Pauli (σ) and Dirac (γ) matrices

1. **Pauli matrices**: Consider the following 2×2 matrices:

$$\sigma_1 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\;\; \sigma_2 \;=\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;,\;\; \sigma_3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;. \tag{1}$$

   a. Show that these matrices are both Hermitian, and Unitary.

   b. Show that $\sigma_i^2 = 1$ for any $i = 1, 2, 3$ where 1 represents the 2×2 identity matrix. (Note that in general 1 will represent the corresponding N×N identity matrix where N is the dimension of the representation that fits the situation.)

   c. Establish the commutation relationship: $[\sigma_i, \sigma_j] \;=\; 2i\epsilon_{ijk}\sigma_k$.

   d. Establish the anticommutation relationship: $\{\sigma_i, \sigma_j\} \;=\; 2\delta_{ij}$.

   e. Show that $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$.

   f. Find the determinant and the trace of each Pauli matrix.

   g. What is the trace of $\sigma_i\sigma_j$? (Hint: use the cyclic property of matrix traces: $\mathrm{Tr}[AB] = \mathrm{Tr}[BA]$, $\mathrm{Tr}[AB...YZ] = \mathrm{Tr}[ZAB...Y]$, etc., where $A, B, ...$ are matrices.)

   f. What is the trace of $\sigma_i\sigma_j\sigma_k$? Do you notice a patern?

   The symbols δ_{ij} and ϵ_{ijk} are very useful. δ_{ij} is known as the Kronecker delta. If i and j are equal then the delta is 1, or otherwise it is 0, i.e. $\delta_{12} \;=\; 0$, but $\delta_{11} \;=\; 1$. The Levi-Civita symbol (ϵ_{ijk}) is even more interesting. It is non-zero only if all three indices (i, j, k) are unequal, and it is $+1(-1)$ when i, j, k is an even(odd) permutation of $1, 2, 3$, i.e. $\epsilon_{223} \;=\; 0, \epsilon_{123} \;=\; 1, \epsilon_{213} \;=\; -1$.

2. **More fun with Pauli matrices**: The three Pauli matrices together with the 2×2 identity matrix 1 form a basis for all 2×2 matrices. In this problem we want to show this. There are two steps:

   a. Show that the four matrices (1 and σ_i for $i = 1, 2, 3$) are linearly independent. (Hint: You have worked the math out in problem 1. Here you simply have to identify the relevant steps.)

   b. Consider the most general 2×2 matrix:

$$M \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;, \tag{2}$$

   where a, b, c, d are in general four different complex numbers. Show, by explicit construction, that the matrix M can be expressed as a unique linear superposition of the 4 matrices. (Hint: Write $M = A\,1 + B_i \cdot \sigma_i$ and solve for the four coefficients)

3. **Dirac γ matrices**: The equation of motion for a fermion, also known as the Dirac equation, is:

$$(i\gamma^\mu\partial_\mu - m)\psi \;\;=\;\; 0 \;. \tag{3}$$

In four spacetime dimensions (Minkowski spacetime), it turns out that each component of the four vector represented by γ^μ has to be (at least) a 4×4 matrix. It is possible to have different representations of these matrices, but for now we will consider the following representation (known as the Chiral or Weyl basis):

$$\gamma^0 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\quad \gamma^i \;=\; \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \;. \tag{4}$$

A shorthand notation for the four matrices is:

$$\gamma^\mu \;=\; \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \;, \qquad \sigma^\mu \equiv (1, \sigma^i) \;,\quad \overline{\sigma}^\mu \equiv (1, -\sigma^i) \;. \tag{5}$$

(Note that in the above notation 0 and 1 represent 2×2 matrices.) With the above notation in mind show the following properties of γ matrices:

a. What is $(\gamma^\mu)^2$? (Work it out for each value that μ can take.)

b. Show that $(\gamma^\mu)^\dagger \;=\; \gamma^0\gamma^\mu\gamma^0$.

c. Show that $\{\gamma^\mu, \gamma^\nu\} \;=\; 2g^{\mu\nu}$.

d. Define a new object $\sigma^{\mu\nu} \;\equiv\; \dfrac{i}{2}[\gamma^\mu, \gamma^\nu]$. Show that $\gamma^\mu\gamma^\nu \;=\; g^{\mu\nu} + i\sigma^{\nu\mu}$

e. Define another new object $\gamma^5 \;=\; i\gamma^0\gamma^1\gamma^2\gamma^3$. Express this as a simple 4×4 matrix.

f. Find the properties of γ^5 : What is $(\gamma^5)^2$, $\{\gamma^\mu, \gamma^5\}$, and $(\gamma^5)^\dagger$?

g. What is the determinant and the trace of each of the five matrices (γ^μ and γ^5)?

h. What is the trace of $\gamma^\mu\gamma^\nu$? (Hint: Remember the cyclic property of matrix traces.)

i. Consider the operators $P_L \;=\; (1-\gamma^5)/2$, $P_R \;=\; (1+\gamma^5)/2$. Find their properties. (Find P_L^2, P_R^2, P_LP_R. Do you notice anything interesting? Do these (anti)commute with the four gamma matrices? Find the (anti)commutation relationships.)

j.* What is the trace of $\gamma^\mu\gamma^\nu\gamma^5$?

k.* What is the trace of $\gamma^\mu\gamma^\nu\gamma^\lambda$?

l.* What is the trace of $\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma$?

* These problems need more advanced work. Work on these for extra credit!

The answer for (3l.) is as follows:

We will assume the following relationships: $\mathrm{Tr}[ABCD] \;=\; \mathrm{Tr}[DABC]$, $\{\gamma^\mu, \gamma^\nu\} \;=\; 2g^{\mu\nu}$ and $\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right] \;=\; 4g^{\mu\nu}$. We then get:

$$\begin{aligned}
\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\right] &= \mathrm{Tr}\left[(2g^{\mu\nu} - \gamma^\nu\gamma^\mu)\gamma^\lambda\gamma^\sigma\right] \;, \\
&= 2g^{\mu\nu}\mathrm{Tr}\left[\gamma^\lambda\gamma^\sigma\right] - \mathrm{Tr}\left[\gamma^\nu(2g^{\mu\lambda} - \gamma^\lambda\gamma^\mu)\gamma^\sigma\right] \;, \\
&= 8g^{\mu\nu}g^{\lambda\sigma} - 2g^{\mu\lambda}\mathrm{Tr}\left[\gamma^\nu\gamma^\sigma\right] + \mathrm{Tr}\left[\gamma^\nu\gamma^\lambda(2g^{\mu\sigma} - \gamma^\sigma\gamma^\mu)\right] \;, \\
&= 8(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) - \mathrm{Tr}[\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu] \;, \\
&= 8(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) - \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] \;, \qquad (6) \\
\Rightarrow \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\right] &= 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) \;. \qquad (7)
\end{aligned}$$