The R_K and $R_{D^{(*)}}$ Puzzles

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1 Introduction

The standard model of particle physics has been an excellent way to predict decays and different types of processes that occur at the subatomic level. However, recent experimental findings have challenged the notion that the standard model is the best way to predict what we see in the real world and in experiments. The certain decays of the B meson could give us some hints into the new physics present beyond the standard model. The R_K and $R_{D^{(*)}}$ are two ratios of these B meson decays that show some puzzling experimental results. The R_K decay is defined as $\mathcal{B}(B^+ \to K^+\mu^+\mu^-)/\mathcal{B}(B^+ \to K^+e^+e^-)$ The experimental value found by the LHCb Collaboration differs from the standard model prediction by 2.6σ [5] The next puzzle is the $R_{D^{(*)}}$ puzzle which is defined as $\mathcal{B}(\bar{B} \to D^{(*)}\tau^-\overline{\nu}_\tau/\mathcal{B}(\bar{B} \to D^{(*)}l^-\overline{\nu}_l$ where ($l = \mu$ or e) this ratio has been measured by the LHCb Collaboration, BaBar, and Belle. The ratio R_D differs from the standard model by 1.7σ and the $R_{D^{(*)}}$ differs by 3.1σ. It must be mentioned that both of these puzzles require non-lepton universality, however this paper will not seek to explore this requirement. We will focus on analyzing the possible NP contributions to the Wilson Coefficients arising from the effective Hamiltonian of the $b \to s\mu^+\mu^-$ transition. The complete Hamiltonian is as follows: [5].

$$H = -\frac{\alpha G_f}{\sqrt{2}\pi} V_{tb}V_{ts}^* \sum_{a=9,10} (C_a O_a + C_a' O_a') \tag{1}$$

This paper will take the Hamiltonian and use the Wilson Coefficients found in the branching ratio of the $B_s \to \mu^+\mu^-$ decay. The paper begins by first exploring and computing different observable and particulars of the $\pi^+ \to l^+\nu_l$ decay. (Where $l = \mu$ or e) This will provide us with a background in manipulating and understanding the theoretical expression for branching ratios and decays. It will also serve the reader as a guide ot the format of the rest of the paper. In section 3 the Wilson coefficients from the branching ratio of the $B_s \to \mu^+\mu^-$ decay will be extracted and constrained with the experimental numbers. We will use the Python package Flavio to generate our data and construct our plots. Flavio's packages focus on flavor physics and allows the user to find the contributions of different NP Wilson Coefficients for varying obserables. Finally in section 4 plots of the Branching ratio based upon the Wilson Coefficients will be shown and fits to the data will be provided. We conclude the paper in section 5.

2 The branching Ratio of a Pion

In order to calculate the Branching ratio of a charged pion, one must be familiar with the Feynman Rules of calculating amplitudes and the trace identities.

We must start with recognizing that the pion decay is a charged, weak, interaction which arises from the fact that a pion is made of quarks, and the decay is mediated by a massive W boson.

A diagram of the decay may been seen below:

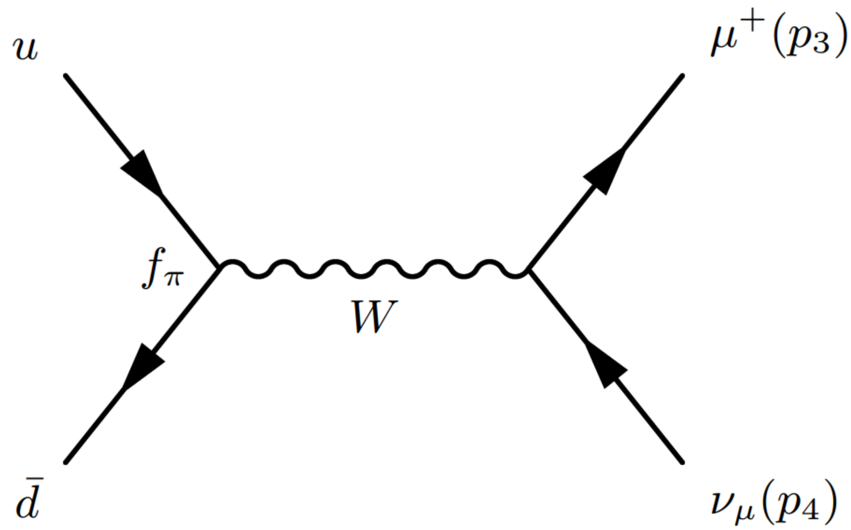

Figure 1: Pion Decay

Picture courtesy of Qora.com

Where the up and ani-down quark are the pion and the muon and muon neutrino are on the right. However, the muon and the muon neutrino could be the electron and electron neutrino.

The formula given to describe the branching ratio of the decay is given by [1]:

$$\Gamma = \frac{S|\mathbf{p}|}{8\pi\hbar m_1^2 c}|\mathcal{M}|^2 \tag{2}$$

Where $|\mathbf{p}|$ is the outgoing momentum, S is the product of statistical factors (in our case it will be equal to 1), and $\mathcal{M}$ is the Feynman amplitude. In our notation, we will use the Natural Units so our expression becomes:

$$\Gamma = \frac{S|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2 \tag{3}$$

The next step is to determine the Feynman Amplitude which can be done in a few steps In the charged weak lepton decays, we have different notation for the vertices and propagators.

1. For each vertex add a factor of $\frac{-ig_w}{2\sqrt{2}}(\gamma^\nu(1-\gamma^5))$ where $g_w = \sqrt{4\pi\alpha_w}$
2. For each propagator we add a factor of $\frac{-ig_{\mu\nu} - \frac{q_\mu q_\nu}{m^2}}{q^2 - m^2}$ where m is the mass of the boson. In our case, $m_w \gg q$ so the expression simplifies to $\frac{ig_{\mu\nu}}{m_w^2}$

From these rules and Figure 1 we are able to calculate the value of $\mathcal{M}$

$$-i\mathcal{M} = \left[\overline{u}(3)\left(\frac{-ig_w}{2\sqrt{2}}(\gamma^\nu(1-\gamma^5)\right)v(2)\right]\left[\frac{ig_{\mu\nu}}{m_w^2}\right]\left[\frac{-ig_w}{2\sqrt{2}}F^\mu\right] \tag{4}$$

We have a factor of -i with the $\mathcal{M}$ so that we obtain the real part of the expression and F^{μ} is the form factor of the coupling of the pion to the W boson. F^{μ} has the form of $f_{\pi}p^{\mu}$

$$\mathcal{M} = \frac{g_w^2}{8m_w^2}[\overline{u}(3)(\gamma^{\mu}(1-\gamma^5))v(2)]F^{\mu} \tag{5}$$

In order to square the amplitude we do the following:

$$\langle|\mathcal{M}^2|\rangle = \left(\frac{g_w^2}{8m_w^2}f_{\pi}\right)^2 Tr[(\overline{u}(3)(\gamma^{\mu}(1-\gamma^5))v(2)p_{\mu}(\overline{v}(2)(\gamma^{\nu}(1-\gamma^5)u(3))]p_{\nu} \tag{6}$$

$$\langle|\mathcal{M}^2|\rangle = \left(\frac{g_w^2}{8m_w^2}f_{\pi}\right)^2 p_{\mu}p_{\nu}(8[p_3^{\mu}p_2^{\nu} + p_2^{\mu}p_3^{\nu} - (p_3 \cdot p_2)g^{\mu\nu}] + 8i\epsilon^{\mu\lambda\nu\sigma}p_{3\lambda}p_{2\sigma} \tag{7}$$

Summing over the spins gives us:

$$\langle|\mathcal{M}^2|\rangle = 8\left(\frac{g_w^2}{8m_w^2}f_{\pi}\right)^2 [2(p_1 \cdot p_2)(p_1 \cdot p_3) - p^2(p_2 \cdot p_3)] \tag{8}$$

Since $p = p_2 + p_3$, we can simplify the equation further
For simplicity and consistency we will use the following notation: $p_1 = p_{\pi}, p_2 = p_l, p_3 = p_{\nu_l}$
We begin with showing the value of the 4-momentum squared:

$$\begin{aligned} p_1 &= (E, \vec{p}_1) && (9)\\ (p_1)^2 &= p_{\mu}p_{\nu}g_{\mu\nu} \\ (p_1)^2 &= p_1p_1(1) + p_2p_2(-1) + ... \\ (p_1)^2 &= E^2 - (\vec{p}_1)^2 \\ (p_1)^2 &= m_1^2 && (10) \end{aligned}$$

This can be also shown to be true for the other 4-Momenta thus:
$(p_1)^2 = (p_{\pi})^2 = m_{\pi}^2, (p_2)^2 = (p_l)^2 = m_l^2, (p_3)^2 = (p_{\nu_l})^2 = (m_{\nu_l})^2 = 0$
Using this we can further simplify Equation 7:

$$\frac{1}{2}[(m_{\pi})^2 - (m_l)^2] = (p_2 \cdot p_3) \tag{11}$$

$$\frac{1}{2}[(m_{\pi})^2 - (m_l)^2] = (p_1 \cdot p_3) \tag{12}$$

$$\frac{1}{2}[(m_{\pi})^2 + (m_l)^2] = (p_1 \cdot p_2) \tag{13}$$

Returning to Equation 7 we now have

$$\langle|\mathcal{M}^2|\rangle = 8\left(\frac{g_w^2}{8m_w^2}f_{\pi}\right)^2 [\frac{1}{2}(m_l)^2((m_{\pi})^2 - (m_l)^2)] \tag{14}$$

In this way we are able to calculate the branching ratio of a pion, since we know have the Feyneman Amplitude we simply return to equation (112)

$$\Gamma = \frac{S|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2 \tag{15}$$

$$\Gamma = \frac{S|\mathbf{p}|}{\pi m_\pi^2}\left(\frac{g_w^2}{8m_w^2}f_\pi\right)^2\left[\frac{1}{2}(m_l)^2((m_\pi)^2-(m_l)^2)\right] \tag{16}$$

We can remove S because in this case S=1
We then need to find the value of $|\mathbf{p}|$

$$\begin{aligned}
|\mathbf{p}| &= \frac{\sqrt{(m_1+m_2+m_3)(m_1-m_2-m_3)(m_1+m_2-m_3)(m_1-m_2+m_3)}}{2m_1} && (17)\\
|\mathbf{p}| &= \frac{\sqrt{m_1^4+m_2^4+m_3^4-2m_1^2m_2^2-2m_1^2m_3^2-2m_2^2m_3^2}}{2m_1} \\
|\mathbf{p}| &= \frac{\sqrt{m_1^4+m_2^4-m_1^2m_2^2}}{2m_1} && (18)
\end{aligned}$$

(Because the neutrino is massless)

$$|\mathbf{p}| = \frac{\sqrt{(m_1^2-m_2^2)^2}}{2m_1} \tag{19}$$

For our case $m_1 = m_\pi$ and $m_2 = m_l$

$$|\mathbf{p}| = \frac{(m_\pi^2-m_l^2)}{2m_\pi} \tag{20}$$

Combining with Equation 14:

$$\begin{aligned}
\Gamma &= \frac{|\mathbf{p}|}{2\pi m_1^2}\left(\frac{g_w^2}{8m_w^2}f_\pi\right)^2\left[(m_l)^2((m_\pi)^2-(m_l)^2)\right] && (21)\\
\Gamma &= \frac{1}{4\pi}\left(\frac{g_w^2 f_\pi}{8m_w^2}\right)^2\frac{1}{m_\pi^3}m_l^2(m_\pi^2-m_l^2)^2 && (22)
\end{aligned}$$

Because $\frac{g_w^2}{8m_w^2} = \frac{G_f}{\sqrt{2}}$ we are able to say:

$$\Gamma = \frac{1}{8\pi}(G_f f_\pi)^2\frac{1}{m_\pi^3}m_l^2(m_\pi^2-m_l^2)^2 \tag{23}$$

Expanding on this idea, we are able to graph $\frac{\Gamma_l}{\Gamma_\pi}$ by:

$$\Gamma = \frac{1}{8\pi}(G_f f_\pi)^2 m_\pi^3\left(\frac{m_l}{m_\pi}\right)^2\left(1-\left(\frac{m_l}{m_\pi}\right)^2\right)^2 \tag{24}$$

In order to find $\frac{\Gamma_l}{\Gamma_\pi}$ we need to divide the above expression by $\frac{1}{\tau_\pi}$ but normalize it with $\hbar$ so our final expression is:

$$\frac{\Gamma_l}{\Gamma_\pi} = \frac{1}{8\pi}(G_f f_\pi)^2 m_\pi^3\left(\frac{\tau_\pi}{\hbar}\right)\left(\frac{m_l}{m_\pi}\right)^2\left(1-\left(\frac{m_l}{m_\pi}\right)^2\right)^2 \tag{25}$$

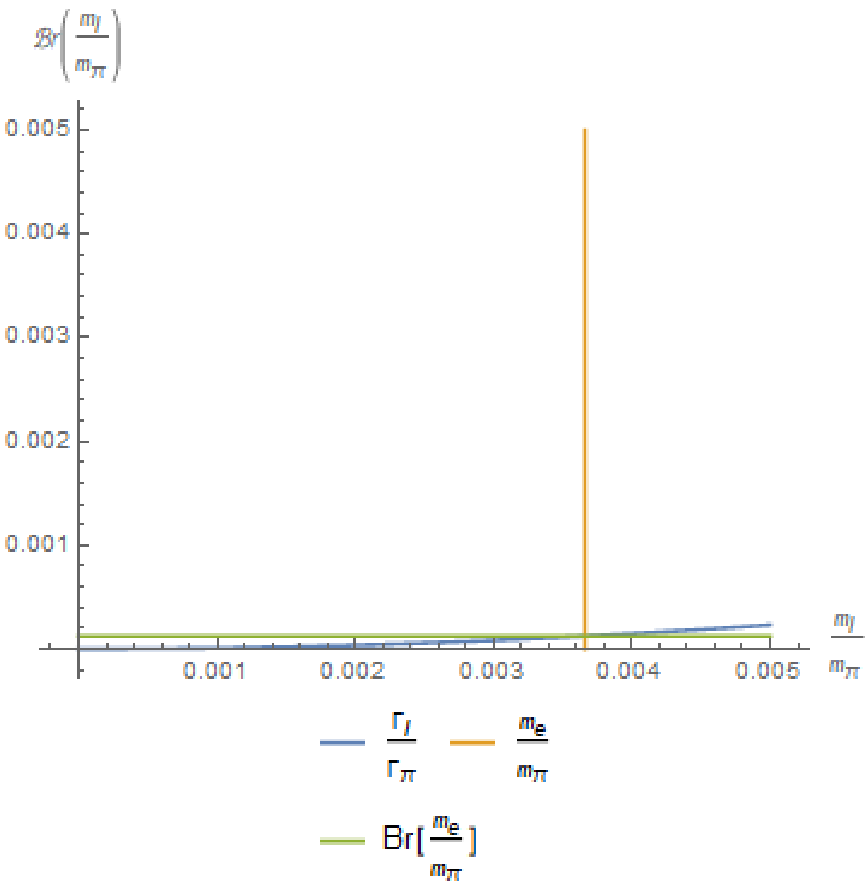

Figure 2: Graph of $\frac{\Gamma_{m_e}}{\Gamma_\pi}$ and the plots of $\frac{m_e}{m_\pi}$ and the value of $Br(\pi^- \to e^- + \nu_e)$

The graph of the equation (with $m_l = m_e$) is:

Using the values from PDG [2] we can do calculations with the branching ratios.

Observables	e	μ	π
τ	6.6×10^{28} yr	$2.1969811(22) \times 10^{-}6$ s	$2.6033(5) \times 10^{-}8$ s
Mass(MeV)	0.5109989461(31)	105.6583745(24)	139.57061(24)

If we would like to calculate the ratio of the $\pi^- \to e^- + \nu_e$ and $\pi^- \to \mu^- + \nu_\mu$ we simply do the following:

$$\frac{\Gamma_e}{\Gamma_\mu} = \frac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2} \tag{26}$$

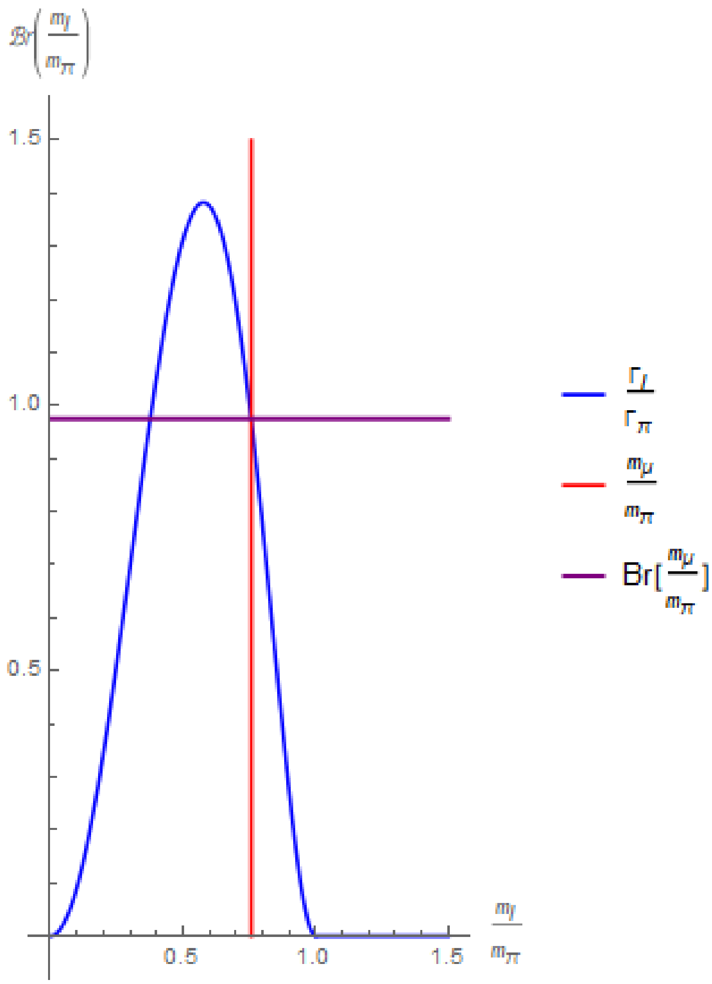

Figure 3: Graph of $\frac{\Gamma_{m_\mu}}{\Gamma_\pi}$ and the plots of $\frac{m_\mu}{m_\pi}$ and the value of $Br(\pi^- \to \mu^- + \nu_\mu)$

$$\frac{\Gamma_e}{\Gamma_\mu} = 1.28334(73) \times 10^{-4} \tag{27}$$

This is an interesting observation because the value of $\frac{\Gamma_e}{\Gamma_\mu}$ suggests that the probability of $\pi^- \to \mu^- + \nu_\mu$ is higher than $\pi^- \to e^- + \nu_e$. This is somewhat striking because the mass of a muon is greater than the mass of an electron, indicating that the pion does not decay into the lightest particle most frequently.

3 The Branching Ratio of the $B_s \to \mu^+\mu^-$ decay

Following the procedure outlined above and comparing to [3] we are able to find an expression for the decay rate of the $B_s \to \mu^+\mu^-$ with some coefficients of new physics included.
The decay rate is as follows:

$$\mathcal{B}(B_s \to \mu^+\mu^-) = \frac{G_F^2 \alpha_{em}^2 m_{B_s}^5 f_{B_s}^2 \tau_{B_s}}{64\pi^3} \sqrt{1 - \frac{4m_\mu^2}{m_{B_s}^2}} \times$$

$$\left\{\left(1-\frac{4m_\mu^2}{m_{B_s}^2}\right)\left|\zeta\frac{C_S-C_S^{'}}{m_b+m_s}\right|^2+\left|\zeta\frac{C_P-C_P^{'}}{m_b+m_s}+\frac{2m_\mu}{m_{B_s}^2}[|V_{tb}V_{ts}^*|C_{10}+\zeta(C_A-C_A^{'})]\right|^2\right\} \quad (28)$$

Where $\zeta \equiv (\frac{g_{NP}^2}{\Lambda^2})(\frac{\sqrt{2}}{4G_F})(\frac{4\pi}{\alpha_{em}})$

Since we seek to find the constraints on the parameters $C_S, C_S^{'}, C_P, C_P^{'}, C_A$, and $C_A^{'}$ we set the equation equal to the branching ratio given in [4] and using the values for $C_{10}, |V_{tb}V_{ts}^*|, g_{NP}$, and Λ given in [3] thus we are able to find the constraints on the parameters. In order to solve for one parameter, we allow the other two to be equal to 0, this will simply our calculations.We will solve each coefficient by the upper and lower limits of the branching ratio then take the average by setting the expression with the unknown coefficient equal to the branching ratio, the upper limit of the branching ratio, and its lower limit.

The values for the Branching ratio of $B_s \to \mu^+\mu^-$ decay is equal to $2.4^{+0.9}_{-0.7} \times 10^{-9}$ [2]

$$\begin{aligned} C_S &= \pm 1.6215i \times 10^{-4} \text{ For the lower BR value} \\ &= \pm 4.63981 \times 10^{-5} \text{ For the upper BR value} \end{aligned}$$

$$\begin{aligned} C_P &= 6.3832 \times 10^{-5} \text{ or } 4.1125 \times 10^{-4} \text{ For the lower BR value} \\ &= -4.482 \times 10^{-6} \text{ or } 4.7957 \times 10^{-4} \text{ For the upper BR value} \end{aligned}$$

$$\begin{aligned} C_A &= 2.0348 \times 10^{-3} \text{ or } 1.31095 \times 10^{-2} \text{ For the lower BR value} \\ &= 1.48237 \times 10^{-4} \text{ or } 1.5287 \times 10^{-2} \text{ For the upper BR value} \end{aligned}$$

Notes about the Wilson Coefficients:
The C_S coefficient is equal to $\pm(0.811i + 0.232) \times 10^{-4}$ (We are able to calculate the mean by a simply average)

The C_P has two values the first value is equal to 2.967×10^{-5} and the second value is equal to 4.454×10^{-4}

The last coefficient C_A also has two values, the first one is equal to 1.089×10^{-3} and the second value is equal to 1.420×10^{-2}

After finding the Wilson Coefficients, I was able to use the Python package Flavio [4] to compute the NP values of $B(B_s \to \mu^+\mu^-)$

We again only apply one Wilson Coefficient at a time to find the NP branching ratio:
For the C_S:

$$C_S = \pm(0.811i + 0.232) \times 10^{-4}$$
Flavio Prediction: BR $= 3.610 \times 10^{-9}$

For C_P:

$$C_P = 2.967 \times 10^{-5}$$
$$\text{Flavio Prediction: } Br = 3.603 \times 10^{-9}$$
$$C_P = 4.454 \times 10^{-4}$$
$$\text{Flavio Prediction: } Br = 3.508 \times 10^{-9}$$

For C_A:

$$C_A = 1.089 \times 10^{-3}$$
$$\text{Flavio Prediction: } Br = 3.608 \times 10^{-9}$$
$$C_A = 1.420 \times 10^{-2}$$
$$\text{Flavio Prediction: } Br = 3.584 \times 10^{-9}$$

4 Plots of $Br(WC)$

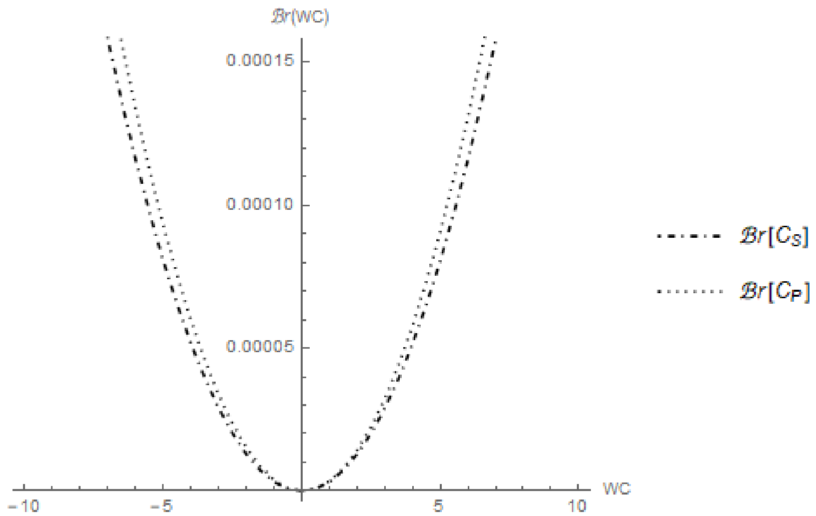

Figure 4: Graph of $Br(B_s \to \mu^+\mu^-)$ as functions of the C_P and C_S Wilson Coefficients

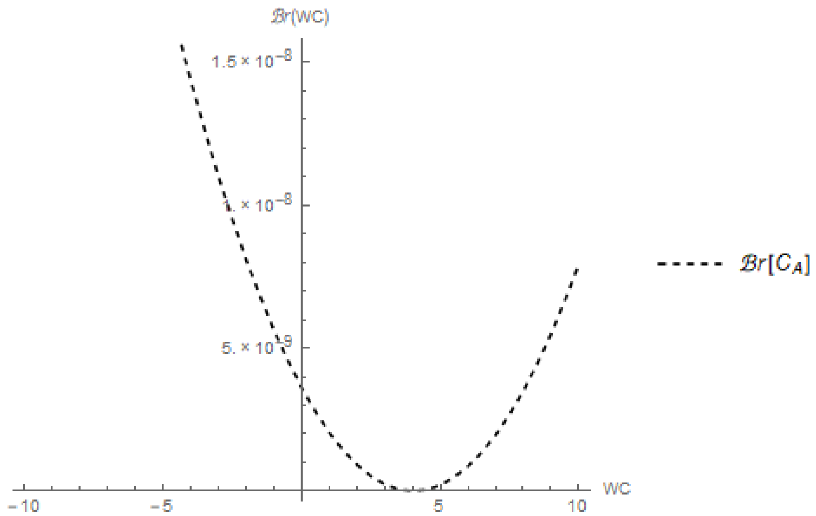

Figure 5: Graph of $Br(B_s \to \mu^+\mu^-)$ as functions of the C_A Wilson Coefficient

The above graphs gives an indication of how drastically the variation of a Wilson Coefficient could change the value of the Branching ratio.
For each of the plots I have conducted a fit to the lines in order to find the coefficient values to the lines.
$Br(C_S) = 3.22046 \times 10^{-6}x^2$
$Br(C_P) = 3.57491 \times 10^{-6}x^2$
$Br(C_A) = 2.79479 \times 10^{-10}x^2$

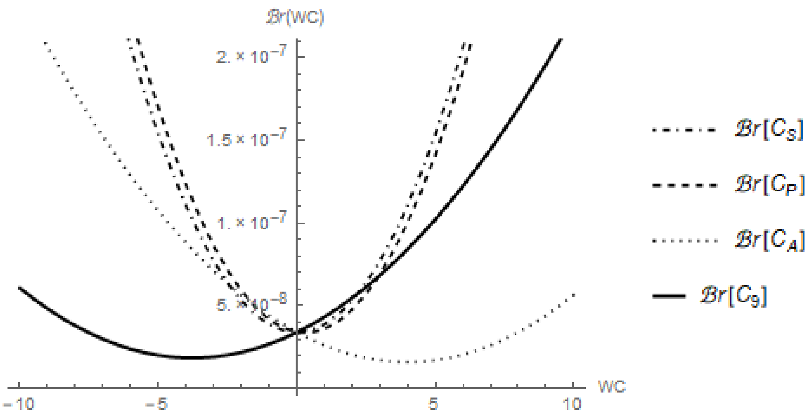

Figure 6: Graph of $Br(B \to K\mu^+\mu^-)$ as functions of the Wilson Coefficients

Here, in this decay I was also able to compute fits to the plots.

$Br(C_S) = 5.73539 \times 10^{-9}x^2$
$Br(C_P) = 5.81325 \times 10^{-9}x^2$
$Br(C_A) = 1.7963 \times 10^{-9}x^2$
$Br(C_9) = 1.75406 \times 10^{-9}x^2$

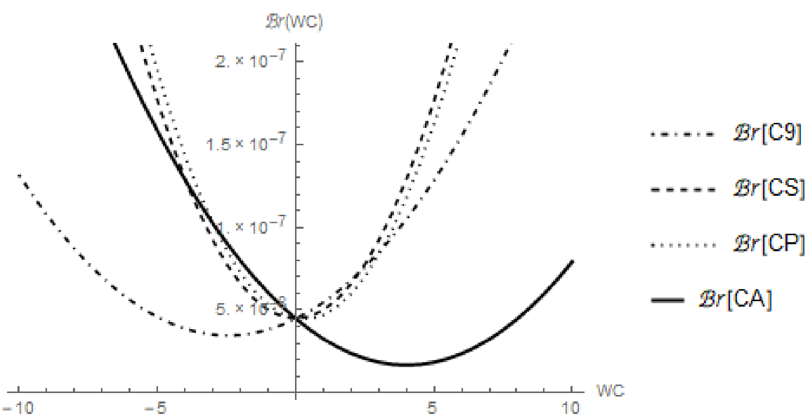

Figure 7: Graph of $Br(B \to K^*\mu^+\mu^-)$ as functions of the Wilson Coefficients

The fits for the plots are as follows:
$Br(C_S) = 5.97644 \times 10^{-9}x^2$
$Br(C_P) = 6.056 \times 10^{-9}x^2$
$Br(C_A) = 2.45199 \times 10^{-9}x^2$
$Br(C_9) = 2.40283 \times 10^{-9}x^2$

5 Conclusion

As one can see, the work done with the Wilson Coefficients gives new suggestions to the possibility of new physics. In the future, I would like to continue to explore these possibilities. Some work that was not included in this paper is working to use a chi squared minimization technique to help limit and constrain the coefficients. The chi squared technique would use the observables that have been calculated experimentally. Although I was not able to accomplish this during the summer, I hope to continue working on this is the future and produce significant results.

6 Acknowledgements

I would like to thank the Department of Physics and Astronomy at Wayne State University for this opportunity to conduct this research. I would also like to thank Dr.Bhubanjyoti Bhattacharya for his guidance and suggestions and doctoral student Cody Grant who's advice and discussions were invaluable. Finally I would like to thank my parents Andrew and Liz Walz for giving me the love and support to pursue my interests and begin my career as a physicist.

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7 Appendix

Practice with Spinors

Notation:

1. $S = \overline{u}u$
2. $P = \overline{u}\gamma^5 u$
3. $V^\mu = \overline{u}\gamma^\mu u$
4. $A^\mu = \overline{u}\gamma^\mu\gamma^5 u$
5. $T^{\mu\nu} = \overline{u}\sigma^{\mu\nu} u$

It was necessary to practice with spinor notation and the different mathematical techniques before starting this project. This appendix summarizes what I was able to do in that regard.

1. $(\overline{u}_1\gamma^\mu u_2)^* \ = ?$

   Note: $(\gamma^0)^\dagger \ = \ \gamma^0$ and $(\gamma^\mu)^\dagger \ = \ \gamma^0\gamma^\mu\gamma^0$

   $(\overline{u}_1\gamma^\mu u_2)$ is a 1×1 matrix. Therefore, its complex conjugate is the same as its Hermitian conjugate, i.e. if we call $V^\mu = (\overline{u}_1\gamma^\mu u_2)$, then $(V^\mu)^* = (V^\mu)^\dagger$. We can then express this quantity as follows:

$$\begin{aligned} V^\mu &= \overline{u}_1\gamma^\mu u_2 \ , \qquad (29) \\ \Rightarrow (V^\mu)^* &= (V^\mu)^\dagger \ , \\ &= (\overline{u}_1\gamma^\mu u_2)^\dagger \ , \\ &= ((u_1)^\dagger\gamma^0\gamma^\mu u_2)^\dagger \quad \text{using } (A\ldots Z)^\dagger = Z^\dagger\ldots A^\dagger \ , \\ &= (u_2^\dagger)(\gamma^\mu)^\dagger(\gamma^0)^\dagger(u_1) \\ &= (u_2^\dagger)\gamma^0\gamma^\mu\gamma^0\gamma^0(u_1) \\ &= (u_2^\dagger)\gamma^0\gamma^\mu(u_1) \\ &= \overline{u}_2\gamma^\mu(u_1) \qquad (30) \end{aligned}$$

   Therefore $(\overline{u}_1\gamma^\mu u_2)^* \ = \ \overline{u}_2\gamma^\mu u_1$. To solve for $|V^\mu|^2$ we simply use $|V^\mu|^2 \ = \ \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1]$.

   Note: $\mathrm{Tr}[\gamma^\mu\gamma^\nu] \ = \ 4g^{\mu\nu}$, $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$, The trace over the product of an odd number of gamma matrices is zero.

$$\begin{aligned} |V^\mu|^2 &= \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1] \qquad (31) \\ &= \mathrm{Tr}[\overline{u}_1\gamma^\mu(\not{p}_2 + m)\gamma^\nu u_1] \\ &= \mathrm{Tr}[u_1\overline{u}_1\gamma^\mu(\not{p}_2 + m)\gamma^\nu] \\ &= \mathrm{Tr}[(\not{p}_1 + m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu] \\ &= \mathrm{Tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] + m[\mathrm{Tr}(\gamma^\mu\not{p}_1\gamma^\nu) + \mathrm{Tr}(\gamma^\mu\gamma^\nu\not{p}_2)] + m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] \\ &= \mathrm{Tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] + m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Tr}[(p_1)_\lambda \gamma^\lambda \gamma^\mu (p_2)_\sigma \gamma^\sigma \gamma^\nu] + 4m^2 g^{\mu\nu} \\
&= (p_1)_\lambda (p_2)_\sigma \mathrm{Tr}[\gamma^\lambda \gamma^\mu \gamma^\sigma \gamma^\nu] + 4m^2 g^{\mu\nu} \\
&= (p_1)_\lambda (p_2)_\sigma 4(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda}) + 4m^2 g^{\mu\nu} \\
&= 4[p_1^\mu p_2^\nu - g^{\mu\nu}(p_1 \cdot p_2) + p_2^\mu p_1^\nu] + 4m^2 g^{\mu\nu} \qquad (32)
\end{aligned}
$$

2. $(\overline{u}_1 \gamma^\mu \gamma^5 u_2)^*$ is also a 1×1 Matrix so the same reasoning applies as above in 1. Note: $(\gamma^5)^\dagger = \gamma^5$ We define: A^μ as $\overline{u}_1 \gamma^\mu \gamma^5 u_2$ thus:

$$
\begin{aligned}
(A^\mu)^* &= (A^\mu)^\dagger \qquad (33) \\
&= (\overline{u}_1 \gamma^\mu \gamma^5 u_2)^\dagger \\
&= ((u_1)^\dagger \gamma^0 \gamma^\mu \gamma^5 u_2)^\dagger \\
&= (u_2^\dagger)(\gamma^5)^\dagger (\gamma^\mu)^\dagger (\gamma^0)^\dagger (u_1) \\
&= (u_2^\dagger)\gamma^5 \gamma^0 \gamma^\mu \gamma^0 \gamma^0 u_1 \\
&= (u_2^\dagger)\gamma^5 \gamma^0 \gamma^\mu (1) u_1 \\
&= -(u_2^\dagger)\gamma^0 \gamma^5 \gamma^\mu u_1 \\
&= -\overline{u}_2 \gamma^5 \gamma^\mu u_1 \\
&= \overline{u}_2 \gamma^\mu \gamma^5 u_1 \qquad (34)
\end{aligned}
$$

Therefore $(\overline{u}_1 \gamma^\mu \gamma^5 u_2)^* \; = \; \overline{u}_2 \gamma^\mu \gamma^5 u_1$

We also are able to calculate $|A^\mu|^2$

$$
\begin{aligned}
|A^\mu|^2 &= \mathrm{Tr}[(\overline{u}_1 \gamma^\mu \gamma^5 u_2)(\overline{u}_2 \gamma^\nu \gamma^5 u_1)] \qquad (35) \\
&= \mathrm{Tr}[\overline{u}_1 \gamma^\mu \gamma^5 (\not{p}_2 + m) \gamma^\nu \gamma^5 u_1] \\
&= \mathrm{Tr}[u_1 \overline{u}_1 \gamma^\mu \gamma^5 (\not{p}_2 + m) \gamma^\nu \gamma^5] \\
&= \mathrm{Tr}[(\not{p}_1 + m) \gamma^\mu \gamma^5 (\not{p}_2 + m) \gamma^\nu \gamma^5] \\
&= \mathrm{Tr}[\not{p}_1 \gamma^\mu \gamma^5 \not{p}_2 \gamma^\nu \gamma^5 + m(\not{p}_1 \gamma^\mu \gamma^5 \gamma^\nu \gamma^5 + \gamma^\mu \gamma^5 \not{p}_2 \gamma^\nu \gamma^5) + m^2(\gamma^\mu \gamma^5 \gamma^\nu \gamma^5)] \\
&= \mathrm{Tr}[\not{p}_1 \gamma^\mu \gamma^5 \not{p}_2 \gamma^\nu \gamma^5 + m(\not{p}_1 \gamma^\mu (-\gamma^5 \gamma^5) \gamma^\nu + \not{p}_2 \gamma^\mu (-\gamma^5 \gamma^5) \gamma^\nu) + m^2(\gamma^\mu \gamma^5 \gamma^\nu \gamma^5)] \\
&= \mathrm{Tr}[\not{p}_1 \gamma^\mu \gamma^5 \not{p}_2 \gamma^\nu \gamma^5 - m(\not{p}_1 \gamma^\mu \gamma^\nu + \not{p}_2 \gamma^\mu \gamma^\nu) + m^2(\gamma^\mu \gamma^5 \gamma^\nu \gamma^5)] \\
&= \mathrm{Tr}[\not{p}_1 \gamma^\mu \gamma^5 \not{p}_2 \gamma^\nu \gamma^5] - mTr[\not{p}_1 \gamma^\mu \gamma^\nu] - mTr[\not{p}_2 \gamma^\mu \gamma^\nu] + m^2(\gamma^\mu \gamma^5 \gamma^\nu \gamma^5)] \\
&= \mathrm{Tr}[\not{p}_1 \gamma^\mu \gamma^5 \not{p}_2 \gamma^\nu \gamma^5 + m^2(\gamma^\mu \gamma^5 \gamma^\nu \gamma^5)] \\
&= \mathrm{Tr}[(p_1)_\lambda \gamma^\lambda \gamma^\mu \gamma^5 (p_2)_\sigma \gamma^\sigma \gamma^\nu \gamma^5] + m^2 \mathrm{Tr}[\gamma^\mu \gamma^5 \gamma^\nu \gamma^5] \\
&= (p_1)_\lambda (p_2)_\sigma \mathrm{Tr}[\gamma^\lambda \gamma^\mu \gamma^5 \gamma^\sigma \gamma^\nu \gamma^5] - m^2 \mathrm{Tr}[\gamma^\mu \gamma^5 \gamma^5 \gamma^\nu] \\
&= (p_1)_\lambda (p_2)_\sigma \mathrm{Tr}[\gamma^\lambda \gamma^\mu \gamma^5 \gamma^5 \gamma^\sigma \gamma^\nu] - m^2 \mathrm{Tr}[\gamma^\mu \gamma^\nu] \\
&= (p_1)_\lambda (p_2)_\sigma \mathrm{Tr}[\gamma^\lambda \gamma^\mu \gamma^\sigma \gamma^\nu] - m^2 (g^{\mu\nu}) \\
&= (p_1)_\lambda (p_2)_\sigma 4(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda}) - 4m^2 g^{\mu\nu} \\
&= 4[p_1^\mu p_2^\nu - g^{\mu\nu}(p_1 \cdot p_2) + p_2^\mu p_1^\nu] - 4m^2 g^{\mu\nu} \qquad (36)
\end{aligned}
$$

3. $(\overline{u}_1 u_2)^* \; = \; ?$ We let $S \; = \; \overline{u}_1 u_2$

$$
(S)^* = (S)^\dagger \qquad (37)
$$

$$
\begin{aligned}
&= (\overline{u}_1 u_2)^\dagger \ , \\
&= ((u_1)^\dagger \gamma^0 u_2)^\dagger \\
&= (u_2)^\dagger (\gamma^0)^\dagger (u_1) \\
&= (u_2)^\dagger \gamma^0 (u_1) \\
&= \overline{u}_2 (u_1) \qquad (38)
\end{aligned}
$$

Therefore$(\overline{u}_1 u_2)^* \ = \ \overline{u}_2 u_1$. In order to find $|S|^2$ we simply do the following:

$$
\begin{aligned}
|S|^2 &= \mathrm{Tr}[\overline{u}_1 u_2 \overline{u}_2 u_1] \qquad (39) \\
&= \mathrm{Tr}[(\not{p}_1 + m)(\not{p}_2 + m)] \\
&= \mathrm{Tr}[\not{p}_1 \not{p}_2 + m(\not{p}_1 + \not{p}_2) + m^2] \\
&= \mathrm{Tr}[\not{p}_1 \not{p}_2] + \mathrm{Tr}[m(\not{p}_1 + \not{p}_2)] + \mathrm{Tr}[m^2] \\
&= \mathrm{Tr}[\not{p}_1 \not{p}_2] + m(\mathrm{Tr}[\not{p}_1] + \mathrm{Tr}[\not{p}_2]) + m^2 \mathrm{Tr}[1] \\
&= \mathrm{Tr}[\not{p}_1 \not{p}_2] + 4m^2 \\
&= 4(p_1 \cdot p_2) + 4m^2 \qquad (40)
\end{aligned}
$$

4. By the same reasoning as shown above it can be shown that $(\overline{u}_1 \gamma^5 u_2)^* \ = \ \overline{u}_2 \gamma^5 u_1$
If we let $P \ = \ \overline{u}_1 \gamma^5 u_2$ then:

$$
\begin{aligned}
(P)^* &= (P)^\dagger \qquad (41) \\
&= (\overline{u}_1 \gamma^5 u_2)^\dagger \ , \\
&= ((u_1)^\dagger \gamma^0 \gamma^5 u_2)^\dagger \\
&= (u_2)^\dagger (\gamma^5)^\dagger (\gamma^0)^\dagger (u_1) \\
&= (u_2)^\dagger (\gamma^5) \gamma^0 (u_1) \\
&= -(u_2)^\dagger \gamma^0 \gamma^5 (u_1) \\
&= -\overline{u}_2 \gamma^5 (u_1) \qquad (42)
\end{aligned}
$$

Therefore $(\overline{u}_1 \gamma^5 u_2)^* \ = \ -\overline{u}_2 \gamma^5 u_1$

In order to square P we do the following:

$$
\begin{aligned}
|P|^2 &= \mathrm{Tr}[\overline{u}_1 \gamma^5 u_2 (-\overline{u}_2 \gamma^5 u_1)] \qquad (43) \\
&= \mathrm{Tr}[u_1 \overline{u}_1 \gamma^5 (-\not{p}_2 - m) \gamma^5] \\
&= \mathrm{Tr}[(\not{p}_1 + m) \gamma^5 (-\not{p}_2 - m) \gamma^5] \\
&= \mathrm{Tr}[((p_1)_\mu \gamma^\mu + m) \gamma^5 ((-p_2)_\nu \gamma^\nu - m) \gamma^5] \\
&= \mathrm{Tr}[((p_1)_\mu \gamma^\mu \gamma^5 + m\gamma^5)((-p_2)_\nu \gamma^\nu \gamma^5 - m\gamma^5)] \\
&= \mathrm{Tr}[(p_1)_\mu \gamma^\mu \gamma^5 (-p_2)_\nu \gamma^\nu \gamma^5 + m\gamma^5 (-p_2)_\nu \gamma^\nu \gamma^5 - m\gamma^5 (p_1)_\mu \gamma^\mu \gamma^5 - \gamma^5 \gamma^5 m^2)] \\
&= \mathrm{Tr}[(p_1)_\mu \gamma^\mu \gamma^5 (-p_2)_\nu \gamma^\nu \gamma^5] + Tr[m\gamma^5 (-\gamma^5)(-p_2)_\nu \gamma^\nu] - Tr[m\gamma^5 (-\gamma^5)(p_1)_\mu \gamma^\mu] - Tr[m^2] \\
&= \mathrm{Tr}[(p_1)_\mu \gamma^\mu \gamma^5 (-p_2)_\nu \gamma^\nu \gamma^5] - Tr[m(-p_2)_\nu \gamma^\nu] + Tr[m(p_1)_\mu \gamma^\mu] - 4m^2 \\
&= \mathrm{Tr}[(p_1)_\mu \gamma^\mu \gamma^5 (-p_2)_\nu \gamma^\nu \gamma^5] - 4m^2 \\
&= (p_1)_\mu (-p_2)_\nu \mathrm{Tr}[\gamma^\mu \gamma^5 \gamma^\nu \gamma^5] - 4m^2
\end{aligned}
$$

$$
\begin{aligned}
&= (p_1)_\mu(-p_2)_\nu \mathrm{Tr}[\gamma^\mu\gamma^5(-\gamma^5\gamma^\nu)] - 4m^2 \\
&= (p_1)_\mu(-p_2)_\nu(-\mathrm{Tr}[\gamma^\mu\gamma^\nu]) - 4m^2 \\
&= (p_1)_\mu(-p_2)_\nu(-4g^{\mu\nu}) - 4m^2 \\
&= 4(p_1)(p_2) - 4m^2 \qquad (44)
\end{aligned}
$$

5. While the above identities could be shown to be trivial, the identity: $(\overline{u}_1\sigma^{\mu\nu}u_2)^* = \overline{u}_2\sigma^{\nu\mu}u_1$ is more difficult to solve. The identity:$(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$ is needed

$$
\begin{aligned}
(\sigma^{\mu\nu})^\dagger &= (\frac{i}{2}[\gamma^\mu,\gamma^\nu])^\dagger \qquad (45) \\
&= -\frac{i}{2}([\gamma^\mu,\gamma^\nu])^\dagger \\
&= -\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)^\dagger \\
&= -\frac{i}{2}((\gamma^\nu)^\dagger(\gamma^\mu)^\dagger - (\gamma^\mu)^\dagger(\gamma^\nu)^\dagger) \\
&= -\frac{i}{2}(\gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^\mu\gamma^0 - \gamma^0\gamma^\mu\gamma^0\gamma^0\gamma^\nu\gamma^0) \\
&= -\frac{i}{2}(\gamma^0\gamma^\nu\gamma^\mu\gamma^0 - \gamma^0\gamma^\mu\gamma^\nu\gamma^0) \\
&= -\frac{i}{2}((-1)^2\gamma^\nu\gamma^\mu - (-1)^2\gamma^\mu\gamma^\nu) \\
&= -\frac{i}{2}(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu) \\
&= -\sigma^{\nu\mu} \qquad (46)
\end{aligned}
$$

After showing $(\sigma^{\mu\nu})^\dagger = -\sigma^{\nu\mu}$ is true it is trivial to show $(\overline{u}_1\sigma^{\mu\nu}u_2)^* = -\overline{u}_2\sigma^{\nu\mu}u_1$. We let $T^{\mu\nu} = \overline{u}_1\sigma^{\mu\nu}u_2$

$$
\begin{aligned}
(T^{\mu\nu})^* &= (T^{\mu\nu})^\dagger\ , \qquad (47) \\
&= (\overline{u}_1\sigma^{\mu\nu}u_2)^\dagger \\
&= ((u_1)^\dagger\gamma^0\sigma^{\mu\nu}u_2)^\dagger \\
&= (u_2^\dagger)(\sigma^{\mu\nu})^\dagger(\gamma^0)^\dagger(u_1) \\
&= (u_2^\dagger)(-\sigma^{\nu\mu})\gamma^0(u_1) \\
&= (u_2^\dagger)(-\gamma^0)(-\sigma^{\nu\mu})(u_1) \\
&= \overline{u}_2\sigma^{\mu\nu}(u_1) \qquad (48)
\end{aligned}
$$

In order to find the value of $|T^{\mu\nu}|^2$ one needs to find the value of $Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}]$

$$
\begin{aligned}
Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] &= Tr[\frac{i}{2}(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma)\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)] \qquad (49) \\
&= Tr[\frac{i}{2}(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma)\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)] \\
&= -\frac{1}{4}Tr[(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma)(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)]
\end{aligned}
$$

$$= -\frac{1}{4}Tr[\gamma^\sigma\gamma^\lambda\gamma^\mu\gamma^\nu] + \frac{1}{4}Tr[\gamma^\sigma\gamma^\lambda\gamma^\nu\gamma^\mu] + \frac{1}{4}Tr[\gamma^\lambda\gamma^\sigma\gamma^\mu\gamma^\nu] - \frac{1}{4}Tr[\gamma^\lambda\gamma^\sigma\gamma^\nu\gamma^\mu]$$

Here we must label each of the traces individually:

$$\begin{aligned} A &= -\frac{1}{4}Tr[\gamma^\sigma\gamma^\lambda\gamma^\mu\gamma^\nu] \\ &= -(g^{\sigma\lambda}g^{\mu\nu} - g^{\sigma\mu}g^{\lambda\nu} + g^{\sigma\nu}g^{\lambda\mu}) \end{aligned} \tag{50}$$

$$\begin{aligned} B &= +\frac{1}{4}Tr[\gamma^\sigma\gamma^\lambda\gamma^\nu\gamma^\mu] \\ &= +(g^{\sigma\lambda}g^{\nu\mu} - g^{\sigma\nu}g^{\lambda\mu} + g^{\sigma\mu}g^{\lambda\nu}) \end{aligned} \tag{51}$$

$$\begin{aligned} C &= \frac{1}{4}Tr[\gamma^\lambda\gamma^\sigma\gamma^\mu\gamma^\nu] \\ &= (g^{\lambda\sigma}g^{\mu\nu} - g^{\lambda\mu}g^{\sigma\nu} + g^{\lambda\nu}g^{\sigma\mu}) \end{aligned} \tag{52}$$

$$\begin{aligned} D &= -\frac{1}{4}Tr[\gamma^\lambda\gamma^\sigma\gamma^\nu\gamma^\mu] \\ &= -(g^{\lambda\sigma}g^{\nu\mu} - g^{\lambda\nu}g^{\sigma\mu} + g^{\lambda\mu}g^{\sigma\nu}) \end{aligned} \tag{53}$$

$$\begin{aligned} Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] &= A + B + +C + D \\ &= 2g^{\sigma\mu}g^{\lambda\nu} - 2g^{\sigma\nu}g^{\lambda\mu} - 2g^{\lambda\mu}g^{\sigma\nu} + 2g^{\lambda\nu}g^{\sigma\mu}) \end{aligned} \tag{54}$$

In order to find the value of $|T^{\mu\nu}|^2$ one needs to do the following:

$$\begin{aligned} |T^{\mu\nu}|^2 &= \mathrm{Tr}[\overline{u}_1\sigma^{\mu\nu}u_2\overline{u}_2\sigma^{\sigma\lambda}u_1] \qquad (55) \\ &= \mathrm{Tr}[\overline{u}_1\sigma^{\mu\nu}(\not{p}_2 + m)\sigma^{\sigma\lambda}u_1] \\ &= \mathrm{Tr}[(\not{p}_1 + m)\sigma^{\mu\nu}(\not{p}_2 + m)\sigma^{\sigma\lambda}] \\ &= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa + m)\sigma^{\mu\nu}((p_2)_\gamma\gamma^\gamma + m)\sigma^{\sigma\lambda}] \\ &= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu} + m\sigma^{\mu\nu})((p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda} + m\sigma^{\sigma\lambda})] \\ &= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda} + (p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}m\sigma^{\sigma\lambda} + (p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}m\sigma^{\mu\nu} + m\sigma^{\sigma\lambda}m\sigma^{\mu\nu})] \\ &= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}] + Tr[(p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}m\sigma^{\sigma\lambda} + (p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}m\sigma^{\mu\nu}] + Tr[m\sigma^{\sigma\lambda}m\sigma^{\mu\nu}] \\ &= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}] + Tr[m\sigma^{\sigma\lambda}m\sigma^{\mu\nu}] \\ &= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}] + m^2Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] \text{ Let B } = \mathrm{m}^2\mathrm{Tr}[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] \\ &= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}] + B \\ &= (p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[\gamma^\kappa\sigma^{\mu\nu}\gamma^\gamma\sigma^{\sigma\lambda}] + B \\ &= (p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[\gamma^\kappa(\frac{i}{2}((\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma^\gamma(\frac{i}{2}(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma))] + B \\ &= -\frac{1}{4}(p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[\gamma^\kappa(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma^\gamma(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma)] + B \text{ (Let A } = -\frac{1}{4}(\mathrm{p}_1)_\kappa(\mathrm{p}_2)_\gamma) \end{aligned}$$

$$
\begin{aligned}
&= (A)\mathrm{Tr}[(\gamma^\kappa\gamma^\mu\gamma^\nu - \gamma^\kappa\gamma^\nu\gamma^\mu)(\gamma^\gamma\gamma^\sigma\gamma^\lambda - \gamma^\gamma\gamma^\lambda\gamma^\sigma)] + B \\
&= (A)\mathrm{Tr}[(\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda - \gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\lambda\gamma^\sigma - \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda + \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\lambda\gamma^\sigma)] + B \\
&= (A)\mathrm{Tr}[(\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda - \gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma(-\gamma^\sigma\gamma^\lambda) - \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda + \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma(-\gamma^\sigma\gamma^\lambda))] + B \\
&= (A)\mathrm{Tr}[(\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda + \gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda - \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda - \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda))] + B \\
&= (A)\mathrm{Tr}[2\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda - 2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda] + B \\
&= (A)\mathrm{Tr}[2\gamma^\kappa(-\gamma^\nu\gamma^\mu)\gamma^\gamma\gamma^\sigma\gamma^\lambda - 2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda] + B \\
&= (A)\mathrm{Tr}[-2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda - 2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda] + B \\
&= (A)\mathrm{Tr}[-4\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda] + B \\
&= -\frac{1}{4}(p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[-4\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda] + B \\
&= (p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[\gamma^\kappa\gamma^\gamma] + B \\
&= 4(p_1)_\kappa(p_2)_\gamma g^{\kappa\gamma} + B \\
&= 4p_1p_2 + m^2Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] \\
&= 4p_1p_2 + m^2 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) \qquad (56)
\end{aligned}
$$