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QF603 Quantitative Analysis for Financial Markets
Assignment 3

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Problem 1

a) the estimate for b

$$\hat{b} = \frac{\sum_{i=1}^{60}(X_i-\bar{X})(Y_i-\bar{Y})}{\sum_{i=1}^{60}(X_i-\bar{X})^2} = \frac{\sum_{i=1}^{60}(X_iY_i - X_i\bar{Y} - \bar{X}Y_i + \bar{X}\bar{Y})}{\sum_{i=1}^{60}(X_i^2 - 2X_i\bar{X} + \bar{X}^2)}$$

$$= \frac{0.005 - 60\times0.005\times0.003 - 60\times0.005\times0.003 + 60\times0.005\times0.003}{0.004 - 2\times60\times0.005^2 + 60\times0.005^2}$$

$$= 1.64$$

the estimate for a

$$\bar{Y} = \hat{a} + \hat{b}\bar{X}$$

$$\therefore\ 0.003 = \hat{a} + 1.64\times0.005$$

$$\hat{a} = -0.0052$$

b) $$t_{58}(\hat{a}) = \frac{\hat{a}-0}{SE(\hat{a})}$$

$$= \frac{\hat{a}}{\hat{\sigma}_e\sqrt{\frac{1}{n} + \frac{\bar{X}^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2}}}$$

$$= \frac{-0.0052}{\sqrt{\frac{1}{58}\times5.8\times\frac{1}{10^5}}\sqrt{\frac{1}{60} + \frac{0.005^2}{0.004-2\times60\times0.005^2+60\times0.005^2}}}$$

$$\doteqdot -31.843$$

$t_{58,0.995} \doteqdot 2.66$, $t_{58,0.005} \doteqdot -2.66$

$-31.843 < -2.66$

$\therefore$ at 1% significance level, we can reject the null hypothesis that $a=0$.
It means that the equity portfolio has abnormal extra return / loss to the market's return at a given level of risk.

c) To test null hypothesis $H_0: b=1$.

$$\text{test statistics} = \frac{\hat{b}-1}{SE(\hat{b})}$$

$$= \frac{\hat{b}-1}{\hat{\sigma}_e\sqrt{\frac{1}{\sum_{i=1}^{n}(X_i-\bar{X})^2}}}$$

$$= \frac{1.64-1}{\sqrt{\frac{1}{58}\times 5.8\times\frac{1}{10^5}}\times\sqrt{\frac{1}{0.004-2\times 60\times 0.005^2+60\times 0.005^2}}}$$

$= 32$.

$t_{58,0.995} \doteqdot 2.66$, $t_{58,0.005} \doteqdot -2.66$

$32 > 2.66$

$\therefore$ At 1% significance level, we can reject the null hypothesis that $b=1$
It means that the variation of the equity portfolio return is not exactly same as the variation of the market return

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Problem 2.

a) $cov(a + br_{mt} + e_{it}, e_{it})$

$= cov(br_{mt}, e_{it}) + cov(e_{it}, e_{it})$

$= 0 + \sigma_e^2$

Since $r_{it}$ has linear correlation with $e_{it}$, these two variables are not independent of each other.

b) When it comes to $\hat{b}$, the student's interpretation is solid. How we get beta in CAPM corresponds to how we calculate $\hat{b}$ in a regression model. Both are determined by below formula.

$$\beta = \hat{b} = \frac{cov(\text{stock i's return, market portfolio's return})}{VAR(\text{market portfolio's return})}$$

Therefore, we can say that $\hat{b}$ denotes how strongly stock i's return is related to the market return; $\hat{b}$ is proportionate to CAPM's notion of systematic risk of stock i.

In case of $\hat{a}$, the situation is different. According to the given regression model in the question, $r_{it} = \hat{a} + \hat{b}\, r_{mt}$.
On the other hand, in CAPM the regression model is about market risk premium and excessive return of an individual stock. Therefore, the formula is as below.

$r_i - r_f = \alpha + \beta(r_m - r_f)$ where, $r_f$ is risk-free rate

If we rearrange the formula, we get

$r_i = (1-\beta) r_f + \alpha + \beta r_m$

Combining two formulas, $\hat{a} = (1-\beta) r_f + \alpha$ where, $\alpha$ is Jensen's alpha.
In conclusion, we cannot say that $\hat{a}$ is equal to Jensen's alpha, even though they are related each other.

c) To tell the conclusion first, the answer is no. This is because the performance of a portfolio has a relationship not only with $\hat{a}$, but also with $\hat{b}$, or beta.
For example, in a bear market, high beta portfolio is likely to underperform the market even if the $\hat{a}$ is positive.
Likewise, low beta portfolio in a bull market would have lower return rate than the market return in many cases.
Therefore, composing a portfolio with positive $\hat{a}$ stocks does not lead to higher return than the market return.

Problem 3

a) 6 by 2

$$X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \quad X' = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

$$\therefore X'X = \begin{pmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{pmatrix} = \begin{pmatrix} 6 & 45 \\ 45 & 355 \end{pmatrix}$$

$n = 6.$

b) $\hat{\beta} = (X'X)^{-1}X'y$

$$= \frac{1}{105}\begin{pmatrix} 355 & -45 \\ -45 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{105}\begin{pmatrix} 400 \\ -51 \end{pmatrix}$$

slope of simple linear regression is $-\frac{17}{35}$

c) From (a), we know that $\sum_{i=1}^{6} x_i = 45.$

$$\bar{x} = \frac{45}{6} = \frac{15}{2}$$

d) Variance of explanatory variable

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i^2 - n\hat{\mu}^2}{n-1}$$

$$= \frac{355 - 6\times\left(\frac{45}{6}\right)^2}{5} = \frac{7}{2}$$

e) $Cov(\hat{a}, \hat{b}) = -\sigma_e^2 \left( \frac{\bar{x}}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \right)$

$$-\frac{3}{28} = -\sigma_e^2 \times \frac{\frac{15}{2}}{\frac{35}{12} \times 6} \qquad \therefore \sigma_e^2 = \frac{1}{4}$$

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f) t statistic of the y-intercept

$$= \frac{\hat{a} - 0}{\hat{\sigma}_e \sqrt{\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}} = \frac{\frac{80}{21}}{\frac{1}{2} \times \sqrt{\frac{1}{6} + \frac{\left(\frac{15}{2}\right)^2}{\frac{35}{12} \times 6}}}$$

$$= 4.14$$

g) i) $\hat{y}_{n+1} = \hat{a} + \hat{b} x_{n+1}$

$$= \frac{80}{21} - \frac{17}{35} \times \frac{3}{2}$$

$$\fallingdotseq 3.08$$

ii) $\hat{y}_{n+1}$'s prediction interval

$$= \hat{\beta}' x \pm q\hat{\sigma}_u \sqrt{1 + x'(X'X)^{-1}x}$$

$$= \begin{pmatrix} \frac{80}{21} & -\frac{17}{35} \end{pmatrix} \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} \pm t_{4,0.975} \frac{1}{2} \sqrt{1 + (1, 1.5) \begin{pmatrix} \frac{11}{21} & -\frac{3}{7} \\ -\frac{3}{7} & \frac{2}{35} \end{pmatrix} \begin{pmatrix} 1 \\ 1.5 \end{pmatrix}}$$

$$= \frac{80}{21} - 1.5 \times \frac{17}{35} \pm 2.7764 \cdot \frac{1}{2} \sqrt{1 + 2.2238}$$

$$= 3.0810 \pm 2.4925$$

$$\therefore \quad 0.5885 \leq \hat{y}_{n+1} \leq 5.5735$$

- The upper bound of the prediction interval at the 5% level of significance is 5.57

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