Goal-oriented adaptivity with multiple dual problems

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Introduction of the Method

We consider the abstract variational formulation and its discrete version (Galerkin approximation):

Find $u^* \in \mathbb{H}$ and $u_h^* \in \mathbb{H}_h$ such that

$$b(u^*, v) = f(v), \quad \forall v \in \mathbb{H}; \qquad b(u_h^*, v) = f(v), \quad \forall v \in \mathbb{H}_h.$$

The goal is to estimate

$$l(u^* - u_h^*) = l(e_h)$$

where $l \in \mathcal{L}(\mathbb{H})$ is the Quantity of Interest (QoI).

The classical approach uses the dual problem and its discrete version:

Find $v^* \in \mathbb{H}$ and $v_h^* \in \mathbb{H}_h$ such that

$$b(u, v^*) = l(u), \quad \forall u \in \mathbb{H}; \qquad b(u, v_h^*) = l(u), \quad \forall u \in \mathbb{H}_h.$$

Our method introduces an arbitrary bilinear form $\hat{b}$ to define an alternative dual problem:

Find $\hat{v}^* \in \mathbb{H}$ such that

$$\hat{b}(u, \hat{v}^*) = l(u), \quad \forall u \in \mathbb{H}.$$

Which method provides the best a posteriori estimate?

Classical Estimate

We have the following error representations:

Find $e_h \in \mathbb{H}$, $\varepsilon_h \in \mathbb{H}$ such that

$$b(e_h, v) = f(v) - b(u_h, v), \quad \forall v \in \mathbb{H},$$
$$b(u, \varepsilon_h) = l(u) - b(u, v_h), \quad \forall u \in \mathbb{H}.$$

A posteriori estimate

$$|l(e_h)| \leqslant \left|\left(b_K(e_h, \varepsilon_h)\right)_K\right|_1.$$

Alternative Estimate

We use these alternative error representations:

Find $e_h \in \mathbb{H}$, $\hat{\varepsilon}_h \in \mathbb{H}$ such that

$$b(e_h, v) = f(v) - b(u_h, v), \quad \forall v \in \mathbb{H},$$
$$\hat{b}(u, \hat{\varepsilon}_h) = l(u) - b(u, v_h), \quad \forall u \in \mathbb{H}.$$

A posteriori estimate

$$|l(e_h)| \leqslant \left|\left(\hat{b}_K(e_h, \hat{\varepsilon}_h)\right)_K\right|_1.$$

Toy problem using Finite Elements Method

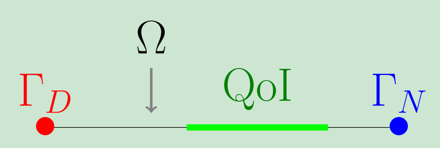
Find $u^* \in H^1(\Omega)$, $\Omega = (0,1)^d$, such that, for $k \in \mathbb{R}^+$,

$$\begin{cases} -\Delta u^* - (2\pi k)^2 u^* = g & \text{on } \Omega, \\ u^* = 0 & \text{on } \Gamma_D, \\ \partial_{\vec{n}} u^* = h & \text{on } \Gamma_N. \end{cases}$$

$$l(u) = \langle \mathbb{1}_{QoI}, u \rangle_{L^2(\Omega)}; \qquad \hat{b} = \langle \nabla \cdot, \nabla \cdot \rangle_{L^2(\Omega)}.$$

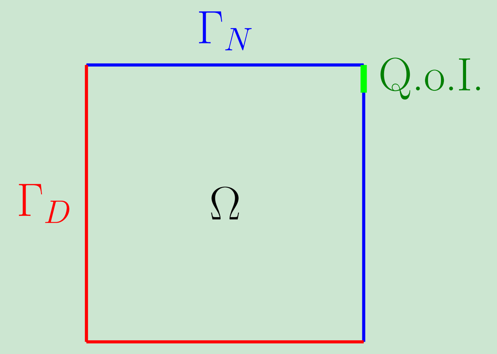
1D

$g \equiv 1; \quad h \equiv \frac{1}{2}.$



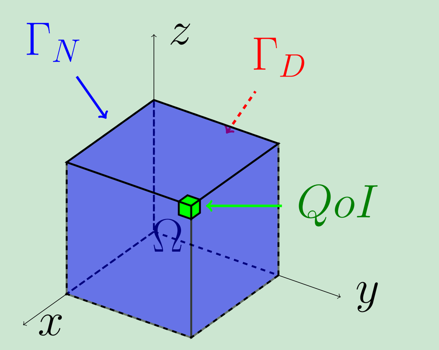
2D

$g \equiv 0; \quad h \equiv (2\pi k) \cdot \vec{n}.$



3D

$g \equiv 0; \quad h \equiv (2\pi k) \cdot \vec{n}.$



1D numerical results

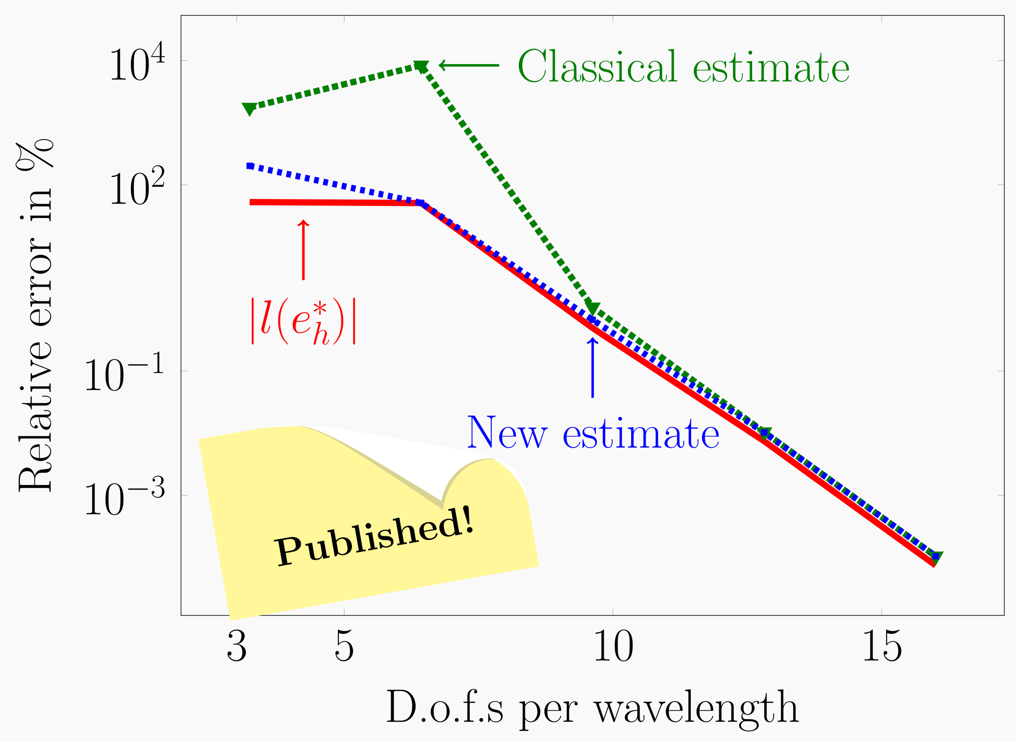


Fig. 1: Uniform p-refinements for $k \simeq 20$ wavelengths, $h(2k\pi) \gg 1$.

2D numerical results

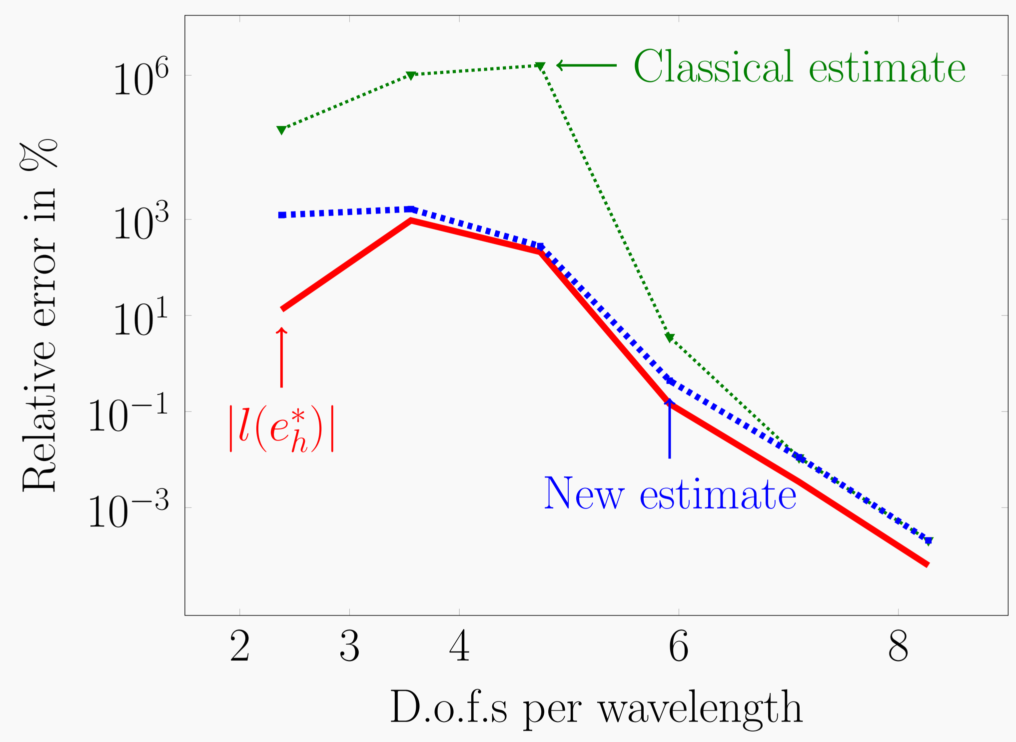


Fig. 2: Uniform p-refinements, for $k = 30$ wavelengths, $h(2k\pi) \gg 1$.

3D numerical results

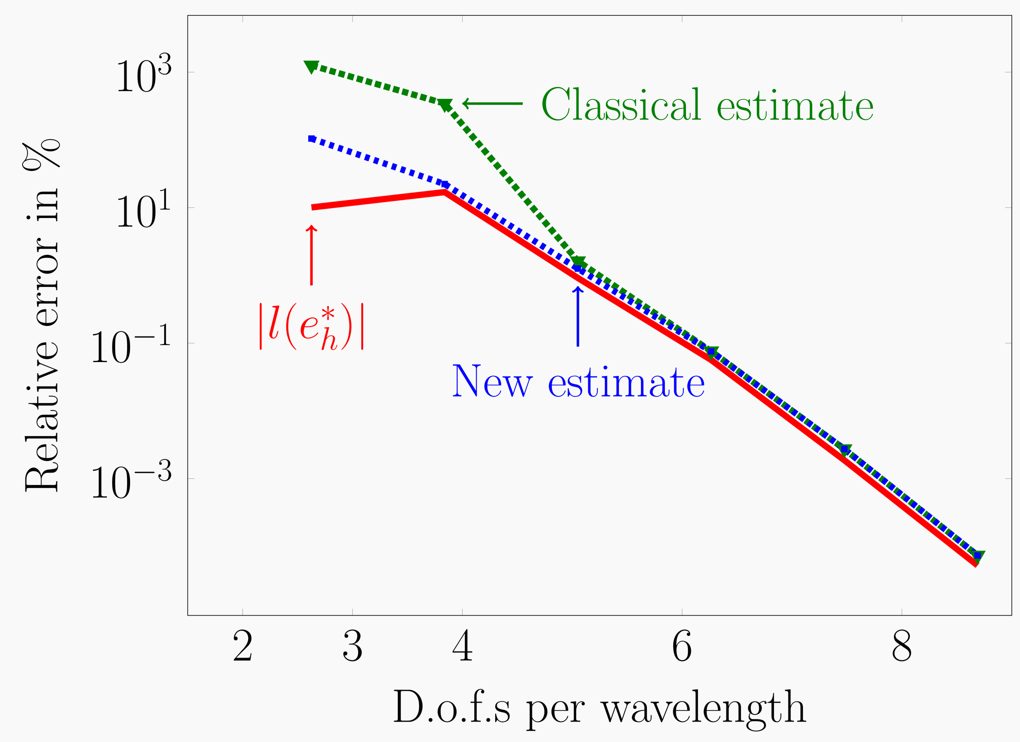


Fig. 3: Uniform p-refinements, for $k = 3$ wavelengths, $h(2k\pi) \gg 1$.

Formations

- Teaching (192h eq. TD in 2014-2015);
- Various formations on scientific computing (python, fortran, MPI);
- Languages certifications (DELE-C1 and CAE).

Work in progress and future work

- Multi-D Goal-oriented Adaptivity;
- Theoretical proof of if there exists a $\hat{b}$ such that the alternative estimate is sharper than the classical one;
- Widen the study to other kind of problems (eg. diffusion-convection problems);
- Application to non-linear quantities of interest.

Mobility

- Bilbao (18 months);
- Pau (12 months);
- Valparaiso (6 months).