Lattice fundamentals

Fernando Virdia — `https://fundamental.domains`
EPFL-ETH Summer School on Lattice-based Cryptography, July 2025

Housekeeping

Interaction

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Slides

You can find them on
`https://github.com/fvirdia/lattice-summer-school-2025-fundamentals`

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Sources

These notes have been adapted from pre-existing material, mainly [1], [2].
References at the end of the deck.

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- Today, an extremely popular subject.
- They are used to build PQC, FHE, iO, proof systems...

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Enough said, let's start.

Definitions and basic properties

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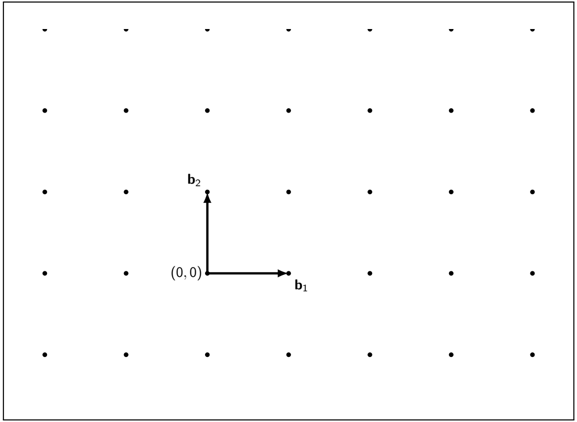
Defining a lattice

- Informally, lattices are a discrete equivalent of vector spaces



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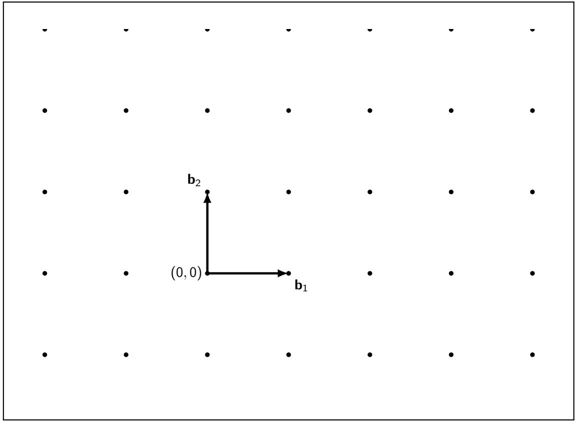

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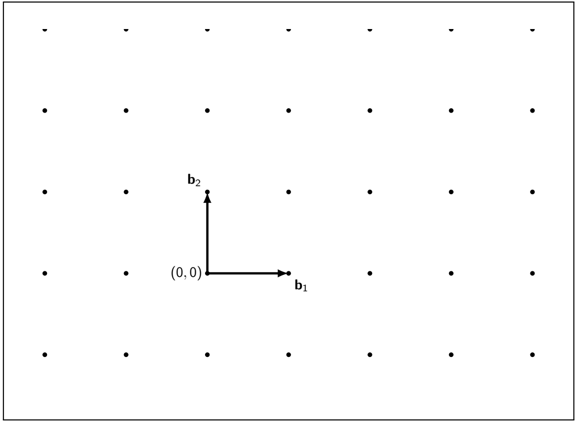

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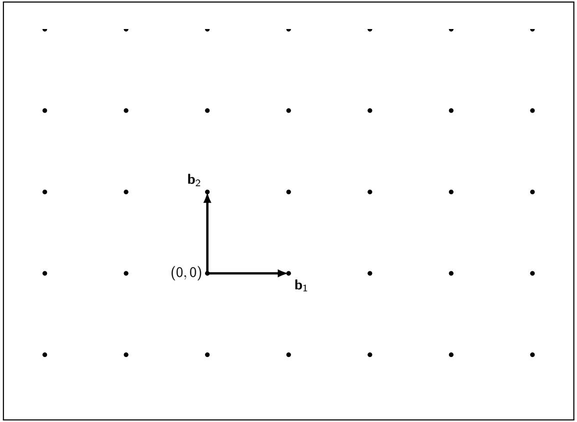
- Other lattices are *linear transformations* of $\mathbb{Z}^n$.



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Definition

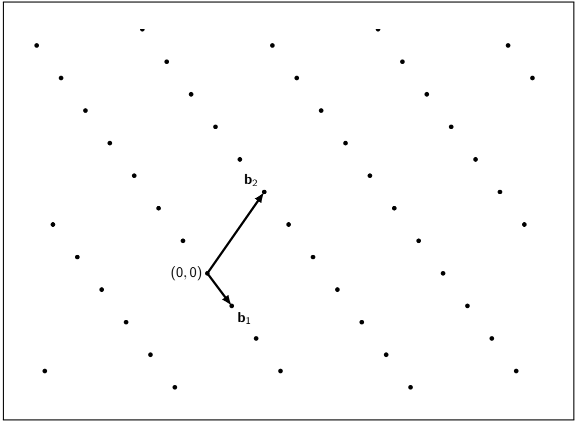
Let $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^d$ be lin. indep.

$$\mathbf{B} := \begin{bmatrix} \text{---} \ \mathbf{b}_1 \ \text{---} \\ \vdots \\ \text{---} \ \mathbf{b}_n \ \text{---} \end{bmatrix}.$$

We say that their integer span

$$\begin{aligned} \Lambda = \Lambda(\mathbf{B}) &:= \operatorname{span}_{\mathbb{Z}}(\mathbf{b}_1, \ldots, \mathbf{b}_n) \\ &= \{x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n : x_i \in \mathbb{Z}\} \\ &= \{\mathbf{xB} : \mathbf{x} \in \mathbb{Z}^n\} \subset \mathbb{R}^d, \end{aligned}$$

is a *real lattice* of rank, n.



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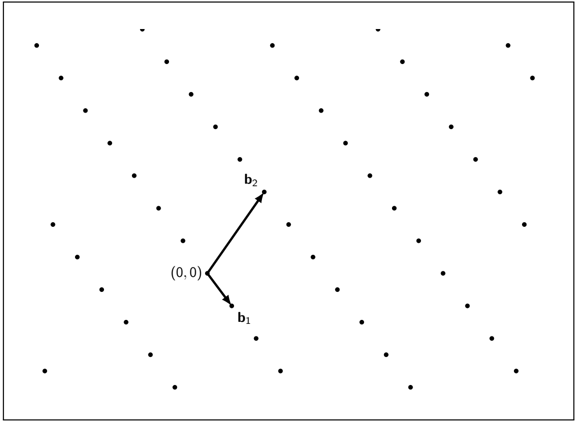

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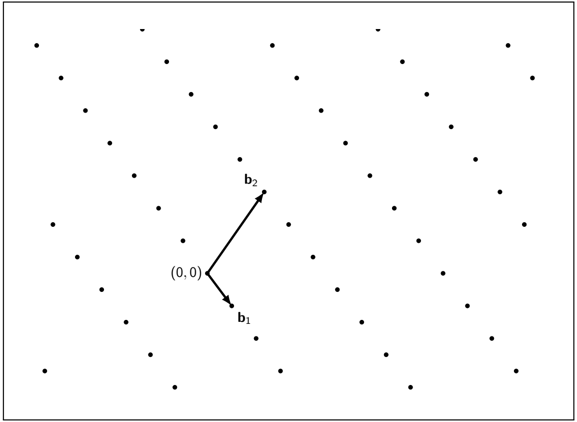

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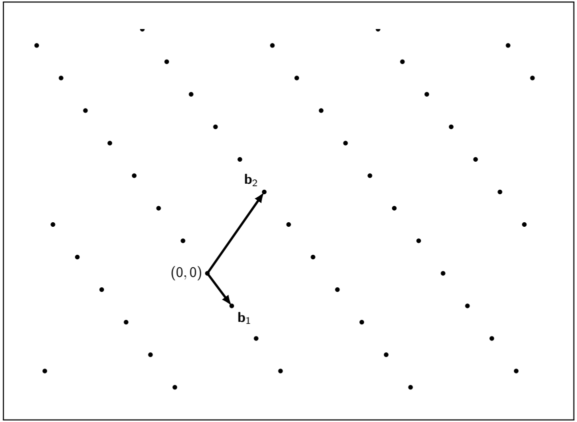

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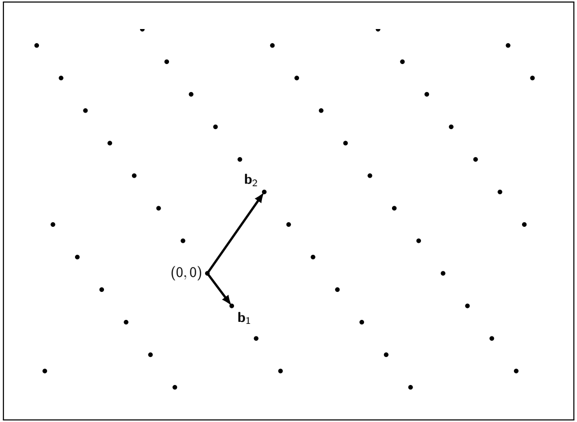

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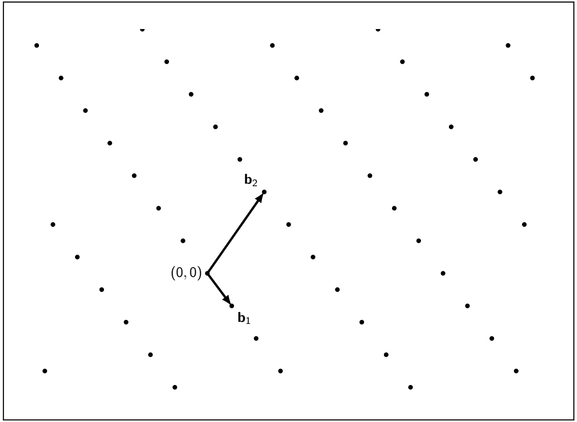
Definition (Sublattices)

Let $\Lambda \subset \mathbb{R}^n$ be a real lattice of rank n. We call any subgroup $\Lambda' \subset \Lambda$ a sublattice of Λ.



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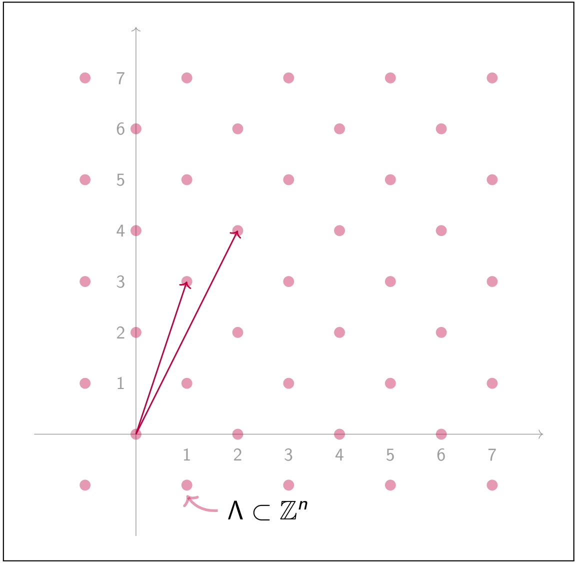

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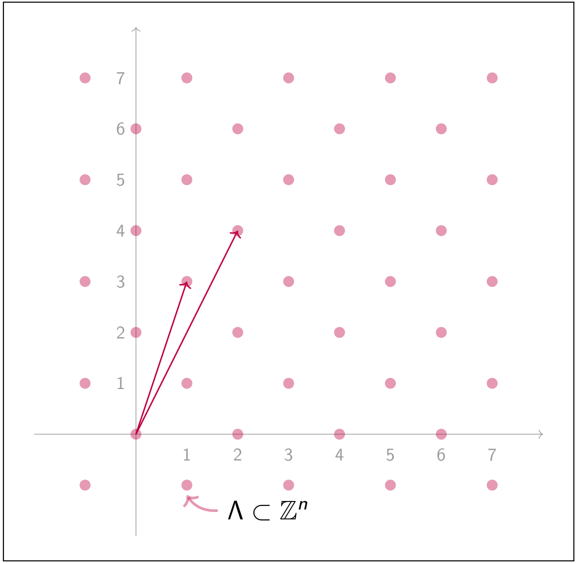

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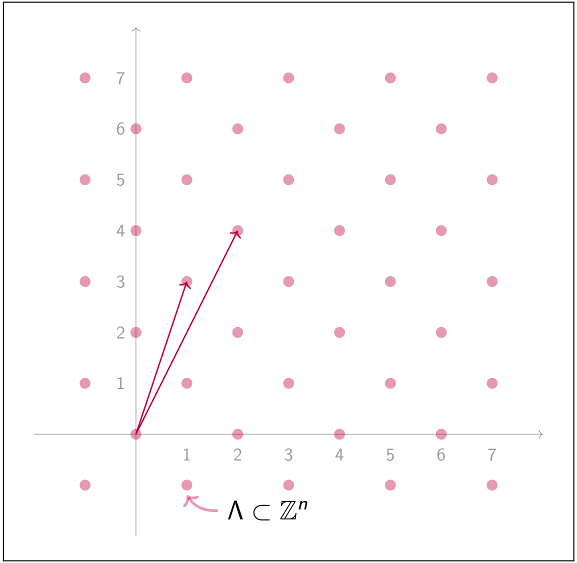

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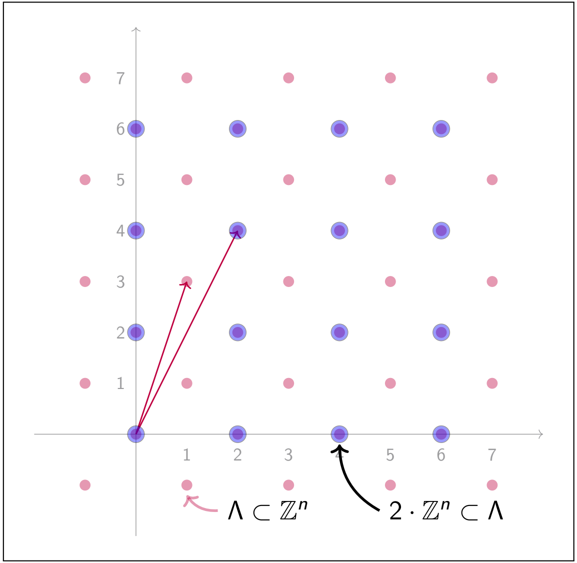

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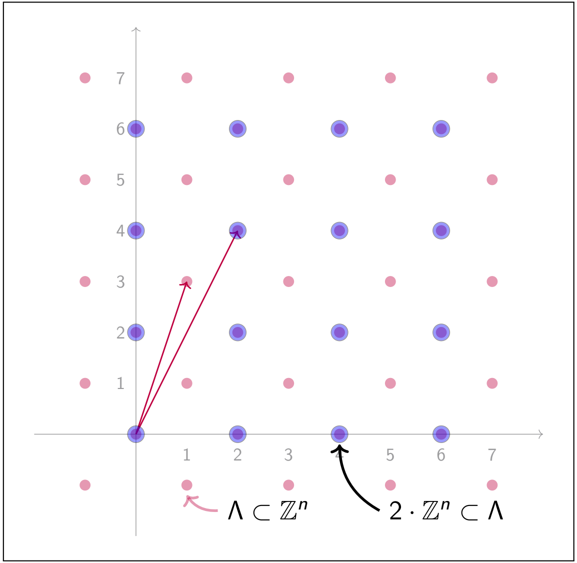

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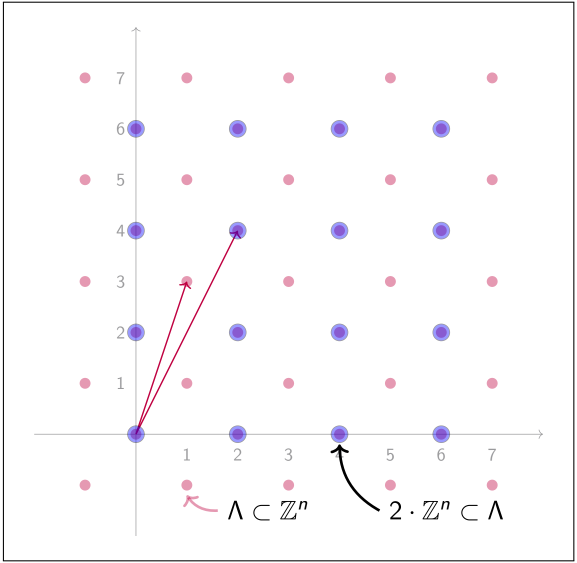

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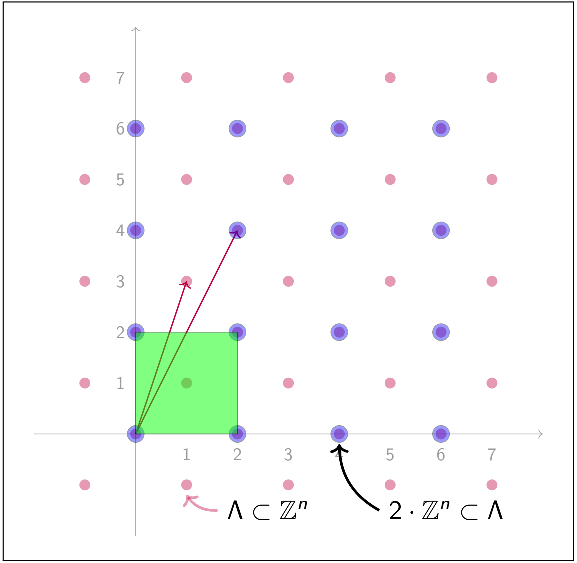
- This allows us to sample $U(\Lambda \bmod q)$.



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Lemma

$\Lambda(\mathbf{B}) = \Lambda(\mathbf{B}')$ *if and only if* $\mathbf{B}' = \mathbf{U}\mathbf{B}$ *where* $\mathbf{U}$ *is unimodular* ($\mathbf{U} \in \mathbb{Z}^{n \times n}$ *with* $\det(\mathbf{U}) = \pm 1$) $\square$

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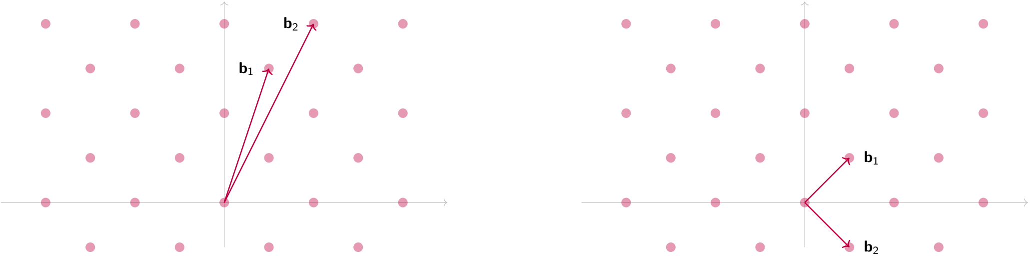
Unimodular matrices $\mathbf{U} \in \mathrm{GL}_n(\mathbb{Z})$ give bijections $\mathbb{Z}^n \to \mathbb{Z}^n$ between coefficient vectors.



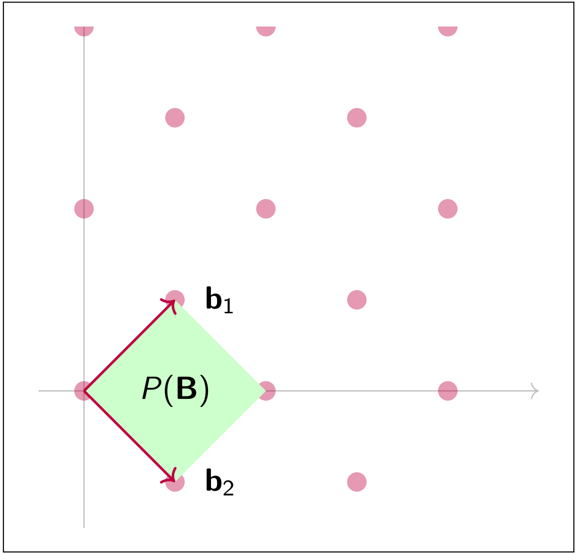
$$\mathbf{B} = \begin{bmatrix} (1, & 3) \\ (2, & 4) \end{bmatrix} \xmapsto{\mathbf{U} = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix}} \mathbf{B}' = \begin{bmatrix} (1, & 1) \\ (1, & -1) \end{bmatrix}$$

Given a lattice basis, we can define its *fundamental parallelepiped* $P(\mathbf{B})$.

Definition

Given a lattice basis $\mathbf{B}$, its fundamental parallelepiped is the set

$$P(\mathbf{B}) := \{x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n : x_i \in [0,1)\}.$$



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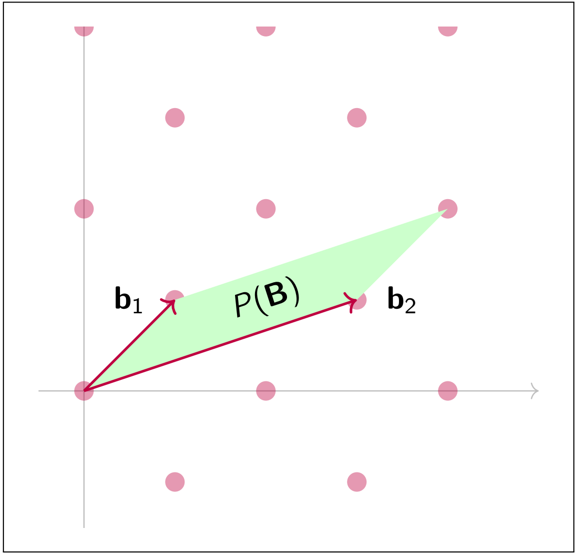

We now define a lattice invariant.

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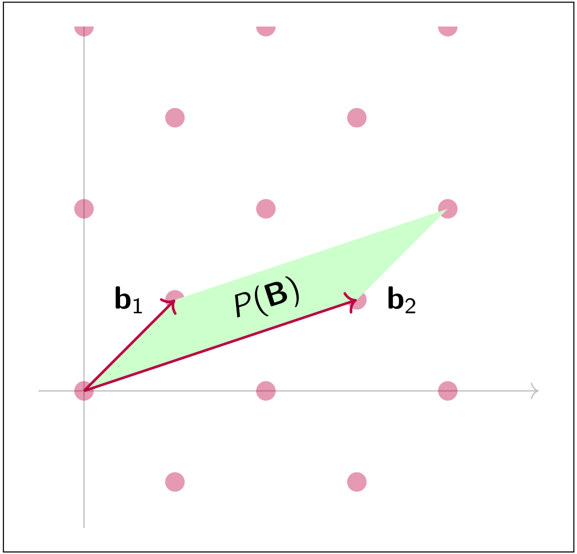
Given any basis $\mathbf{B}$ the *volume* of Λ is

$$\mathrm{vol}(\Lambda) := \sqrt{\det(\mathbf{B}\mathbf{B}^t)}$$



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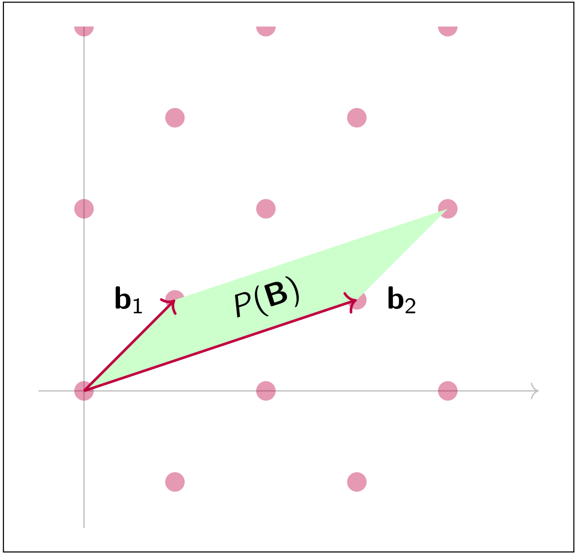
Lemma

- $\mathrm{Vol}(\Lambda) = \mathrm{Vol}(P(\Lambda)) = \int_{P(\Lambda)} d\mathbf{v}$
- *If* $\mathbf{B} \in \mathbb{Z}^{n \times n}$, *then* $\mathrm{Vol}(\Lambda) = |\det(\mathbf{B})|$



Sublattices may potentially have a smaller rank or a larger volume than Λ.

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Example

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WLOG, we can choose to work only with unit-volume lattices.

- Any lattice keeps the same "structure" when scaled down by $\mathrm{Vol}(\Lambda)^{1/n}$ in all directions.

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- $\Lambda = \mathbb{Z} \times (2\mathbb{Z})$ has $\lambda_1(\Lambda) = 1$ and $\lambda_2(\Lambda) = 2$.

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Definition (Hermite's constant [3])

Let $\mathcal{L}_n$ be the set of real lattices of rank n. Then Hermite's constant for rank n lattices, γ_n, is

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Theorem (Hermite's inequality [3])

Let $n \geq 2$ be an integer. Then $\gamma_n \leq \gamma_2^{n-1}$.

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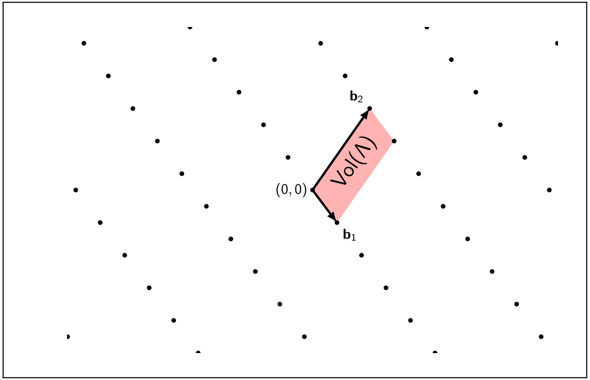

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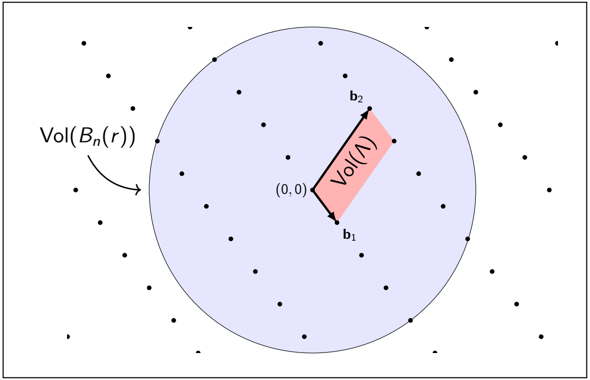

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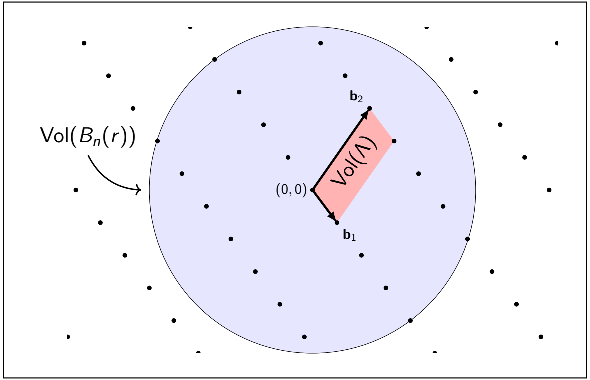

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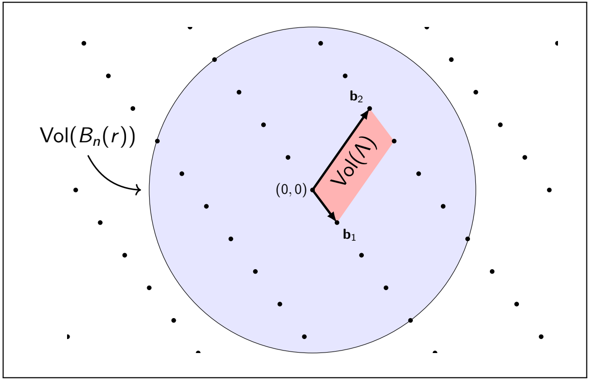

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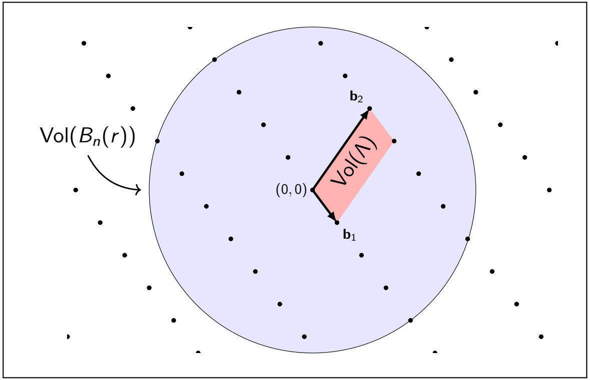

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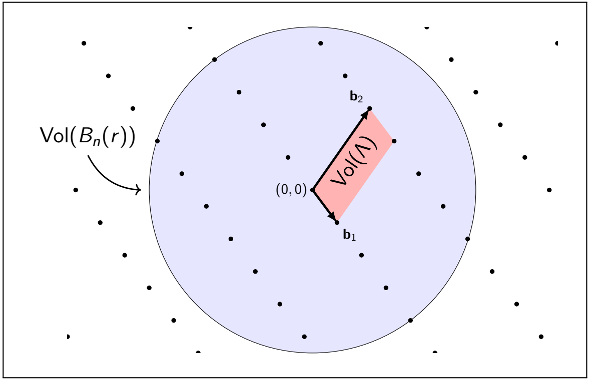

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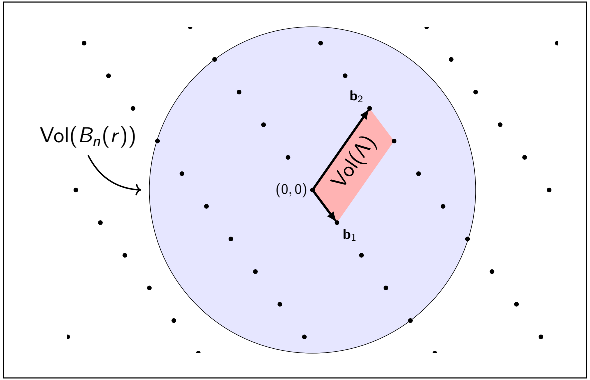

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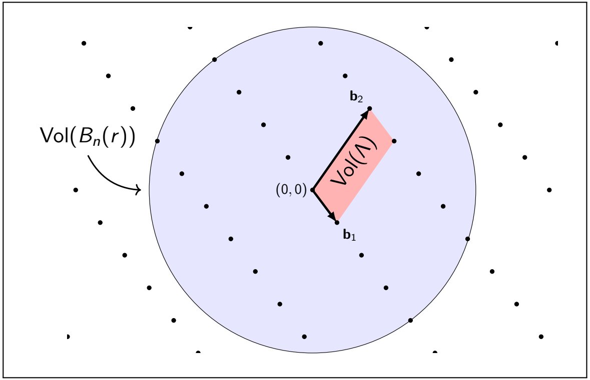

While estimating λ_1 is generally easy, finding a vector realising λ_1 is generally hard!

Computational problems and hardness assumptions

Fernando Virdia — `https://fundamental.domains`
EPFL-ETH Summer School on Lattice-based Cryptography, July 2025

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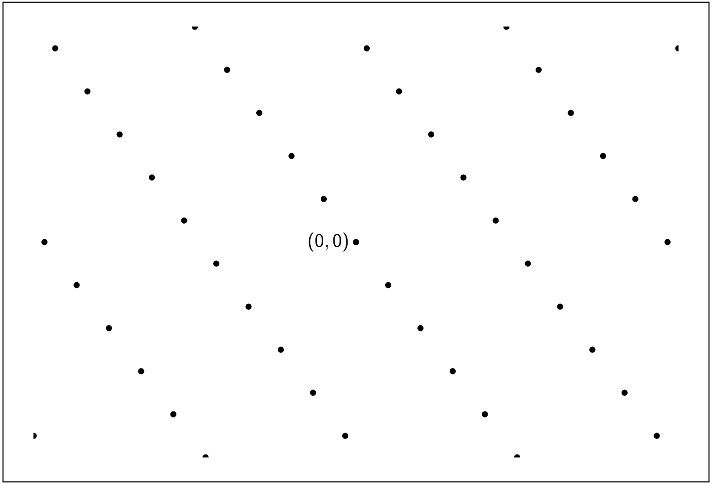
- What about lattices as sources of computational problems?
- Various questions can be asked of a lattice, many giving rise to problems hard in the worst case.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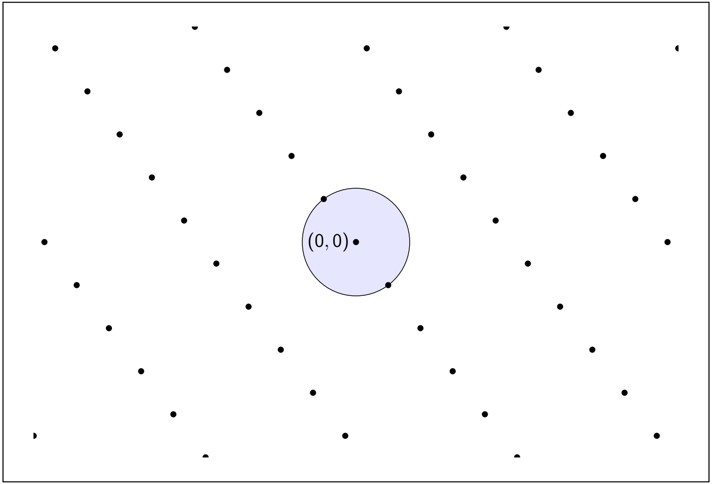
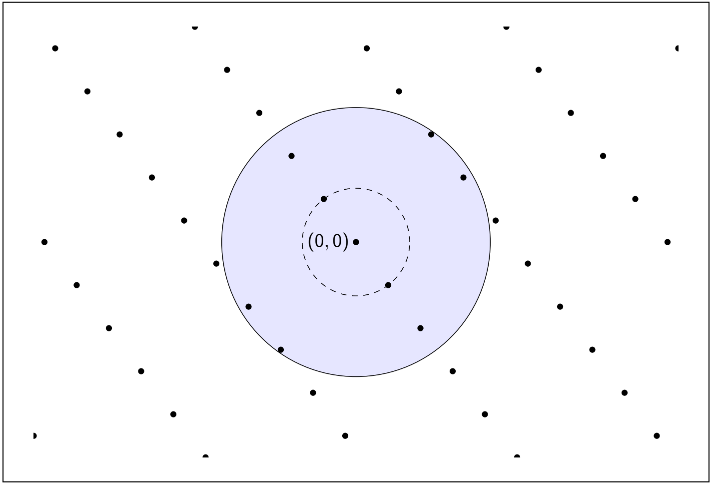


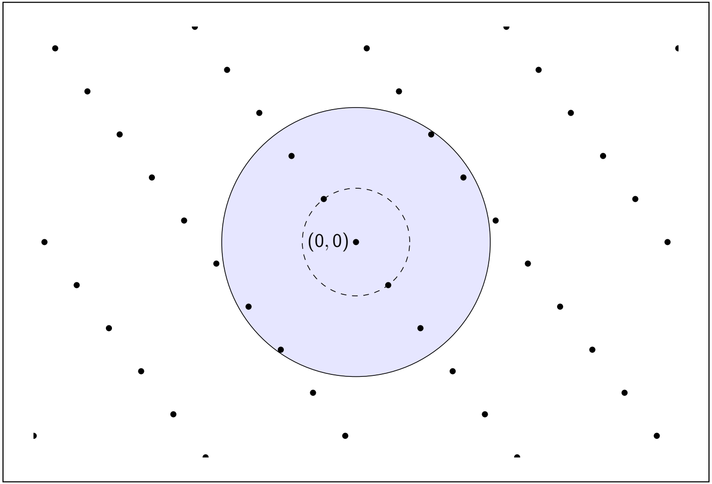
Definition (Shortest Vector Problem (SVP))

Given a lattice Λ find a vector $\mathbf{v} \in \Lambda$ of norm $\lambda_1(\Lambda)$.



Definition (γ-approximate Shortest Vector Problem (approx-SVP$_\gamma$))

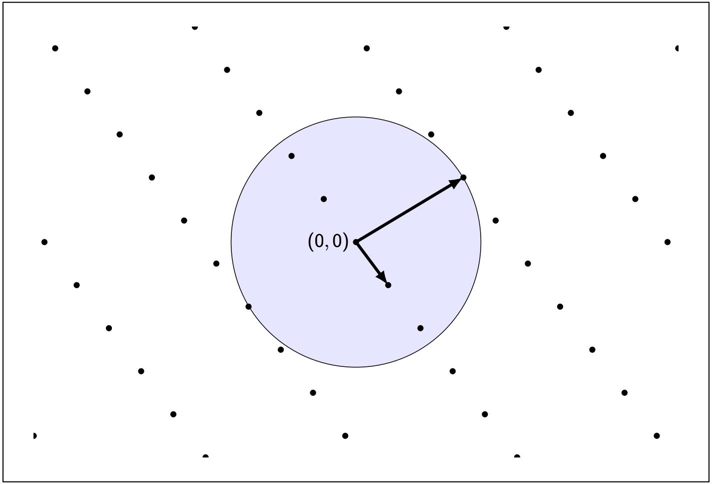
Given a lattice Λ, find a non-zero vector $\mathbf{v} \in \Lambda$ of norm $\leq \gamma \cdot \lambda_1(\Lambda)$.



Definition (γ-Hermite Shortest Vector Problem (Hermite-SVP$_\gamma$))

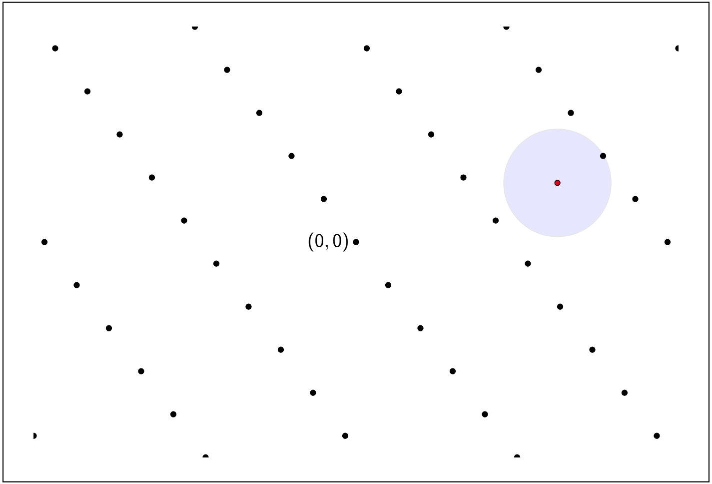
Given a lattice Λ, find a non-zero vector $\mathbf{v} \in \Lambda$ of norm $\leq \gamma \cdot \mathrm{vol}(\Lambda)^{1/n}$.

Approx-SVP is *relative* to Λ, Hermite-SVP is *absolute*: every Λ can be scaled to $\mathrm{Vol}(\Lambda) = 1$.



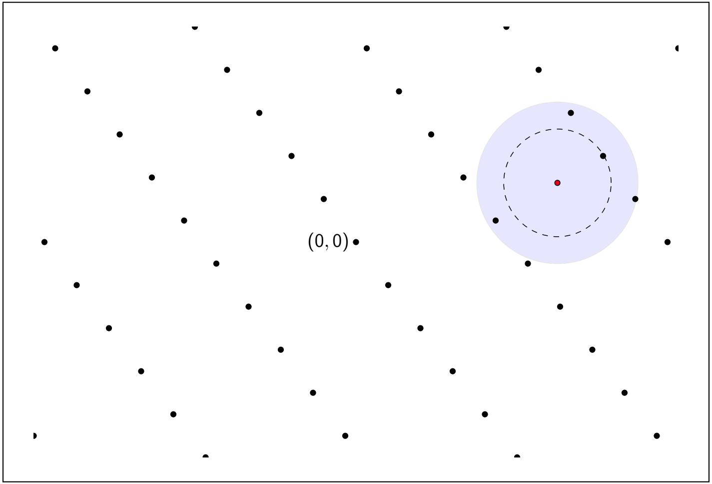
Definition (γ-Shortest Independent Vectors Problem (SIVP$_\gamma$))

Given a lattice Λ of rank n, find n linearly independent lattice vectors $\mathbf{v}_i \in \Lambda$ of norm at most $\gamma \cdot \lambda_n(\Lambda)$.



Definition (γ-Closest Vector Problem (CVP_γ))

Given a lattice basis $\mathbf{B}$ and a vector $\mathbf{v} \in \mathrm{span}_{\mathbb{R}}(\mathbf{B})$, find a lattice point $\mathbf{u} \in \Lambda(\mathbf{B})$ such that $\|\mathbf{v} - \mathbf{u}\| \leq \gamma \cdot \min_{\mathbf{x} \in \Lambda(\mathbf{B})} \|\mathbf{v} - \mathbf{x}\|$.



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$$\text{dist}(\mathbf{v}, \Lambda) := \min_{\mathbf{x} \in \Lambda} \|\mathbf{v} - \mathbf{x}\| < \gamma \cdot \lambda_1(\Lambda),$$

find $\mathbf{t} \in \Lambda$ such that $\|\mathbf{v} - \mathbf{t}\| = \text{dist}(\mathbf{v}, \Lambda)$.

Note: $\mathbf{t}$ is unique (up to $\pm$) if $\gamma < 1/2$.

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- These are relevant when a short vector is "planted" into a lattice.

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In the *minicrypt* corner

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In the *minicrypt* corner	In the *cryptomania* corner
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Homogeneous SIS is the SIS problem when $\mathbf{b} = \mathbf{0}$.

Inhomogeneous SIS (I-SIS) is the SIS problem when $\mathbf{b} \xleftarrow{\$} U(\mathbb{Z}_q^n)$.

Most commonly, $p = 2$ or $p = \infty$.

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$\Rightarrow$ $\mathbf{x}-\mathbf{x}'$ is an $\mathrm{SIS}_{B,\infty}$ solution. $\square$

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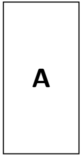

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return $(\mathbf{a}_i, b_i)$

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Search-LWE

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- For any SIS instance, we can use the Gaussian heuristic to estimate the norm of solutions.

$$\begin{bmatrix} \mathbf{A}_0 & \mathbf{A}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{y} \end{bmatrix} \Leftrightarrow \begin{bmatrix} \mathbf{I}_n & \mathbf{A}_0^{-1} \times \mathbf{A}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0 \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbf{A}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{y}} \end{bmatrix}$$

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- Let $\mathbf{A} = [\mathbf{A}_0 \mid \mathbf{A}_1] \leftarrow U(\mathbb{Z}_q^{n\times n} \times \mathbb{Z}_q^{n\times(m-n)})$, with high probability, $\det(\mathbf{A}_0) \neq 0 \bmod q$.
- Pick an usnusually short $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1) \in \mathbb{Z}^n \times \mathbb{Z}^{m-n}$ and let $\mathbf{y} := \mathbf{A}\mathbf{x} \bmod q$.
- Multiply through by $\mathbf{A}_0^{-1} \bmod q$. Let $\widetilde{\mathbf{A}} = \mathbf{A}_0^{-1} \times \mathbf{A}_1$ and $\widetilde{\mathbf{y}} = \mathbf{A}_0^{-1}\mathbf{y}$.

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- $(\widetilde{\mathbf{A}}, \widetilde{\mathbf{y}} = \mathbf{A}_0^{-1}\mathbf{y})$ are n LWE samples with secret $\mathbf{x}_1 \in \mathbb{Z}^{m-n}$, and error vector $\mathbf{x}_0$.

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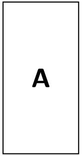

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$$\implies \mathbf{e} \text{ is a solution to the I-SIS instance } \mathbf{A}^{\perp}\mathbf{x} = \mathbf{y} \bmod q \text{ where } \mathbf{y} := \mathbf{A}^{\perp}\mathbf{b} \sim U(\mathbb{Z}_q^n)$$

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- IND-CPA follows from each ciphertext using its own pad.

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Proof.

$$
\begin{array}{rlrrr}
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for $\mathbf{u}_i, \mathbf{w}_i, w' \overset{\text{iid}}{\sim} U(\mathbb{Z}_q)$. $\square$

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  - From worst-case $\mathrm{BDD}_{n^{-1/2}}$ to average-case LWE with $m \in \mathrm{poly}(n)$, χ_e discrete Gaussian with $\sigma/q \in (0, 1)$ [10].

Solving SIS and LWE

Fernando Virdia — `https://fundamental.domains`
EPFL-ETH Summer School on Lattice-based Cryptography, July 2025

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- Now we will talk about such attacks.

Linear algebra

- Given a basis $\mathbf{B}$, we can derive an orthogonal basis $\mathbf{B}^*$ via the Gram–Schmidt process.
- The rows of $\mathbf{B}^*$ are

$$\mathbf{b}_i^* = \mathbf{b}_i - \sum_{j<i} \mu_{i,j} \mathbf{b}_j^* \quad \text{for} \quad i \in [d], \quad \text{where} \quad \mu_{i,j} = \langle \mathbf{b}_i, \mathbf{b}_j^* \rangle / \|\mathbf{b}_j^*\|^2 \quad \text{for} \quad i > j.$$

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- In matrix form,

$$\begin{bmatrix} \text{---} \; \mathbf{b}_1 \; \text{---} \\ \vdots \\ \text{---} \; \mathbf{b}_d \; \text{---} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ \mu_{2,1} & 1 & & \\ \vdots & \ddots & \ddots & \\ \mu_{d,1} & \cdots & \mu_{d,d-1} & 1 \end{bmatrix} \begin{bmatrix} \text{---} \; \mathbf{b}_1^* \; \text{---} \\ \vdots \\ \text{---} \; \mathbf{b}_d^* \; \text{---} \end{bmatrix}.$$

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Remark

Recall that the volume of a lattice $\Lambda(\mathbf{B})$ is $|\det(\mathbf{B})|$. Given the Gram-Schmidt orthogonalisation $\mathbf{b}_1^*, \ldots, \mathbf{b}_n^*$ of the basis $\mathbf{B}$, $|\det(\mathbf{B})| = \prod_{i=1}^{n} \|\mathbf{b}_i^*\|$.

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  - The *flatter* a basis profile is, the closer to orthogonal is the basis.

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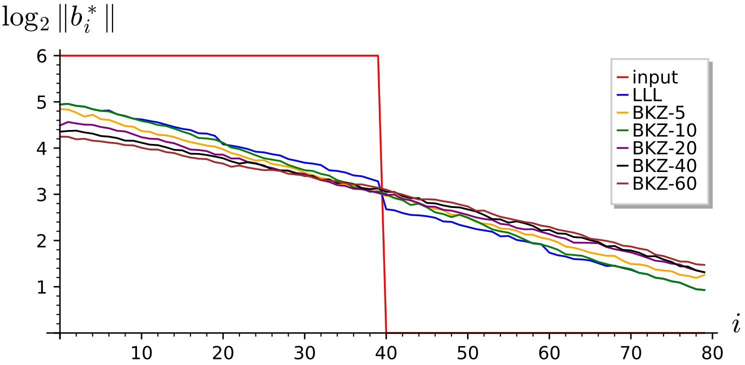

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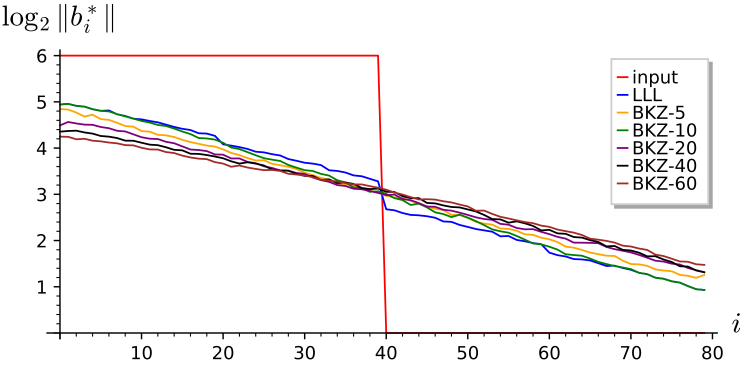

Remark

Interestingly, it would appear that the log-plot of a reduced basis profile forms a straight line.

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- LLL has root-Hermite factor $\delta = 1.02$, BKZ-β has $\delta = \left((\pi\beta)^{1/\beta}\frac{\beta}{2\pi e}\right)^{\frac{1}{2(\beta-1)}}$, independent of n.

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- The GSA captures the straightness of the log-plot of the basis profiles we saw before.
  - $\log \|\mathbf{b}_i^*\| = (i-1) \cdot \log \alpha + \log \|\mathbf{b}_1\|$ is a straight line with slope $\log \alpha$.

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Lemma

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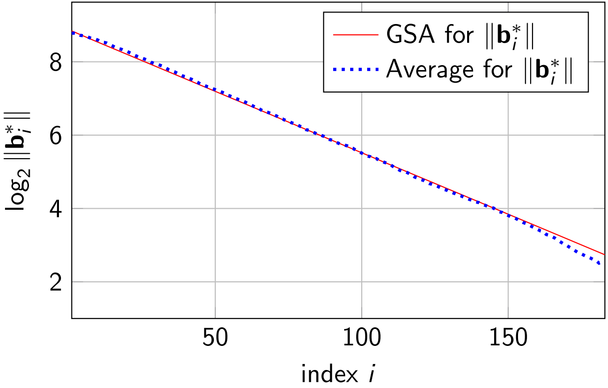

Figure 1: Comparison of a GSA prediction for the profile of a BKZ-56-reduced basis.

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Solving SIS via lattice reduction

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- First construct a basis for the right-kernel of $\mathbf{A}$ over $\mathbb{Z}_q$.
  - Put $\mathbf{A}$ into reduced echelon form $\begin{bmatrix} \mathbf{I}_n & \hat{\mathbf{A}} \end{bmatrix}$ where $\hat{\mathbf{A}} \in \mathbb{Z}_q^{n\times(m-n)}$.
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- With a basis $\mathbf{B}$ for our integer lattice, we can then use strong lattice reduction to recover a short vector in the kernel of $\mathbf{A}$, solving $\mathbf{A}\mathbf{x} = \mathbf{0} \bmod q$.

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- Essentially, Search-LWE is an average-case form of BDD.

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- Let $\mathbf{v} = \mathbf{t} + \mathbf{e} \in \mathrm{span}_{\mathbb{R}}(\mathbf{B})$, where $\mathbf{t} = \mathbf{x}\mathbf{B} \in \Lambda(\mathbf{B})$, and let $\mathbf{e}$ be short.

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- And since $\mathbf{e}$ is short enough (BDD), this is an instance of unique-SVP!
- Hence by solving uSVP we recover $\mathbf{e}$, from which we recover $\mathbf{t} = \mathbf{v} - \mathbf{e}$, solving BDD.

- Recall that we want to solve BDD for

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  - If lattice reduction is strong enough, projections $\pi_k(\mathbf{e}, 1)$ will not respect the GSA.
  - Eventually, $\pi_k(\mathbf{e}, 1)$ can be recovered as $\mathbf{b}_i^*$ and recovery of $(\mathbf{e}, 1)$ becomes easy [18].

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  - To find $\mathbf{x}$, follow the SIS methodology.

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- Suppose $\Pr[T_i = b] \geq \frac{1}{2} + \varepsilon$.

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  - LWE case: $\mathbf{x}^T\mathbf{b} = \mathbf{x}^T(\mathbf{A}\mathbf{s}+\mathbf{e}) = \cancel{(\mathbf{x}^T\mathbf{A})}\mathbf{s} + \mathbf{x}^T\mathbf{e} \bmod q \implies \left\|x^T\mathbf{b}\right\| = \left\|\mathbf{x}^T\mathbf{e}\right\| \leq \|x\|\,\|e\| \ll q$
  - Uniform case: $\mathbf{x}^T\mathbf{b}$ will be uniformly distributed
- This is a probabilistic distinguisher: the shorter $\mathbf{x}$ the better the guess.
  - But the harder to find $\mathbf{x}$.

Accuracy/cost trade-off

- Let $b = 0$ if "LWE", and $b = 1$ if "uniform".
- Re-test N times by re-randomizing the basis of $\Lambda_q^\perp(\mathbf{A})$.
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- Use F4/Groebner bases to solve for $\mathbf{s}$.
- Asymptotically the best approach whenever χ_e has width $O(\sqrt{n})$!
- Practically inefficient

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- Overall, an impractical attack.

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Thank you

Lab? `https://github.com/fvirdia/lattice-summer-school-2025-fundamentals`

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