Practice with Spinors

I Algebra with Dirac γ matrices

1. $(\overline{u}_1\gamma^\mu u_2)^* \ = ?$

   Note: $(\gamma^0)^\dagger \ = \ \gamma^0$ and $(\gamma^\mu)^\dagger \ = \ \gamma^0\gamma^\mu\gamma^0$

   $(\overline{u}_1\gamma^\mu u_2)$ is a 1×1 matrix. Therefore, its complex conjugate is the same as its Hermitian conjugate, i.e. if we call $L^\mu = (\overline{u}_1\gamma^\mu u_2)$, then $(L^\mu)^* = (L^\mu)^\dagger$. We can then express this quantity as follows:

$$
\begin{aligned}
L^\mu &= \overline{u}_1\gamma^\mu u_2 \ , \\
\Rightarrow (L^\mu)^* &= (L^\mu)^\dagger \ , \\
&= (\overline{u}_1\gamma^\mu u_2)^\dagger \ , \\
&= ((u_1)^\dagger\gamma^0\gamma^\mu u_2)^\dagger \quad \text{using } (A\ldots Z)^\dagger = Z^\dagger\ldots A^\dagger \ , \\
&= (u_2^\dagger)(\gamma^\mu)^\dagger(\gamma^0)^\dagger(u_1) \\
&= (u_2^\dagger)\gamma^0\gamma^\mu\gamma^0\gamma^0(u_1) \\
&= (u_2^\dagger)\gamma^0\gamma^\mu(u_1) \\
&= \overline{u}_2\gamma^\mu(u_1)
\end{aligned}
$$

   Therefore $(\overline{u}_1\gamma^\mu u_2)^* \ = \ \overline{u}_2\gamma^\mu u_1$

2. $(\overline{u}_1\gamma^\mu\gamma^5 u_2)^*$ is also a 1×1 Matrix so the same reasoning applies as above in 1. Note: $(\gamma^5)^\dagger = \gamma^5$

   We define: R^μ as $\overline{u}_1\gamma^\mu\gamma^5 u_2$ thus:

$$
\begin{aligned}
\Rightarrow (R^\mu)^* &= (R^\mu)^\dagger \\
&= (\overline{u}_1\gamma^\mu\gamma^5 u_2)^\dagger \\
&= ((u_1)^\dagger\gamma^0\gamma^\mu\gamma^5 u_2)^\dagger \\
&= (u_2^\dagger)(\gamma^5)^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger(u_1) \\
&= (u_2^\dagger)\gamma^5\gamma^0\gamma^\mu\gamma^0\gamma^0 u_1 \\
&= (u_2^\dagger)\gamma^5\gamma^0\gamma^\mu(1)u_1 \\
&= -(u_2^\dagger)\gamma^0\gamma^5\gamma^\mu u_1 \\
&= -\overline{u}_2\gamma^5\gamma^\mu u_1 \\
&= \overline{u}_2\gamma^\mu\gamma^5 u_1
\end{aligned}
$$

   Therefore $(\overline{u}_1\gamma^\mu\gamma^5 u_2)^* \ = \ \overline{u}_2\gamma^\mu\gamma^5 u_1$

3. $(\overline{u}_1 u_2)^* \ = \ ?$

   We let:$P \ = \ \overline{u}_1 u_2$

$$
\Rightarrow (P)^* \ = \ (P)^\dagger \ ,
$$

$$
\begin{aligned}
&= (\overline{u}_1 u_2)^\dagger \ , \\
&= ((u_1)^\dagger \gamma^0 u_2)^\dagger \\
&= (u_2)^\dagger (\gamma^0)^\dagger (u_1) \\
&= (u_2)^\dagger \gamma^0 (u_1) \\
&= \overline{u}_2 (u_1)
\end{aligned}
$$

Therefore$(\overline{u}_1 u_2)^* \ = \ \overline{u}_2 u_1$

4. By the same reasoning as shown above it can be shown that $(\overline{u}_1 \gamma^5 u_2)^* \ = \ \overline{u}_2 \gamma^5 u_1$
   If we let $T \ = \ \overline{u}_1 \gamma^5 u_2$ then:

$$
\begin{aligned}
\Rightarrow (T)^* &= (T)^\dagger \ , \\
&= (\overline{u}_1 \gamma^5 u_2)^\dagger \ , \\
&= ((u_1)^\dagger \gamma^0 \gamma^5 u_2)^\dagger \\
&= (u_2)^\dagger (\gamma^5)^\dagger (\gamma^0)^\dagger (u_1) \\
&= (u_2)^\dagger (-\gamma^5) \gamma^0 (u_1) \\
&= (u_2)^\dagger \gamma^0 \gamma^5 (u_1) \\
&= \overline{u}_2 \gamma^5 (u_1)
\end{aligned}
$$

   Therefore $(\overline{u}_1 \gamma^5 u_2)^* \ = \ \overline{u}_2 \gamma^5 u_1$

5. While the above identities could be shown to be trivial, the identity: $(\overline{u}_1 \sigma^{\mu\nu} u_2)^* \ = \ \overline{u}_2 \sigma^{\mu\nu} u_1$ is more difficult to solve
   The identity:$(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$ is needed

$$
\begin{aligned}
(\sigma^{\mu\nu})^\dagger &= (\frac{i}{2}[\gamma^\mu, \gamma^\nu])^\dagger \\
&= (u_2)^\dagger (\sigma^{\mu\nu})^\dagger (\gamma^0)^\dagger u_1 \\
&= \frac{i}{2}([\gamma^\mu, \gamma^\nu])^\dagger \\
&= \frac{i}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)^\dagger \\
&= \frac{i}{2}((\gamma^\mu)^\dagger (\gamma^\nu)^\dagger - (\gamma^\nu)^\dagger (\gamma^\mu)^\dagger) \\
&= \frac{i}{2}(\gamma^0 \gamma^\mu \gamma^0 \gamma^0 \gamma^\nu \gamma^0 - \gamma^0 \gamma^\nu \gamma^0 \gamma^0 \gamma^\mu \gamma^0) \\
&= \frac{i}{2}(\gamma^0 \gamma^\mu \gamma^\nu \gamma^0 - \gamma^0 \gamma^\nu \gamma^\mu \gamma^0) \\
&= \frac{i}{2}((-1)^2 \gamma^\mu \gamma^\nu - (-1)^2 \gamma^\nu \gamma^\mu) \\
&= \frac{i}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \\
&= \sigma^{\mu\nu} \qquad (-34)
\end{aligned}
$$

   After showing $(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$ is true it is trivial to show $(\overline{u}_1 \sigma^{\mu\nu} u_2)^* \ = \ \overline{u}_2 \sigma^{\mu\nu} u_1$
   We let $B^{\mu\nu} \ = \ \overline{u}_1 \sigma^{\mu\nu} u_2$

$$
\Rightarrow (B^{\mu\nu})^* \ = \ (B^{\mu\nu})^\dagger \ ,
$$

$$
\begin{aligned}
&= (\overline{u}_1\sigma^{\mu\nu}u_2)^\dagger \ , \\
&= ((u_1)^\dagger\gamma^0\sigma^{\mu\nu}u_2)^\dagger \\
&= (u_2^\dagger)(\sigma^{\mu\nu})^\dagger(\gamma^0)^\dagger(u_1) \\
&= (u_2^\dagger)\sigma^{\mu\nu}\gamma^0(u_1) \\
&= (u_2^\dagger)(\gamma^0)\sigma^{\mu\nu}(u_1) \\
&= \overline{u}_2\sigma^{\mu\nu}(u_1)
\end{aligned}
$$

An interesting thing to note is that the expression $\overline{u}\sigma^{\mu\nu}\gamma^5 u$ is not an independent quantity. Since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ it follows that the product of $\sigma^{\mu\nu}$ and γ^5 can be simplified to an expression with only 2 γ matrices which has been defined as a pusedoscalar. For example, let $\mu = 0$ and $\nu = 1$:

$$
\begin{aligned}
\overline{u}\sigma^{01}\gamma^5 u &= \overline{u}\sigma^{01}(i\gamma^0\gamma^1\gamma^2\gamma^3)u \\
&= \overline{u}((\frac{i}{2})(\gamma^0\gamma^1 - \gamma^1\gamma^0))(i\gamma^0\gamma^1\gamma^2\gamma^3)u \\
&= \overline{u}(\frac{i}{2})[\gamma^0\gamma^1(i\gamma^0\gamma^1\gamma^2\gamma^3) - \gamma^1\gamma^0(i\gamma^0\gamma^1\gamma^2\gamma^3)]u \\
&= \overline{u}(\frac{-1}{2})[\gamma^0\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 - \gamma^1\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3)]u \\
&= \overline{u}(\frac{-1}{2})[-\gamma^2\gamma^3 - \gamma^2\gamma^3)]u \\
&= \overline{u}(\frac{-1}{2})[-2\gamma^2\gamma^3]u \\
&= 2\overline{u}\gamma^2\gamma^3 u
\end{aligned}
$$

This expression (because it contains two gamma matrices) is a puesdoscalar. Any values of μ and ν can be shown to be similar to this because of the communal and indentity properties of the gamma matrices.

II Squaring Expressions

6. $|\ \overline{u}_1\gamma^\mu u_2\ |^2 = (\overline{u}_1\gamma^\mu u_2)(\overline{u}_1\gamma^\mu u_2)^*$

   $|\ \overline{u}_1\gamma^\mu u_2\ |^2 = (\overline{u}_1\gamma^\mu u_2)(\overline{u}_2\gamma^\mu u_1)$

   I know that both of the expressions in the parentheses are 1x1 matrices but I fail to see how it can be simplified anymore.

   Or is this the way to proceed: $|\ \overline{u}_1\gamma^\mu u_2\ |^2 = |\ \overline{u}_1\ |^2|\ \gamma^\mu\ |^2|\ u_2\ |^2$

   $|\ \overline{u}_1\gamma^\mu u_2\ |^2 = (\overline{u}_1)(\overline{u}_1)^*\gamma^\mu(\gamma^\mu)^*(u_2)(u_2)^*$

   $|\ \overline{u}_1\gamma^\mu u_2\ |^2 = (\overline{u}_1)(\overline{u}_1)^*\gamma^\mu\gamma^\mu(u_2)(u_2)^*$

   $|\ \overline{u}_1\gamma^\mu u_2\ |^2 = (\overline{u}_1)\overline{u}_1)^*(u_2)(u_2)^*$