Lattice fundamentals

Fernando Virdia — `https://fundamental.domains`

EPFL-ETH Summer School on Lattice-based Cryptography, July 2025

Housekeeping

Interaction

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Sources

These notes have been adapted from pre-existing material, mainly [1], [2].
References at the end of the deck.

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- Today, an extremely popular subject.
- They are used to build PQC, FHE, iO, proof systems...

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Enough said, let's start.

Definitions and basic properties

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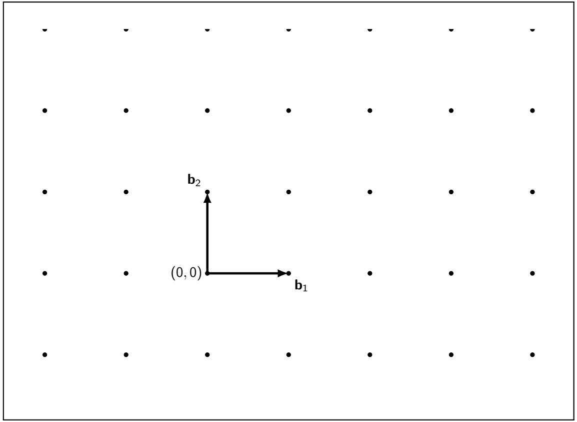
Defining a lattice

- Informally, lattices are a discrete equivalent of vector spaces



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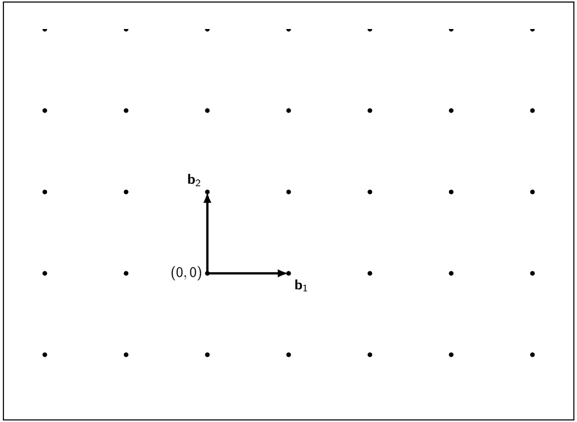

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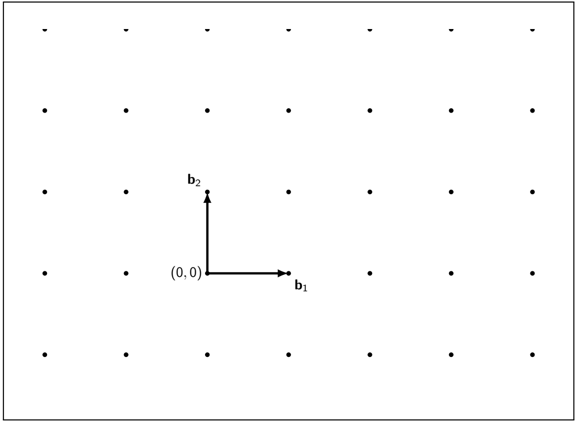

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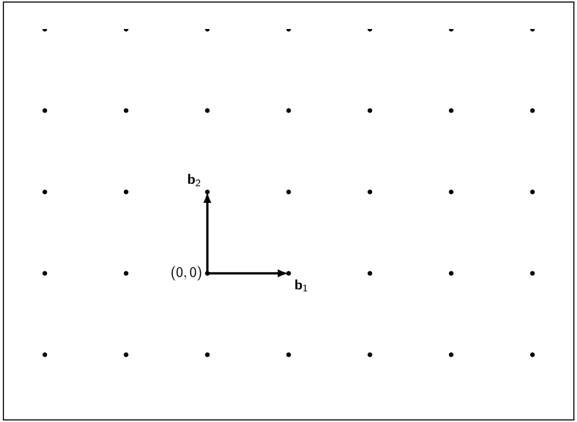
- Other lattices are *linear transformations* of $\mathbb{Z}^n$.



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Definition

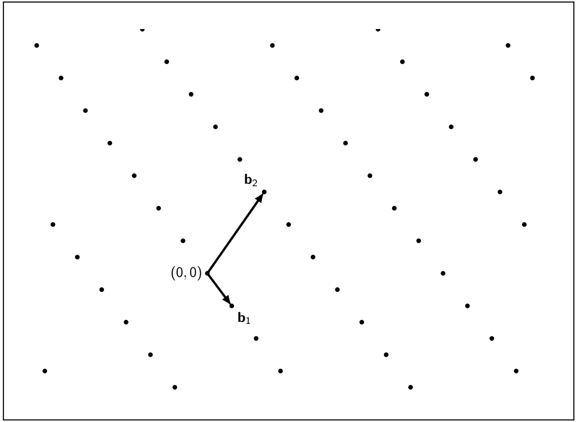
Let $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^d$ be lin. indep.

$$\mathbf{B} := \begin{bmatrix} \text{---} \ \mathbf{b}_1 \ \text{---} \\ \vdots \\ \text{---} \ \mathbf{b}_n \ \text{---} \end{bmatrix}.$$

We say that their integer span

$$\begin{aligned} \Lambda = \Lambda(\mathbf{B}) &:= \mathrm{span}_{\mathbb{Z}}(\mathbf{b}_1, \ldots, \mathbf{b}_n) \\ &= \{x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n : x_i \in \mathbb{Z}\} \\ &= \{\mathbf{xB} : \mathbf{x} \in \mathbb{Z}^n\} \subset \mathbb{R}^d, \end{aligned}$$

is a *real lattice* of rank, n.



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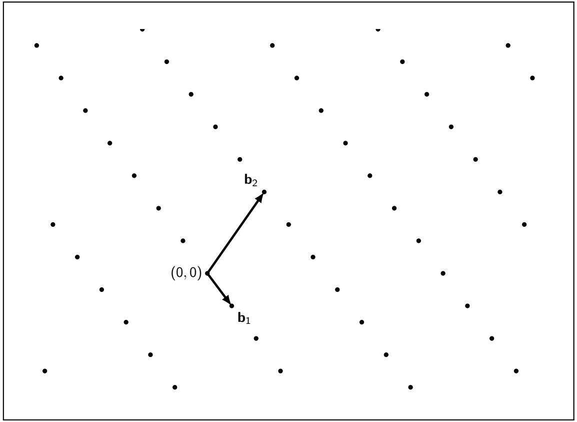

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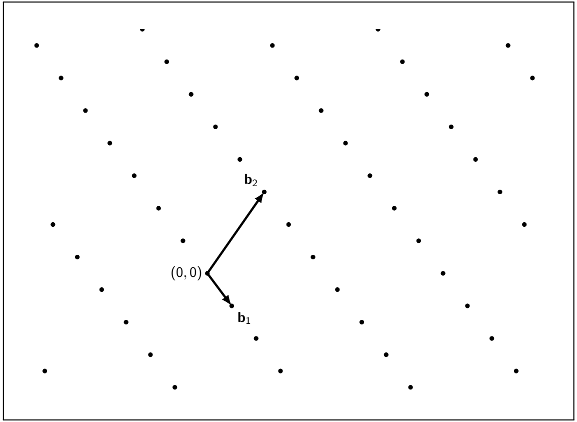

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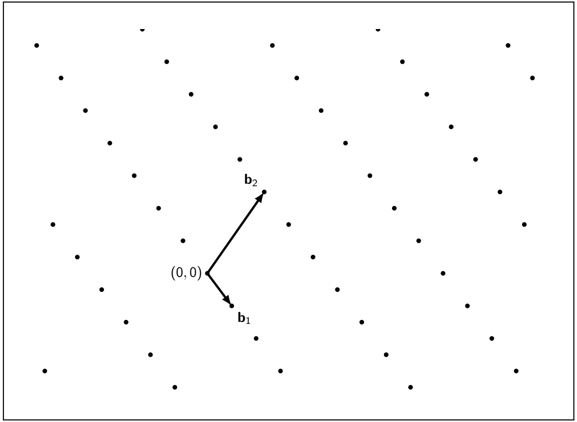

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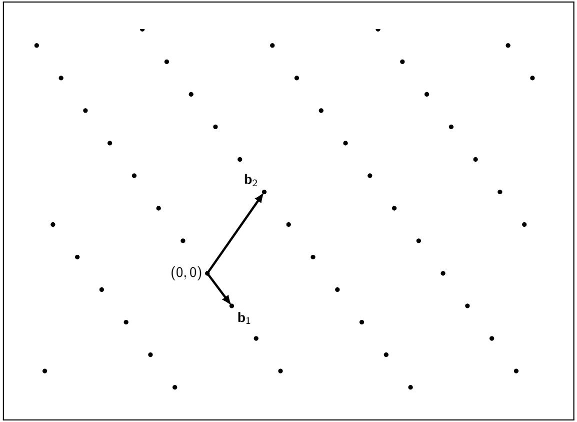

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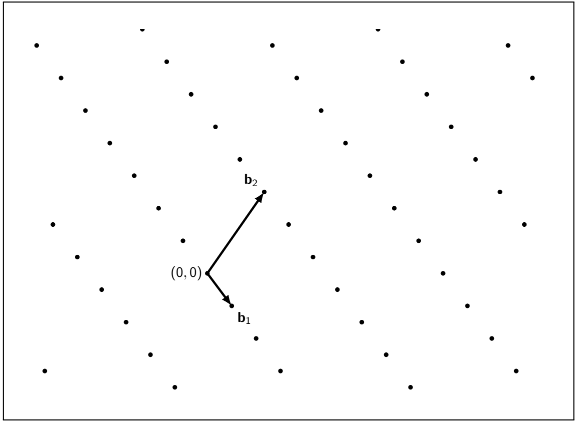
Definition (Sublattices)

Let $\Lambda \subset \mathbb{R}^n$ be a real lattice of rank n. We call any subgroup $\Lambda' \subset \Lambda$ a sublattice of Λ.



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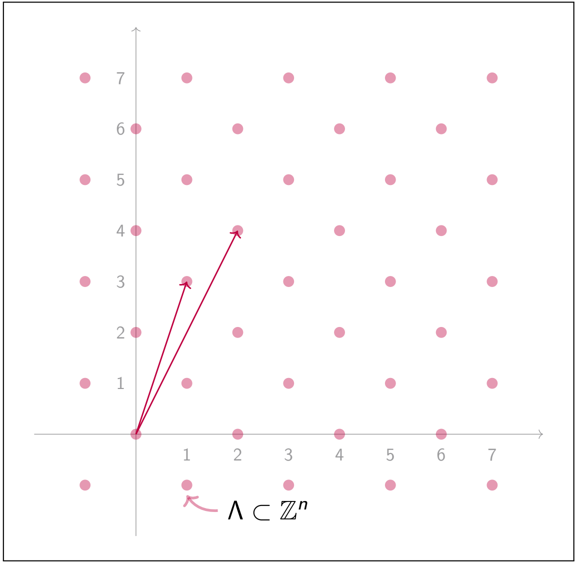

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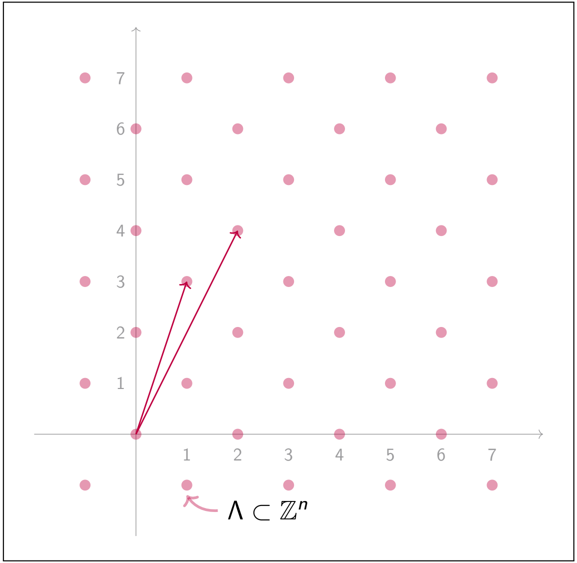

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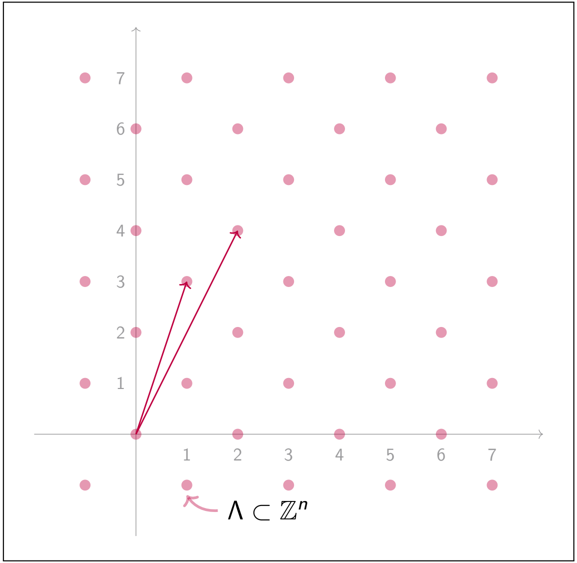

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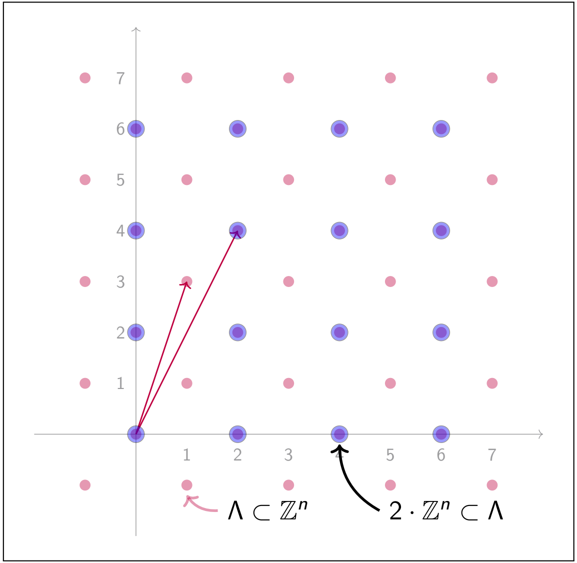

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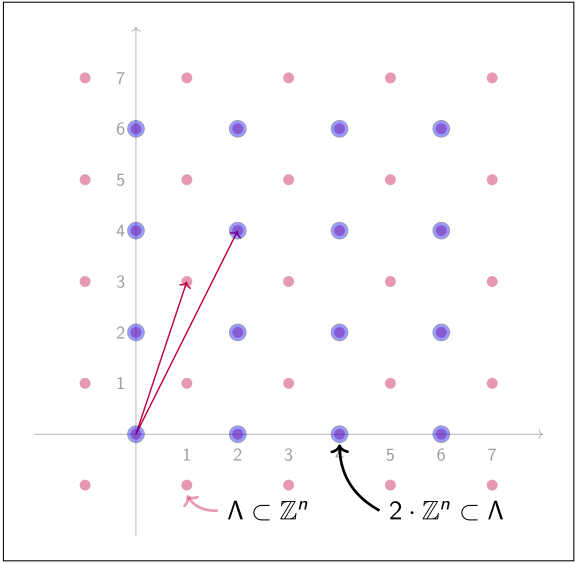

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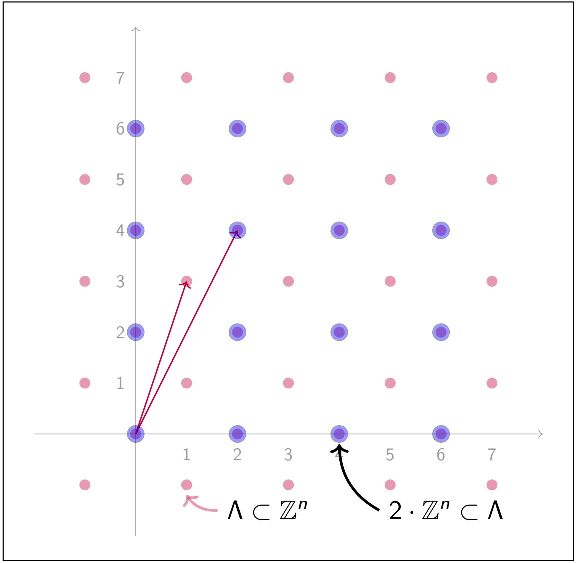

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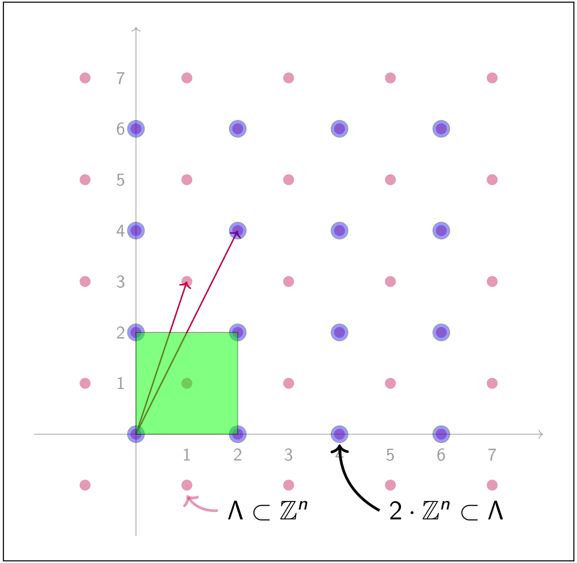
- This allows us to sample $U(\Lambda \bmod q)$.



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Lemma

$\Lambda(\mathbf{B}) = \Lambda(\mathbf{B}')$ *if and only if* $\mathbf{B}' = \mathbf{U}\mathbf{B}$ *where* $\mathbf{U}$ *is unimodular* ($\mathbf{U} \in \mathbb{Z}^{n \times n}$ *with* $\det(\mathbf{U}) = \pm 1$) $\square$

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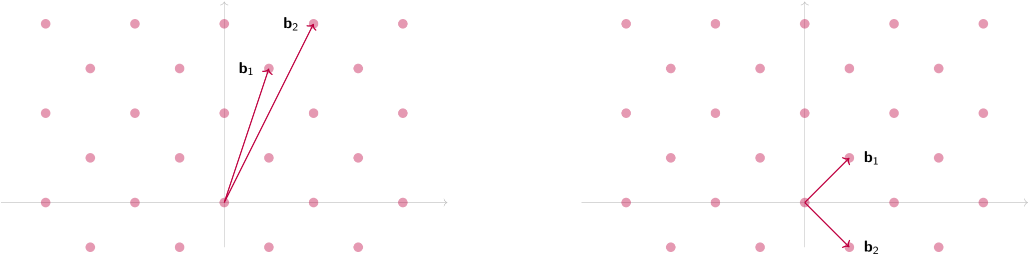
Unimodular matrices $\mathbf{U} \in \mathrm{GL}_n(\mathbb{Z})$ give bijections $\mathbb{Z}^n \to \mathbb{Z}^n$ between coefficient vectors.



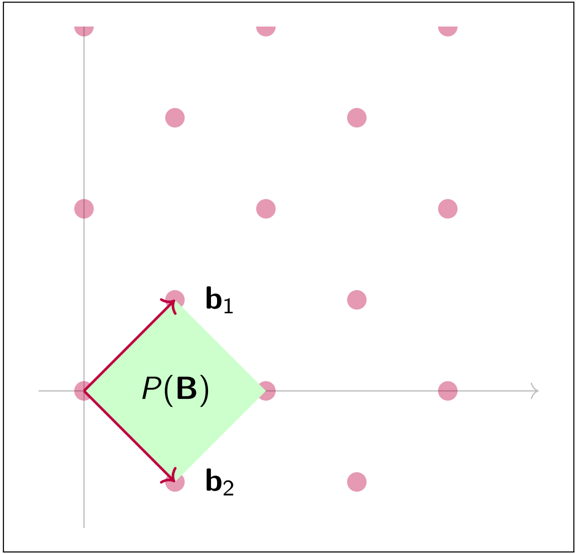
$$\mathbf{B} = \begin{bmatrix} (1, & 3) \\ (2, & 4) \end{bmatrix} \xmapsto{\mathbf{U} = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix}} \mathbf{B}' = \begin{bmatrix} (1, & 1) \\ (1, & -1) \end{bmatrix}$$

Given a lattice basis, we can define its *fundamental parallelepiped* $P(\mathbf{B})$.

Definition

Given a lattice basis $\mathbf{B}$, its fundamental parallelepiped is the set

$$P(\mathbf{B}) := \{x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n \colon x_i \in [0,1)\}.$$



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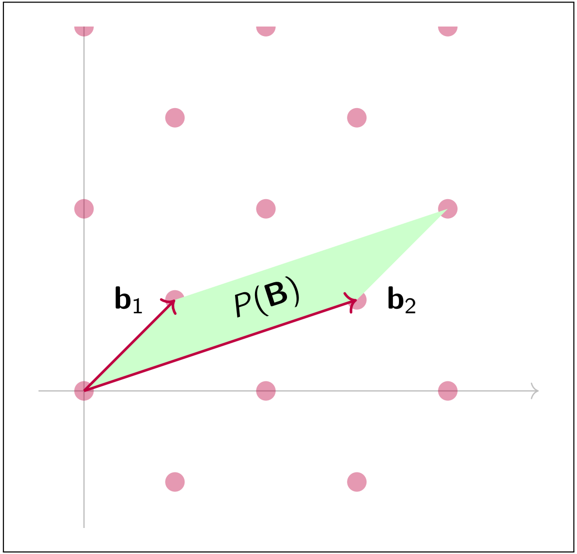
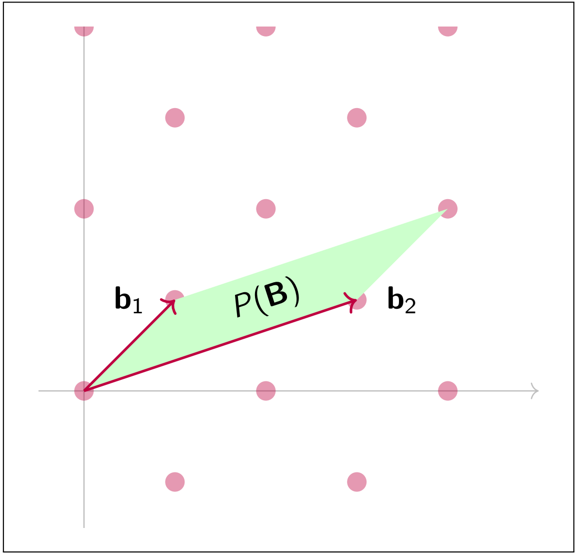


We now define a lattice invariant.

Definition

Given any basis $\mathbf{B}$ the *volume* of Λ is

$$\mathrm{vol}(\Lambda) := \sqrt{\det(\mathbf{B}\mathbf{B}^t)}$$



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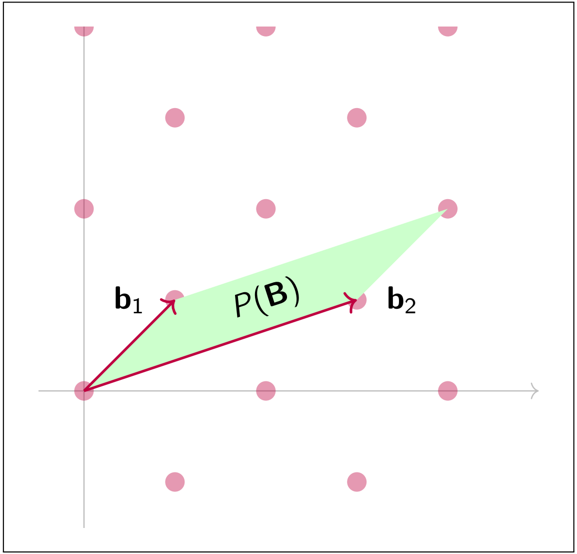
Lemma

- $\mathrm{Vol}(\Lambda) = \mathrm{Vol}(P(\Lambda)) = \int_{P(\Lambda)} d\mathbf{v}$
- *If* $\mathbf{B} \in \mathbb{Z}^{n\times n}$, *then* $\mathrm{Vol}(\Lambda) = |\det(\mathbf{B})|$



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Example

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WLOG, we can choose to work only with unit-volume lattices.

- Any lattice keeps the same "structure" when scaled down by $\mathrm{Vol}(\Lambda)^{1/n}$ in all directions.

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- $\Lambda = \mathbb{Z} \times \mathbb{Z}$ has $\lambda_1(\Lambda) = \lambda_2(\Lambda) = 1$.
- $\Lambda = \mathbb{Z} \times (2\mathbb{Z})$ has $\lambda_1(\Lambda) = 1$ and $\lambda_2(\Lambda) = 2$.

Estimating λ_1: worst-case.

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Definition (Hermite's constant [3])

Let $\mathcal{L}_n$ be the set of real lattices of rank n. Then Hermite's constant for rank n lattices, γ_n, is

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Theorem (Hermite's inequality [3])

Let $n \geq 2$ be an integer. Then $\gamma_n \leq \gamma_2^{n-1}$.

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Estimating λ_1: worst-case.

Definition (Hermite's constant [3])

Let $\mathcal{L}_n$ be the set of real lattices of rank n. Then Hermite's constant for rank n lattices, γ_n, is

$$\gamma_n := \sup_{\Lambda \in \mathcal{L}_n} \frac{\lambda_1(\Lambda)^2}{\mathrm{vol}(\Lambda)^{2/n}}.$$

Hermite's constant is known for $n \leq 8$ and $n = 24$ [4, § 6].

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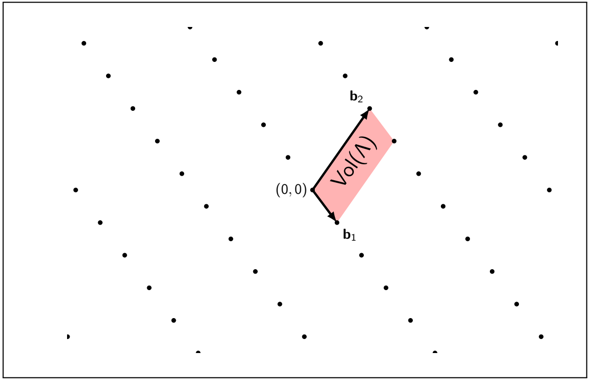
Let $n \geq k \geq 2$ be integers. Then $\gamma_n^{k-1} \leq \gamma_k^{n-1}$.

Estimating λ_1: average-case.



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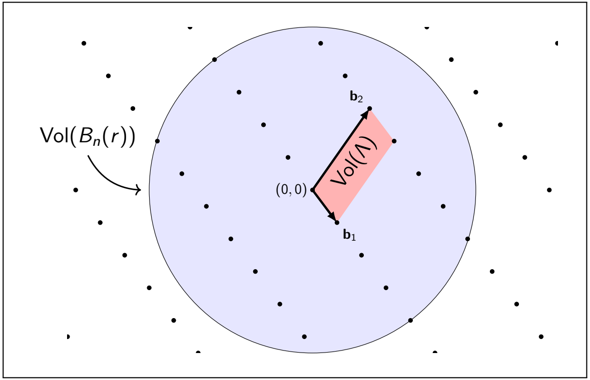

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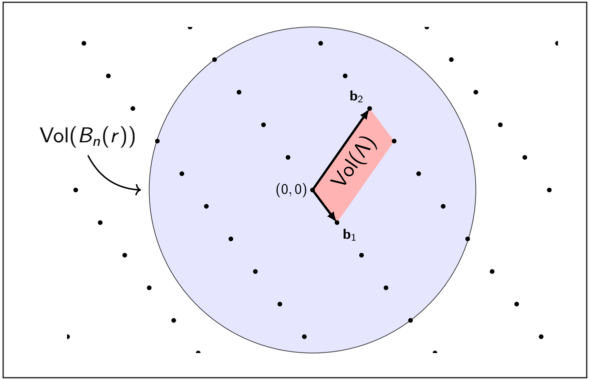

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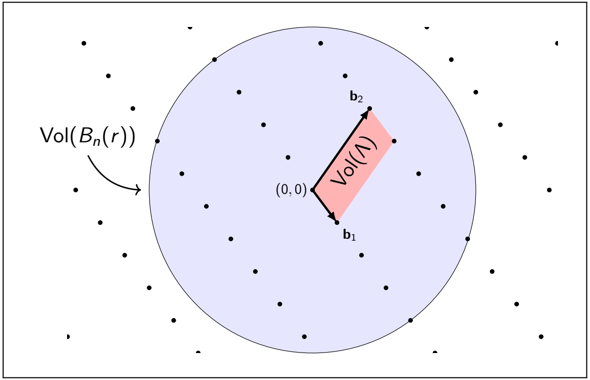

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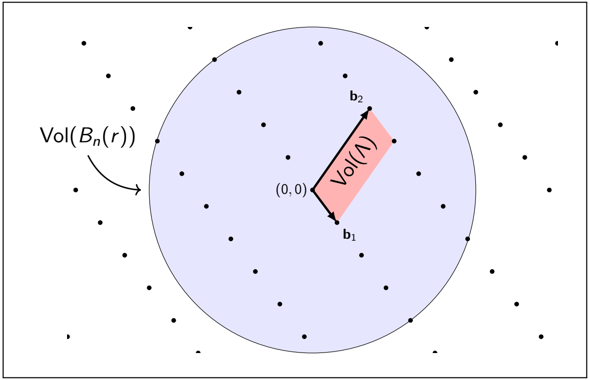

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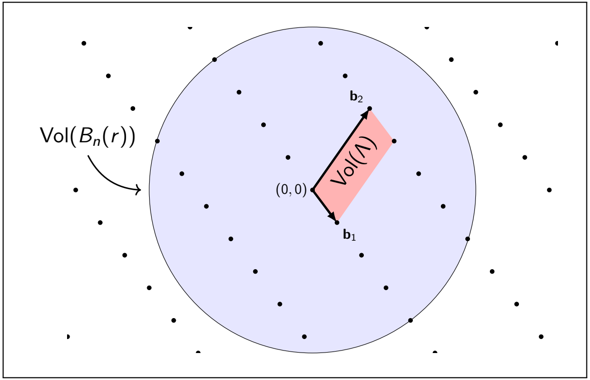

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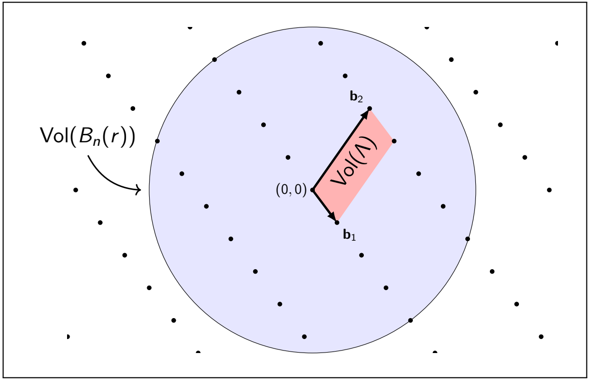

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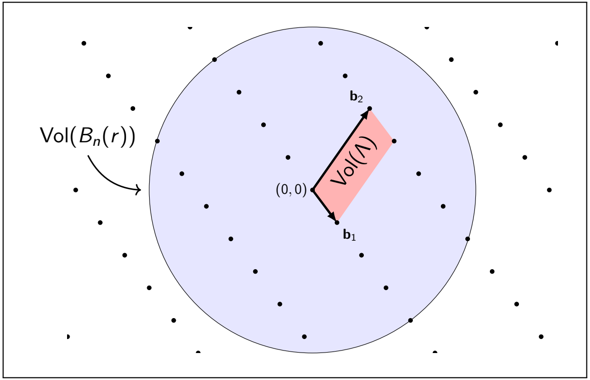

While estimating λ_1 is generally easy, finding a vector realising λ_1 is generally hard!

Computational problems and hardness assumptions

Fernando Virdia — `https://fundamental.domains`
EPFL-ETH Summer School on Lattice-based Cryptography, July 2025

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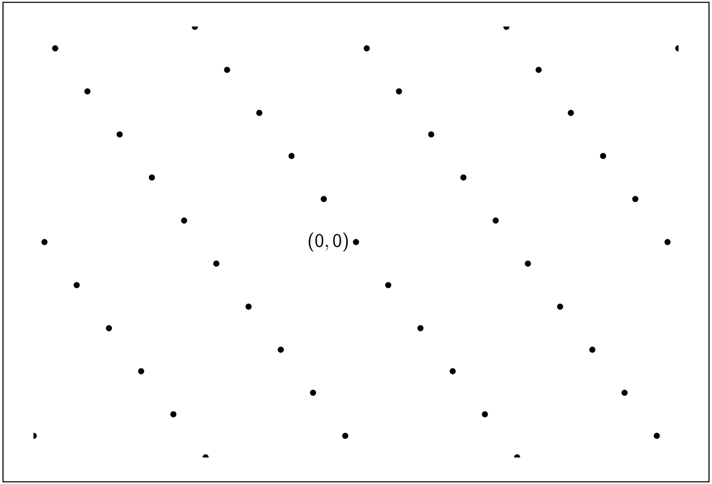
- What about lattices as sources of computational problems?
- Various questions can be asked of a lattice, many giving rise to problems hard in the worst case.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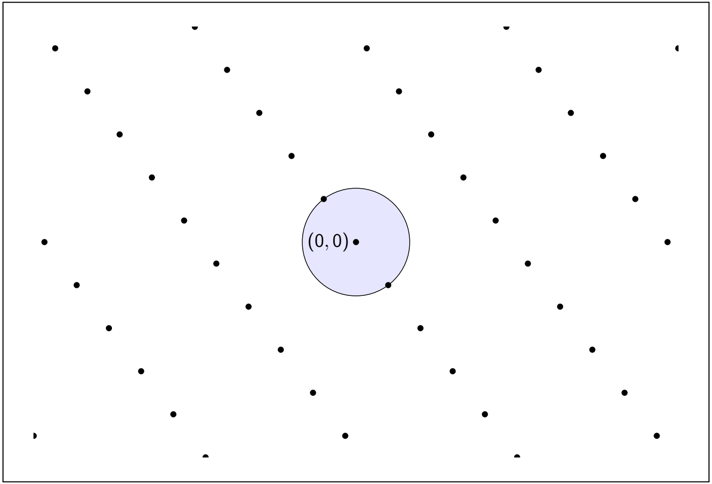
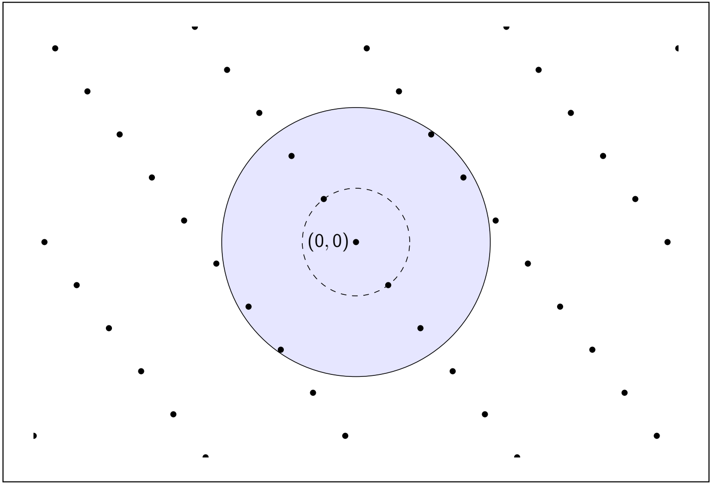


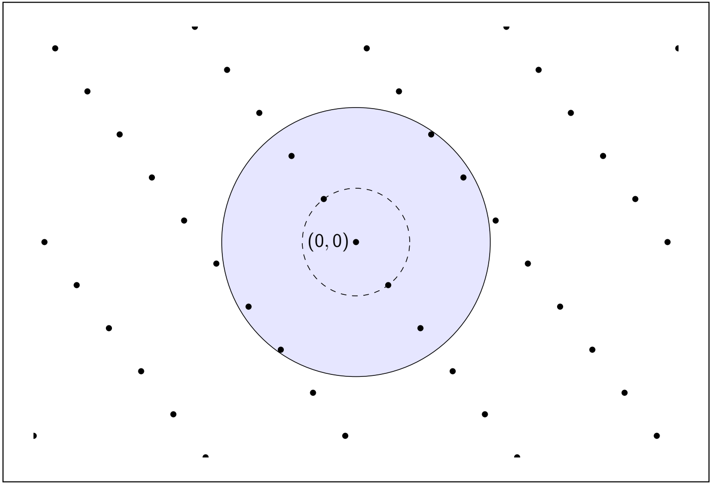
Definition (Shortest Vector Problem (SVP))

Given a lattice Λ find a vector $\mathbf{v} \in \Lambda$ of norm $\lambda_1(\Lambda)$.



Definition (γ-approximate Shortest Vector Problem (approx-SVP$_\gamma$))

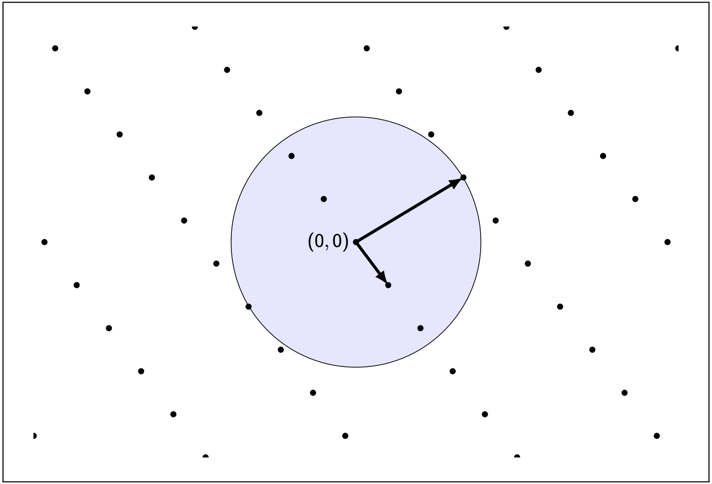
Given a lattice Λ, find a non-zero vector $\mathbf{v} \in \Lambda$ of norm $\leq \gamma \cdot \lambda_1(\Lambda)$.



Definition (γ-Hermite Shortest Vector Problem (Hermite-SVP$_\gamma$))

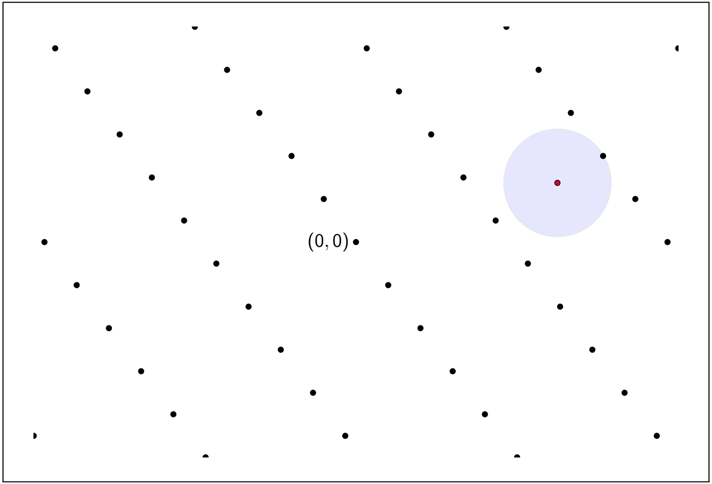
Given a lattice Λ, find a non-zero vector $\mathbf{v} \in \Lambda$ of norm $\leq \gamma \cdot \mathrm{vol}(\Lambda)^{1/n}$.

Approx-SVP is *relative* to Λ, Hermite-SVP is *absolute*: every Λ can be scaled to $\mathrm{Vol}(\Lambda) = 1$.



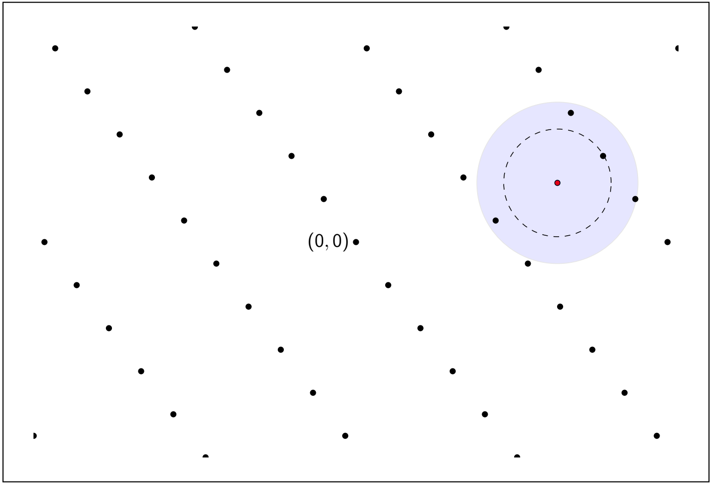
Definition (γ-Shortest Independent Vectors Problem (SIVP_γ))

Given a lattice Λ of rank n, find n linearly independent lattice vectors $\mathbf{v}_i \in \Lambda$ of norm at most $\gamma \cdot \lambda_n(\Lambda)$.



Definition (γ-Closest Vector Problem (CVP$_\gamma$))

Given a lattice basis $\mathbf{B}$ and a vector $\mathbf{v} \in \mathrm{span}_{\mathbb{R}}(\mathbf{B})$, find a lattice point $\mathbf{u} \in \Lambda(\mathbf{B})$ such that $\|\mathbf{v} - \mathbf{u}\| \leq \gamma \cdot \min_{\mathbf{x} \in \Lambda(\mathbf{B})} \|\mathbf{v} - \mathbf{x}\|$.



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Given a lattice Λ and a vector $\mathbf{v} \in \mathrm{span}_{\mathbb{R}}(\Lambda) \setminus \Lambda$ such that

$$\mathrm{dist}(\mathbf{v}, \Lambda) := \min_{\mathbf{x} \in \Lambda} \|\mathbf{v} - \mathbf{x}\| < \gamma \cdot \lambda_1(\Lambda),$$

find $\mathbf{t} \in \Lambda$ such that $\|\mathbf{v} - \mathbf{t}\| = \mathrm{dist}(\mathbf{v}, \Lambda)$.

Note: $\mathbf{t}$ is unique (up to $\pm$) if $\gamma < 1/2$.

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- These are relevant when a short vector is "planted" into a lattice.

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In the *minicrypt* corner	In the *cryptomania* corner
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Homogeneous SIS is the SIS problem when $\mathbf{b} = \mathbf{0}$.

Inhomogeneous SIS (I-SIS) is the SIS problem when $\mathbf{b} \xleftarrow{\$} U(\mathbb{Z}_q^n)$.

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$\Rightarrow \mathbf{x} - \mathbf{x}'$ is an $\mathrm{SIS}_{B,\infty}$ solution. $\square$

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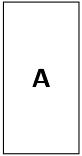

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$\mathbf{a}_i \xleftarrow{\$} \mathbb{Z}_q^n, \quad e_i \xleftarrow{\$} \chi_e$

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return $(\mathbf{a}_i, b_i)$

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Definition

Decision-LWE

Search-LWE

Learning With Errors

- Let n, q be positive integers,
- χ_e be a probability distribution on $\mathbb{Z}$
- χ_s be a probability distribution on $\mathbb{Z}_q$.
  - Let $\mathbf{s} \leftarrow \chi_s^n$ be a secret vector in $\mathbb{Z}_q^n$.

$$\mathbf{A} = \begin{bmatrix} \text{---} \ \mathbf{a}_1 \ \text{---} \\ \vdots \\ \text{---} \ \mathbf{a}_m \ \text{---} \end{bmatrix}$$

$$\boxed{\mathbf{b}} = \boxed{\mathbf{A}}\ \boxed{\mathbf{s}} + \boxed{\mathbf{e}} \quad \bmod q,$$

Sample $L_{\mathbf{s},\chi_e,q}$

$\mathbf{a}_i \xleftarrow{\$} \mathbb{Z}_q^n, \quad e_i \xleftarrow{\$} \chi_e$

$b_i \leftarrow \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i \bmod q$

return $(\mathbf{a}_i, b_i)$

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Decision-LWE Guess whether $\{(\mathbf{a}_i, b_i)\}_{i=1}^m \leftarrow L_{\mathbf{s},\chi_e,q}$ or $\{(\mathbf{a}_i, b_i)\}_{i=1}^m \leftarrow U(\mathbb{Z}_q^n \times \mathbb{Z}_q)$.

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Search-LWE Given $\{(\mathbf{a}_i, b_i)\}_{i=1}^m \leftarrow L_{\mathbf{s},\chi_e,q}$, recover $\mathbf{s}$.

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$$f_{D_{\mu,\sigma}} : \mathbb{Z} \to [0,1], \quad \text{where} \quad f_{D_{\mu,\sigma}}(x) := \frac{f_{N(\mu,\sigma^2)}(x)}{f_{N(\mu,\sigma^2)}(\mathbb{Z})} = \frac{f_{N(\mu,\sigma^2)}(x)}{\sum_{y\in\mathbb{Z}} f_{N(\mu,\sigma^2)}(y)}.$$

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- *Use the Search-LWE solver to attempt to recover* $\mathbf{s}$ *(and hence* $\mathbf{e}$*).*
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- *Works because for a random, a short* $\mathbf{e}$ *satisfying the equation is very unlikely to exist.*

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- *By repeating this* $O(q \cdot n)$ *times, we recover* $\mathbf{s}$.

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I-SIS with “planted” solution as LWE.

- Let $\mathbf{A} = [\mathbf{A}_0 \mid \mathbf{A}_1] \leftarrow U(\mathbb{Z}_q^{n\times n} \times \mathbb{Z}_q^{n\times(m-n)})$, with high probability, $\det(\mathbf{A}_0) \neq 0 \bmod q$.
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Inhomogeneous-SIS (I-SIS) with "planted" solutions as LWE

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- $(\widetilde{\mathbf{A}}, \widetilde{\mathbf{y}} = \mathbf{A}_0^{-1}\mathbf{y})$ are n LWE samples with secret $\mathbf{x}_1 \in \mathbb{Z}^{m-n}$, and error vector $\mathbf{x}_0$.

LWE as I-SIS with "planted" solutions

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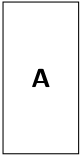

LWE as I-SIS with “planted” solution.

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$$\implies \mathbf{e} \text{ is a solution to the I-SIS instance } \mathbf{A}^\perp\mathbf{x} = \mathbf{y} \bmod q \quad \text{where} \quad \mathbf{y} := \mathbf{A}^\perp\mathbf{b} \sim U(\mathbb{Z}_q^n)$$

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- IND-CPA follows from each ciphertext using its own pad.

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Similarities with DDH

- To build in cryptomania, we needed extra functionality, such as DDH:
  - Distinguish (g, g^x, g^y, g^{xy}) from (g, g^x, g^y, g^z), where $g^x, g^y, g^z \overset{\text{iid}}{\sim} U(\mathbb{G})$.

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Proof.

$$
\begin{array}{rcccc}
 & (\mathbf{A}, & \mathbf{A}\mathbf{s}+\mathbf{e}, & \mathbf{z}^T\mathbf{A}+\mathbf{f}, & \mathbf{z}^T(\mathbf{A}\mathbf{s}+\mathbf{e})+e') \\
\stackrel{c}{\approx} & (\mathbf{A}, & \mathbf{u}, & \mathbf{z}^T\mathbf{A}+\mathbf{f}, & \mathbf{z}^T\mathbf{u}+e') \\
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for $\mathbf{u}_i, \mathbf{w}_i, w' \stackrel{\text{iid}}{\sim} U(\mathbb{Z}_q)$. □

Why SIS and LWE are “lattice-based”: the provable angle

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  - From worst-case $\mathrm{BDD}_{n^{-1/2}}$ to average-case LWE with $m \in \mathrm{poly}(n)$, χ_e discrete Gaussian with $\sigma/q \in (0, 1)$ [10].

Solving SIS and LWE

Fernando Virdia — `https://fundamental.domains`
EPFL-ETH Summer School on Lattice-based Cryptography, July 2025

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- Now we will talk about such attacks.

Linear algebra

- Given a basis $\mathbf{B}$, we can derive an orthogonal basis $\mathbf{B}^*$ via the Gram–Schmidt process.
- The rows of $\mathbf{B}^*$ are

$$\mathbf{b}_i^* = \mathbf{b}_i - \sum_{j<i} \mu_{i,j} \mathbf{b}_j^* \quad \text{for} \quad i \in [d], \quad \text{where} \quad \mu_{i,j} = \langle \mathbf{b}_i, \mathbf{b}_j^* \rangle / \|\mathbf{b}_j^*\|^2 \quad \text{for} \quad i > j.$$

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- In matrix form,

$$\begin{bmatrix} \text{---} & \mathbf{b}_1 & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{b}_d & \text{---} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ \mu_{2,1} & 1 & & \\ \vdots & \ddots & \ddots & \\ \mu_{d,1} & \dots & \mu_{d,d-1} & 1 \end{bmatrix} \begin{bmatrix} \text{---} & \mathbf{b}_1^* & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{b}_d^* & \text{---} \end{bmatrix}.$$

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Remark

Recall that the volume of a lattice $\Lambda(\mathbf{B})$ is $|\det(\mathbf{B})|$. Given the Gram-Schmidt orthogonalisation $\mathbf{b}_1^*, \dots, \mathbf{b}_n^*$ of the basis $\mathbf{B}$, $|\det(\mathbf{B})| = \prod_{i=1}^{n} \|\mathbf{b}_i^*\|$.

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  - The *flatter* a basis profile is, the closer to orthogonal is the basis.

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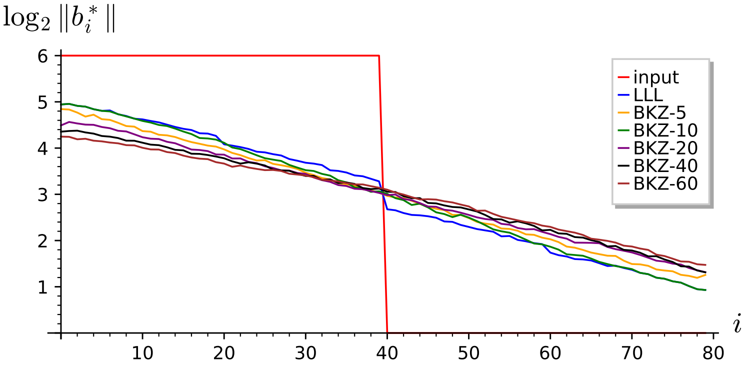

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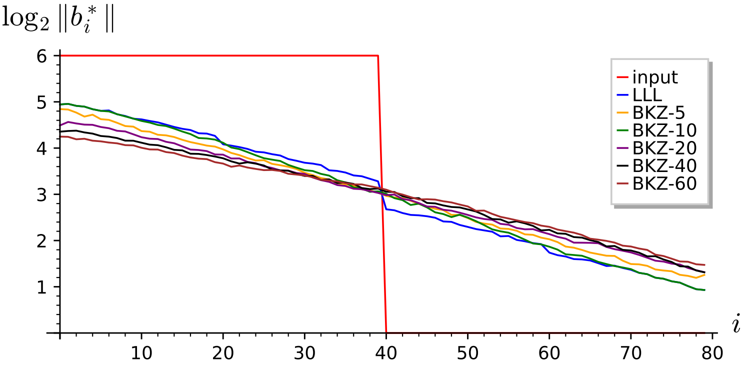

Remark

Interestingly, it would appear that the log-plot of a reduced basis profile forms a straight line.

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- The GSA captures the straightness of the log-plot of the basis profiles we saw before.
  - $\log \|\mathbf{b}_i^*\| = (i-1) \cdot \log \alpha + \log \|\mathbf{b}_1\|$ is a straight line with slope $\log \alpha$.

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Lemma

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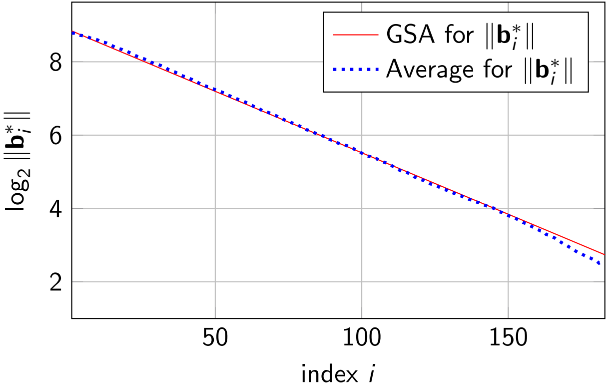

Figure 1: Comparison of a GSA prediction for the profile of a BKZ-56-reduced basis.

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- With a basis $\mathbf{B}$ for our integer lattice, we can then use strong lattice reduction to recover a short vector in the kernel of $\mathbf{A}$, solving $\mathbf{A}\mathbf{x} = \mathbf{0} \bmod q$.

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Solving BDD via lattice reduction

- Let $\mathbf{v} = \mathbf{t} + \mathbf{e} \in \mathrm{span}_{\mathbb{R}}(\mathbf{B})$, where $\mathbf{t} = \mathbf{xB} \in \Lambda(\mathbf{B})$, and let $\mathbf{e}$ be short.

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- And since $\mathbf{e}$ is short enough (BDD), this is an instance of unique-SVP!
- Hence by solving uSVP we recover $\mathbf{e}$, from which we recover $\mathbf{t} = \mathbf{v} - \mathbf{e}$, solving BDD.

- Recall that we want to solve BDD for

$$\Lambda_q(\mathbf{A}) = \{\mathbf{y} \in \mathbb{Z}^m \mid \exists\ \mathbf{x} \in \mathbb{Z}^n \text{ such that } \mathbf{y} = \mathbf{A}\mathbf{x} \bmod q\} \subset \mathbb{Z}^m$$

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  - If lattice reduction is strong enough, projections $\pi_k(\mathbf{e}, 1)$ will not respect the GSA.
  - Eventually, $\pi_k(\mathbf{e}, 1)$ can be recovered as $\mathbf{b}_i^*$ and recovery of $(\mathbf{e}, 1)$ becomes easy [18].

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  - To find $\mathbf{x}$, follow the SIS methodology.

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- Suppose $\Pr[T_i = b] \geq \frac{1}{2} + \varepsilon$.

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  - Uniform case: $\mathbf{x}^T\mathbf{b}$ will be uniformly distributed
- This is a probabilistic distinguisher: the shorter $\mathbf{x}$ the better the guess.
  - But the harder to find $\mathbf{x}$.

Accuracy/cost trade-off

- Let $b = 0$ if "LWE", and $b = 1$ if "uniform".
- Re-test N times by re-randomizing the basis of $\Lambda_q^\perp(\mathbf{A})$.
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- Set up a polynomial system over $\mathbb{Z}_q$,

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- Use F4/Groebner bases to solve for **s**.
- Asymptotically the best approach whenever χ_e has width $O(\sqrt{n})$!
- Practically inefficient

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- Overall, an impractical attack.

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Thank you

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