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On the Solution of an Elliptical Inhomogeneity in Plane Elasticity by the Equivalent Inclusion Method

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Abstract Stress analysis of an elliptical inhomogeneity in an infinite isotropic elastic plane is a classical elasticity problem, which is usually solved by means of the complex variable formulation. In this work, we demonstrate that an alternative method of solution for such a problem, via the equivalent inclusion method, may be more convenient and straightforward without recourse to complex potentials or curvilinear coordinates. The explicit analytical solution can be derived through simple algebraic manipulation, although the longitudinal eigenstrain component should be handled with care in the case of plane strain. Since the exterior Eshelby tensor for an elliptical inclusion is available in closed-form, the present study provides a full field stress solution expressed in Cartesian coordinates. Furthermore, the in-plane stress components are represented in terms of Dundurs' parameters. The solution methodology and the convenient formulae of the stress concentration may be of practical use to the engineers in developing benchmarks for design evaluation.

Keywords Elliptical inhomogeneity · Analytical solution · Stress concentration factor · Equivalent inclusion method

Mathematics Subject Classification (2010) 75B05 · 74M25 · 74G05 · 74G70

1 Introduction

During the last century, two masterpieces in the classical theory of elasticity were attributed to Eshelby and Dundurs. In a pioneering work, Eshelby [1] proved elegantly that the elastic fields inside an ellipsoidal inhomogeneity are uniform, when the ellipsoidal is perfectly bonded to an infinitely extended matrix with a uniform load applied at infinity. In another ingenious study, Dundurs [2] insightfully showed that the stress state in a bi-material compos-

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ite under plane deformation depends on only two combinations of elastic constants, which are called Dundurs' parameters. The present work in a large part is inspired by the above works and aims to develop a closed-form solution to the stresses due to the disturbance of an elliptical inhomogeneity, in terms of Dundurs' parameters. Stress analysis of an elliptical inhomogeneity has been a classical problem of the theory of elasticity. For a given uniform remote load, it is a known fact [3] that the stress and strain fields inside the elliptical inhomogeneity are uniform, whose magnitudes only depend on the material constants and the shape of the ellipse. Most of the existing solutions (cf. [3–8], to name a few) take advantage of the complex variable techniques, as culminated in Muskhelishvili's monograph [9]. However, as noted by Kushch et al. [10], this methodology for solving the plane inhomogeneity case requires more analytical effort than solving plane cavity problems [11], because finding the complex potential with matching interface conditions for the former is relatively complicated.

Since the problem under investigation is a particular limiting case of the more general three-dimensional (3D) problem considered by Eshelby [1] using the equivalent inclusion method (EIM), it is natural to examine whether EIM can be effective for handling the degenerated 2D plane inhomogeneity problem. To the best of the authors' knowledge, surprisingly, such a solution has not been documented in the literature. Mura, the most influential advocator of Eshelby's theory, also admitted [12, p. 41] that for plane strain or plane stress inhomogeneity problems, Muskhelishvili's complex potential method is more effective than the EIM. As a matter of fact, the only 2D example of EIM in Mura's monograph [12] considers a plane strain inhomogeneity of a circular shape. Those formulations are quite complicated, yet the results are far from complete since the only solution obtained are for the symmetric case with no attempt of the exterior field solution. Jaswon and Bhargava [4] commented that Eshelby's solutions "involve analytically intractable integrals of a formidable nature", and "this applies even in two-dimensional situations". Nonetheless, the current work demonstrates that the EIM is a useful tool at least for solving the classical elliptical inhomogeneity problem, and the analytical formulation presented here tends to be fairly succinct and show ease of practical applications, since the in-plane stress components are represented in Cartesian coordinates in terms of Dundurs' parameters. As Eshelby [1] remarked, the EIM can circumvent the mathematical complexities such as curvilinear coordinates or complex potential theory. The simplicity and the advantages of the solution methodology are also manifested by the fact that the interface continuity and even the elastic equilibrium equations are already invoked implicitly through application of EIM.

The major difficulty in applying EIM to the current 2D problem is the exterior field solution of the associated inclusion problem. Since such an elliptical inclusion solution [13] has been obtained recently, the obstacle is overcome and hence the EIM solution is meaningful. In the case of plane strain, an error often seen in formulating the 2D EIM is due to the longitudinal eigenstrain component, which usually does not vanish unless the inhomogeneity and matrix share the same Poisson's ratio or the rigidity of the inhomogeneity vanishes, i.e., a cavity. It is further noted that the formulation of the EIM in terms of eigenstress, rather than the usual eigenstrain, may greatly simplify the algebraic manipulation involved in the present analysis.

The paper is organized as follows. In Sect. 2, a general formulation of the EIM is discussed for a three-dimensional ellipsoidal inhomogeneity. In Sect. 3, a plane elliptical inclusion solution is presented for both interior and exterior fields, where explicit closed-form expressions are given for either plane strain or plane stress. In Sect. 4, the EIM formulation is implemented for plane elasticity. By taking the eigenstress components as the fundamental unknowns, the equations for consistency conditions are solved in closed form, and the

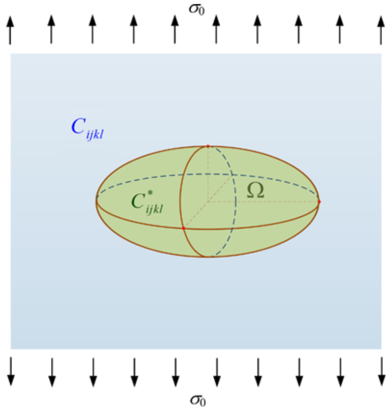


Fig. 1 The inhomogeneity problem: a material inhomogeneity in an infinitely extended isotropic elastic medium, which is subjected to remote applied load

resulting elastic fields are represented in terms of Dundurs' parameters. Finally, the concluding remarks are given in Sect. 5, and the major results are summarized in the Appendix for convenience.

2 Equivalent Inclusion Method

Although the EIM is a well-established approach, the details are retained here for completeness and also for clarity of a variant version used in the later section. Let us consider an infinitely extended matrix material with elastic moduli C_{ijkl}, containing a perfectly bonded subdomain, Ω, with different elastic moduli C^*_{ijkl}. Under the application of any external loads, the presence of inhomogeneity Ω will result in a disturbance of the elastic field, particularly pronounced in the vicinity of the inhomogeneity. In contrast, if the subdomain Ω undergoes transformation strains (eigenstrains) but has identical elastic moduli as those of the matrix, then it is called an "inclusion" in this context. The elastic field caused by an inclusion is due to constraint from the surrounding matrix which prevents the geometrically incompatible deformation of the eigenstrain, and the resulting self-equilibrated stress field is called the eigenstress. The equivalent inclusion method conceives that the disturbance of an applied stress due to the presence of inhomogeneity can be simulated by an eigenstress field generated by an inclusion when the eigenstrain is chosen properly.

For simplicity of presentation, let us distinguish a few types of solutions: homogeneous material solution, inhomogeneity solution, disturbance solution, and inclusion solution. Here, depending on whether or not the inhomogeneity is of the same material as the matrix, the resulting elastic field under the external load is termed homogeneous material solution or inhomogeneity solution, respectively. The disturbance solution refers to the deviation of the inhomogeneity solution from the homogeneous material solution. Lastly, the inclusion solution refers to the elastic field solely caused by the inclusion. The essence of the EIM establishes the equivalence between the disturbance solution and the inclusion solution. Therefore, in view of EIM, an inhomogeneity solution (Fig. 1) may be obtained by summing a homogeneous material solution (Fig. 2a) and a corresponding inclusion solution (Fig. 2b). Note that the equivalency here means that the actual stress and strain fields in Fig. 1 are identical to the resultant stress and *total strain* fields in Fig. 2.

Using Cartesian tensor notation, the homogeneous material solution (Fig. 2a) is

$$\sigma^0_{ij} = C_{ijkl}\varepsilon^0_{kl}, \tag{1}$$

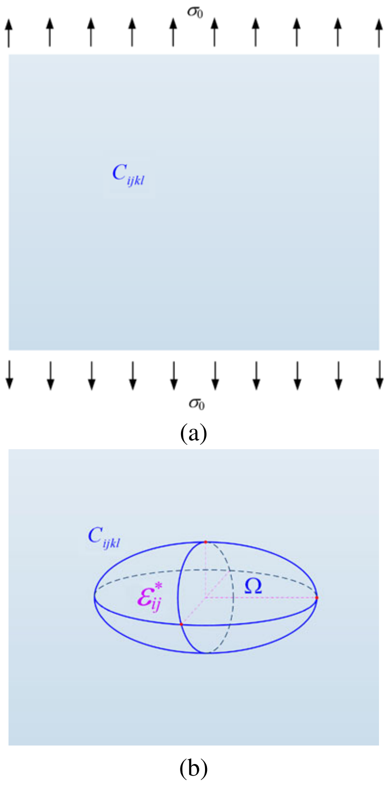


Fig. 2 Schematic illustration of the equivalent inclusion method. Solutions to an inhomogeneity problem may be decomposed as the summation of two auxiliary solutions: (**a**) A homogeneous solution: a homogeneous material, i.e., in the absence of the inhomogeneity, subjected to the applied load; and (**b**) A corresponding inclusion solution: The disturbance of the applied stress caused by the presence of the inhomogeneity is equivalently simulated by the eigenstress field due to a corresponding inclusion

where σ^0_{ij} and ε^0_{kl} are the stress and strain fields due to the remote load in the absence of the inhomogeneity. The inclusion solution (Fig. 2b) formulates the relation between the total strain, ε_{ij}, and the eigenstrain, ε^*_{ij}:

$$\varepsilon_{ij} = S_{ijkl}\varepsilon^*_{kl}, \tag{2}$$

where S_{ijkl} is the so-called Eshelby tensor. Note the total strain consists of eigenstrain and the elastic strain, e_{ij}. The stress, σ_{ij}, is related to the elastic strain by Hooke's law,

$$\sigma_{ij} = C_{ijkl}e_{kl} = C_{ijkl}\left(\varepsilon_{kl} - \varepsilon^*_{kl}\right) = C_{ijkl}\left(S_{klmn}\varepsilon^*_{mn} - \varepsilon^*_{kl}\right). \tag{3}$$

To summarize (Figs. 2a–b), the resultant stress and *total strain* inside Ω are respectively:

$$\sigma^0_{ij} + \sigma_{ij} = C_{ijkl}\left(\varepsilon^0_{kl} + S_{klmn}\varepsilon^*_{mn} - \varepsilon^*_{kl}\right), \tag{4}$$

and

$$\varepsilon^0_{ij} + \varepsilon_{ij} = \varepsilon^0_{ij} + S_{ijkl}\varepsilon^*_{kl}. \tag{5}$$

The equivalency condition implies that Eqs. (4) and (5) are respectively the actual stress and strain fields of the inhomogeneity problem (Fig. 1), upon which Hooke's law applies:

$$C_{ijkl}\left(\varepsilon^0_{kl} + S_{klmn}\varepsilon^*_{mn} - \varepsilon^*_{kl}\right) = C^*_{ijkl}\left(\varepsilon^0_{kl} + S_{klmn}\varepsilon^*_{mn}\right) \quad \text{in } \Omega. \tag{6}$$

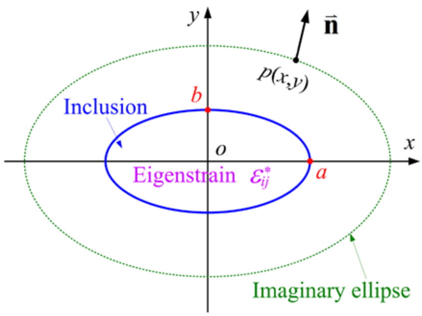


Fig. 3 Schematic illustration of an elliptical inclusion, the imaginary ellipse and its outward unit normal vector $\vec{\mathbf{n}}$

The consistency condition of Eq. (6) is necessary and sufficient for the implementation of EIM. It can be proved by the uniqueness theorem [14, p. 75] that the equivalency of the actual elastic field will be automatically satisfied in the matrix domain $(D - \Omega)$, provided that equivalency of Eq. (6) is established inside Ω.

When the inclusion is of ellipsoidal shape and subjected to uniform eigenstrain, the Eshelby tensor S_{ijkl} for the interior field, i.e., inside Ω, is constant depending only on the elastic moduli and the shape (relative length of the axes) of the ellipsoid. In Eq. (6), the equivalent eigenstrains, ε_{ij}^*, are the only unknowns to be determined. Moreover, the EIM is also effective for treating the extreme cases when the inhomogeneity becomes either a cavity or a rigid body. The former case may be readily obtained by substituting $C_{ijkl}^* = 0$ into Eq. (6), leading to

$$\varepsilon_{kl}^0 + S_{klmn}\varepsilon_{mn}^* - \varepsilon_{kl}^* = 0 \quad \text{in } \Omega. \tag{7}$$

Often in numerical computation, when the rigidity of the inhomogeneity is much greater than that of the matrix, it might be more convenient to represent Hooke's law in terms of the compliance moduli M_{ijkl}, i.e., $M_{ijkl} = C_{ijkl}^{-1}$. Let the compliance moduli for the matrix and the inhomogeneity be M_{ijkl} and M_{ijkl}^*, respectively. The counterpart equivalency condition follows from a similar manner and is derived as:

$$M_{ijkl}^*\left(\sigma_{kl}^0 + \sigma_{kl}\right) = M_{ijkl}\left(\sigma_{kl}^0 + \sigma_{kl}\right) + \varepsilon_{ij}^*, \tag{8}$$

where the eigenstress σ_{kl} is determined in Eq. (3). For a rigid inhomogeneity, D_{ijkl}^* vanishes and Eq. (8) becomes

$$M_{ijkl}\left(\sigma_{kl}^0 + \sigma_{kl}\right) + \varepsilon_{ij}^* = 0. \tag{9}$$

3 Plane Elliptical Inclusion Solution

For plane strain or plane stress problems, only eigenstrain components $(\varepsilon_x^*, \varepsilon_y^*, \varepsilon_z^*, \gamma_{xy}^*)$ are of concern. The formulation of EIM discussed in the preceding section can be expressed most simply using matrix notation, e.g., $[\boldsymbol{\sigma}] = [\sigma_x \ \sigma_y \ \sigma_z \ \tau_{xy}]^T$, $[\boldsymbol{\varepsilon}] = [\varepsilon_x \ \varepsilon_y \ \varepsilon_z \ \gamma_{xy}]^T$ and $[\boldsymbol{\varepsilon}^*] = [\varepsilon_x^* \ \varepsilon_y^* \ \varepsilon_z^* \ \gamma_{xy}^*]^T$, where the superscript T denotes the transpose. Here in order to preserve the symmetry property of the stress expressions in matrix form, the engineering shear component, γ_{xy} (or γ_{xy}^*), instead of the tensor shear component, ε_{12} (or ε_{12}^*), is adopted henceforth. Note that the former is equal to two times the latter, e.g., $\gamma_{xy}^* = 2\varepsilon_{12}^*$. Consider an infinite isotropic homogeneous elastic plane containing an elliptical inclusion with uniform eigenstrain. The ellipse with semi-axes of a and b is centered at the origin (Fig. 3). Young's modulus and Poisson's ratio of the material are E and ν, respectively. In the following, both the interior field and exterior field solutions will be discussed. Note that the interior solution will be used to determine the equivalent eigenstrain, while the exterior solution will be employed to compute the final stress field in the matrix material.

3.1 Interior Field Solution

The interior inclusion solution in the case of plane strain (an elliptic cylindrical inclusion), has been well documented [12, p. 80]; and it can be readily extended to plane stress, noting in which case that the longitudinal eigenstrain, ε_z^*, is a stress free component. For a unified presentation of the plane strain and plane stress problems, the two independent elastic constants are chosen to be shear modulus, μ, and Kolosov's constant, κ, with $\mu = E/2(1+\nu)$, and

$$\begin{cases} \kappa = 3 - 4\nu, & \text{in plane strain;} \\ \kappa = \dfrac{3-\nu}{1+\nu}, & \text{in plane stress.} \end{cases} \tag{10}$$

Then at any point (x, y) inside the elliptical inclusion, i.e., $(x/a)^2 + (y/b)^2 < 1$, the eigenstress components are

$$\left[\boldsymbol{\sigma}^*\right] = \begin{bmatrix} \sigma_x^* \\ \sigma_y^* \\ \sigma_z^* \\ \tau_{xy}^* \end{bmatrix} = [\mathbf{T}]\left[\boldsymbol{\varepsilon}^*\right] = \frac{-4\mu t}{(\kappa+1)(1+t)^2} \begin{bmatrix} 1+\frac{2}{t} & 1 & T_{13} & 0 \\ 1 & 1+2t & T_{23} & 0 \\ T_{31} & T_{32} & T_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_x^* \\ \varepsilon_y^* \\ \varepsilon_z^* \\ \gamma_{xy}^* \end{bmatrix}, \tag{11}$$

where $t = b/a$, and

$$\begin{cases} T_{13} = T_{31} = \dfrac{3-\kappa}{2}\left(1+\dfrac{1}{t}\right), \qquad T_{23} = T_{32} = \dfrac{3-\kappa}{2}(1+t), \qquad T_{33} = \dfrac{2(1+t)^2}{t}, \\ \quad \text{for plane strain,} \\ T_{13} = T_{31} = T_{23} = T_{32} = T_{33} = 0, \\ \quad \text{for plane stress.} \end{cases} \tag{12}$$

Moreover, the total strain components of the interior field are

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \end{bmatrix} = [\mathbf{S}]\left[\boldsymbol{\varepsilon}^*\right]$$
$$= \frac{1}{(\kappa+1)(1+t)^2}$$
$$\times \begin{bmatrix} t[t+3+\kappa(1+t)] & t[1+3t-\kappa(1+t)] & S_{13} & 0 \\ t+3-\kappa(1+t) & 1+3t+\kappa(1+t) & S_{23} & 0 \\ S_{31} & S_{32} & S_{33} & 0 \\ 0 & 0 & 0 & (1-t)^2+\kappa(1+t)^2 \end{bmatrix} \begin{bmatrix} \varepsilon_x^* \\ \varepsilon_y^* \\ \varepsilon_z^* \\ \gamma_{xy}^* \end{bmatrix}, \tag{13}$$

where

$$\begin{cases} S_{13} = t(1+t)(3-\kappa), & S_{23} = (1+t)(3-\kappa), \\ S_{31} = S_{32} = S_{33} = 0, & & \text{for plane strain,} \\ S_{31} = (1+t)(3-\kappa), & S_{32} = t(1+t)(3-\kappa), \\ S_{33} = (1+t)^2(\kappa+1), & S_{13} = S_{23} = 0, & \text{for plane stress.} \end{cases} \tag{14}$$

It is also noted that in Eq. (11), a superscript star is imposed on the eigenstress components. This is to indicate that the interior eigenstresses are uniform, similar to the eigenstrains. It will be demonstrated in Sect. 4, the computations involved in EIM will be significantly simplified if these eigenstresses, rather than the eigenstrain components, are selected as the basic unknowns. To this end, it is noted that the $[\mathbf{T}]$ matrix in Eq. (11) is invertible in the case of *plane strain*; therefore, the eigenstrain matrix, $[\boldsymbol{\varepsilon}^*]$, may be expressed in terms of the eigenstress matrix, $[\boldsymbol{\sigma}^*]$, i.e.,

$$[\boldsymbol{\varepsilon}^*] = \begin{bmatrix} \varepsilon_x^* \\ \varepsilon_y^* \\ \varepsilon_z^* \\ \gamma_{xy}^* \end{bmatrix} = [\mathbf{T}]^{-1}[\boldsymbol{\sigma}^*] = \frac{1}{E}\begin{bmatrix} -1-\nu_c t & \nu_c - 1 & \nu & 0 \\ \nu_c - 1 & -1-\frac{\nu_c}{t} & \nu & 0 \\ \nu & \nu & -1 & 0 \\ 0 & 0 & 0 & \frac{-\nu_c(1+t)^2}{t} \end{bmatrix}\begin{bmatrix} \sigma_x^* \\ \sigma_y^* \\ \sigma_z^* \\ \tau_{xy}^* \end{bmatrix}, \tag{15}$$

where $\nu_c = 2(1-\nu^2)$. It is further derived from Eqs. (13) and (15),

$$\begin{aligned} [\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \end{bmatrix} &= [\mathbf{S}][\boldsymbol{\varepsilon}^*] = [\mathbf{S}][\mathbf{T}]^{-1}[\boldsymbol{\sigma}^*] \\ &= \frac{-1}{4\mu}\begin{bmatrix} t(1+\kappa) & 1-\kappa & 0 & 0 \\ 1-\kappa & (1+\kappa)/t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-t)^2+\kappa(1+t)^2}{2t} \end{bmatrix}\begin{bmatrix} \sigma_x^* \\ \sigma_y^* \\ \sigma_z^* \\ \tau_{xy}^* \end{bmatrix}. \end{aligned} \tag{16}$$

Similarly for *plane stress*, the $[\mathbf{S}]$ matrix in Eq. (14) is invertible, and consequently the following relation is derived

$$\begin{aligned} [\boldsymbol{\sigma}^*] = \begin{bmatrix} \sigma_x^* \\ \sigma_y^* \\ \sigma_z^* \\ \tau_{xy}^* \end{bmatrix} &= [\mathbf{T}][\boldsymbol{\varepsilon}^*] = [\mathbf{T}][\mathbf{S}]^{-1}[\boldsymbol{\varepsilon}] \\ &= \frac{-\mu}{\kappa}\begin{bmatrix} (1+\kappa)/t & \kappa-1 & 0 & 0 \\ \kappa-1 & t(1+\kappa) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{8\kappa t}{(1-t)^2+\kappa(1+t)^2} \end{bmatrix}\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \end{bmatrix}. \end{aligned} \tag{17}$$

3.2 Exterior Field Solution

The expressions for the exterior elastic field tend to be more complicated; however, they can still be written in explicit closed-form [13]. The detailed results [13] are briefly summarized here. First, for any point $p(x, y)$ located outside the ellipse, i.e., $(x/a)^2 + (y/b)^2 > 1$, an imaginary ellipse as shown in Fig. 3 is constructed

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1, \tag{18}$$

where

$$\lambda = \frac{1}{2}\left[x^2 + y^2 - a^2 - b^2 + \sqrt{\left(x^2 + y^2 - a^2 + b^2\right)^2 + 4\left(a^2 - b^2\right)y^2}\right]. \tag{19}$$

When presenting the exterior field solution, it is necessary to introduce two auxiliary quantities, ρ_a and ρ_b, defined as follows,

$$\rho_a = \frac{a}{\sqrt{a^2+\lambda}}, \qquad \rho_b = \frac{b}{\sqrt{b^2+\lambda}}. \tag{20}$$

Moreover, the outward unit normal vector $\vec{\mathbf{n}} = \vec{\mathbf{n}}(n_x, n_y)$ at $p(x, y)$ on the imaginary ellipse can be calculated as

$$n_x = \frac{m_x}{\sqrt{m_x^2 + m_y^2}}, \qquad n_y = \frac{m_y}{\sqrt{m_x^2 + m_y^2}}, \tag{21}$$

in which

$$m_x = \frac{x}{a^2+\lambda}, \qquad m_y = \frac{y}{b^2+\lambda}. \tag{22}$$

According to [13], the stress components of the exterior field are

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \end{bmatrix} = [\mathbf{H}]\left[\boldsymbol{\varepsilon}^*\right] = \frac{4\mu\rho_a\rho_b}{\kappa+1} \begin{bmatrix} H_{11} & H_{12} & \nu H_{13} & H_{14} \\ H_{21} & H_{22} & \nu H_{23} & H_{24} \\ \nu H_{31} & \nu H_{32} & 0 & \nu H_{34} \\ H_{41} & H_{42} & \nu H_{43} & H_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_x^* \\ \varepsilon_y^* \\ \varepsilon_z^* \\ \gamma_{xy}^* \end{bmatrix}, \tag{23}$$

where

$$\begin{aligned}
H_{14} &= H_{41} = n_x n_y\left[1 - 3\rho_a^2 + \left(6\rho_a^2 + 2\rho_b^2 + T_6\right)n_x^2\right]; \\
H_{24} &= H_{42} = n_x n_y\left[1 - 3\rho_b^2 + \left(2\rho_a^2 + 6\rho_b^2 + T_6\right)n_y^2\right]; \\
H_{11} &= \frac{\rho_a b(a\rho_b + b\rho_a + 2a\rho_a^2\rho_b + b\rho_a^3)}{(a\rho_b + b\rho_a)^2} + n_x^2\left[2 - 6\rho_a^2 + \left(8\rho_a^2 + T_6\right)n_x^2\right]; \\
H_{22} &= \frac{\rho_b a(a\rho_b + b\rho_a + 2b\rho_a\rho_b^2 + a\rho_b^3)}{(a\rho_b + b\rho_a)^2} + n_y^2\left[2 - 6\rho_b^2 + \left(8\rho_b^2 + T_6\right)n_y^2\right]; \\
H_{12} &= H_{21} = H_{44} = \frac{a^2\rho_a^2\rho_b^2 + b^2\rho_a^2 + ab\rho_a\rho_b}{(a\rho_b + b\rho_a)^2} - \rho_b^2 n_x^2 - \rho_a^2 n_y^2 + \left(4\rho_a^2 + 4\rho_b^2 + T_6\right)n_x^2 n_y^2,
\end{aligned} \tag{24}$$

wherein the above equation,

$$T_6 = \rho_a^2 + \rho_b^2 - 4\rho_a^2 n_x^2 - 4\rho_b^2 n_y^2 - 4. \tag{25}$$

Note that a coefficient "ν" is factored out from the entries on the third row and the third column of the square matrix in Eq. (23). For plane stress, these entries vanish, i.e., $H_{13} = H_{31} = H_{23} = H_{32} = H_{34} = H_{43} = 0$. However, for plane strain, they are obtained as

$$
\begin{aligned}
H_{13} &= H_{31} = H_{11} + H_{21} = 2\left(\frac{b\rho_a}{a\rho_b + b\rho_a} - n_x^2\right);\\
H_{23} &= H_{32} = H_{12} + H_{22} = 2\left(\frac{a\rho_b}{a\rho_b + b\rho_a} - n_y^2\right);\\
H_{34} &= H_{43} = H_{14} + H_{24} = -2n_x n_y.
\end{aligned}
\tag{26}
$$

Now all the H_{ij} $(i, j = 1, 2, 3, 4)$, in Eqs. (24) and (26) are independent of the elastic constants. From Eqs. (11) and (23), it is also noted that both [**T**] and [**H**] are symmetric matrices. Their longitudinal-related components, viz. entries on the third row and column, vanish in the case of plane stress. In the interior field solution, the normal and the shear components are decoupled, as opposed to the exterior field solution. The exterior strain field may be obtained from the stress solution of Eq. (23) in a straightforward manner through Hooke's law, and is therefore omitted here, but the interested readers are referred to [13] for detailed expression.

4 Implementation of EIM for Plane Problem

The consistency condition of EIM is established inside the elliptical inclusion domain, where the eigenstrains exist. Therefore, it is desirable to generalize the two dimensional Hooke's law to accommodate the eigenstrain:

$$
\begin{bmatrix} \varepsilon_x - \varepsilon_x^* \\ \varepsilon_y - \varepsilon_y^* \\ \gamma_{xy} - \gamma_{xy}^* \end{bmatrix} = \frac{1}{8\mu} \begin{bmatrix} \kappa+1 & \kappa-3 & 0 \\ \kappa-3 & \kappa+1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}. \tag{27}
$$

The above equation is valid for both plane strain and plane stress cases. The left-hand side of Eq. (27) implies that Hooke's law expresses the relationship between the elastic strain and the stress [12, p. 5]. The longitudinal components, such as ε_z, ε_z^*, and σ_z, are not explicitly shown in Eq. (27); however, they should be interpreted with care:

(i) For plane strain, $\varepsilon_z = 0, \sigma_z = \nu(\sigma_x + \sigma_y) - E\varepsilon_z^*$;
(ii) For plane stress, $\sigma_z = 0, \varepsilon_z = \varepsilon_z^* + \frac{\nu}{1-\nu}(\varepsilon_x^* + \varepsilon_y^* - \varepsilon_x - \varepsilon_y)$.

When all the eigenstrain components vanish (e.g., Fig. 5a), the preceding discussions recover to the usual Hooke's law for plane elasticity. For example, Eq. (27) becomes

$$
\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{8\mu} \begin{bmatrix} \kappa+1 & \kappa-3 & 0 \\ \kappa-3 & \kappa+1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}; \tag{28}
$$

and the longitudinal stress component in plane strain becomes

$$
\sigma_z = \nu(\sigma_x + \sigma_y). \tag{29}
$$

Now consider an infinitely extended plane containing an elliptical inhomogeneity (Fig. 4) under uniform remote stresses, $[\sigma_x^0 \ \sigma_y^0 \ \tau_{xy}^0]$. The Young's modulus and Poisson's ratio are

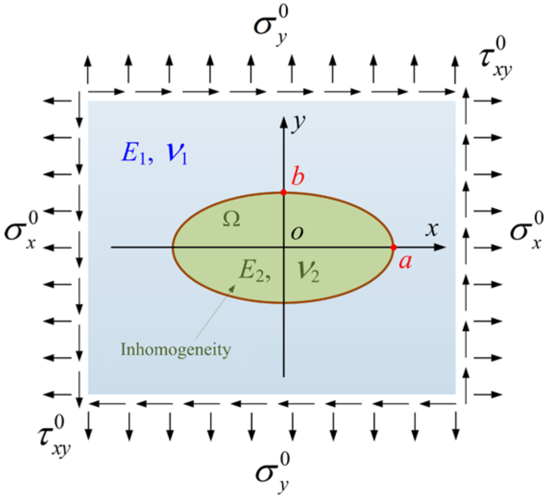


Fig. 4 The elliptical inhomogeneity problem in plane elasticity: an elliptical inhomogeneity in an infinite elastic plane, which is subjected to uniform load applied at infinity

respectively E_1 and ν_1 for the matrix material, and E_2 and ν_2 for the inhomogeneity. Here the subscripts '1' and '2' denote the material properties of the matrix and inhomogeneity respectively. When using the EIM to determine the elastic fields, the inhomogeneity will be replaced by an inclusion, and the solution will be obtained as the superposition of Figs. 5a and 5b. The following discussion of EIM will be mainly focused on the plane strain case, since the planes stress case may be solved in a similar manner but with relatively simpler algebraic manipulation.

4.1 EIM in Plane Strain

From Eq. (28), the homogeneous solution (Fig. 5a) yields

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \frac{1}{8\mu_1} \begin{bmatrix} \kappa_1+1 & \kappa_1-3 & 0 \\ \kappa_1-3 & \kappa_1+1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} \sigma_x^0 \\ \sigma_y^0 \\ \tau_{xy}^0 \end{bmatrix}. \tag{30}$$

For the inclusion solution (Fig. 5b), the relation between the total strain and the eigenstress follows from Eq. (16):

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{-1}{4\mu_1} \begin{bmatrix} t(1+\kappa_1) & 1-\kappa_1 & 0 \\ 1-\kappa_1 & (1+\kappa_1)/t & 0 \\ 0 & 0 & \frac{(1-t)^2+\kappa_1(1+t)^2}{2t} \end{bmatrix} \begin{bmatrix} \sigma_x^* \\ \sigma_y^* \\ \tau_{xy}^* \end{bmatrix}. \tag{31}$$

In view of the EIM, the total strain inside the inhomogeneity (Fig. 4) is obtained by adding Eq. (30) to (31). Also the total stress in the inhomogeneity is equal to the summation of $[\sigma_x^0\ \sigma_y^0\ \tau_{xy}^0]$ (Fig. 5a) and $[\sigma_x^*\ \sigma_y^*\ \tau_{xy}^*]$ (Fig. 5b). Therefore, applying Hooke's law, Eq. (28), to the inhomogeneity (Fig. 4) yields

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{8\mu_2} \begin{bmatrix} \kappa_2+1 & \kappa_2-3 & 0 \\ \kappa_2-3 & \kappa_2+1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} \sigma_x^0+\sigma_x^* \\ \sigma_y^0+\sigma_y^* \\ \tau_{xy}^0+\tau_{xy}^* \end{bmatrix}. \tag{32}$$

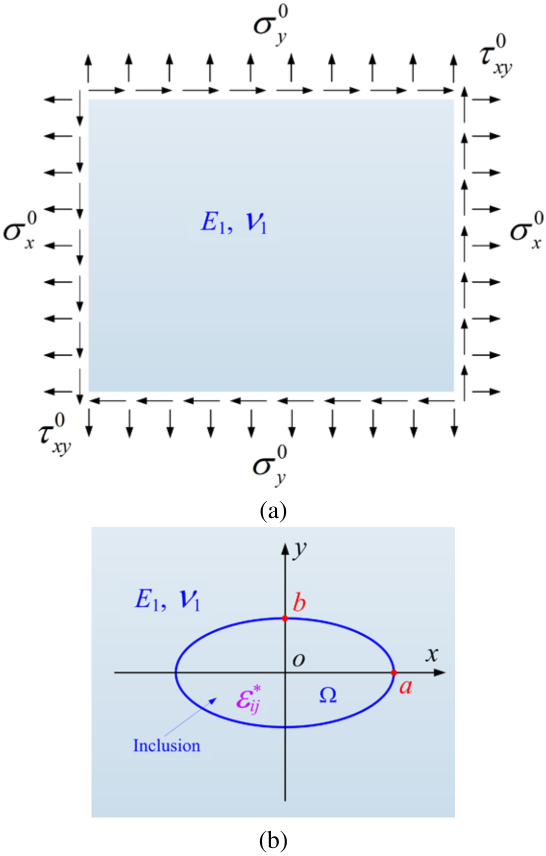


Fig. 5 In view of the equivalent inclusion method, solutions to an elliptical inhomogeneity problem in plane elasticity may be decomposed as summation of the solutions of two auxiliary problems: (**a**) A homogeneous plane subjected to the uniform load applied at infinity; and (**b**) A plane inclusion with equivalent eigenstrain

Substitution of Eqs. (30) and (31) into (32) gives algebraic equations in the unknowns of eigenstress:

$$\left(\frac{1}{4\mu_1}\begin{bmatrix} t(1+\kappa_1) & 1-\kappa_1 & 0 \\ 1-\kappa_1 & (1+\kappa_1)/t & 0 \\ 0 & 0 & \frac{(1-t)^2+\kappa_1(1+t)^2}{2t} \end{bmatrix} + \frac{1}{8\mu_2}\begin{bmatrix} \kappa_2+1 & \kappa_2-3 & 0 \\ \kappa_2-3 & \kappa_2+1 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right)\begin{bmatrix} \sigma_x^* \\ \sigma_y^* \\ \tau_{xy}^* \end{bmatrix}$$
$$= \left(\frac{1}{8\mu_1}\begin{bmatrix} \kappa_1+1 & \kappa_1-3 & 0 \\ \kappa_1-3 & \kappa_1+1 & 0 \\ 0 & 0 & 8 \end{bmatrix} - \frac{1}{8\mu_2}\begin{bmatrix} \kappa_2+1 & \kappa_2-3 & 0 \\ \kappa_2-3 & \kappa_2+1 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right)\begin{bmatrix} \sigma_x^0 \\ \sigma_y^0 \\ \tau_{xy}^0 \end{bmatrix}. \tag{33}$$

It is seen that the normal and shear eigenstress components are decoupled:

$$\left(\frac{1}{4\mu_1}\begin{bmatrix} t(1+\kappa_1) & 1-\kappa_1 \\ 1-\kappa_1 & (1+\kappa_1)/t \end{bmatrix} + \frac{1}{8\mu_2}\begin{bmatrix} \kappa_2+1 & \kappa_2-3 \\ \kappa_2-3 & \kappa_2+1 \end{bmatrix}\right)\begin{bmatrix} \sigma_x^* \\ \sigma_y^* \end{bmatrix}$$
$$= \left(\frac{1}{8\mu_1}\begin{bmatrix} \kappa_1+1 & \kappa_1-3 \\ \kappa_1-3 & \kappa_1+1 \end{bmatrix} - \frac{1}{8\mu_2}\begin{bmatrix} \kappa_2+1 & \kappa_2-3 \\ \kappa_2-3 & \kappa_2+1 \end{bmatrix}\right)\begin{bmatrix} \sigma_x^0 \\ \sigma_y^0 \end{bmatrix}; \tag{34}$$

and

$$\left[\frac{(1-t)^2+\kappa_1(1+t)^2}{8\mu_1 t}+\frac{1}{\mu_2}\right]\tau_{xy}^*=\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)\tau_{xy}^0. \tag{35}$$

Equation (35) may be readily solved for τ_{xy}^*. To facilitate the solution of Eq. (34), Dundurs parameters [15], α and β, are introduced:

$$\begin{cases}\alpha=\left(\dfrac{\kappa_1+1}{\mu_1}-\dfrac{\kappa_2+1}{\mu_2}\right)\Big/\left(\dfrac{\kappa_1+1}{\mu_1}+\dfrac{\kappa_2+1}{\mu_2}\right),\\ \beta=\left(\dfrac{\kappa_1-1}{\mu_1}-\dfrac{\kappa_2-1}{\mu_2}\right)\Big/\left(\dfrac{\kappa_1+1}{\mu_1}+\dfrac{\kappa_2+1}{\mu_2}\right).\end{cases} \tag{36}$$

After some algebraic simplification, the eigenstress solution is obtained as follows:

$$\begin{gathered}\tau_{xy}^*=\frac{4t(\alpha-\beta)}{(1+t)^2+\alpha(1-t)^2+4t\beta}\tau_{xy}^0;\\ \begin{bmatrix}\sigma_x^*\\ \sigma_y^*\end{bmatrix}=\frac{1}{\Delta}\begin{bmatrix}c_{xx} & c_{xy}\\ c_{yx} & c_{yy}\end{bmatrix}\begin{bmatrix}\sigma_x^0\\ \sigma_y^0\end{bmatrix},\end{gathered} \tag{37}$$

where

$$\begin{aligned}c_{xx}&=-2\left(\alpha^2+\alpha-3t\alpha\beta+t\beta+4t\beta^2\right);\\ c_{xy}&=2(1+\alpha)(\alpha-2\beta-t\beta);\\ c_{yx}&=2t(1+\alpha)(t\alpha-\beta-2t\beta);\\ c_{yy}&=-2t\left(t\alpha^2+t\alpha-3\alpha\beta+\beta+4\beta^2\right);\\ \Delta&=\left(\alpha^2-1\right)(1-t)^2+4t(\beta+1)(2\beta-\alpha-1).\end{aligned} \tag{38}$$

The stress field inside the inhomogeneity is obtained by adding $[\,\sigma_x^0\ \sigma_y^0\ \tau_{xy}^0\,]$ (Fig. 5a) to Eq. (37), and the detailed expressions are listed in the Appendix. It is evident that α and β are the only elastic parameters associated with the in-plane stress components. The longitudinal normal stress component in Figs. 4 and 5a can be readily obtained (cf. Eq. (29)). Therefore, the longitudinal eigenstress (Fig. 5b) is determined:

$$\sigma_z^*=(\nu_2-\nu_1)\left(\sigma_x^0+\sigma_y^0\right)+\nu_2\left(\sigma_x^*+\sigma_y^*\right). \tag{39}$$

The eigenstrain components (Fig. 5b) are found by substituting Eqs. (37)–(39) into (15). In particular, the longitudinal eigenstrain is

$$\varepsilon_z^*=\frac{(\nu_2-\nu_1)(1+\alpha)}{E_1\Delta}\left(d_{zx}\sigma_x^0+d_{zy}\sigma_y^0\right), \tag{40}$$

where

$$\begin{aligned}d_{zx}&=(1+\alpha)(1+t)^2+4t^2(\beta-\alpha);\\ d_{zy}&=(1+\alpha)(1+t)^2+4(\beta-\alpha).\end{aligned} \tag{41}$$

Therefore, in the case of plane strain, any remote uniform in-plane load will not cause the longitudinal eigenstrain, provided that either $\nu_2=\nu_1$ or $\alpha=-1$ (elliptical cavity) holds. Note also that d_{zx} and d_{zy} can not both be zero, since $-0.5\le\beta\le 0.5$.

To solve the exterior field of the inclusion problem (Fig. 5b), Eqs. (23) and (15) are used to represent the stress in terms of eigenstress:

$$[\boldsymbol{\sigma}] = [\mathbf{H}]\left[\boldsymbol{\varepsilon}^*\right] = [\mathbf{H}][\mathbf{T}]^{-1}\left[\boldsymbol{\sigma}^*\right] = [\mathbf{D}]\left[\boldsymbol{\sigma}^*\right] = \rho_a\rho_b \begin{bmatrix} D_{11} & D_{12} & 0 & D_{14} \\ D_{21} & D_{22} & 0 & D_{24} \\ D_{31} & D_{32} & 0 & D_{34} \\ D_{41} & D_{42} & 0 & D_{44} \end{bmatrix} \begin{bmatrix} \sigma_x^* \\ \sigma_y^* \\ \sigma_z^* \\ \tau_{xy}^* \end{bmatrix}. \tag{42}$$

It is seen that the third column of the [**D**] matrix vanishes, implying that the longitudinal eigenstress has no contribution to the exterior stresses. The other entries are

$$\begin{aligned} D_{11} &= \frac{H_{12}}{2} - \left(t + \frac{1}{2}\right)H_{11}; \qquad & D_{12} &= \frac{H_{11}}{2} - \left(\frac{1}{t} + \frac{1}{2}\right)H_{12}; \\ D_{21} &= \frac{H_{22}}{2} - \left(t + \frac{1}{2}\right)H_{12}; \qquad & D_{22} &= \frac{H_{12}}{2} - \left(\frac{1}{t} + \frac{1}{2}\right)H_{22}; \end{aligned} \tag{43}$$

$$\begin{aligned} D_{14} &= \frac{-(1+t)^2}{t}H_{14}; \qquad & D_{24} &= \frac{-(1+t)^2}{t}H_{24}; \\ D_{41} &= \frac{H_{24}}{2} - \left(t + \frac{1}{2}\right)H_{14}; \qquad & D_{42} &= \frac{H_{14}}{2} - \left(\frac{1}{t} + \frac{1}{2}\right)\frac{H_{24}}{t}; \end{aligned} \tag{44}$$

$$\begin{aligned} D_{31} &= \nu_1(D_{11} + D_{21}) = \frac{2\nu_1(a\rho_b n_x^2 - b\rho_a n_y^2)(a+b)}{(a\rho_b + b\rho_a)a}; \\ D_{32} &= \nu_1(D_{12} + D_{22}) = \frac{-2\nu_1(a\rho_b n_x^2 - b\rho_a n_y^2)(a+b)}{(a\rho_b + b\rho_a)b}; \\ D_{34} &= \nu_1(D_{13} + D_{23}) = 2\nu_1 n_x n_y \frac{(a+b)^2}{ab}. \end{aligned} \tag{45}$$

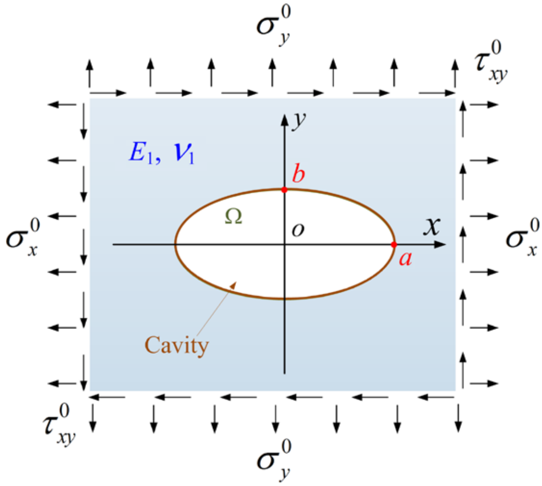
It is seen that the entries in Eqs. (43) and (44) are independent of the elastic constants. The explicit closed-form solution for the exterior eigenstress field (Fig. 5b) may be obtained from Eqs. (37), (38), (42)–(45). It is therefore concluded that the Dundurs parameters α and β are the only elastic parameters associated with all the in-plane components of the full stress field.

4.2 Elliptical Cavity

The solution of an elliptical cavity (Fig. 6) may be obtained as a special case from the above solution by setting $\alpha = -1$. In this case, the eigenstress of Eq. (37) becomes

$$\begin{bmatrix} \sigma_x^* \\ \sigma_y^* \\ \tau_{xy}^* \end{bmatrix} = -\begin{bmatrix} \sigma_x^0 \\ \sigma_y^0 \\ \tau_{xy}^0 \end{bmatrix}. \tag{46}$$

Fig. 6 An elastic plane containing an elliptical cavity subjected to remote uniform loading



The above Eq. (46) verifies that the resultant stresses (Fig. 4) inside the ellipse vanish. Moreover, the stresses outside the elliptical cavity are formulated in a straightforward manner:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \sigma_x^0 \\ \sigma_y^0 \\ \tau_{xy}^0 \end{bmatrix} - \begin{bmatrix} D_{11} & D_{12} & D_{14} \\ D_{21} & D_{22} & D_{24} \\ D_{41} & D_{42} & D_{44} \end{bmatrix} \begin{bmatrix} \sigma_x^0 \\ \sigma_y^0 \\ \tau_{xy}^0 \end{bmatrix}. \tag{47}$$

It is thus seen that D_{ij} defined in Eqs. (43) and (44) have clear physical meaning in representing the stress field caused by an elliptical cavity. Note that the elasticity solution of an infinite plate with an elliptical hole was first given in elliptical coordinates by Inglis [16] with rather complicated calculations [17, p. 112]. The present calculation is performed in a straightforward manner with simple algebraic manipulation, and the current result (47) provides a complementary interpretation of Inglis' solution in Cartesian coordinates.

To this end, a benchmark problem is solved to demonstrate the application of Eq. (47). Consider the stress field of a plane strain elliptical cavity (cf. Fig. 6) subjected to a remote uniaxial loading σ_y^0, while $\sigma_x^0 = \tau_{xy}^0 = 0$. For any point on the Ox-axis ($x > a$, $y = 0$), the expressions are greatly simplified:

$$\lambda = x^2 - a^2, \qquad n_x = 1, \qquad n_y = 0, \qquad \rho_a = a/x, \qquad \rho_b = b/\omega, \tag{48}$$

where

$$\omega = \sqrt{x^2 - a^2 + b^2}. \tag{49}$$

The stresses given by Eq. (47) becomes

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \sigma_y^0 \begin{bmatrix} D_{12} \\ 1 - D_{22} \\ D_{42} \end{bmatrix}, \tag{50}$$

where

$$
\begin{aligned}
D_{12} &= \frac{x[a\omega^2(\omega - x) + (a-b)b^2 x]}{b(a-b)^2\omega^2};\\
D_{22} &= \frac{x[(a-2b)\omega^2(\omega - x) - (a-b)b^2 x]}{b(a-b)^2\omega^2};\\
D_{42} &= 0.
\end{aligned}
\tag{51}
$$

One can readily verify that the above results, Eqs. (50) and (51), agree with the known solution [17, p. 121].

4.3 Plane Stress Results

For a plane stress problem, the longitudinal eigenstrain, ε_z^*, does not cause stress. Also, the longitudinal stress (or eigenstress) component, σ_z, vanishes in all the three solutions (Figs. 4 and 5a–b). Therefore, the longitudinal components can be safely omitted, and only the in-plane ones are involved in the EIM formulation. The results for plane stress presented in Sect. 3 are sufficient to solve the elastic fields by EIM, following a similar procedure discussed in Sect. 4.1. However, to minimize duplicate efforts, we may deduce the plane stress solution directly from the results of plane strain, since the two solutions possess quite similar structures. It will be elucidated below that the solutions of the in-plane components of the stress (or eigenstress) in the two problems are identical in form. To be specific, the plane strain solutions of Eqs. (37), (38), (43), (44), (46)–(51) apply exactly to the plane stress problem. Note that Eq. (42) is applicable to plane stress also if all the longitudinal related components are disregarded.

From Eq. (17) for plane stress, the eigenstress is related to the total strain as:

$$
\begin{bmatrix} \sigma_x^* \\ \sigma_y^* \\ \tau_{xy}^* \end{bmatrix} = \frac{-\mu_1}{\kappa_1} \begin{bmatrix} (1+\kappa_1)/t & \kappa_1 - 1 & 0 \\ \kappa_1 - 1 & t(1+\kappa_1) & 0 \\ 0 & 0 & \frac{8\kappa_1 t}{(1-t)^2 + \kappa_1(1+t)^2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}. \tag{52}
$$

One may readily verify mathematically that inversion of Eq. (52) yields Eq. (31); therefore, one sees the mathematical equivalence between the plane strain (31) and the plane stress (52). Since Eqs. (30), (31) and (32) are used to formulate plane strain EIM, while Eqs. (30), (32) and (52) are used for the plane stress case, the two sets of equations should yield exactly the same results. Based on the above arguments, it is concluded that Eqs. (37) and (38) are also the interior eigenstress solution of the plane stress problem.

The exterior stress field is obtained analogous to Eq. (42). By deleting those longitudinal related components, $[\mathbf{T}]^{-1}$ in plane stress is

$$
[\mathbf{T}]^{-1} = \frac{\kappa + 1}{4\mu} \begin{bmatrix} -\frac{1}{2} - t & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} - \frac{1}{t} & 0 \\ 0 & 0 & \frac{-(1+t)^2}{t} \end{bmatrix}. \tag{53}
$$

The exterior stresses are calculated by multiplying $[\mathbf{H}]$ comprising only the in-plane elements (cf. Eq. (24)) and $[\mathbf{T}]^{-1}$ of Eq. (53). One can easily verify that the multiplication yields the same expressions as Eqs. (43) and (44). Finally, if the cavity example (Fig. 6) is solved for plane stress, Eqs. (46)–(51) are reproduced.

5 Concluding Remarks

The elliptical inhomogeneity problem in plane elasticity was previously solved by means of Muskhelishvili's complex variable techniques, where final solutions of such cases, e.g., [3, 7, 8], were typically reported at the level of the complex potentials with no readily available stress formulation given in explicit form. However, for the elliptic *hole* problem, Gao [11] has derived an explicit closed form solution for the stress components in *elliptic-hyperbolic* coordinates. Since the complex potentials for an elliptical inhomogeneity are much lengthier, compared with those for the elliptical hole problem, the deduction of a compact explicit formulation for stress components in *Cartesian* coordinates, which are preferred for engineering visualization, should involve significant analytical effort, if it is at all possible and worthwhile.

This work demonstrates that the EIM may be an effective alternative. The present solution indicates that the exterior stress formula for the inhomogeneity problem has a similar structure to that of the cavity problem, except that the interior eigenstresses are readily available (Eq. (46)) for the latter while they can be represented in an explicit closed-form solution in terms of Dundurs' parameters (Eq. (37)) for the former. In the case of plane strain, the longitudinal component of the equivalent eigenstrain usually does not vanish, but the contrary example occurs once Poisson's ratio of the inhomogeneity and the matrix are of the same value, or the inhomogeneity degenerates to a cavity. The EIM formulation for plane stress is apparently simpler than for the plane strain case, although the final elasticity solution to the in-plane stress components of the two problems share the same expression using Kolosov's constant. Compared with the previous complex potential solution, the analytical representation here features Cartesian coordinates and Dundurs' parameters and is thus more viable for engineering practice.

For the case of plane strain, it is also possible to solve for the eigenstrains directly from Eq. (6), with the assistance of Eqs. (30), (2), and (16). However, this procedure will lead to the inversion of a 3×3 matrix. The present work formulates the EIM in terms of the *eigenstress* components, and the only laborious part is to invert the 2×2 matrix of Eq. (34), therefore reducing the involved algebraic complexity. Moreover, it should be noted that even for plane stress, the in-plane *eigenstrain* components cannot be represented only in the two Dundurs parameters, and hence an additional combination of elastic constants must be introduced. All the above facts suggest that the present EIM formulation in terms of eigenstresses will be more convenient for solving the classical plane elliptical inhomogeneity problem than the most widely known versions based on the eigenstrains.

Several advantages of the present analytical solution are noted: (1) The present solution is obtained in a straightforward manner, without resort to elliptical coordinates or complex potentials to match the stress and displacement at the interface; (2) All the elastic field components are given in Cartesian coordinates for ease of engineering applications; (3) The in-plane stress components are represented in Dundurs' parameters to facilitate the study of material combinations; (4) The stress concentration formulae are given in explicit closed form and the solution for an elliptical cavity is directly given in a matrix form; and (5) The connections and subtle differences in formulating EIM for both the plane strain and plane stress are elaborated.

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Appendix: Summary of the Results for Plane Strain or Plane Stress

For convenience of reference, the main results of this paper are summarized as follows. Kolosov's constant

$$\begin{cases} \kappa = 3 - 4\nu, & \text{in plane strain;} \\ \kappa = \dfrac{3-\nu}{1+\nu}, & \text{in plane stress.} \end{cases} \tag{54}$$

Dundurs' parameters

$$\alpha = \frac{\mu_2(\kappa_1+1) - \mu_1(\kappa_2+1)}{\mu_2(\kappa_1+1) + \mu_1(\kappa_2+1)}, \qquad \beta = \frac{\mu_2(\kappa_1-1) - \mu_1(\kappa_2-1)}{\mu_2(\kappa_1+1) + \mu_1(\kappa_2+1)}, \tag{55}$$

where the subscripts '1' and '2' denote the material properties of the matrix and inhomogeneity, respectively.

The stress field inside the inhomogeneity (Fig. 4) is uniform, i.e., for $(x/a)^2 + (y/b)^2 < 1$

$$\begin{gathered} \tau_{xy} = \frac{(1+\alpha)(1+t)^2}{(1+t)^2 + \alpha(1-t)^2 + 4t\beta}\tau_{xy}^0; \\ \begin{bmatrix} \sigma_x \\ \sigma_y \end{bmatrix} = \frac{1+\alpha}{\Delta} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \begin{bmatrix} \sigma_x^0 \\ \sigma_y^0 \end{bmatrix}, \end{gathered} \tag{56}$$

where

$$\begin{gathered} f_{xx} = -(1+t)^2(1+\alpha) + 2t(t\alpha+\beta); \qquad f_{xy} = 2(\alpha - 2\beta - t\beta); \\ f_{yx} = 2t(t\alpha - \beta - 2t\beta); \qquad f_{yy} = -(1+t)^2(1+\alpha) + 2(t\beta+\alpha); \\ t = b/a; \qquad \Delta = \big(\alpha^2-1\big)(1-t)^2 + 4t(\beta+1)(2\beta-\alpha-1). \end{gathered} \tag{57}$$

Note that Δ has been given in Eq. (38) and is repeated here for convenience.

For any exterior point, i.e., $(x/a)^2 + (y/b)^2 > 1$, the in-plane stresses (Fig. 4) are

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \sigma_x^0 \\ \sigma_y^0 \\ \tau_{xy}^0 \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{14} \\ D_{21} & D_{22} & D_{24} \\ D_{41} & D_{42} & D_{44} \end{bmatrix} \begin{bmatrix} \sigma_x^* \\ \sigma_y^* \\ \tau_{xy}^* \end{bmatrix}, \tag{58}$$

where the entries D_{ij} are given in Eqs. (43) and (44), and the eigenstresses are given in Eqs. (37) and (38). Particularly, the stresses at the vertices of the ellipse are of practical interest in determining the stress concentration factor. At $(a, 0)$ of the matrix side, the stresses σ_x and τ_{xy} are the same as those given by Eqs. (56) and (57), while

$$\sigma_y(a_+, 0) = \frac{1}{\Delta}\big[k_{ax}\sigma_x^0 + k_{ay}\sigma_y^0\big], \tag{59}$$

where

$$\begin{gathered} k_{ax} = 2(1+\alpha)\big[(3t+2)\beta - (2t+1)\alpha\big] + 8t\beta(\alpha-\beta); \\ k_{ay} = \big(\alpha^2-1\big)(1+t)^2 + (6t+16)\beta(\beta-\alpha) + 2\alpha(\alpha+1) + 2t\beta(\beta+1). \end{gathered} \tag{60}$$

Similarly, at $(0, b)$ of the matrix side, the stresses σ_y and τ_{xy} are the same as those given by Eqs. (56) and (57), while

$$\sigma_x(0, b_+) = \frac{1}{\Delta}\left[k_{bx}\sigma_x^0 + k_{by}\sigma_y^0\right], \tag{61}$$

where

$$\begin{aligned} k_{bx} &= \left(\alpha^2 - 1\right)(1+t)^2 + (6+16t)t\beta(\beta-\alpha) + 2t^2\alpha(\alpha+1) + 2t\beta(\beta+1); \\ k_{by} &= 2t(\alpha+1)\left[(3+2t)\beta - (t+2)\alpha\right] + 8t\beta(\alpha-\beta). \end{aligned} \tag{62}$$

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