Sketch of Stochastic Integral Properties Proof

In this note we shall take a look at the sketch of the proof of the 2 stochastic integral properties without invoking measure theory and σ-algebra. This will help us build up our intuition and understanding of stochastic integrals. Suppose f is a bounded function. Let I denote the stochastic integral

$$I = \int_0^T f(t, W_t)\, dW_t.$$

The two key properties of the stochastic integral I are

1. Stochastic integrals have 0 mean:

$$\mathbb{E}[I] = 0$$

2. Itô's Isometry:

$$\mathbb{E}\left[I^2\right] = \mathbb{E}\left[\int_0^T f(t, W_t)^2\, dt\right]$$

1 Stochastic integrals have 0 mean

Recall that our first principle definition of the stochastic integrals is given by

$$\int_0^T f(t, W_t)\, dW_t = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, W_{t_i}) \times (W_{t_{i+1}} - W_{t_i})$$

Note that Brownian motion's increments are independent, so $(W_{t_{i+1}} - W_{t_i})$ is independent from W_{t_i}. Hence, we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^T f(t, W_t)\, dW_t\right] &= \mathbb{E}\left[\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, W_{t_i}) \times (W_{t_{i+1}} - W_{t_i})\right] \\
&= \lim_{N\to\infty} \sum_{i=0}^{N-1} \mathbb{E}\left[f(t_i, W_{t_i}) \times (W_{t_{i+1}} - W_{t_i})\right] \\
&= \lim_{N\to\infty} \sum_{i=0}^{N-1} \mathbb{E}\left[f(t_i, W_{t_i})\right] \times \mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})\right] \\
&= \lim_{N\to\infty} \sum_{i=0}^{N-1} \mathbb{E}\left[f(t_i, W_{t_i})\right] \times 0 = 0
\end{aligned}$$

2 Itô Isometry

Let $\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}$ and $\Delta W_{t_j} = W_{t_{j+1}} - W_{t_j}$, note that

$$\mathbb{E}\Big[f(t_i, W_{t_i}) \cdot f(t_j, W_{t_j}) \cdot \Delta W_{t_i} \cdot \Delta W_{t_j}\Big] = \begin{cases} \mathbb{E}[f(t_i, W_{t_i})^2] \times (t_{i+1} - t_i) & i = j \\ 0 & i \neq j \end{cases}$$

We have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T f(t,W_t)\,dW_t\right)^2\right] &= \mathbb{E}\left[\int_0^T f(s,W_s)\,dW_s \times \int_0^T f(u,W_u)\,dW_u\right] \\
&= \lim_{N\to\infty} \sum_{i=0,j=0}^{N} \mathbb{E}\Big[f(t_i,W_{t_i}) \cdot f(t_j,W_{t_j}) \cdot \Delta W_{t_i} \cdot \Delta W_{t_j}\Big] \\
&= \lim_{N\to\infty} \sum_{i=0}^{N} \mathbb{E}[f(t_i,W_{t_i})^2] \times (t_{i+1}-t_i) \\
&= \mathbb{E}\left[\int_0^T f(t,W_t)^2\,dt\right]
\end{aligned}$$