A Painless Automatic hp-Adaptive Coarsening Strategy For Indefinite Problems: A Goal-Oriented Approach

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A painless hp-adaptive strategy

Let $\mathcal{F} = \{\phi_i\}_{i=1}^{n_\mathcal{F}}$ be a set of basis functions ϕ_i, $\mathbb{H}_\mathcal{F} \coloneqq \text{span}\{\phi_1, \dots, \phi_{n_\mathcal{F}}\}$, and $n_\mathcal{F} = \dim(\mathbb{H}_\mathcal{F})$. For a given bilinear continuous form b, we consider the abstract variational formulation and its discrete version:

Find $u \in \mathbb{H}$ and $u_\mathcal{F} \in \mathbb{H}_\mathcal{F}$ such that

$$b(u,\phi) = f(\phi), \quad \forall \phi \in \mathbb{H}; \qquad b(u_\mathcal{F},\phi_\mathcal{F}) = f(\phi_\mathcal{F}), \quad \forall \phi_\mathcal{F} \in \mathbb{H}_\mathcal{F},$$

where $\mathbb{H}$ is a Hilbert functional space and $\mathbb{H}_\mathcal{F}$ is a finite element discretization $\mathcal{T}$ of $\mathbb{H}$, such that $\mathbb{H}_\mathcal{F} \subset \mathbb{H}$.

For any element K, we denote by $\mathcal{R}_K$ the set of *removable* basis functions associated to K, by $|\mathcal{R}_K|$ its cardinality, and by $\mathbb{H}_{\mathcal{R}_K}$ its associated space.

We express any $u_\mathcal{F} \in \mathbb{H}_\mathcal{F}$, as

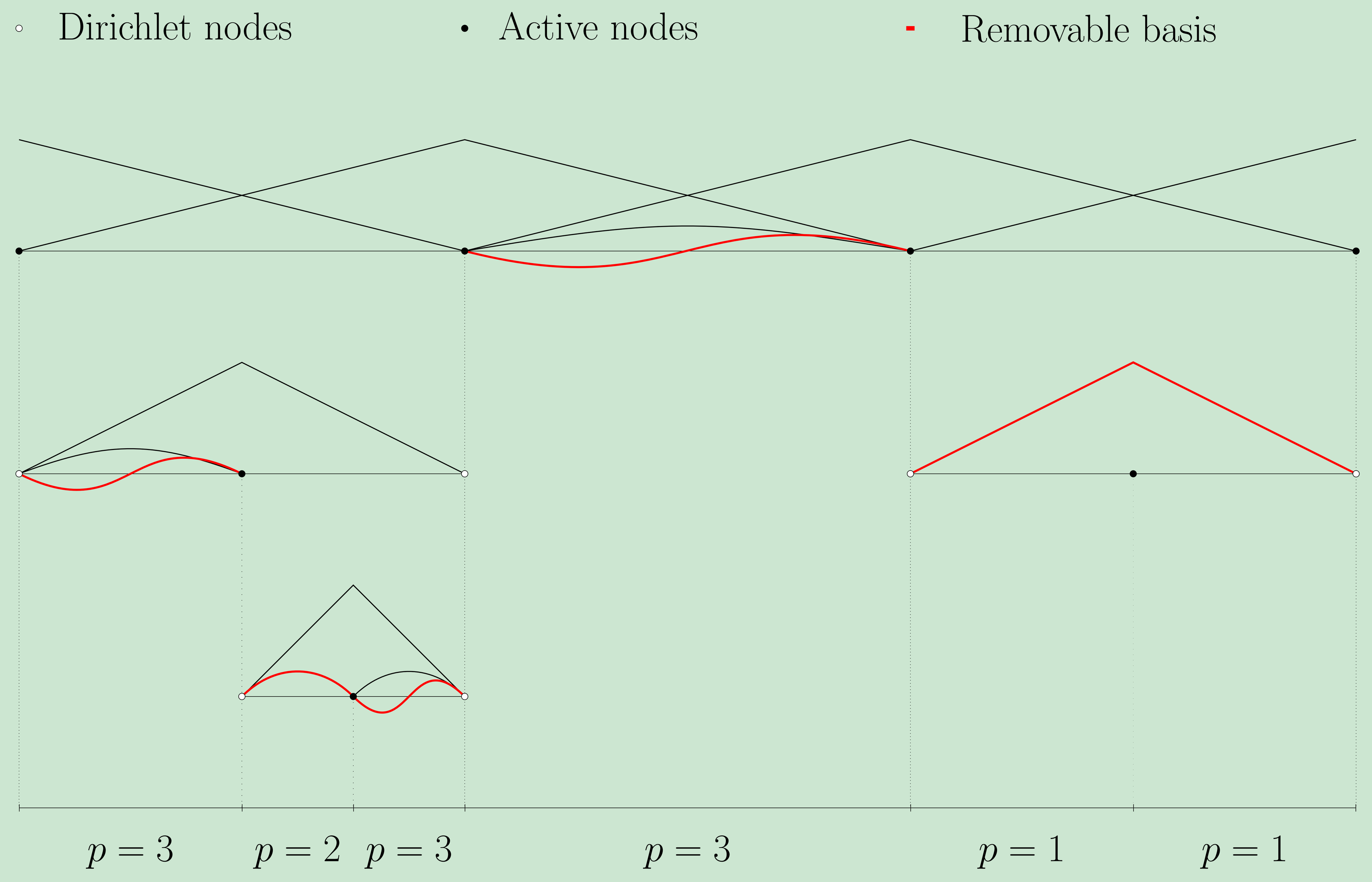
$$u_\mathcal{F} = \Pi_\mathcal{F}^{\mathcal{E}_K} u_\mathcal{F} + \Pi_\mathcal{F}^{\mathcal{R}_K} u_\mathcal{F},$$

where $\mathcal{E}_K \coloneqq \mathcal{F} \setminus \mathcal{R}_K$ is the subset of *essential* basis functions.

Since we consider a single mesh at a time, the solution $u_{\mathcal{E}_K}$ in $\mathcal{E}_K$ is never computed. Instead, we employ the projection of $u_\mathcal{F}$ into $\mathcal{E}_K$ to approximate it when necessary.

Data structures

We illustrate a 1D multi-level hp-grid with hierarchical basis functions proposed by Zander and collaborators.



Extension to goal-oriented adaptivity

We introduce the following adjoint problem and its discrete counterpart as follows:

Find $v \in \mathbb{H}$ and $v_\mathcal{F} \in \mathbb{H}_\mathcal{F}$ such that

$$b(\phi,v) = l(\phi), \quad \forall \phi \in \mathbb{H}; \qquad b(\phi_\mathcal{F},v_\mathcal{F}) = l(\phi_\mathcal{F}), \quad \forall \phi_\mathcal{F} \in \mathbb{H}_\mathcal{F},$$

where $v_\mathcal{F}$ stands for the Galerkin approximation of the solution v to the adjoint problem associated with the space $\mathbb{H}_\mathcal{F}$.

The objective is to control

$$\left| l(u_\mathcal{F}) - l(u_{\mathcal{E}_K}) \right|, \quad \forall K \in \mathcal{T},$$

where $l : \mathbb{H} \longrightarrow \mathbb{R}$ is a linear continuous form.

Error indicators

As a consequence of assuming that $\mathcal{E}_K$ is (quasi) b-orthogonal to $\mathcal{R}_K$ due to the (quasi)-orthogonality assumption of the basis functions, then

$$\left| l(u_\mathcal{F}) - l(u_{\mathcal{E}_K}) \right| \simeq \left| b(\Pi_\mathcal{F}^{\mathcal{R}_K} u_\mathcal{F}, \Pi_\mathcal{F}^{\mathcal{R}_K} v_\mathcal{F}) \right| \leqslant \left| a(\Pi_\mathcal{F}^{\mathcal{R}_K} u_\mathcal{F}, \Pi_\mathcal{F}^{\mathcal{R}_K} v_\mathcal{F}) \right|.$$

We define the element-wise indicators as

$$\eta_K \coloneqq \left| a(\Pi_\mathcal{F}^{\mathcal{R}_K} u_\mathcal{F}, \Pi_\mathcal{F}^{\mathcal{R}_K} v_\mathcal{F}) \right|, \quad \forall K \in \mathcal{T}.$$

Indefinite problem

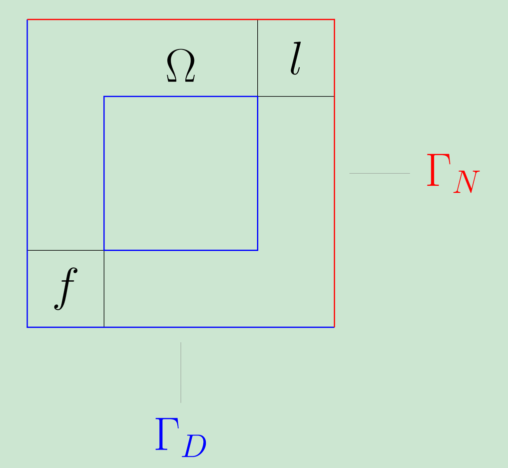
Find u such that,

$$\begin{aligned} -\nabla\cdot(\sigma\nabla u) - k^2 u &= \left\langle \mathbb{1}_{(0,\frac{1}{4})^d}, \cdot \right\rangle_{L^2(\Omega)} && \text{in } \Omega \subset \mathbb{R}^d,\\ u &= 0 && \text{on } \Gamma_D,\\ \nabla u \cdot \vec{n} &= 0 && \text{on } \Gamma_N. \end{aligned}$$

$$l(\cdot) = \left\langle \mathbb{1}_{(\frac{3}{4},1)^d}, \cdot \right\rangle_{L^2(\Omega)}; \qquad a(\cdot,\cdot) = \sum_K \left| \langle \nabla\cdot, \nabla\cdot \rangle_{L^2(K)} \right| + \left| k^2 \right| \left| \langle \cdot, \cdot \rangle_{L^2(K)} \right|.$$

2D

$$k = (8 \cdot 2\pi, 2\pi).$$

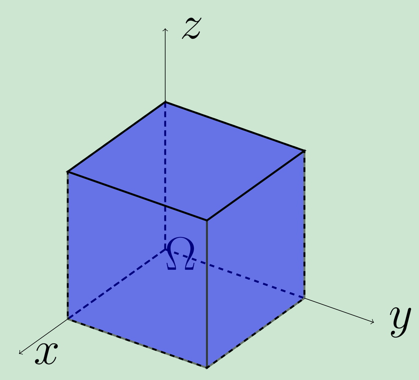


3D

$$k = (4 \cdot 2\pi, 2\pi).$$

$\Gamma_D =$ faces whose intersection is (0,0,0) .

$\Gamma_N =$ faces whose intersection is (1,1,1) .



2D numerical results

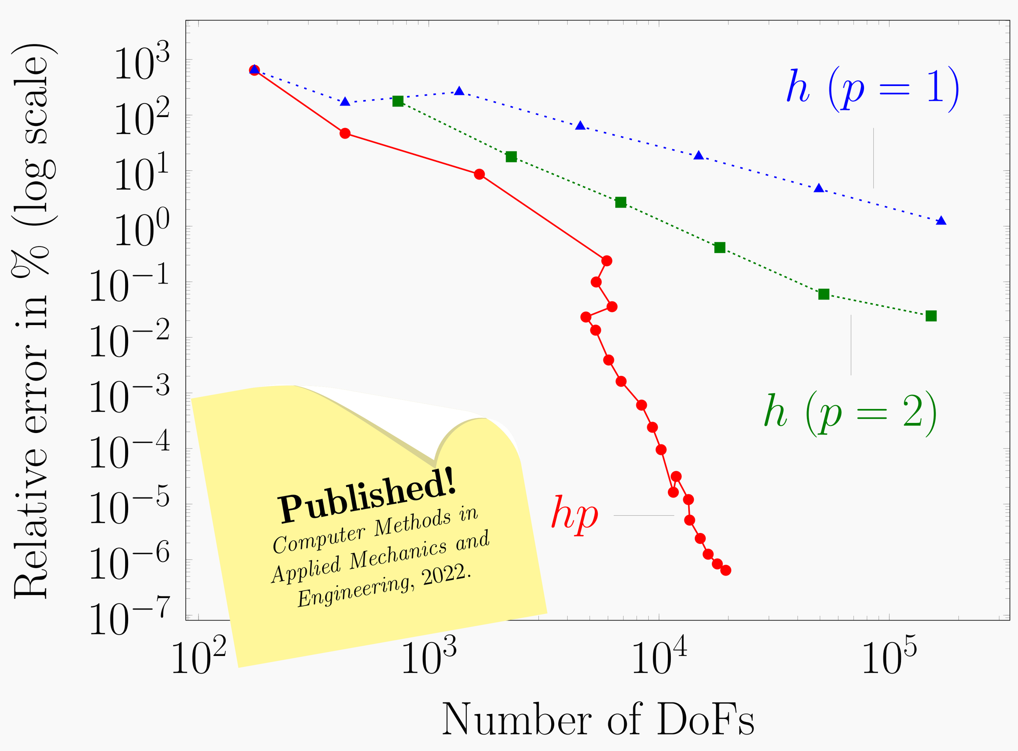


Fig. 1: Evolution of the error in the QoI.

3D numerical results

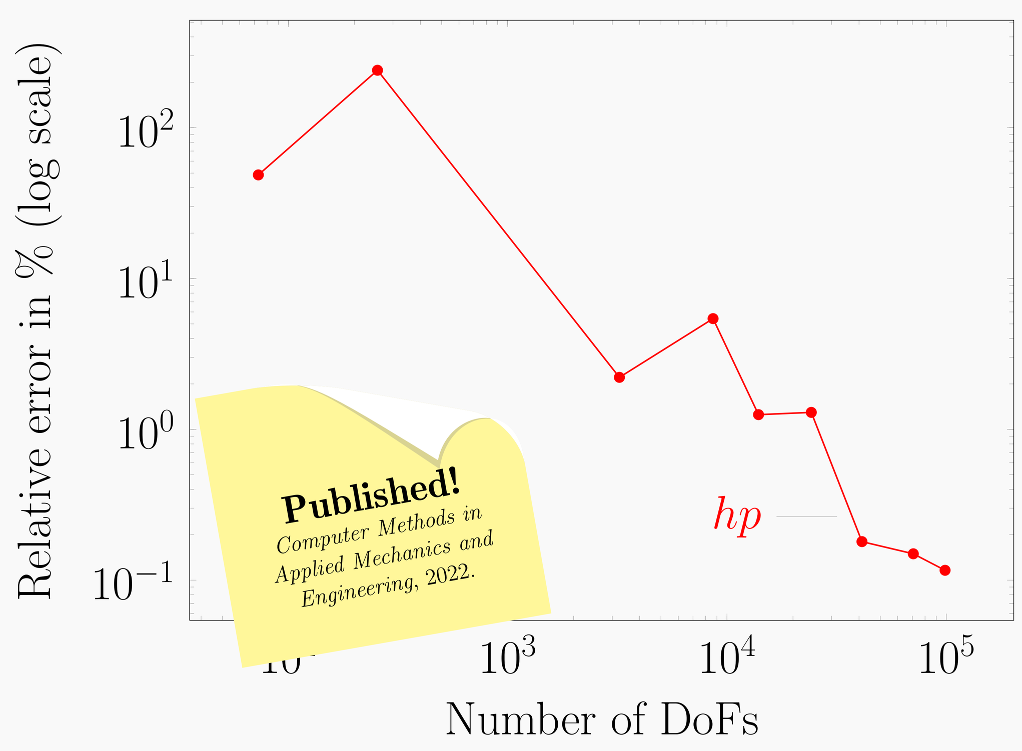


Fig. 2: Evolution of the error in the QoI.

Work in progress

- Multi-adaptive goal-oriented;
- Magnetotellurics;
- Controlled Source Electromagnetics;