Multi-Period Asset Pricing

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Multi-Period Setting – Part 1

- Consider investor with T-period planning horizon and time-separable utility of consumption:

$$V(C_0, \ldots, C_T) = E\left[\sum_{t=0}^{T} \delta^t U(C_t)\right]$$

- Here $\delta \in (0,1)$ is subjective discount factor that reflects investor's rate of time preference, while U is (strictly increasing and concave) utility function of consumption
- Investor is endowed with initial wealth of W_0 and trades in n risky assets with random return of $R_{i,t+1}$ over time interval from t to $t+1$, for $i = 1, \ldots, n$ and $t = 0, \ldots, T-1$

Multi-Period Setting – Part 2

- Investor can rebalance portfolio at start of each time interval
- Investor allocates proportion $w_{i,t}$ of savings of $(W_t - C_t)$ to i'th asset at time t, subject to $\sum_{i=1}^{n} w_{i,t} = 1$:

$$W_{t+1} = (W_t - C_t) \sum_{i=1}^{n} w_{i,t} R_{i,t+1}$$

- Investor must choose consumption and asset allocation for each time period to maximise lifetime expected utility
- In principle, problem can be solved using recursive procedure known as **dynamic programming**

Single-Period Asset Pricing

- Intertemporal allocation condition for optimal consumption:

$$U'(C_t) = \delta E_t\left[U'(C_{t+1})\, R_{t+1}\right]$$

- Here $E_t[\cdot]$ is expectation conditional on information at time t
- Divide through to get single-period asset pricing formula:

$$E_t\left[\delta \frac{U'(C_{t+1})}{U'(C_t)} R_{t+1}\right] = E_t[M_{t+1} R_{t+1}] = 1$$

- Here $M_{t+1} = \delta U'(C_{t+1}) / U'(C_t)$ is investor's intertemporal marginal rate of substitution over time interval from t to $t+1$, which represents single-period pricing kernel

Dividend Discount Model – Part 1

- Consider long-lived asset with price of P_{t+i} for $i = 0, \ldots, T$, and which pays regular dividend of D_{t+i} for $i = 1, \ldots, T$
- Hence holding period return over first time interval:

$$R_{t+1} = \frac{D_{t+1} + P_{t+1}}{P_t}$$

- Apply single-period asset pricing formula to get price of long-lived asset at time t:

$$P_t = E_t[M_{t+1}\,(D_{t+1} + P_{t+1})]$$

Dividend Discount Model – Part 2

- By extension, price of long-lived asset at time $t+1$:

$$P_{t+1} = E_{t+1}[M_{t+2}\,(D_{t+2} + P_{t+2})]$$

- Substitute for P_{t+1} and use law of iterated expectations:

$$\begin{aligned} P_t &= E_t[M_{t+1}D_{t+1} + M_{t+1}E_{t+1}[M_{t+2}\,(D_{t+2} + P_{t+2})]] \\ &= E_t[M_{t+1}D_{t+1} + M_{t,t+2}\,(D_{t+2} + P_{t+2})] \end{aligned}$$

- Here $M_{t,t+2} = M_{t+1}M_{t+2}$ is two-period pricing kernel for pricing assets over time interval from time t to $t+2$

Dividend Discount Model – Part 3

- Repeat process to get general pricing formula:

$$P_t = E_t\left[\sum_{i=1}^{T} M_{t,t+i}D_{t+i} + M_{t,t+T}P_{t+T}\right]$$

- Here $M_{t,t+i} = M_{t+1} \cdots M_{t+i}$ is multi-period pricing kernel, for pricing assets over time interval from t to $t+i$
- Pricing kernel is given by investor's intertemporal marginal rate of substitution: $M_{t,t+i} = \delta^i U'(C_{t+i})\,/\,U'(C_t)$
- What if investor has infinite lifetime, and long-lived asset has no fixed maturity date?

Dividend Discount Model – Part 4

- Assume no price "bubbles", so $\lim_{T\to\infty} E_t[M_{t,t+T}P_{t+T}] = 0$
- Let $T \to \infty$ to get "discounted dividend" pricing formula:

$$P_t = E_t\left[\sum_{i=1}^{\infty} M_{t,t+i}D_{t+i}\right]$$

- Infinite sum will converge to finite value as long as dividends do not grow too quickly
- From economic perspective, dividends for single asset cannot consistently grow faster than overall economy, or else single asset will become larger than entire economy!

Representative Investor

- If asset market is "complete" and frictionless, then there is unique pricing kernel that prices all assets
- Even if individual investors have different utility functions, complete market allows perfect risk sharing
- Hence individual investors will shift consumption across time periods until all investors become marginally homogenous:

$$\delta_i \frac{U_i'(C_{i,t+1})}{U_i'(C_{i,t})} = \delta_j \frac{U_j'(C_{j,t+1})}{U_j'(C_{j,t})} = \delta \frac{U'(C_{t+1})}{U'(C_t)} = M_{t+1}$$

- Same pricing kernel as economy with **representative investor** who consumes (per capita) aggregate consumption

Lucas (1978) "Fruit Tree" Economy

- Endowment economy with fixed supply of risky assets and exogenous real output process
- Can interpret output process as fruit tree that grows (and hence produces more fruit) at random rate
- No reinvestment or depreciation (since output process grows exogenously) and no storage technology, so aggregate consumption must be equal to output in every time period
- Complete and frictionless market for risky assets
- Shares of risky assets represent ownership claim on output, so aggregate dividend (of market portfolio) must be equal to aggregate consumption in every time period

Market Equilibrium

- Asset market is in equilibrium when supply of risky assets is in balance with demand for risky assets
- Supply of risky assets is fixed in endowment economy, so asset prices must adjust to bring supply and demand into balance
- Opposite is true for production economy, where asset prices are fixed and supply must adjust to balance demand
- Use pricing kernel to relate future dividends to current prices
- Pricing kernel is given by representative investor's intertemporal marginal rate of substitution, so must specify representative investor's utility function

Power Utility

- Suppose investor has power utility: $U(C_t) = (1-\gamma)^{-1} C_t^{1-\gamma}$, then pricing kernel: $M_{t+i} = \delta^i \left(C_{t+i}/C_t\right)^{-\gamma}$
- Hence price-dividend ratio of market portfolio, given that aggregate consumption is same as aggregate dividend:

$$\frac{P_t}{D_t} = E_t\left[\sum_{i=1}^{\infty} \delta^i \left(\frac{D_{t+i}}{D_t}\right)^{1-\gamma}\right]$$

- Must specify distribution of dividend (or consumption) growth rate in order to solve for price-dividend ratio
- Can solve algebraically for simple distributions, but need numerical simulation for more complicated distributions

Lognormal Growth: Economic Environment

- Suppose that natural log of aggregate consumption evolves as random walk with drift:

$$\ln C_{t+1} = \ln C_t + \mu_c + \sigma_c \epsilon_{t+1}$$

- Here $\epsilon_t \sim N(0,1)$ is i.i.d. random variable that represents effect of economic fluctuations on consumption growth
- Hence continuously compounded consumption growth rate has normal distribution over every time interval
- Then μ_c represents average long-run consumption growth rate, while σ_c represents volatility of economic fluctuations
- Let $\rho = -\ln \delta$ be investor's rate of time preference

Lognormal Growth: Market Portfolio – Part 1

- Consider price-dividend ratio of equity claim that gives ownership of aggregate dividend for next time period only:

$$\frac{P_{1,t}}{D_t} = E_t\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right] = e^{-\rho+(1-\gamma)\mu_c+\frac{1}{2}(1-\gamma)^2\sigma_c^2} = \theta$$

- Since consumption growth is i.i.d., price-dividend ratio of equity claim on aggregate dividend for any future time period:

$$\frac{P_{i,t}}{D_t} = E_t\left[\delta^i\left(\frac{C_{t+i}}{C_t}\right)^{1-\gamma}\right] = \prod_{\tau=t}^{t+i-1} E_t\left[\delta\left(\frac{C_{\tau+1}}{C_\tau}\right)^{1-\gamma}\right] = \theta^i$$

Lognormal Growth: Market Portfolio – Part 2

- Market portfolio consists of sum of all one-period equity claims, so has finite constant price-dividend ratio when $\theta < 1$:

$$\frac{P_t}{D_t} = \sum_{i=1}^{\infty} \frac{P_{i,t}}{D_t} = \sum_{i=1}^{\infty} \theta^i = \frac{\theta}{1-\theta}$$

- Expected return for market portfolio:

$$\begin{aligned} E_t[R_{t+1}] &= E_t\left[\frac{D_{t+1} + P_{t+1}}{P_t}\right] \\ &= \frac{D_t}{P_t} E_t\left[\frac{D_{t+1}}{D_t}\left(1 + \frac{P_{t+1}}{D_{t+1}}\right)\right] \end{aligned}$$

Lognormal Growth: Market Portfolio – Part 3

- Substitute for price-dividend ratio of market portfolio:

$$\begin{aligned} E_t[R_{t+1}] &= \frac{1-\theta}{\theta} E_t\left[\frac{D_{t+1}}{D_t}\left(1+\frac{\theta}{1-\theta}\right)\right] \\ &= \frac{1}{\theta} E_t\left[\frac{C_{t+1}}{C_t}\right] = e^{\rho+\gamma\mu_c-\frac{1}{2}\gamma^2\sigma_c^2+\gamma\sigma_c^2} \end{aligned}$$

- Since investor's lifetime utility is time-separable and consumption growth rate is i.i.d., market portfolio has same expected return as next-period equity claim:

$$E_t[R_{t+1}] = \frac{D_t}{P_{1,t}} E_t\left[\frac{D_{t+1}}{D_t}\right] = E_t\left[\frac{D_{t+1}}{P_{1,t}}\right] = E_t[R_{1,t+1}]$$

Lognormal Growth: Equity Premium

- Suppose there exists asset that provides riskless payoff of one unit of output in next time period:

$$R_{f,t} = \frac{1}{P_{f,t}} = E_t\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]^{-1} = e^{\rho+\gamma\mu_c-\frac{1}{2}\gamma^2\sigma_c^2} = R_f$$

- Continuously compounded equity premium:

$$\ln E_t[R_{t+1}] - \ln R_f = \gamma\sigma_c^2$$

- Consumption growth is smooth: $\sigma_c \approx 2\%$ per year, so equity premium is too small for "reasonable" levels of risk aversion

Rare Disasters: Economic Environment

- Assume that natural log of aggregate consumption evolves as "augmented" random walk with drift:

$$\ln C_{t+1} = \ln C_t + \mu_c + \sigma_c \epsilon_{t+1} + \nu_{t+1}$$

- Here ν_t is i.i.d. random variable that represents effect of rare disaster on consumption growth:

$$\nu_t = \begin{cases} \ln \phi & \text{with probability } \pi \\ 0 & \text{with probability } 1 - \pi \end{cases}$$

- Then $\phi \in (0, 1)$ represents fraction of aggregate consumption that remains in event of disaster

Rare Disasters: Market Return

- Price-dividend ratio for next-period equity claim:

$$\frac{P_{1,t}}{D_t} = \theta E_t\left[e^{(1-\gamma)\nu_{t+1}}\right] = \left(1 + \pi\left(\phi^{1-\gamma} - 1\right)\right)\theta$$

- Same expected return for market portfolio and next-period equity claim, so use $\ln(1+x) \approx x$ for $x \ll 1$ to get continuously compounded expected market return:

$$E_t[R_{t+1}] = \frac{D_t}{P_{1,t}} E_t\left[\frac{C_{t+1}}{C_t}\right] = \frac{\left(1+\pi\left(\phi - 1\right)\right) e^{\mu_c + \frac{1}{2}\sigma_c^2}}{\left(1 + \pi\left(\phi^{1-\gamma} - 1\right)\right)\theta}$$

$$\ln E_t[R_{t+1}] \approx \rho + \gamma\mu_c - \frac{1}{2}\gamma^2\sigma_c^2 + \gamma\sigma_c^2 - \pi\phi\left(\phi^{-\gamma} - 1\right)$$

Rare Disasters: Equity Premium

- Continuously compounded risk-free rate and equity premium:

$$R_f = \left(1 + \pi\left(\phi^{-\gamma} - 1\right)\right)^{-1} e^{\rho + \gamma\mu_c - \frac{1}{2}\gamma^2\sigma_c^2}$$

$$\ln R_f \approx \rho + \gamma\mu_c - \frac{1}{2}\gamma^2\sigma_c^2 - \pi\left(\phi^{-\gamma} - 1\right)$$

$$\ln E_t[R_{t+1}] - \ln R_f \approx \gamma\sigma_c^2 + \pi\left(1 - \phi\right)\left(\phi^{-\gamma} - 1\right)$$

- Barro (2006) finds that $\pi = 0.017$ and $\phi = 0.65$ during 20th century, so for relative risk aversion of $\gamma = 6$:

$$\ln \frac{E_t[R_{t+1}]}{R_f} \approx 0.0024 + 0.00595 \times \left(\frac{1}{0.65^6} - 1\right) = 7.5\%$$