<u>Practice with Spinors</u>

I Algebra with Dirac γ matrices

Notation: $S = \overline{u}u$
$P = \overline{u}\gamma^5 u$
$V^\mu = \overline{u}\gamma^\mu u$
$A^\mu = \overline{u}\gamma^\mu\gamma^5 u$
$T^{\mu\nu} = \overline{u}\sigma^{\mu\nu}u$

1. $(\overline{u}_1\gamma^\mu u_2)^* \ = ?$

   Note: $(\gamma^0)^\dagger \ = \ \gamma^0$ and $(\gamma^\mu)^\dagger \ = \ \gamma^0\gamma^\mu\gamma^0$

   $(\overline{u}_1\gamma^\mu u_2)$ is a 1×1 matrix. Therefore, its complex conjugate is the same as its Hermitian conjugate, i.e. if we call $L^\mu = (\overline{u}_1\gamma^\mu u_2)$, then $(L^\mu)^* = (L^\mu)^\dagger$. We can then express this quantity as follows:

$$\begin{aligned} L^\mu &= \overline{u}_1\gamma^\mu u_2 \ , && (1) \\ \Rightarrow (L^\mu)^* &= (L^\mu)^\dagger \ , \\ &= (\overline{u}_1\gamma^\mu u_2)^\dagger \ , \\ &= ((u_1)^\dagger\gamma^0\gamma^\mu u_2)^\dagger \quad \text{using } (A\ldots Z)^\dagger = Z^\dagger\ldots A^\dagger \ , \\ &= (u_2^\dagger)(\gamma^\mu)^\dagger(\gamma^0)^\dagger(u_1) \\ &= (u_2^\dagger)\gamma^0\gamma^\mu\gamma^0\gamma^0(u_1) \\ &= (u_2^\dagger)\gamma^0\gamma^\mu(u_1) \\ &= \overline{u}_2\gamma^\mu(u_1) && (2) \end{aligned}$$

   Therefore $(\overline{u}_1\gamma^\mu u_2)^* \ = \ \overline{u}_2\gamma^\mu u_1$. To solve for $(L^\mu)^2$ we simply use $(L^{\mu\nu})^2 = Tr[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1]$

   Note: $\mathrm{Tr}[\gamma^\mu\gamma^\nu] \ = \ 4g^{\mu\nu}$, $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$, The trace of an odd number gamma matrix product is 0

$$\begin{aligned} (L^{\mu\nu})^2 &= Tr[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1] && (3) \\ &= Tr[\overline{u}_1\gamma^\mu(\not{p}_2 + m)\gamma^\nu u_1] \\ &= Tr[u_1\overline{u}_1\gamma^\mu(\not{p}_2 + m)\gamma^\nu] \\ &= Tr[(\not{p}_1 + m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu] \\ &= Tr[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] + m[Tr(\gamma^\mu\not{p}_1\gamma^\nu) + Tr(\gamma^\mu\gamma^\nu\not{p}_2)] + m^2Tr[\gamma^\mu\gamma^\nu] \\ &= Tr[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] + m^2Tr[\gamma^\mu\gamma^\nu] \\ &= Tr[(p_1)_\lambda\gamma^\lambda\gamma^\mu(p_2)_\sigma\gamma^\sigma\gamma^\nu] + 4m^2g^{\mu\nu} \\ &= (p_1)_\lambda(p_2)_\sigma Tr[\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu] + 4m^2g^{\mu\nu} \\ &= (p_1)_\lambda(p_2)_\sigma 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) + 4m^2g^{\mu\nu} \\ &= 4[p_1^\mu p_2^\nu - g^{\mu\nu}(p_1\cdot p_2) + p_2^\mu p_1^\nu] + 4m^2g^{\mu\nu} \\ &= 4[p_1^\mu p_2^\nu - 4g^{\mu\nu}(p_1\cdot p_2 + m^2) + p_2^\mu p_1^\nu] && (4) \end{aligned}$$

2. $(\overline{u}_1\gamma^\mu\gamma^5u_2)^*$ is also a 1×1 Matrix so the same reasoning applies as above in 1. Note: $(\gamma^5)^\dagger = \gamma^5$
   We define: R^μ as $\overline{u}_1\gamma^\mu\gamma^5u_2$ thus:

$$\begin{aligned}
\Rightarrow (R^\mu)^* &= (R^\mu)^\dagger && (5)\\
&= (\overline{u}_1\gamma^\mu\gamma^5u_2)^\dagger\\
&= ((u_1)^\dagger\gamma^0\gamma^\mu\gamma^5u_2)^\dagger\\
&= (u_2^\dagger)(\gamma^5)^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger(u_1)\\
&= (u_2^\dagger)\gamma^5\gamma^0\gamma^\mu\gamma^0\gamma^0u_1\\
&= (u_2^\dagger)\gamma^5\gamma^0\gamma^\mu(1)u_1\\
&= -(u_2^\dagger)\gamma^0\gamma^5\gamma^\mu u_1\\
&= -\overline{u}_2\gamma^5\gamma^\mu u_1\\
&= \overline{u}_2\gamma^\mu\gamma^5u_1 && (6)
\end{aligned}$$

   Therefore $(\overline{u}_1\gamma^\mu\gamma^5u_2)^* = \overline{u}_2\gamma^\mu\gamma^5u_1$
   We also are able to calculate $(R^\mu)^2$

$$\begin{aligned}
(R^{\mu\nu})^2 &= Tr[\overline{u}_1\gamma^\mu\gamma^5u_2\overline{u}_2\gamma^\nu\gamma^5u_1 && (7)\\
&= Tr[\overline{u}_1\gamma^\mu\gamma^5(\not{p}_2+m)\gamma^\nu\gamma^5u_1]\\
&= Tr[u_1\overline{u}_1\gamma^\mu\gamma^5(\not{p}_2+m)\gamma^\nu\gamma^5]\\
&= Tr[(\not{p}_1+m)\gamma^\mu\gamma^5(\not{p}_2+m)\gamma^\nu\gamma^5]\\
&= Tr[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5+m(\not{p}_1\gamma^\mu\gamma^5\gamma^\nu\gamma^5+\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5)+m^2(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)]\\
&= Tr[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5+m^2(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)]\\
&= Tr[(p_1)_\lambda\gamma^\lambda\gamma^\mu\gamma^5(p_2)_\sigma\gamma^\sigma\gamma^\nu\gamma^5]+m^2Tr[\gamma^\mu\gamma^5\gamma^\nu\gamma^5]\\
&= (p_1)_\lambda(p_2)_\sigma Tr[\gamma^\lambda\gamma^\mu\gamma^5\gamma^\sigma\gamma^\nu\gamma^5]-m^2Tr[\gamma^\mu\gamma^5\gamma^5\gamma^\nu]\\
&= (p_1)_\lambda(p_2)_\sigma Tr[\gamma^\lambda\gamma^\mu\gamma^5\gamma^5\gamma^\sigma\gamma^\nu]-m^2Tr[\gamma^\mu\gamma^\nu]\\
&= (p_1)_\lambda(p_2)_\sigma Tr[\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu]-m^2(g^{\mu\nu})\\
&= (p_1)_\lambda(p_2)_\sigma 4(g^{\mu\nu}g^{\lambda\sigma}-g^{\mu\lambda}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\lambda})-4m^2g^{\mu\nu}\\
&= 4[p_1^\mu p_2^\nu-g^{\mu\nu}(p_1\cdot p_2)+p_2^\mu p_1^\nu]-4m^2g^{\mu\nu}\\
&= 4[p_1^\mu p_2^\nu-4g^{\mu\nu}(p_1\cdot p_2-m^2)+p_2^\mu p_1^\nu] && (8)
\end{aligned}$$

3. $(\overline{u}_1u_2)^* = ?$
   We let:$P = \overline{u}_1u_2$

$$\begin{aligned}
\Rightarrow (P)^* &= (P)^\dagger\ , && (9)\\
&= (\overline{u}_1u_2)^\dagger\ ,\\
&= ((u_1)^\dagger\gamma^0u_2)^\dagger\\
&= (u_2)^\dagger(\gamma^0)^\dagger(u_1)\\
&= (u_2)^\dagger\gamma^0(u_1)\\
&= \overline{u}_2(u_1) && (10)
\end{aligned}$$

Therefore$(\overline{u}_1 u_2)^* \;=\; \overline{u}_2 u_1$
In order to find $(P)^2$ we simply do the following:

$$\begin{aligned}
(P)^2 &= Tr[\overline{u}_1 u_2 \overline{u}_2 u_1] &(11)\\
&= Tr[(\not{p}_1 + m)(\not{p}_2 + m)]\\
&= Tr[\not{p}_1\not{p}_2 + m(\not{p}_1 + \not{p}_2) + m^2]\\
&= Tr[\not{p}_1\not{p}_2] + Tr[m(\not{p}_1 + \not{p}_2)] + Tr[m^2]\\
&= Tr[\not{p}_1\not{p}_2] + m(Tr[\not{p}_1] + Tr[\not{p}_2]) + m^2 Tr[1]\\
&= Tr[\not{p}_1\not{p}_2] + 4m^2\\
&= 4(p_1 \cdot p_2) + 4m^2 &(12)
\end{aligned}$$

4. By the same reasoning as shown above it can be shown that $(\overline{u}_1\gamma^5 u_2)^* \;=\; \overline{u}_2\gamma^5 u_1$
If we let $T \;=\; \overline{u}_1\gamma^5 u_2$ then:

$$\begin{aligned}
\Rightarrow (T)^* &= (T)^\dagger\ , &(13)\\
&= (\overline{u}_1\gamma^5 u_2)^\dagger\ ,\\
&= ((u_1)^\dagger\gamma^0\gamma^5 u_2)^\dagger\\
&= (u_2)^\dagger(\gamma^5)^\dagger(\gamma^0)^\dagger(u_1)\\
&= (u_2)^\dagger(-\gamma^5)\gamma^0(u_1)\\
&= (u_2)^\dagger\gamma^0\gamma^5(u_1)\\
&= \overline{u}_2\gamma^5(u_1) &(14)
\end{aligned}$$

Therefore $(\overline{u}_1\gamma^5 u_2)^* \;=\; \overline{u}_2\gamma^5 u_1$

In order to square T we do the following:

$$\begin{aligned}
T^2 &= Tr[\overline{u}_1\gamma^5 u_2\overline{u}_2\gamma^5 u_1] &(15)\\
&= Tr[u_1\overline{u}_1\gamma^5\not{p}_2\gamma^5]\\
&= Tr[\not{p}_1\gamma^5\not{p}_2\gamma^5]\\
&= Tr[(p_1)_\mu\gamma^\mu\gamma^5(p_2)_\nu\gamma^\nu\gamma^5]\\
&= (p_1)_\mu(p_2)_\nu Tr[\gamma^\mu\gamma^5\gamma^\nu\gamma^5]\\
&= -(p_1)_\mu(p_2)_\nu Tr[\gamma^\mu\gamma^5\gamma^5\gamma\nu]\\
&= -(p_1)_\mu(p_2)_\nu Tr[\gamma^\mu\gamma^\nu]\\
&= -(p_1)_\mu(p_2)_\nu(4g^{\mu\nu})\\
&= -4(p_1)(p_2)
\end{aligned}$$
$$(16)$$

5. While the above identities could be shown to be trivial, the identity: $(\overline{u}_1\sigma^{\mu\nu}u_2)^* \;=\; \overline{u}_2\sigma^{\mu\nu}u_1$ is more difficult to solve
The identity:$(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$ is needed

$$(\sigma^{\mu\nu})^\dagger \;=\; (\frac{i}{2}[\gamma^\mu,\gamma^\nu])^\dagger \qquad (17)$$

$$
\begin{aligned}
&= (u_2)^\dagger(\sigma^{\mu\nu})^\dagger(\gamma^0)^\dagger u_1 \\
&= \frac{i}{2}([\gamma^\mu,\gamma^\nu])^\dagger \\
&= \frac{i}{2}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)^\dagger \qquad (18)\\
&= \frac{i}{2}((\gamma^\mu)^\dagger(\gamma^\nu)^\dagger-(\gamma^\nu)^\dagger(\gamma^\mu)^\dagger) \\
&= \frac{i}{2}(\gamma^0\gamma^\mu\gamma^0\gamma^0\gamma^\nu\gamma^0-\gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^\mu\gamma^0) \\
&= \frac{i}{2}(\gamma^0\gamma^\mu\gamma^\nu\gamma^0-\gamma^0\gamma^\nu\gamma^\mu\gamma^0) \\
&= \frac{i}{2}((-1)^2\gamma^\mu\gamma^\nu-(-1)^2\gamma^\nu\gamma^\mu) \\
&= \frac{i}{2}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu) \\
&= \sigma^{\mu\nu} \qquad (19)
\end{aligned}
$$

After showing $(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$ is true it is trivial to show $(\overline{u}_1\sigma^{\mu\nu}u_2)^* = \overline{u}_2\sigma^{\mu\nu}u_1$
We let $B^{\mu\nu} = \overline{u}_1\sigma^{\mu\nu}u_2$

$$
\begin{aligned}
\Rightarrow (B^{\mu\nu})^* &= (B^{\mu\nu})^\dagger\ , \qquad (20)\\
&= (\overline{u}_1\sigma^{\mu\nu}u_2)^\dagger\ , \\
&= ((u_1)^\dagger\gamma^0\sigma^{\mu\nu}u_2)^\dagger \\
&= (u_2^\dagger)(\sigma^{\mu\nu})^\dagger(\gamma^0)^\dagger(u_1) \\
&= (u_2^\dagger)\sigma^{\mu\nu}\gamma^0(u_1) \\
&= (u_2^\dagger)(\gamma^0)\sigma^{\mu\nu}(u_1) \\
&= \overline{u}_2\sigma^{\mu\nu}(u_1) \qquad (21)
\end{aligned}
$$

An interesting thing to note is that the expression $\overline{u}\sigma^{\mu\nu}\gamma^5 u$ is not an independent quantity. Since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ it follows that the product of $\sigma^{\mu\nu}$ and γ^5 can be simplified to an expression with only 2 γ matrices which has been defined as a pusedoscalar.
For example, let $\mu = 0$ and $\nu = 1$:

$$
\begin{aligned}
\overline{u}\sigma^{01}\gamma^5 u &= \overline{u}\sigma^{01}(i\gamma^0\gamma^1\gamma^2\gamma^3)u \qquad (22)\\
&= \overline{u}((\frac{i}{2})(\gamma^0\gamma^1-\gamma^1\gamma^0))(i\gamma^0\gamma^1\gamma^2\gamma^3)u \\
&= \overline{u}(\frac{i}{2})[\gamma^0\gamma^1(i\gamma^0\gamma^1\gamma^2\gamma^3)-\gamma^1\gamma^0(i\gamma^0\gamma^1\gamma^2\gamma^3)]u \\
&= \overline{u}(\frac{-1}{2})[\gamma^0\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3-\gamma^1\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3)]u \\
&= \overline{u}(\frac{-1}{2})[-\gamma^2\gamma^3-\gamma^2\gamma^3)]u \\
&= \overline{u}(\frac{-1}{2})[-2\gamma^2\gamma^3]u \\
&= 2\overline{u}\gamma^2\gamma^3 u \qquad (23)
\end{aligned}
$$

This expression (because it contains two gamma matrices) is a puesdoscalar. Any values of μ and ν can be shown to be similar to this because of the communal and indentity properties of the gamma matrices.

In order to find the value of $(B^{\mu\nu})^2$ one needs to do the following:

$$\begin{aligned}
(B^{\mu\nu\sigma\lambda})^2 &= Tr[\overline{u}_1\sigma^{\mu\nu}u_2\overline{u}_2\sigma^{\sigma\lambda}u_1] && (24)\\
&= Tr[\overline{u}_1\sigma^{\mu\nu}\not{p}_2\sigma^{\sigma\lambda}u_1] \\
&= Tr[\not{p}_1\sigma^{\mu\nu}\not{p}_2\sigma^{\sigma\lambda}] \\
&= Tr[p_\kappa\gamma^\kappa\sigma^{\mu\nu}p_\gamma\gamma^\gamma\sigma^{\sigma\lambda}] && (25)\\
&= p_\kappa p_\gamma Tr[\gamma^\kappa\sigma^{\mu\nu}\gamma^\gamma\sigma^{\sigma\lambda}] && (26)\\
&= -\frac{1}{2}p_\kappa p_\gamma Tr[\gamma^\kappa(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)\gamma^\gamma(\gamma^\sigma\gamma^\lambda-\gamma^\lambda\gamma^\sigma)] && (27)\\
&= -\frac{1}{2}p_\kappa p_\gamma Tr[2\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda-2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda] && (28)\\
&= -\frac{1}{2}p_\kappa p_\gamma Tr[4\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda] && (29)\\
&= -2p_\kappa p_\gamma Tr[\gamma^\kappa\gamma^\gamma] \text{ Using a similar identity as shown in eq. 23} && (30)\\
&= -8p_\kappa p_\gamma g^{\kappa\gamma} && (31)
\end{aligned}$$