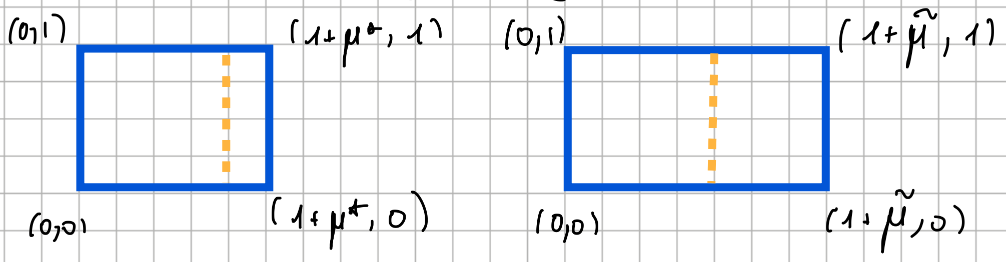
GEOMETRIC PARAMETRIZATION

Let us assume $\mu \in \mathcal{P}$ our input parameter.
The parameter can change the physics of the domain, but also the geometry. The μ-dependent domain is $\tilde{\Omega}(\mu)$



Two evaluations of $\tilde{\Omega}(\mu)$

$\tilde{\Omega}(\mu)$ is the ORIGINAL DOMAIN.
To solve the problem we need to define a REFERENCE DOMAIN Ω, that is μ-independent!
The original domain $\tilde{\Omega}(\mu)$ is obtained by means of a parametric map $\phi : \Omega \times \mathcal{P} \to \tilde{\Omega}$ s.t. $x \mapsto \tilde{x}(\mu)$ and, namely,

$$\boxed{\tilde{\Omega}(\mu) = \phi(\Omega; \mu), \quad \forall \mu \in \mathcal{P}}.$$

Let us specify the nature of μ:
$\mu = (\mu_1^{ph}, \dots, \mu_p^{ph}, \mu_g)$ where μ_g is the only parameter interacting with the domain (THIS IS A SIMPLIFICATION)
Our PDE problem will be: given $\mu \in \mathcal{P}$, find $\tilde{u}(\mu) \in \tilde{V}(\mu_g)$ s.t.

$$\tilde{a}(\tilde{u}(\mu), v; \mu) = \tilde{f}(v; \mu) \quad \forall v \in \tilde{V}(\mu_g).$$

Thanks to the map ϕ, we can PULL-BACK the problem onto the reference domain Ω: given $\mu \in \mathcal{P}$, find $u(\mu) \in V$ s.t.

$$a(u(\mu), v; \mu) = f(v; \mu) \quad \forall v \in V.$$

⚠ The two problems are different!!! There are some geometric factors induced by the map ϕ.

Let us recall that $x \in \Omega$ and $\tilde{x} \in \tilde{\Omega}(\mu)$.
Moreover, both the subdomains are subsets of $\mathbb{R}^d$.
We can define the JACOBIAN MATRIX of ϕ, $\mathbb{J}_\phi(x,\mu) \in \mathbb{R}^{d \times d}$ as

$$[\mathbb{J}_\phi(x,\mu)]_{k\ell} = \frac{\partial \phi_k(x,\mu)}{\partial x_\ell}(x) \qquad k,\ell = 1,\dots,d.$$

If for any $\mu \in \mathcal{P}$, the determinant $|\mathbb{J}_\phi(x;\mu)| \neq 0$, then ϕ is well-posed!

For any $\tilde{\varphi} : \tilde{\Omega} \to \mathbb{R}$ it holds:

$$\int_{\tilde{\Omega}(\mu)} \tilde{\varphi}(\tilde{x})\, d\tilde{x} = \int_{\Omega} \varphi(x)\, |\mathbb{J}_\phi(x;\mu)|\, dx,$$

where $\varphi = \tilde{\varphi} \circ \phi$.

AFFINE TRANSFORMATION!

We will always assume ϕ being an affine map of the form

$$\phi(x;\mu) = \mathbb{A}(\mu)x + c(\mu) \quad \text{with } \mathbb{A}(\mu) \in \mathbb{R}^{d \times d} \text{ and } c(\mu) \in \mathbb{R}^d.$$

In this case $\mathbb{J}_\phi(x;\mu) = \mathbb{A}(\mu)$ and the determinant depends only on the parameter μ.
If ϕ is well-posed, $\phi^{-1} : \tilde{\Omega} \to \Omega$ is defined.

$$\boxed{\text{RECALL: } \Omega = \phi^{-1}(\tilde{\Omega}(\mu);\mu)} \quad \text{and}$$

$$\mathbb{J}_{\phi^{-1}}(\tilde{x},\mu) = \underbrace{(\mathbb{J}_\phi(x;\mu))^{-1}} \quad \text{for } \tilde{x} = \phi(x;\mu)$$

By means of the inverse function theorem: ϕ continuous and non-singular in x the ϕ is invertible in a neighborhood of x and $\mathbb{J}_{\phi^{-1}} \circ \phi = (\mathbb{J}_\phi)^{-1}$.

Namely, the determinant verifies:

$$|J_{\phi^{-1}}(\tilde{x};\mu)| = \frac{1}{|J_\phi(x;\mu)|}.$$

Let us do some examples in the change of variables of problems with derivatives.

First of all we recall the chain rule

$$\frac{\partial \tilde{\varphi}(\tilde{x})}{\partial \tilde{x}_i} = \sum_{j=1}^{d} \frac{\partial \varphi(x)}{\partial x_j} \frac{\partial \phi_j^{-1}(\tilde{x};\mu)}{\partial \hat{x}_i}.$$

$$\tilde{\varphi} = \varphi \circ \phi^{-1}$$

In compact expression:

$$\begin{aligned} \nabla_{\tilde{x}} \hat{\varphi}(\tilde{x}) &= \left[J_{\phi^{-1}}(\tilde{x};\mu)\right]^T \nabla_x \varphi(x) \\ &= \left[J_\phi(x;\mu)\right]^{-T} \nabla_x \varphi(x) \end{aligned}$$

Some useful exemples:

STIFFNESS MATRIX

$$\int_{\tilde{\Omega}(\mu)} \nabla_{\hat{x}} \hat{\varphi} \cdot \nabla_{\hat{x}} \tilde{\varphi}\, dx = \int_{\Omega} \left(J_\phi^{-T}(x;\mu) \nabla_x \varphi(x)\right) \cdot \left(J_\phi^{-T}(x;\mu) \nabla_x \varphi(x) \right) |J_\phi(x;\mu)| d$$

That in compact form reads:

$$\int_{\tilde{\Omega}(\mu)} \nabla_{\hat{x}} \tilde{\varphi} \cdot \nabla_{\hat{x}} \tilde{\varphi}\, dx = \sum_{k,l=1}^{d} \int_{\Omega} \frac{\partial \varphi}{\partial x_k} T_{kl} \frac{\partial \varphi}{\partial x_l} dx,$$

where $T(x;\mu) : \mathbb{R}^d \times \mathbb{P} \to \mathbb{R}^{d\times d}$ and is

$$T = J_\phi^{-1}(x;\mu) J_\phi^{-T}(x;\mu) |J_\phi(x;\mu)|.$$

REACTION TERM

$$\int \nabla_{\tilde{x}} \tilde{\varphi}\, \tilde{\varphi}\, d\tilde{x} = \sum_{k,l=1}^{d} \int_{\Omega} \eta_{kl} \frac{\partial \varphi}{\partial x_l} \varphi \, dx \quad \text{where} \quad \eta : \mathbb{R}^d \times \Theta \to \mathbb{R}^{d \times d}$$

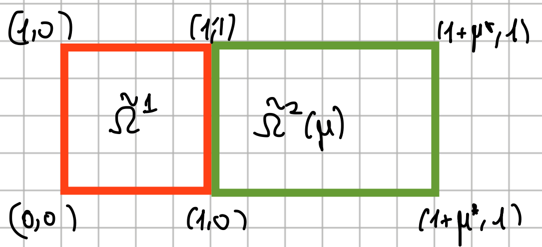
and η is defined as $\eta(x;\mu) = J_{\phi}^{-T}(x;\mu)\, |J_{\phi}(x;\mu)|$.

SOME PRACTICAL EXAMPLE!

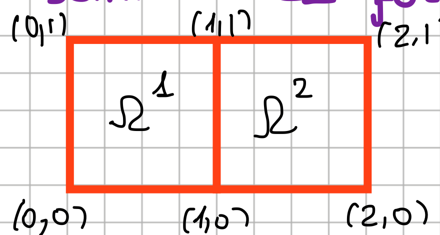
We want to solve the following problem: given $\mu \in [1, 3.5]$, find $\tilde{u}(\mu) \in H_0^1(\tilde{\Omega}(\mu))$ s.t.

$$\begin{cases} -\Delta \tilde{u}(\mu) = \tilde{f}(\mu) & \text{in } \tilde{\Omega}(\mu) \\ \tilde{u}(\mu) = 0 & \text{on } \partial\tilde{\Omega}(\mu) \end{cases}$$

where $\tilde{\Omega}(\mu)$ is the following domain:



WE CHOOSE THE REFERENCE DOMAIN Ω for $\mu = 1$



The bilinear form is

$$\tilde{a}(\tilde{u},\tilde{v};\mu) = \tilde{f}(\tilde{v};\mu) \quad \text{where}$$

$$\tilde{a}(\tilde{u},\tilde{v};\mu) = \sum_{r=1}^{2} \int_{\tilde{\Omega}^r} \nabla\tilde{u} \cdot \nabla\tilde{v}\, d\tilde{x} \quad \text{and} \quad \tilde{f}(\tilde{v};\mu) = \sum_{r=1}^{2} \int_{\tilde{\Omega}^r} \tilde{f}\tilde{v}\, dx$$

We notice that the first subdomain is fixed!

$\Rightarrow \phi(x;\mu) = \mathbb{I}x + 0$, namely $A(\mu) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $c(\mu) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

In this case $|J_{\phi}(x;\mu)| = |A(\mu)| = 1$.

In $\tilde{\Omega}^2(\mu)$, the map is:

$$\phi(x;\mu) = A(\mu)x + c(\mu) \quad \text{with}$$

$$A(\mu) = \begin{bmatrix} \mu & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad c(\mu) = \begin{bmatrix} 1-\mu \\ 0 \end{bmatrix}$$

The map transforms 2 in $1+\mu$ for the first coordinate and 1 stays in 1.

No changes are performed for the second coordinate.

In this case

$$|J\phi(x,\mu)| = \mu \qquad J\phi^{-1}(x,\mu) = \begin{bmatrix} 1/\mu & 0 \\ 0 & 1 \end{bmatrix}.$$

FORCING TERM

IDENTITY

$$\tilde{f}(\tilde{v};\mu) = \int_{\tilde{\Omega}^1} \tilde{f}\tilde{v}\,d\tilde{x} + \int_{\tilde{\Omega}^2} \tilde{f}\tilde{v}\,d\tilde{x} = \int_{\Omega^1} f v\,dx + \mu \int_{\Omega^2} f v\,dx$$

determinant of the Jacobian

Stiff ness matrix

$$\tilde{a}(\tilde{u},\tilde{v};\mu) = \underbrace{\int_{\tilde{\Omega}^1} \nabla_{\tilde{x}}\tilde{u}\,\nabla_{\tilde{x}}\tilde{v}\,d\tilde{x}}_{\int_{\Omega^1} \nabla_x u \cdot \nabla_x v\,dx} + \underbrace{\int_{\tilde{\Omega}^2} \nabla_{\tilde{x}}\tilde{u}\cdot\nabla_{\tilde{x}}\tilde{v}\,d\tilde{x}}_{\sum_{k,\ell=1}^{d} \int_{\Omega^2} \frac{\partial u}{\partial x_k} T_{k\ell} \frac{\partial v}{\partial x_\ell}\,dx}$$

$$T = J_\phi^{-1}(x,\mu)\,J_\phi^{-T}(x,\mu)\,|J\phi(x,\mu)|$$

$$= \begin{bmatrix} 1/\mu & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\mu & 0 \\ 0 & 1 \end{bmatrix} \mu$$

$$= \begin{bmatrix} 1/\mu^2 & 0 \\ 0 & 1 \end{bmatrix} \mu$$

$$T_{12} = 0 = T_{21}$$

The only terms surviving are T_{11} and T_{22}

$$= \frac{1}{\mu}\int_{\Omega^2} \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx + \mu\int_{\Omega^2} \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\,dx$$

At the reduced level, I solve on the reference domain Ω the following problem:

$$\sum_{i=1}^{3} \vartheta_i^a(\mu)\, a_i(u,v) = \sum_{J=1}^{2} \vartheta_J^f(\mu)\, f(v)$$

$$\vartheta_1^a(\mu) = 1 \qquad a_1 = \int_{\Omega^1} \nabla u \cdot \nabla v \, dx$$

$$\vartheta_2^a(\mu) = \frac{1}{\mu} \qquad a_2 = \int_{\Omega^2} \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} \, dx$$

$$\vartheta_3^a(\mu) = \mu \qquad a_3 = \int_{\Omega^2} \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_2} \, dx$$

$$\vartheta_1^f(\mu) = 1 \qquad f_1 = \int_{\Omega_1} f v \, dx$$

$$\vartheta_2^f(\mu) = \mu \qquad f_2 = \int_{\Omega^2} f v \, dx$$