Part IV - Decompounded Options

Question 1

Starting from the generic contract valuation formula, we were able to obtain the static replication formula for the contract in question by first applying Leibniz's Rule on the IRR Payer and Receiver swaption formulas twice, after which integration by parts was carried out twice on the integrals in the generic contract valuation formula.

First, applying Leibniz's rule on the IRR Payer and Receiver swaption formulas twice yields:

Payer IRR Swaption

$$V^{pay}(K) = D(0,T)\int_K^\infty IRR(S)\cdot(S-K)\cdot f(S)dS$$

$$\frac{\partial^2 V^{pay}(K)}{\partial K^2} = D(0,T)\cdot IRR(K)\cdot f(K)$$

Receiver IRR Swaption

$$V^{rec}(K) = D(0,T)\int_0^F IRR(S)\cdot(K-S)\cdot f(S)dS$$

$$\frac{\partial^2 V^{rec}(K)}{\partial K^2} = D(0,T)\cdot IRR(K)\cdot f(K)$$

We are now able to denote the generic contract valuation formula as such:

$$\begin{aligned} V_0 &= D(0,T)\mathbb{E}\left[g(S)\right] \\ &= D(0,T)\int_0^\infty g(K)f(K)dK \\ &= \int_0^F h(K)\frac{\partial^2 V^{rec}(K)}{\partial K^2}dK + \int_F^\infty h(K)\frac{\partial^2 V^{pay}(K)}{\partial K^2}dK \end{aligned}$$

Integration by parts twice yields:

$$V_0 = D(0,T)g(F) + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK$$

Where:

$$F = S_{n,N}(0), \quad n = 5, \quad N = 15, \quad T = 5$$

$$g(K) = K^{\frac{1}{p}} - 0.04^{\frac{1}{q}} = K^{\frac{1}{4}} - 0.2, \quad g'(K) = \frac{1}{4}K^{-\frac{3}{4}}, \quad g''(K) = -\frac{3}{16}K^{-\frac{7}{4}}$$

$$h(K) = \frac{g(K)}{IRR(K)}, \quad h'(K) = \frac{IRR(K)g'(K) - g(K)IRR'(K)}{IRR(K)^2}$$

$$h''(K) = \frac{IRR(K)g''(K) - IRR''(K)g(K) - 2IRR'(k)g'(K)}{IRR(K)^2} + \frac{2IRR'(K)^2 g(K)}{IRR(K)^3}$$

$$V^{rec} = D(0,T)\cdot IRR(S_{n,N}(0))\cdot \text{Black76Put}(S_{n,N}(0), K, \sigma_{\text{SABR}}, T)$$

$$V^{pay} = D(0,T)\cdot IRR(S_{n,N}(0))\cdot \text{Black76Call}(S_{n,N}(0), K, \sigma_{\text{SABR}}, T)$$

Using these parameters, we were able to obtain a Present Value (PV) for the V_0 of **0.2334**. V_0 can also be seen as a forward contract on the 10-year CMS rate with forward price set at 0.0016.

Question 2

The contract (V_0^+) can be valued as though it is a CMS caplet when the payoff is $(S_T^{1/4} - 0.04^{1/2})^+$.

For $S_T^{1/4} - 0.04^{1/2}$ to be positive:

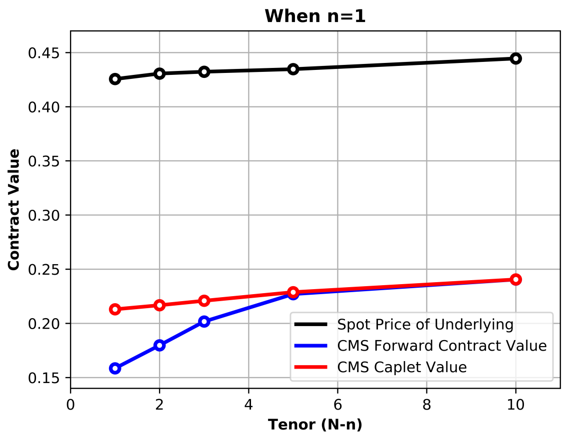
$$\begin{aligned} S_T^{1/4} &> 0.2 \\ S_T &> 0.0016 = L \end{aligned}$$

Thus, we can see V_0^+ as a CMS caplet struck at $L = 0.0016$:

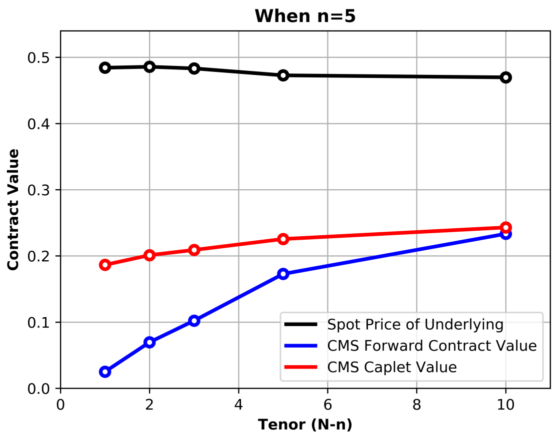
$$\begin{aligned} V_0^+ &= D(0,T)\int_L^\infty g(K)f(K)dK \\ &= \int_L^\infty h(K)\frac{\partial^2 V^{pay}(K)}{\partial K^2}dK \\ &= h'(L)V^{pay}(L) + \int_L^\infty h''(K)V^{pay}(K)dK \end{aligned}$$

Using this valuation formula and relevant parameters from Question 1, we were able to obtain a Present Value (PV) for $\mathbf{V_0^+}$ **of 0.2430**, which is higher than $\mathbf{V_0}$**'s value of 0.2334**.

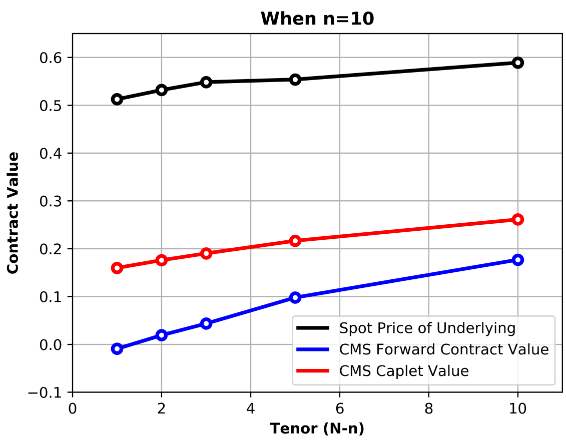
This is intuitive as V_0^+ omits the negatively valued region of V_0 where $S_T < L$ and as such should be valued higher than V_0. It also follows that $V_0 < V_0^+ < \mathbb{E}\left[S_T^{1/4}\right]$ (spot price of the underlying), as this is the model-free no-arbitrage boundary that the three products must satisfy. Running further diagnostics, it can be seen that V_0^+ is consistently more highly priced than V_0 across the given values of N (swap end date) for every n (swap start date):



(a)



(b)



(c)