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<u>STOKES</u> Find $\underline{u} = (u_1, u_2)$, p s.t.

$$\begin{cases} -\nu \Delta \underline{u} + \Delta p = \underline{f} & \text{on } \Omega \\ \nabla \cdot \underline{u} = 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Delta \underline{u} = \nabla(\nabla \cdot \underline{u}) - \nabla \times (\nabla \times \underline{u})$

$$\Rightarrow \nabla \times \underline{u} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ u_1 & u_2 & 0 \end{vmatrix} = (0, \ 0, \ \partial_x u_2 - \partial_y u_1)$$

$$\nabla \times (\nabla \times \underline{u}) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \partial_x u_2 - \partial_y u_1 \end{vmatrix} = (\partial_x \partial_y u_2 - \partial_y^2 u_1, \partial_x \partial_y u_1 - \partial_x^2 u_2, 0)$$

If $\nabla \cdot \underline{u} = 0 \Rightarrow \partial_x u_1 = -\partial_y u_2$, thus

$$\Delta \underline{u} = (-\partial_x^2 u_1 - \partial_y^2 u_1, -\partial_x^2 u_2 - \partial_y^2 u_2, 0) = $$
$$= (\Delta u_1, \Delta u_2)$$

To find the weak formulation, find $\underline{u} \in [H_0^1(\Omega)]^2$
$p \in L_0^2(\Omega)$

$$\begin{cases} -\int_\Omega \nu \Delta \underline{u} \cdot \underline{v} + \int_\Omega \nabla p \cdot \underline{v} = \int_\Omega \underline{f} \cdot \underline{v} & \forall \underline{v} \in [H_0^1(\Omega)]^2 \\ \int_\Omega \nabla \cdot \underline{u} \, q = 0 & \forall q \in L_0^2(\Omega) \end{cases}$$

We recall the Green Formula for functions $w, s \in H_0^1(\Omega)$

$$\int_\Omega \Delta w \, s + \int_\Omega \nabla w \cdot \nabla s = \cancel{\int_{\partial\Omega} \nabla w \cdot \underline{n} \, s} = 0$$

then if $\nabla s = p$ and $\nabla w = \underline{v}$

$$\int_\Omega \nabla p \cdot \underline{v} = -\int_\Omega \nabla \cdot \underline{v} \, p$$

For Tensors we recall that:

$$\nabla \underline{u} = \nabla \underline{u}^T = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} (u_1 \ u_2) = \begin{pmatrix} \partial_x u_1 & \partial_x u_2 \\ \partial_y u_1 & \partial_y u_2 \end{pmatrix}$$

thus the Green's Formula becomes (neglect the boundary)

$$\int_\Omega \Delta \underline{u} \cdot \underline{v} + \int_\Omega \nabla \underline{u} : \nabla \underline{v} = 0$$

where $\underline{\underline{A}} : \underline{\underline{B}}$ is the DOUBLE INNER PRODUCT defined as

$$\underline{\underline{A}} : \underline{\underline{B}} = \sum_{i,j=0}^{n-1} a_{ij} b_{ij} = a_{11} b_{11} + a_{12} b_{12} + a_{21} b_{21} + a_{22} b_{22}$$

Thus the weak form becomes

$$\begin{cases} \int_\Omega \nu \nabla \underline{u} : \nabla \underline{v} - \int_\Omega p \nabla \cdot \underline{v} = \int_\Omega \underline{f} \cdot \underline{v} & \forall \underline{v} \in [H_0^1(\Omega)]^2 \\ -\int_\Omega q \nabla \cdot \underline{u} = 0 & \forall q \in L_0^2(\Omega) \end{cases}$$

which becomes a SADDLE POINT problem

$$(1) \quad \begin{cases} a(\underline{u}, \underline{v}) + b(\underline{v}, p) = (\underline{f}, \underline{v})_\Omega & \forall \underline{v} \in V \\ b(\underline{u}, q) = 0 & \forall q \in Q \end{cases}$$

where $a(\underline{u}, \underline{v}) = \int_\Omega \nu \nabla \underline{u} : \nabla \underline{v}$, $b(\underline{v}, p) = -\int_\Omega p \nabla \cdot \underline{v}$,
$V = [H_0^1(\Omega)]$ and $Q = L_0^2(\Omega)$.

<u>Theorem</u>: Problem (1) has solution $\Longleftrightarrow$

1) $a(\cdot,\cdot)$ is coercive on V^0
where
$$V^0 = \{ \underline{v} \in V : b(\underline{v}, q) = 0 \ \forall q \in Q \}$$
2) $a(\cdot,\cdot)$ is continuous on V
3) $b(\cdot,\cdot)$ is continuous, $\exists \gamma > 0$ s.t. $\forall \underline{v} \in V, q \in Q$
$$|b(\underline{v}, q)| \le \gamma \|\underline{v}\|_V \|q\|_Q$$
4) $b(\cdot,\cdot)$ is STABLE, $\exists \beta_0 > 0$ s.t.
$$\beta = \inf_{q \in Q} \sup_{\underline{v} \in V} \frac{b(\underline{v}, q)}{\|q\|_Q \|\underline{v}\|_V} \ge \beta_0$$
also rewritten as $\forall q \in Q, \exists \underline{v} \in V, \underline{v} \neq \underline{0}$ s.t.
$$b(\underline{v}, q) \ge \beta_0 \|q\|_Q \|\underline{v}\|_V$$
and β_0 is the INF-SUP constant

In the high-fidelity space the situation is similar, thus the problem becomes:
given $V_h \subset V$ and $Q_h \subset Q$ find $\underline{u}_h \in V_h$ and $p_h \in Q_h$ s.t.

$$\forall \underline{v}_h \in V_h, \ q_h \in Q_h \quad \begin{cases} a(\underline{u}_h, \underline{v}_h) + b(\underline{v}_h, p) = (\underline{f}, \underline{v}_h)_\Omega \\ b(\underline{u}_h, q_h) = 0 \end{cases}$$

rewritten in matrix form as a saddle point

$$\underbrace{\begin{bmatrix} \underline{\underline{A}} & \underline{\underline{B}}^T \\ \underline{\underline{B}} & \underline{\underline{0}} \end{bmatrix}}_{\underline{\underline{J}}} \underbrace{\begin{bmatrix} \underline{u}_h \\ \underline{p}_h \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} \underline{f}_h \\ \underline{0} \end{bmatrix}}_{\underline{q}}$$

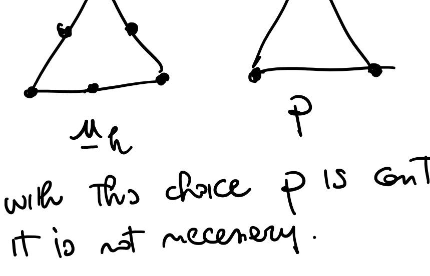
The solution is guaranteed if the Theorem we show before is satisfied for the finite spaces; thus we have to find V_h and Q_h s.t.

1) $\exists \alpha_h > 0$ s.t. $a(\underline{v}_h, \underline{v}_h) \ge \alpha \|\underline{v}_h\|_{V_h} \ \forall \underline{v}_h \in V_h^0$
with $V_h^0 = \{ \underline{v}_h \in V_h : b(\underline{v}_h, q_h) = 0 \ \forall q_h \in Q_h \}$

2) $\exists \beta_h > 0$ s.t. $b(\underline{v}_h, q_h) \ge \beta_h \|\underline{v}_h\|_{V_h} \|q_h\|_{Q_h}$
$\forall q_h \in Q_h, \exists \underline{v}_h \in V_h, \underline{v}_h \neq \underline{0}$

From (2) we can say that if $b(\underline{v}_h, q_h) = 0 \ \forall \underline{v}_h \in V_h$ then necessarily $q_h = 0$.
If (2) does not hold, then $\exists q_h \neq 0$ s.t. $b(\underline{v}_h, q_h) = 0$ $\forall \underline{v}_h \in V_h$. The q_h is called SPURIOUS PRESSURE.
Thus to satisfy (2) we can
1) choose V_h, Q_h properly;
2) stabilize the spaces, removing the spurious pressures.

We see today the first choice, choosing
$V_h = [\{\text{FEM order } 2\}]^2$ $Q_h = \{\text{FEM order } 1\}$



$\mathbb{P}_2 - \mathbb{P}_1$

<u>NOTE</u>: with this choice p is continuous, even if it is not necessary.

Thus, matrix $\underline{\underline{J}}$ can be computed choosing a base for V_h and for Q_h

- BASE for V_h: $\{ [\varphi_i^1, 0], [0, \varphi_i^2] \} = \{ \underline{\psi}_j \}_{j=0 \dots 2N_{DOFs}}$
where $\{\varphi_i\}_{i=0,\dots,N_{DOFs}}$ is the standard base for FEM $\mathbb{P}_2(\Omega)$.
- BASE for Q_h: standard base for FEM $\mathbb{P}_1(\Omega)$ $\{\varepsilon_k\}$

Then, we can compute the matrices $\underline{\underline{A}}, \underline{\underline{B}}$ by:

$$\underline{\underline{A}}_{ij} = \int_\Omega \nu \nabla \underline{\psi}_j : \nabla \underline{\psi}_i = \int_\Omega \nu \begin{bmatrix} \partial_x \varphi_j^1 & 0 \\ \partial_y \varphi_j^1 & 0 \end{bmatrix} : \begin{bmatrix} \partial_x \varphi_i^1 & 0 \\ \partial_y \varphi_i^1 & 0 \end{bmatrix} +$$
$$+ \int_\Omega \nu \begin{bmatrix} 0 & \partial_x \varphi_j^2 \\ 0 & \partial_y \varphi_j^2 \end{bmatrix} : \begin{bmatrix} 0 & \partial_x \varphi_i^2 \\ 0 & \partial_y \varphi_i^2 \end{bmatrix} =$$
$$= \underbrace{\int_\Omega \nu \nabla \varphi_j^1 \cdot \nabla \varphi_i^1}_{\underline{\underline{A}}^1} + \underbrace{\int_\Omega \nu \nabla \varphi_j^2 \cdot \nabla \varphi_i^2}_{\underline{\underline{A}}^2}$$

$$\underline{\underline{B}}_{kj} = \int_\Omega \nabla \cdot \underline{\psi}_j \, \varepsilon_k = \int_\Omega \partial_x \varphi_j^1 \varepsilon_k + \int_\Omega \partial_y \varphi_j^2 \varepsilon_k =$$
$$= \underbrace{\int_\Omega [1, 0] \cdot \nabla \varphi_j^1 \varepsilon_k}_{\underline{\underline{B}}^1} + \underbrace{\int_\Omega [0, 1] \cdot \nabla \varphi_j^2 \varepsilon_k}_{\underline{\underline{B}}^2}$$

and for $(\underline{f}, \underline{v}_h)_\Omega$ we compute

$$(\underline{f}, \underline{\psi}_i) = \int_\Omega \underline{f} \cdot \underline{\psi}_i = \underbrace{\int_\Omega f_1 \varphi_i^1}_{\underline{f}_1} + \underbrace{\int_\Omega f_2 \varphi_i^2}_{\underline{f}_2}$$