Part III - Convexity Correction

Present value of CMS product

To calculate PV of leg receiving CMS10y semi-annually over the next 5 years, we need to find SABR paramters at different expiries in order to price each CMS rate. To this end, cubic spline interpolation is used between α ν, and ρ of $1y \times 10y$, $5y \times 10y$ and $10y \times 10y$ SABR models we have calibrated. Since there was no expiry lower than 1y for us to interpolate, parameters for 0.5y expiry follows those of 1y expiry.

Interpolation profiles are given as follow:

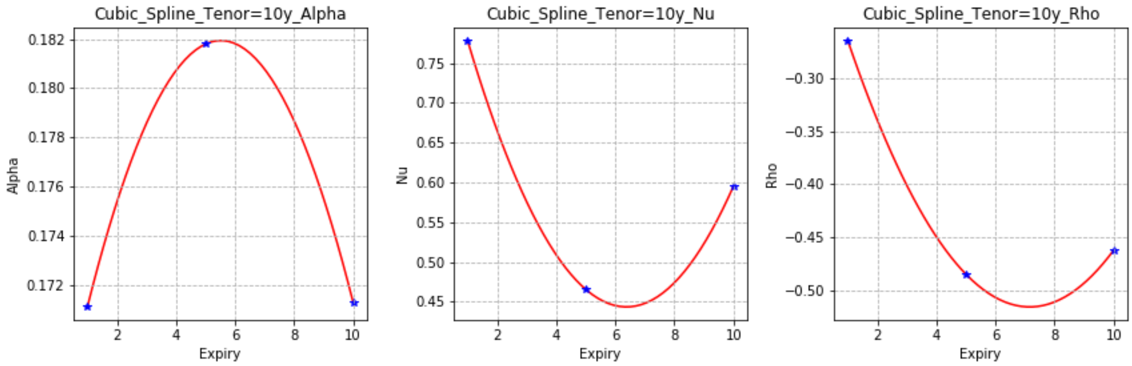


Figure 1: Interpolation of SABR parameters - CMS10Y

After interpolating all the SABR parameters, static replication is used to price each CMS rate and PV is the sum of the discounted values of all CMS rates, multiplied by the day count fraction. Here goes the mathematical form:

$$
\begin{aligned}
PV_{CMS10y} &= D(0,6m) \times 0.5 \times E^T[S_{6m,10y6m}(6m)] \\
&+ D(0,1y) \times 0.5 \times E^T[S_{1y,11y}(1y)] \\
&+ \cdots + D(0,5y) \times 0.5 \times E^T[S_{5y,15y}(5y)] \\
&= 0.213606
\end{aligned}
$$

Similarly, for CMS2y processed quarterly, α ν, and ρ can be interpolated between $1y \times 2y$, $5y \times 2y$,$10y \times 2y$, whose profiles are demonstrated below:

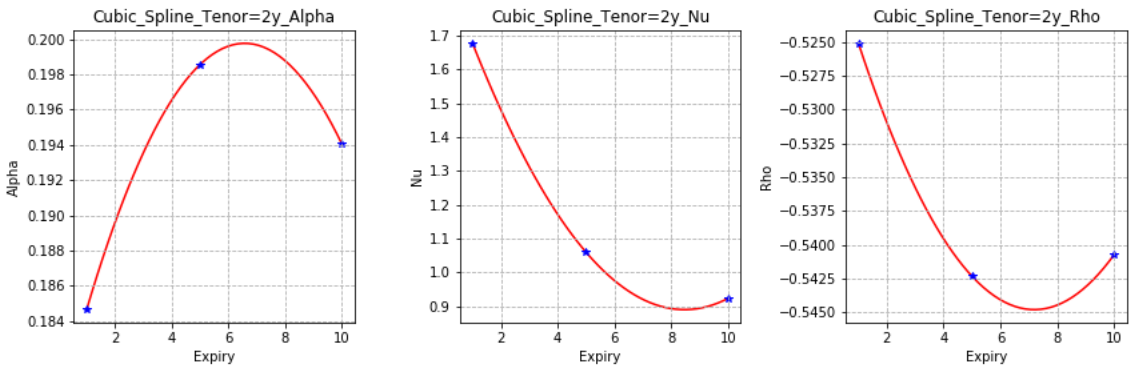


Figure 2: Interpolation of SABR parameters - CMS2Y

In addition to SABR parameters interpolation, due to quarterly arrangement, more discrete OIS discount rates and Libor discount rates are interpolated based on DF calculated in Section 1. After getting all the inputs, we can calculate PV of CMS2y as follow:

$$
\begin{aligned}
PV_{CMS2y} &= D(0,3m) \times 0.25 \times E^T[S_{3m,2y3m}(3m)] \\
&+ D(0,6m) \times 0.25 \times E^T[S_{6m,2y6m}(6m)] \\
&+ \cdots + D(0,10y) \times 0.25 \times E^T[S_{10y,12y}(10y)] \\
&= 0.504841
\end{aligned}
$$

CMS VS Par Swap Rate

Through trial and error, we found out that the CMS rates can become unlikely large numbers or even drop below par swap rate when upper bound for payer swaption integral is set as a large number or infinity. To figure out the optimal upper bound that not only covers most of the cases but generates plausible CMS rates, we calculated pure $f(K)$ in CMS rate formula by setting the payoff function as constant 1. When the upper bound is 0.85, no value exceeded 1, and CMS rates converged on reasonable readings.

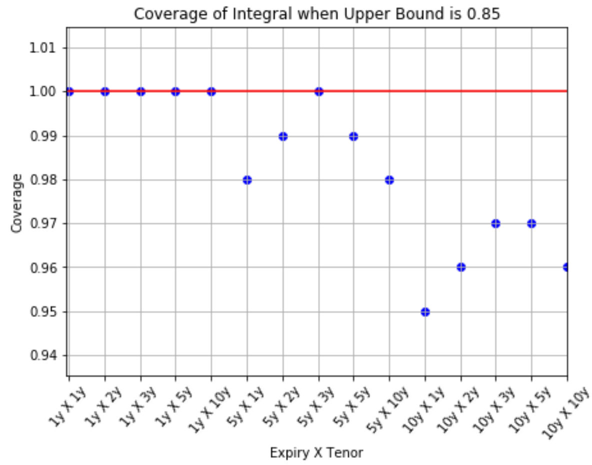


Figure 3: Coverage of Integral inside of CMS rate when Upper Bound is 0.85

Tables presented below show CMS rates for each maturity and tenor.

Tenor	CMS Rate
1x1	0.032787
1x2	0.034371
1x3	0.034913
1x5	0.035680
1x10	0.039031

Tenor	CMS Rate
5x1	0.055005
5x2	0.055665
5x3	0.054539
5x5	0.049968
5x10	0.048699

Tenor	CMS Rate
10x1	0.068968
10x2	0.079980
10x3	0.090381
10x5	0.094072
10x10	0.120374

Figure 4: CMS rates

Comparing CMS rates with forward swap rates of corresponding expiry and tenor which are derived from Part 1, we can recognise that the difference between CMS and forward swap rate increases as the expiry lengthens. It means that the longer expiry becomes, the greater the magnitude of convexity correction grows. On the contrary, the influence of tenor on the convexity correction is irregular. This phenomenon is presumed to be a result of volatility smile.

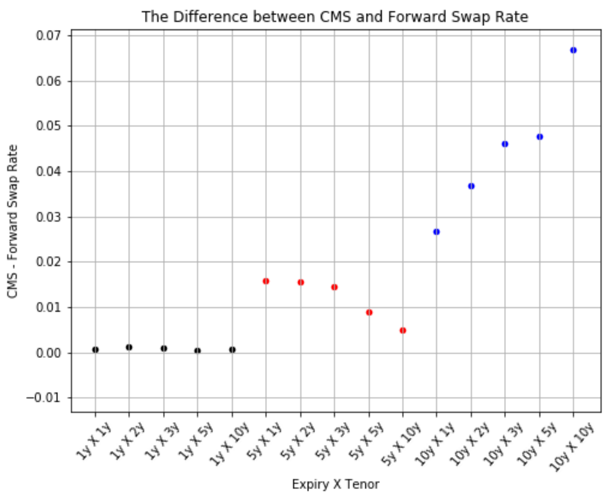


Figure 5: Delta Profile for Up-and-In Barrier Option of Given Condition

Part IV - Decompounded Options

Question 1

Starting from the generic contract valuation formula, we were able to obtain the static replication formula for the contract in question by first applying Leibniz's Rule on the IRR Payer and Receiver swaption formulas twice, after which integration by parts was carried out twice on the integrals in the generic contract valuation formula.

First, applying Leibniz's rule on the IRR Payer and Receiver swaption formulas twice yields:

Payer IRR Swaption

$$V^{pay}(K) = D(0,T)\int_K^\infty IRR(S)\cdot(S-K)\cdot f(S)dS$$

$$\frac{\partial^2 V^{pay}(K)}{\partial K^2} = D(0,T)\cdot IRR(K)\cdot f(K)$$

Receiver IRR Swaption

$$V^{rec}(K) = D(0,T)\int_0^F IRR(S)\cdot(K-S)\cdot f(S)dS$$

$$\frac{\partial^2 V^{rec}(K)}{\partial K^2} = D(0,T)\cdot IRR(K)\cdot f(K)$$

We are now able to denote the generic contract valuation formula as such:

$$\begin{aligned} V_0 &= D(0,T)\mathbb{E}\left[g(S)\right] \\ &= D(0,T)\int_0^\infty g(K)f(K)dK \\ &= \int_0^F h(K)\frac{\partial^2 V^{rec}(K)}{\partial K^2}dK + \int_F^\infty h(K)\frac{\partial^2 V^{pay}(K)}{\partial K^2}dK \end{aligned}$$

Integration by parts twice yields:

$$V_0 = D(0,T)g(F) + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK$$

Where:

$$F = S_{n,N}(0), \quad n=5, \quad N=15, \quad T=5$$

$$g(K) = K^{\frac{1}{p}} - 0.04^{\frac{1}{q}} = K^{\frac{1}{4}} - 0.2, \quad g'(K) = \frac{1}{4}K^{-\frac{3}{4}}, \quad g''(K) = -\frac{3}{16}K^{-\frac{7}{4}}$$

$$h(K) = \frac{g(K)}{IRR(K)}, \quad h'(K) = \frac{IRR(K)g'(K) - g(K)IRR'(K)}{IRR(K)^2}$$

$$h''(K) = \frac{IRR(K)g''(K) - IRR''(K)g(K) - 2IRR'(k)g'(K)}{IRR(K)^2} + \frac{2IRR'(K)^2 g(K)}{IRR(K)^3}$$

$$V^{rec} = D(0,T)\cdot IRR(S_{n,N}(0))\cdot \text{Black76Put}(S_{n,N}(0), K, \sigma_{\text{SABR}}, T)$$

$$V^{pay} = D(0,T)\cdot IRR(S_{n,N}(0))\cdot \text{Black76Call}(S_{n,N}(0), K, \sigma_{\text{SABR}}, T)$$

Using these parameters, we were able to obtain a Present Value (PV) for the V_0 of **0.2334**. V_0 can also be seen as a forward contract on the 10-year CMS rate with forward price set at 0.0016.

Question 2

The contract (V_0^+) can be valued as though it is a CMS caplet when the payoff is $(S_T^{1/4} - 0.04^{1/2})^+$.

For $S_T^{1/4} - 0.04^{1/2}$ to be positive:

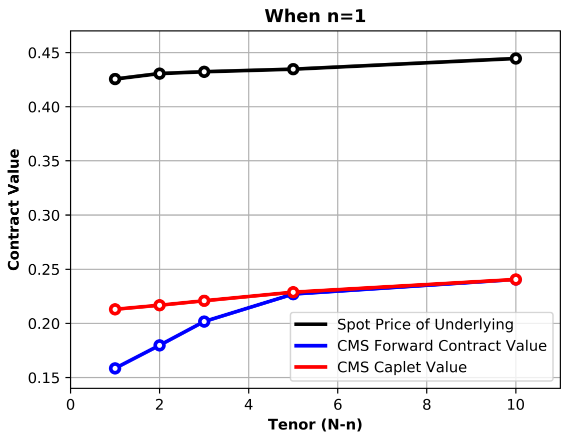
$$\begin{aligned} S_T^{1/4} &> 0.2 \\ S_T &> 0.0016 = L \end{aligned}$$

Thus, we can see V_0^+ as a CMS caplet struck at $L = 0.0016$:

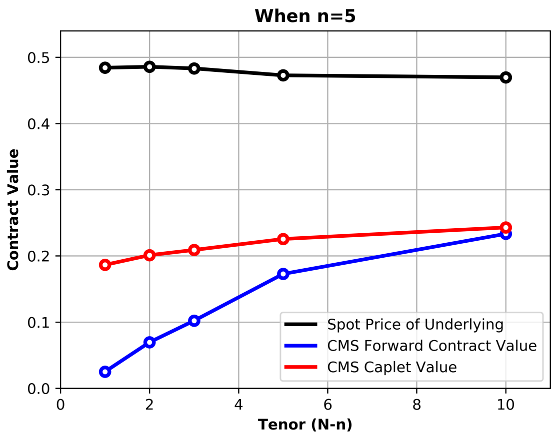
$$\begin{aligned} V_0^+ &= D(0,T)\int_L^{\infty} g(K)f(K)dK \\ &= \int_L^{\infty} h(K)\frac{\partial^2 V^{pay}(K)}{\partial K^2}dK \\ &= h'(L)V^{pay}(L) + \int_L^{\infty} h''(K)V^{pay}(K)dK \end{aligned}$$

Using this valuation formula and relevant parameters from Question 1, we were able to obtain a Present Value (PV) for $\mathbf{V_0^+}$ **of 0.2430**, which is higher than $\mathbf{V_0}$**'s value of 0.2334**.

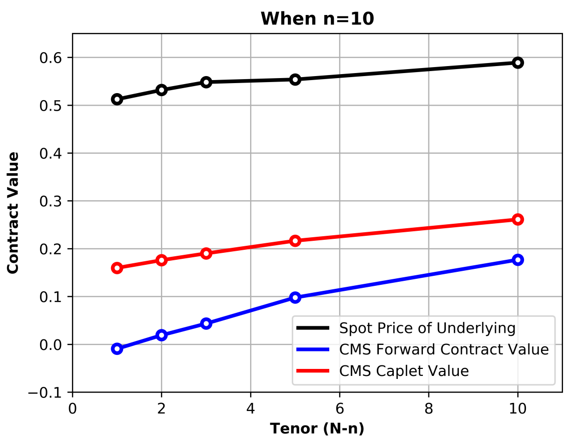
This is intuitive as V_0^+ omits the negatively valued region of V_0 where $S_T < L$ and as such should be valued higher than V_0. It also follows that $V_0 < V_0^+ < \mathbb{E}\left[S_T^{1/4}\right]$ (spot price of the underlying), as this is the model-free no-arbitrage boundary that the three products must satisfy. Running further diagnostics, it can be seen that V_0^+ is consistently more highly priced than V_0 across the given values of N (swap end date) for every n (swap start date):



(a)



(b)



(c)