QF605
Fixed Income Securities
Group Project

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Part I: Bootstrapping Swap Curves
OIS Discount Factors

With the provided OIS rates data, we proceeded to use the following methodology to bootstrap the OIS discount factor curve.

$$PV_{fix} = PV_{float}$$
$$D(0,1y) * OIS_{1y} = D(0,1y) * [(1+\frac{f_0}{360})^{360} - 1]$$
$$[D(0,1y) + D(0,2y)] * OIS_{2y} = D(0,1y) * [(1+\frac{f_0}{360})^{360} - 1] + D(0,2y) * [(1+\frac{f_1}{360})^{360} - 1]$$
$$\vdots$$
$$[D(0,1y) + \cdots + D(0,20y)] * OIS_{20y} = D(0,1y) * [(1+\frac{f_0}{360})^{360} - 1] + \cdots + D(0,20y) * [(1+\frac{f_{19}}{360})^{360} - 1]$$

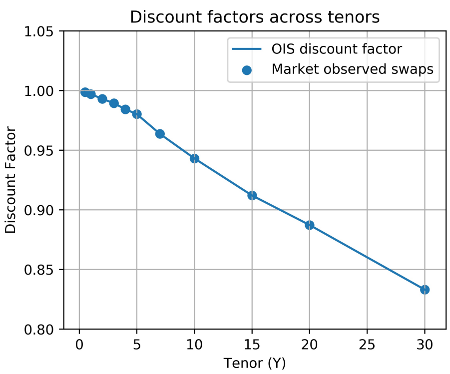
Due to only a handful of OIS swaps observable in the market, we can only then use the OIS swaps of varying tenor [6m,1y,2y,3y,5y,7y,10y,20y] to bootstrap the whole OIS discount curve while linearly interpolate for the rest of the "gap" discount factors. In order to solve for all discount factors, we will have to adopt the following:

$$PV_{fix} = PV_{float}$$
$$[D(0,1y) + \cdots + D(0,7y)] * OIS_{7y} = D(0,1y) * [(1+\frac{f_0}{360})^{360} - 1] + \cdots + D(0,7) * [(1+\frac{f_6}{360})^{360} - 1]$$

* Assuming all prior discount factors have been bootstrapped, we can then subtitute the following into above equation to help isolate D(0,7y), and then derive D(0,7y):

$$f_6 = 360 * [D(0,7y)^{-\frac{1}{360*7}} - 1]$$
$$D(0,6y) = \frac{[D(0,7y) - D(0,5y)]}{2} * 1 + D(0,5y)$$
$$f_5 = 360 * [[\frac{[D(0,7y) - D(0,5y)]}{2} * 1 + D(0,5y)]^{-\frac{1}{360*6}} - 1]$$

Proceeding to do the similar for all OIS swaps, we derive the following OIS discount factors results and graph.



(a) OIS Discount Curve

Tenor	OIS rate	Overnight rate	OIS DF
0.5	0.00250	0.002497	0.998752
1.0	0.00300	0.002996	0.997009
2.0	0.00325	0.003495	0.993035
3.0	0.00335	0.003545	0.989422
4.0	0.00350	0.003946	0.984339
5.0	0.00360	0.003996	0.980216
7.0	0.00400	0.005281	0.963709
10.0	0.00450	0.005858	0.943106
15.0	0.00500	0.006135	0.912079
20.0	0.00525	0.005980	0.887281
30.0	0.00550	0.006085	0.833138

(b) OIS Discount Factors

Figure 1: OIS Results

LIBOR Discount Factors

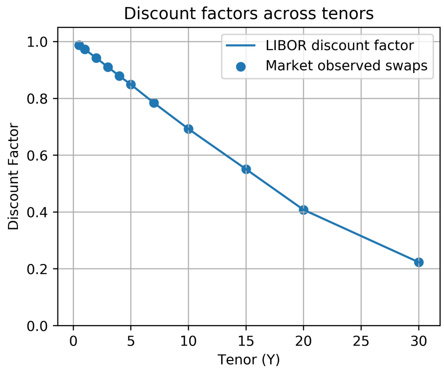
Similarly for LIBOR discount factors, we will adopt the same approach using the OIS discount factors to derive with the forward libor rates and LIBOR discount factors.

$$PV_{fix} = PV_{float}$$
$$0.5 * [D_{OIS}(0,0.5y) + D_{OIS}(0,1y)] * IRS_{1y} = 0.5 * [D_{OIS}(0,6m) * L(0,6m) + D_{OIS}(0,1y) * L(6m,1y)]$$
$$\vdots$$
$$0.5 * [D_{OIS}(0,0.5y) + \cdots + D_{OIS}(0,20y)] * IRS_{20y} = 0.5 * [D_{OIS}(0,0.5y) * L(0,6m) + \cdots + D_{OIS}(0,20y) * L(19.5y,20y)]$$

Likewise, we will also substitute the following equations to solve for one unknown (ie. LIBOR discount factor) for each of the above equation starting from 0.5y. . . 20y.In this example, we will use 7 years IRS to be consistent with our OIS approach

$$
\begin{aligned}
D(0,5.5y) &= [\frac{[D(0,7y)-D(0,5y)]}{4} * 1 + D(0,5y)] * \frac{1}{\frac{1}{2}} \\
D(0,6y) &= [\frac{[D(0,7y)-D(0,5y)]}{4} * 2 + D(0,5y)] * \frac{1}{\frac{1}{2}} \\
D(0,6.5y) &= [\frac{[D(0,7y)-D(0,5y)]}{4} * 3 + D(0,5y)] * \frac{1}{\frac{1}{2}} \\
L(5y,5.5y) &= \frac{D(0,5y)-D(0,5.5y)}{D(0,5.5y)} * \frac{1}{\frac{1}{2}} \\
L(5.5y,6y) &= \frac{D(0,5.5y)-D(0,6y)}{D(0,6y)} * \frac{1}{\frac{1}{2}} \\
L(6y,6.5y) &= \frac{D(0,6y)-D(0,6.5y)}{D(0,6.5y)} * \frac{1}{\frac{1}{2}} \\
L(6.5y,7y) &= \frac{D(0,6.5y)-D(0,7y)}{D(0,7y)} * \frac{1}{\frac{1}{2}}
\end{aligned}
$$

Proceeding to execute the same approach for all IRS , we derive the following LIBOR discount factors results and graph.



(a) LIBOR Discount Curve

Tenor	LIBOR IRS	Forward Libor	LIBOR DF
0.5	0.0250	0.025000	0.987654
1.0	0.0280	0.031005	0.972577
2.0	0.0300	0.032264	0.942178
3.0	0.0315	0.034815	0.910480
4.0	0.0325	0.035842	0.878976
5.0	0.0330	0.035329	0.848982
7.0	0.0350	0.041329	0.784164
10.0	0.0370	0.044084	0.692570
15.0	0.0400	0.051483	0.550788
20.0	0.0450	0.070136	0.407786
30.0	0.0500	0.082638	0.223275

(b) LIBOR Discount Factors

Figure 2: LIBOR Results

Forward Swap Rates

With all the necessary OIS discount factors and Forward LIBOR rates, we can go on to derive the Forward Swap rates:

Tenor	Forward Swap Rates
1x1	0.032008
1x2	0.033260
1x3	0.034012
1x5	0.035259
1x10	0.038436

Tenor	Forward Swap Rates
5x1	0.039302
5x2	0.040103
5x3	0.040105
5x5	0.041129
5x10	0.043676

Tenor	Forward Swap Rates
10x1	0.042245
10x2	0.043174
10x3	0.044157
10x5	0.046313
10x10	0.053545

Figure 3: Forward Swap rates

Part II: Swaption Calibration
Model Calibration

Calibrated Displaced-Diffsion Model Parameters

Sigma

Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	0.254272	0.315158	0.319319	0.272275	0.251499
5Y	0.300943	0.312254	0.308568	0.270108	0.247218
10Y	0.295093	0.297198	0.295274	0.266388	0.242929

Beta

Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	0.089747	0.081814	0.111088	0.155968	0.219184
5Y	0.047096	0.079271	0.146513	0.244734	0.336325
10Y	0.170571	0.11628	0.172804	0.328284	0.26183

(a) Displaced-Diffsion Model

Calibrated SABR Model Parameters

Alpha

Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	0.139067	0.184647	0.19685	0.178062	0.171145
5Y	0.166427	0.199486	0.210327	0.19118	0.17709
10Y	0.177375	0.195093	0.206595	0.201625	0.180628

Nu

Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	2.049654	1.677466	1.43821	1.064937	0.777669
5Y	1.339878	1.062712	0.937505	0.671783	0.497149
10Y	1.008059	0.926327	0.869024	0.720847	0.579857

Rho

Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y	-0.63326	-0.52512	-0.48286	-0.41448	-0.26497
5Y	-0.58484	-0.547	-0.5498	-0.512	-0.43794
10Y	-0.5457	-0.54425	-0.54963	-0.56287	-0.50973

(b) SABR Model

Figure 4: Parameter Calibration

Pricing swaptions using the calibrated model

Price of 2y x 10y payer swaption		
Strikes	SABR	Displaced-Diffusion
1.00%	2883.8565	2866.3810
2.00%	1977.9057	1937.4855
3.00%	1152.3756	1120.7630
4.00%	524.1811	528.5040
5.00%	215.0259	197.4940
6.00%	107.8104	58.1529
7.00%	66.3351	13.6305
8.00%	46.4321	2.5904

(a) payer 2y x 10y

Price of 8y x 10y receiver swaption		
Strikes	SABR	Displaced-Diffusion
1.00%	190.2511	122.4097
2.00%	378.4785	264.1790
3.00%	599.4857	496.1299
4.00%	880.4752	833.9424
5.00%	1263.6682	1282.7001
6.00%	1796.8569	1837.2748
7.00%	2483.9513	2485.1275
8.00%	3274.8412	3209.9318

(b) receiver 8y x 10y

Figure 5: Swaption price

We are calibrating sigma and beta in the meantime since both Black Scholes and Bachelier model assumes constant σ. To this end, we applied least square method to find optimal sigma and beta such that deviation of price given by DD-model and actual price is minimized.

In order to price payer swap 2y*10y, we interpolate sigma and beta between 1y*10y and 5y*10y DD-model after which calibrated model is used for pricing. When pricing through SABR, we interpolate alpha, rho and nu to get the right model for 2y expiry. We do the same for 8y receiver swap expect for interpolation is conducted between 5y expiry and 10 expiry now.

Fitting curve

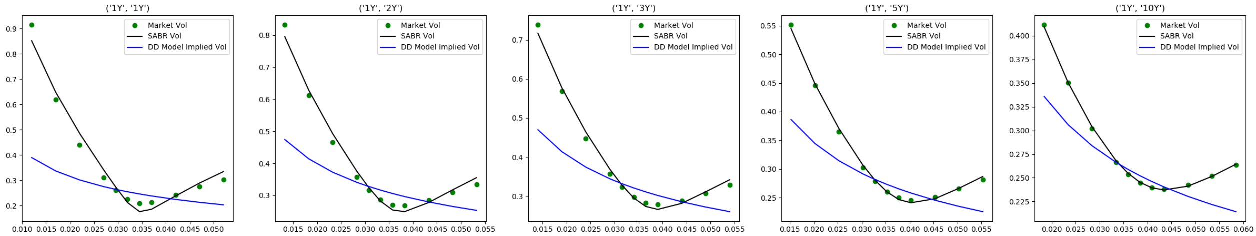


Figure 6: 1y expiry swaption: Tenor from 1y to 10y

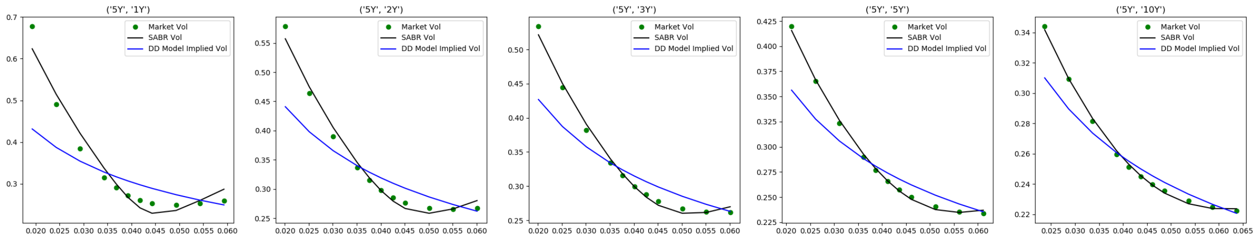


Figure 7: 5y expiry swaption: Tenor from 1y to 10y

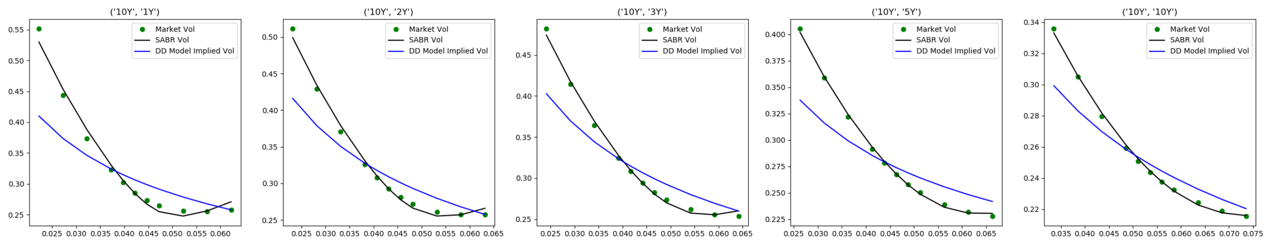


Figure 8: 10y expiry swaption: Tenor from 1y to 10y

According to the fitting curve, the SABR model can fitting the volatility "smile" closely for every expiry and tenor type of swaption, but for Displaced-Diffusion Model, the least sqaue method calibrated constant σ and β make the implied volatiliy only fit the market volatility roughly and couldn't keep trace to the existing smlie.

Given the same expiry, the longer the tenor is, the more implied volatility close to lognormal distributed. And assume the same tenor, β also goes up with expiry of the contract increase.

Part III - Convexity Correction

Present value of CMS product

To calculate PV of leg receiving CMS10y semi-annually over the next 5 years, we need to find SABR paramters at different expiries in order to price each CMS rate. To this end, cubic spline interpolation is used between α ν, and ρ of $1y \times 10y$, $5y \times 10y$ and $10y \times 10y$ SABR models we have calibrated. Since there was no expiry lower than 1y for us to interpolate, parameters for 0.5y expiry follows those of 1y expiry.

Interpolation profiles are given as follow:

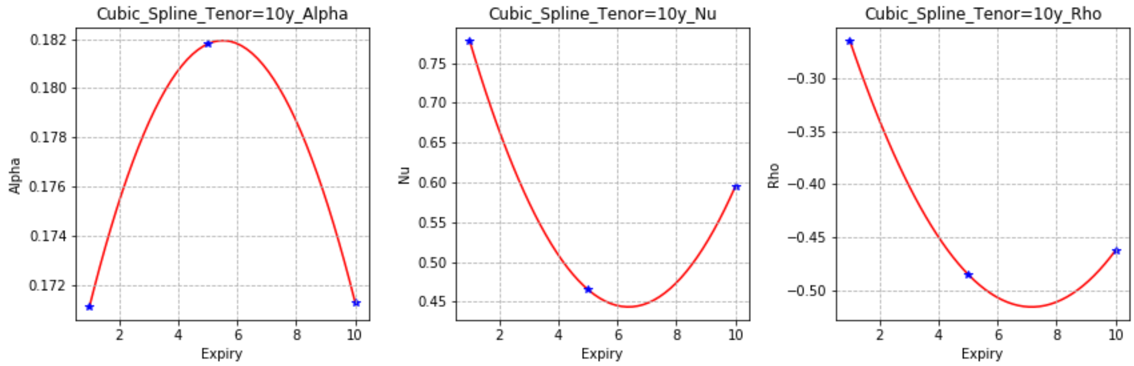


Figure 9: Interpolation of SABR parameters - CMS10Y

After interpolating all the SABR parameters, static replication is used to price each CMS rate and PV is the sum of the discounted values of all CMS rates, multiplied by the day count fraction. Here goes the mathematical form:

$$\begin{aligned} PV_{CMS10y} &= D(0,6m) \times 0.5 \times E^T[S_{6m,10y6m}(6m)] \\ &+ D(0,1y) \times 0.5 \times E^T[S_{1y,11y}(1y)] \\ &+ \cdots + D(0,5y) \times 0.5 \times E^T[S_{5y,15y}(5y)] \\ &= 0.213606 \end{aligned}$$

Similarly, for CMS2y processed quarterly, α ν, and ρ can be interpolated between $1y \times 2y$, $5y \times 2y$,$10y \times 2y$, whose profiles are demonstrated below:

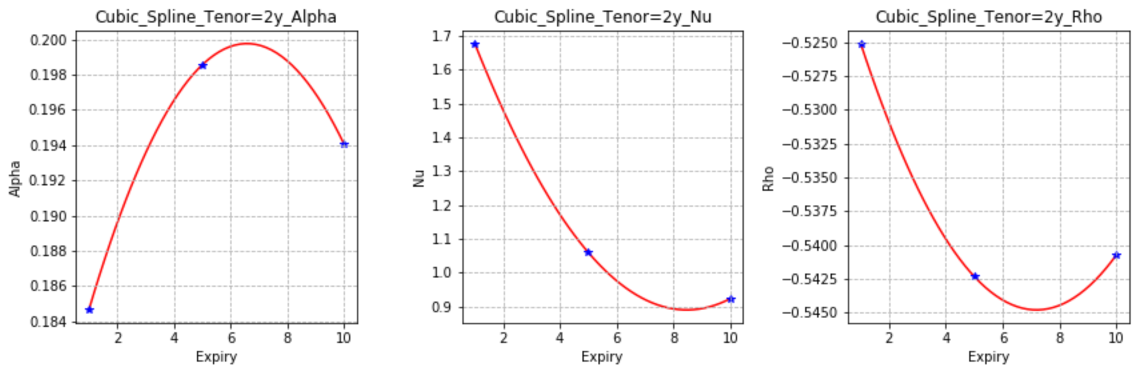


Figure 10: Interpolation of SABR parameters - CMS2Y

In addition to SABR parameters interpolation, due to quarterly arrangement, more discrete OIS discount rates and Libor discount rates are interpolated based on DF calculated in Section 1. After getting all the inputs, we can calculate PV of CMS2y as follow:

$$\begin{aligned} PV_{CMS2y} &= D(0,3m) \times 0.25 \times E^T[S_{3m,2y3m}(3m)] \\ &+ D(0,6m) \times 0.25 \times E^T[S_{6m,2y6m}(6m)] \\ &+ \cdots + D(0,10y) \times 0.25 \times E^T[S_{10y,12y}(10y)] \\ &= 0.504841 \end{aligned}$$

CMS VS Par Swap Rate

Through trial and error, we found out that the CMS rates can become unlikely large numbers or even drop below par swap rate when upper bound for payer swaption integral is set as a large number or infinity. To figure out the optimal upper bound that not only covers most of the cases but generates plausible CMS rates, we calculated pure $f(K)$ in CMS rate formula by setting the payoff function as constant 1. When the upper bound is 0.85, no value exceeded 1, and CMS rates converged on reasonable readings.

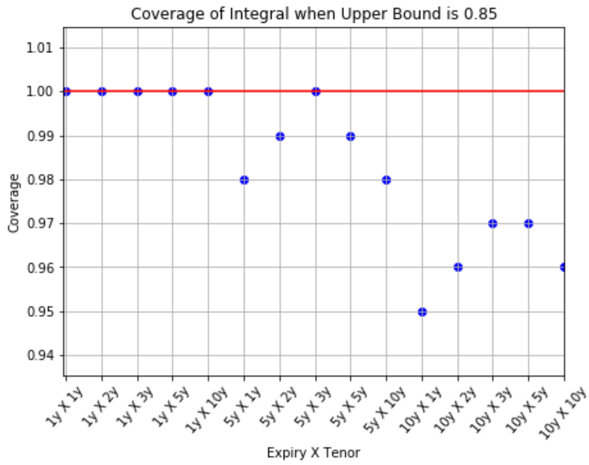


Figure 11: Coverage of Integral inside of CMS rate when Upper Bound is 0.85

Tables presented below show CMS rates for each maturity and tenor.

Tenor	CMS Rate
1x1	0.032787
1x2	0.034371
1x3	0.034913
1x5	0.035680
1x10	0.039031

Tenor	CMS Rate
5x1	0.055005
5x2	0.055665
5x3	0.054539
5x5	0.049968
5x10	0.048699

Tenor	CMS Rate
10x1	0.068968
10x2	0.079980
10x3	0.090381
10x5	0.094072
10x10	0.120374

Figure 12: CMS rates

Comparing CMS rates with forward swap rates of corresponding expiry and tenor which are derived from Part 1, we can recognise that the difference between CMS and forward swap rate increases as the expiry lengthens. It means that the longer expiry becomes, the greater the magnitude of convexity correction grows. On the contrary, the influence of tenor on the convexity correction is irregular. This phenomenon is presumed to be a result of volatility smile.

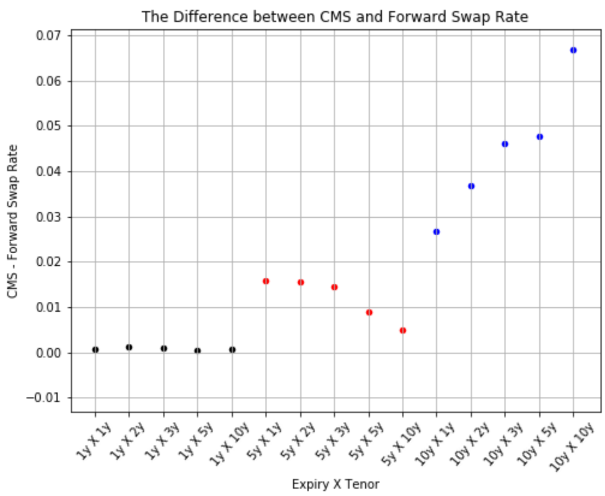


Figure 13: Delta Profile for Up-and-In Barrier Option of Given Condition

Part IV - Decompounded Options

Question 1

Starting from the generic contract valuation formula, we were able to obtain the static replication formula for the contract in question by first applying Leibniz's Rule on the IRR Payer and Receiver swaption formulas twice, after which integration by parts was carried out twice on the integrals in the generic contract valuation formula.

First, applying Leibniz's rule on the IRR Payer and Receiver swaption formulas twice yields:

Payer IRR Swaption

$$V^{pay}(K) = D(0,T)\int_K^{\infty} IRR(S)\cdot(S-K)\cdot f(S)dS$$

$$\frac{\partial^2 V^{pay}(K)}{\partial K^2} = D(0,T)\cdot IRR(K)\cdot f(K)$$

Receiver IRR Swaption

$$V^{rec}(K) = D(0,T)\int_0^{F} IRR(S)\cdot(K-S)\cdot f(S)dS$$

$$\frac{\partial^2 V^{rec}(K)}{\partial K^2} = D(0,T)\cdot IRR(K)\cdot f(K)$$

We are now able to denote the generic contract valuation formula as such:

$$\begin{aligned} V_0 &= D(0,T)\mathbb{E}\left[g(S)\right] \\ &= D(0,T)\int_0^{\infty} g(K)f(K)dK \\ &= \int_0^F h(K)\frac{\partial^2 V^{rec}(K)}{\partial K^2}dK + \int_F^{\infty} h(K)\frac{\partial^2 V^{pay}(K)}{\partial K^2}dK \end{aligned}$$

Integration by parts twice yields:

$$V_0 = D(0,T)g(F) + \int_0^F h''(K)V^{rec}(K)dK + \int_F^{\infty} h''(K)V^{pay}(K)dK$$

Where:

$$F = S_{n,N}(0), \quad n=5, \quad N=15, \quad T=5$$

$$g(K) = K^{\frac{1}{p}} - 0.04^{\frac{1}{q}} = K^{\frac{1}{4}} - 0.2, \quad g'(K) = \frac{1}{4}K^{-\frac{3}{4}}, \quad g''(K) = -\frac{3}{16}K^{-\frac{7}{4}}$$

$$h(K) = \frac{g(K)}{IRR(K)}, \quad h'(K) = \frac{IRR(K)g'(K) - g(K)IRR'(K)}{IRR(K)^2}$$

$$h''(K) = \frac{IRR(K)g''(K) - IRR''(K)g(K) - 2IRR'(k)g'(K)}{IRR(K)^2} + \frac{2IRR'(K)^2 g(K)}{IRR(K)^3}$$

$$V^{rec} = D(0,T)\cdot IRR(S_{n,N}(0))\cdot \text{Black76Put}(S_{n,N}(0), K, \sigma_{\text{SABR}}, T)$$

$$V^{pay} = D(0,T)\cdot IRR(S_{n,N}(0))\cdot \text{Black76Call}(S_{n,N}(0), K, \sigma_{\text{SABR}}, T)$$

Using these parameters, we were able to obtain a Present Value (PV) for the V_0 of **0.2334**. V_0 can also be seen as a forward contract on the 10-year CMS rate with forward price set at 0.0016.

Question 2

The contract (V_0^+) can be valued as though it is a CMS caplet when the payoff is $(S_T^{1/4} - 0.04^{1/2})^+$.

For $S_T^{1/4} - 0.04^{1/2}$ to be positive:

$$\begin{aligned} S_T^{1/4} &> 0.2 \\ S_T &> 0.0016 = L \end{aligned}$$

Thus, we can see V_0^+ as a CMS caplet struck at $L = 0.0016$:

$$\begin{aligned} V_0^+ &= D(0,T)\int_L^\infty g(K)f(K)dK \\ &= \int_L^\infty h(K)\frac{\partial^2 V^{pay}(K)}{\partial K^2}dK \\ &= h'(L)V^{pay}(L) + \int_L^\infty h''(K)V^{pay}(K)dK \end{aligned}$$

Using this valuation formula and relevant parameters from Question 1, we were able to obtain a Present Value (PV) for $\mathbf{V_0^+}$ **of 0.2430**, which is higher than $\mathbf{V_0}$**'s value of 0.2334**.

This is intuitive as V_0^+ omits the negatively valued region of V_0 where $S_T < L$ and as such should be valued higher than V_0. It also follows that $V_0 < V_0^+ < \mathbb{E}\left[S_T^{1/4}\right]$ (spot price of the underlying), as this is the model-free no-arbitrage boundary that the three products must satisfy. Running further diagnostics, it can be seen that V_0^+ is consistently more highly priced than V_0 across the given values of N (swap end date) for every n (swap start date):