PROPERTIES OF EXPECTATION

In General:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

Mean:

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

Variance:

$$E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

Stirling's Approximation:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

NORMAL DISTRIBUTION

PDF, where $N(\mu,\sigma^2)$:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

CDF:

$$F(x) = \int_{-\infty}^{x} f(u)du$$

MGF:

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

STANDARD NORMAL DIST.

PDF, where $N(0,1)$: $\phi(x)$

CDF: $\Phi(x)$

EXPONENTIAL DISTRIBUTION

PDF:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

CDF:

$$F(x) = \int_{-\infty}^{x} f(u)du = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Mean & Variance:

$$E[X] = \frac{1}{\lambda}, \qquad V[X] = \frac{1}{\lambda^2}$$

1-D RANDOM WALK

When $p = q = 0.5$:

$$f(k) = \frac{k}{N}$$

$$g(k) = k(N-k)$$

When $p \neq q$:

$$f(k) = \frac{\left(\frac{q}{p}\right)^k - 1}{\left(\frac{q}{p}\right)^N - 1}$$

$$g(k) = \frac{k}{q-p} - \frac{N}{q-p} f(k)$$

	0^o	45^o	90^o	135^o	180^o	225^o	270^o	270^o	360^o
	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\sin x$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0
$\cos x$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1

Variance:

$$Var(X) = E[(X-\mu)^2] = E[X^2] - \mu^2$$

$$Var(aX+b) = a^2 Var(X), \qquad Var(a) = 0$$

$$Var(aX+bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X,Y)$$

$$Var(aX-bY) = a^2Var(X) + b^2Var(Y) - 2abCov(X,Y)$$

Covariance:

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

$$Cov(X,Y) = E[XY] - \mu_X\mu_Y$$

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$Cov(X,X) = V[X]$$

$$Cov(a+X,Y) = Cov(X,Y)$$

$$Cov(aX,bY) = abCov(X,Y)$$

$$Cov(X,Y+Z) = Cov(X,Y) + Cov(X,Z)$$

Correlation:

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

$$\hat{\beta} = \frac{\rho(X,Y)\sigma_Y}{\sigma_X}, \quad \hat{\beta} = \text{slope of lin reg curve}$$

NORMAL DISTRIBUTION

$$Mean = E[X] = \frac{a+b}{2}, \qquad Variance = V[X] = \frac{(b-a)^2}{12}$$

JOINT DISTRIBUTIONS

Joint CDF:

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$$

Marginal PDFs:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

Conditional Distribution:

$$f_{x|y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

Joint PDF, When x and y are Independent:

$$f(x,y) = f_X(x)f_Y(y)$$

$$E[XY] = E[X]E[Y]$$

POISSON PROCESS

$$P(N_t - N_0 = n) = \frac{(t\lambda)^n e^{-t\lambda}}{n!}$$

POISSON DISTRIBUTION

$\lambda = np$, the exp no. of events over t

When $n \to \infty, p \to 0$

PDF:

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

LAW OF ITERATED EXPECTATION

$$E[X] = E[E[X|Y]] = \begin{cases} \int_{-\infty}^{\infty} E[X|Y]f_Y(y)dy, & Y \text{ is continuous} \\ \sum_i E[X|Y_i]P(Y_i), & Y \text{ is discrete} \end{cases}$$

Skewedness, the Standardised 3rd Central Moment:

$$\gamma = \frac{E[(X-\mu)^3]}{\sigma^3}$$

Kurtosis, Related to 4th Central Moment:

$$\kappa = \frac{E[(X-\mu)^4]}{\sigma^4}$$

Excess Kurtosis:

$$\kappa_{ex} = \kappa - 3$$

PROBABILITY THEORY

Total Probability Law:

$$P(B) = \sum_i P(A_i \cap B)$$

$$E[X] = \sum_{i=1}^{n} E[X|B_i]P(B_i)$$

Bayes' Theorem:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})} = \frac{P(A\cap B)}{P(B)}$$

Other:

$$P(A\cap B) = P(A|B)P(B) = P(B|A)P(A)$$

If A and B are Independent:

$$P(A\cap B) = P(A)P(B)$$

PROPERTIES OF EXPECTATION

$$E[aX+b] = aE[X] + b$$

$$E[aX+bY] = aE[X] + bE[Y]$$

MOMENT GEN. FUNCTIONS

$$M_X(t) = E[E^{tX}] = \int_{-\infty}^{\infty} e^{tX} f(x)dx$$

$$E[X^n] = \frac{d^n M_X(t)}{dt^n}, t = 0$$

$$M'_X(0) = E[X] = \mu_X$$

$$M''_X(0) - [M'_X(0)]^2 = E[X^2] - E[X]^2 = \sigma_X^2$$

BINOMIAL DISTRIBUTION

Number of Permutations:

$$\frac{n!}{x!\,(n-x)!} = \binom{n}{x}$$

Probability of x successes in n trials:

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Mean & Variance:

$$E[X] = np, \qquad V[X] = np(1-p)$$

MGF:

$$M_X(t) = (pe^t + q)^n$$

BOLD PLAY

Where $f(0) = 0, f(1) = 1$:

$$f(k) = \begin{cases} pf(2x), & x \in [0, 0.5] \\ p + qf(2x-1), & x \in [0.5, 1] \end{cases}$$

Where $g(0) = 0, g(1) = 0$:

$$g(k) = \begin{cases} 1 + pg(2x), & x \in [0, 0.5] \\ 1 + qg(2x-1), & x \in [0.5, 1] \end{cases}$$

OPTIMAL EXECUTION

Growth Rate for Bet Size x:

$$G(x) = plog(1+x) + (1-p)log(1-x)$$

Kelly's Criterion:

$$x = 2p - 1$$

LOGARITHMS

$$ln\, xy = ln\, x + ln\, y, \qquad ln\, x^y = yln\, x$$

$$ln\, e^x = x, \qquad e^{lnx} = x$$

$$ln\, e = 1, \qquad ln1 = 0$$

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COPULA

Sklar's Theorem:

$$F_{XY}(x,y) = C(F_x(x), F_Y(y)) = C(u,v)$$

Frechet's Bounds:

$$Upper = min\{F_x(x), F_Y(y)\}$$

$$Lower = max\{F_x(x) + F_Y(y) - 1, 0\}$$

$$Lower \le C(u,v) \le Upper$$

When X & Y are Independent:

$$F_{XY}(x,y) = F_x(x)F_Y(y)$$

$$C_I(u,v) = uv$$

Gaussian Copula:

$$C(u,v;\rho) = \Phi(\Phi^{-1}(u), \Phi^{-1}(v); \rho)$$

Survival Copula:

$$\bar{C}(1-u, 1-v) = 1 - u - v + C(u,v)$$

First-to-Default Swap:

$$Lower = u + v - min\{u,v\}$$

$$Upper = u + v - max\{u + v - 1, 0\}$$

PORTFOLIO STATISTICS

Expected Return:

$$E[R_p] = \sum_{i=1}^{n} w_i E[R_i]$$

Portfolio Variance:

$$V[R_p] = \sum_{i=1}^{n} w_i \sigma^2(R_i) + 2\sum_{i=1}^{n}\sum_{i<j}^{n} w_i w_j Cov(R_i, R_j)$$

For Equal Weighted Portfolio, Indp. Assets:

$$V[R_p] = \frac{1}{n}\bar{\sigma}_i^2$$

For Equal Weighted Portfolio, Non-Indp. Assets:

$$V[R_p] = \frac{1}{n}\bar{\sigma}_i^2 + \frac{n-1}{n}\bar{\sigma}_{ij}$$

Implied Average Correlation:

$$\bar{\rho}_{im} = \frac{\sigma_p^2 - \sum_{i=1}^{N} w_i^2 \sigma_i^2}{2\sum_{i=1}^{N}\sum_{j>i}^{N} w_i w_j \sigma_i \sigma_j}$$

Weighted Average Correlation:

$$\bar{\rho}_{avg} = \frac{2\sum_{i=1}^{N}\sum_{j>i}^{N} w_i w_j \rho_{ij}}{1 - \sum_{i=1}^{N} w_i^2}$$

Marginal Rate of Change in Risk Per Unit i:

$$\frac{\partial \sigma_p}{\partial \sigma_i} = \frac{1}{\sigma_p}\left[w_i \sigma_i^2 + \sum_{j \ne i}^{n} w_j Cov(R_i, R_j)\right]$$

Marginal Contribution to Overall Risk from i:

$$w_i \frac{\partial \sigma_p}{\partial \sigma_i}$$

Sample Coefficient of Variation: $s/\bar{X}$

Population Coefficient of Variation: σ/μ

Sharpe Ratio:

$$\frac{R - R_f}{s}$$

Trader's Sharpe Ratio:

$$\frac{P\&L}{s_{P\&L}}$$

Information Ratio:

$$IR = \frac{\bar{R}_p - \bar{R}_b}{\sigma_{p-b}}$$

HYPOTHESIS TESTING

Mean:

$$t_{n-1} = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}, \quad df = n - 1$$

Mean Differences:

$$d_i = x_i - y_i$$

$$\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i$$

$$s_d^2 = \frac{1}{n-1}\sum_{i=1}^{n}(d_i - \bar{d})^2$$

$$s_{\bar{d}} = \frac{s_d}{\sqrt{n}}$$

$$t = \frac{\bar{d} - \mu_{d0}}{s_{\bar{d}}}, \quad df = n - 1$$

Variance:

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}, \quad df = n - 1$$

2 Variances:

$$F = \frac{s_1^2}{s_2^2}, \quad F \ge 1$$

$$df_1 = n_1 - 1, \quad df_2 = n_2 - 1$$

CONFIDENCE INTERVAL

For Z Distribution:

$$\bar{X} \pm z_{a/2}\frac{\sigma}{\sqrt{n}}$$

For t Distribution:

$$\bar{X} \pm t_{a/2}\frac{s}{\sqrt{n}}$$

RADON-NIKODYM PROCESS

Radon-Nikodym Derivative:

$$\frac{dQ}{dP} \; or \; \frac{dP}{dQ}$$

$$P = Physical\ Prob, Q = Risk\ Neutral\ Prob$$

Conversion:

$$E^P\left[X\frac{dQ}{dP}\right] = E^Q[X]$$

$$E^Q\left[X\frac{dP}{dQ}\right] = E^P[X]$$

INTEGRALS (PART I)

$$\int x^n\, dx = \frac{1}{n+1}x^{n+1} + c, \quad n \ne -1$$

$$\int x^{-1}\, dx = ln|x| + c$$

$$\int \frac{1}{ax+b}\, dx = \frac{1}{a}ln|ax+b| + c$$

$$\int ln\, u\, du = u\, ln(u) - u + c, \quad \int e^u\, du = e^u + c$$

$$\int cos\, u\, du = sin\, u + c, \quad \int sin\, u\, du = -cos\, u + c$$

UNBIASEDNESS

$$E[\bar{X}] = \mu$$

$$E[s^2] = \sigma^2$$

CONSISTENCY

$$V[\bar{X}] = \frac{\sigma^2}{n}, V[\bar{X}] \to 0 \text{ when } n \to \infty$$

$$V[s^2] = \frac{V[X_i^2]}{n}, V[s^2] \to 0 \text{ when } n \to \infty$$

NO ARBITRAGE FRAMEWORK

Stock Forward Price:

$$K = E[S_T] = S_0 e^{rT}$$

Relationship:

$$0 < d < 1 + r < u$$

Recombining Tree:

$$d = \frac{1}{u}$$

Risk Neutral Probabilities:

$$p^* = \frac{(1+r) - d}{u - d}, \quad q^* = \frac{u - (1+r)}{u - d}$$

Delta Hedging Formula:

$$\triangle_0 = \frac{V_1(U) - V_1(D)}{S(U) - S_1(D)}$$

DERIVATIVES

Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Product Rule:

$$(fg)' = f'g + fg'$$

Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Chain Rule:

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))g'(x)$$

Common Derivatives:

$$\frac{d}{dx}(sin\, x) = cos\, x$$

$$\frac{d}{dx}(cos\, x) = -sin\, x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = a^x ln(a)$$

$$\frac{d}{dx}(ln(x)) = \frac{1}{x}, \quad x > 0$$

INTEGRALS (PART II)

Substitution when $g(x) = u$:

$$\int f(u)\, du = \int f(g(x))g'(x)dx$$

Integration by Parts:

$$\int fg'\, dx = fg - \int f'g\, dx$$

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