Sample Final Exam Paper

Date / Start Time	-
Course	QF620 Stochastic Modelling
Instructor	Tee Chyng Wen

INSTRUCTIONS TO CANDIDATES

1. The time allowed for this examination paper is 3 hours.
2. This is a closed-book exam (no formula sheet).
3. This examination paper contains a total of 10 questions and comprises 11 pages including this instruction sheet.
4. You are allowed to bring along your calculators.
5. You are required to return the full set of question paper and the solutions at the end of the examination.
6. You are encouraged to indicate techniques that may solve questions where you only provide partial answers.
7. If you are uncertain about the context of a question, please write down the assumptions made before your solution.

1. (a) Let W_t denote a standard Brownian motion. Use Itô formula to solve the following stochastic differential equations, and proceed to evaluate the expectation $\mathbb{E}[X_T]$:

   i. $dX_t = X_t dt + dW_t$, ***hint:*** $Y_t = f(t, X_t)$, ***where*** $f(t, x) = e^{-t}x$

   ii. $dX_t = -X_t dt + e^{-t} dW_t$

   iii. $dX_t = (X_t + 10) dW_t$

   (b) Let W_t denote a standard Brownian motion. Suppose t and s are two timestamps, with $t > s$. Determine the following variances:

   i. $V[W_t - W_s]$

   ii. $V[(W_t - W_s)^2]$

2. (a) Let W_t denote a standard Brownian motion. Derive the stochastic differential equations for dX_t using Itô's formula for the following processes:

   i. $X_t = e^{W_t}$

   ii. $X_t = \log(W_t)$

   iii. $X_t = t^2 + W_t^3$

   iv. $X_t = \frac{W_t^2}{Z_t}$ where $dW_t dZ_t = \rho\ dt$

   (b) Let W_t denote a standard Brownian motion. Evaluate

   i. $\mathbb{E}[W_t^2]$

   ii. $\mathrm{Cov}(W_t, W_s)$ where $s < t$

   iii. $\rho(W_t, W_s)$ where $s < t$

3. (a) For a standard Brownian motion W_t, show that

$$\int_0^T W_t^2 \, dW_t = \frac{1}{3} W_T^3 - \int_0^T W_t \, dt.$$

(b) Consider the Vasicek interest rate model

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*.$$

i. Explain how does this model produce the mean reversion characteristic.

ii. Solve the stochastic differential equation, and evaluate the following limits

$$\lim_{t\to\infty} \mathbb{E}^*[r_t], \qquad \lim_{t\to\infty} V[r_t].$$

4. (a) Let W_t be a standard Brownian motion. Evaluate the expectation $\mathbb{E}[|W_T|]$, where $|\cdot|$ denote the absolute value.

(b) Consider a cash-or-nothing digital option paying

$$\mathbb{1}_{S_T>K} = \begin{cases} 1, & S_T > K \\ 0, & \text{otherwise.} \end{cases}$$

Assume the following process:

$$dS_t = rdt + \sigma dW_t$$

Derive a valuation formula for this option.

5. Let S_t denote the stock price at time t. Solve the following stochastic differential equations:

(a)

$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$

(b)

$$dS_t = \sigma[\beta S_T + (1-\beta)S_0]dW_t^*$$

And proceed to evaluate the expectation

$$\mathbb{E}^*[S_T]$$

6. A European exotic option pays the following at time T

$$\begin{cases} \log(S_T), & K_1 < S_T < K_2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Using the Black-Scholes model of

$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$

where W_t^* is a $\mathbb{Q}^*$-Brownian motion, derive the valuation formula for this option.

(b) Starting from (assuming $F < K_1$)

$$\int_0^\infty h(K) \frac{\partial^2 C(K)}{\partial K^2} dK,$$

where $h(\cdot)$ is the function describing the option's payoff, apply the static replication approach to value this exotic option.

7. The SABR stochastic volatility model is defined as

$$\begin{aligned} dF_t &= \alpha_t F_t^{\beta} dW_t^F \\ d\alpha_t &= \nu \alpha_t dW_t^{\alpha} \end{aligned}$$

where $dW_t^F dW_t^{\alpha} = \rho \ dt$. Briefly discuss how does the model parameters ν and ρ affect the profile of the implied volatility smile.

8. Consider an asset-or-nothing put option with the following payoff at time T:

$$V_T = S_T \mathbb{1}_{S_T \leq K}.$$

Starting with

$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$

where W_t^* is a standard Brownian motion under $\mathbb{Q}^*$, the risk-neutral measure associated to the risk-free bond numeraire (B_t). Show how we can use Radon-Nikodym derivative and change-of-measure theorem to evaluate the following expectation under the $\mathbb{Q}^S$ measure associated to the stock numeraire, and proceed to derive the valuation formula:

$$V_0 = e^{-rT}\mathbb{E}^*[S_T \mathbb{1}_{S_T \leq K}].$$

9. Consider a stock following the lognormal process

$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$

where W_t^* is a Brownian motion under the risk-neutral measure associated to the risk-free bond numeraire $\mathbb{Q}^*$. Evaluate the following expectation:

$$e^{-rT}\mathbb{E}^*[(K - \log S_T)^+]$$

10. Let W_t denote a standard Brownian motion in the real world probability measure $\mathbb{P}$. A stock follows the process

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

in the $\mathbb{P}$ measure. In the market there is also a risk-free money market account with 0 interest rate, i.e. $dB_t = 0$. Derive a valuation formula for an European option paying $(S_T^2 - K)$.

Suggested Solutions

1. (a) i. Let $Y_t = f(t, X_t) = e^{-t}X_t$, we have the following partial derivatives:

$$f_t(t,x) = -e^{-t}x, \qquad f_x(t,x) = e^{-t}, \qquad f_{xx}(t,x) = 0.$$

Using Itô's formula, we obtain the following stochastic differential equation:

$$\begin{aligned} dY_t &= -e^{-t}X_t dt + e^{-t}dX_t \\ &= -e^{-t}X_t dt + e^{-t}\left(X_t dt + dW_t\right) \\ &= e^{-t}dW_t \end{aligned}$$

Integrating both sides from 0 to t yields

$$\begin{aligned} Y_t - Y_0 &= \int_0^t e^{-u} dW_u \\ e^{-t}X_t &= X_0 + \int_0^t e^{-u} dW_u \\ X_t &= X_0 e^t + \int_0^t e^{t-u} dW_u \quad \triangleleft \end{aligned}$$

ii. Based on the hint given in previous question, let $Y_t = f(t, X_t) = e^t X_t$, we have the following partial derivatives:

$$f_t(t,x) = e^t x, \qquad f_x(t,x) = e^t, \qquad f_{xx}(t,x) = 0.$$

Using Itô's formula, we obtain the following stochastic differential equation:

$$\begin{aligned} dY_t &= e^t X_t dt + e^t\left(-X_t dt + e^{-t}dW_t\right) \\ &= dW_t \end{aligned}$$

Integrating both sides from 0 to t yields

$$\begin{aligned} Y_t - Y_0 &= \int_0^t dW_u \\ Y_t &= Y_0 + W_t \\ e^t X_t &= X_0 + W_t \\ X_t &= X_0 e^{-t} + e^{-t} W_t \quad \triangleleft \end{aligned}$$

iii. Let $Y_t = \log(X_t + 10) = f(X_t)$, we have the following partial derivatives

$$f'(x) = \frac{1}{x+10}, \qquad f''(x) = -\frac{1}{(x+10)^2}$$

Using Itô's formula, we obtain the following stochastic differential equation

$$\begin{aligned} dY_t &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 \\ &= dW_t - \frac{1}{2}dt \end{aligned}$$

Integrating both sides and rearranging, we obtain

$$X_T = (X_0 + 10)e^{-\frac{1}{2}T + W_T} - 10 \quad \triangleleft$$

(b) First, we note that:

$$\mathbb{E}[(W_t - W_s)^2] = t - s, \qquad \mathbb{E}[(W_t - W_s)^4] = 3(t-s)^2.$$

i.

$$V[W_t - W_s] = \mathbb{E}[(W_t - W_s)^2] = t - s \quad \lhd$$

ii.

$$\begin{aligned} V[(W_t - W_s)^2] &= \mathbb{E}[(W_t - W_s)^4] - \mathbb{E}[(W_t - W_s)]^2 \\ &= 3(t-s)^2 - (t-s)^2 \\ &= 2(t-s)^2 \quad \lhd \end{aligned}$$

2. (a) i. We have $f(W_t) = e^{W_t}$, using Itô's formula, we obtain:

$$dX_t = \frac{1}{2}X_t dt + X_t dW_t \quad \triangleleft$$

ii. We have $f(W_t) = \log(W_t)$, using Itô's formula, we obtain:

$$dX_t = -\frac{1}{2W_t^2}dt + \frac{1}{W_t}dW_t \quad \triangleleft$$

iii. We have $f(t, W_t) = t^2 + W_t^3$, using Itô's formula, we obtain:

$$dX_t = (2t + 3W_t)dt + 3W_t^2 dW_t \quad \triangleleft$$

iv. We have $f(W_t, Z_t) = \frac{W_t^2}{Z_t}$, using Itô's formula, we obtain:

$$dX_t = \left(\frac{1}{Z_t} + \frac{W_t^2}{Z_t^3} - \frac{2W_t\rho}{Z_t}\right)dt + \frac{2W_t}{Z_t}dW_t - \frac{W_t^2}{Z_t^2}dZ_t \quad \triangleleft$$

(b) i.

$$\mathbb{E}[W_t^2] = t \quad \triangleleft$$

ii. Given that $s < t$, we have

$$\text{Cov}(W_t, W_s) = s \quad \triangleleft$$

iii. Given that $s < t$, we have

$$\rho(W_t, W_s) = \frac{\text{Cov}(W_t, W_s)}{\sqrt{t} \times \sqrt{s}} = \sqrt{\frac{s}{t}} \quad \triangleleft$$

3. (a) Let $X_t = f(W_t) = W_t^3$, the partial derivatives are given by

$$f'(W_t) = 3W_t^2, \qquad f''(W_t) = 6W_t.$$

Applying Itô's Lemma, we obtain

$$dX_t = 3W_t^2 dW_t + \frac{1}{2} \cdot 6 \cdot W_t dt = 3W_t dt + 3W_t^2 dW_t.$$

Integrating both sides from 0 to T, we obtain

$$X_T - X_0 = 3\int_0^T W_t dt + 3\int_0^T W_t^2 dW_t$$
$$\int_0^T W_t^2 dW_t = \frac{W_T^3}{3} - \int_0^T W_t dt. \quad \triangleleft$$

(b) i. We note that

$$\mathbb{E}^*[dr_t] = \kappa(\theta - r_t)dt,$$

so if short rate is lower than the long run mean (i.e. $\theta > r_t$), the drift term is positive, and dr_t is expected to drift upward, while if short rate is higher than the long run mean (i.e. $\theta < r_t$), the drift term is negative, and dr_t is expected to drift downward. When $\theta = r_t$, the drift term is zero, and so dr_t is expected to remain at the same level. κ is a positive multiplier and controls the mean reversion speed. $\triangleleft$

ii. Consider the function $e^{\kappa t} r_t$, the total derivative is given by

$$\begin{aligned} d(e^{\kappa t} r_t) &= \kappa e^{\kappa t} r_t dt + e^{\kappa t} dr_t \\ &= \kappa\theta e^{\kappa t} dt + \sigma e^{\kappa t} dW_t. \end{aligned}$$

Integrating both sides from 0 to t, we can obtain a solution to the stochastic differential equation

$$\int_0^t d(e^{\kappa u} r_u) = \int_0^t \kappa\theta e^{\kappa u} du + \int_0^t \sigma e^{\kappa u} dW_u$$
$$r_t = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma \int_0^t e^{\kappa(u-t)} dW_u.$$

Taking expectation on both sides gives us the mean

$$\mathbb{E}[r_t] = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}).$$

Applying Itô's Isometry theorem, we can evaluate the variance of the short rate

$$\begin{aligned} V[r_t] &= \mathbb{E}\left[\left(\sigma \int_0^t e^{\kappa(u-t)} dW_u\right)^2\right] \\ &= \mathbb{E}\left[\sigma^2 \int_0^t e^{2\kappa(u-t)} du\right] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right). \end{aligned}$$

The distribution of r_t is therefore given by

$$r_t \sim N\left(r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}), \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})\right).$$

So we have

$$\lim_{t\to\infty} \mathbb{E}^*[r_t] = \lim_{t\to\infty}\left(r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t})\right) = \theta, \quad \triangleleft$$
$$\lim_{t\to\infty} V[r_t] = \lim_{t\to\infty}\left(\frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})\right) = \frac{\sigma^2}{2\kappa}. \quad \triangleleft$$

4. (a) First we note that $W_T \sim N(0,T)$. Let $X \sim N(0,1)$, we have

$$\begin{aligned}
\mathbb{E}\Big[|W_T|\Big] &= \mathbb{E}\Big[\sqrt{T}|X|\Big] \\
&= \frac{\sqrt{T}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x| e^{-\frac{x^2}{2}} \, dx \\
&= \frac{\sqrt{T}}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} -x e^{-\frac{x^2}{2}} \, dx + \int_{0}^{\infty} x e^{-\frac{x^2}{2}} \, dx \right] \\
&= \frac{\sqrt{T}}{\sqrt{2\pi}} \times 2 \times \underbrace{\int_{0}^{\infty} x e^{-\frac{x^2}{2}} \, dx}_{1} \\
&= \sqrt{\frac{2T}{\pi}}. \quad \lhd
\end{aligned}$$

(b) Solving the SDE, we obtain

$$S_T = S_0 + \sigma W_T.$$

We are looking for

$$S_T = S_0 + \sigma\sqrt{T}x > 0 \qquad \Rightarrow \qquad x > -\frac{S_0}{\sigma\sqrt{T}} = x^*.$$

Hence

$$\begin{aligned}
V_0 = e^{-rT}\mathbb{E}\left[\mathbb{1}_{S_T > K}\right] &= e^{-rT} \frac{1}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{x^2}{2}} \, dx \\
&= e^{-rT}\Phi(-x^*) = e^{-rT}\Phi\left(\frac{S_0}{\sigma\sqrt{T}}\right).
\end{aligned}$$

5. (a) Solving, we obtain

$$S_T = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^*}$$

Taking expectation yields

$$\mathbb{E}^*[S_T] = S_0 e^{rT} \quad \triangleleft$$

(b) Solving the displaced-diffusion stochastic differential equation, we obtain

$$S_T = \frac{S_0}{\beta} e^{-\frac{1}{2}\beta^2\sigma^2 T + \beta\sigma W_T^*} - \frac{1-\beta}{\beta} S_0.$$

Taking the expectation, we obtain

$$\begin{aligned}
\mathbb{E}^*[S_T] &= \mathbb{E}\left[\frac{S_0}{\beta} e^{-\frac{1}{2}\beta^2\sigma^2 T + \beta\sigma W_T} - \frac{1-\beta}{\beta} S_0\right] \\
&= \frac{S_0}{\beta} - \frac{1-\beta}{\beta} S_0 \\
&= S_0 \quad \triangleleft
\end{aligned}$$

6. (a) The solution of the stochastic differential equation is given by

$$S_T = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}.$$

Payoff is non-zero when

$$\begin{array}{ccccc} K_1 & < & S_T & < & K_2 \\ K_1 & < & S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x} & < & K_2 \\ x_l^* = \frac{\log\frac{K_1}{S_0}-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} & < & x & < & \frac{\log\frac{K_2}{S_0}-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x_h^* \end{array}$$

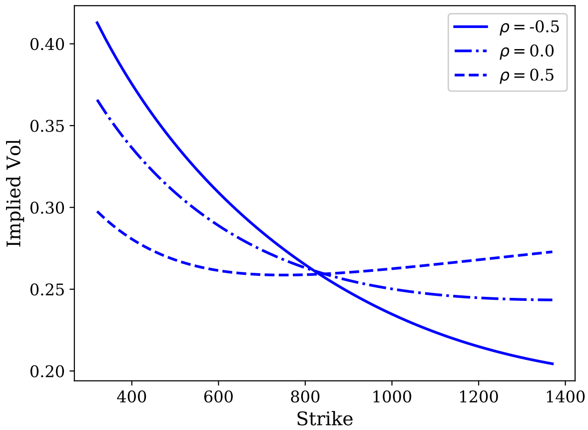
Under Black-Scholes model, the option can be valued as

$$\begin{aligned} V_0 &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x_l^*}^{x_h^*} \left(\log S_0 + \left(r-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x\right) e^{-\frac{x^2}{2}}\,dx \\ &= e^{-rT}\left[\log S_0 + \left(r-\frac{\sigma^2}{2}\right)T\right]\left(\Phi(x_h^*)-\Phi(x_l^*)\right) + \frac{e^{-rT}}{\sqrt{2\pi}}\sigma\sqrt{T}\int_{x_l^*}^{x_h^*} x e^{-\frac{x^2}{2}}\,dx \\ &= e^{-rT}\left[\log S_0 + \left(r-\frac{\sigma^2}{2}\right)T\right]\left(\Phi(x_h^*)-\Phi(x_l^*)\right) + \frac{e^{-rT}}{\sqrt{2\pi}}\sigma\sqrt{T}\left[e^{-\frac{(x_l^*)^2}{2}} - e^{-\frac{(x_h^*)^2}{2}}\right]. \quad \triangleleft \end{aligned}$$

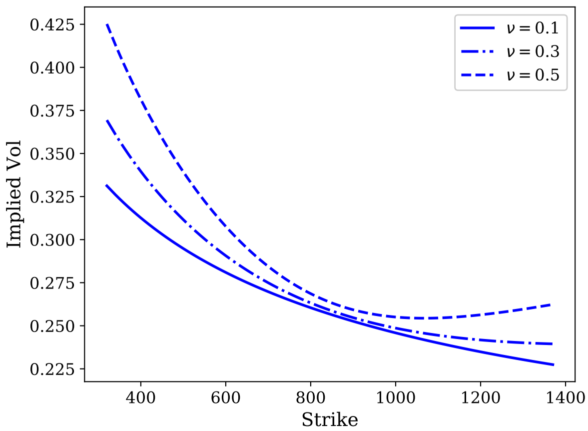
(b) Using integrating by parts, we obtain

$$\begin{aligned} \int_0^\infty h(K)\frac{\partial^2 C(K)}{\partial K^2}dK &= \int_{K_1}^{K_2} \log(K)\frac{\partial^2 C(K)}{\partial K^2}dK \\ &= \left[\log(K)\cdot\frac{\partial C(K)}{\partial K}\right]_{K_1}^{K_2} - \int_{K_1}^{K_2}\frac{1}{K}\frac{\partial C(K)}{\partial K}dK \\ &= \left[\log(K_2)\cdot\frac{\partial C(K_2)}{\partial K} - \log(K_1)\cdot\frac{\partial C(K_1)}{\partial K}\right] - \left[\frac{C(K)}{K}\right]_{K_1}^{K_2} \\ &\quad + \int_{K_1}^{K_2}\frac{C(K)}{K^2}dK \\ &= \left[\log(K_2)\cdot\frac{\partial C(K_2)}{\partial K} - \log(K_1)\cdot\frac{\partial C(K_1)}{\partial K}\right] - \left[\frac{C(K_2)}{K} - \frac{C(K_1)}{K}\right] \\ &\quad + \int_{K_1}^{K_2}\frac{C(K)}{K^2}dK. \quad \triangleleft \end{aligned}$$

7. Positive correlation between stock and volatility is associated with positive skew in return distribution. Negative correlation between stock and volatility is associated with negative skew in return distribution. Negative correlation increases the price of out-of-the-money put options and decreases the price of out-of-the-money call options.



Increasing volatility-of-volatility has the effect of increasing the kurtosis of return. When the volatility-of-volatility parameter is 0, volatility will be deterministic. Larger volatility-of-volatility ν increases the price of out-of-the-money call and put options.



8. We use the following Radon-Nikodym derivative to change the measure from $\mathbb{Q}^*$ to $\mathbb{Q}^S$

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}^*} = \frac{S_T/S_0}{e^{rT}/1} = e^{-\frac{1}{2}\sigma^2 T + \sigma W_T^*}.$$

Comparing this to the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \kappa_t dW_t - \frac{1}{2}\int_0^T \kappa_t^2 dt\right),$$

we note that $\mathbb{Q}^S$ is equivalent to $\mathbb{Q}^*$, and $W_t^S = W_t^* + \kappa t$ is a $\mathbb{Q}^S$-Brownian motion (where $\kappa = -\sigma$). Substituting, we have

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t^* \\ &= (r + \sigma^2) S_t dt + \sigma S_t dW_t^S \end{aligned}$$

The solution is given by

$$S_T = S_0 \exp\left[\left(r + \frac{1}{2}\sigma^2\right) T + \sigma W_T^S\right].$$

$$\begin{aligned} V_0 &= S_0 \mathbb{E}^*\left[\frac{S_T/S_0}{e^{rT}/1} \mathbb{1}_{S_T \le K}\right] \\ &= S_0 \mathbb{E}^S\left[\mathbb{1}_{S_T \le K}\right] \\ &= S_0 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{1}_{S_T \le K} e^{-\frac{x^2}{2}} dx \\ &= S_0 \int_{-\infty}^{x^*} e^{-\frac{x^2}{2}} dx \\ &= S_0 \Phi\left(-\frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}\right) \quad \triangleleft \end{aligned}$$

9. Logarithm of S_T yields a normal process, giving rise to a Bachelier-like option formula. We have

$$\log S_T = \log S_0 + \left(r - \frac{\sigma^2}{2}\right) T + \sigma\sqrt{T}x, \quad x \sim N(0,1)$$

The exercise region with non-zero payoff is

$$\begin{aligned} & K - \log S_0 - \left(r - \frac{\sigma^2}{2}\right) T - \sigma\sqrt{T}x > 0 \\ \Rightarrow \quad & x^* = \frac{K - \log S_0 - \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} > x \end{aligned}$$

Hence, the expectation can be evaluated as

$$\begin{aligned} e^{-rT}\mathbb{E}^*[(K - \log S_T)^+] &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{x^*} \left[K - \log S_0 - \left(r - \frac{\sigma^2}{2}\right) T - \sigma\sqrt{T}x\right] e^{-\frac{x^2}{2}} dx \\ &= e^{-rT} \left[K - \log S_0 - \left(r - \frac{\sigma^2}{2}\right) T\right] \Phi(x^*) + \frac{e^{-rT}}{\sqrt{2\pi}} \sigma\sqrt{T} e^{-\frac{(x^*)^2}{2}} \\ &= e^{-rT} \left[K - \log S_0 - \left(r - \frac{\sigma^2}{2}\right) T\right] \Phi(x^*) + e^{-rT}\sigma\sqrt{T}\phi(x^*) \quad \triangleleft \end{aligned}$$

10. Starting with the real-world process

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

we write down the stochastic differential equation for $d\left(\frac{S_t}{B_t}\right)$

$$\begin{aligned} d\left(\frac{S_t}{B_t}\right) &= -\frac{S_t}{B_t}dB_t + \frac{1}{B_t}dS_t \\ &= \mu\frac{S_t}{B_t}dt + \sigma\frac{S_t}{B_t}dW_t \\ &= \sigma\frac{S_t}{B_t}\left(dW_t + \frac{\mu}{\sigma}dt\right). \end{aligned}$$

Under the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^t \kappa^2 du - \int_0^t \kappa dW_u\right), \qquad \kappa = \frac{\mu}{\sigma},$$

we have

$$W_t^* = W_t + \frac{\mu}{\sigma}t \quad \Rightarrow \quad dW_t^* = dW_t + \frac{\mu}{\sigma}dt$$

as a $\mathbb{Q}^*$ martingale. Hence, the stochastic process followed by the stock price can be written as

$$dS_t = \mu S_t dt + \sigma S_t\left(dW_t^* - \frac{\mu}{\sigma}dt\right) = \sigma S_t dW_t^*.$$

The solution is given by

$$S_T = S_0 e^{-\frac{\sigma^2 T}{2} + \sigma W_T^*}.$$

Since $r = 0$, a contract paying $S_T^2 - K$ can be valued as

$$\begin{aligned} V_0 &= e^{-0\times T}\mathbb{E}^*[S_T^2 - K] \\ &= \mathbb{E}^*\left[S_0 e^{-\sigma^2 T + 2\sigma W_T^*} - K\right] \\ &= \left(S_0 e^{-\sigma^2 T + 2\sigma^2 T} - K\right) \\ &= \left(S_0 e^{\sigma^2 T} - K\right). \quad \triangleleft \end{aligned}$$