QF620 Stochastic Modelling in Finance
Solution to Assignment 1/4

1. (a)

$$\begin{aligned}\mathbb{P}(W_2 < 0 | W_1 > 0) &= \mathbb{P}(|W_2 - W_1| > |W_1 - W_0|) \times \mathbb{P}(W_2 < W_1) \\ &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \quad \triangleleft\end{aligned}$$

(b)

$$\begin{aligned}\mathbb{P}(W_1 \times W_2 < 0) &= \mathbb{P}(W_1 < 0, W_2 > 0) + \mathbb{P}(W_1 > 0, W_2 < 0) \\ &= \mathbb{P}(W_2 > 0 | W_1 < 0)\mathbb{P}(W_1 < 0) + \mathbb{P}(W_2 < 0 | W_1 > 0)\mathbb{P}(W_1 > 0) \\ &= \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{4} \quad \triangleleft\end{aligned}$$

(c)

$$\begin{aligned}\mathbb{P}(W_1 < 0 \cap W_2 < 0) &= \mathbb{P}(W_2 < 0 | W_1 < 0)\mathbb{P}(W_1 < 0) \\ &= \frac{3}{4} \times \frac{1}{2} = \frac{3}{8} \quad \triangleleft\end{aligned}$$

2. We know that $W_{t+\Delta t} - W_t \sim N(0, \Delta t)$. Let $X \sim N(0,1)$, we have

$$\begin{aligned}\mathbb{E}[|W_{t+\Delta t} - W_t|] = \sqrt{\Delta} t \mathbb{E}[|X|] &= \sqrt{\frac{\Delta t}{2\pi}} \int_{-\infty}^{\infty} |x| e^{-\frac{x^2}{2}} \, dx \\ &= 2\sqrt{\frac{\Delta t}{2\pi}} \int_0^{\infty} x e^{-\frac{x^2}{2}} \, dx \\ &= 2\sqrt{\frac{2\Delta t}{2\pi}} \int_0^{\infty} e^{-u} du = \sqrt{\frac{2\Delta t}{\pi}} \quad \triangleleft\end{aligned}$$

3. First we note that if $X \sim N(0,1)$, then we have the following (using Moment Generating Function)

$$\mathbb{E}[X] = 0, \quad \mathbb{E}[X^2] = 1, \quad \mathbb{E}[X^3] = 0, \quad \mathbb{E}[X^4] = 3.$$

Next, note that

$$\begin{aligned}(W_t - W_s)^2 &\sim N(0, (t-s))^2 = (t-s)N(0,1)^2 = (t-s)X^2 \\ (W_t - W_s)^4 &\sim N(0, (t-s))^4 = (t-s)^2 N(0,1)^4 = (t-s)^2 X^4\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}[(W_t - W_s)^2] &= \mathbb{E}[(t-s)X^2] = (t-s) \\ \mathbb{E}[(W_t - W_s)^4] &= \mathbb{E}[(t-s)^2 X^4] = 3(t-s)^2\end{aligned}$$

So we have

$$\begin{aligned}V[(W_t - W_s)^2] &= \mathbb{E}[(W_t - W_s)^4] - \mathbb{E}[(W_t - W_s)^2]^2 \\ &= 3(t-s)^2 - (t-s)^2 = 2(t-s)^2 \quad \triangleleft\end{aligned}$$

4. (a) $dX_t = dt + 2W_t dW_t$
   (b) $dX_t = \left(1 + \frac{1}{2}e^{W_t}\right) dt + e^{W_t} dW_t$
   (c) $dX_t = 3(W_t^2 - t)dW_t$
   (d) $dX_t = \frac{3}{2}X_t dt + X_t dW_t$
   (e) $dX_t = \cos(W_t)e^{\frac{t}{2}} dW_t$
   (f) $dX_t = e^{W_t - \frac{t}{2}} dW_t$

5. Applying product rule for stochastic calculus, we have

$$\begin{aligned} dX_t = d(Y_t Z_t) &= Y_t dZ_t + Z_t dY_t + b(t)Y_t B(t)dt \\ &= Y_t(A(t)dt + B(t)dW_t) + Z_t b(t)Y_t dW_t + b(t)Y_t B(t)dt \end{aligned}$$

   Collecting dt and dW_t terms together, we obtain:

$$dX_t = (A(t)Y_t + b(t)Y_t B(t))dt + (Y_t B(t) + X_t b(t))dW_t$$

6. Let $Y_t = \frac{W_t}{\tilde{W}_t} = f(W_t, \tilde{W}_t)$, we have By Itô's lemma

$$dY_t = \frac{1}{\tilde{W}_t} dW_t - \frac{W_t}{\tilde{W}_t^2} d\tilde{W}_t + \frac{W_t}{\tilde{W}_t^3} dt \quad \triangleleft$$

7. First we integrate the arithmetic stochastic differential equation from 0 to t:

$$\int_0^t dr_u = \int_0^t \theta \; du + \int_0^t \sigma \; dW_u \qquad \Rightarrow \qquad r_t = r_0 + \theta t + \sigma W_t$$

   Next, we integrate r_t from 0 to T to obtain

$$\begin{aligned} \int_0^T r_t \; dt &= r_0 T + \int_0^T \theta t \; dt + \int_0^T \sigma W_t \; dt \\ &= r_0 T + \frac{\theta T^2}{2} + \sigma \int_0^T W_t \; dt \end{aligned}$$

   (a) We have

$$\mathbb{E}\left[\int_0^T r_t \; dt\right] = r_0 T + \frac{\theta T^2}{2} \quad \triangleleft$$

   since we have established that

$$\int_0^T W_t \; dt \sim N\left(0, \frac{T^3}{3}\right)$$

   (b) We have

$$\begin{aligned} V\left[\int_0^T r_t \; dt\right] &= V\left[\sigma \int_0^T W_t \; dt\right] \\ &= \sigma^2 V\left[\int_0^T W_t \; dt\right] \\ &= \frac{\sigma^2 T^3}{3} \quad \triangleleft \end{aligned}$$