

Easy-to-implement hp -adaptivity for non-elliptic goal-oriented problems

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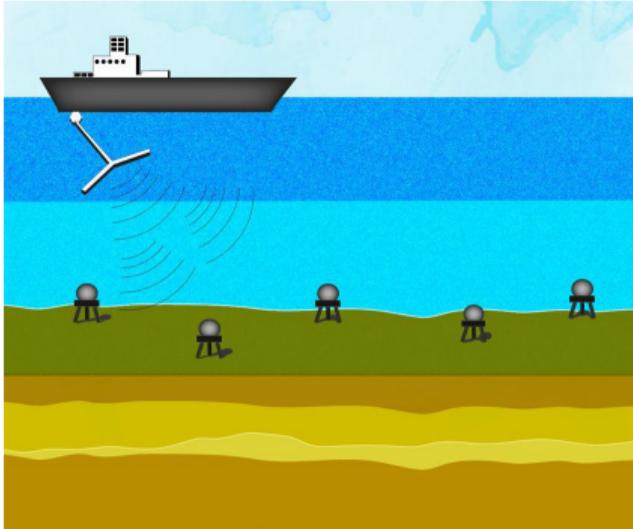
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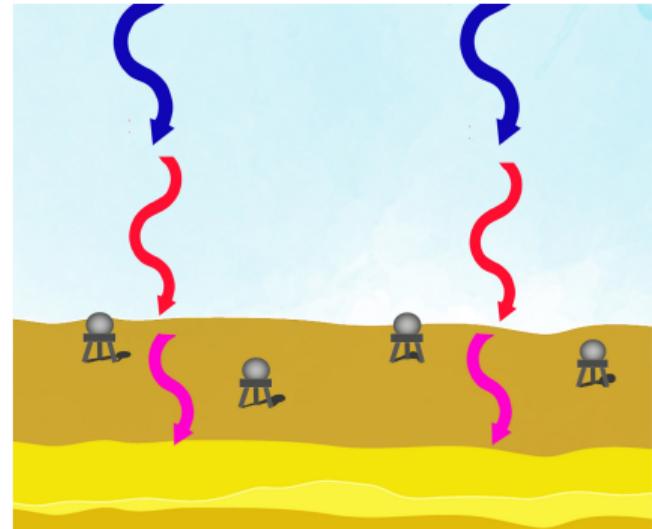
29 November 2023, Leioa



Electromagnetic (EM) Applications



(a) CSEM (artificial source)

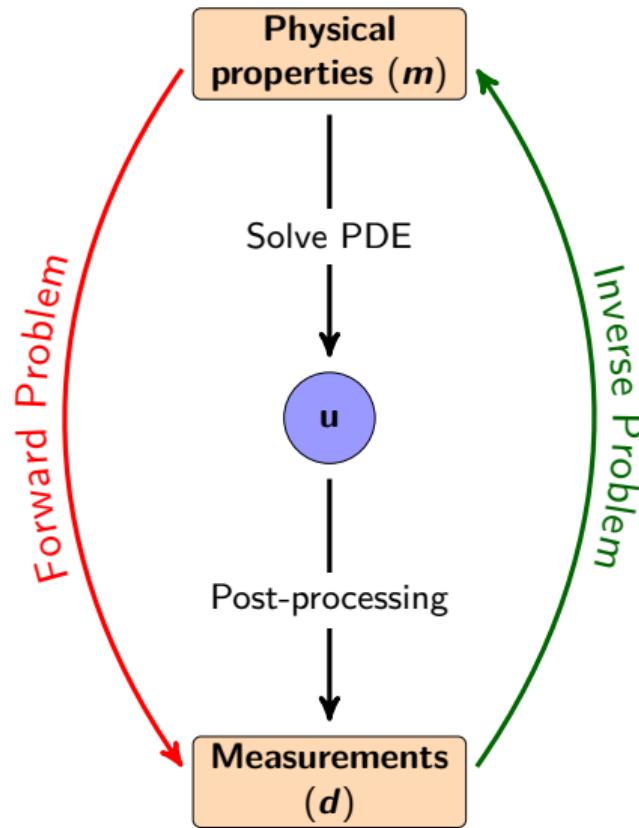


(b) MT (natural source)

Objective

Objective: To obtain the **conductivity/resistivity** distribution of the Earth's subsurface.

Overview of Forward and Inverse Problems



Definitions

\mathcal{F} : Forward Operator
 $\mathbf{m} \mapsto \mathbf{d}$

\mathcal{I} : Inverse Operator
 $\mathbf{d} \mapsto \mathbf{m}$

\mathcal{I}_ϕ : Neural Network approximation of \mathcal{I}

Generation of Massive Databases

Objective

Building the **Inverse Operator** (not just evaluating it).

Loss Function and Training

Find \mathcal{I}_{ϕ^*} such that

$$\phi^* = \arg \min_{\phi \in \Phi} \sum \|(\mathcal{F} \circ \mathcal{I}_\phi)(\mathbf{d}_i) - \mathbf{d}_i\|^2$$

where evaluating \mathcal{F} is **expensive!**

Difficulties

- ① We need a Forward Solver \mathcal{F} for **any parameterization** (model).
- ② A **huge number** of evaluations needed **to train** the DNN.

Solution

- ① Approximate the Forward Operator with a **Neural Network**.

Traditional Numerical Methods in Geophysics

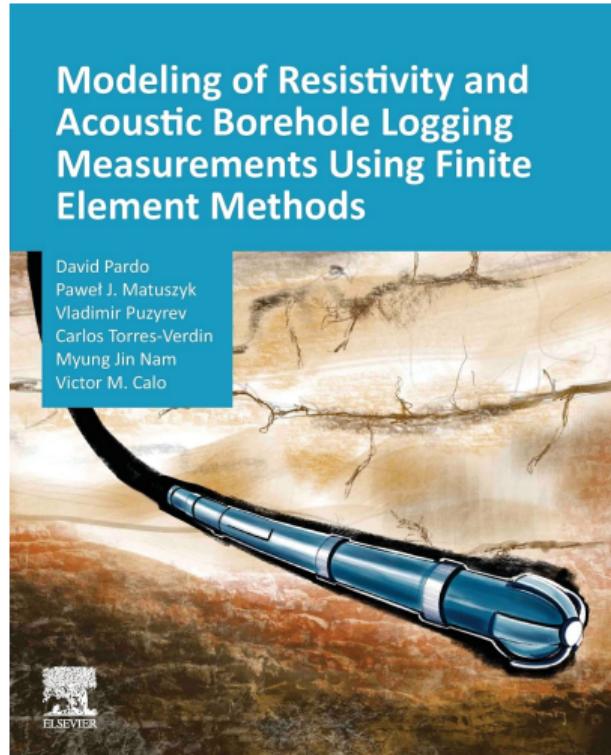


Figure: Published in 2021

Numerical Methods

- Finite Element method
- Finite Difference method
- Finite Volumes method
- Integral methods
- Semi-analytical methods

Outline

- ① Goal-Oriented hp -adaptivity for non-parametric PDEs
 - Why hp -Adaptivity?
 - Goal-Oriented coarsening strategy
 - 1D Numerical results for Goal-Oriented h - and p -adaptivity
 - 2D Numerical results for hp -adaptivity
 - 3D Numerical results for hp -adaptivity
- ② Goal-Oriented hp -adaptivity for parametric PDEs.
 - Database generation for DL inversion
- ③ Main Achievements
- ④ Conclusions and Future Work

Goal-Oriented hp -adaptivity for non-parametric PDEs

Why hp -Adaptivity?

Advantages

hp -adaptive FEM achieves exponential convergence rates

Precise mesh refinement near singularities

Smaller meshes with higher accuracy per number of degrees of freedom

Limitations

Solution accuracy heavily mesh-dependent, requiring precise mesh design

Increasing accuracy leads to prohibitive computational costs

Main Ingredients

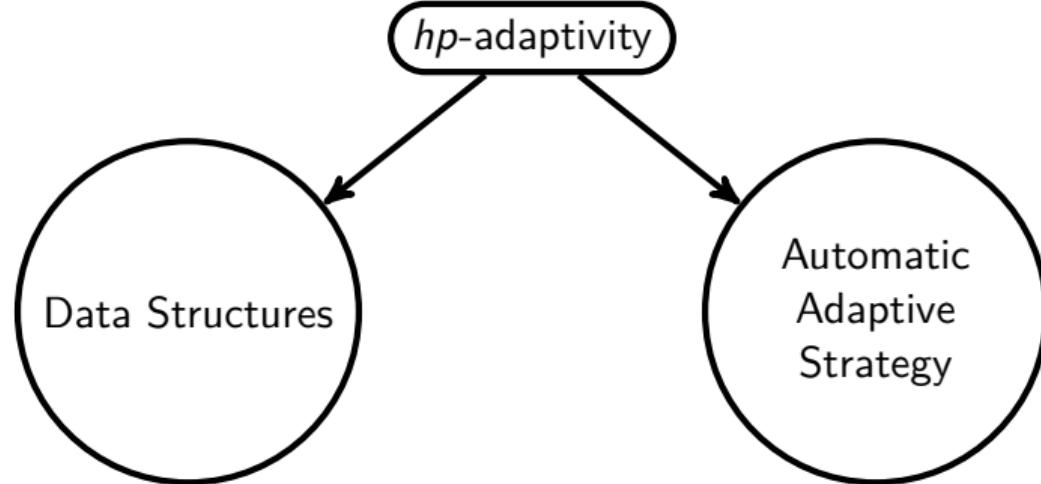
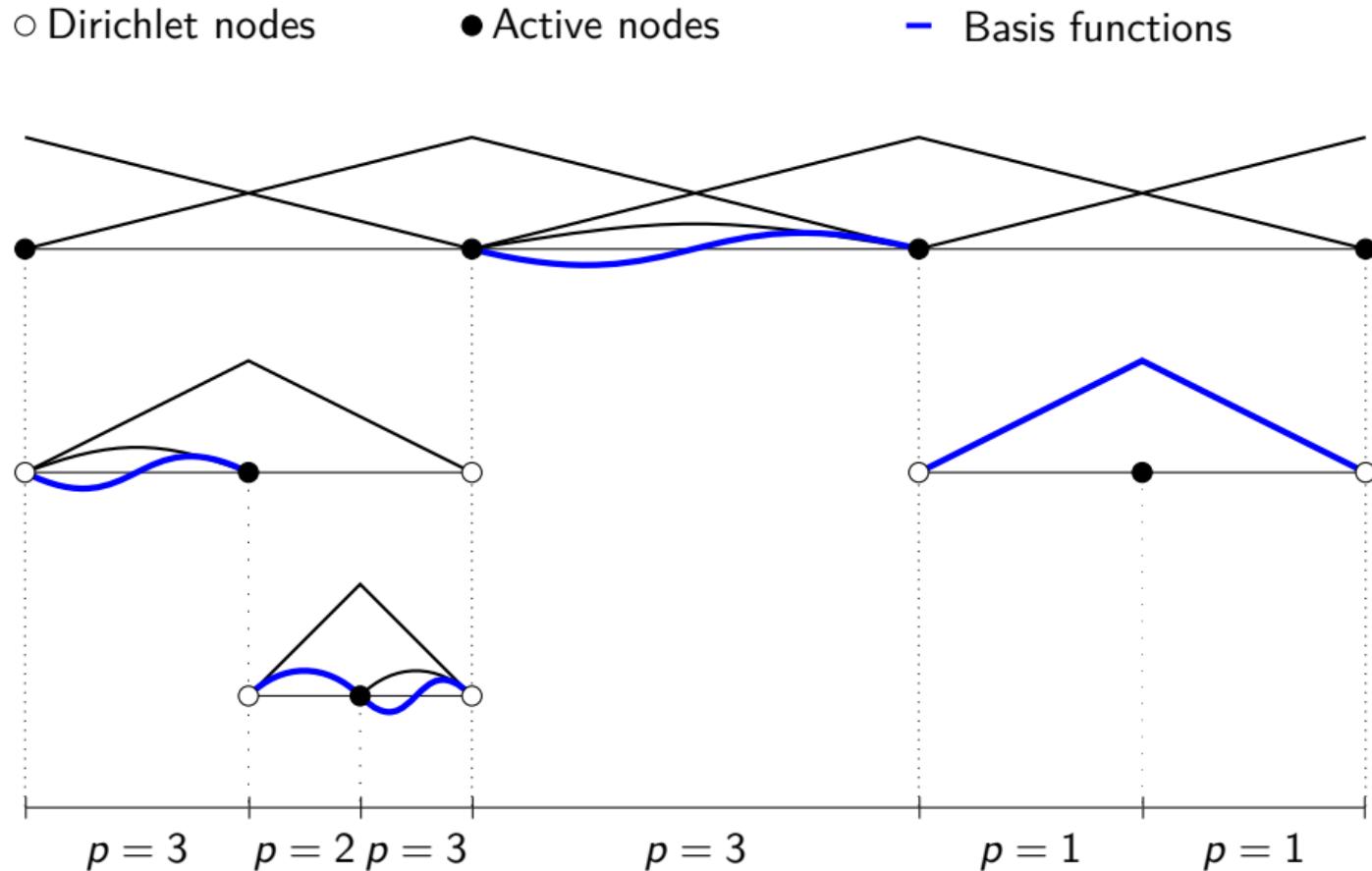


Figure: Main ingredients

Multi-Level Mesh Data Structure 1D



Multi-Level Mesh Data Structure 2D

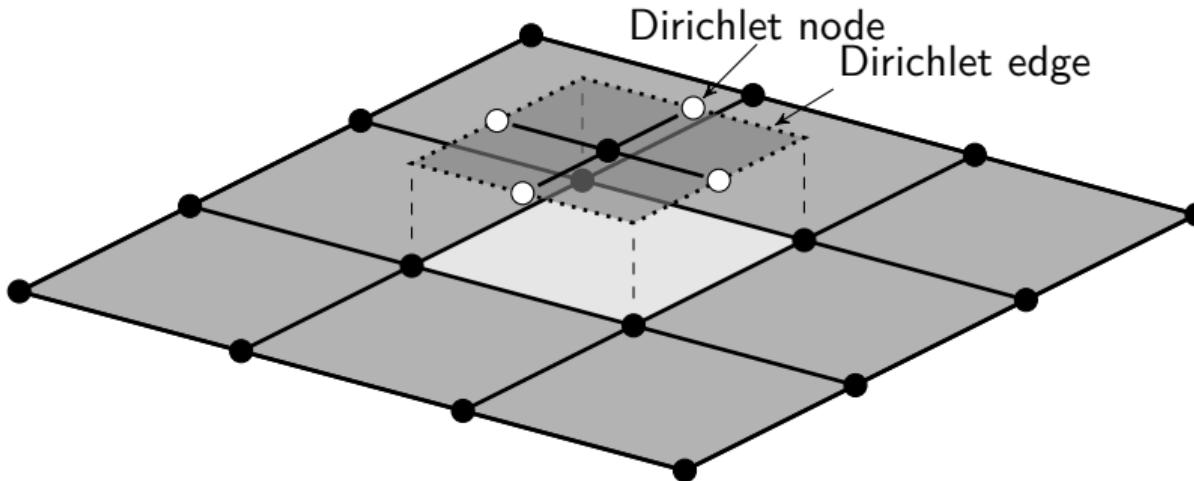


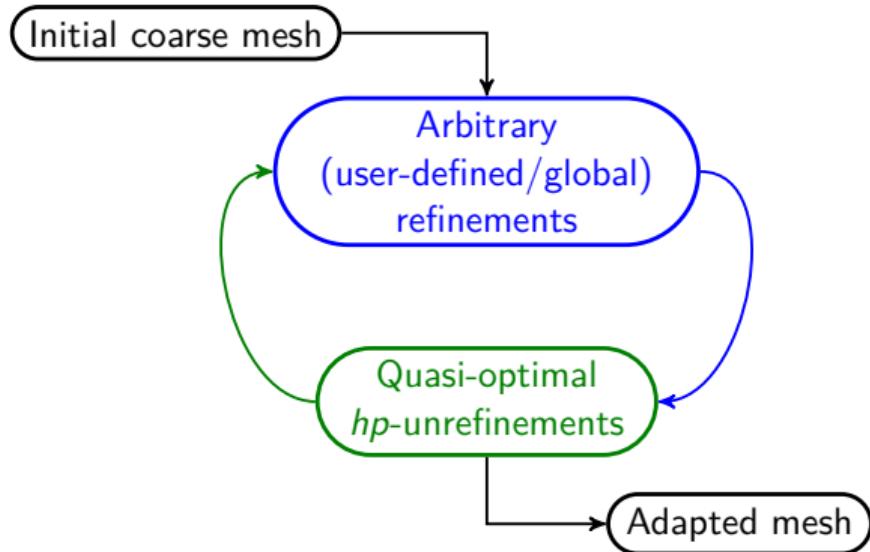
Figure: Multi-level 2D mesh without constraints on hanging nodes using Dirichlet nodes. The bubble basis functions are at the lowest level of each family



N. Zander, T. Bog, S. Kollmannsberger, D. Schillinger, and E. Rank

Multi-level *hp*-adaptivity: high-order mesh adaptivity without the difficulties of constraining hanging nodes
Computational Mechanics, 55(3):499–517, Mar 2015

A Painless Automatic Adaptive Strategy



V. Darrigrand, D. Pardo, T. Chaumont-Frelet, I. Gómez-Revuelto, L. E. García-Castillo

A painless automatic *hp*-adaptive strategy for elliptic problems

Finite Elements in Analysis and Design, 2020.



F. V. Caro, V. Darrigrand, J. Alvarez-Aramberri, E. Alberdi, D. Pardo

A painless multi-level automatic goal-oriented *hp*-adaptive coarsening strategy for elliptic and non-elliptic problems

Computer Methods in Applied Mechanics and Engineering, 2022.

Adaptive Mesh Refinement Algorithm

Algorithm 1: Adaptive process

Input: A given initial mesh

Output: A final hp -adapted mesh

while error above tolerance **do**

Perform a global and uniform (h or p) refinement;

Execute a (quasi)-optimal hp -coarsening step (Algorithm 2) to the mesh;

Update error;

end

hp-Unrefinement Policy

Algorithm 2: *hp*-unrefinement policy

Input: A given mesh

Output: An *hp*-unrefined mesh

do

 Compute the solution on the current mesh;

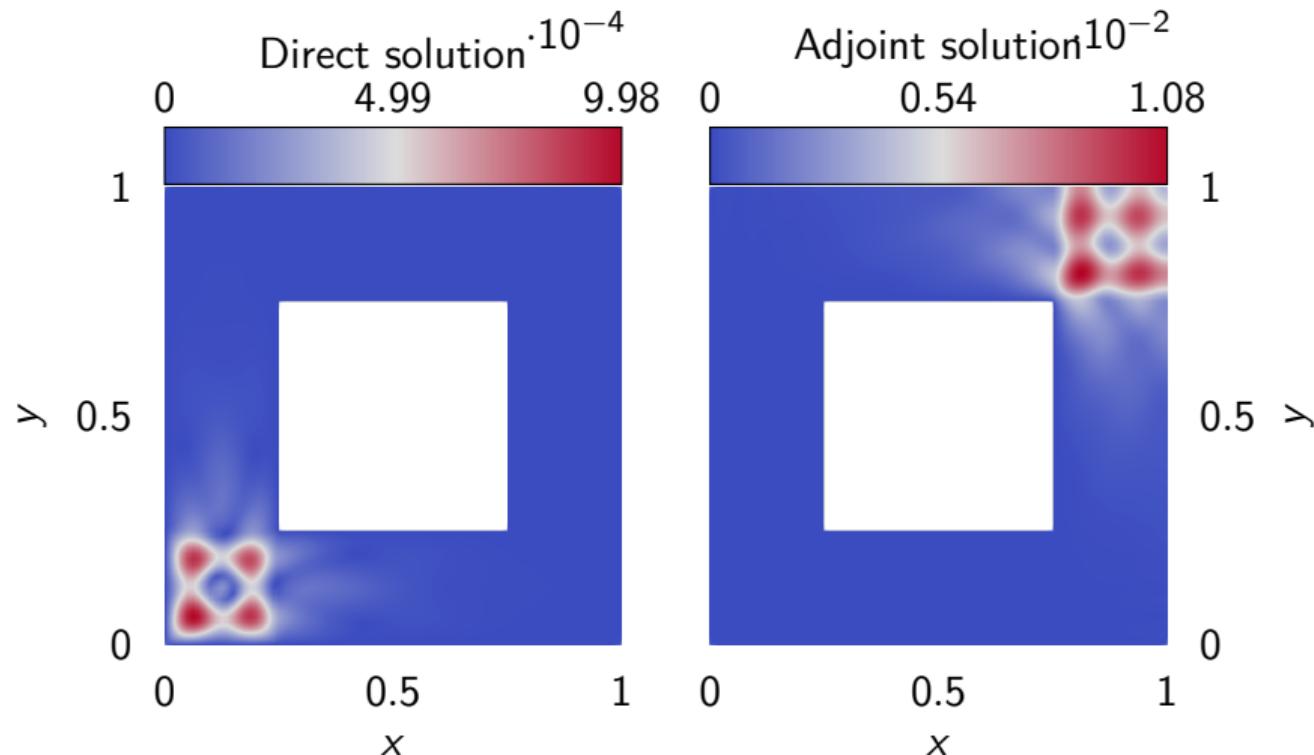
 Compute the element-wise error indicators;

 Unrefine the mesh by eliminating the *removable* basis functions with low error
 indicators;

 When no contributions are below a given tolerance, exit;

end;

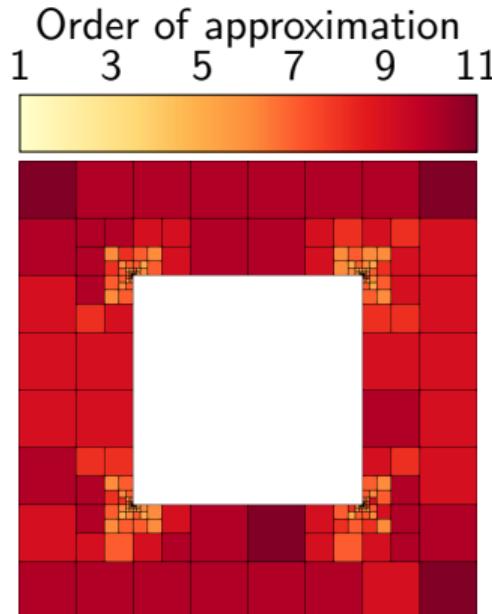
Illustrating the Goal-Oriented hp -Adaptive Strategy



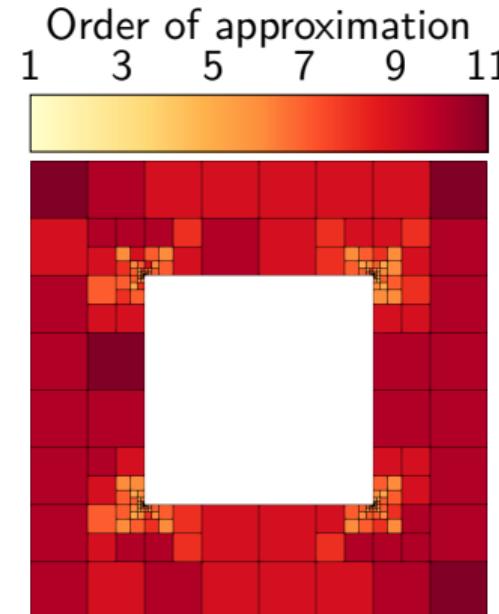
(a) Solution to the direct problem.

(b) Solution to the adjoint problem.

Illustrating the Goal-Oriented hp -Adaptive Strategy



(a) Polynomial orders p in the
x-direction



(b) Polynomial orders p in the
y-direction

Figure: Final hp -adapted meshes after the adaptivity

Goal-Oriented coarsening strategy

Main Achievements

Main Achievements

Peer-Reviewed Publications

-  F. V. Caro, V. Darrigrand, J. Alvarez-Aramberri, and D. Pardo. "A Multi-Adaptive-Goal-Oriented Strategy to Generate Massive Databases of Parametric PDEs," To be submitted to *Computer Methods in Applied Mechanics and Engineering*, December 2023.
-  F. V. Caro, V. Darrigrand, J. Alvarez-Aramberri, E. Alberdi, and D. Pardo. "A Painless Multi-Level Automatic Goal-Oriented hp -Adaptive Coarsening Strategy for Elliptic and Non-Elliptic Problems," *Computer Methods in Applied Mechanics and Engineering*, vol. 401, 115641, 2022. <https://doi.org/10.1016/j.cma.2022.115641>
-  F. V. Caro, V. Darrigrand, J. Alvarez-Aramberri, E. A. Celaya, and D. Pardo. "1D Painless Multi-Level Automatic Goal-Oriented h and p Adaptive Strategies Using a Pseudo-Dual Operator," In *Computational Science – ICCS 2022*, pp. 347–357, 2022.
https://doi.org/10.1007/978-3-031-08754-7_43

Main Achievements

Conference Talks

- [1] F. V. Caro, V. Darrigrand, J. Alvarez-Aramberri, and D. Pardo.
Databases for Deep Learning Inversion Using A Goal-Oriented hp-Adaptive Strategy.
XI International Conference on Adaptive Modeling and Simulation,
Gothenburg, Sweden, June 19-21, 2023.
- [2] F. V. Caro, V. Darrigrand, J. Alvarez-Aramberri, E. Alberdi, and D. Pardo.
*A Painless Automatic hp-Adaptive Coarsening Strategy For Non-SPD problems:
A Goal-Oriented Approach.* 15th World Congress on Computational Mechanics
& 8th Asian Pacific Congress on Computational Mechanics,
Yokohama, Japan, July 31 - August 5, 2022.
- [3] F. V. Caro, V. Darrigrand, J. Alvarez-Aramberri, E. Alberdi, and D. Pardo.
*1D Painless Multi-Level Automatic Goal-Oriented h and p Adaptive Strategies using
a Pseudo-Dual Operator.* 22nd International Conference on Computational Science,
London, United Kingdom, June 21-23, 2022.

Main Achievements

Conference Talks

- [4] F. V. Caro, V. Darrigrand, J. Alvarez-Aramberri, E. Alberdi, and D. Pardo.
Goal-Oriented hp-Adaptive Finite Element Methods: A Painless Multilevel Automatic Coarsening Strategy For Non-SPD Problems. 8th European Congress on Computational Methods in Applied Sciences and Engineering, Oslo, Norway, June 5-9, 2022.
- [5] F. V. Caro, V. Darrigrand, J. Alvarez-Aramberri, E. Alberdi, and D. Pardo.
A Painless Goal-Oriented hp-Adaptive Strategy for Indefinite Problems.
16th U.S. National Congress on Computational Mechanics,
Chicago, U.S.A, July 25-29, 2021.
- [6] F. V. Caro, V. Darrigrand, J. Alvarez-Aramberri, E. Alberdi, and D. Pardo.
Goal-Oriented hp-Adaptive Finite Element Methods: A Painless Multi-level Automatic Coarsening Strategy. 10th International Conference on Adaptive Modeling and Simulation, Gothenburg, Sweden, June 21-23, 2021.

Main Achievements

Research Stays

FEB. 2023 – MAR. 2023
(2 months) University of Science and Technology (AGH),
Krakow (Poland).
Supervisor: Maciej Paszynski.

SEP. 2021 – Nov. 2021
(2 months) CNRS-IRIT-ENSEEIHT (N7),
Toulouse (France).

Supervisor: Vincent Darrigrand.

Nov. 2020 – DEC. 2020
(1 month) CNRS-IRIT-ENSEEIHT (N7),
Toulouse (France).

Supervisor: Vincent Darrigrand.

Main Achievements

Bilbao



Toulouse



Kraków



Conclusions and Future Work

Conclusions

- We employ hierarchical basis functions that effectively address the challenge of *hanging nodes*.
- We have developed **simple-to-implement** h - and p -GOA strategies that use an unconventional symmetric and positive definite bilinear form for possibly non-elliptic goal-oriented problems.
- We have expanded upon a painless automatic hp strategy, initially developed for energy-norm adaptivity, to both non-elliptic and goal-oriented problems.
- We have extended the applicability of a coarsening strategy to encompass parametric PDEs.

Future Work

- Extend algorithms to address multi-physics problems, notably $H(\text{curl})$ and $H(\text{div})$.
- Validate the efficacy of our algorithms in real-world scenarios such as Magnetotellurics, Controlled Sources, and Logging While Drilling.
- Analyze the impact of the nature and distribution of various random samples on DL inversion for optimization.
- Enhance parallelization and factorization techniques to reduce computational resource requirements in future applications.