OPTIMAL CONTROL PROBLEMS (OCPs)

In this context, the PDE is a state system that needs to be optimized by means of some external variables called controls.

GOAL: minimize an objective functional to reach a beneficial configuration.

Let us define the PDE problem as:
for a given CONTROL $u \in U$, for a suited Hilbert space U, the STATE VARIABLE $y(u) \in Y$, Y suited Hilbert space, verifies

$$\underbrace{G(y(u), u; \mu) = 0}_{\text{Strong form}} \quad \text{in } Y', \tag{1}$$

where $\mu \in \mathcal{P}$ can be a physical or geometrical parameter.

The main goal is to minimize over $Y \times U$
$J : Y \times U \times \mathcal{P} \longrightarrow \mathbb{R}$ defined as

$$J(y(u), u; \mu) = \frac{1}{2} \| y(u) - y_d \|^2_{Y_d} + \frac{\alpha}{2} \| u \|^2_U$$

s.t. (1) holds true.

⚠ QUADRATIC FUNCTIONAL!
Needed to guarantee the existence of a minimum.

where

- $Y_d \supset Y$ is the OBSERVATION SPACE
- $\alpha > 0$ is a penalization parameter

Abstractely, an OCP can b written as

$$\boxed{\begin{aligned} &\text{find } u^* = \arg\min_{u \in \mathcal{U}} J(y(u), u; \mu) \\ &\text{s.t. } G(y(u), u; \mu) = 0 \end{aligned}} \qquad (2)$$

u^* is the optimal control, while $y(u^*)$ is the optimal state.

Under ellipticity assumptions on $G(y(u), u; \mu)$, with $J(y, u; \mu)$ quadratic and $\alpha > 0$ you can prove existence and uniqueness of the solution!

↳ If interested:

- linear
- nonlinear
- parabolic

Hinze, Pinnau, Ulbrich, Ulbrich

OPTIMIZATION TOTALLY FITS WITH MANY-QUERY PROBLEMS!

From now on we consider linear PDEs.

Namely: (dropping "u" for the sake of notation)

$$G(y, u; \mu) = K(\mu) y - C(\mu) u - f(\mu)$$

$C(\mu) : \mathcal{U} \to Y'$, continuous

$K(\mu) : Y \to Y'$, continuous and positive definite

$f(\mu) \in Y'$ (coercive problem)

The related bilinear forms are:

$a : Y \times Y \to \mathbb{R}$ $\quad c : \mathcal{U} \times Y \to \mathbb{R}$ $\quad f : Y \to \mathbb{R}$

The state equation in weak form reads

$$\underbrace{a(y,v;\mu) - c(u,v;\mu) - f(v;\mu) = 0 \qquad \forall v \in Y}_{g((y,u),v;\mu)=0 \qquad \forall v \in Y}$$

Also the functional can be written in weak form as:

$$J(y,u;\mu) = \frac{1}{2} m(y-y_d, y-y_d;\mu) + \frac{\alpha}{2} n(u,u;\mu) - \frac{1}{2} m(y_d,y_d)$$

where $m: Y_d \times Y_d \longrightarrow \mathbb{R}$ is the Y_d-inner product bilinear form

equivalent

and

$n: U \times U \longrightarrow \mathbb{R}$ is the U-inner product bilinear form

How do we solve the problem?

LAGRANGIAN APPROACH

Let us define $p \in Y$ (we will clarify later why) and define the LAGRANGIAN FUNCTIONAL

$\mathcal{L}(y,u,p;\mu) : Y \times U \times Y \times \mathcal{P} \longrightarrow \mathbb{R}$ s.t.

$$\mathcal{L}(y,u,p;\mu) = J(y,u;\mu) + g((y,u),p;\mu) \quad (3)$$

p is an ADJOINT VARIABLE and has the role of LAGRANGIAN MULTIPLIER.

It is known that a constrained minimum of (2) is a unconstrained minimum for (3).

We can derive the NECESSARY conditions differentiating the Lagrangian and imposing the quantities to zero, i.e.

Let us recall the explicit Lagrangian

$$\mathcal{L}(y,u,p;\mu) = \frac{1}{2} m(y-y_d, y-y_d) + \frac{\alpha}{2} n(u,u) + a(y,p;\mu) - c(u,p;\mu) - f(p;\mu) - \frac{1}{2} m(y_d,y_d)$$

$$= \frac{1}{2} m(y,y) - m(y_d,y) + \cancel{\frac{1}{2} m(y_d,y_d)} + a(y,p;\mu) - c(v,p;\mu) - f(p;\mu) - \cancel{\frac{1}{2} m(y_d,y_d)}$$

simmetric (inner-product)

The conditions are three:

1) $D_y \mathcal{L}(y,u,p;\mu)[q] =$

$$\frac{1}{2} m(q,y) + \frac{1}{2} m(y,q) - m(y_d,q) + a(q,p;\mu)$$

$$= m(y,q) - m(y_d,q) + a^*(p,q;\mu)$$

where $a^*(p,q;\mu) = a(q,p;\mu)$ is the ADJOINT OPERATOR.

2) $D_u \mathcal{L}(y,u,p;\mu)[v] =$

$$\frac{\alpha}{2} n(v,u) + \frac{\alpha}{2} n(u,v) - c(v,p;\mu)$$

$$= \alpha n(u,v) - c^*(p,v;\mu)$$

3) $D_p \mathcal{L}(y,u,p;\mu)[z] = a(y,z;\mu) - c(u,z;\mu) - f(z;\mu)$

To define a stationary point, I have to solve

1) $= 0 \quad \forall q \in Y$ ADJOINT EQUATION
2) $= 0 \quad \forall v \in \mathcal{U}$ CONTROL EQUATION
3) $= 0 \quad \forall z \in Y$ STATE EQUATION

Namely,

$$\begin{cases} m(y,q) + a^*(p,q;\mu) = m(y_d, q) & \forall q \in Y \\ \alpha\, n(u,v) - c^*(p,v;\mu) = 0 & \forall v \in \mathcal{U} \\ a(y,z;\mu) - c(u,z;\mu) = f(z;\mu) & \forall z \in Y \end{cases}$$

Matrix-wise (FE-wise)

$$\begin{cases} \mathbb{M}_\delta(\mu) y + \mathbb{K}^T(\mu) p = \mathbb{M}_\delta(\mu) y_d \\ \alpha\, \mathbb{N}_\delta(\mu) u - \mathbb{C}^T(\mu) p = 0 \\ \mathbb{K}(\mu) y - \mathbb{C}_\delta(\mu) u = f_h(\mu) \end{cases}$$

In compact form we can write

$$\begin{bmatrix} \mathbb{M}_\delta(\mu) & 0 & \mathbb{K}^T_\delta(\mu) \\ 0 & \alpha\, \mathbb{N}_\delta(\mu) & \mathbb{C}^T_\delta(\mu) \\ \mathbb{K}_\delta(\mu) & -\mathbb{C}_\delta(\mu) & 0 \end{bmatrix} \begin{bmatrix} y \\ u \\ p \end{bmatrix} = \begin{bmatrix} \mathbb{M}_\delta(\mu) y_{d_\delta} \\ 0 \\ f_\delta \end{bmatrix}.$$

SADDLE POINT STRUCTURE

to prove inf-sup condition over $\mathbb{B}$ $\rightarrow$

well-posed if $\underbrace{p \in Y}$ Same space of the state

$$\begin{bmatrix} \mathbb{A}(\mu) & \mathbb{B}^T(\mu) \\ \mathbb{B}(\mu) & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} g_h \\ f_h \end{bmatrix}$$

$$x = \begin{bmatrix} y \\ u \end{bmatrix} \qquad g_h = \begin{bmatrix} \mathbb{M}_\delta(\mu) y_d \\ 0 \end{bmatrix}$$

MODEL ORDER REDUCTION

• **POD** → collect three sets of snapshots and build three different basis functions $\mathbb{V}_y$, $\mathbb{V}_u$ and $\mathbb{V}_p$

→ $y \in \mathbb{V}_y$
$u \in \mathbb{V}_u$
$p \in \mathbb{V}_p \neq \mathbb{V}_y$!!!

How can I deal with reduced inf-sup?

ACCREGATED SPACE

We define $\mathbb{V}_{yp} = \mathbb{V}_y \oplus \mathbb{V}_p$
and we consider
$y \in \mathbb{V}_{yp}$ and $p \in \mathbb{V}_{yp}$

Once we have the basis functions, assuming affinity, we can pre-compute

$$
\begin{aligned}
\mathbb{M}_N &= \mathbb{V}_{yp}^T \mathbb{M}_\delta \mathbb{V}_{yp} \\
\mathbb{N}_N &= \mathbb{V}_u^T \mathbb{N}_\delta \mathbb{V}_u^T \\
\mathbb{K}_N &= \mathbb{V}_{yp}^T \mathbb{K}_\delta \mathbb{V}_{yp} \\
\mathbb{C}_N &= \mathbb{V}_{yp}^T \mathbb{C}_\delta \mathbb{V}_q
\end{aligned}
\xrightarrow{\text{and solve}}
\begin{pmatrix} \mathbb{M}_N & 0 & \mathbb{K}_N^T \\ 0 & \alpha \mathbb{N}_N & \mathbb{C}_N^T \\ \mathbb{K}_N & \mathbb{C}_N & 0 \end{pmatrix}
\begin{bmatrix} y_N \\ u_N \\ p_N \end{bmatrix}
=
\begin{bmatrix} \mathbb{M}_N y_N \\ 0 \\ f_N \end{bmatrix}
$$

$$
y_N = \mathbb{V}_{yp}^T y_{d\delta}
$$
$$
f_N = \mathbb{V}_{yp}^T f_\delta
$$

— • — •

• GREEDY

To apply the algorithm we need an ERROR ESTIMATOR

Applying classical arguments of RB strategies we have that

$$\sqrt{\|y_\delta - y_N\|_Y^2 + \|u_\delta - u_N\|_U^2 + \|p_\delta - p_N\|_Y^2} \leq \frac{\|R\|^{*}}{\beta(\mu)}$$

where $\beta(\mu)$ is the inf-sup constant of the WHOLE optimality system and R = whole RESIDUAL

$$\hookrightarrow \inf_{p} \sup_{(y,u)} \frac{a(y,z;\mu) - c(u,z;\mu) + m(y,q;\mu) + a^*(p,q;\mu) + \alpha m(u,v;\mu) - c^*(p,v;\mu)}{\sqrt{\|y\|_Y^2 + \|p\|_Y^2 + \|u\|_U^2}}$$

The algorithm works in the following way:

1. define $\mathcal{P}_h \subset \mathcal{P}$ and pick $\mu_0 \in \mathcal{P}_h$
2. Initialize $Y_N^y = \{y(\mu_0)\}$, $Y_N^p = \{p(\mu_0)\}$ and $U_N = \{u(\mu_0)\}$ and $Y_N = \{y(\mu_0), p(\mu_0)\}$
3. $$\mu_1 = \arg\max_{\mathcal{P}_h} \Delta_N(\mu) = \arg\max_{\mathcal{P}_h} \frac{\|R\|^{*}}{\beta(\mu)}$$
4. Enlarge the spaces and apply aggregated spaces, i.e.
$$Y_N = \{y(\mu_0), y(\mu_1), p(\mu_0), p(\mu_1)\}$$
$$U_N = \{u(\mu_0), u(\mu_1)\}$$
5. Continue until $\Delta_N(\mu) \leq \tau$, for τ tolerance.

A VERY SIMPLE PRACTICAL EXAMPLE:

$\Omega = (0,1) \times (0,1)$ $\quad$ $y_d \in L^2(\Omega_{obs})$ $\quad$ $\Omega_{obs} \subseteq \Omega$

$\mu \in \mathcal{P}$ $\quad$ $\alpha > 0$

given $\mu \in \mathcal{P}$, find $(y,u) \in H^1_0(\Omega) \times L^2(\Omega)$ s.t. they verify

$$\min_{(y,u)} \frac{1}{2} \| y - y_d \|^2_{L^2(\Omega_{obs})} + \frac{\alpha}{2} \|u\|^2_{L^2(\Omega)}$$

s.t.

$$\begin{cases} -\mu \Delta y = u & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega \end{cases}$$

Let us define $p \in H^1_0(\Omega)$ and the Lagrangian

$$\mathcal{L}(y,u,p;\mu) = \frac{1}{2} \int_{\Omega_{obs}} (y - y_d)^2 \, dx + \frac{\alpha}{2} \int_\Omega u^2 \, dx + \mu \int_\Omega \nabla y \nabla p \, dx - \int_\Omega u p \, dx$$

$$D_y \mathcal{L}(y,u,p;\mu)[q] = \int_{\Omega_{obs}} (y - y_d) q \, dx + \underbrace{\mu \int_\Omega \nabla q \cdot \nabla p \, dx}_{\text{self-adjoint}}$$

$a(q,p;\mu) = a^*(p,q;\mu) = a(p,q;\mu)$

$$D_u \mathcal{L}(y,u,p;\mu)[v] \quad \alpha \int_\Omega u v \, dx - \underbrace{\int_\Omega v p \, dx}_{\text{self-adjoint}}$$

$$D_p \mathcal{L}(y,u,p;\mu)[z] = \int_\Omega \nabla y \nabla z \, dx - \int_\Omega u z \, dx$$

⚠ If you have an advection term, you have

$$a(y,z;\mu) = \int_\Omega \beta \cdot \nabla y \, z \, dx = \int_\Omega \operatorname{div}(\beta y z) dx - \int \beta \cdot \nabla z \, y \, dx$$

$$\int_{\Omega} \mathrm{div}(\beta y z)\,dx = \int_{\partial\Omega} \cancel{\beta \cdot y z n}\, ds$$

$$\Rightarrow -\int \beta \cdot \nabla z \, y \, dx = a^{*}(z, y ; \mu)$$

↑

change of sign!