Easy-to-implement *hp*-adaptivity for non-elliptic goal-oriented problems

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Outline

1. Goal-Oriented *hp*-adaptivity for non-parametric PDEs
   - Why *hp*-adaptivity?
   - Goal-Oriented coarsening strategy
   - 1D Numerical results for Goal-Oriented *h*- and *p*-adaptivity
   - 2D Numerical results for *hp*-adaptivity
   - 3D Numerical results for *hp*-adaptivity
2. Goal-Oriented *hp*-adaptivity for parametric PDEs.
   - Database generation for DL inversion
3. Main Achievements
4. Conclusions and Future Work

Goal-Oriented *hp*-adaptivity for non-parametric PDEs

Goal-Oriented coarsening strategy

1D Numerical results for Goal-Oriented h- and p-adaptivity

2D Numerical results for *hp*-adaptivity

2D Numerical results for *hp*-adaptivity

Singular Poisson example

Find u satisfying:

$$-\Delta u = \mathbb{1}_{\Omega_f} \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{2}$$

Domain Definitions:

- $\Omega_f = \left(\frac{1}{4}, \frac{1}{2}\right)^2 \subset \Omega$,
- $\Omega_l = \left(\frac{1}{2}, \frac{3}{4}\right)^2 \subset \Omega$,
- $a(\cdot,\cdot) := \langle \nabla \cdot \,, \nabla \cdot \rangle_{L^2(\Omega)}$.

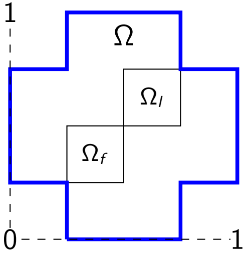
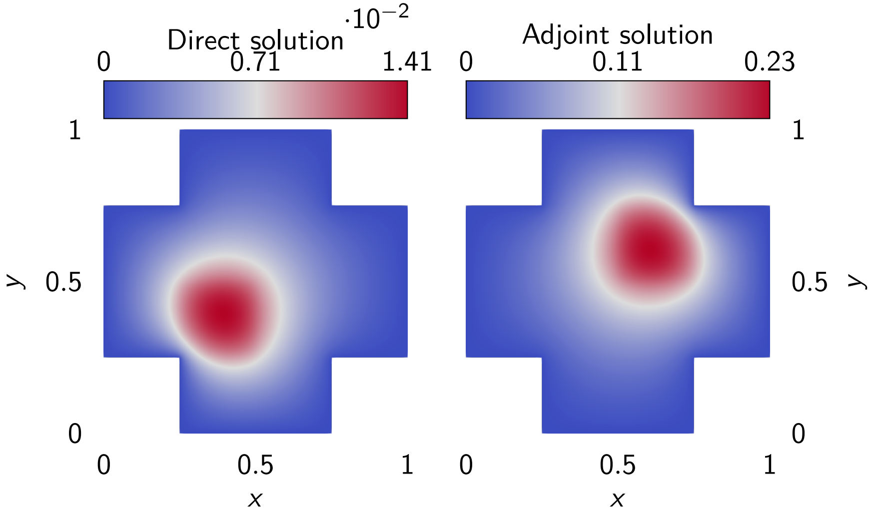


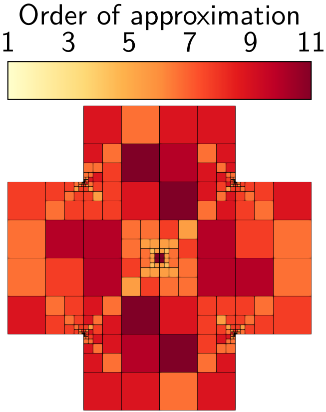
Figure: Domain
$\Omega = \left((0,1) \times \left(\frac{1}{4}, \frac{3}{4}\right)\right) \cup \left(\left(\frac{1}{4}, \frac{3}{4}\right) \times (0,1)\right)$.

Singular Poisson example

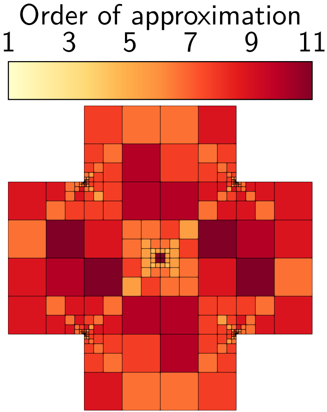


(a) Solution to the direct problem. **(b)** Solution to the adjoint problem.

Singular Poisson example



(a) Final *hp*-adapted mesh with polynomial orders in the x-direction.



(b) Final *hp*-adapted mesh with polynomial orders in the y-direction.

Singular Poisson Example

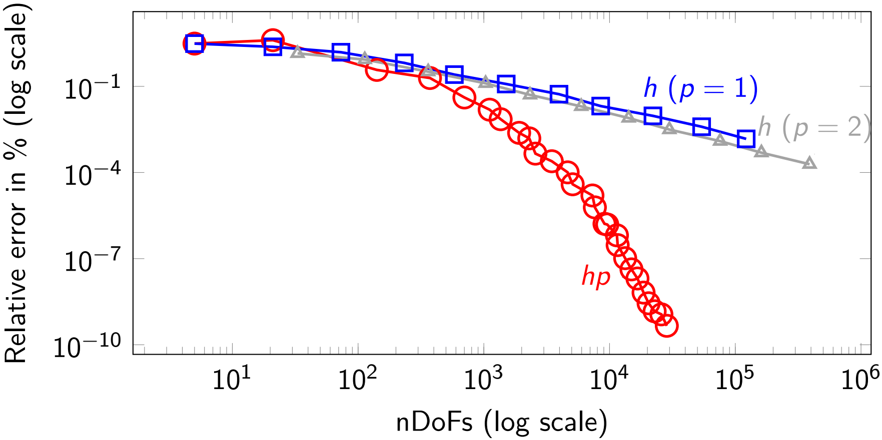


Figure: Evolution of $e_{\mathrm{rel}}^{\mathrm{QoI}}$ in the adaptive process.

Wave propagation problem

Find u satisfying:

$$-\Delta u - k^2 u = \mathbb{1}_{\Omega_f} \quad \text{in } \Omega, \tag{3}$$

$$u = 0 \quad \text{on } \Gamma_D, \tag{4}$$

$$\nabla u \cdot \vec{n} = 0 \quad \text{on } \Gamma_N. \tag{5}$$

Domain Definitions:

- $\Omega_f = \left(0, \frac{1}{4}\right)^2 \subset \Omega$,
- $k = (8 \cdot 2\pi, 2\pi)$,
- $\Omega_I = \left(\frac{3}{4}, 1\right)^2 \subset \Omega$,
- $a(\cdot,\cdot) := \left|\langle \nabla\cdot, \nabla\cdot \rangle_{L^2(\Omega)}\right| + |k^2| \left|\langle \cdot, \cdot \rangle_{L^2(\Omega)}\right|$.

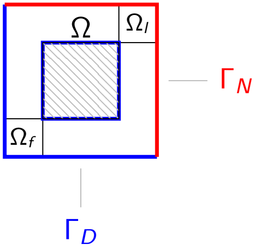
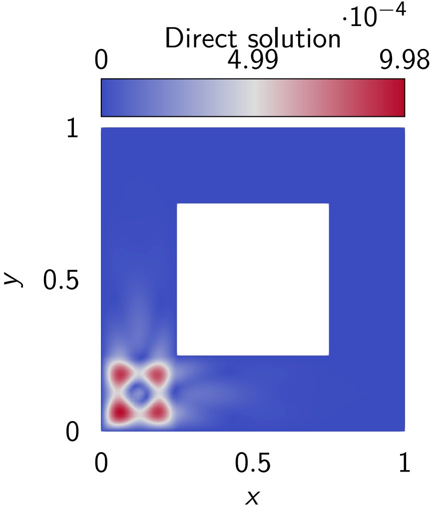
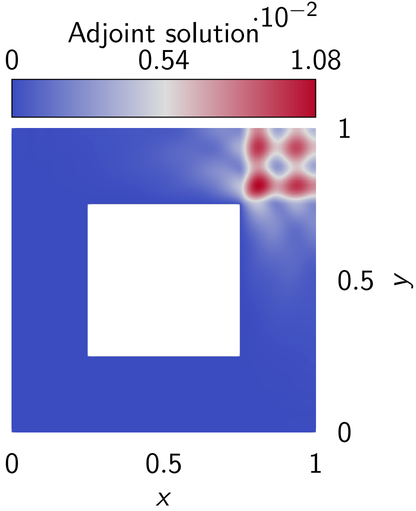


Figure: Domain $\Omega = (0,1)^2 \setminus \left(\frac{1}{4}, \frac{3}{4}\right)^2 \subset \mathbb{R}^2$.

Wave propagation problem

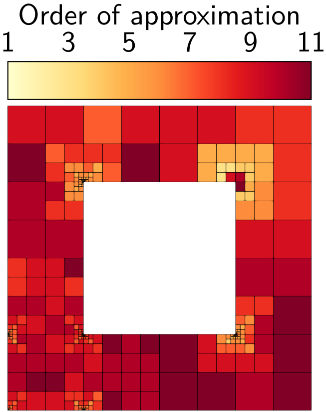


(a) Solution to the direct problem.

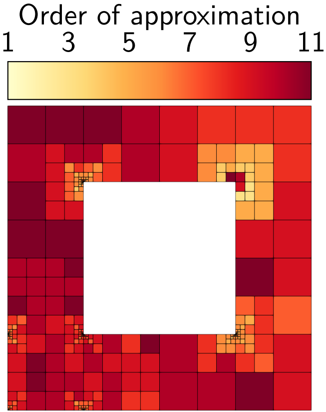


(b) Solution to the adjoint problem.

Energy-norm adaptivity



(a) Final *hp*-adapted mesh with polynomial orders in the x-direction.



(b) Final *hp*-adapted mesh with polynomial orders in the y-direction.

Energy-norm adaptivity

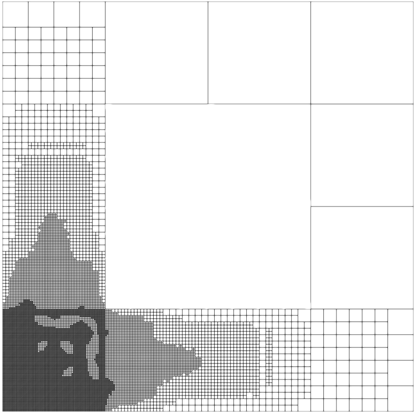
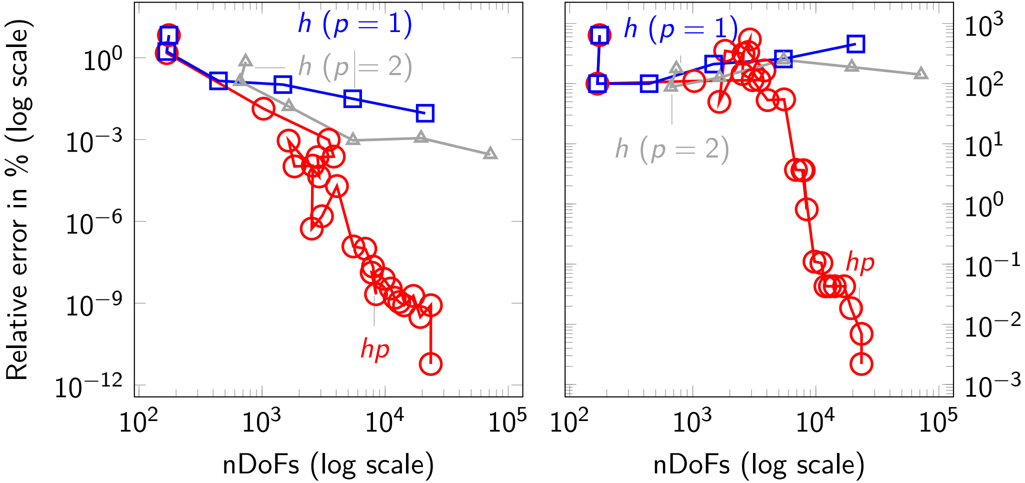


Figure: Final h-adapted mesh for $p = 1$.

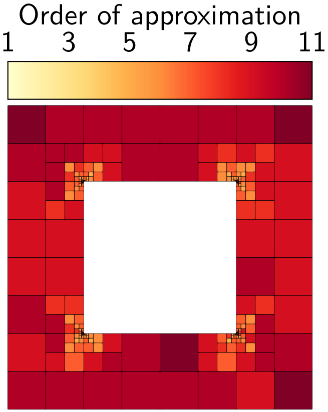
Energy-norm adaptivity



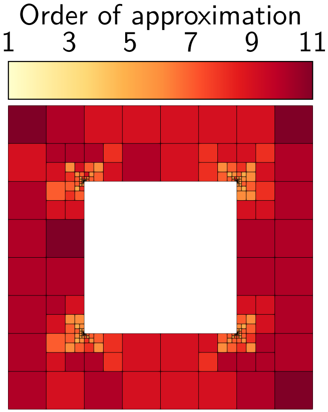
(a) Evolution of $\tilde{e}_{\mathrm{rel}}^{\mathrm{energy}}$ in the process.

(b) Evolution of $e_{\mathrm{rel}}^{\mathrm{QoI}}$ in the process.

Goal-Oriented adaptivity



(a) Final *hp*-adapted mesh with polynomial orders in the x-direction.



(b) Final *hp*-adapted mesh with polynomial orders in the y-direction.

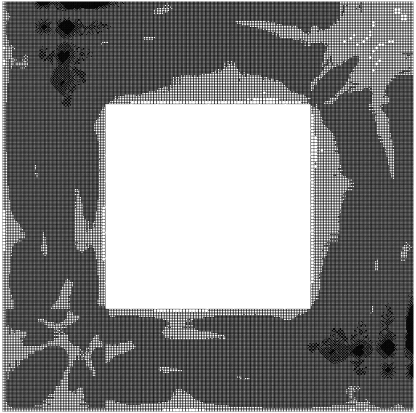
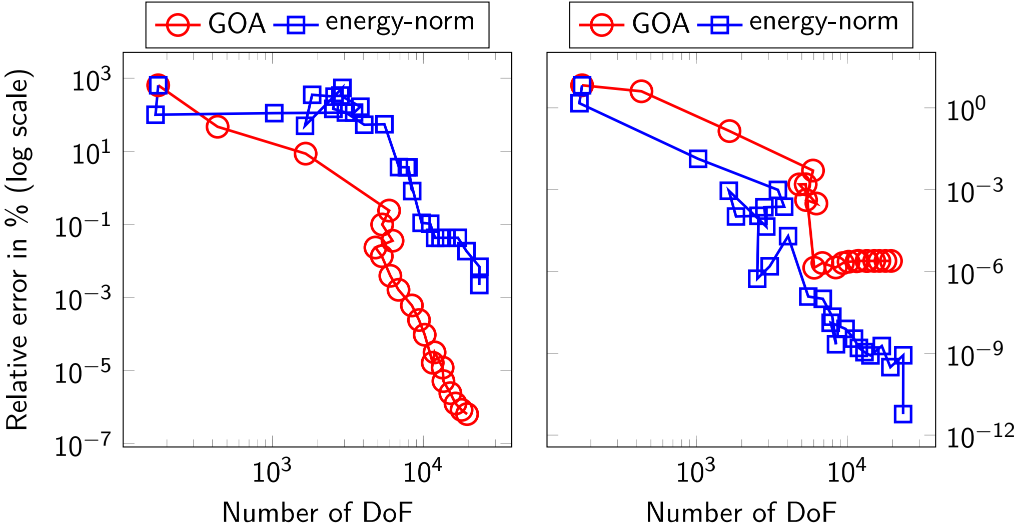


Figure: Final h-adapted mesh for $p = 1$.

Energy-norm and Goal-Oriented *hp*-adaptive strategy



(a) Evolution of goal-oriented adaptivity. **(b)** Evolution of energy-norm adaptivity.

Convection-dominated diffusion: example 1

Find u satisfying:

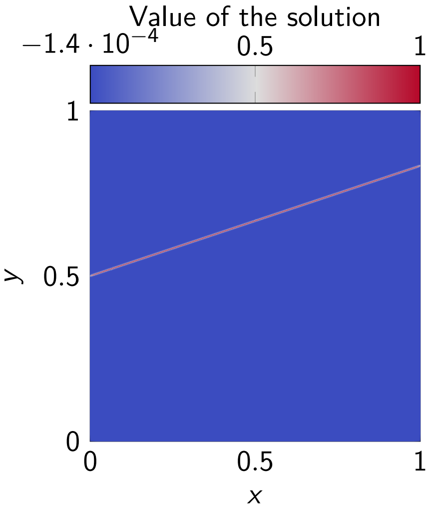
$$-\varepsilon\Delta u + \sigma \cdot \nabla u = f \text{ in } \Omega, \tag{6}$$

$$u = 0 \text{ on } \partial\Omega. \tag{7}$$

Definitions:

- $\Omega = (0,1)^2 \subset \mathbb{R}^2$,
- $\sigma = (3,1)^{\mathsf{T}}$,
- $\varepsilon = 10^{-3}$,
- $u\,(x,y) = e^{\frac{\varepsilon}{x(x-1)}} \cosh\left(500(\frac{1}{2} + \sigma^{-1}\,(x,-y)\,)\right)^{-2}$,
- $a\,(\cdot,\cdot) := (\varepsilon + C)\,\langle\nabla\cdot\,,\nabla\cdot\rangle_{L^2(\Omega)}$.

Convection-dominated diffusion: example 1



(a) Solution of the example 1.

Main Achievements

Main Achievements

Peer-Reviewed Publications

- F. V. Caro, V. Darrigrand, J. Alvarez-Aramberri, and D. Pardo. "A Multi-Adaptive-Goal-Oriented Strategy to Generate Massive Databases of Parametric PDEs," To be submitted to *Computer Methods in Applied Mechanics and Engineering*, December 2023.
- F. V. Caro, V. Darrigrand, J. Alvarez-Aramberri, E. Alberdi, and D. Pardo. "A Painless Multi-Level Automatic Goal-Oriented *hp*-Adaptive Coarsening Strategy for Elliptic and Non-Elliptic Problems," *Computer Methods in Applied Mechanics and Engineering*, vol. 401, 115641, 2022. `https://doi.org/10.1016/j.cma.2022.115641`
- F. V. Caro, V. Darrigrand, J. Alvarez-Aramberri, E. A. Celaya, and D. Pardo. "1D Painless Multi-Level Automatic Goal-Oriented *h* and *p* Adaptive Strategies Using a Pseudo-Dual Operator," In *Computational Science – ICCS 2022*, pp. 347–357, 2022. `https://doi.org/10.1007/978-3-031-08754-7_43`

Main Achievements

Conference Talks

[1] F. V. Caro, V. Darrigrand, J. Alvarez-Aramberri, and D. Pardo.
Databases for Deep Learning Inversion Using A Goal-Oriented hp-Adaptive Strategy.
XI International Conference on Adaptive Modeling and Simulation,
Gothenburg, Sweden, June 19-21, 2023.

[2] F. V. Caro, V. Darrigrand, J. Alvarez-Aramberri, E. Alberdi, and D. Pardo.
A Painless Automatic hp-Adaptive Coarsening Strategy For Non-SPD problems: A Goal-Oriented Approach. 15th World Congress on Computational Mechanics & 8th Asian Pacific Congress on Computational Mechanics,
Yokohama, Japan, July 31 - August 5, 2022.

[3] F. V. Caro, V. Darrigrand, J. Alvarez-Aramberri, E. Alberdi, and D. Pardo.
1D Painless Multi-Level Automatic Goal-Oriented h and p Adaptive Strategies using a Pseudo-Dual Operator. 22nd International Conference on Computational Science,
London, United Kingdom, June 21-23, 2022.

Conference Talks

[4] <u>F. V. Caro</u>, V. Darrigrand, J. Alvarez-Aramberri, E. Alberdi, and D. Pardo. *Goal-Oriented hp-Adaptive Finite Element Methods: A Painless Multilevel Automatic Coarsening Strategy For Non-SPD Problems.* 8th European Congress on Computational Methods in Applied Sciences and Engineering, Oslo, Norway, June 5-9, 2022.

[5] <u>F. V. Caro</u>, V. Darrigrand, J. Alvarez-Aramberri, E. Alberdi, and D. Pardo. *A Painless Goal-Oriented hp-Adaptive Strategy for Indefinite Problems.* 16th U.S. National Congress on Computational Mechanics, Chicago, U.S.A, July 25-29, 2021.

[6] <u>F. V. Caro</u>, V. Darrigrand, J. Alvarez-Aramberri, E. Alberdi, and D. Pardo. *Goal-Oriented hp-Adaptive Finite Element Methods: A Painless Multi-level Automatic Coarsening Strategy.* 10th International Conference on Adaptive Modeling and Simulation, Gothenburg, Sweden, June 21-23, 2021.

Main Achievements

Research Stays

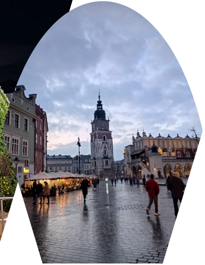
FEB. 2023 – MAR. 2023 (2 months)	University of Science and Technology (AGH), Krakow (Poland). **Supervisor:** Maciej Paszynski.
SEP. 2021 – NOV. 2021 (2 months)	CNRS-IRIT-ENSEEIHT (N7), Toulouse (France). **Supervisor:** Vincent Darrigrand.
NOV. 2020 – DEC. 2020 (1 month)	CNRS-IRIT-ENSEEIHT (N7), Toulouse (France). **Supervisor:** Vincent Darrigrand.

Main Achievements

Bilbao

Kraków

Toulouse



Conclusions and Future Work

Conclusions

- We employ hierarchical basis functions that effectively address the challenge of *hanging nodes*.
- We have developed **simple-to-implement** *h*- and *p*-GOA strategies that use an unconventional symmetric and positive definite bilinear form for possibly non-elliptic goal-oriented problems.
- We have expanded upon a painless automatic *hp* strategy, initially developed for energy-norm adaptivity, to both non-elliptic and goal-oriented problems.
- We have extended the applicability of a coarsening strategy to encompass parametric PDEs.

Future Work

- Extend algorithms to address multi-physics problems, notably $H(\text{curl})$ and $H(\text{div})$.
- Validate the efficacy of our algorithms in real-world scenarios such as Magnetotellurics, Controlled Sources, and Logging While Drilling.
- Analyze the impact of the nature and distribution of various random samples on DL inversion for optimization.
- Enhance parallelization and factorization techniques to reduce computational resource requirements in future applications.