SMU
SINGAPORE MANAGEMENT UNIVERSITY

Session 1: Brownian Motion and Martingale

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QF620 Stochastic Modelling in Finance

Random Walk on a Number Line

A random walk of n steps is defined as:

$$S_n = \sum_{i=1}^{n} X_i, \quad X_i = \pm 1$$

For a random walk of n steps, there are a total of $N = 2^n$ paths.

Definition of Symmetrical Random Walk

Consider a series of X's, each of which can only take the value of -1 or $+1$, and are independent and identically distributed. We further define a probability measure whereby either movement is equally likely, i.e.

$$\forall i: \quad \mathbb{P}(X_i = +1) = \frac{1}{2}, \qquad \mathbb{P}(X_i = -1) = \frac{1}{2}.$$

Random Walk on a Number Line

Let $S_n = \sum_{i=1}^{n} X_i$ where $n \in \mathbb{N}$, we note that S_n can be visualised as an n-step random walk. Furthermore, we can show that S_n exhibits the following properties:

1. $\mathbb{E}[S_n] = 0$
2. $V[S_n] = n$
3. If $n, m \in \mathbb{N}$ and $m < n$, $\mathbb{E}_m[S_n] = S_m$
4. If $n, m \in \mathbb{N}$ and $m < n$, $\mathrm{Cov}(S_n, S_m) = m$

The covariance property can be shown as follow:

$$
\begin{aligned}
\mathrm{Cov}(S_n, S_m) &= \mathbb{E}[S_n S_m] - \mathbb{E}[S_n]\mathbb{E}[S_m] \\
&= \mathbb{E}[\mathbb{E}_m[S_n S_m]] \\
&= \mathbb{E}[S_m \mathbb{E}_m[S_n]] \\
&= \mathbb{E}[S_m^2] = m.
\end{aligned}
$$

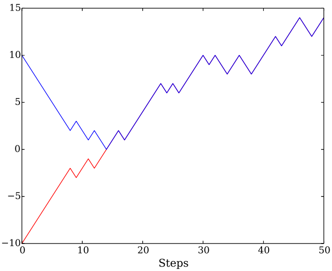
Reflection Principle of Random Walk

Reflection Principle

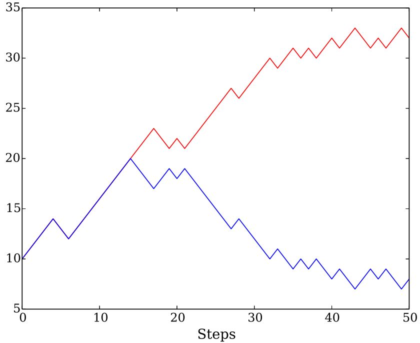
Let $A = (0, \alpha)$ and $B = (n, \beta)$ with $\alpha, \beta > 0$. Let $A' = (0, -\alpha)$ be the reflection of A through the x-axis, then the number of paths from A to B that touch the x-axis is the same as the number of paths from A' to B. The argument is straightforward: consider the first point P where a path from A to B hits the x-axis, and reflect the part of the path from A to P across the x-axis, and we obtain a path from A' to B.

The reflection principle is an important insight that allows us to formulate and solve a large number of random walk problems.

Example: $\alpha = 10$, $\beta = 14$, $n = 50$.



Reflection Principle of Random Walk



Turning the argument around, we can also say that: for every path that starts at 10, hits the barrier 20, and ends up above the barrier, there is a corresponding path that hits the barrier at 20 and ends up below the barrier.

Reflection Principle of Random Walk

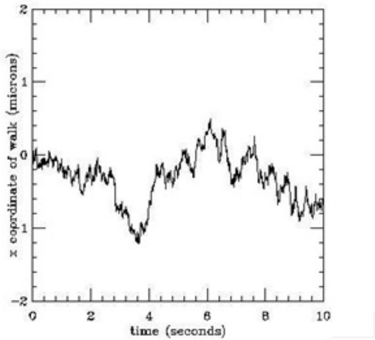
Example Consider the following 2 digital call options with the same maturity:

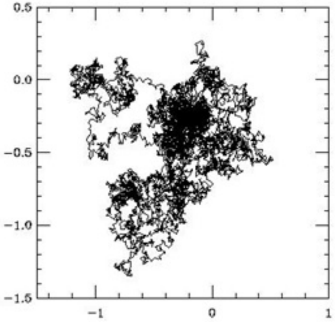
1. Pays \$1 if the stock ends up greater than \$20 on maturity
2. Pays \$1 if the stock hits \$20 before maturity, and the contract terminates

Assume interest rate is negligible. What is the relationship between the prices of the two options?

These motions were such as to satisfy me, after frequently repeated observation, that they arose neither from currents in the fluid, nor from its gradual evaporation, but belonged to the particle itself.

— Robert Brown (1773–1858)





Brownian Motion

...the phenomenon of Brownian motion, the apparently erratic movement of tiny particles suspended in a liquid: Einstein showed that these movements satisfied a clear statistical law.

— C. P. Snow (1905–1980)

Si, à l'égard de plusieurs questions traitées dans cette étude, j'ai comparé les résultats de l'observation à ceux de la théorie, ce n'est pas pour vérifier des formules établies par des méthodes mathématiques, mais pour montrer seulement que le marché, à son insu, obéit à une loi qui le domine: la loi de la probabilité.

— Louis Bachelier, Théorie de la spéculation, 1900

Rough translation:

If, regarding several questions analysed in this study, I compared the observed results to those of the theory, it is not to verify the formulas obtained by mathematical methods, but only to show that the market, unwittingly, complies to a law that dominates it: the law of probability.

THÉORIE

DE

LA SPÉCULATION,

PAR M. L. BACHELIER.

Fig. 5.

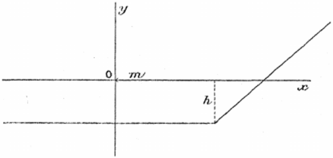


Fig. 7.

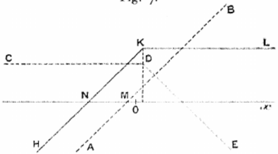
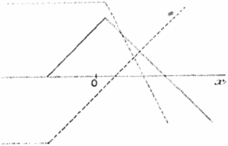


Fig. 10.



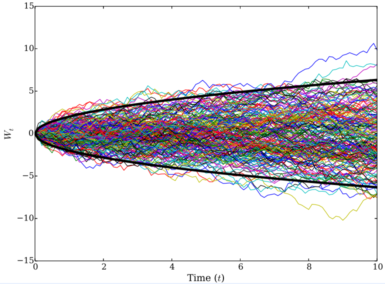
Probabilité de l'opération en blanc........... 0,30

» de bénéfice...................... 0,45

» de perte......................... 0,25

Brownian Motion

- As early as 1900, Louis Bachelier, in his thesis “La Théorie de la Spéculation”, proposed Brownian motion as a model of the fluctuations of stock prices.
- Even today it is the building block from which we construct the basic reference model for a continuous time market.
- We shall approach this topic by considering Brownian motion as an “infinitesimal” random walk in which smaller and smaller steps are taken at ever more frequent time intervals.



Binomial Process to Brownian Motion

Let's start with a definition of random walk. For a positive integer n, define the binomial process $W_n(t)$ to have the following properties

1. $W_n(0) = 0$
2. time spacing is $\frac{1}{n}$
3. up and down jumps equal and of size $\frac{1}{\sqrt{n}}$
4. measure $\mathbb{P}$, given by up and down probabilities everywhere equal to $\frac{1}{2}$

In other words, $X_1, X_2, \ldots$ is a sequence of independent binomial random variables taking values $+1$ or -1 with equal probability, then the value of W_n on the i^{th} step is defined by

$$W_n\left(\frac{i}{n}\right) = W_n\left(\frac{i-1}{n}\right) + \frac{X_i}{\sqrt{n}}, \quad 1 \leq i.$$

When n becomes large, W_n will not blow out due to the scaling of $\frac{1}{\sqrt{n}}$. It can be shown that

$$W_n(t) = \sqrt{t}\left(\frac{\sum_{i=1}^{nt} X_i}{\sqrt{nt}}\right).$$

Binomial Process to Brownian Motion

- The distribution in the brackets tends to $N(0,1)$ by central limit theorem. Hence the distribution of $W_n(t)$ tends to $N(0,t)$.
- All the marginal distributions and conditional marginal distributions tend towards the same underlying normal distribution.
- Each random walk W_n has the property that its future movements away from a particular position are independent of where that position is, and indeed independent of its entire history of movements up to that time.
- In other words, Brownian motion is the limit of a scaled random walk. If $X_i = \pm 1$ with equal probability, and define the path $S_n = \sum_{i=1}^{nt} X_i$, then Brownian motion is defined as the limit

$$W_t = \lim_{n\to\infty} \frac{1}{\sqrt{n}} S_n.$$

Mathematical Definition of Brownian Motion

Named after Robert Brown, a British botanist, who observed the random movement of pollens of grains in water. This random movement process was analysed mathematically by the American mathematician Norbert Wiener, so is also called a Wiener process.

The randomly generated Brownian path is continuous. The definition of Brownian motion is as follows:

1. The process starts at 0.
2. The path increments are independent normal, with zero mean and variance equal to the temporal extension of the increment. Suppose $0 \leq t_1 \leq t_2$, we have

$$W_{t_2} - W_{t_1} \sim N(0, t_2 - t_1),$$

3. $W_{t_2} - W_{t_1}$ is independent of $W_{t_1} - W_{t_0}$.

These are necessary and sufficient conditions for a process to be identified as a Brownian motion.

Mathematical Definition of Brownian Motion

Example If $Z \sim N(0,1)$, then $X_t = \sqrt{t}Z$ is continuous, and is marginally distributed as $N(0,t)$. Is X_t a Brownian motion?

Solution The increment $X_{s+t} - X_s$ is normally distributed with a mean of 0 and a variance of

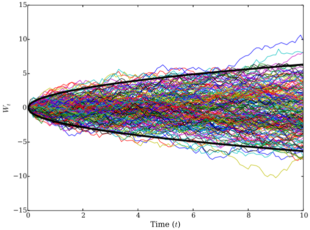
$$2s + t - 2\sqrt{s(s+t)}.$$

- For the process to be a Brownian motion, the increment needs to be of variance t.
- In addition, the increment is not independent of X_s.
- So X_t is not a Brownian motion. $\triangleleft$

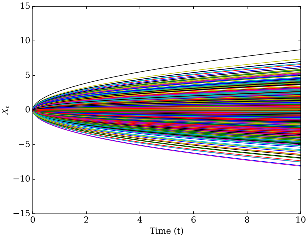
Mathematical Definition of Brownian Motion

- Consider $s < t$, it is important to realize that the condition that the distribution is normal of variance $t - s$ for every t and s and independent of the path up to time s, is much stronger than requiring W_t to be normally distributed with variance t for every t.

- For example, if we let $X_t = \sqrt{t}Z$, where Z is the same draw from a normal distribution for all t then we have that X_t is normally distributed with variance t.



vs



Mathematical Definition of Brownian Motion

The definition does not guarantee the existence of such a process. However, Paul Lévy has provided a very nice way of constructing such a process which enables us to simulate trajectories. In other words, the Brownian motion exists in a mathematical sense.

Note:

- It should be clear in the previous example that the independent increment condition is not a restatement of the $N(0,t)$ marginal requirement.
- Many processes that have marginals $N(0,t)$ are not Brownian motion.
- In the continuous world, it's not just the marginals conditional on the process's value at time 0 that counts, but all marginals conditional on all the histories.

Mathematical Definition of Brownian Motion

Given this formal definition, we can proceed to derive the following statistical properties based on the properties of normal distribution:

$$
\begin{aligned}
\mathbb{E}[W_t] &= 0, \quad W_t \sim N(0,t) \\
\mathbb{E}[W_t^2] &= t \\
\mathbb{E}[W_t - W_s] &= 0 \\
V[W_t - W_s] &= \mathbb{E}[(W_t - W_s)^2] = t - s \\
\mathrm{Cov}(W_s, W_t) &= s, \quad s < t \\
\mathrm{Corr}(W_s, W_t) &= \sqrt{\frac{s}{t}}, \quad s < t.
\end{aligned}
$$

Proof: Note that

$$
V[W_T] = V[W_T - W_0] = T, \quad V[W_{t+\Delta t} - W_t] = \Delta t.
$$

Also note that

$$
\begin{aligned}
\mathrm{Cov}(W_s, W_t) &= \mathrm{Cov}(W_s, W_t - W_s + W_s) \\
&= \mathrm{Cov}(W_s, W_s) + \mathrm{Cov}(W_s, W_t - W_s) = V[W_s] = s,
\end{aligned}
$$

since W_s and $W_t - W_s$ are uncorrelated. $\triangleleft$

Mathematical Definition of Brownian Motion

Example Let W_t denote a Brownian process. Conditional on $W_1 > 0$, what is the probability that $W_2 < 0$?

Example Let W_t denote a Brownian process. Determine the probability of $\mathbb{P}(W_1 \times W_2 > 0)$.

Brownian Martingales

If M_t is a stochastic process, we say that it is a martingale if

$$\forall T \geq t : \quad M_t = \mathbb{E}_t[M_T].$$

In words, the expected value of this process, conditional on information up to t, is equal to its value taken at time t.

Example Let W_t be a Brownian motion, show that W_t is a martingale.

Solution Consider $0 \leq s \leq t$, we can show that

$$\begin{aligned} \mathbb{E}_s[W_t] &= \mathbb{E}_s[W_t - W_s + W_s] \\ &= \mathbb{E}_s[W_t - W_s] + \mathbb{E}_s[W_s] \\ &= W_s. \quad \triangleleft \end{aligned}$$

Brownian Martingales

Example Let W_t be a Brownian motion, show that $W_t^2 - t$ is a martingale.

Solution Consider $0 \leq s \leq t$, we can show that

$$\begin{aligned}\mathbb{E}_s[W_t^2 - t] &= \mathbb{E}_s[(W_t - W_s + W_s)^2] - t \\ &= W_s^2 - s. \quad \triangleleft\end{aligned}$$

Brownian Martingales

Example If W_t is a Brownian motion, show that W_t^3 isn't a martingale.

Solution To show this, we check the conditional expectation

$$\begin{aligned}\mathbb{E}_s[W_t^3] &= \mathbb{E}_s[(W_t - W_s + W_s)^3] \\ &= \mathbb{E}_s[(W_t - W_s)^3 + 3(W_t - W_s)^2 W_s + 3(W_t - W_s)W_s^2 + W_s^3] \\ &= 3(t-s)W_s + W_s^3.\end{aligned}$$

From here we observe that W_t^3 doesn't satisfy the definition of a standard Brownian motion. $\triangleleft$

Brownian Martingales

Example If W_t is a Brownian motion, show that $\exp\left(\theta W_t - \frac{\theta^2 t}{2}\right)$ is an exponential martingale.

Solution

$$\begin{aligned}
\mathbb{E}_s\left[e^{\sigma W_t - \frac{1}{2}\sigma^2 t}\right] &= \mathbb{E}_s\left[e^{\sigma(W_t - W_s)} \cdot e^{\sigma W_s - \frac{1}{2}\sigma^2 t}\right] \\
&= e^{\sigma W_s - \frac{1}{2}\sigma^2 t} \cdot \mathbb{E}_s\left[e^{\sigma(W_t - W_s)}\right] \\
&= e^{\sigma W_s - \frac{1}{2}\sigma^2 t} \cdot e^{\frac{1}{2}\sigma^2 (t-s)} \\
&= e^{\sigma W_s - \frac{1}{2}\sigma^2 s}. \quad \triangleleft
\end{aligned}$$

Brownian Motion Properties

Here's a list of Brownian motion properties that might seem odd at first glance:

- Although W_t is continuous everywhere, it is differentiable nowhere.
- Brownian motion will eventually hit any and every real value no matter how large or how negative. No matter how far above the axis, it will (with probability one) be back down to zero at some later time.
- Once Brownian motion hits a value, it immediately hits it again infinitely often, and will continue to return after arbitrarily large times.
- It doesn't matter what scale you examine Brownian motion on - it looks just the same. Brownian motion is fractal.

Brownian Motion Properties

Question Having shorted an at-the-money call option, I hedge by buying one stock when the option moves into-the-money ($S > K$), and selling that stock when the option moves out-of-the-money ($S \leq K$).

Is this hedging strategy valid?

Brownian Motion Properties

We can use Brownian motion to describe infinitesimal evolution of an asset price process.

Let S_t denote a stock price process. It would appear that stochastic calculus is the natural candidate to model the evolution of the price process:

$$\underbrace{S_{t+\Delta t}}_{\text{future price}} = \underbrace{S_t}_{\text{present price}} + \underbrace{\mu(t, S_t)\Delta t}_{\text{deterministic}} + \underbrace{\sigma(t, S_t)(W_{t+\Delta t} - W_t)}_{\text{stochastic}}$$

Taking the limit as $\Delta t \to 0^+$, we get the stochastic differential equation

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t$$

To solve the SDE, we wish we could rewrite as follow

$$\frac{dS_t}{dt} = \mu(t, S_t) + \sigma(t, S_t)\frac{dW_t}{dt}.$$

Question Can we proceed to apply ordinary calculus to solve this 1^{st}-order differential equation?

Scaling and Time Inversion

A number of transformations will not affect the properties of a Brownian motion:

1. $W_t :\to -W_t$, reflection along the x-axis (multiplied by -1) will not affect the properties of W_t as a Brownian motion since the distribution is symmetric around 0.
2. $W_t :\to \alpha W\left(\frac{t}{\alpha^2}\right)$, scaling by $\alpha \in \mathbb{R}^+$, we note that
$$\alpha W\left(\frac{t}{\alpha^2}\right) - \alpha W\left(\frac{s}{\alpha^2}\right) \sim N(0, t-s)$$
3. Time Inversion: we obtain a Brownian motion by the following transformation
$$W_t :\to \begin{cases} 0, & t = 0 \\ tW_{\frac{1}{t}}, & t \in \mathbb{R}^+ \end{cases}$$
We can prove this as follow:
$$\begin{aligned} \text{Cov}\left[tW\left(\frac{1}{t}\right), (t+s)W\left(\frac{1}{t+s}\right)\right] &= t(t+s)\,\text{Cov}\left[W\left(\frac{1}{t}\right), W\left(\frac{1}{t+s}\right)\right] \\ &= t(t+s)\left[\frac{1}{t+s}\right] = t \end{aligned}$$

Quadratic Variation

Strong Law of Large Numbers

Kolmogorov's strong law states that the average of a sequence of independent random variables having a common distribution will with probability 1 converge to the mean of that distribution:

$$n \to \infty \quad \Rightarrow \quad \mathbb{P}\left(\frac{X_1 + X_2 + X_3 + \cdots + X_n}{n} = \mu\right) = 1.$$

For example, if $X_1, X_2, \ldots$ is a sequence of independent binomial random variables taking values $+1$ or -1 with equal probability, then the Strong Law states that

$$\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = 0.$$

If W_t is a Brownian motion, under the time partition $\{t_0, t_1, t_2, \ldots, t_n\}$, define $\Delta_i = W_{t_i} - W_{t_{i-1}}$, then the Strong Law says with probability 1 that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Delta_i^2 = t.$$

Stopping Time

A <u>random time</u> is a random variable τ with values in $[0, \infty]$. It is called a <u>stopping time</u> if the event set $\{\tau \leq t\}$ is in $\mathcal{F}_t$ for every $t \in \mathbb{R}_+$.

A filtration $(\mathcal{F}_t)$ is defined as the information available up to time t. The increasing feature of the filtration $\mathcal{F}_s \subset \mathcal{F}_t$ for $0 < s < t$ means that information can only increase as time goes on.

$\Rightarrow$ We often use $(\Omega, \mathcal{F}, \mathbb{P})$ to denote a given probability space.

A stopping time τ for the process W_t is a random time such that for each t, the event $(\tau \leq t)$ depends only on the history of the process up to and including time t.

We shall encounter stopping times only in the context of hitting times. For a fixed a, the hitting time of level a is defined by

$$\tau_a = \inf_t \{t \geq 0 : \ W_t = a\}.$$

We take $\tau_a = \infty$ if a is never reached.

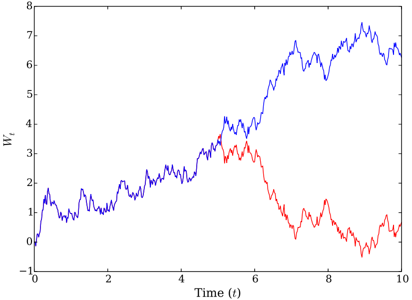
Reflection Principle

Reflection Principle

Let W_t denote a Brownian motion, and let τ be a stopping time. Define:

$$\tilde{W}_t = \begin{cases} W_t, & t \leq \tau \\ 2W_\tau - W_t, & t > \tau \end{cases}$$

then $\tilde{W}_t$ is also a Brownian motion.



Reflection Principle

Example A direct implication of the reflection principle is that if $a > 0$, then

$$\mathbb{P}(\tau_a < t) = 2 \times \mathbb{P}(W_t > a)$$

In words, let W_t denote a standard Brownian motion, this says that the probability of W_t hitting a is twice the probability of W_t exceeding a.

Why is this true?

Reflection Principle

Let W_t be a Brownian motion, and m_T be the minimum value of W_t over the interval $[0, T]$. We want to compute the probability of the event defined by

$$m_T = \min_{t \in [0,T]} W_t, \qquad m_T < y,\ W_T > x$$

for $x > y$ and $y < 0$. The Brownian motion needs to descend at least as far as y and then comes back up to level x. Let τ denote the time the Brownian motion hit the level y. We apply reflection principle at the level y and define a second random process:

$$\tilde{W}_t = \begin{cases} W_t, & t \le \tau \\ 2y - W_t, & t > \tau \end{cases}$$

The event $W_T > x$ becomes $\tilde{W}_T < 2y - x$. The crucial point here is that the event $\tilde{W}_T < 2y - x$ can only occur if $m_T < y$ also occurs. Thus the event is now much simpler

$$\mathbb{P}(W_T > x, m_T < y) = \mathbb{P}(W_T < 2y - x)$$

Building Stochastic Models

Example Let W_t denote a Brownian motion. We know that just W_t itself isn't going to be a particularly good model for a stock, since we would like to be able to scale its volatility, and also to add a deterministic drift term. To that end, with t denoting time,

$$S_t = S_0 + \mu t + \sigma W_t, \quad \mu \in \mathbb{R}, \ \sigma \in \mathbb{R}.$$

looks to be a more reasonable model for a stock. Discuss how is S_t distributed.

Solution The stock price process is normally distributed as $S_t \sim N(S_0 + \mu t, \sigma^2 t)$. You can verify this:

$$\mathbb{E}[S_t] = S_0 + \mu t.$$

$$V[S_t] = V[\sigma W_t] = \sigma^2 t.$$

Building Stochastic Models

Example Consider the stock model

$$S_t = S_0 + \mu t + \sigma W_t$$

described in the previous page. Show that in this model, there is a non-zero probability for S_t to take on negative values.

Solution Since $S_t \sim N(S_0 + \mu t, \sigma^2 t)$, its probability density function is given by

$$f(s) = \frac{1}{\sqrt{2\pi t}\sigma} \exp\left[-\frac{(s - S_0 - \mu t)^2}{2\sigma^2 t}\right].$$

The probability of the event $S_t < 0$ can then be evaluated as

$$\mathbb{P}(S_t < 0) = \frac{1}{\sqrt{2\pi t}\sigma} \int_{-\infty}^{0} e^{-\frac{(s-S_0-\mu t)^2}{2\sigma^2 t}} ds.$$

If $\mu > 0$, then as t increases this probability will decrease, but it remains a non-zero positive value, since the density for $S_t < 0$ is non-zero.

Building Stochastic Models

A stochastic process is a continuous process that can be written either in the integral form

$$S_t = S_0 + \int_0^t \sigma \, dW_u + \int_0^t \mu \, du,$$

or equivalently in the differential form

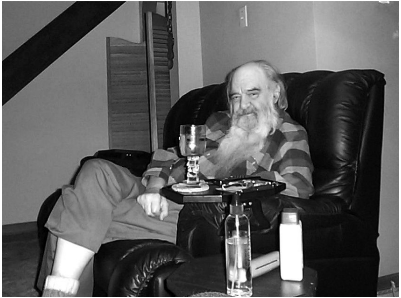
$$dS_t = \mu \, dt + \sigma \, dW_t.$$

In most cases, especially in finance, σ and μ are functions of S_t and t only, so we write

$$dS_t = \mu(t, S_t) \, dt + \sigma(t, S_t) \, dW_t.$$

Suppose μ and σ are both constants, such that $\mu(t, S_t) = \mu$ and $\sigma(t, S_t) = \sigma$, then we can solve the SDE for this simple case and obtain

$$S_t = S_0 + \mu t + \sigma W_t.$$



A stochastic process is about the results of convolving probabilities – which is just what management is about, as well.

— Stafford Beer (1926 – 2002)