QF620 Additional Examples
Session 3: Stochastic Differential Equations

1 Examples

1. Solve the following stochastic differential equation
$$dX_t = X_t dW_t, \quad X_0 = 1.$$
Hint: consider the process of $\log(X_t)$.

2. Solve the following stochastic differential equation
$$dX_t = (a + X_t) dW_t, \quad X_0 = 0.$$
Hint: consider the process of $\log(a + X_t)$.

3. Solve the following stochastic differential equation
$$dX_t = rX_t dt + \sigma dW_t.$$
Hint: use "integrating factor" e^{-rt}.

4. Consider the stochastic differential equation
$$dS_t = \sigma S_0 dW_t,$$
where σ is the volatility and S_0 is the stock price today. What is the mean and variance of S_T? Is it normally distributed?

5. Consider the stochastic differential equation
$$dS_t = rS_t dt + \sigma S_t dW_t.$$
What is the mean and variance of S_T? Is it normally distributed?

6. Consider the stochastic differential equation
$$dS_t = rS_t dt + \sigma S_t dW_t.$$
What is the mean and variance of $\log S_T$? Is it normally distributed?

7. Consider the stochastic differential equation
$$dF_t = \sigma(F_t + a) dW_t.$$
Solve for F_T.

8. A stochastic process for a stock price is given by

$$S_t = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right].$$

Use Itô's Formula to derive the stochastic differential equation for dS_t.

9. The stochastic differential equation for the forward price is given by

$$dF_t = \sigma F_t dW_t.$$

Show that $\mathbb{E}[F_T] = F_0$.

10. We know that $\mathbb{E}[e^{\sigma W_T}] = e^{\frac{\sigma^2 T}{2}}$. What about $\mathbb{E}[e^{-\sigma W_T}]$?

11. If we use the following stochastic differential equation to model the evolution of the stock price:

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

What can we say about the process $d\frac{1}{S_t}$? What is $\mathbb{E}\left[\frac{1}{S_T}\right]$?

12. Suppose we use the following stochastic differential equations to model 2 stock price processes (X_t and Y_t):

$$\begin{cases} dX_t = rX_t dt + \sigma_X X_t dW_t \\ dY_t = rY_t dt + \sigma_Y Y_t d\tilde{W}_t \end{cases}$$

where $W_t \perp \tilde{W}_t$. What can we say about the stochastic differential equation $dZ_t = d(X_t Y_t)$?

13. Consider the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

What is the stochastic differential equation for dS_t^2? What is $\mathbb{E}[S_T^2]$?

2 Suggested Solutions

1. By Itô's formula, consider $Y_t = f(X_t)$, where $f : \mathbb{R} \to \mathbb{R},\ f(x) = \log(x)$, we have

$$
\begin{aligned}
dY_t &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 \\
&= \frac{1}{X_t}X_t dW_t - \frac{1}{2}dt \\
\therefore \quad Y_T &= W_T - \frac{T}{2} \\
\log(X_T) &= W_T - \frac{T}{2} \quad \Rightarrow \quad X_T = e^{W_T - \frac{T}{2}}
\end{aligned}
$$

2. The derivatives of the function f are given by

$$
f'(X_t) = \frac{1}{a + X_t}, \qquad f''(X_t) = -\frac{1}{(a + X_t)^2}.
$$

Let $Y_t = f(X_t) = \log(a + X_t)$. By Itô's formula, we have

$$
\begin{aligned}
dY_t &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 \\
&= \frac{1}{a + X_t}(a + X_t)dW_t - \frac{1}{2}\frac{1}{(a + X_t)^2}(a + X_t)^2 dt \\
&= dW_t - \frac{1}{2}dt
\end{aligned}
$$

Integrating both sides from 0 to T, we obtain

$$
\begin{aligned}
\int_0^T dY_t &= \int_0^T dW_t - \int_0^T \frac{1}{2}dt \\
Y_T - Y_0 &= W_T - \frac{T}{2} \\
\log(a + X_T) - \log(a + X_0) &= W_T - \frac{T}{2} \\
\log\left(\frac{a + X_T}{a}\right) &= W_T - \frac{T}{2} \\
X_T &= a\left(e^{W_T - \frac{T}{2}} - 1\right)
\end{aligned}
$$

3. Using the "integrating factor" e^{-rt}, we consider the process $Y_t = X_t e^{-rt} = f(t, X_t)$, where

$$
f : \mathbb{R}^2 \to \mathbb{R},\ f(t, x) = xe^{-rt}.
$$

Its derivatives are given by

$$
f_t = -rxe^{-rt}, \qquad f_x = e^{-rt}, \qquad f_{xx} = 0.
$$

By Itô's formula, we have

$$
\begin{aligned}
dY_t &= f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, X_t)(dX_t)^2 \\
&= -rX_t e^{-rt}dt + e^{-rt}(rX_t dt + \sigma dW_t) \\
&= \sigma e^{-rt}dW_t
\end{aligned}
$$

Now integrating both sides from 0 to T, we obtain

$$\begin{aligned}
\int_0^T dY_t &= \sigma \int_0^T e^{-rt} dW_t \\
Y_T &= Y_0 + \sigma \int_0^T e^{-rt} dW_t \\
X_T e^{-rT} &= X_0 + \sigma \int_0^T e^{-rt} dW_t \\
X_T &= X_0 e^{rT} + \sigma \int_0^T e^{-r(T-t)} dW_t
\end{aligned}$$

4. This simple stochastic differential equation can be readily solved by integrating both sides from 0 to T:

$$\begin{aligned}
&\int_0^T dS_u = \sigma S_0 \int_0^T dW_u \\
&S_T - S_0 = \sigma S_0 W_T \\
&\Rightarrow \quad S_T = S_0 + \sigma S_0 W_T.
\end{aligned}$$

   Hence the mean of S_T is given by

$$\mathbb{E}[S_T] = \mathbb{E}[S_0 + \sigma S_0 W_T] = S_0,$$

   while the variance of S_T is given by

$$V[S_T] = V[S_0 + \sigma S_0 W_T] = \sigma^2 S_0^2 V[W_T] = \sigma^2 S_0^2 T.$$

   S_T is normally distributed and $S_T \sim N(S_0, \sigma^2 S_0^2 T)$.

5. First, apply Itô's Formula to solve the stochastic differential equation, and obtain the stochastic process S_T as the solution

$$S_T = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T}.$$

   The mean is given by

$$\begin{aligned}
\mathbb{E}[S_T] &= \mathbb{E}\left[S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T}\right] \\
&= S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T} \mathbb{E}\left[e^{\sigma W_T}\right] \\
&= S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T} e^{\frac{\sigma^2 T}{2}} \\
&= S_0 e^{rT}.
\end{aligned}$$

   The variance is given by

$$\begin{aligned}
V[S_T] &= \mathbb{E}[S_T^2] - \mathbb{E}[S_T]^2 \\
&= \mathbb{E}\left[S_0^2 e^{\left(2r - \sigma^2\right)T + 2\sigma W_T}\right] - \mathbb{E}\left[S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T}\right]^2 \\
&= S_0^2 e^{\left(2r - \sigma^2\right)T} \mathbb{E}\left[e^{2\sigma W_T}\right] - S_0^2 e^{2rT} \\
&= S_0^2 e^{\left(2r - \sigma^2\right)T} e^{\frac{4\sigma^2 T}{2}} - S_0^2 e^{2rT} \\
&= S_0^2 e^{(2r + \sigma^2)T} - S_0^2 e^{2rT} \\
&= S_0^2 e^{2rT} \left(e^{\sigma^2 T} - 1\right).
\end{aligned}$$

S_T is not normally distributed (it is lognormally distributed).

6. Applying Itô's Formula to the function $X_t = f(S_t)$, where $f : \mathbb{R} \to \mathbb{R},\ f(x) = \log(x)$, we can derive the stochastic differential equation for $d\log S_t$ as follow:

$$d\log S_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dW_t.$$

Integrating both sides from 0 to T, we obtain

$$\begin{aligned}\int_0^T d\log S_u &= \left(r - \frac{\sigma^2}{2}\right) \int_0^T du + \sigma \int_0^T dW_u \\ \log S_T - \log S_0 &= \left(r - \frac{\sigma^2}{2}\right) T + \sigma W_T \\ \log S_T &= \log S_0 + \left(r - \frac{\sigma^2}{2}\right) T + \sigma W_T.\end{aligned}$$

The mean of $\log S_T$ is given by

$$\begin{aligned}\mathbb{E}[\log S_T] &= \mathbb{E}\left[\log S_0 + \left(r - \frac{\sigma^2}{2}\right) T + \sigma W_T\right] \\ &= \log S_0 + \left(r - \frac{\sigma^2}{2}\right) T.\end{aligned}$$

The variance of $\log S_T$ is given by

$$\begin{aligned}V[\log S_T] &= V\left[\log S_0 + \left(r - \frac{\sigma^2}{2}\right) T + \sigma W_T\right] \\ &= V[\sigma W_T] = \sigma^2 T.\end{aligned}$$

Yes $\log S_T$ is normally distributed and $\log S_T \sim N\left(\log S_0 + \left(r - \frac{\sigma^2}{2}\right) T, \sigma^2 T\right)$.

7. Consider the function $X_t = \log(F_t + a) = f(F_t)$, where $f : \mathbb{R} \to \mathbb{R},\ f(x) = \log(x + a)$, its partial derivatives are given by

$$f'(F_t) = \frac{1}{F_t + a}, \qquad f''(F_t) = -\frac{1}{(F_t + a)^2}.$$

Applying Itô's Formula to f, we obtain

$$\begin{aligned}dX_t &= f'(F_t)dF_t + \frac{1}{2} f''(F_t)(dF_t)^2 \\ &= \frac{1}{F_t + a}\sigma(F_t + a)dW_t - \frac{1}{2}\frac{1}{(F_t + a)^2}\sigma^2(F_t + a)^2 dt \\ &= -\frac{\sigma^2}{2}dt + \sigma dW_t.\end{aligned}$$

Integrating both sides from 0 to T, we obtain

$$\begin{aligned}\int_0^T dX_u &= -\frac{\sigma^2}{2}\int_0^T du + \sigma \int_0^T dW_u \\ X_T - X_0 &= -\frac{\sigma^2 T}{2} + \sigma W_T \\ \log(F_T + a) - \log(F_0 + a) &= -\frac{\sigma^2 T}{2} + \sigma W_T \\ \Rightarrow \quad F_T &= (F_0 + a)e^{-\frac{\sigma^2 T}{2} + \sigma W_T} - a.\end{aligned}$$

8. Consider the function $S_t = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t} = f(t, W_t)$, where $f : \mathbb{R}^2 \to \mathbb{R},\ f(t,x) = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma x}$, the partial derivatives are given by

$$\frac{\partial f}{\partial t}(t, W_t) = \left(r - \frac{\sigma^2}{2}\right) S_t, \qquad \frac{\partial f}{\partial x}(t, W_t) = \sigma S_t, \qquad \frac{\partial^2 f}{\partial x^2}(t, W_t) = \sigma^2 S_t.$$

Applying Itô's Formula, we obtain

$$\begin{aligned} dS_t &= \frac{\partial f}{\partial t}(t, W_t)dt + \frac{\partial f}{\partial x}(t, W_t)dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, W_t)(dW_t)^2 \\ &= \left(r - \frac{\sigma^2}{2}\right) S_t dt + \sigma S_t dW_t + \frac{1}{2}\sigma^2 S_t dt \\ &= r S_t dt + \sigma S_t dW_t. \end{aligned}$$

9. Consider the function $X_t = \log F_t = f(F_t)$, where $f : \mathbb{R} \to \mathbb{R},\ f(x) = \log(x)$, the partial derivatives are given by

$$f'(F_t) = \frac{1}{F_t}, \qquad f''(F_t) = -\frac{1}{F_t^2}.$$

Applying Itô's Formula, we obtain

$$\begin{aligned} dX_t &= f'(F_t)dF_t + \frac{1}{2}f''(F_t)(dF_t)^2 \\ &= \frac{1}{F_t}\sigma F_t dW_t - \frac{1}{2}\frac{1}{F_t^2}\sigma^2 F_t^2 dt \\ &= -\frac{\sigma^2}{2}dt + \sigma dW_t. \end{aligned}$$

Integrating both sides from 0 to T

$$\begin{aligned} \int_0^T dX_u &= -\frac{\sigma^2}{2}\int_0^T du + \sigma \int_0^T dW_u \\ X_T - X_0 &= -\frac{\sigma^2 T}{2} + \sigma W_T \\ \log\frac{F_T}{F_0} &= -\frac{\sigma^2 T}{2} + \sigma W_T \\ F_T &= F_0 e^{-\frac{\sigma^2 T}{2} + \sigma W_T}. \end{aligned}$$

We can show that

$$\mathbb{E}[F_T] = \mathbb{E}\left[F_0 e^{-\frac{\sigma^2 T}{2}+\sigma W_T}\right] = F_0 e^{-\frac{\sigma^2 T}{2}}\mathbb{E}\left[e^{\sigma W_T}\right] = F_0 e^{-\frac{\sigma^2 T}{2}} e^{\frac{\sigma^2 T}{2}} = F_0.$$

10. Using the "completing the square" method, noting that $W_T \sim N(0,T) \sim \sqrt{T}N(0,1)$, we have

$$
\begin{aligned}
\mathbb{E}[e^{-\sigma W_T}] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\sigma\sqrt{T}x} e^{-\frac{x^2}{2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2+2\sigma\sqrt{T}x}{2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2+2\sigma\sqrt{T}x+\sigma^2T-\sigma^2T}{2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x+\sigma\sqrt{T})^2}{2}} e^{\frac{\sigma^2T}{2}}\, dx \\
&= e^{\frac{\sigma^2T}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x+\sigma\sqrt{T})^2}{2}}\, dx \\
&= e^{\frac{\sigma^2T}{2}}.
\end{aligned}
$$

11. Consider $X_t = \frac{1}{S_t} = f(S_t)$, where $f : \mathbb{R} \to \mathbb{R},\ f(x) = \frac{1}{x}$, the partial derivatives are given by

$$
f'(S_t) = -\frac{1}{S_t^2}, \qquad f''(S_t) = \frac{2}{S_t^3}.
$$

Applying Itô's Formula, we obtain

$$
\begin{aligned}
d\frac{1}{S_t} = dX_t &= f'(S_t)dS_t + \frac{1}{2}f''(S_t)(dS_t)^2 \\
&= -\frac{1}{S_t^2}(rS_tdt + \sigma S_tdW_t) + \frac{1}{2}\frac{2}{S_t^3}\sigma^2 S_t^2 dt \\
&= (\sigma^2 - r)X_tdt - \sigma X_tdW_t.
\end{aligned}
$$

In order to be able to evaluate the expectation $\mathbb{E}\left[\frac{1}{S_T}\right]$, we first need to solve the stochastic differential equation for $X_t = \frac{1}{S_t}$. Now consider $Y_t = \log X_t = g(X_t)$, where $g : \mathbb{R} \to \mathbb{R},\ g(x) = \log(x)$, the partial derivatives are given by

$$
g'(X_t) = \frac{1}{X_t}, \qquad g''(X_t) = -\frac{1}{X_t^2}.
$$

Applying Itô's Formula to this function g, we obtain

$$
\begin{aligned}
dY_t &= g'(X_t)dX_t + \frac{1}{2}g''(X_t)(dX_t)^2 \\
&= \frac{1}{X_t}\left[(\sigma^2 - r)X_tdt - \sigma X_tdW_t\right] - \frac{1}{2}\frac{1}{X_t^2}\sigma^2 X_t^2 dt \\
&= \left(\frac{\sigma^2}{2} - r\right)dt - \sigma dW_t.
\end{aligned}
$$

Integrating both sides from 0 to T

$$
\begin{aligned}
\int_0^T dY_u &= \left(\frac{\sigma^2}{2} - r\right)\int_0^T du - \sigma \int_0^T dW_u \\
Y_T - Y_0 &= \left(\frac{\sigma^2}{2} - r\right)T - \sigma W_T \\
\log\frac{X_T}{X_0} &= \left(\frac{\sigma^2}{2} - r\right)T - \sigma W_T \\
X_T &= X_0 e^{\left(\frac{\sigma^2}{2}-r\right)T - \sigma W_T} \\
\Rightarrow \quad \frac{1}{S_T} &= \frac{1}{S_0} e^{\left(\frac{\sigma^2}{2}-r\right)T - \sigma W_T}.
\end{aligned}
$$

Taking expectation, we obtain

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{S_T}\right] &= \mathbb{E}\left[\frac{1}{S_0}e^{\left(\frac{\sigma^2}{2}-r\right)T-\sigma W_T}\right] \\
&= \frac{1}{S_0}e^{\left(\frac{\sigma^2}{2}-r\right)T}\mathbb{E}\left[e^{-\sigma W_T}\right] \\
&= \frac{1}{S_0}e^{\left(\frac{\sigma^2}{2}-r\right)T}e^{\frac{\sigma^2 T}{2}} \\
&= \frac{1}{S_0}e^{\left(\sigma^2-r\right)T}
\end{aligned}$$

12. We shall apply chain rule for stochastic calculus to derive the stochastic differential equation for dZ_t. X_t and Y_t are adapted to two independent Brownian motions W_t and $\tilde{W}_t$, Itô's Formula would yield (show this)

$$dZ_t = X_t dY_t + Y_t dX_t.$$

Substituting for dX_t and dY_t, we obtain

$$\begin{aligned}
dZ_t = d(X_t Y_t) &= X_t(rY_t dt + \sigma_Y Y_t d\tilde{W}_t) + Y_t(rX_t dt + \sigma_X X_t dW_t) \\
&= 2rZ_t dt + Z_t(\sigma_Y d\tilde{W}_t + \sigma_X dW_t).
\end{aligned}$$

13. Consider $X_t = S_t^2 = f(S_t)$, where $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$, its partial derivatives are given by

$$f'(S_t) = 2S_t, \qquad f''(S_t) = 2.$$

Applying Itô's Formula to the function f, we obtain

$$\begin{aligned}
dS_t^2 = dX_t &= f'(S_t)dS_t + \frac{1}{2}f''(S_t)(dS_t)^2 \\
&= 2S_t(rS_t dt + \sigma S_t dW_t) + \frac{1}{2}\cdot 2\cdot \sigma^2 S_t^2 dt \\
&= (2r+\sigma^2)X_t dt + 2\sigma X_t dW_t.
\end{aligned}$$

We proceed to solve this stochastic differential equation. Consider $Y_t = \log X_t = g(X_t)$, where $g : \mathbb{R} \to \mathbb{R},\ g(x) = \log(x)$, its partial derivatives are given by

$$g'(X_t) = \frac{1}{X_t}, \qquad g''(X_t) = -\frac{1}{X_t^2}.$$

Applying Itô's Formula to g, we have

$$\begin{aligned}
d\log X_t = dY_t &= g'(X_t)dX_t + \frac{1}{2}g''(X_t)(dX_t)^2 \\
&= \frac{1}{X_t}[(2r+\sigma^2)X_t dt + 2\sigma X_t dW_t] - \frac{1}{2}\cdot\frac{1}{X_t^2}\cdot 4\cdot\sigma^2 X_t^2 dt \\
&= (2r-\sigma^2)dt + 2\sigma dW_t.
\end{aligned}$$

Integrating both sides from 0 to T

$$\begin{aligned}
\int_0^T dY_u &= (2r-\sigma^2)\int_0^T du + 2\sigma\int_0^T dW_u \\
Y_T - Y_0 &= (2r-\sigma^2)T + 2\sigma W_T \\
\log\frac{X_T}{X_0} &= (2r-\sigma^2)T + 2\sigma W_T \\
X_T &= X_0 e^{(2r-\sigma^2)T + 2\sigma W_T}.
\end{aligned}$$

And hence

$$\begin{aligned}\mathbb{E}[S_T^2] = \mathbb{E}[X_T] &= \mathbb{E}\left[X_0 e^{(2r-\sigma^2)T + 2\sigma W_T}\right] \\ &= X_0 e^{(2r-\sigma^2)T}\mathbb{E}\left[e^{2\sigma W_T}\right] \\ &= X_0 e^{(2r-\sigma^2)T} e^{\frac{4\sigma^2 T}{2}} \\ &= S_0^2 e^{(2r+\sigma^2)T}.\end{aligned}$$