Tight Lower Bounds for the Fractional Pebble Game

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Abstract. Fractional pebbling is a generalization of black-white pebbling introduced recently. Here we solve an open problem by proving a tight lower bound on the pebble weight required to fractionally pebble a balanced d-ary tree of height h. This bound has a close connection with the nondeterministic space complexity of the tree evaluation problem.

1 Introduction

1.1 Motivation

The black pebble game was introduced by Paterson and Hewitt [PH70] to compare the power of programming languages. Motivated by the problem of separating the complexity classes **P** (polynomial time) from **NL** (nondeterministic logarithmic space), it was generalized to the black-white pebble game by Cook and Sethi [CS76] to help in the proof of a super-logarithmic nondeterministic space lower bound for solving a specific problem in **P** (Path Systems) on a restricted computation model. Recently in [CMW$^+$12] it was further generalized to the fractional pebble game in order study the space complexity of the Tree Evaluation Problem (TEP), another candidate for separating **P** and **NL**.

The input to the TEP is a rooted, balanced d-ary tree of height h (denoted T_d^h), whose internal nodes are labeled with d-ary functions on $[k] = \{1, \ldots, k\}$, and whose leaves are labeled with elements of $[k]$. Each node obtains a value in $[k]$ equal to its d-ary function applied to the values of its d children. The output is the value of the root. In [CMW$^+$12] the space complexity of TEP is studied by measuring the number of states required by k-way branching programs to solve TEP. There it is shown that the optimal way of black pebbling the binary tree of height h (which requires h pebbles) yields a deterministic k-way branching program (k-BP) with $\Theta(k^h)$ states, and no k-BP with fewer states solving this problem is known. A proof of a lower bound on the number of states required, even if the lower bound was a much smaller function $k^{r(h)}$, where $r(h)$ is any unbounded function, would separate **L** from **P**.

In [CMW$^+$12] a simple semantic restriction called *thrifty* on k-BPs solving TEP was introduced and it was shown that the k-BPs mentioned above (coming from black pebbling) for solving TEP are thrifty, and in fact their state bound of $\Theta(k^h)$ is the smallest possible among all thrifty k-BPs solving TEP. (A k-BP solving TEP is *thrifty* if for every computation on a labeled input tree, for every query to the function $f_v(x_1, \ldots, x_k)$ labelling a node v, the values of the arguments $x_1, \ldots, x_k$ for the query must be the actual values of the children of v.)

In [CMW$^+$12] nondeterministic k-BPs solving the Boolean TEP problem (BTEP) are also studied, where BTEP has the same input as TEP, but the problem is to determine whether the root has value 1. There the following result is proved by implementing the algorithm suggested by fractionally pebbling the tree T_d^h.

Proposition 1. *There is a nondeterministic thrifty k-BP with $\Theta(k^{p_{frac}(h,d)})$ states which solves BTEP, where $p_{frac}(h,d)$ is the pebble weight required to fractionally pebble T_d^h.*

In fact no nondeterministic k-BP (thrifty or not) with fewer than $\Theta(k^{p_{frac}(h,d)})$ states solving BTEP is known. Similar to the deterministic case, a proof of even a much weaker lower bound on the number of states required would separate **NL** from **P**.

In the present paper we establish exact values for $p_{frac}(h,d)$ for all $h, d \geq 2$. Although approximate values were known previously (see Proposition 3), the complicated proof depends on Klawe's [Kla85] elaborate lower bound proof for black-white pebbling of DAGs. We present a direct proof by induction on the height of the tree.

A major motivation for giving our direct proof for tight lower bounds on fractional pebbling numbers is the hope of using ideas in the proof to establish that the bound given in Proposition 1 for nondeterministic thrifty BPs solving BTEP is tight. Such a tight lower bound has been proved for the deterministic case (see Theorem 33 in [CMW$^+$12]), and the proof proceeds by associating a black pebbling with every computation of a deterministic thrifty BP that solves TEP, and showing that if the pebbling uses p pebbles then the BP must have at least k^p states. The hope (not yet realized) is to associate a fractional pebbling with every accepting computation of a nondeterministic thrifty BP solving BTEP so as to get an analogous result.

1.2 Pebbling

Here we are interested in the pebble game on rooted d-ary trees, although there is a natural generalization to rooted DAGs. We define three versions. The first is the simple 'black pebble' game: A black pebble can be placed on any leaf, and in general if all children of a node i have pebbles, then one of the pebbles on the children can be slid to i (this is a "black sliding move')'. Any black pebble can be removed at any time. The goal is to pebble the root, using as few pebbles as possible.

The second version is (whole) black-white pebbling with the restriction that we do not allow "white sliding moves". Thus if node i has a white pebble and each child of i has a pebble (either black or white) then the white pebble can be removed. (A white sliding move would apply if one of the children had no pebble, and the white pebble on i was slid to the empty child. We do not allow this, because there is no corresponding move in a k-way branching program solving TEP.) A white pebble can be placed on any node at any time. The goal is to start and end with no pebbles, and to have a pebble on the root at some time.

The third version is fractional pebbling, which generalizes whole black-white pebbling by allowing the black and white pebble weight of a node to be any real number between 0 and 1. However the total pebble weight of each child of a node i must be 1 before the black weight of i is increased or the white weight of i is decreased.

Figure 1 illustrates an optimal fractional pebbling with pebble weight 2.5 of the binary tree of height three.

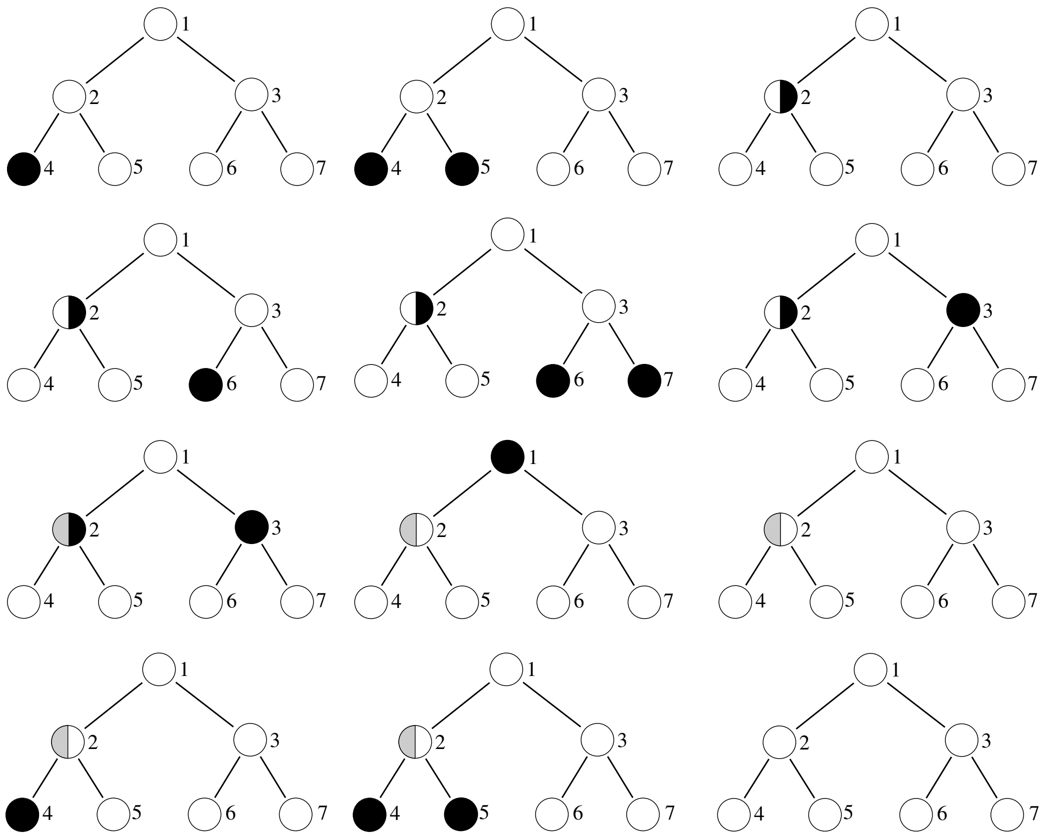

Fig. 1. An optimal fractional pebbling sequence for the height 3 tree using 2.5 pebbles, all configurations included (except the empty starting config). The grey half circle means the *white* weight of that node is .5, whereas unshaded area means absence of pebble weight. So for example in the seventh configuration, node 2 has black weight .5 and white weight .5, node 3 has black weight 1, and the remaining nodes all have black and white weight 0.

The motivation for choosing these definitions is that we want pebbling algorithms for trees to closely correspond to k-way branching program algorithms for solving TEP. The idea is that a whole black pebble on a node means that the corresponding branching program state 'knows' the value of the node, and a whole white pebble (applicable to nondeterministic BPs) means that that state has a specific conjecture for the value of the node (which must later be verified). A fractional pebble w, $0 \leq w \leq 1$, on a node means that the current state knows or conjectures a fraction w of the $\log k$ bits needed to specify the value of the node.

The *pebble weight* of a pebble configuration is the sum over all nodes i of the pebble weight of i. The *pebble weight* of a pebbling of a tree is the maximum, over all configurations c of the pebbling, of the pebble weight of c.

Recall that T_d^h is the balanced d-ary tree of height h (i.e. h levels).

Definition 1. *$p_{black}(h,d)$ (resp. $p_{bw}(h,d)$, $p_{frac}(h,d)$) is the minimum pebble weight over all black (resp. black-white, fractional) pebblings of T_d^h.*

The exact values of p_{black}, and for many cases of p_{bw}, have been known for some time [Nor10,CMW+12].

Proposition 2.

$$p_{black}(h,d) = (d-1)h - d + 2 \tag{1}$$

$$p_{bw}(h,d) = \lceil (d-1)h/2 \rceil + 1 \textit{ for } d = 2 \textit{ and } d \textit{ odd} \tag{2}$$

From [CMW+12] we have the following bounds for fractional pebbling.

Proposition 3.

$$(d-1)h/2 - d/2 \leq p_{frac}(h,d) \leq (d-1)h/2 + 1 \tag{3}$$

$$p_{frac}(3,d) = (3/2)d - 1/2 \tag{4}$$

$$p_{frac}(4,2) = 3 \tag{5}$$

The upper and lower bounds for $p_{black}(h,d)$ are easily proved by induction on the height h of the tree. The bounds for black-white pebbling $p_{bw}(h,2)$ for $d=2$ can also be proved by induction on h, but the proof is a little more complicated in that it involves two base cases (for $h=2$ and $h=3$) and the induction step is $h \to h+2$.

Here we give a simple proof of the fractional pebbling upper bound (3) for the case that the degree $d=2$. (See Figure 1 for the case $h=3$.)

Proposition 4. $p_{frac}(h,2) \leq h/2 + 1$

Proof. We prove by induction on the height h that there is a fractional pebbling of T_2^h for which the maximum pebble weight of any configuration is $min_h = h/2 + 1$ and the pebbling starts and ends with pebble weight 0, and further there is a time t_h in the pebbling at which the root of T_2^h has a whole black pebble and the total pebble weight at t_h is $min_h - 1$.

The base case $h = 2$ is trivial: two black pebbles suffice.

For the induction step $h \to h+1$, start by applying the induction hypothesis to the left principal subtree of T_2^{h+1}, but at the time t_h place a half black pebble on the root of the subtree (instead of a whole black pebble), and keep the half pebble there until this pebbling is completed. At this point there is a half black pebble on the left child of the root of T_2^{h+1}, and no pebble weight anywhere else. Now apply the procedure of the induction hypothesis to the right principal subtree until time t_h. At this time there is a whole black pebble on the right child of the root of T_2^{h+1}, and the total pebble weight is $min_h - .5$ (counting the half black pebble on the left child). (This is the 6th configuration in Figure 1.)

Now place a half white pebble on the left child (to give that node pebble weight 1), slide the black pebble on the right child to the root, and remove the half black pebble on the left child. This is the time t_{h+1}: the total pebble weight is $min_h - .5 = min_{h+1} - 1$ (this is the 8th configuration in Figure 1.) Next remove the black pebble from the root, and complete the pebbling of the right principal subtree (to remove any remaining white pebble weight there). During this completion, the maximum pebble weight on the whole tree is at most min_h (for the right principal subtree) plus .5 for the half white pebble on the left child of the root, making a total of at most min_{h+1}.

Finally remove the half white pebble on the left child of the root by applying the induction hypothesis to the left principal subtree, but at the step when a black pebble would be placed on the root, remove the half white pebble instead.

The generalization of this proof to obtain an upper bound of $(d-1)h/2+1$ for fractionally pebbling T_d^h for degree $d > 2$ is straightforward: see Theorem 15 in [CMW$^+$12] (preliminary version). It is interesting to note that even for the general case, whole and half pebbles suffice: no other fraction is needed to obtain this (optimal) upper bound (even though arbitrary fractions are allowed).

Proving good lower bounds on fractional pebbling for trees is much more difficult than for black-white pebbling. The best previous lower bounds are stated in Proposition 3. In particular, the lower bound in (3) from [CMW$^+$12] is not tight, and (as mentioned before) the complicated proof depends on Klawe's [Kla85] elaborate lower bound proof for black-white pebbling of DAGs.

The purpose of the present paper is to prove the following tight lower bound.

Theorem 1. *Every fractional pebbling of T_d^h (the balanced d-ary tree of with h levels) requires at least pebble weight* $(d-1)h/2+1$.

Since this lower bound matches the upper bound (3), it is tight.

Corollary 1. $p_{frac}(h,d) = (d-1)h/2+1$.

The proof of Theorem 1 is by induction on the height h. The induction hypothesis and the proof are complicated because fractions of pebbles allow for many very different pebbling strategies. These possible pebblings must be taken into account in the induction step.

2 The Proof

We start by introducing notation and giving precise definitions.

Definition 2 (Pebbling). *A* fractional pebble configuration *on a rooted d-ary tree T is an assignment of a pair of real numbers $(b(i), w(i))$ to each node i of the tree, where*

$$0 \leq b(i), w(i) \tag{6}$$

$$b(i) + w(i) \leq 1 \tag{7}$$

Here $b(i)$ and $w(i)$ are the **black pebble weight** *and the* **white pebble weight***, respectively, of i, and $b(i) + w(i)$ is the* pebble weight *of i. The* **pebble weight** *of a configuration is the sum over all nodes i of the pebble weight of i. The legal pebble moves are as follows (always subject to maintaining the constraints (6), (7)): (i) For any node i, decrease $b(i)$ arbitrarily, (ii) For any node i, increase $w(i)$ arbitrarily, (iii) For every node i, if each child of i has pebble weight 1, then decrease $w(i)$ to 0, increase $b(i)$ arbitrarily, and simultaneously decrease the black pebble weight of the children of i arbitrarily.*

A **fractional pebbling** *π is a sequence $m_1, m_2, \ldots$ of fractional pebble moves resulting in a sequence $c_0, c_1, c_2, \ldots$, of fractional pebble configurations, where c_0 is the initial configuration, and for $t > 0$, c_t is the configuration after move m_t. We refer to a configuration c_t as the* **time** *t.*

The **weight***, $w_\pi(t)$, of π at time t is sum of the pebble weights on T in configuration c_t. The* **subtree weight***, $sw_\pi(t)$, of π at time t is the sum of the pebble weights in the principal subtrees of T in configuration c_t. The* **white subtree weight** *$w.sw_\pi(t)$ (resp.* **black subtree weight** *$b.sw_pi(t)$) of π at time t is the sum of the white (resp. black) pebble weights in the principal subtrees of T in configuration c_t. The* **root weight***, $rw_\pi(t)$, of π at time t is the pebble weight on the root of T in configuration c_t. The* **white root weight** *$w.rw_\pi(t)$ (resp.* **black root weight** *$b.rw_\pi(t)$) of π at time t is the white (resp. black) pebble weight on the root of T in configuration c_t.*

Square brackets after the symbols defined above are used to indicate in which tree or subtree the pebble weight is located. For example, the symbol $b.rw_\pi(t)[P_1]$ would be used to specify some amount of black pebble weight on the root of the tree P_1 at time t.

Definition 3. *A* **root-pebbling** *is a pebbling π such that the initial and final pebble weights of π are 0, and $rw_\pi(t) = 1$ at some time t. A* **sub-pebbling** *is a pebbling that may start and end with nonzero pebble weight, but it must end with white pebble weight 0. A* **root sub-pebbling** *is a sub-pebbling such that $rw_\pi(t) = 1$ at some time t. A* **sub-root sub-pebbling** *is a sub-pebbling such that for some time t all the principal subtrees of T have $rw_\pi(t) = 1$.*

We are now ready to prove Theorem 1, which we restate using the above definitions.

Main Theorem
Let $min_h = (d-1)h/2+1$. For every root-pebbling π of T_d^h there is a time t such that $w_\pi(t) \geq min_h$.

The proof is obvious for $h = 2$. The proof for $h \geq 3$ is by induction on h.

Here we state the induction hypothesis and main lemmas needed for the proof, but omit some details of the proof and sometimes treat only the case that the degree $d = 2$. Full details can be found in the appendix, and in [Van11].

Induction Hypothesis [IH(h)]: Let π be a **sub-root sub-pebbling** of T_d^h, with $h \geq 3$. Let $t_{root}{}^*$ be a time such that $rw_\pi(t_{root}{}^*) = 1$ for all principal subtrees. If there exists $\epsilon \in (-0.5, 0.5]$ such that $b.sw_\pi(0) \leq 1 - \epsilon$, $b.rw_\pi(0)$ = arbitrary, and π is such that $sw_\pi(t) \leq min_h - \epsilon$ for $t \leq t_{root}{}^*$, then there is a time $t_b{}^* > t_{root}{}^*$ such that $sw_\pi(t_b{}^*) \geq min_h + \epsilon$ and $w.sw_\pi(t) \geq 0.5 + \epsilon$ for t in $[t_{root}{}^*, t_b{}^*]$.

initial conditions	additional conditions	consequences
$b.sw_\pi(0) \leq 1 - \epsilon$	$sw_\pi(t) \leq min_h - \epsilon$ for $t \leq t_{root}{}^*$	$sw_\pi(t_b{}^*) \geq min_h + \epsilon$
$b.rw_\pi(0)$ = arbitrary		$w.sw_\pi(t) \geq 0.5 + \epsilon$ for t in $[t_{root}{}^*, t_b{}^*]$

Note that ϵ can be positive, negative, or 0. For positive ϵ the IH says that if the pebbling uses less than min_h pebble weight on the subtrees before $t_{root}{}^*$ then it must use more than min_h after $t_{root}{}^*$. For negative ϵ it says that if more than min_h pebble weight is used before then there is a lesser lower bound on the pebble weight after $t_{root}{}^*$.

The Induction Hypothesis implies the theorem. A root-pebbling must have a time t_{root} with pebble weight 1 on the root. If at t_{root} the root has any black pebble weight there must be a time $t_{root}{}^*$ to place this black pebble weight. If it has only white pebble weight at t_{root}, there must be a time $t_{root}{}^*$ to remove this white pebble weight. The IH then implies that there must be a time either before or after $t_{root}{}^*$ with pebble weight at least min_h.

Proof of the Base Case of the Induction Hypothesis (for $d = 2$)

In this case $h = 3$, so $min_h = min_3 = 2.5$.

Let the nodes v_2 and v_3 be the two children of the root.

Case I : The black pebble weight on the v_i is never increased at any time t such that $t \leq t_{root}{}^*$.

Then the total black pebble weight of the v_i at $t_{root}{}^*$ is at most $1 - \epsilon$, so the white pebble weight for these nodes at $t_{root}{}^*$ must be at least $2 - (1 - \epsilon) = 1 + \epsilon$.

Let $t_b{}^*$ be the first time we remove white pebble weight after $t_{root}{}^*$. Since we must have pebble weight 1 on all of the children to remove white pebble weight we have that the total pebble weight required to remove white pebble weight is at least $2 + (1 + \epsilon) = 3 + \epsilon > 2.5 + \epsilon = min_h + \epsilon$ at time $t_b{}^*$.

Also, during the interval [$t_{root}{}^*$, $t_b{}^*$], $w.sw_\pi(t) \geq 1 + \epsilon > 0.5 + \epsilon$, as required.

Thus the IH is satisfied in this case.

Case II : The black pebble weight on the nodes v_i is increased at some time t such that $t \leq t_{root}{}^*$.

Let t_a* be one step before the last time of such an increase. Let α be the total black pebble weight of the v_i at time t_a*. Then the total subtree pebble weight at time t_a* is at least $2 + \alpha$, which by assumption is at most $min_h - \epsilon$. Therefore, $2 + \alpha \leq 2.5 - \epsilon$, and hence

$$\alpha \leq 0.5 - \epsilon \tag{8}$$

After this increase at time t_a* the total black pebble weight of the v_i is at most 1 + α. Hence the white pebble weight of the v_i at $t_{root}{}^*$ satisfies $w.sw_\pi(t_{root}{}^*) \geq 2 - (1 + \alpha) = 1 - \alpha$.

Let $t_b{}^*$ be the time just before the first time after $t_{root}{}^*$ that this white pebble weight is decreased. Since we need pebble weight 2 on the leaves at such a time, $sw_\pi(t_b{}^*) \geq 2 + (1 - \alpha) = 3 - \alpha \geq 3 - 0.5 + \epsilon$ (by 8)
= $min_h + \epsilon$ as required.

Also, $t_b{}^* > t_{root}{}^*$, since at $t_{root}{}^*$ the pebble weight on the v_i is 2, thus we could not have had the required pebble weight on the children due to the restriction on total pebble weight.

Finally, during the interval $[t_{root}{}^*, t_b{}^*]$, $w.sw_\pi(t) \geq 1-\alpha \geq 1-(0.5-\epsilon) = 0.5+\epsilon$, as required. Thus the IH is satisfied in this case. ■

The next two lemmas will be used in the proof of the induction step. They will be applied to the subtrees of the root.

Lemma 1. *Let π be a* **root sub-pebbling** *of T_d^h. Let t_{root} be any time such that $rw_\pi(t_{root}) = 1$.*

It follows from the IH for height h, that if $E \in [0.0, 0.5)$, $b.sw_\pi(0) \leq 0.5 + E$, $b.rw_\pi(0) \leq 2E$ and π is such that $sw_\pi(t) \leq min_h - 0.5 + E$ for $t \leq t_{root}$, then there is a time $t_b{}^{**}$, such that $t_{root} < t_b{}^{**}$, $w_\pi(t_b{}^{**}) \geq min_h + 0.5 - E$ and $w.w_\pi(t) \geq 1 - 2E$ for t in $[t_{root}, t_b{}^{**}]$.

initial conditions	additional conditions	consequences
$b.sw_\pi(0) \leq 0.5 + E$	$sw_\pi(t) \leq min_h - 0.5 + E$ for $t \leq t_{root}$	$w_\pi(t_b{}^{**}) \geq min_h + 0.5 - E$
$b.rw_\pi(0) \leq 2E$		$w.w_\pi(t) \geq 1 - 2E$ for t in $[t_{root}, t_b{}^{**}]$

Lemma 2. *Let π be a* **root sub-pebbling** *of T_d^h. Let t_{root} be any time such that $rw_\pi(t_{root}) = 1$.*

It follows from the IH for height h, that if $E \in [0, 1)$, $b.sw_\pi(0) \leq 0.5 + E$, at some time t_0, $0 \leq t_0 \leq t_{root}$, $b.rw_\pi(t_0) \leq E$ and π is such that $w_\pi(t) \leq min_h - 0.5 + E$ for $t \leq t_{root}$, then there is a time $t_b{}^{**}$, such that $t_{root} < t_b{}^{**}$, $w_\pi(t_b{}^{**}) \geq min_h + 0.5 - E$ and $w.w_\pi(t) \geq 1 - E$ for t in $[t_{root}, t_b{}^{**}]$.

initial conditions	additional conditions	consequences
$b.sw_\pi(0) \leq 0.5 + E$	$w_\pi(t) \leq min_h - 0.5 + E$ for $t \leq t_{root}$	$w_\pi(t_b{}^{**}) \geq min_h + 0.5 - E$
$b.rw_\pi(t_0) \leq E$, $t_0 \leq t_{root}$		$w.w_\pi(t) \geq 1 - E$ for t in $[t_{root}, t_b{}^{**}]$

Observe that in **Lemma 1** additional initial black pebble weight on the root allows us to use less white pebble weight for t in $[t_{root}, t_b{}^{**}]$ than in **Lemma 2**. In **Lemma 2** we introduce a time t_0. There may be more black pebble weight on the root before time t_0, however, it cannot help us achieve the specified t_{root} if it is removed before t_{root}.

The IH implies conditions on the subtree pebble weight while the lemmas imply conditions on pebble weight anywhere.

Proof. (of Lemma 1)

We must reach a time $t_{root}{}^*$, either to add black pebble weight to reach t_{root} or to remove white pebble weight added to reach t_{root}. Since times $t_{root}{}^*$ exist, π is also a **sub-root sub-pebbling**. Thus we will apply the IH at these points denoted $t_{root}{}^*$.

Case 1 : $\exists t_{root}{}^*, t_{root}{}^* \leq t_{root}$.

By IH with $\epsilon = 0.5 - E$, since by assumption $sw_\pi(t) \leq min_h - 0.5 + E$ for $t \leq t_{root}$ and $b.sw_\pi(0) \leq 0.5 + E$, then at some time $t_b{}^{**} = t_b{}^*$, $sw_\pi(t_b{}^{**}) \geq min_h + 0.5 - E$ and $w.w_\pi(t) \geq 1 - E$ for t in [$t_{root}{}^*$, $t_b{}^{**}$]. Also, $1 - E \geq 1 - 2E$ since $E \geq 0$.

Since $min_h + 0.5 - E > min_h - 0.5 + E$ for all allowed E, we have not been allotted enough pebbles before t_{root} and $t_{root} < t_b{}^{**}$.

Thus the conditions of the lemma are satisfied.

Case 2 : $\forall t_{root}{}^*, t_{root} < t_{root}{}^*$. Then, to reach t_{root} we must use white pebble weight. Since $b.rw_\pi(0) \leq 2E$, $w.rw_\pi(t_{root}) \geq 1 - 2E$. We must then reach a $t_{root}{}^*$ to remove this white pebble weight. Let $t_{root}{}^{*First}$ be the first such $t_{root}{}^*$. Thus,

$$w.rw_\pi(t) \geq 1 - 2E \text{ for } t \text{ in } [t_{root}, t_{root}{}^{*First}] \tag{9}$$

Case 2-A : $\exists t, t \in (t_{root}, t_{root}{}^{*First}]$ and $sw_\pi(t) \geq min_h - 0.5 + E$

Choose $t_b{}^{**}$ to be the first such t. Then $w_\pi(t_b{}^{**}) \geq min_h + 0.5 - E$ and $w.w_\pi(t) \geq 1 - 2E$ for times t in [t_{root}, $t_b{}^{**}$] since we have yet to remove the white pebble weight on the root (9). Thus the lemma is satisfied in this case.

Case 2-B : $\forall t, if\ t \ \in (t_{root}, t_{root}{}^{*First}]$ then $sw_\pi(t) < min_h - 0.5 + E$

Then $sw_\pi(t) \leq min_h - 0.5 + E$ for t in [0, $t_{root}{}^{*First}$]. By IH with $\epsilon = 0.5 - E$, we have some time $t_b{}^* > t_{root}{}^{*First}$ such that $sw_\pi(t_b{}^*) \geq min_h + 0.5 - E$ and $w.w_\pi(t) \geq 1 - E$ for t in [$t_{root}{}^{*First}$, $t_b{}^*$]. We choose $t_b{}^{**} = t_b{}^*$.

$w.w_\pi(t) \geq 1 - 2E$ for times t in [t_{root}, $t_{root}{}^{*First}$] (9). Thus, $w.w_\pi(t) \geq 1 - 2E$ for t in $[t_{root}, t_b{}^{**}]$. Thus, all conditions are met and the lemma is satisfied in this case.

Thus **Lemma 1** is satisfied in all cases.

$Proof.$ (of Lemma 2) (See Appendix)

Induction step : We prove the induction hypothesis for h+1 assuming it for $h', 3 \leq h' \leq h$.

Here we concentrate on the case $d = 2$*.*

Let $\pi = 0, ..., t_{root}{}^*, ...$ to be a **sub-root sub-pebbling** of T_2^{h+1} where $rw_\pi(t_{root}{}^*) = 1$ for each principal subtree and

$$sw_\pi(t) \leq min_{h+1} - \epsilon = (h+1)/2 + 1 - \epsilon = min_h + 0.5 - \epsilon \; for\ t\ in\ [0, t_{root}{}^*] \tag{10}$$

Further, we assume,

$$\epsilon \in (-0.5, 0.5] \tag{11}$$

$$b.sw_\pi(0) \leq 1 - \epsilon \tag{12}$$

Let P_1 and P_2 be the principal subtrees of T_2^{h+1}. The restriction of π to each of these subtrees is a valid pebbling of that subtree.

Case 1 : $\forall t, \forall i$, if $t \leq t_{root}{}^*$ then sw_π(t)[P_i] $< min_h - 0.5$

For each principal subtree we will apply **Lemma 1**. We will then show that taken together this implies the desired bounds.

In this case, the subtree pebble weight of all subtrees P_i is less than $min_h - 0.5$.

We have at most $1-\epsilon$ initial black pebble weight in the P_i by assumption (12). We will separate this pebble weight between the subtrees and apply **Lemma 1** to each subtree. Let us have $b.w_\pi(0)[P_i] = 2E_i$. We choose to express the amount this way since it resemble amounts expressed in **Lemma 1**.

It is the case that $E_i \geq 0$ since pebble weight is non-negative.

If $0 \leq E_i < 0.5$ we may apply **Lemma 1** to the i^{th} subtree. Let G be the set of all i such that $0 \leq E_i < 0.5$. Note, G is not the empty set since $b.sw_\pi(0) \leq 1-\epsilon$ and we have two subtrees.

We have $\Sigma_{i\in G}(1-2E_i) \geq \Sigma_{i=1}^2(1-2E_i)$ since $0 \geq 1-2E_i$ for $i \notin G$.

By the above and since $\Sigma_{i=1}^2 2E_i \leq 1-\epsilon$, we have

$$\Sigma_{i\in G}(1-2E_i) \geq 1+\epsilon \tag{13}$$

For each subtree, we take t_{root} in the lemma to be the time $t_{root}{}^*$. This is possible since $rw_\pi(t_{root}{}^*)[P_i] = 1$ as required by **Lemma 1**.

We apply **Lemma 1** to P_i, $i \in G$, taking E in the lemma to be E_i and with $t_b[P_i] := t_b{}^{**}$. Then, $t_b[P_i] > t_{root}{}^*$, $w_\pi(t_b[P_i])[P_i] \geq min_h + 0.5 - E_i$ and $w.w_\pi(t)[P_i] \geq 1-2E_i$ for t in $[t_{root}{}^*, t_b[P_i]]$.

We let $t_b{}^* = min(t_b[P_i])$ for $i \in G$. We define $first$ to be this i. It is the first $t_b[P_i]$ we reach in π. Then we require $min_h + 0.5 - E_{first}$ in P_{first} while maintaining at least $1-2E_i$ in the remaining P_i, $i \in G$ and $i \neq first$. Then,
$sw_\pi(t_b{}^*) \geq min_h + 0.5 - E_{first} + \Sigma_{i\in G, i\neq first}(1-2E_i)$
$\geq min_h + 0.5 - 2E_{first} + \Sigma_{i\in G, i\neq first}(1-2E_i)$ (since, $0 \geq -E_{first}$)
$= min_h - 0.5 + \Sigma_{i\in G}(1-2E_i)$
$\geq min_h - 0.5 + 1 + \epsilon$ (by 13)
$= min_{h+1} + \epsilon$

Additionally,
$w.sw_\pi(t) \geq \Sigma_{i\in G}(1-2E_i) \geq 1+\epsilon$ for t in $[t_{root}{}^*, t_b{}^*]$ since we maintain the sum of the white pebble weights from the lemma until $t_b{}^*$ by the lemma.

Thus the IH is satisfied in **Case 1**.

Case 2 : $\exists t$, $\exists i$, $t \leq t_{root}{}^*$ and $sw_\pi(t)[P_i] \geq min_h - 0.5$.

For each principal subtree we will again apply one of the lemmas. We will then show that taken together this implies the desired bounds. Also recall that we have fixed $\pi = 0, ..., t_{root}{}^*, ...$.

WLOG, suppose $sw_\pi(t)[P_i] \geq min_h - 0.5$ for the last time before $t_{root}{}^*$ in the subtree P_1. Let this time be t_{last}. Then $t_{last} \leq t_{root}{}^*$ and

$$sw_\pi(t_{last})[P_1] \geq min_h - 0.5 \tag{14}$$

For any value r, define t_r to be the last time in $[0, t_{root}{}^*]$ such that $sw_\pi(t_r)[P_2] \geq min_h - 0.5 + r$ or the initial time if no such time exists.

Define R to be the maximum r such that $w_\pi(t)[P_2] \geq 2r$ for times t in $[t_r, t_{root}{}^*]$.

There is always a time $t_{root}{}^*$ since π is a **sub-root sub-pebbling**. The described condition is true for some value of r as it is true for $r = 0$ and this is the smallest value possible. There is therefore always a time t_R. Thus,

$$R \geq 0 \tag{15}$$

By definition of t_R and t_{last},

$$t_R < t_{last} \tag{16}$$

This is a result of the restriction on total pebble weight (10) and having at least $min_h - 0.5$ pebble weight in P_1 at t_{last}.

In summary, the choice of R implies the following,

$$sw_\pi(t_R)[P_2] \geq min_h - 0.5 + R \; or \; t_R = \mathbf{0} \tag{17}$$

$$w_\pi(t)[P_2] \geq 2R \; for \; t \; in \; [t_R, t_{root}{}^*] \tag{18}$$

Definition 4. *Define $t_{P2-init}$ to be a time such that $w_\pi(t_{P2-init})[P_2] \leq 2R$ and $sw_\pi(t)[P_2] \leq min_h - 0.5 + R$ fot t in [$t_{P2-init}$, $t_{root}{}^*$].*

This will be useful since we wish to apply **Lemma 1** to P_2 later for R and initial time $t_{P2-init}$. We show such a time always exists.

Case I : $w_\pi(t)[P_2] = 2R$ for some t in [t_R, $t_{root}{}^*$]. We let this time be $t_{P2-init}$.

Case II : $w_\pi(t)[P_2] > 2R$ for all times t in [t_R, $t_{root}{}^*$].

Then $sw_\pi(t_R)[P_2] = min_h - 0.5 + R$. If this was not the case, the conditions would be true for a greater value of R and we would have a contradiction. For similar reasons, t_R is not the initial time else the condition would be true for a larger value of R.

Let $t_{before-R}$ be the last time such that $sw_\pi(t_{before-R})[P_2] > min_h - 0.5 + R$ or the initial time if no such time exists. Then $t_{before-R} < t_R$. There must have been a time, $t_{P2-init}$, in [$t_{before-R}$, t_R] such that $w_\pi(t_{P2-init})[P_2] \leq 2R$. If this were not the case, the conditions would be true for a greater value of R since we would have $w_\pi(t)[P_2] > 2R$ for t in [$t_{before-R}$, $t_{root}{}^*$] using the assumption in **Case II**. Thus, the chosen $t_{P2-init}$ satisfies the necessary conditions.

Thus in all cases, such a $t_{P2-init}$ exists.

We use $sw_\pi(t_{last})[P_1] \geq min_h - 0.5$ (14) while maintaining at least $2R$ in P_2 at time t_{last} (17). Thus, $min_h - 0.5 + 2R \leq min_h + 0.5 - \epsilon$ due to the restriction on total pebble weight (10). Then $1 - \epsilon - 2R$ is the maximum amount of pebble weight at t_{last} on the root of P_1. It is the difference between the maximum pebble weight and the pebble weight elsewhere.

We denote this upper bound $Rmax$ except we take min(2R,1) to simplify calculations later.

$$1 - \epsilon - 2R \leq 1 - \epsilon - min(2R, 1) \tag{19}$$

$$Rmax = 1 - \epsilon - min(2R, 1) \tag{20}$$

Case 2A : $Rmax \geq 1$.

We apply **Lemma 1** to P_2 as in **Case 1**. (See the Appendix for details.)

Case 2B : $Rmax < 1$ ($Rmax = 1 - \epsilon - min(2R, 1)$).

We wish to eventually apply **Lemma 2** to P_1 for $E = Rmax$. To do this we take t_R (17 and 18) to be the initial time and t_{last} to be the time t_0 in the lemma. To apply **Lemma 2**, we must show upper bounds on $b.sw_\pi(t_R)[P_1]$, $w_\pi(t)[P_1]$ for t in $[t_R, {t_{root}}^*]$, $b.rw_\pi(t_{last})[P_1]$ and we must show $Rmax \in$ [0,1).

(See the Appendix for the proof of these bounds.)

If $0 \leq R < 0.5$ we may apply **Lemma 1** to P_2 with initial time $t_{P2-init}$.

Since ${t_{root}}^*$ occurs when $rw_\pi({t_{root}}^*)[P_1] = 1$ and $rw_\pi({t_{root}}^*)[P_2] = 1$, we apply **Lemma 2** and **Lemma 1**, respectively, taking ${t_{root}}^*$ as the time t_{root} in the lemmas.

We apply **Lemma 2** to P_1 with $t_b[P_1] := {t_b}^{**}$ from the lemma. Then, $t_b[P_1] > {t_{root}}^*$,
$w_\pi(t_b[P_1])[P_1] \geq min_h + 0.5 - Rmax$
$= min_h - 0.5 + \epsilon + min(2R, 1)$ (by 20)
and
$w.w_\pi(t)[P_1] \geq 1 - Rmax$
$= \epsilon + min(2R, 1)$ (by 20) for t in [${t_{root}}^*$, $t_b[P_1]$].

If $R < 0.5$, we apply **Lemma 1** to P_2 taking the initial time in the lemma to be $t_{P2-init}$, taking E in the lemma to be R and with $t_b[P_2] := {t_b}^{**}$ from the lemma. We may do this since $b.sw_\pi(0) \leq 2R \leq 0.5 + R$ and $b.rw_\pi(0) \leq 2R$. Then, $t_b[P_2] > {t_{root}}^*$, $w_\pi(t_b[P_2])[P_2] \geq min_h + 0.5 - R$ and $w.w_\pi(t)[P_2] \geq 1 - 2R$ for t in [${t_{root}}^*$, $t_b[P_2]$].

We choose ${t_b}^* = min(t_b[P_1],\ t_b[P_2])$ if $t_b[P_2]$ exists else we choose ${t_b}^* = min(t_b[P_1])$.

Case 2B-1 : ${t_b}^* = t_b[P_1]$. Note, $1 - min(2R, 1) = 0$ if there is no $t_b[P_2]$. Then,
$sw_\pi({t_b}^*) \geq min_h - 0.5 + \epsilon + min(2R, 1) + 1 - min(2R, 1)$
$= min_{h+1} + \epsilon$
We add the pebble weight in P_2 since we had yet to reach $t_b[P_2]$ if it exists. This matches the lower bounds required by the IH.

Also, we have white pebble weight as follows between [${t_{root}}^*$, ${t_b}^*$],
$w.sw_\pi(t) \geq \epsilon + min(2R, 1) + 1 - min(2R, 1)$
$\geq 0.5 + \epsilon$ as required. Thus we exceed or match the minimum pebble weight allotted by the IH.

Case 2B-2 : ${t_b}^* = t_b[P_2]$.

Then $R < 0.5$ and
$sw_\pi({t_b}^*) \geq min_h + 0.5 - R + \epsilon + 2R$
$\geq min_{h+1} + \epsilon$

As in **Case 2B-1**, we have the same amount of white pebble weight until this time. Thus the IH is satisfied in this case, and therefore in all cases.

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