FRACTIONAL PEBBLING GAME LOWER BOUNDS

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1 Introduction

The primary result in this paper involves the *fractional pebbling game*. The origins of the *fractional pebbling game* is the *black pebbling game*. The *black pebbling game* was introduced by Paterson and Hewitt [Paterson & Hewitt, 1970] to compare the power of programming languages. Since this time, variations of the pebbling game have been used in many areas of Computer Science.

The pebbling game and related terms are more rigorously defined in Section 2. The definitions presented in this paper are reinterpretations of the definitions for pebbling games presented by Cook, McKenzie, Wehr, Braverman, and Santhanam [Cook et al., 2012]. Surveys of the pebbling games are available, [Pippenger, 1982] and [Nordström, 2010].

The game is played on DAGs. Each node in the DAG may have up to one pebble. Configurations are allocations of pebbles to nodes. There is one distinguished node. The goal is to reach a configuration that has a pebble on the distinguished node while the final configuration must end with no pebbles in the DAG. Configurations of pebbles are changed from one to another via the following moves :

- Place or remove a black pebble on a leaf node

- Place a black pebble on a node that has all children pebbled

- Remove a black pebble from a node

For the *black pebbling game*, lower and upper bounds for balanced trees are given in [Cook et al., 2012]. Similar, motivational, lower and upper bounds are replicated in Section 3.

Each node can be thought of as having a value and a method of determining that value from the value of its children. Black pebbles can be thought of as values deterministically computed from previous values. This analogy is essentially the *tree evaluation problem* [Cook et al., 2012].

Branching programs are a nonuniform model of a Turing machines. Branching programs are directed multi-graphs whose nodes are states. Every edge is labelled with a value. There is one initial state from which the computation starts. Every state queries a variable and branches to new states along edges labelled with that value. These computations may eventually reach accepting or rejecting states.

A state in a branching program corresponds to a turing machine configuration. Thus L $\neq$ P if we can show the branching programs solving a problem in P requires a superpolynomial number of states.

With this goal, [Cook et al., 2012] examined a restricted class of branching programs. A thrifty branching program for the *tree evaluation problem* must query the value of the functions only at the correct value of the children. The thrifty hypothesis states that thrifty branching programs are optimal among all branching programs.

Under the thrifty hypothesis, *black pebbling game* lower bounds allow for a proof of deterministic branching program lower bounds which separate L from P [Cook et al., 2012]. It is hoped that *fractional pebbling game* lower bounds allow for a similar proof for nondeterministic branching

programs, which under the thrifty hypothesis, would separate NL from P.

Another variation of the pebbling game is the whole *black-white pebbling game*. It was introduced by Cook and Sethi [Cook & Sethi, 1976] in an attempt to separate NL and P. It is similar to the *black pebbling game* except the rules for changing one configuration to another are the following :

- Place or remove a pebble on a leaf node

- Place a black pebble on a node that has all children pebbled

- Remove a black pebble from a node

- Place a white pebble on a node

- Remove a white pebble from a node that has all children pebbled

White pebbles can be thought of as non-deterministic guesses for values. When we removed them we have essentially justified those guesses.

The pebbling games are important due to their relation to propositional proof complexity, particularly resolution. For this purpose, the whole *black-white pebbling game* is usually used. Aspects of the game are encoded as a CNF formulas. Properties of the formulas are then argued based on properties of the pebbling game. [Nordström, 2010] produced a survey of how the pebbling games relate to proof complexity.

Aleknovich showed a separation between regular and general resolution using a problem that is a modified version of the whole *black-white pebbling game* [Alekhnovich et al., 2002].

Using the pebbling contradiction problem derived from the pebbling game, Nordstrom showed resolution refutations of small widths may have large space requirements [Nordstrom, 2005]. Ben-Sasson showed, using the same pebbling contradictions, trade-offs between time size space and width of resolution [Ben-Sasson, 2002].

Motivated by proving lower bounds for branching programs [Cook et al., 2012] recently introduced the *fractional pebbling game*. The *fractional pebbling game* is a generalization of the whole *black-white pebbling game*.

The rules are similar to those presented in the whole *black-white pebbling game* except we now allow for fraction of pebbles.

The *fractional pebbling game* should better represent the non-deterministic approach to the problem than the whole *black-white pebbling game*. Fractions of pebbles can be thought of as partially specifying the possible values of a node. This intuitively is helpful and seems less restrictive than the whole *black-white pebbling game*. We confirm that this is helpful by showing smaller lower bounds for the *fractional pebbling game* than are possible for the whole *black-white pebbling game*. These lower bounds match upper bounds presented in [Cook et al., 2012] for the *fractional pebbling game*.

The main theorem we show in this paper (Section 5.3) relies on the $fractional\ pebbling\ game$. Let T_d^h be the balanced d-ary tree of height h. Let $min_h = (d-1) * h/2 + 1$. Let the root node be the node that must be pebbled.

Main Theorem
In every fractional pebbling of T_d^h, where the distinguished node is the root, there is a configuration such that the number of pebbles is greater than or equal to min_h.

Loose lower bounds for this problem were presented in [Cook et al., 2012] and tight lower bounds were left as an open problem. In that case the lower bounds for the problem came from a reduction to a paper by Klawe [Klawe, 1985] which proves the bounds for pyramid graphs rather than balanced trees. Accuracy is lost in the reduction. We present tight lower bounds for balanced trees of any degree by taking a more direct approach.

We will solve this problem using a $shifting$ argument. The idea in our shifting argument is that if we use less pebbles before placing a pebble on the root we use more pebbles after placing a pebble on the root. We proceed in this manner since we must cover a larger range of pebbling strategies once we allow for fractional pebbles.

1.1 Organization

The organization of this paper is as follows. Section 2 defines the pebbling game and associated terms. It first defines the $black\ pebbling\ game$ and the whole $black\text{-}white\ pebbling\ game$. It then defines the $half\ pebbling\ game$ and the $fractional\ pebbling\ game$ as modifications of the whole $black\text{-}white\ pebbling\ game$. Further we define terms related to all games. Section 3 first demonstrates upper bounds for the $black\ pebbling\ game$. It then demonstrates lower bounds for the $black\ pebbling\ game$. In Section 4 we show upper and lower bounds for the whole $black\text{-}white$ $pebbling\ game$. Section 5 shows upper bounds for the half and $fractional\ pebbling\ game$s and concludes by showing $fractional\ pebbling\ game$ lower bounds.

2 Preliminaries

In Section 3 we examine the *black pebbling game*. We next present definitions and rules needed for the *black pebbling game* played on DAGs.

Definition 2.0.1 *A **black pebble configuration** on a DAG is an assignment of values b(i) to each node i of the tree, where*
b(i) = 0 or b(i) = 1
*We let b(i) represent the **black pebble weight value** of i.*

Definition 2.0.2 *A **black pebble move** changes one black pebble configuration into another. Possible black pebble moves are :*
(i) For any node i, decrease b(i) from 1 to 0
(ii) For any node i, if each child of i has pebble value 1, increase b(i) to 1, and optionally decrease any of the black pebble values of the children of i to 0
(iii) For each leaf node i, increase b(i) to 1

For (ii), if we choose to decrease the black pebble value of the children it is done simultaneously, this is called a **black sliding move**.

Definition 2.0.3 *A **black pebbling** π is a sequence $m_1, m_2, \ldots$ of black pebble moves resulting in a sequence $c_0, c_1, c_2, \ldots$, of black pebble configurations, where c_0 is the initial configuration, and for $t > 0$, c_t is the configuration after move m_t.*

We next present definitions needed for the *whole black-white pebbling game*. Upper bounds for this game are presented in Section 4.

Definition 2.0.4 *A **whole black-white pebble configuration** on a DAG, is an assignment of a pair of numbers (b(i),w(i)) to each node i of the tree, where*
b(i) = 0 or b(i) = 1,
w(i) = 0 or w(i) = 1 and
b(i) + w(i) $\leq$ 1
*Here b(i) and w(i) are the **black pebble weight value** and the **white pebble weight value**, respectively, of node i, and b(i) + w(i) is the **pebble weight** of node i.*

Definition 2.0.5 *A **whole black-white pebbling move** changes one whole black-white pebble configuration into another. Possible whole black-white pebble moves are :*
(i) For any node i, set b(i) to 0
(ii) For any node i, if each child of i has pebble value 1, set w(i) to 0, increase b(i) to 1, and optionally decrease any of the black pebble weight values of the children of i to 0
(iii) For any node i, increase w(i) to 1
(iiii) For each leaf node i, increase b(i) to 1

Definition 2.0.6 *A* **whole black-white pebbling** π *is a sequence* $m_1, m_2, \ldots$ *of whole black-white pebble moves resulting in a sequence* $c_0, c_1, c_2, \ldots$, *of whole black-white pebble configurations, where* c_0 *is the initial configuration, and for* $t > 0$, c_t *is the configuration after move* m_t.

In Section 5.1 we use a variation of the whole *black-white pebbling game* wherein we additionally allow b(i) and w(i) to be 0.5. We call this variation the *half pebbling game*. This closely resembles the *fractional pebbling game* defined next.

In Section 5.2 and 5.3 we use a variation of the whole *black-white pebbling game* that allows b(i) and w(i) to be any real number in [0,1]. We call this variation the *fractional pebbling game*:

Definition 2.0.7 *A* **fractional pebble configuration** *on a DAG, is an assignment of a pair of real numbers (b(i),w(i)) to each node i of the tree, where*
$0 \leq b(i), w(i)$ *and*
$b(i) + w(i) \leq 1$
Here b(i) and w(i) are the **black pebble weight value** *and the* **white pebble weight value**, *respectively, of node i, and b(i) + w(i) is the* **pebble weight** *of node i.*

Definition 2.0.8 *A* **fractional pebble move** *changes one fractional pebble configuration into another. Possible fractional pebble moves are :*
(i) For any node i, decrease b(i) arbitrarily
(ii) For any node i, if each child of i has pebble value 1, decrease w(i) to 0, increase b(i) arbitrarily, and optionally decrease the black pebble weight values of the children of i arbitrarily
(iii) For any node i, increase w(i) such that b(i) + w(i) = 1
(iiii) For each leaf node i, increase b(i) arbitrarily

Definition 2.0.9 *A* **fractional pebbling** π *is a sequence* $m_1, m_2, \ldots$ *of fractional pebble moves resulting in a sequence* $c_0, c_1, c_2, \ldots$, *of fractional pebble configurations, where* c_0 *is the initial configuration, and for* $t > 0$, c_t *is the configuration after move* m_t.

We additionally define the following terms and symbols important to all variations of the games.

Definition 2.0.10 *We refer to a configuration* c_t *as the* **time** *t.*

Definition 2.0.11 *We let* **0** *denote the initial configuration, equivalently the* **initial time**.

Definition 2.0.12 *The* **weight**, $w_\pi(t)$, *of* π *at time* t *is sum of the pebble weights on* T *in configuration* c_t. *The* **subtree weight**, $sw_\pi(t)$, *of* π *at time* t *is the sum of the pebble weights in the principal subtrees of* T *in configuration* c_t. *The* **white subtree weight**, $w.sw_\pi(t)$, *of* π *at time* t *is the sum of the white pebble weights in the principal subtrees of* T *in configuration* c_t. *The* **black subtree weight**, $b.sw_\pi(t)$, *of* π *at time* t *is the sum of the black pebble weights in the principal subtrees of*

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*T in configuration c_t. The **root weight**, $rw_\pi(t)$, of π at time t is the pebble weight on the root of T in configuration c_t. The **black root weight**, $b.rw_\pi(t)$, of π at time t is the black pebble weight on the root of T in configuration c_t. The **white root weight**, $w.rw_\pi(t)$, of π at time t is the white pebble weight on the root T in configuration c_t.*

Square brackets after the symbols defined above are used to indicate in which tree or subtree the pebble weight is located. For example, the symbol $b.rw_\pi(t)[P_{last}]$ would be used to specify some amount of black pebble weight on the root of the tree P_{last} at time t. If it is not specified, the symbol is assumed to pertain to the entire tree.

Definition 2.0.13 *A **root-pebbling** is a pebbling that requires that the initial and final pebble weights of π are 0, and $rw_\pi(t) = 1$ at some time t.*
*A **sub-pebbling** is a pebbling that may start or end with pebble weight. It may initially have arbitrary white pebble weight and at the end of the pebbling it may have arbitrary black pebble weight. It may also have some specified initial black pebble weight. At the end of the pebbling it has no white pebble weight.*
*A **root sub-pebbling** is a sub-pebbling such that $rw_\pi(t) = 1$ at some time t.*
*Similarly, a **sub-root sub-pebbling** is a sub-pebbling such that the subtrees of T have $rw_\pi(t)=1$ at some time t.*

Lemma 2.0.14 *If π_1 is a sub-pebbling with initial white and black pebble weight, and $w_{\pi_1}(t) \leq P$ for all times t then there exists a sub-pebbling π_2 with the same initial black pebble weight and no white pebble weight such that $w_{\pi_2}(t) \leq P$ for all times t.*

We show such a π_2. The first steps is to place the same white pebble weight on the same nodes as initially in π_1. We then could follow the sub-pebbling π_1. Since we have less pebble weight before we add the white pebble weight, $w_{\pi_2}(t) \leq P$ for all times t. ∎

This lemma indicates that initial white pebble weight is not helpful.

In all pebbling games we allow for a **black sliding** move. This is pebble move (ii) in all games. Rule (ii) is sometimes alternatively written as follows :
(ii) For any node i, if each child of i has pebble value 1, increase b(i) arbitrarily.
This would be the case if we did not allow for black sliding moves. This decouples increasing pebble weight and removing pebble weight from the children.

Observation 2.0.15 *A pebbling with **black sliding** moves can be converted to a pebbling without black sliding moves which requires at most 1 more pebble weight.*

This is simply the result of changing a black sliding move to two subsequent moves. We allow **sliding** moves in our proofs.

Definition 2.0.16 *We let T_d^h represent the balanced d-ary tree of height h.*

3 Black Pebbling Game

3.1 Black Pebbling Game Upper Bounds

We prove the following theorem which shows an upper bound for the *black pebbling game* defined in Section 2. Similar results can be found in [Cook et al., 2012].

Theorem 3.1.1 *Let $min_h = h$. There exists a black pebbling game* **root-pebbling** π *of T_2^h, $h \geq 2$, such that for all times t, $w_\pi(t) \leq min_h$.*

To show this we use induction.

Base Case : h = 2.

There are 2 children of the root. We place a black pebble weight on each leaf and slide a black pebble weight to the root. Thus, $sw_\pi \leq 2$ at this time and all previous times. Thus the IH is satisfied in the base case.

Induction step : We prove for $h + 1$ assuming for h', $3 \leq h' \leq h$.
Note $min_{h+1} = min_h + 1$.

There are two subtrees of the root. Using min_h pebble weight we pebble the first subtree root using the pebbling in the IH for height h. We then remove black pebble weight that is not on the root of the subtree such that we have only this 1 pebble weight.

We next use min_h pebble weight to pebble the second subtree root using the pebbling in the IH for height h. At this time we maintain one pebble weight in the first subtree. We thus use $sw_\pi \leq min_h + 1$.

We now have a pebble on each subtree root and slide a pebble to the root. Thus, $sw_\pi \leq min_h + 1 = min_{h+1}$ at all times.

Thus, the IH is satisfied.

To show this for d-ary balanced trees we would iteratively pebble the children of the root using the pebbling in the IH. Each time leaving a pebble. This would result in an upper bound of $(d-1)*(h-1)+1$.

The key insight is that we had to leave some pebble weight in one subtree while we proceeded with the pebbling in another subtree. This idea is important to all subsequent proofs.

3.2 Black Pebbling Game Lower Bounds

We prove the following theorem which shows a lower bound for the *black pebbling game* defined in Section 2. Combined with the previous section we have a tight bound on the number of pebbles taken to complete the *black pebbling game* for balanced trees of degree 2. Similar results have

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been shown in [Cook et al., 2012].

Theorem 3.2.1 *Let $min_h = h$. For every black pebbling game* **root-pebbling** π *of* T_2^h, $h \geq 2$, *there is a time t such that $w_\pi(t) \geq min_h$.*

To show this we use induction.

Base Case : h = 2.

There are 2 children of the root. To place a pebble on the root we must pebble these 2 nodes. The IH is then satisfied in the base case.

Induction step : We prove for $h + 1$ assuming for h', $3 \leq h' \leq h$.
Note $min_{h+1} = min_h + 1$.

There are 2 subtrees of the root. There must be a time before we pebble the root that we have a pebble on each subtree root if we are to place a pebble on the root. Thus, by IH, there must be a last time we use pebble weight min_h in one of the subtrees. Let this time be t_{last}.

At t_{last}, suppose for contradiction we did not have one pebble in the other subtree. Having less than one black pebble on any node does not allow us to apply any of the pebbling rules and is thus equivalent to having no pebble weight.

To pebble the root we must have a pebble on each of the subtree roots. Thus if we had less than one pebble in any subtree we must place a pebble on the root of that subtree before we pebble the root. To do this we require min_h pebble weight by IH. This would contradict t_{last} being the last time we use pebble weight min_h.

Thus we maintain at least one pebble in the other subtree at t_{last} and $w_\pi(t_{last}) \geq min_h + 1 = min_{h+1}$ as required.

Thus, the IH is satisfied.

To show this for d-ary balanced trees we would look at the last time we use min_h in any tree and argue that we need 1 pebble in each other subtree at this time.

This would result in an lower bound of $(d - 1) * (h - 1) + 1$. The proofs in this section result in a tight lower bound for the *black pebbling game* on balanced binary trees. We will show a tight lower bound for the *fractional pebbling game*.

4 Whole Black-White Pebbling Game

4.1 Whole Black-White Pebbling Game Upper Bounds

We prove the following theorem which shows an upper bound for the whole *black-white pebbling game* defined in Section 2. Similar results can be found in [Cook et al., 2012].

Theorem 4.1.1 *Let $min_h = \lceil h/2 \rceil + 1$. There exists a whole black-white pebbling game* **root-pebbling** π *of T_2^h, $h \geq 2$, such that for all times t, $w_\pi(t) \leq min_h$.*

To show this we use induction. We show this only for the even height cases and it follows for the odd height cases since we can extract a pebbling for an odd height from the larger even height pebbling.

Induction Hypothesis [IH(h)]:
Let $min_h = h/2 + 1$.
For even $h \geq 2$ there exist a *whole black-white pebbling game* **root-pebbling** π of T_2^h and a time t_{root} such that $sw_\pi \leq min_h$ at all times. Additionally,
  (1) $b.rw_\pi(t_{root}) = 1$
  (2) $w.w_\pi(t_{root}) \leq min_h - 2$
  (3) White pebble weight at t_{root} can be removed using $w_\pi(t) \leq min_h$ for $t > t_{root}$

Condition (1) specifies that the root weight at t_{root} is black. Condition (2) specifies that there is not too much white pebble weight at t_{root}.

Base Case : h = 2.
There are 2 children of the root. We use 2 pebble weight on the leaves and slide it to the root. Thus, $sw_\pi \leq 2$ at this time and all previous times. Condition (2) and (3) are satisfied since we have no white pebble weight. Thus the IH is satisfied in the base case.

Induction step : We prove the induction hypothesis for $h + 2$ assuming it for h', $2 \leq h' \leq h$. Note $min_{h+2} = min_h + 1$.

We let the children of the root be p_2 and p_3. We call the children of these v_1, v_2, v_3 and v_4 as in the following figure.
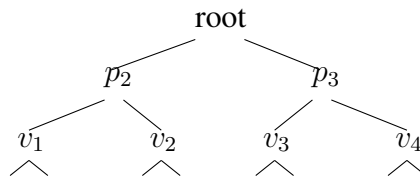

Figure 1: Our labeling of the nodes of T_2^h.

We simulate the pebbling in the IH for height h in the subtree rooted at v_1. We modify the pebbling to leave the pebble on the root. This requires at most $min_h + 1$ pebble weight.

We then simulate the pebbling in the subtree rooted at v_2. We interrupt the pebbling when v_2 is pebbled. We use $sw_\pi \leq min_h + 1$ at all times before this point.

We remove all other black pebble weight in v_2 such that we have $min_h - 2$ white pebble weight in the subtree rooted at v_2 by condition (2) and an additional pebble on v_2. At this time we maintain one pebble weight on v_1. We then have min_h pebble weight in the tree.

We then slide a pebble to p_2. We then place a white pebble on p_3. We may then slide a pebble to the root. At this point we have 1 pebble on the root, 1 pebble on p_3 and $min_h - 2$ in the subtree rooted at v_2. By sliding the pebble to the root we satisfy condition (1).

We then remove all black pebble weight and have white pebble weight $min_h - 1$, satisfying (2). We have yet to exceed min_{h+2}. We only have white pebble weight present at t_{root} thus removing it will show (3).

We remove the $min_h - 2$ white pebble weight in the subtree rooted at v_2. This takes min_h by condition (3) of the IH. The only other pebble weight is on p_3. Thus condition (3) has yet to be violated and we still have not exceeded min_{h+2}.

We simulate the pebbling in the subtree rooted at v_3 and interrupt it when there is a pebble on v_3. We remove all black pebble weight other than on the v_3. At this point there is 1 pebble on p_3, 1 pebble on v_3, and $min_h - 2$ white pebbles in the subtree rooted at v_3. We then place a white pebble on v_4. Thus we have yet to exceed min_{h+2}.

We remove the pebble on p_3 and the black pebble on v_3. We then remove the white pebble weight in the subtree rooted at v_3 using (3) from the IH.

To remove the white pebble on v_4 we simulate the pebbling for h but remove the white pebble instead of placing a black pebble. We remove the resulting white pebble weight and the pebbling is complete. At no point in removing the white pebble weight that was present at t_{root} have we used more that $min_h + 1$, thus condition (3) and the IH are satisfied.

This shows the power of white pebbles. We next show that the upper bound for fractional pebbling can be obtained using only half pebbles. However, in Section 5 we show that fractional pebbles allow for a multitude of pebbling strategies.

4.2   Whole Black-White Pebbling Game Lower Bounds

We prove the following theorem which shows a lower bound for the whole *black-white pebbling game* defined in Section 2. Combined with the previous section we have a tight bound on the number of pebbles taken to complete the whole *black-white pebbling game* for balanced trees of degree 2. Similar results have been shown in [Cook et al., 2012].

Theorem 4.2.1 *Let* $min_h = \lceil h/2 \rceil + 1$. *For every whole black-white pebbling game* **root-pebbling** π *of* T_2^h, $h \geq 2$, *there is a time t such that* $w_\pi(t) \geq min_h$.

We show this by induction :

Base Case : h = 2

We must show that for $h = 2$, if π is a *whole black-white pebbling game* **root-pebbling** of T_2^2, then there is a time t such that $sw_\pi(t) \geq 2$. This is trivially true.

Base Case : h = 3

We need to show that if π is a *whole black-white pebbling game* **root-pebbling** of T_2^3, then there is a time t such that $sw_\pi(t) \geq 3$.

If we ever use a white pebble we must use at least 3 pebbles at the time before we remove it. Thus we may not use white pebbles if we wish to use less than pebble weight 3.

Then, if we used less than 3 pebble weight, we would contradict Theorem 3.2.1.

Induction step : Assuming the theorem is true for h', $2 \leq h' \leq h$, it is sufficient to prove the following.

Lemma 4.2.2 *For* $h \geq 2$, *if* π *is a whole black-white pebbling game* **root-pebbling** *of* T_2^{h+2} *then there is a time t such that* $sw_\pi(t) \geq min_{h+2}$.

Proof:

Note $min_{h+2} = min_h + 1$. For the sake of contradiction, suppose $sw_\pi(t) < min_h + 1$ or equivalently $sw_\pi(t) \leq min_h$ for all times t.

Since there is a time where the root is pebbled there must be a time where the children of the root are pebbled to add black pebble weight or to remove white pebble weight from the root. Let $t_{root}{}^*$ be a time such that $rw_\pi(t_{root}{}^*)=1$ for both principal subtrees.

By the same logic we must pebble v_1, v_2, v_3 and v_4 (Figure 1). Thus, by the IH, it is the case that at some time we must use min_h pebble weight in the subtrees rooted at these nodes. Note there may be more than one time fitting this description for each tree rooted at the v_i.

If two or more of these times occur before $t_{root}{}^*$ then at the last time there must be no pebble weight elsewhere in the tree. Thus we must again use min_h in the subtrees that are not the last subtree. Thus we will need to use min_h in at least three subtrees after $t_{root}{}^*$ (subtrees rooted at v_1, v_2, v_3 or v_4).

When we use min_h in the first such subtree after $t_{root}{}^*$ there can be no pebbles elsewhere. This would indicate we no longer need to reach such a time in any other subtrees. This is a contradiction since we need to use min_h in at least three subtrees after $t_{root}{}^*$. Thus at some some time t, $sw_\pi(t) > min_h$ and $sw_\pi(t) \geq min_{h+2}$ as desired.

∎

The previous proof is much simpler than the proof of the main theorem we will show later. This is due to the limited number of strategies possible when using whole pebbles.

5 Fractional Pebbling Game

5.1 Half Pebbling Game Upper Bounds

We prove the following theorem which shows an upper bound for the $half\ pebbling\ game$ defined in Section 2. Similar results can be found in [Cook et al., 2012].

Theorem 5.1.1 *Let $min_h = h/2 + 1$. There exists a $half\ pebbling\ game$ **root-pebbling** π of T_2^h, $h \geq 2$, such that for all times t, $w_\pi(t) \leq min_h$.*

To show this we use induction.

Induction Hypothesis [IH(h)]:
Let $min_h = h/2 + 1$. Let t_{root} be a time such that $rw_\pi(t_{root}) = 1$.
For $h \geq 2$ there exist a $half\ pebbling\ game$ **root-pebbling** π of T_2^h such that $sw_\pi \leq min_h$ at all times. Additionally,
   (1) $b.rw_\pi(t_{root}) = 1$
   (2) $w.w_\pi(t_{root}) \leq min_h - 2$
   (3) White pebble weight at t_{root} can be removed using $w_\pi(t) \leq min_h$ for $t > t_{root}$

Base Case : h = 2.
   There are 2 children of the root. We place 2 black pebble weight on the leaves and slide it to the root. Thus, $sw_\pi \leq 2$ at this time and all previous times. Condition (2) and (3) are satisfied since we have no white pebble weight. Thus the IH is satisfied in the base case.

Induction step : We prove the induction hypothesis for $h + 1$ assuming it for h', $2 \leq h' \leq h$. Let P_2 and P_3 be the principal subtrees. Note $min_{h+1} = min_h + 0.5$.

   We simulate the pebbling in the IH for height h in P_2. We modify the pebbling to leave half a black pebble on the root. This requires at most half a pebble more or min_{h+1} pebble weight.
   We then simulate the pebbling in the IH for height h in P_3. We interrupt the pebbling when the root of P_3 is pebbled. We use $sw_\pi \leq min_{h+1}$ at all times before this point.
   We remove all other black pebble weight in P_3 such that we have $min_h - 2$ white pebble weight in the subtree P_3 by condition (2) and an additional pebble on the root of P_3.
   We next add half a white pebble to the root of P_2 and slide a pebble from the root of P_3 to the root. Thus condition (1) is satisfied. We remove all black pebble weight and have half a white pebble on the root of P_2 and $min_h - 2$ white pebble weight in P_3. We thus satisfy condition (2). Additionally, we only have white pebble weight present at this t_{root} and removing it will show condition (3).

   We remove the $min_h - 2$ white pebble weight in P_3. This takes min_h pebble weight by condition (3) of the IH. The only other pebble weight is the half pebble on the root of P_2.

We simulate the pebbling from the IH for height h in P_2. Instead of placing a black pebble we remove the white pebble on the root. This takes min_h while maintaining the half a white pebble on the root of P_2. Thus condition (3) is not violated.

Thus the IH is satisfied.

We next show we can not do better using fractional pebbles. However, we also show there are strategies not available using only half pebbles.

5.2 Fractional Pebbling Game Upper Bounds

We prove the following theorem which shows an upper bound for the $fractional\ pebbling\ game$ defined in Section 2. Similar results have been known since [Cook et al., 2012].

Theorem 5.2.1 *Let $min_h = (d-1) * h/2 + 1$. There exists a $fractional\ pebbling\ game$ **root-pebbling** π of T_d^h such that for all times t, $w_\pi(t) \leq min_h$.*

To show this we use induction.

Induction Hypothesis [IH(h)]:
Let $min_h = (d-1) * h/2 + 1$. Let $t_{root}{}^*$ be a time such that $rw_\pi(t_{root}{}^*) = 1$ for all principal subtrees.
For $h \geq 3$, $\epsilon \in [-0.5, 0.5]$, there exists a $fractional\ pebbling\ game$ **sub-root pebbling** π of T_d^h such that the following conditions are true.
    (0) there exists a time $t_{root}{}^*$ such that $rw_\pi(t_{root}{}^*) = 1$ for all subtrees
    (1) $sw_\pi(t) \leq min_h - \epsilon$ for $t \leq t_{root}{}^*$
    (2) $w.w_\pi(t_{root}{}^*) \leq min_h + \epsilon - d$
    (3) Any white pebble weight at $t_{root}{}^*$ can be removed using $sw_\pi(t) \leq min_h + \epsilon$ for $t > t_{root}{}^*$
    (4) $b.rw_\pi(t_{root}{}^*) = 1$ for at least one subtree
    (5) $sw_\pi(t) \leq min_h + \epsilon$ for $t > t_{root}{}^*$

Observation 5.2.2 *The previous IH resembles the IH for the lower bound to be proved later.*

The next two lemmas are to be used in the proof of the Induction hypothesis. They are to be applied to the subtrees of the root. They deal with leaving black pebble weight and removing white pebble weight.

Lemma 5.2.3 *It follows from the IH for height h, for $E \in (0, 0.5]$, that there exists a **pebbling** π with $w_\pi(t) \leq min_h + E$ for all times t and $w_\pi(0) = 0$, that ends with $b.rw_\pi = 2E$ and $sw_\pi = 0$.*

Lemma 5.2.4 *It follows from the IH for height h, for $E \in (0, 0.5]$, that there exists a **pebbling** π with $w_\pi(t) \leq min_h + E$ for all times t, $w.rw_\pi(0) = 2E$ and $sw_\pi(0) = 0$, that ends with $w_\pi = 0$.*

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Proof of Lemma 5.2.3

We modify the pebbling in the IH with $\epsilon = -E$. We slide 2E black pebble weight to the root a step after t_{root}^*. This does not exceed $min_h + E$ weight since we use the same weight as at t_{root}^*, $w_\pi(t_{root}^*) \le min_h + E$.

We remove all black pebble weight and we use $sw_\pi \le min_h - E$ to remove the remaining white pebble weight by condition 3 of IH. Thus for $t > t_{root}^*$, since we maintain $b.rw_\pi(t) = 2E$, we use $w_\pi(t) \le min_h + E$. Thus we use $w_\pi(t) \le min_h + E$ for all times t and have satisfied the conditions of the lemma.

∎

Proof of Lemma 5.2.4

Given the white pebble weight on the root we follow the pebbling in the IH with $\epsilon = E$. We modify the pebbling by removing the pebble weight on the root at time t_{root}^*. We use $sw_\pi(t) \le min_h - E$ for $t \le t_{root}^*$ while maintaining $w.rw_\pi(t) = 2E$.

We then remove all black pebble weight and use $sw_\pi(t) \le min_h + E$ for $t > t_{root}^*$ to remove the white pebble weight by the IH. Thus we use $w_\pi(t) \le min_h + E$ for all t.

∎

Proof of the Induction Hypothesis
Base Case : h = 3

In this case $min_h = min_3 = 3/2 * (d-1) + 1$.

Let the nodes v_i be the children of the root, $i \in [d]$. Let v_{last} be the last node enumerated in this way.

For the first (d-1) v_i, place $(d-1)/2 - \epsilon$ black pebble weight between them. This value is the amount in excess of d, the amount needed to pebble the leaves of the final subtree. Do this by placing d pebble weight on the leaves and sliding the largest possible portion of this amount to the subtree root (at most 1 per subtree root). Next, remove black pebble weight not on the subtree roots. Repeat starting with the first subtree until we place $(d-1)/2 - \epsilon$ black pebble weight.

There are enough children of the root which are not v_{last} to leave this amount since $(d-1)/2 - \epsilon \le (d-1)/2 + (d-1)/2 = (d-1)$.

We must use d pebble weight on the leaves each time we leave a fraction of a black pebble on a v_i. However, $(d-1)/2 - \epsilon + d = 3/2(d-1) - \epsilon + 1 = min_3 - \epsilon$. Thus we do not violate (1) in the IH when leaving $(d-1)/2 - \epsilon$ black pebble weight on the first (d-1) v_i.

We then use d pebble weight on the leaves of v_{last}. We then slide one pebble weight to v_{last} and remove the weight on the leaves.

We then add $(d-1)/2 + \epsilon$ white pebble weight to the first (d-1) v_i to reach t_{root}^*.
At this time we have d pebble weight, thus we have not violated (1).
In this way $sw_\pi(t) \le min_3 - \epsilon$ for $t \le t_{root}^*$ thus π satisfies (1).
Since at this time v_{last} is black pebbled (4) is satisfied.
Also $w.w_\pi(t_{root}^*) = (d-1)/2 + \epsilon = min_3 - d$, thus (2) is satisfied.
We then remove all black pebble weight.

We may then remove any of this white pebble weight using d pebble weight.

When we remove this white pebble weight we have $sw_\pi \le (d-1)/2 + \epsilon + d = 3/2(d-1) + \epsilon + 1 = min_3 + \epsilon$ as required. Thus (3) is satisfied.

Since this is all we must do and this is the most we use after $t_{root}{}^*$, condition (5) is satisfied.

Thus the specified π satisfies all conditions and the IH is satisfied.

Induction step : We prove the induction hypothesis for $h+1$ assuming it for h', $3 \le h' \le h$. Note $min_{h+1} = min_h + (d-1)/2$.

Let P_i be the the subtrees of the root, $i \in [d]$. Let P_{last} be the last subtree enumerated in this way.

Using **Lemma 5.2.3** we leave $(d-1)/2 - \epsilon$ pebble weight on the root of the first (d-1) subtrees.

If $(d-1)/2 - \epsilon \le 1$. We leave $(d-1)/2 - \epsilon$ pebble weight on the last of the first (d-1) P_i. To do so we require $w_\pi \le min_h + ((d-1)/2 - \epsilon)/2$ by **Lemma 5.2.3**. In the other subtrees we leave no pebble weight. Thus we do not exceed $min_h + (d-1)/2 - \epsilon$ and do not violate (1).

If $(d-1)/2 - \epsilon > 1$. We leave one pebble weight on the root of the last of the first (d-1) P_i. Thus we require $min_h + 0.5$ by **Lemma 5.2.3**. At this time we have $(d-1)/2 - \epsilon - 1$ on the root of the other P_i. In the prior trees we require at most the same pebble weight while maintaining less in the other trees at that time. Thus we do not exceed $min_h + (d-1)/2 - \epsilon$ and do not violate (1).

For the final subtree, we use the pebbling in the IH for height h, with $\epsilon = 0$, except we modify the pebbling to slide a pebble in the step after $t_{root}{}^*$. A slidable pebble exists by condition (4). We then remove all black pebbles in P_{last} other than the black pebble on the root, leaving $min_h - d$ white pebble weight. Since we do not use more than pebble weight min_h in P_{last} while maintaining $(d-1)/2 - \epsilon$ in the other subtrees, we do not violate (1).

We then use $(d-1)/2 + \epsilon$ white pebble weight on the root of the other P_i to reach $t_{root}{}^*$. At this time we have d pebble weight on the subtree roots while having $min_h - d$ white pebble weight in P_{last}. We thus have min_h total pebble weight at this time and do not violate (1).

Thus, condition (1) is satisfied as we have $sw_\pi(t) \le min_h - \epsilon$ for all $t \le t_{root}{}^*$.

At this time we have $b.rw_\pi(t_{root}{}^*)[P_{last}] = 1$, thus (4) is satisfied.

We then remove all black pebble weight.

We have $(d-1)/2 + \epsilon$ white pebble weight on the roots of the subtrees while having $w.w_\pi(t_{root}{}^*)[P_{last}] = min_h - d$. Thus we have $w.w_\pi(t_{root}{}^*) \le min_h + (d-1)/2 + \epsilon - d = min_{h+1} + \epsilon - d$ and (2) is satisfied.

We first remove the white pebble weight from the subtree P_{last}. By IH, this requires $sw_\pi[P_{last}] \le min_h$ while maintaining $(d-1)/2 + \epsilon$ pebble weight in the other subtrees. Thus, to remove this white pebble weight we require $sw_\pi(t) \le min_{h+1} + \epsilon$ for $t > t_{root}{}^*$.

We next remove white pebble weight from the first subtree with white pebble weight on the root, P_{first}. Suppose, $w.rw_\pi(t_{root}{}^*)[P_{first}] = 2E$. Using **lemma 5.2.4** we can remove the white pebble weight using $w_\pi(t)[P_{first}] \le min_h + E$. At this time we have less than $(d-1)/2 + \epsilon - 2E$ pebble weight in the other trees. Thus $sw_\pi(t) \le min_{h+1} + \epsilon$. We then remove the white pebble weight on the root of any remaining subtree in the same way.

Thus to remove the white pebble weight we required $sw_\pi(t) \le min_{h+1} + \epsilon$ for $t > t_{root}{}^*$ and condition (3) is satisfied. Also, all times $t > t_{root}{}^*$, $sw_\pi(t) \le min_{h+1} + \epsilon$ and (5) is satisfied.

Thus the specified pebbling π satisfies all conditions and the IH is satisfied.

This result is obviously not possible without the use of fractional pebbles. Thus fractional pebbles allow for a large number of strategies that are not possible in other pebbling games. This gives us the intuition as to why we need a stronger induction hypothesis in the proof of the main lemma.

5.3 Fractional Pebbling Game Lower Bounds

We now prove the main theorem, which we state formally as :

Main Theorem
Let $min_h = (d-1)h/2 + 1$. For every **root-pebbling** π of T_d^h there is a time t such that $w_\pi(t) \ge min_h$.

The proof is simple for $h = 2$. The proof for $h \ge 3$ is by induction on h.

When Combined with the previous section we have a tight bound on the number of pebbles taken to complete the *fractional pebbling game* for balanced d-ary trees. The result is new. Similar, but loose, lower bounds can be found in [Cook et al., 2012]. In [Cook et al., 2012], they are the result of a reduction to a similar problem [Klawe, 1985], we take a more direct approach.

The theorem is shown using the following induction hypothesis.

Induction Hypothesis [IH(h)]: Let π be a **sub-root sub-pebbling** of T_d^h. Let $t_{root}{}^*$ be a time such that $rw_\pi(t_{root}{}^*)$=1 for all principal subtrees.

If $h \ge 3$, $\epsilon \in (-0.5, 0.5]$, $b.sw_\pi(0) \le 1 - \epsilon$, $b.rw_\pi(0) =$ arbitrary, and π is such that $sw_\pi(t) \le min_h - \epsilon$ for $t \le t_{root}{}^*$, then there is a time $t_b{}^* > t_{root}{}^*$ such that $sw_\pi(t_b{}^*) \ge min_h + \epsilon$ and $w.sw_\pi(t) \ge 0.5 + \epsilon$ for t in $[t_{root}{}^*, t_b{}^*]$.

initial conditions	additional conditions	consequences
$b.sw_\pi(0) \le 1 - \epsilon$	$sw_\pi(t) \le min_h - \epsilon$ for $t \le t_{root}{}^*$	$sw_\pi(t_b{}^*) \ge min_h + \epsilon$
$b.rw_\pi(0) =$ arbitrary		$w.sw_\pi(t) \ge 0.5 + \epsilon$ for t in $[t_{root}{}^*, t_b{}^*]$

The Induction Hypothesis can be interpreted as indicating that we require more after if we use less before.

Observation 5.3.1 *The Induction Hypothesis implies the theorem. This is the case since we must at some time, t_{root}, have pebble weight 1 on the root in a* **root-pebbling***. If at t_{root} the root has any black pebble weight we must have reached a time $t_{root}{}^*$ to place this black pebble weight. If it has only white pebble weight at t_{root}, we must reach a time $t_{root}{}^*$ to remove this white pebble weight. White pebble weight must be removed to satisfy the conditions of a* **root-pebbling***. It is therefore impossible to always use less than min_h since by the Induction hypothesis we would need to use more than min_h after $t_{root}{}^*$.*

Proof of the Base Case of the Induction Hypothesis (h = 3)

In this case $min_h = min_3 = 3/2(d-1) + 1 = 3/2d - 1/2$.

Let the nodes v_i be the children of the root.

Case I : The black pebble weight on the v_i is never increased at any time t such that $t \leq t_{root}{}^*$.

Then the total black pebble weight of the v_i at $t_{root}{}^*$ is at most $1 - \epsilon$, so the white pebble weight for these nodes at $t_{root}{}^*$ must be at least $d - (1 - \epsilon) = d - 1 + \epsilon$.

Let $t_b{}^*$ be the first time we remove white pebble weight after $t_{root}{}^*$. Since we must have pebble weight 1 on all of the children to remove white pebble weight we have that the total pebble weight required to remove white pebble weight is at least $d + (d - 1 + \epsilon) = 2d - 1 + \epsilon > 3/2d - 1/2 + \epsilon = min_h + \epsilon$ at time $t_b{}^*$.

$t_b{}^* > t_{root}{}^*$, since at $t_{root}{}^*$ the pebble weight on the v_i is d, thus at this time we could not have had the required pebble weight on the children due to the restriction on total pebble weight.

Also, during the interval $[t_{root}{}^*, t_b{}^*]$, $w.sw_\pi(t) \geq (d - 1) + \epsilon > 0.5 + \epsilon$, as required.

Thus the IH is satisfied in this case.

Case II : The black pebble weight on the nodes v_i is increased at some time t such that $t \leq t_{root}{}^*$.

Let $t_a{}^*$ be one step before the last time of such an increase. Let α be the total black pebble weight of the v_i at time $t_a{}^*$. Then the total subtree pebble weight at time $t_a{}^*$ is at least d+α, which by assumption is at most $min_h - \epsilon$. Therefore, d+ $\alpha \leq$ 3/2d -1/2 - ϵ, and hence

$$\alpha \leq 1/2d - 1/2 - \epsilon \tag{1}$$

After this increase at time $t_a{}^*$ the total black pebble weight of the v_i is at most $1 + \alpha$. Hence the white pebble weight of the v_i at $t_{root}{}^*$ satisfies $w.sw_\pi(t_{root}{}^*) \geq$ d-(1 + α) = d-1-α.

Let $t_b{}^*$ be the time just before the first time after $t_{root}{}^*$ that this white pebble weight is decreased. Since we need d pebble weight on the leaves at such a time,

$sw_\pi(t_b{}^*) \geq$ d+(d-1-α)

= 2d -1-α

$\geq$ 2d -1 - 1/2d + 1/2 + ϵ (by 1)

= 3/2d - 1/2 + ϵ

= $min_h + \epsilon$, as required.

Also, $t_b{}^* > t_{root}{}^*$, since at $t_{root}{}^*$ the pebble weight on the v_i is d, thus we could not have had the required pebble weight on the children due to the restriction on total pebble weight.

Finally, during the interval $[t_{root}{}^*, t_b{}^*]$, $w.sw_\pi(t) \geq d - 1 - \alpha \geq d - 1 - (1/2d - 1/2 - \epsilon) = 1/2d - 1/2 + \epsilon \geq 0.5 + \epsilon$, as required $(d \geq 2)$. Thus the IH is satisfied in this case.

Thus, in the base case the IH is satisfied.

The next two lemmas are to be used in the proof of the induction step. They are to be applied to the subtrees of the root.

Lemma 5.3.2 *Let π be a* **root sub-pebbling** *of T_d^h. Let t_{root} be any time such that $rw_\pi(t_{root})$ = 1.*

It follows from the IH for height h, that if $E \in [0.0, 0.5)$, $b.sw_\pi(0) \leq 0.5 + E$, $b.rw_\pi(0) \leq 2E$ and π is such that $sw_\pi(t) \leq min_h - 0.5 + E$ for $t \leq t_{root}$, then there is a time $t_b{}^{**}$, such that $t_{root} < t_b{}^{**}$, $w_\pi(t_b{}^{**}) \geq min_h + 0.5 - E$ and $w.w_\pi(t) \geq 1 - 2E$ for t in $[t_{root}, t_b{}^{**}]$.

initial conditions	additional conditions	consequences
$b.sw_\pi(0) \leq 0.5 + E$	$sw_\pi(t) \leq min_h - 0.5 + E$ for t $\leq t_{root}$	$w_\pi(t_b{}^{**}) \geq min_h + 0.5 - E$
$b.rw_\pi(0) \leq 2E$		$w.w_\pi(t) \geq 1 - 2E$ for t in $[t_{root}, t_b{}^{**}]$

Lemma 5.3.3 *Let π be a* **root sub-pebbling** *of T_d^h. Let t_{root} be any time such that $rw_\pi(t_{root})$ = 1.*

It follows from the IH for height h, that if $E \in [0, 1)$, $b.sw_\pi(0) \leq 0.5 + E$, at some time t_0, $0 \leq t_0 \leq t_{root}$, $b.rw_\pi(t_0) \leq E$ and π is such that $w_\pi(t) \leq min_h - 0.5 + E$ for $t \leq t_{root}$, then there is a time $t_b{}^{**}$, such that $t_{root} < t_b{}^{**}$, $w_\pi(t_b{}^{**}) \geq min_h + 0.5 - E$ and $w.w_\pi(t) \geq 1 - E$ for t in $[t_{root}, t_b{}^{**}]$.

initial conditions	additional conditions	consequences
$b.sw_\pi(0) \leq 0.5 + E$	$w_\pi(t) \leq min_h - 0.5 + E$ for t $\leq t_{root}$	$w_\pi(t_b{}^{**}) \geq min_h + 0.5 - E$
$b.rw_\pi(t_0) \leq E$, $t_0 \leq t_{root}$		$w.w_\pi(t) \geq 1 - E$ for t in $[t_{root}, t_b{}^{**}]$

We make the following observations :

Observation 5.3.4 *In* **Lemma 5.3.2** *additional initial black pebble weight on the root allows us to use less white pebble weight for t in $[t_{root}, t_b{}^{**}]$ than in* **Lemma 5.3.3**.

Observation 5.3.5 *In* **Lemma 5.3.3** *we introduce a time t_0. There may be more black pebble weight on the root before time t_0, however, it can not help us achieve the specified t_{root} if it is removed before t_{root}.*

Observation 5.3.6 *The IH implies conditions on the subtree pebble weight while the lemmas imply conditions on pebble weight anywhere.*

Observation 5.3.7 *The IH allows for arbitrary black root weight. Given the allowed pebbling moves, black root weight can not help us achieve $t_{root}{}^*$. This is not the case in the lemmas, it is possible that black root weight helps us attain t_{root}.*

Proof of Lemma 5.3.2

Lemma 5.3.2 will be used in the induction step since it is possible to leave some pebble weight on one subtree and proceed with the pebbling in the other subtrees.

We must reach a time $t_{root}{}^*$, either to add black pebble weight to reach t_{root} or to remove white pebble weight added to reach t_{root}. Since times $t_{root}{}^*$ exist, π is also a **sub-root sub-pebbling**. Thus we will apply the IH at these points denoted $t_{root}{}^*$.

Case 1 : $\exists t_{root}{}^*, t_{root}{}^* \leq t_{root}$.

By IH with $\epsilon = 0.5 - E$, since by assumption $sw_\pi(t) \leq min_h - 0.5 + E$ for $t \leq t_{root}$ and $b.sw_\pi(0) \leq 0.5 + E$, then at some time $t_b{}^{**} = t_b{}^*$, $sw_\pi(t_b{}^{**}) \geq min_h + 0.5 - E$ and $w.w_\pi(t) \geq 1 - E$ for t in $[t_{root}{}^*, t_b{}^{**}]$. Also, $1 - E \geq 1 - 2E$ since $E \geq 0$.

Since $min_h + 0.5 - E > min_h - 0.5 + E$ for all allowed E, we have not been allotted enough pebbles before t_{root} and $t_{root} < t_b{}^{**}$.

Thus the conditions of the lemma are satisfied.

Case 2 : $\forall t_{root}{}^*, t_{root} < t_{root}{}^*$. Then, to reach t_{root} we must use white pebble weight. Since $b.rw_\pi(0) \leq 2E$, $w.rw_\pi(t_{root}) \geq 1 - 2E$. We must then reach a $t_{root}{}^*$ to remove this white pebble weight. Let $t_{root}{}^{*First}$ be the first such $t_{root}{}^*$. Thus,

$$w.rw_\pi(t) \geq 1 - 2E \text{ for t in } [t_{root}, t_{root}{}^{*First}] \tag{2}$$

Case 2-A : $\exists t, t \in (t_{root}, t_{root}{}^{*First}]$ and $sw_\pi(t) \geq min_h - 0.5 + E$

Choose $t_b{}^{**}$ to be the first such t. Then $w_\pi(t_b{}^{**}) \geq min_h + 0.5 - E$ and $w.w_\pi(t) \geq 1 - 2E$ for times t in $[t_{root}, t_b{}^{**}]$ since we have yet to remove the white pebble weight on the root (2). Thus the lemma is satisfied in this case.

Case 2-B : $\forall t, if\ t \in (t_{root}, t_{root}{}^{*First}]$ then $sw_\pi(t) < min_h - 0.5 + E$

Then $sw_\pi(t) \leq min_h - 0.5 + E$ for t in $[0, t_{root}{}^{*First}]$. By IH with $\epsilon = 0.5 - E$, we have some time $t_b{}^* > t_{root}{}^{*First}$ such that $sw_\pi(t_b{}^*) \geq min_h + 0.5 - E$ and $w.w_\pi(t) \geq 1 - E$ for t in $[t_{root}{}^{*First}, t_b{}^*]$. We choose $t_b{}^{**} = t_b{}^*$.

$w.w_\pi(t) \geq 1 - 2E$ for times t in $[t_{root}, t_{root}{}^{*First}]$ (2). Thus, $w.w_\pi(t) \geq 1 - 2E$ for t in $[t_{root}, t_b{}^{**}]$. Thus, all conditions are met and the lemma is satisfied in this case.

Thus **Lemma 5.3.2** is satisfied in all cases.

$\blacksquare$

Proof of Lemma 5.3.3

Lemma 5.3.3 is to be used in the induction step when we increase the pebble weight on the root of the subtrees.

We must reach a time t_{root}^*, either to add black pebble weight to reach t_{root} or to remove white pebble weight added to reach t_{root}. Since these times exist, π is also a **sub-root sub-pebbling**. Thus we will apply the IH at these times denoted t_{root}^*.

Case 1 : $t_{root}^* \leq t_{root} < t_b^*$ for some t_{root}^* and corresponding t_b^*.
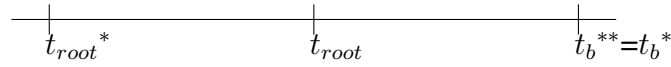
Figure 2: Timeline for Case 1. In this case we reach t_{root}^* before t_{root} and do not reach the corresponding t_b^* until after t_{root}

By IH, taking ϵ to be $0.5 - E$, taking $t_b^{**} = t_b^*$, since $sw_\pi(t) \leq min_h - 0.5 + E$ for $t \leq t_{root}^*$ and $b.sw_\pi(0) \leq 0.5 + E$, then $sw_\pi(t_b^{**}) \geq min_h + 0.5 - E$ and $w.w_\pi(t) \geq 1 - E$ for t in $[t_{root}^*, t_b^{**}]$. By assumption we also have $t_{root} < t_b^{**}$. Thus in this case the lemma is satisfied.

Case 2 : $\forall\, t_{root}^*$, $t_{root} < t_{root}^*$.

Figure 3: Setup for Case 2.

Then we use white pebble weight to reach t_{root},

$$w.rw_\pi(t) = 1 - E \text{ for t in } [t_{root}, t_{root}^{*First}] \tag{3}$$

Let t_{root}^{*First} be the first t_{root}^*.

Case 2-A : $\exists\, t,\ t \in (t_{root}, t_{root}^{*First}]$ and $sw_\pi(t) \geq min_h - 0.5$

We let t_b^{**} be such a time t. Then we meet the criteria in the lemma since we have $w_\pi(t_b^{**}) \geq min_h + 0.5 - E$ and $w.w_\pi(t) \geq 1 - E$ for t in $[t_{root}, t_b^{**}]$ (3). Thus the lemma is satisfied in this case.

Case 2-B : $\forall\, t$, if $t \in (t_{root}, t_{root}^{*First}]$ then $sw_\pi(t) < min_h - 0.5$

$min_h - 0.5 \leq min_h - 0.5 + E$ for all allowed E. We have used $sw_\pi(t) \leq min_h - 0.5 + E$ for t in $[0, t_{root}^{*First}]$. By the IH, taking ϵ to be $0.5 - E$, letting $t_b^{**} = t_b^*$, we must use $sw_\pi(t_b^{**}) \geq min_h + 0.5 - E$ at $t_b^{**} > t_{root}^{*First}$.

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Also by the IH $w.w_\pi$(t) $\geq 1 - E$ for t in $[t_{root}{}^{*First}, t_b{}^{**}]$. $w.rw_\pi$(t) $\geq$ 1-E for t in $[t_{root}, t_{root}{}^{*First}]$ (3), thus $w.w_\pi$(t) $\geq 1 - E$ for t in $[t_{root}, t_b{}^{**}]$. Thus the lemma is satisfied in this case.
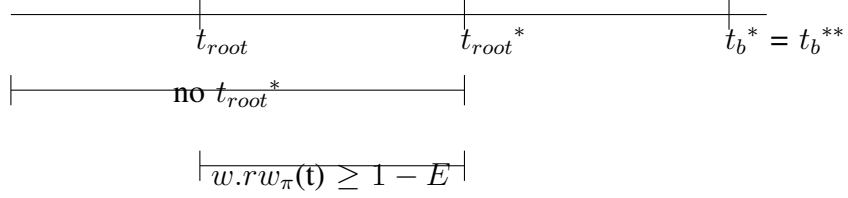

Figure 4: Timeline for Case 2. As mentioned, 1-E pebble weight is on the root between t_{root} and $t_{root}{}^*$.

Case 3 : $t_{root}{}^* < t_b{}^* \leq t_{root}$ for the last $t_{root}{}^*$ and corresponding $t_b{}^*$ before t_{root}.
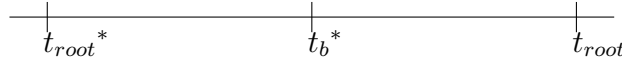
Figure 5: Setup for Case 3.

Case 3-A : $E < 0.5$. By IH, taking ϵ to be $0.5 - E$, since sw_π(t) $\leq min_h - 0.5 + E$ for $t \leq t_{root}{}^*$ and $b.sw_\pi(0) \leq 0.5 + E$, then $sw_\pi(t_b{}^*) \geq min_h + 0.5 - E$. However, $min_h + 0.5 - E > min_h - 0.5 + E$. Thus we have not been allotted enough pebble weight before t_{root} and we must proceed past t_{root} before we may reach $t_b{}^*$. Thus when $0.5 > E$, **Case 3** is not possible.

Case 3-B : $E \geq 0.5$.

By IH, taking ϵ to be $0.5 - E$, we must have a $t_b{}^*$ such that $sw_\pi(t_b{}^*) \geq min_h + 0.5 - E$.

At this time, $b.rw_\pi(t_b{}^*) \leq 2E - 1 < 1$ due to the restriction on total pebble weight before t_{root}. Since the chosen $t_{root}{}^*$ was the last before t_{root} we must use white pebble weight to reach t_{root}, $w.rw_\pi(t_{root}) \geq 2 - 2E$.

Since this is not 0 we will need to reach another $t_{root}{}^*$ after t_{root} to remove this white pebble weight. Since 2-2E $\geq$ 1-E, this case follows by the same argument in **Case 2-A** and **Case 2-B** .

Thus in all cases Lemma 5.3.3 follows from IH.

■

Induction step : We prove the induction hypothesis for h+1 assuming it for h', $3 \leq h' \leq h$.

Fix $\pi = 0, ..., t_{root}{}^*, ...$ to be a **sub-root sub-pebbling** of T_d^{h+1} with $t_{root}{}^*$ such that $rw_\pi(t_{root}{}^*)$=1 for all principal subtrees, and with

$$sw_\pi(t) \leq min_{h+1} - \epsilon = (d-1)(h+1)/2 + 1 - \epsilon = min_h + (d-1)/2 - \epsilon \; for \; t \; in \; [0, t_{root}{}^*] \; (4)$$

Further, we assume,

$$\epsilon \in (-0.5, 0.5] \tag{5}$$

$$b.sw_\pi(0) \leq 1 - \epsilon \tag{6}$$

Let P_i be the principal subtrees of T_d^{h+1}. The restriction of π to each of these subtrees is a valid pebbling of that subtree.

Case 1 : $\forall t, \forall i$, if $t \leq t_{root}{}^*$ then $sw_\pi(t)[P_i] < min_h - 0.5$

For each principal subtree we will apply **Lemma 5.3.2**. We will show that if we consider all subtrees this implies the desired bounds.

In this case, the subtree pebble weight of all subtrees P_i is less than $min_h - 0.5$.

We have at most $1 - \epsilon$ initial black pebble weight in the P_i by assumption (6). We will separate this pebble weight between the subtrees and apply **Lemma 5.3.2** to each subtree. Let us have $b.w_\pi(0)[P_i] = 2E_i$. We choose to express the amount this way since it resemble amounts expressed in **Lemma 5.3.2**.

It is the case that $E_i \geq 0$ since pebble weight is non-negative.

If $0 \leq E_i < 0.5$ we may apply **Lemma 5.3.2** to the ith subtree. Let G be the set of all i such that $0 \leq E_i < 0.5$. We have $\Sigma_{i \in G} 1 - 2E_i \geq \Sigma_{i=1}^d 1 - 2E_i$ since $0 \geq 1 - 2E_i$ for i $\notin$ G.

The way in which we will use G will affirm that maintaining more than 1 black pebble weight in any tree is useless.

Note, G is not the empty set since $b.sw_\pi(0) \leq 1 - \epsilon$ and d ≥ 2.

Note,
$\Sigma_{i=1}^d 2E_i \leq 1 - \epsilon$, by construction,
$-\Sigma_{i=1}^d 2E_i \geq -1 + \epsilon$, then,
$\Sigma_{i=1}^d 1 - 2E_i \geq d - 1 + \epsilon$, then,

$$\Sigma_{i \in G}(1 - 2E_i) \geq d - 1 + \epsilon \tag{7}$$

For each subtree, we take t_{root} in the lemma to be the time $t_{root}{}^*$. This is possible since $rw_\pi(t_{root}{}^*)[P_i]=1$ as required by **Lemma 5.3.2**.

We apply **Lemma 5.3.2** to P_i, i $\in$ G, taking E in the lemma to be E_i and with $t_b[P_i] := t_b{}^{**}$. Then, $t_b[P_i] > t_{root}{}^*$, $w_\pi(t_b[P_i])[P_i] \geq min_h + 0.5 - E_i$ and $w.w_\pi(t)[P_i] \geq 1 - 2E_i$ for t in $[t_{root}{}^*, t_b[P_i]]$.

We let $t_b{}^*=min(t_b[P_i])$ for $i \in$ G.

We define $first$ to be this i. It is the first $t_b[P_i]$ we reach in π. Then we require $min_h + 0.5 - E_{first}$ in P_{first} while maintaining at least 1-$2E_i$ in the remaining P_i, $i \in$ G and $i \neq first$. Then,
$sw_\pi(t_b{}^*) \geq min_h + 0.5 - E_{first} + \Sigma_{i \in G, i \neq first}(1 - 2E_i)$
$\geq min_h + 0.5 - 2E_{first} + \Sigma_{i \in G, i \neq first}(1 - 2E_i)$ (since, $0 \geq -E_{first}$)
$= min_h - 0.5 + \Sigma_{i \in G}(1 - 2E_i)$

$\geq min_h - 0.5 + d - 1 + \epsilon$ (by 7)
$\geq min_h - (d - 1)/2 + (d - 1) + \epsilon$ (since $d \geq 2$)
$= min_h + (d - 1)/2 + \epsilon$
$= min_{h+1} + \epsilon$

Additionally,
$w.sw_\pi(t) \geq \Sigma_{i \in G}(1 - 2E_i) \geq d - 1 + \epsilon \geq 1 + \epsilon$ for t in $[t_{root}^*, t_b^*]$ (by 7).

Thus the IH is satisfied in **Case 1**.

Case 2 : $\exists t, \exists i, t \leq t_{root}^*$ and $sw_\pi(t)[P_i] \geq min_h - 0.5$.

For each principal subtree we will try to apply one of the lemmas. We will then show that taken together this results in the desired bounds. Also recall that we fixed $\pi = 0, ..., t_{root}^*, ...$.

Suppose $sw_\pi(t) \geq min_h - 0.5$ for the last time before t_{root}^* in the subtree P_{last}. Let this time be t_{last}. Then $t_{last} \leq t_{root}^*$ and

$$sw_\pi(t_{last})[P_{last}] \geq min_h - 0.5 \tag{8}$$

For any value r_i, for all $i \neq last$, define t_{r_i} to be the last time in $[0, t_{root}^*]$ such that $sw_\pi(t_{r_i})[P_i] \geq min_h - 0.5 + r_i$ or the initial time if no such time exists.

Define R_i to be the max r_i such that $w_\pi(t)[P_i] \geq 2r_i$ for times t in $[t_{r_i}, t_{root}^*]$.

There is always a time t_{root}^* since π is a **sub-root sub-pebbling**. The described condition is true for some value of r_i as it is true for $r_i = 0$ and this is the smallest value possible. There is therefore always a time t_{R_i} for each principal subtree. Thus,

$$R_i \geq 0 \tag{9}$$

By definition of t_{R_i} and t_{last},
$$t_{R_i} < t_{last} \tag{10}$$

This is a result of the restriction on total pebble weight (4) and having at least $min_h - 0.5$ pebble weight in P_{last} at t_{last}. We show that we must have less pebble weight than $min_h - 0.5$ in the other subtrees at t_{last}. Suppose we did not, we then have at least $2min_h - 1$ total pebble weight.

$sw_\pi(t) \geq 2min_h - 1$
$= min_h + (d - 1)h/2 + 1 - 1$
$= min_h + (d - 1)h/2$
$> min_h + (d - 1)$ (Since $h > 2$)
$\geq min_h + (d - 1)/2 - \epsilon$
$= min_{h+1} - \epsilon$

This would contradict the assumption for total subtree pebble weight (4). Thus t_{last} is the last time in π we use the amount described at t_{R_i} and (10) holds.

In summary, the choice of R_i implies the following,

$$sw_\pi(t_{R_i})[P_i] \geq min_h - 0.5 + R_i \ or \ t_{R_i} = 0 \tag{11}$$

$$w_\pi(t)[P_i] \geq 2R_i \ for \ t \ in \ [t_{R_i}, t_{root}{}^*] \tag{12}$$

Definition 5.3.8 *For each $i \neq last$, define $t_{Pi-init}$ to be a time such that $w_\pi(t_{Pi-init})[P_i] \leq 2R_i$ and $sw_\pi(t)[P_i] \leq min_h - 0.5 + R_i$ fot t in [$t_{Pi-init}$, $t_{root}{}^*$].*

This will be useful since we wish to apply **Lemma 5.3.2** to P_i later with E = R_i and initial time $t_{Pi-init}$. We show such a time always exists.
Case I : $w_\pi(t)[P_i] = 2R_i$ for some t in $[t_{R_i}, t_{root}{}^*]$. We let this time be $t_{Pi-init}$.

t_{R_i} $t_{Pi-init}$

$sw_\pi(t_{R_i})[P_i] \geq min_h - 0.5 + R_i$ $w_\pi(t)[P_i] = 2R_i$
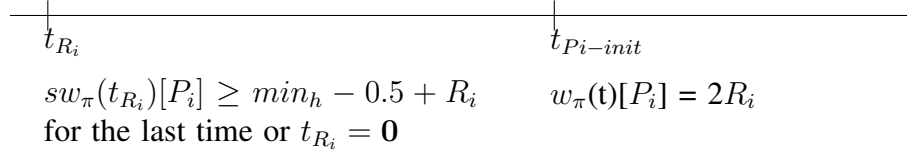for the last time or $t_{R_i} = 0$

Figure 6: Depicts the situation in P_i for **Case I**.

Case II : $w_\pi(t)[P_i] > 2R_i$ for all times t in $[t_{R_i}, t_{root}{}^*]$.
Then $sw_\pi(t_{R_i})[P_i] = min_h - 0.5 + R_i$. If this was not the case, the conditions would be true for a greater value of R_i and we would have a contradiction. For similar reasons, t_{R_i} is not the initial time else the condition would be true for a larger value of R_i.

Let $t_{before-R_i}$ be the last time such that $sw_\pi(t_{before-R_i})[P_i] > min_h - 0.5 + R_i$ or the initial time if no such time exists. Then $t_{before-R_i} < t_{R_i}$. There must have been a time, $t_{Pi-init}$, in $[t_{before-R_i}, t_{R_i}]$ such that $w_\pi(t_{Pi-init})[P_i] \leq 2R_i$. If this were not the case, the conditions would be true for a greater value of R_i since we would have $w_\pi(t)[P_i] > 2R_i$ for t in $[t_{before-R_i}, t_{root}{}^*]$ using the assumption in **Case II**. Thus, the chosen $t_{Pi-init}$ satisfies the necessary conditions.
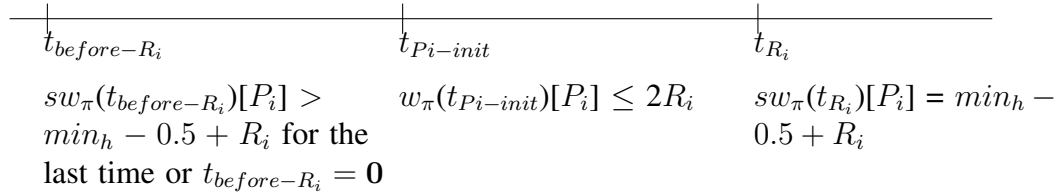

$t_{before-R_i}$ $t_{Pi-init}$ t_{R_i}

$sw_\pi(t_{before-R_i})[P_i] >$ $w_\pi(t_{Pi-init})[P_i] \leq 2R_i$ $sw_\pi(t_{R_i})[P_i] = min_h -$
$min_h - 0.5 + R_i$ for the $0.5 + R_i$
last time or $t_{before-R_i} = 0$

Figure 7: Depicts the situation in P_i for **Case II**.

Thus in all cases, such a $t_{Pi-init}$ exists.

Let G be the set of all i such that $0 \leq R_i < 0.5$, $i \neq last$. Since $2R_i \geq 1$ for $i \notin G$,

$$\Sigma_{i=1, i \neq last}^d 2R_i \geq (d - 1 - |G|) + \Sigma_{i \in G} 2R_i \tag{13}$$

We will apply **Lemma 5.3.2** to P_i for $i \in G$, taking the initial time in the lemma to be $t_{Pi-init}$ and taking E in the lemma to be R_i.

We use $sw_\pi(t_{last})[P_{last}] \geq min_h - 0.5$ (8) while maintaining $\Sigma_{i=1,i\neq last}^d 2R_i$ in the other P_i at time t_{last} (11). Thus, $min_h - 0.5 + \Sigma_{i=1,i\neq last}^d 2R_i \leq min_h + (d-1)/2 - \epsilon$ due to the restriction on total pebble weight (4). Then $(d-1)/2 - \epsilon + 0.5 - \Sigma_{i=1,i\neq last}^d 2R_i$ is the maximum amount of pebble weight at t_{last} on the root of P_{last}. It is the difference between the maximum pebble weight and the pebble weight elsewhere.

It is also the case that,

$(d-1)/2 - \epsilon + 0.5 - \Sigma_{i=1,i\neq last}^d 2R_i \leq 0.5 + (d-1)/2 - \epsilon - (d-1-|G|) - \Sigma_{i\in G}2R_i$ (by 13)
$= 0.5 + (d-1)/2 - \epsilon - (d-1) + |G| - \Sigma_{i\in G}2R_i$
$= 0.5 - (d-1)/2 - \epsilon + |G| - \Sigma_{i\in G}2R_i$

We denote this quantity $Rmax$.

$$Rmax = 0.5 - (d-1)/2 - \epsilon + |G| - \Sigma_{i\in G}2R_i \tag{14}$$

Thus $Rmax$ is an upper bound on the maximum amount of pebble weight at t_{last} on the root of P_{last}. It is a measure dependent on the pebble weight maintained in the other subtrees.

Case 2A : $Rmax \geq 1$.

Note by assumption for **Case 2A** and (14),

$$- \Sigma_{i\in G}2R_i \geq 0.5 + (d-1)/2 + \epsilon - |G| \tag{15}$$

This will be used later in this **Case**.

In this case we have not left enough pebble weight in the P_i, $i \neq last$.

Also in this case G is not the empty set. For contradiction, suppose it was. Then, $Rmax = 0.5 - (d-1)/2 - \epsilon + |G| - \Sigma_{i\in G}2R_i = 0.5 - (d-1)/2 - \epsilon \geq 1$. However, this is not possible since d ≥ 2 and $\epsilon \in (-0.5, 0.5]$ (5).

If $0 \leq R_i < 0.5$ we may apply **Lemma 5.3.2** to the ith subtree at $t_{Pi-init}$.

Thus, we apply **Lemma 5.3.2** to P_i, $i \in$ G, taking the initial time in the lemma to be $t_{Pi-init}$, taking E in the lemma to be R_i and with $t_b[P_i] := t_b^{**}$ from the lemma. Then, $t_b[P_i] > t_{root}^*$, $w_\pi(t_b[P_i])[P_i] \geq min_h + 0.5 - R_i$ and $w.w_\pi(t)[P_i] \geq 1 - 2R_i$ for t in $[t_{root}^*, t_b[P_i]]$.

We choose t_b^*=min$(t_b[P_i])$, $i \in$ G. This is the first $t_b[P_i]$ which is reached in π. Let this $i = first$. Then we add $\Sigma_{i\in G,i\neq first}(1 - 2R_i)$ since we had yet to remove the pebble weight from the other P_i, $i \in G$,

$sw_\pi(t_b^*) \geq min_h + 0.5 - D_{first} + \Sigma_{i\in G,i\neq first}(1 - 2R_i)$
$\geq min_h + 0.5 - 2D_{first} + \Sigma_{i\in G,i\neq first}(1 - 2R_i)$
$\geq min_h - 0.5 + \Sigma_{i\in G}(1 - 2R_i)$
$= min_h - 0.5 + |G| - \Sigma_{i\in G}2R_i$
$\geq min_h - 0.5 + |G| + 0.5 + (d-1)/2 + \epsilon - |G|$ by (15).
$= min_h + (d-1)/2 + \epsilon$
$= min_{h+1} + \epsilon$

Thus we exceed or match the minimum pebble weight allotted by the IH.

Also, we have $w.sw_\pi(t) \geq \Sigma_{i \in G}(1 - 2R_i)$ for t in $[t_{root}^*, t_b^*]$ since we have yet to remove the weight from any of the P_i.

$w.sw_\pi(t) \geq \Sigma_{i \in G}(1 - 2R_i)$

$= |G| - \Sigma_{i \in G} 2R_i$

$\geq |G| + 0.5 + (d - 1)/2 + \epsilon - |G|$ by (15).

$= 0.5 + (d - 1)/2 + \epsilon$

$> 0.5 + \epsilon$ as required.

Thus in this case the IH is satisfied.

Case 2B : $Rmax < 1$ (14)

Let $t_{D-before-last}$ be the last of the t_{R_i} (see 11 and 12). If all t_{R_i} are the initial time, choose any one arbitrarily as $t_{D-before-last}$. Let $P_{before-last}$ be the subtree associated with $t_{D-before-last}$ in the definition.

We wish to eventually apply **Lemma 5.3.3** to P_{last} for E = $Rmax$. To do this we take $t_{D-before-last}$ to be the initial time and t_{last} to be the time t_0 in the lemma. To apply **Lemma 5.3.3**, we must show upper bounds on $b.sw_\pi(t_{D-before-last})[P_{last}]$, $w_\pi(t)[P_{last}]$ for t in $[t_{D-before-last}, t_{root}^*]$, $b.rw_\pi(t_{last})[P_{last}]$ and we must show $Rmax \in [0,1)$.

We first show $b.sw_\pi(t_{D-before-last})[P_{last}] \leq 0.5 + Rmax$. This is divided into cases.

Case I : $t_{D-before-last}$ was the initial time

If $t_{D-before-last}$ was the initial time, due to the restriction on initial black pebble weight (6) and due to the pebble weight in the other subtrees (11),

$b.sw_\pi(t_{D-before-last})[P_{last}] \leq 1 - \epsilon - \Sigma_{i=1, i \neq last}^d 2R_i$

$\leq 1 - \epsilon - (d - 1 - |G|) - \Sigma_{i \in G} 2R_i$ (by 13)

$\leq 0.5 + (d - 1)/2 - \epsilon - (d - 1 - |G|) - \Sigma_{i \in G} 2R_i$ ($d \geq 2$)

$= 0.5 - (d - 1)/2 - \epsilon + |G| - \Sigma_{i \in G} 2R_i$

$= Rmax$ (by 14)

$\leq 0.5 + Rmax$ as required.

Case II : $t_{D-before-last}$ was not the initial time

If $t_{D-before-last}$ was not the initial time, due to the restrictions on total pebble weight (4), the amount in $P_{before-last}$ (11) and the pebble weight in the other subtrees,

$b.sw_\pi(t_{D-before-last})[P_{last}] \leq (d - 1)/2 - \epsilon - D_{before-last} + 0.5 - \Sigma_{i=1, i \neq last, i \neq before-last}^d 2R_i$

Case IIA : $before - last$ is in G, therefore $D_{before-last} < 0.5$.

There are $(d - 1 - |G|)$ other subtrees not in G since $before - last$ is in G. Thus if we continue from the above,

$b.sw_\pi(t_{D-before-last})[P_{last}] \leq (d-1)/2 - \epsilon - D_{before-last} + 0.5 - (d-1-|G|) - \Sigma_{i \in G, i \neq before-last} 2R_i$ (similar to 13)

$\leq (d - 1)/2 - \epsilon - 2D_{before-last} + 1 - (d - 1 - |G|) - \Sigma_{i \in G, i \neq before-last} 2R_i$

$= (d - 1)/2 - \epsilon + 1 - (d - 1 - |G|) - \Sigma_{i \in G} 2R_i$

$= 1 - (d-1)/2 - \epsilon + |G| - \Sigma_{i \in G} 2R_i$
$= 0.5 + Rmax$ (by 14) as required.

Case IIB : $before - last$ is not in G, therefore $D_{before-last} \geq 0.5$.

There are $(d - 2 - |G|)$ subtrees not in G other than $before - last$, since $before - last$ is not in G. Thus if we continue from what was described at the beginning of **Case II**,
$b.sw_\pi(t_{D-before-last})[P_{last}] \leq (d-1)/2 - \epsilon - D_{before-last} + 0.5 - \Sigma_{i \in G} 2R_i - (d - 2 - |G|)$ (similar to 13)
$\leq (d-1)/2 - \epsilon - 0.5 + 0.5 - \Sigma_{i \in G} 2R_i - (d - 2 - |G|)$
$= (d-1)/2 - \epsilon - \Sigma_{i \in G} 2R_i - (d - 2 - |G|)$
$= (d-1)/2 - \epsilon - \Sigma_{i \in G} 2R_i - (d - 2 - |G| + 1 - 1)$
$= (d-1)/2 - \epsilon - \Sigma_{i \in G} 2R_i - (d - 1 - |G|) + 1$
$= 1 - (d-1)/2 - \epsilon + |G| - \Sigma_{i \in G} 2R_i$
$= 0.5 + Rmax$ (by 14) as required.
Thus in all cases the condition is met for the $b.sw_\pi$.

We next show $w_\pi(t)[P_{last}] \leq min_h - 0.5 + Rmax$ for t in $[t_{D-before-last}, t_{root}{}^*]$. We use at most $w_\pi(t)[P_{last}] \leq min_h + (d-1)/2 - \epsilon - \Sigma_{i=1, i \neq last}^d 2R_i$ for t in $[t_{D-before-last}, t_{root}{}^*]$ due to the pebble weight elsewhere (12) and the restriction on total pebble weight before $t_{root}{}^*$ (4).
$w_\pi(t)[P_{last}] \leq min_h + (d-1)/2 - \epsilon - \Sigma_{i=1, i \neq last}^d 2R_i$
$\leq min_h + (d-1)/2 - \epsilon - \Sigma_{i \in G}^d 2R_i - (d - 1 - |G|)$ (by 13)
$= min_h - (d-1)/2 - \epsilon + |G| - \Sigma_{i \in G} 2R_i$
$= min_h - 0.5 + Rmax$ (by 14) as required.

We know by construction $b.rw_\pi(t_{last})[P_{last}] \leq Rmax$.

Finally we show $Rmax \in [0,1)$. We use $sw_\pi(t_{last})[P_{last}] \geq min_h - 0.5$ (8) while maintaining $\Sigma_{i=1, i \neq last}^d 2R_i$ in the other subtrees at time t_{last} (12). Thus $min_h - 0.5 + \Sigma_{i=1, i \neq last}^d 2R_i \leq min_h + (d-1)/2 - \epsilon$ due to the restriction on total pebble weight (4). Then,
$0 \leq (d-1)/2 - \epsilon + 0.5 - \Sigma_{i=1, i \neq last}^d 2R_i$
$\leq 0.5 + (d-1)/2 - \epsilon - (d - 1 - |G|) - \Sigma_{i \in G} 2R_i$ (by 13)
$= Rmax$
Using this and the assumption, $Rmax \in [0,1)$, as required.

Thus we have shown all the necessary conditions to apply **Lemma 5.3.3** to P_{last}.

If $0 \leq R_i < 0.5$ we may apply **Lemma 5.3.2** to the ith subtree at $t_{Pi-init}$.

Since $t_{root}{}^*$ occurs when $rw_\pi(t_{root}{}^*)[P_{last}]=1$ and $rw_\pi(t_{root}{}^*)[P_i]=1$, we apply **Lemma 5.3.3** and **Lemma 5.3.2**, respectively, taking $t_{root}{}^*$ as the time t_{root} in the lemmas.

We apply **Lemma 5.3.3** to P_{last} with $t_b[P_{last}] := t_b{}^{**}$ from the lemma. Then, $t_b[P_{last}] > t_{root}{}^*$,
$w_\pi(t_b[P_{last}])[P_{last}] \geq min_h + 0.5 - Rmax$
$= min_h + 0.5 - 0.5 + (d-1)/2 + \epsilon - |G| + \Sigma_{i \in G} 2R_i$ (by 14)
$= min_h + (d-1)/2 + \epsilon - |G| + \Sigma_{i \in G} 2R_i$

and
$w.w_\pi(t)[P_{last}] \geq 1 - Rmax$
$= 1 - 0.5 + (d-1)/2 + \epsilon - |G| + \Sigma_{i \in G} 2R_i$ (by 14)
$= 0.5 + (d-1)/2 + \epsilon - |G| + \Sigma_{i \in G} 2R_i$ for t in $[t_{root}^*, t_b[P_{last}]]$.

We apply **Lemma 5.3.2** to P_i, $i \in G$, taking the initial time in the lemma to be $t_{Pi-init}$, taking E in the lemma to be R_i and with $t_b[P_i] := t_b^{**}$ from the lemma. We may do this since $b.sw_\pi(0) \leq 2R_i \leq 0.5 + R_i$ and $b.rw_\pi(0) \leq 2R_i$. Then, $t_b[P_i] > t_{root}^*$, $w_\pi(t_b[P_i])[P_i] \geq min_h + 0.5 - R_i$ and $w.w_\pi(t)[P_i] \geq 1 - 2R_i$ for t in $[t_{root}^*, t_b[P_i]]$.

We choose $t_b^*=min(t_b[P_{last}], t_b[P_i])$ for i $\in$ G.

Case 2B-1 : $t_b^* = t_b[P_{last}]$. Then,
$sw_\pi(t_b^*) \geq min_h + (d-1)/2 + \epsilon - |G| + \Sigma_{i \in G} 2R_i + \Sigma_{i \in G}(1 - 2R_i)$
$= min_h + (d-1)/2 + \epsilon - |G| + \Sigma_{i \in G} 2R_i + |G| - \Sigma_{i \in G} 2R_i$
$= min_h + (d-1)/2 + \epsilon$
$= min_{h+1} + \epsilon$
Where we add the pebble weight in the P_is since we had yet to reach the $t_b[P_i]$. Thus we exceed or match the minimum pebble weight allotted by the IH.

Also, we have white pebble weight as follows between $[t_{root}^*, t_b^*]$,
$w.sw_\pi(t) \geq 0.5 + (d-1)/2 + \epsilon - |G| + \Sigma_{i \in G} 2R_i + \Sigma_{i \in G}(1 - 2R_i)$
$= 0.5 + (d-1)/2 + \epsilon - |G| + \Sigma_{i \in G} 2R_i + |G| - \Sigma_{i \in G} 2R_i$
$= 0.5 + (d-1)/2 + \epsilon$
$\geq 0.5 + \epsilon$ as required.

Thus the IH is satisfied in this case.

Case 2B-2 : $t_b^* = t_b[P_i]$, $i \neq last$.
We let this $i = first$. Then,
$sw_\pi(t_b^*) \geq min_h + 0.5 - D_{first} + 0.5 + (d-1)/2 + \epsilon - |G| + \Sigma_{i \in G} 2R_i + \Sigma_{i \in G, i \neq first}(1 - 2R_i)$
$\geq min_h + 1 - 2D_{first} + (d-1)/2 + \epsilon - |G| + \Sigma_{i \in G} 2R_i + \Sigma_{i \in G, i \neq first}(1 - 2R_i)$
$= min_h + (d-1)/2 + \epsilon - |G| + \Sigma_{i \in G} 2R_i + \Sigma_{i \in G}(1 - 2R_i)$
$= min_h + (d-1)/2 + \epsilon - |G| + \Sigma_{i \in G} 2R_i + |G| - \Sigma_{i \in G} 2R_i$
$= min_h + (d-1)/2 + \epsilon$
$= min_{h+1} + \epsilon$
This matches the lower bounds specified in the IH.

As in **Case 2B-1**, we have the same amount of white pebble weight until this time. Thus the IH is satisfied in this case.

Thus the IH holds in all cases. Consequently the main theorem holds as well.

6 Conclusion

We have presented a proof of an open problem given in [Cook et al., 2012]. Fractional pebbles allow for many pebbling strategies. To accommodate for this, we used a *shifting* argument to build a direct proof. Many open problems remain related to the fractional pebbling game.

Branching programs were briefly introduced in the introduction (Section 1). They are nonuniform models of Turing machines. Showing that non-deterministic branching programs require a superpolynomial number of states for a problem in P would separate NL from P.

[Cook et al., 2012] proposed the *tree evaluation problem* as a mean of separating NL from P. The *tree evaluation problem* is similar to the pebbling game except values are attached to each leaf node and functions are attached to each non-leaf node. The value of a node is determined by the value of its function evaluated at the value of its children. The goal is then to determine the value of the root node.

One step towards separating NL from P is to show a superpolynomial lower bound on the number of states for a restricted class of branching programs. A thrifty branching program for the *tree evaluation problem* must query the value of the functions only at the correct value of the children. The thrifty hypothesis states that thrifty branching programs are optimal among all branching programs.

[Cook et al., 2012], under the thrifty hypothesis, showed that deterministic branching programs solving the *tree evaluation problem* required a superpolynomial number of states that would separate L from P. This followed from a proof similar to the one in Section 3.2. Thus we propose the following as an open problem :

Open Problem 1 *Adapt the proof of the Main Theorem to get lower bounds for non-deterministic thrifty branching programs solving the* tree evaluation problem.

Showing this would separate NL from P under the thrifty hypothesis. To show their original result, [Cook et al., 2012] used a non-inductive proof. It seems difficult to instead use an inductive proof, thus the following would be interesting :

Open Problem 2 *Provide an alternative proof, using induction, that under the thrifty hypothesis, deterministic thrifty branching programs solving the* tree evaluation problem *require a super-polynomial number of states which would separate L from P.*

If this could be done without the thrifty hypothesis it would be an even more important result. Similarly, showing that the thrifty hypothesis held or did not is an important open problem.

Klawe showed the lower bound for the whole *black-white pebbling game* for the pyramid graphs [Klawe, 1985]. The advantage of the pyramid graphs is that the number of nodes is polynomial in the height of the tree. Thus for various application of the pebbling game, it is possible that lower bounds for the pyramid graphs could result in better bounds. We thus suggest the following open problem :

Open Problem 3 *Show upper bounds and lower bounds for the* $fractional$ $pebbling$ $game$ *on pyramid graphs.*

Acknowledgements

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