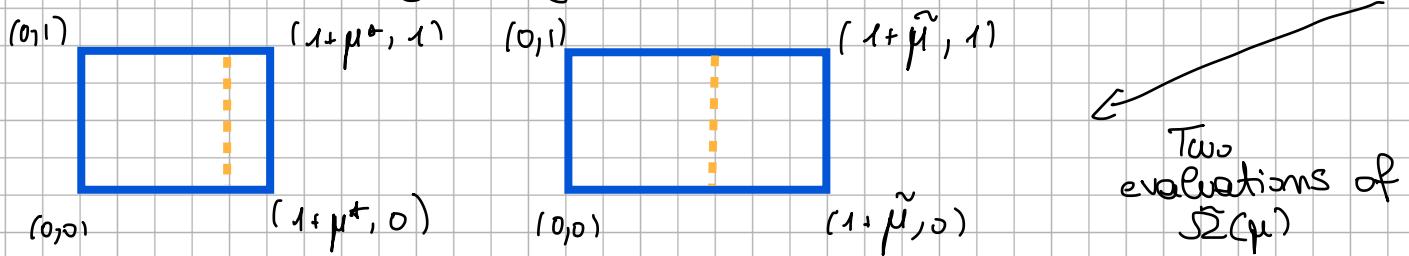


## GEOMETRIC PARAMETRIZATION

Let us assume  $\mu \in \mathcal{P}$  our input parameter.

The parameters can change the physics of the domain, but also the geometry. The  $\mu$ -dependent domain is  $\tilde{\Omega}(\mu)$ .



$\tilde{\Omega}(\mu)$  is the ORIGINAL DOMAIN.

To solve the problem we need to define a REFERENCE DOMAIN  $\Omega$ , that is  $\mu$ -independent!

The original domain  $\tilde{\Omega}(\mu)$  is obtained by means of a parametric map  $\phi : \Omega \times \mathcal{P} \rightarrow \tilde{\Omega}$  s.t.  $x \mapsto \tilde{x}(\mu)$  and, namely,  $\tilde{\Omega}(\mu) = \phi(\Omega; \mu)$ ,  $\forall \mu \in \mathcal{P}$ .

Let us specify the nature of  $\mu$ :

$\mu = (\mu_1^{ph}, \dots, \mu_p^{ph}, \mu_g)$  where  $\mu_g$  is the only parameter interacting with the domain (THIS IS A SIMPLIFICATION)  
Our PDE problem will be: given  $\mu \in \mathcal{P}$ , find  $\tilde{u}(\mu) \in \tilde{V}(\mu_g)$  s.t.  
 $\tilde{a}(\tilde{u}(\mu), v; \mu) = f(v; \mu) \quad \forall v \in V(\mu_g)$ .

Thanks to the map  $\phi$ , we can PULL-BACK the problem onto the reference domain  $\Omega$ : given  $\mu \in \mathcal{P}$ , find  $u(\mu) \in V$  s.t.

$$a(u(\mu), v; \mu) = f(v; \mu) \quad \forall v \in V.$$



The two problems are different!! There are some geometric factors induced by the map  $\phi$ .

Let us recall that  $x \in \Omega$  and  $\tilde{x} \in \tilde{\Omega}(\mu)$ .

Moreover, both the subdomains are subsets of  $\mathbb{R}^d$ .

We can define the JACOBIAN MATRIX  $J$  of  $\phi$ ,  $J_\phi(x; \mu) \in \mathbb{R}^{d \times d}$  as

$$[J_\phi(x; \mu)]_{k\ell} = \frac{\partial \phi_k(x; \mu)}{\partial x_\ell}(x) \quad k, \ell = 1, \dots, d.$$

If for any  $\mu \in \mathcal{P}$ , the determinant  $|J_\phi(x; \mu)| \neq 0$ , then  $\phi$  is well-posed!

For any  $\tilde{\psi}: \tilde{\Omega} \rightarrow \mathbb{R}$  it holds:

$$\int_{\tilde{\Omega}(\mu)} \tilde{\psi}(\tilde{x}) d\tilde{x} = \int_{\Omega} \psi(x) |J_\phi(x; \mu)| dx,$$

where  $\psi = \tilde{\psi} \circ \phi$ .

### AFFINE TRANSFORMATION:

We will always assume  $\phi$  being an affine map of the form

$$\phi(x; \mu) = A(\mu)x + c(\mu) \quad \text{with } A(\mu) \in \mathbb{R}^{d \times d} \text{ and } c(\mu) \in \mathbb{R}^d$$

In this case  $J_\phi(x; \mu) = A(\mu)$  and the determinant depends only on the parameter  $\mu$ .

If  $\phi$  is well-posed,  $\phi^{-1}: \tilde{\Omega} \rightarrow \Omega$  is defined.

RECALL:  $\Omega = \phi^{-1}(\tilde{\Omega}(\mu); \mu)$  and

$$J_{\phi^{-1}}(\tilde{x}, \mu) = \underbrace{(J_\phi(x; \mu))^{-1}}_{\text{for } \tilde{x} = \phi(x; \mu)}$$

By means of the inverse function theorem:  $\phi$  continuous and non-singular in  $x$  the  $\phi$  is invertible in a neighborhood of  $x$  and  $J_{\phi^{-1}} \circ \phi = (J_\phi)^{-1}$ .

Namely, the determinant verifies:

$$|\mathcal{J}_{\phi^{-1}}(\tilde{x}; \mu)| = \frac{1}{|\mathcal{J}\phi(x; \mu)|}.$$

Let us do some examples in the change of variables of problems with derivatives.

First of all we recall the chain rule

$$\frac{\partial \tilde{\psi}(\tilde{x})}{\partial \tilde{x}_i} = \sum_{j=1}^d \frac{\partial \psi(x)}{\partial x_j} \frac{\partial \phi_j^{-1}(\tilde{x}; \mu)}{\partial \tilde{x}_i}$$

$\tilde{\psi} \stackrel{\sim}{=} \psi \circ \phi^{-1}$

In compact expression:

$$\begin{aligned} \nabla_{\tilde{x}} \tilde{\psi}(\tilde{x}) &= [\mathcal{J}_{\phi^{-1}}(\tilde{x}; \mu)]^T \nabla_x \psi(x) \\ &= [\mathcal{J}\phi(x; \mu)]^{-T} \nabla_x \psi(x) \end{aligned}$$

Some useful examples:

### STIFFNESS MATRIX

$$\int_{\tilde{\Omega}(\mu)} \nabla_{\tilde{x}} \tilde{\psi} \cdot \nabla_{\tilde{x}} \tilde{\varphi} d\tilde{x} = \int_{\Omega} (\mathcal{J}\phi^{-T}(x; \mu) \nabla_x \psi(x)) \cdot (\mathcal{J}\phi^{-T}(x; \mu) \nabla_x \varphi(x)) |\mathcal{J}\phi(x; \mu)| dx$$

That in compact form reads:

$$\int_{\tilde{\Omega}(\mu)} \nabla_{\tilde{x}} \tilde{\psi} \cdot \nabla_{\tilde{x}} \tilde{\varphi} d\tilde{x} = \sum_{k, e=1}^d \int_{\Omega} \frac{\partial \psi}{\partial x_k} T_{ke} \frac{\partial \varphi}{\partial x_p} dx,$$

where  $T(x; \mu) : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^{d \times d}$  and is

$$T = \mathcal{J}\phi^{-1}(x; \mu) \mathcal{J}\phi^{-T}(x; \mu) |\mathcal{J}\phi(x; \mu)|.$$

## REACTION TERM

$$\int \nabla \tilde{\psi} \cdot \tilde{\varphi} dx = \sum_{k=1}^d \int_{\Omega} \frac{\partial \psi}{\partial x_k} \varphi dx \quad \text{where } \eta : \mathbb{R}^d \times \mathcal{P} \rightarrow \mathbb{R}^{d \times d}$$

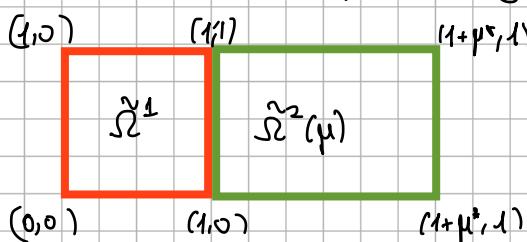
and  $\eta$  is defined as  $\eta(x; \mu) = \mathcal{J}_\phi^T(x; \mu) | \mathcal{J}_\phi(x; \mu) |$ .

## SOME PRACTICAL EXAMPLE!

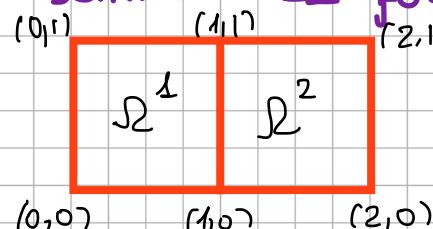
We want to solve the following problem: given  $\mu \in [1, 3.5]$ , find  $\tilde{u}(\mu) \in H_0^1(\tilde{\Omega}(\mu))$  s.t.

$$\begin{cases} -\Delta \tilde{u}(\mu) = f(\mu) & \text{in } \tilde{\Omega}(\mu) \\ \tilde{u}(\mu) = 0 & \text{on } \partial \tilde{\Omega}(\mu) \end{cases}$$

where  $\tilde{\Omega}(\mu)$  is the following domain:



WE CHOOSE THE DOMAIN  $\Omega$  FOR  $\mu = 1$



The bilinear form is

$$\tilde{a}(\tilde{u}, \tilde{v}; \mu) = \tilde{f}(\tilde{v}; \mu) \quad \text{where}$$

$$\tilde{a}(\tilde{u}, \tilde{v}; \mu) = \sum_{r=1}^2 \int_{\tilde{\Omega}^r} \nabla \tilde{u} \cdot \nabla \tilde{v} dx \quad \text{and } \tilde{f}(\tilde{v}; \mu) = \sum_{r=1}^2 \int_{\tilde{\Omega}^r} \tilde{f} \tilde{v} dx$$

We notice that the first subdomain is fixed!

$$\Rightarrow \phi(x; \mu) = Ix + 0, \quad \text{namely } A(\mu) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } c(\mu) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In this case  $|\mathcal{J}_\phi(x; \mu)| = |A(\mu)| = 1$ .

In  $\tilde{\Omega}^2(\mu)$ , the map is:

$$\phi(x; \mu) = A(\mu)x + c(\mu) \quad \text{with}$$

$$A(\mu) = \begin{bmatrix} \mu & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and } c(\mu) = \begin{bmatrix} 1-\mu \\ 0 \end{bmatrix}$$

The map transforms 2 in  $1+\mu$  be the first coordinate  
and 1 stays in 1.

No changes are performed for the second coordinate.

In this case

$$|\mathcal{J}\phi(x, \mu)| = \mu \quad \mathcal{J}\phi^{-1}(x, \mu) = \begin{bmatrix} 1/\mu & 0 \\ 0 & 1 \end{bmatrix}.$$

FORCING TERM

$$\tilde{f}(\tilde{v}; \mu) = \int_{\tilde{\Omega}^1} \tilde{f} \tilde{v} d\tilde{x} + \int_{\tilde{\Omega}^2} \tilde{f} \tilde{v} d\tilde{x} = \int_{\Omega^1} f v dx$$

IDENTITY

$$+ \mu \int_{\Omega^2} f v dx$$

Stiffness matrix

$$\tilde{a}(\tilde{u}, \tilde{v}; \mu) = \int_{\tilde{\Omega}^1} \nabla_{\tilde{x}} \tilde{u} \cdot \nabla_{\tilde{x}} \tilde{v} d\tilde{x} + \int_{\tilde{\Omega}^2} \nabla_{\tilde{x}} \tilde{u} \cdot \nabla_{\tilde{x}} \tilde{v} d\tilde{x}$$

$$\underbrace{\int_{\Omega^1} \nabla_x u \cdot \nabla_x v dx}_{-} + \sum_{k,e=1}^d \int_{\Omega^2} \frac{\partial u}{\partial x_k} T_{ke} \frac{\partial v}{\partial x_e} dx$$

↳ determinant of the Jacobian

$$T = \mathcal{J}_\phi^{-1}(x, \mu) \mathcal{J}_\phi^{-T}(x, \mu) |\mathcal{J}\phi(x, \mu)|$$

$$= \begin{bmatrix} 1/\mu & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\mu & 0 \\ 0 & 1 \end{bmatrix} \mu$$

$$= \begin{bmatrix} 1/\mu^2 & 0 \\ 0 & 1 \end{bmatrix} \mu$$

$$T_{12} = 0 = T_{21}$$

The only terms surviving are  $T_{11}$  and  $T_{22}$

$$= \frac{1}{\mu} \int_{\Omega^2} \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} dx + \mu \int_{\Omega^2} \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_2} dx$$

At the reduced level, I solve on the reference domain  $\Omega$  the following problem:

$$\sum_{i=1}^3 \Theta_i^a(\mu) a_i(u, v) = \sum_{j=1}^2 g_j^f(\mu) f(v)$$

$$\Theta_1^a(\mu) = 1 \quad a_1 = \int_{\Omega^1} \nabla u \cdot \nabla v \, dx$$

$$\Theta_2^a(\mu) = \frac{1}{\mu} \quad a_2 = \int_{\Omega^2} \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} \, dx$$

$$\Theta_3^a(\mu) = \mu \quad a_3 = \int_{\Omega^2} \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_2} \, dx$$

$$\Theta_1^f(\mu) = 1 \quad f_1 = \int_{\Omega^1} f v \, dx$$

$$\Theta_2^f(\mu) = \mu \quad f_2 = \int_{\Omega^2} f v \, dx$$