A painless multi-level automatic goal-oriented hp-adaptive coarsening strategy for elliptic and non-elliptic problems

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Data structures

We define our *projection operator* $\Pi_{\mathcal{F}}^{\mathcal{S}}\colon \mathbb{H}_{\mathcal{F}} \longrightarrow \mathbb{H}_{\mathcal{S}}$ as

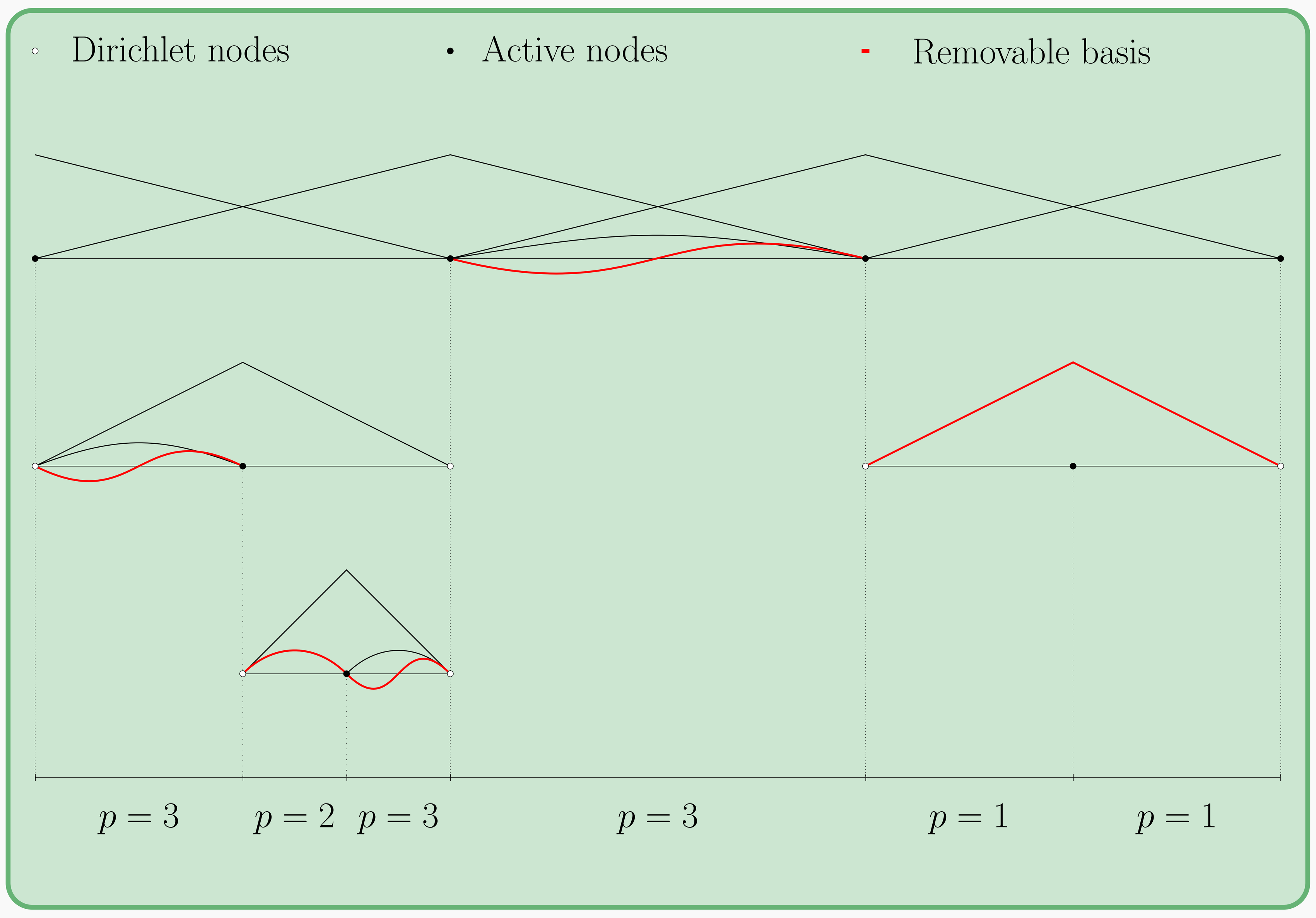
$$\Pi_{\mathcal{F}}^{\mathcal{S}} u_{\mathcal{F}} \coloneqq \sum_{\phi_i \in \mathcal{S}} u_i \phi_i,$$

for a given subset of basis functions $\mathcal{S} \subset \mathcal{F}$.

We express any $u_{\mathcal{F}} \in \mathbb{H}_{\mathcal{F}}$, as

$$u_{\mathcal{F}} = \Pi_{\mathcal{F}}^{\mathcal{E}_K} u_{\mathcal{F}} + \Pi_{\mathcal{F}}^{\mathcal{R}_K} u_{\mathcal{F}},$$

where $\mathcal{E}_K \coloneqq \mathcal{F} \backslash \mathcal{R}_K$ is the subset of *essential* basis functions and $\mathcal{R}_K$ the set of *removable* basis functions associated to K. We illustrate of a 1D multi-level hp-grid with hierarchical basis functions.



Problem settings

We consider the abstract variational formulation and its discrete version:

Find $u \in \mathbb{H}$ and $u_{\mathcal{F}} \in \mathbb{H}_{\mathcal{F}}$ such that

$$b(u,\phi) = f(\phi), \quad \forall \phi \in \mathbb{H}; \qquad b(u_{\mathcal{F}},\phi_{\mathcal{F}}) = f(\phi_{\mathcal{F}}), \quad \forall \phi_{\mathcal{F}} \in \mathbb{H}_{\mathcal{F}}.$$

The objective is to control

$$\left| l(u_{\mathcal{F}}) - l(u_{\mathcal{E}_K}) \right|, \quad \forall K \in \mathcal{T},$$

where $l : \mathbb{H} \longrightarrow \mathbb{R}$ is a linear continuous form.

We introduce the following adjoint problem and its discrete counterpart as follows:

Find $v \in \mathbb{H}$ and $v_{\mathcal{F}} \in \mathbb{H}_{\mathcal{F}}$ such that

$$b(\phi, v) = l(\phi), \quad \forall \phi \in \mathbb{H}; \qquad b(\phi_{\mathcal{F}}, v_{\mathcal{F}}) = l(\phi_{\mathcal{F}}), \quad \forall \phi_{\mathcal{F}} \in \mathbb{H}_{\mathcal{F}}.$$

Error indicators

We define the element-wise indicators for energy-norm based elliptic problems as

$$\eta_K \coloneqq \left\| \Pi_{\mathcal{F}}^{\mathcal{R}_K} u_{\mathcal{F}} \right\|_e^2, \quad \forall K \in \mathcal{T}.$$

We define the element-wise indicators for goal-oriented problems as

$$\eta_K \coloneqq \left| a(\Pi_{\mathcal{F}}^{\mathcal{R}_K} u_{\mathcal{F}}, \Pi_{\mathcal{F}}^{\mathcal{R}_K} v_{\mathcal{F}}) \right|, \quad \forall K \in \mathcal{T}.$$

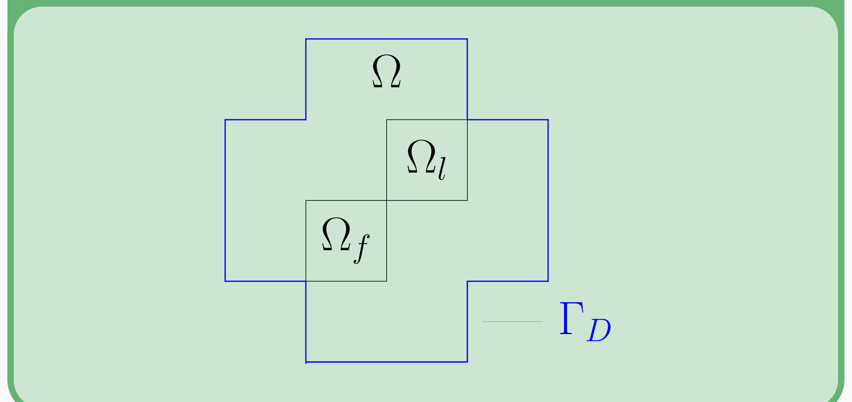
An indefinite problem

Find u such that,

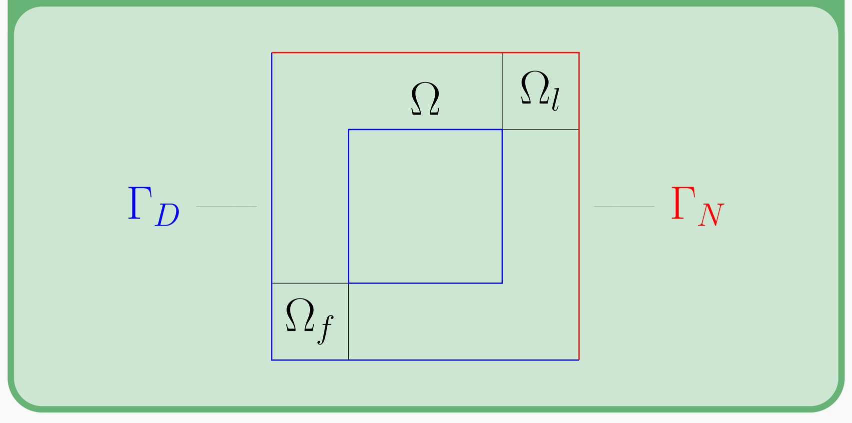
$$\begin{aligned} -\nabla\cdot(\sigma\nabla u) - k^2 u &= \left\langle \mathbb{1}_{(0,\frac{1}{4})^2}, \cdot \right\rangle_{L^2(\Omega)} && \text{in } \Omega \subset \mathbb{R}^2,\\ u &= 0 && \text{on } \Gamma_D,\\ \nabla u \cdot \vec{n} &= 0 && \text{on } \Gamma_N. \end{aligned}$$

$$l(\cdot) = \left\langle \mathbb{1}_{(\frac{3}{4},1)^2}, \cdot \right\rangle_{L^2(\Omega)}; \qquad a(\cdot,\cdot) = \sum_K \left| \langle \nabla\cdot, \nabla\cdot \rangle_{L^2(K)} + \left|k^2\right| \langle \cdot, \cdot \rangle_{L^2(K)} \right|.$$

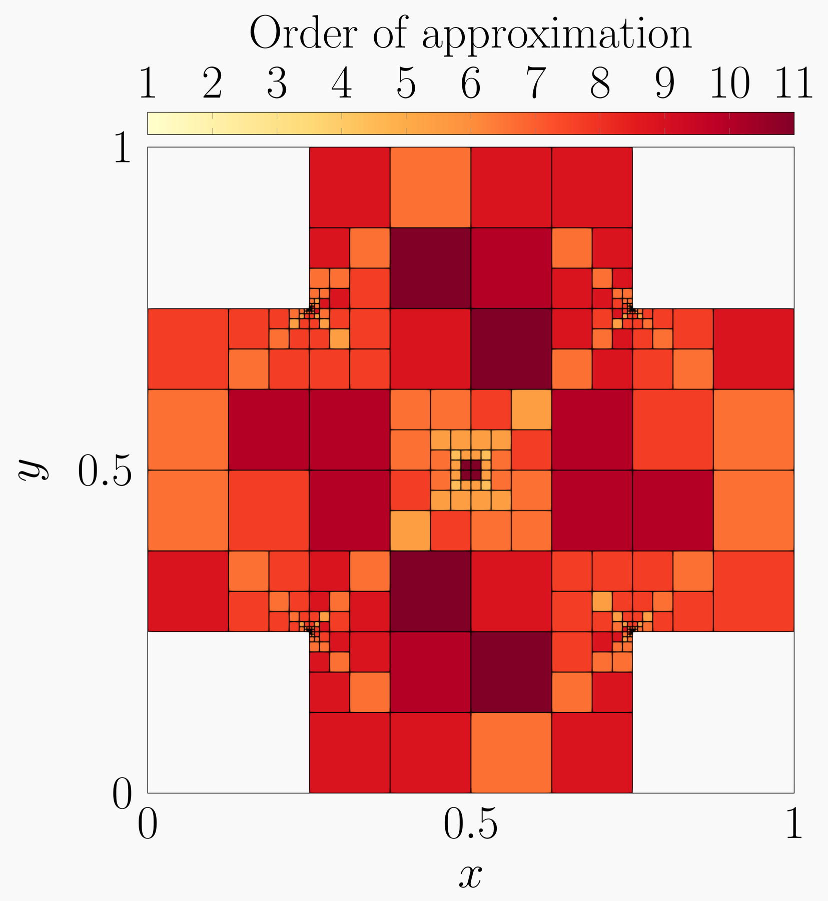
$k = (0,0)$

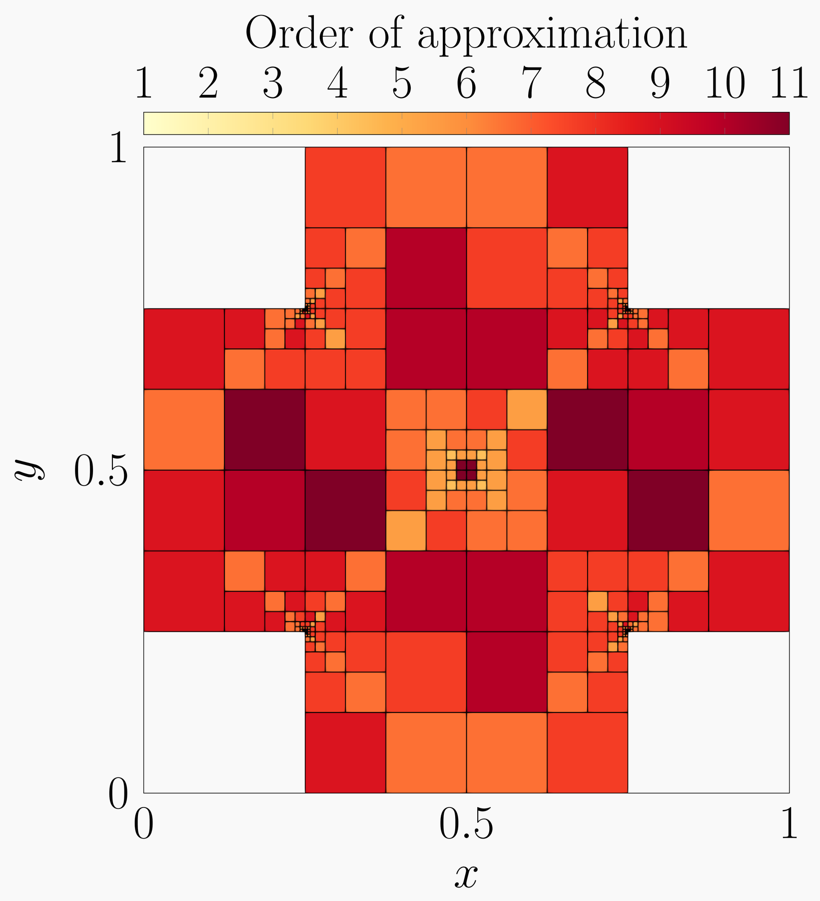


$k = (4\cdot 2\pi, 2\pi)$



Final hp-adapted meshes for Poisson example





Final hp-adapted meshes for Helmholtz example