p:

$$R_p = \mathbf{w}'\mathbf{R} + \left(1 - \mathbf{w}'\mathbf{e}\right) R_f = R_f + \mathbf{w}'\left(\mathbf{R} - R_f\mathbf{e}\right)$$

- Lagrangian for asset allocation problem:

$$\min_{\{\mathbf{w},\lambda\}} \mathcal{L} = \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w} + \lambda\left[R_p - R_f - \mathbf{w}'\left(\mathbf{R} - R_f\mathbf{e}\right)\right]$$

Asset Allocation with Riskless Asset – Part 2

- Use first-order condition to solve for optimal portfolio weights:

$$\mathbf{V}\mathbf{w} - \lambda\left(\mathbf{R} - R_f\mathbf{e}\right) = 0 \quad \Rightarrow \quad \mathbf{w}^* = \lambda\mathbf{V}^{-1}\left(\mathbf{R} - R_f\mathbf{e}\right)$$

- Substitute into condition for expected return of portfolio:

$$\begin{aligned} R_p &= R_f + \lambda\left(\mathbf{R} - R_f\mathbf{e}\right)'\mathbf{V}^{-1}\left(\mathbf{R} - R_f\mathbf{e}\right) \\ &= R_f + \lambda\left(\zeta - 2\alpha R_f + \delta R_f^2\right) \\ \Rightarrow \lambda &= \frac{R_p - R_f}{\zeta - 2\alpha R_f + \delta R_f^2} \end{aligned}$$

- Denominator is strictly positive since **V** is positive definite

Asset Allocation with Riskless Asset – Part 3

- Variance of return for frontier portfolio:

$$\sigma_p^2 = \mathbf{w}'\mathbf{V}\mathbf{w} = \lambda^2 \left(\mathbf{R} - R_f\mathbf{e}\right)' \mathbf{V}^{-1} \left(\mathbf{R} - R_f\mathbf{e}\right)$$
$$= \frac{\left(R_p - R_f\right)^2}{\zeta - 2\alpha R_f + \delta R_f^2}$$

- Portfolio frontier is linear in mean–standard deviation space:

$$R_p = R_f \pm \left(\zeta - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}} \sigma_p$$

- Efficient frontier is upper half of minimum-variance frontier

Portfolio Separation with Riskless Asset – Part 1

- If $R_f < R_{mv} = \frac{\alpha}{\delta}$, then efficient frontier (with riskless asset) is tangent to efficient portion of risky-asset-only frontier
- Use result for orthogonal risky-asset-only frontier portfolios:

$$R_{tg} = R_{mv} - \frac{\zeta\delta - \alpha^2}{\delta^2 \left(R_f - R_{mv}\right)} = \frac{\alpha R_f - \zeta}{\delta R_f - \alpha}$$

- Risk premium for tangency portfolio:

$$R_{tg} - R_f = \frac{\alpha R_f - \zeta}{\delta R_f - \alpha} - R_f = -\frac{\zeta - 2\alpha R_f + \delta R_f^2}{\delta \left(R_f - R_{mv}\right)}$$

Portfolio Separation with Riskless Asset – Part 2

- Variance of return for tangency portfolio:

$$\sigma_{tg}^2 = \frac{1}{\delta} + \frac{\delta\left(R_{tg} - R_{mv}\right)^2}{\zeta\delta - \alpha^2} = \frac{1}{\delta} + \frac{\zeta\delta - \alpha^2}{\delta^3\left(R_f - R_{mv}\right)^2}$$
$$= \frac{1}{\delta}\left[1 + \frac{\zeta\delta - \alpha^2}{\left(\delta R_f - \alpha\right)^2}\right] = \frac{\zeta - 2\alpha R_f + \delta R_f^2}{\delta^2\left(R_f - R_{mv}\right)^2}$$

- Choose square root such that σ_{tg} is positive:

$$\sigma_{tg} = -\frac{\left(\zeta - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}}}{\delta\left(R_f - R_{mv}\right)}$$

Portfolio Separation with Riskless Asset – Part 3

- Sharpe ratio for tangency portfolio:

$$
\begin{aligned}
\frac{R_{tg} - R_f}{\sigma_{tg}} &= \left[-\frac{\zeta - 2\alpha R_f + \delta R_f^2}{\delta \left(R_f - R_{mv} \right)} \right] \left[-\frac{\delta \left(R_f - R_{mv} \right)}{\left(\zeta - 2\alpha R_f + \delta R_f^2 \right)^{\frac{1}{2}}} \right] \\
&= \left(\zeta - 2\alpha R_f + \delta R_f^2 \right)^{\frac{1}{2}}
\end{aligned}
$$

- Hence tangency portfolio lies on efficient frontier, and has highest Sharpe ratio out of all portfolios
- Similar result for $R_f > R_{mv}$, except that tangency portfolio lies on inefficient portion of risky-asset-only frontier (and has lowest Sharpe ratio out of all portfolios)

CARA Utility: Economic Environment

- $n \geq 2$ risky assets and riskless asset with return R_f
- $\tilde{\mathbf{R}}$ is $n \times 1$ vector of (gross) risky asset returns:

$$\tilde{R}_p = R_f + \mathbf{w}' \left(\tilde{\mathbf{R}} - R_f \mathbf{e} \right)$$

- Let $b_r = bW_0$, and assume that investor maximizes expected utility of end-of-period wealth:

$$U\left(\tilde{W}\right) = -e^{-b\tilde{W}} = -e^{-b_r \frac{\tilde{W}}{W_0}} = -e^{-b_r \tilde{R}_p}$$

CARA Utility: Asset Allocation

- Risky asset returns have joint normal distribution, so $U\left(\tilde{W}\right)$ has lognormal distribution:

$$E\left[U\left(\tilde{W}\right)\right] = -e^{-b_r[R_f+\mathbf{w}'(\mathbf{R}-R_f\mathbf{e})]+\frac{1}{2}b_r^2\mathbf{w}'\mathbf{V}\mathbf{w}}$$

- Since $b_r > 0$ and exponential function is monotonically increasing, portfolio choice problem is equivalent to:

$$\max_{\mathbf{w}} \left\{ \mathbf{w}'\left(\mathbf{R} - R_f\mathbf{e}\right) - \frac{1}{2}b_r\mathbf{w}'\mathbf{V}\mathbf{w} \right\}$$

CARA Utility: Optimal Portfolio

- Use first-order condition to solve for optimal portfolio weights:

$$\mathbf{R} - R_f\mathbf{e} - b_r\mathbf{V}\mathbf{w} = 0 \quad \Rightarrow \quad \mathbf{w}^* = \frac{1}{b_r}\mathbf{V}^{-1}\left(\mathbf{R} - R_f\mathbf{e}\right)$$

- Pre-multiply both sides by $W_0\mathbf{e}'$:

$$W_0\mathbf{e}'\mathbf{w}^* = \frac{1}{b}\left(\alpha - \delta R_f\right)$$

- Investor with constant absolute risk aversion invests fixed dollar amount in risky assets (regardless of initial wealth)