# Multi-Factor Models
# Performance Measurement
# Black–Litterman Model

Wang Wei Mun

Lee Kong Chian School of Business
Singapore Management University

October 21, 2018

# Pricing Anomalies

- Empirical research on CAPM has uncovered various systematic pricing "anomalies"
- Size effect: "small-cap" stocks (with low market value) tend to outperform "big-cap" stocks (with high market value)
- Value effect: "value" stocks (with high book-to-market ratio) outperform "growth" stocks (with low book-to-market ratio)
- Momentum effect: "winner" stocks (with recent high returns) tend to outperform "loser" stocks (with recent low returns)
- Pricing anomalies persist even after adjusting for differences in expected return due to differences in exposure to market risk

# Multi-Factor Linear Pricing Models

- Since APT provides no guidance on nature of systematic risk factors, popular approach is to augment market model with additional risk factors to account for pricing anomalies
- Similar to market model, use returns on **factor-mimicking portfolios** to represent effect of risk factors
- In principle, factor-mimicking portfolio has sensitivity of one to specific risk factor and sensitivity of zero to other risk factors
- In practice, factor-mimicking portfolios will not be completely insensitive to other risk factors
- Factor-mimicking portfolios are usually zero-investment portfolios, which reduces correlation among risk factors

# Fama–French 3-Factor Model – Part 1

- Fama and French (1993) use three-factor model for realised returns, with additional risk factors for size and value effects:

$$R_i - R_f = \alpha_i + \beta_i \left(R_m - R_f\right) + \gamma_i \left(R_s - R_b\right) + \delta_i \left(R_h - R_l\right) + \tilde{\epsilon}_i$$

- Risk factor for size effect ("SMB") is return on portfolio that is long on small-cap stocks and short on big-cap stocks
- Risk factor for value effect ("HML") is return on portfolio that is long on value stocks and short on growth stocks

# Fama–French 3-Factor Model – Part 2

- Economic intuition is that "size risk" and "value risk" are separate types of systematic risk that cannot be combined with each other or market risk
- If Fama–French three-factor model provides correct description of realised returns, then $\alpha_i$ will not be significantly different from zero for any asset or (passive) portfolio
- Hence risk premium for any asset is given by:

$$\begin{aligned} E[R_i - R_f] = \beta_i E[R_m - R_f] + \\ \gamma_i E[R_s - R_b] + \delta_i E[R_h - R_l] \end{aligned}$$

# Carhart 4-Factor Model – Part 1

- Carhart (1994) augments Fama–French three-factor model with additional risk factor for momentum effect:

$$\begin{aligned} R_i - R_f = \alpha_i + \beta_i\left(R_m - R_f\right) + \gamma_i\left(R_s - R_b\right) + \\ \delta_i\left(R_h - R_l\right) + \zeta_i\left(R_u - R_d\right) + \epsilon_i \end{aligned}$$

- Risk factor for momentum effect ("UMD") is return on portfolio that is long on winners and short on losers
- Performs better than Fama–French three-factor model for actively-managed portfolios, since fund managers tend to buy winners and sell losers

# Carhart 4-Factor Model – Part 2

- Economic intuition is that "momentum risk" cannot be combined with other types of systematic risk
- If Carhart four-factor model provides correct description of realised returns, then $\alpha_i$ will not be significantly different from zero for any asset or portfolio
- Hence risk premium for any asset is given by:

$$\begin{aligned} E[R_i - R_f] = \beta_i E[R_m - R_f] + \gamma_i E[R_s - R_b] + \\ \delta_i E[R_h - R_l] + \zeta_i E[R_u - R_d] \end{aligned}$$

# Sharpe Ratio

- **Sharpe ratio** is risk premium per unit of standard deviation:

$$SR_i = \frac{E[R_i - R_f]}{\sqrt{\text{Var}[R_i - R_f]}}$$

- Denominator is designed to capture total risk, i.e., both systematic and idiosyncratic risk
- However, denominator ignores higher moments such as skewness and kurtosis, so may not fully reflect risk of investment if return distribution is not normal

# Treynor Ratio

- **Treynor ratio** is risk premium per unit of market risk:

$$TR_i = \frac{E[R_i - R_f]}{\beta_i}$$

- Denominator is designed to capture systematic risk, but cannot easily be extended to account for possibility of multiple sources of systematic risk
- Measures "reward-to-risk" ratio for asset as part of well-diversified portfolio

# Jensen's Alpha

- **Jensen's alpha** is intercept coefficient from market model regression using excess returns:

$$\alpha_i = E[R_i - R_f] - \beta_i E[R_m - R_f]$$

- For passive portfolio, represents pricing error for CAPM
- For actively-managed portfolio, represents "abnormal" or "value added" preemium due to managerial skill
- Can extend to allow for multiple sources of systematic risk by using intercept coefficient from regression using Fama–French three-factor model or Carhart four-factor model

# Information Ratio

- **Information ratio** (or **appraisal ratio**) is mean deviation from target (or benchmark) return, per unit of **tracking error**:

$$IR_i = \frac{E[R_i - R_t]}{\sqrt{\mathrm{Var}[R_i - R_t]}}$$

- Same as Sharp ratio when target return is risk-free rate
- Measures ability of fund manager to exceed target return, relative to amount of additional risk exposure

# Downside Risk

- **Downside risk** of investment is risk that realised return on investment will fall below target return
- One measure of downside risk is **below-target semi-variance**:

$$\mathrm{SV}[R_i; R_t] = E\left[\min\left\{R_i - R_t, 0\right\}^2\right]$$

- Can distinguish between asymmetric return distributions with same variance but different skewness
- Intuition is that investors are only concerned with probability and magnitude of underperformance, not overperformance

# Sortino Ratio

- **Sortino ratio** is mean deviation from target (or benchmark) return, per unit of below-target semi-deviation:

$$StR_i = \frac{E[R_i - R_t]}{\sqrt{\mathrm{SV}[R_i; R_t]}}$$

- Will produce rankings similar to information ratio when return distribution is close to symmetric, and mean asset return is close to mean target return
- May deviate from rankings produced by information ratio when return distribution is highly skewed, or when mean asset return deviates substantially from mean target return

# Issues with Efficient Frontier

- Biggest econometric issue with mean-variance-efficient frontier is difficulty of estimating mean return
- Standard error of sample mean is $\sigma/\sqrt{n}$, where $\sigma$ is standard deviation of return and $n$ is number of observations
- Returns are volatile and data is limited, so estimate of mean return tends to have large standard error
- Related issue is instability of efficient frontier: small change in vector of mean returns can induce large change in composition of frontier portfolios
- Fischer Black and Robert Litterman developed model in 1990 to overcome issues with efficient frontier

# Excess Returns – Part 1

- Excess returns for tradable assets follow normal distribution:

$$\tilde{\mathbf{R}} \sim N(\tilde{\boldsymbol{\mu}}, \boldsymbol{\Sigma})$$

- Vector of (unknown) risk premiums is also random variable:

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{\pi} + \tilde{\boldsymbol{\epsilon}}$$

- Here $\tilde{\boldsymbol{\epsilon}}$ is error term with (independent) normal distribution:

$$\tilde{\boldsymbol{\epsilon}} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_{\mu})$$

# Excess Returns – Part 2

- For simplicity, assume that $\boldsymbol{\Sigma}_\mu$ is proportional of $\boldsymbol{\Sigma}$:

$$\boldsymbol{\Sigma}_\mu = \tau \boldsymbol{\Sigma}$$

- In practice, often set $\tau = 1/n$, where $n$ is number of data points used to estimate $\boldsymbol{\Sigma}$
- Hence distribution of excess returns around $\boldsymbol{\pi}$:

$$\tilde{\mathbf{R}} \sim N\Big(\boldsymbol{\pi}, (1+\tau)\,\boldsymbol{\Sigma}\Big)$$

# Implied Risk Premiums

- Use observed weights of individual assets in market portfolio to determine implied risk premiums:

$$\boldsymbol{\pi} = \lambda \boldsymbol{\Sigma} \mathbf{w}_m$$

- Here $\lambda$ is coefficient of relative risk aversion (based on initial wealth), for investor with constant absolute risk aversion
- Calibrate $\lambda$ using market risk premium or Sharpe ratio:

$$\lambda = \frac{\mathbf{w}_m' \boldsymbol{\pi}}{\mathbf{w}_m' \boldsymbol{\Sigma} \mathbf{w}_m} = \frac{R_m - R_f}{\sigma_m^2} = \frac{S_m}{\sigma_m}$$

# Investor Views

- Black–Litterman model also incorporates investor's "views" on (absolute or relative) expected returns of risky assets
- Suppose that investor has $k \geq 1$ views on expected returns
- Let $\mathbf{P}$ be $k \times n$ vector of asset weights corresponding to investor's views, and let $\mathbf{Q}$ be $k \times 1$ vector of expected returns corresponding to investor's views
- Also let $\boldsymbol{\Omega}$ be $k \times k$ covariance matrix of error terms derived from confidence of investor's views
- For simplicity, require that investor's views be uncorrelated with one another, so that $\boldsymbol{\Omega}$ is diagonal

# Example: Investor Views

- Suppose that investible universe consists of three risky assets (or asset classes, or mutual funds)
- Investor expects first risky asset to earn return of 5% per year
- Investor expects that second risky asset to outperform third risky asset by 100 basis points (i.e., 1%) per year
- Then $\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} 0.05 \\ 0.01 \end{bmatrix}$
- Notice that first row of $\mathbf{P}$ corresponds to absolute view (where weights sum to one), while second row of $\mathbf{P}$ corresponds to relative view (where weights sum to zero)

# Bayes' Theorem

- Bayes' theorem is used to update probability of given hypothesis $H$ when new evidence $E$ is observed:

$$\Pr(H|E) = \frac{\Pr(E|H)}{\Pr(E)} \Pr(H)$$

- $\Pr(H)$ is **prior distribution**, or probability that $H$ is true before any new evidence is observed
- $\Pr(E|H)$ is conditional probability of observing $E$, assuming that $H$ is true, while $\Pr(E)$ is unconditional probability of observing $E$, which is independent of $H$
- $\Pr(H|E)$ is **posterior distribution**, or probability that $H$ is true given that we have observed $E$

# Posterior Distribution of Asset Returns

- Black–Litterman model uses Bayes' theorem to update prior distribution of excess returns using investor's views
- Conditional on investor's views, excess returns have (posterior) normal distribution of $N(\hat{\boldsymbol{\pi}}, \mathbf{M})$, where:

$$\hat{\boldsymbol{\pi}} = \boldsymbol{\pi} + \tau\boldsymbol{\Sigma}\mathbf{P}' \left(\tau\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1} (\mathbf{Q} - \mathbf{P}\boldsymbol{\Pi}),$$

$$\mathbf{M} = \left(\frac{1}{\tau}\boldsymbol{\Sigma}^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1}$$