Discussion 1A

Your Name

February 11, 2026

Contents

1 Introduction

Write your introduction here.

1.1 1 Natural Induction on Inequality

We will proceed with induction.
The base case $n = 0$ holds because $(1 + x)^0 = 1$ is greater than or equal to $1 + 0 \cdot x = 1$ We will now show that if this holds for n, that it holds for $n + 1$. Notice that

$$(1 + x)^{n+1} = (1 + x)^n \cdot (1 + x) \geq (1 + nx)(1 + x)$$

because $x > 0$

The RHS expands to

$$1 + x + nx + nx^2$$

Notice that

$$1 + x + nx + nx^2 = 1 + (n + 1)x + nx^2$$

and that since $n \in \mathbb{N}$ and $x > 0$ that $nx^2 \geq 0$ so we are done because we have that

$$(1 + x)^{n+1} \geq 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x$$

1.2 2 Make It Stronger

Because log is increasing on $\mathbb{Z}^+$, Proving $a_n \leq 3^{2^n}$ is analogous to proving

$$\log_3 a_n \leq 2^n$$

Let's create a sequence $b_1, b_2, b_3, ...$, defined as $b_1 = 0$, and

$$b_i = \log_3 a_i$$

Then, notice that the recurrence is equivalent to

$$a_{n+1} = 3a_n^2$$

After taking log base 3 of both sides, notice that we have

$$\log_3 a_{n+1} = 1 + 2 \log_3 a_n$$

or

$$b_{n+1} = 1 + 2b_n$$

Notice that we wish to prove that $\log_3 a_n \leq 2^n$ or $b_n \leq 2^n$
We have $b_0 = 0, b_1 = 1, b_2 = 3, b_3 = 7, ...$ Notice that $b_0 = 2^0 - 1$
Let's prove that $b_i = 2^i - 1$ by induction, and then we'd be done because $b_i = 2^i - 1 \leq 2^i$

So this recurrence can be written as

$$b_{n+1} = 1 + 2b_n = 1 + 2(2^i - 1) = 2^{i+1} - 1$$

and we are done.

1.3 3 Binary Numbers

Notice that $1 = 2^0 \cdot 1$

Thus, this property holds for $n = 1$ and $k = 1$

Let's show that if every integer from 1 to $2^k - 1$ for $k \in \mathbb{Z}$ can be written in the form $\sum_{i=0}^{k} c_i 2^i$, that every integer from 1 to $2^{k+1} - 1$ can be written in the form $\sum_{i=0}^{k+1} c_i 2^i$

Because for any positive integer n, we'd have that

$$2^{\lfloor \log_2 n \rfloor + 1} > n \geq 2^{\lfloor \log_2 n \rfloor}$$

Suppose that every integer from 1 to $2^k - 1$ can be written in this form. Consider the set of integers 2^k to $2^{k+1} - 1$

These two sets are 1 to 1, as suppose each number from 1 to $2^k - 1$ is denoted as a_i for $i \in 0, 1, ..., 2^k - 1$ and each number 2^k to $2^{k+1} - 1$ can be written as $b_i = 2^i + i - 1 = a_i + i$ for $i \in 0, 1, 2, ..., 2^k - 1$

Thus, we are done.

1.4 4 Fibonacci for Home

We claim the parities cycle odd, odd, even, etc.

Notice that $F_1 = F_2 = 1$ which are both odd.

Notice that $F_2 = 2$, which is even

Suppose that this holds for the first 3n integers. (our base case was $n = 1$)

Then, we have that

$$F_{3n} = F_{3n-1} + F_{3n-1} \equiv 0 + 1 \equiv 1 \mod 2$$

We also have that

$$F_{3n+1} \equiv F_{3n} + F_{3n-1} \equiv 1 + 0 \equiv 1 \mod 2$$

And finally, that

$$F_{3n+2} \equiv F_{3n+1} + F_{3n} \equiv 0 \mod 2$$

And we are done.