Practice with Spinors

I Algebra with Dirac γ matrices

Notation:

1. $S = \overline{u}u$
2. $P = \overline{u}\gamma^5 u$
3. $V^\mu = \overline{u}\gamma^\mu u$
4. $A^\mu = \overline{u}\gamma^\mu\gamma^5 u$
5. $T^{\mu\nu} = \overline{u}\sigma^{\mu\nu}u$

1. $(\overline{u}_1\gamma^\mu u_2)^* \ = ?$

   Note: $(\gamma^0)^\dagger \ = \ \gamma^0$ and $(\gamma^\mu)^\dagger \ = \ \gamma^0\gamma^\mu\gamma^0$

   $(\overline{u}_1\gamma^\mu u_2)$ is a 1×1 matrix. Therefore, its complex conjugate is the same as its Hermitian conjugate, i.e. if we call $V^\mu = (\overline{u}_1\gamma^\mu u_2)$, then $(V^\mu)^* = (V^\mu)^\dagger$. We can then express this quantity as follows:

$$
\begin{aligned}
V^\mu &= \overline{u}_1\gamma^\mu u_2 \ , && (1)\\
\Rightarrow (V^\mu)^* &= (V^\mu)^\dagger \ , && (2)\\
&= (\overline{u}_1\gamma^\mu u_2)^\dagger \ , && (3)\\
&= ((u_1)^\dagger\gamma^0\gamma^\mu u_2)^\dagger \quad \text{using } (A\ldots Z)^\dagger = Z^\dagger\ldots A^\dagger \ , && (4)\\
&= (u_2^\dagger)(\gamma^\mu)^\dagger(\gamma^0)^\dagger(u_1) && (5)\\
&= (u_2^\dagger)\gamma^0\gamma^\mu\gamma^0\gamma^0(u_1) && (6)\\
&= (u_2^\dagger)\gamma^0\gamma^\mu(u_1) && (7)\\
&= \overline{u}_2\gamma^\mu(u_1) && (8)
\end{aligned}
$$

   Therefore $(\overline{u}_1\gamma^\mu u_2)^* \ = \ \overline{u}_2\gamma^\mu u_1$. To solve for $|V^\mu|^2$ we simply use $|V^\mu|^2 \ = \ \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1]$.

   Question:
   Why is $|V^\mu|^2 \ \neq \ \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma_\mu u_1]$?
   Why is $|V^\mu|^2 \ = \ \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1]$?

   You start with one index μ. When you square why do you get two indices and not a sum over two of the same index?
   When you square you have two indices because you must increase the number of components. The number of components when you square should go as n^2 not simply n. When you have one index, you restrict the number of components, because you have 4 components and not 16.

Note: $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$, $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$, The trace over the product of an odd number of gamma matrices is zero.

$$\begin{aligned}
|V^\mu|^2 &= \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1] && (9)\\
&= \mathrm{Tr}[\overline{u}_1\gamma^\mu(\not{p}_2 + m)\gamma^\nu u_1] && (10)\\
&= \mathrm{Tr}[u_1\overline{u}_1\gamma^\mu(\not{p}_2 + m)\gamma^\nu] && (11)\\
&= \mathrm{Tr}[(\not{p}_1 + m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu] && (12)\\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] + m[\mathrm{Tr}(\gamma^\mu\not{p}_1\gamma^\nu) + \mathrm{Tr}(\gamma^\mu\gamma^\nu\not{p}_2)] + m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] && (13)\\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] + m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] && (14)\\
&= \mathrm{Tr}[(p_1)_\lambda\gamma^\lambda\gamma^\mu(p_2)_\sigma\gamma^\sigma\gamma^\nu] + 4m^2g^{\mu\nu} && (15)\\
&= (p_1)_\lambda(p_2)_\sigma\mathrm{Tr}[\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu] + 4m^2g^{\mu\nu} && (16)\\
&= (p_1)_\lambda(p_2)_\sigma 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) + 4m^2g^{\mu\nu} && (17)\\
&= 4[p_1^\mu p_2^\nu - g^{\mu\nu}(p_1\cdot p_2) + p_2^\mu p_1^\nu] + 4m^2g^{\mu\nu} && (18)
\end{aligned}$$

2. $(\overline{u}_1\gamma^\mu\gamma^5u_2)^*$ is also a 1×1 Matrix so the same reasoning applies as above in 1. Note: $(\gamma^5)^\dagger = \gamma^5$ We define: A^μ as $\overline{u}_1\gamma^\mu\gamma^5u_2$ thus:

$$\begin{aligned}
(A^\mu)^* &= (A^\mu)^\dagger && (19)\\
&= (\overline{u}_1\gamma^\mu\gamma^5u_2)^\dagger && (20)\\
&= ((u_1)^\dagger\gamma^0\gamma^\mu\gamma^5u_2)^\dagger && (21)\\
&= (u_2^\dagger)(\gamma^5)^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger(u_1) && (22)\\
&= (u_2^\dagger)\gamma^5\gamma^0\gamma^\mu\gamma^0\gamma^0u_1 && (23)\\
&= (u_2^\dagger)\gamma^5\gamma^0\gamma^\mu(1)u_1 && (24)\\
&= -(u_2^\dagger)\gamma^0\gamma^5\gamma^\mu u_1 && (25)\\
&= -\overline{u}_2\gamma^5\gamma^\mu u_1 && (26)\\
&= \overline{u}_2\gamma^\mu\gamma^5u_1 && (27)
\end{aligned}$$

Therefore $(\overline{u}_1\gamma^\mu\gamma^5u_2)^* = \overline{u}_2\gamma^\mu\gamma^5u_1$

We also are able to calculate $|A^\mu|^2$

$$\begin{aligned}
|A^\mu|^2 &= \mathrm{Tr}[(\overline{u}_1\gamma^\mu\gamma^5u_2)(\overline{u}_2\gamma^\nu\gamma^5u_1)] && (28)\\
&= \mathrm{Tr}[\overline{u}_1\gamma^\mu\gamma^5(\not{p}_2 + m)\gamma^\nu\gamma^5u_1] && (29)\\
&= \mathrm{Tr}[u_1\overline{u}_1\gamma^\mu\gamma^5(\not{p}_2 + m)\gamma^\nu\gamma^5] && (30)\\
&= \mathrm{Tr}[(\not{p}_1 + m)\gamma^\mu\gamma^5(\not{p}_2 + m)\gamma^\nu\gamma^5] && (31)\\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5 + m(\not{p}_1\gamma^\mu\gamma^5\gamma^\nu\gamma^5 + \gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5) + m^2(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)] && (32)\\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5 + m(\not{p}_1\gamma^\mu(-\gamma^5\gamma^5)\gamma^\nu + \not{p}_2\gamma^\mu(-\gamma^5\gamma^5)\gamma^\nu) + m^2(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)]\\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5 - m(\not{p}_1\gamma^\mu\gamma^\nu + \not{p}_2\gamma^\mu\gamma^\nu) + m^2(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)] && (33)\\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5] - mTr[\not{p}_1\gamma^\mu\gamma^\nu] - mTr[\not{p}_2\gamma^\mu\gamma^\nu] + m^2(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)] && (34)
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5 + m^2(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)] && (35)\\
&= \mathrm{Tr}[(p_1)_\lambda\gamma^\lambda\gamma^\mu\gamma^5(p_2)_\sigma\gamma^\sigma\gamma^\nu\gamma^5] + m^2\mathrm{Tr}[\gamma^\mu\gamma^5\gamma^\nu\gamma^5] && (36)\\
&= (p_1)_\lambda(p_2)_\sigma\mathrm{Tr}[\gamma^\lambda\gamma^\mu\gamma^5\gamma^\sigma\gamma^\nu\gamma^5] - m^2\mathrm{Tr}[\gamma^\mu\gamma^5\gamma^5\gamma^\nu] && (37)\\
&= (p_1)_\lambda(p_2)_\sigma\mathrm{Tr}[\gamma^\lambda\gamma^\mu\gamma^5\gamma^5\gamma^\sigma\gamma^\nu] - m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] && (38)\\
&= (p_1)_\lambda(p_2)_\sigma\mathrm{Tr}[\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu] - m^2(g^{\mu\nu}) && (39)\\
&= (p_1)_\lambda(p_2)_\sigma 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) - 4m^2g^{\mu\nu} && (40)\\
&= 4[p_1^\mu p_2^\nu - g^{\mu\nu}(p_1\cdot p_2) + p_2^\mu p_1^\nu] - 4m^2g^{\mu\nu} && (41)
\end{aligned}
$$

3. $(\overline{u}_1u_2)^* \;=\; ?$ We let $S \;=\; \overline{u}_1u_2$

$$
\begin{aligned}
(S)^* &= (S)^\dagger && (42)\\
&= (\overline{u}_1u_2)^\dagger\ , && (43)\\
&= ((u_1)^\dagger\gamma^0u_2)^\dagger && (44)\\
&= (u_2)^\dagger(\gamma^0)^\dagger(u_1) && (45)\\
&= (u_2)^\dagger\gamma^0(u_1) && (46)\\
&= \overline{u}_2(u_1) && (47)
\end{aligned}
$$

Therefore$(\overline{u}_1u_2)^* \;=\; \overline{u}_2u_1$. In order to find $|S|^2$ we simply do the following:

$$
\begin{aligned}
|S|^2 &= \mathrm{Tr}[\overline{u}_1u_2\overline{u}_2u_1] && (48)\\
&= \mathrm{Tr}[(\not{p}_1+m)(\not{p}_2+m)] && (49)\\
&= \mathrm{Tr}[\not{p}_1\not{p}_2 + m(\not{p}_1+\not{p}_2) + m^2] && (50)\\
&= \mathrm{Tr}[\not{p}_1\not{p}_2] + \mathrm{Tr}[m(\not{p}_1+\not{p}_2)] + \mathrm{Tr}[m^2] && (51)\\
&= \mathrm{Tr}[\not{p}_1\not{p}_2] + m(\mathrm{Tr}[\not{p}_1] + \mathrm{Tr}[\not{p}_2]) + m^2\mathrm{Tr}[1] && (52)\\
&= \mathrm{Tr}[\not{p}_1\not{p}_2] + 4m^2 && (53)\\
&= 4(p_1\cdot p_2) + 4m^2 && (54)
\end{aligned}
$$

4. By the same reasoning as shown above it can be shown that $(\overline{u}_1\gamma^5u_2)^* \;=\; \overline{u}_2\gamma^5u_1$
If we let $P \;=\; \overline{u}_1\gamma^5u_2$ then:

$$
\begin{aligned}
(P)^* &= (P)^\dagger && (55)\\
&= (\overline{u}_1\gamma^5u_2)^\dagger\ , && (56)\\
&= ((u_1)^\dagger\gamma^0\gamma^5u_2)^\dagger && (57)\\
&= (u_2)^\dagger(\gamma^5)^\dagger(\gamma^0)^\dagger(u_1) && (58)\\
&= (u_2)^\dagger(\gamma^5)\gamma^0(u_1) && (59)\\
&= -(u_2)^\dagger\gamma^0\gamma^5(u_1) && (60)\\
&= -\overline{u}_2\gamma^5(u_1) && (61)
\end{aligned}
$$

Therefore $(\overline{u}_1\gamma^5u_2)^* \;=\; -\,\overline{u}_2\gamma^5u_1$

In order to square P we do the following:

$$|P|^2 \;=\; \mathrm{Tr}[\overline{u}_1\gamma^5u_2(-\overline{u}_2\gamma^5u_1)] \qquad (62)$$

$$
\begin{aligned}
&= \mathrm{Tr}[u_1\overline{u}_1\gamma^5(-\not{p}_2 - m)\gamma^5] && (63)\\
&= \mathrm{Tr}[(\not{p}_1 + m)\gamma^5(-\not{p}_2 - m)\gamma^5] && (64)\\
&= \mathrm{Tr}[((p_1)_\mu\gamma^\mu + m)\gamma^5((-p_2)_\nu\gamma^\nu - m)\gamma^5] && (65)\\
&= \mathrm{Tr}[((p_1)_\mu\gamma^\mu\gamma^5 + m\gamma^5)((-p_2)_\nu\gamma^\nu\gamma^5 - m\gamma^5)] && (66)\\
&= \mathrm{Tr}[(p_1)_\mu\gamma^\mu\gamma^5(-p_2)_\nu\gamma^\nu\gamma^5 + m\gamma^5(-p_2)_\nu\gamma^\nu\gamma^5 - m\gamma^5(p_1)_\mu\gamma^\mu\gamma^5 - \gamma^5\gamma^5m^2)]\\
&= \mathrm{Tr}[(p_1)_\mu\gamma^\mu\gamma^5(-p_2)_\nu\gamma^\nu\gamma^5] + Tr[m\gamma^5(-\gamma^5)(-p_2)_\nu\gamma^\nu] - Tr[m\gamma^5(-\gamma^5)(p_1)_\mu\gamma^\mu] - Tr[m^2]\\
&= \mathrm{Tr}[(p_1)_\mu\gamma^\mu\gamma^5(-p_2)_\nu\gamma^\nu\gamma^5] - Tr[m(-p_2)_\nu\gamma^\nu] + Tr[m(p_1)_\mu\gamma^\mu] - 4m^2 && (67)\\
&= \mathrm{Tr}[(p_1)_\mu\gamma^\mu\gamma^5(-p_2)_\nu\gamma^\nu\gamma^5] - 4m^2 && (68)\\
&= (p_1)_\mu(-p_2)_\nu\mathrm{Tr}[\gamma^\mu\gamma^5\gamma^\nu\gamma^5] - 4m^2 && (69)\\
&= (p_1)_\mu(-p_2)_\nu\mathrm{Tr}[\gamma^\mu\gamma^5(-\gamma^5\gamma^\nu)] - 4m^2 && (70)\\
&= (p_1)_\mu(-p_2)_\nu(-\mathrm{Tr}[\gamma^\mu\gamma^\nu]) - 4m^2 && (71)\\
&= (p_1)_\mu(-p_2)_\nu(-4g^{\mu\nu}) - 4m^2 && (72)\\
&= 4(p_1)(p_2) - 4m^2 && (73)
\end{aligned}
$$

5. While the above identities could be shown to be trivial, the identity: $(\overline{u}_1\sigma^{\mu\nu}u_2)^* = \overline{u}_2\sigma^{\nu\mu}u_1$ is more difficult to solve. The identity:$(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$ is needed

$$
\begin{aligned}
(\sigma^{\mu\nu})^\dagger &= (\frac{i}{2}[\gamma^\mu,\gamma^\nu])^\dagger && (74)\\
&= -\frac{i}{2}([\gamma^\mu,\gamma^\nu])^\dagger && (75)\\
&= -\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)^\dagger && (76)\\
&= -\frac{i}{2}((\gamma^\nu)^\dagger(\gamma^\mu)^\dagger - (\gamma^\mu)^\dagger(\gamma^\nu)^\dagger) && (77)\\
&= -\frac{i}{2}(\gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^\mu\gamma^0 - \gamma^0\gamma^\mu\gamma^0\gamma^0\gamma^\nu\gamma^0) && (78)\\
&= -\frac{i}{2}(\gamma^0\gamma^\nu\gamma^\mu\gamma^0 - \gamma^0\gamma^\mu\gamma^\nu\gamma^0) && (79)\\
&= -\frac{i}{2}((-1)^2\gamma^\nu\gamma^\mu - (-1)^2\gamma^\mu\gamma^\nu) && (80)\\
&= -\frac{i}{2}(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu) && (81)\\
&= -\sigma^{\nu\mu} && (82)
\end{aligned}
$$

After showing $(\sigma^{\mu\nu})^\dagger = \sigma^{\nu\mu}$ is true it is trivial to show $(\overline{u}_1\sigma^{\mu\nu}u_2)^* = -\overline{u}_2\sigma^{\nu\mu}u_1$. We let $T^{\mu\nu} = \overline{u}_1\sigma^{\mu\nu}u_2$

$$
\begin{aligned}
(T^{\mu\nu})^* &= (T^{\mu\nu})^\dagger\ , && (83)\\
&= (\overline{u}_1\sigma^{\mu\nu}u_2)^\dagger && (84)\\
&= ((u_1)^\dagger\gamma^0\sigma^{\mu\nu}u_2)^\dagger && (85)\\
&= (u_2^\dagger)(\sigma^{\mu\nu})^\dagger(\gamma^0)^\dagger(u_1) && (86)\\
&= (u_2^\dagger)(-\sigma^{\nu\mu})\gamma^0(u_1) && (87)
\end{aligned}
$$

$$
\begin{aligned}
&= (u_2^\dagger)(-\gamma^0)(-\sigma^{\nu\mu})(u_1) && (88)\\
&= \overline{u}_2\sigma^{\mu\nu}(u_1) && (89)
\end{aligned}
$$

In order to find the value of $|T^{\mu\nu}|^2$ one needs to find the value of $Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}]$

$$
\begin{aligned}
Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] &= Tr[\frac{i}{2}(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma)\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)] && (90)\\
&= Tr[\frac{i}{2}(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma)\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)] && (91)\\
&= -\frac{1}{4}Tr[(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma)(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)] && (92)\\
&= -\frac{1}{4}Tr[\gamma^\sigma\gamma^\lambda\gamma^\mu\gamma^\nu] + \frac{1}{4}Tr[\gamma^\sigma\gamma^\lambda\gamma^\nu\gamma^\mu] + \frac{1}{4}Tr[\gamma^\lambda\gamma^\sigma\gamma^\mu\gamma^\nu] - \frac{1}{4}Tr[\gamma^\lambda\gamma^\sigma\gamma^\nu\gamma^\mu]
\end{aligned}
$$

Here we must label each of the traces individually:

$$
\begin{aligned}
A &= -\frac{1}{4}Tr[\gamma^\sigma\gamma^\lambda\gamma^\mu\gamma^\nu] && (93)\\
&= -(g^{\sigma\lambda}g^{\mu\nu} - g^{\sigma\mu}g^{\lambda\nu} + g^{\sigma\nu}g^{\lambda\mu}) && (94)
\end{aligned}
$$

$$
\begin{aligned}
B &= +\frac{1}{4}Tr[\gamma^\sigma\gamma^\lambda\gamma^\nu\gamma^\mu] && (95)\\
&= +(g^{\sigma\lambda}g^{\nu\mu} - g^{\sigma\nu}g^{\lambda\mu} + g^{\sigma\mu}g^{\lambda\nu}) && (96)
\end{aligned}
$$

$$
\begin{aligned}
C &= \frac{1}{4}Tr[\gamma^\lambda\gamma^\sigma\gamma^\mu\gamma^\nu] && (97)\\
&= (g^{\lambda\sigma}g^{\mu\nu} - g^{\lambda\mu}g^{\sigma\nu} + g^{\lambda\nu}g^{\sigma\mu}) && (98)
\end{aligned}
$$

$$
\begin{aligned}
D &= -\frac{1}{4}Tr[\gamma^\lambda\gamma^\sigma\gamma^\nu\gamma^\mu] && (99)\\
&= -(g^{\lambda\sigma}g^{\nu\mu} - g^{\lambda\nu}g^{\sigma\mu} + g^{\lambda\mu}g^{\sigma\nu}) && (100)
\end{aligned}
$$

$$
\begin{aligned}
Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] &= A + B + +C + D && (101)\\
&= 2g^{\sigma\mu}g^{\lambda\nu} - 2g^{\sigma\nu}g^{\lambda\mu} - 2g^{\lambda\mu}g^{\sigma\nu} + 2g^{\lambda\nu}g^{\sigma\mu}) && (102)
\end{aligned}
$$

In order to find the value of $|T^{\mu\nu}|^2$ one needs to do the following:

$$
\begin{aligned}
|T^{\mu\nu}|^2 &= \mathrm{Tr}[\overline{u}_1\sigma^{\mu\nu}u_2\overline{u}_2\sigma^{\sigma\lambda}u_1] && (103)\\
&= \mathrm{Tr}[\overline{u}_1\sigma^{\mu\nu}(\not{p}_2 + m)\sigma^{\sigma\lambda}u_1] && (104)\\
&= \mathrm{Tr}[(\not{p}_1 + m)\sigma^{\mu\nu}(\not{p}_2 + m)\sigma^{\sigma\lambda}] && (105)\\
&= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa + m)\sigma^{\mu\nu}((p_2)_\gamma\gamma^\gamma + m)\sigma^{\sigma\lambda}] && (106)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu} + m\sigma^{\mu\nu})((p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda} + m\sigma^{\sigma\lambda})] && (107)\\
&= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda} + (p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}m\sigma^{\sigma\lambda} + (p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}m\sigma^{\mu\nu} + m\sigma^{\sigma\lambda}m\sigma^{\mu\nu})] && (108)\\
&= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}] + Tr[(p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}m\sigma^{\sigma\lambda} + (p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}m\sigma^{\mu\nu}] + Tr[m\sigma^{\sigma\lambda}m\sigma^{\mu\nu}] \\
&= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}] + Tr[m\sigma^{\sigma\lambda}m\sigma^{\mu\nu}] && (108)\\
&= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}] + m^2Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] \text{ Let B } = \mathrm{m}^2\mathrm{Tr}[\sigma^{\sigma\lambda}\sigma^{\mu\nu}]\\
&= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}] + B\\
&= (p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[\gamma^\kappa\sigma^{\mu\nu}\gamma^\gamma\sigma^{\sigma\lambda}] + B\\
&= (p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[\gamma^\kappa(\frac{i}{2}((\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma^\gamma(\frac{i}{2}(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma))] + B && (106)\\
&= -\frac{1}{4}(p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[\gamma^\kappa(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma^\gamma(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma)] + B \text{ (Let A } = -\frac{1}{4}(\mathrm{p}_1)_\kappa(\mathrm{p}_2)_\gamma)\\
&= (A)\mathrm{Tr}[(\gamma^\kappa\gamma^\mu\gamma^\nu - \gamma^\kappa\gamma^\nu\gamma^\mu)(\gamma^\gamma\gamma^\sigma\gamma^\lambda - \gamma^\gamma\gamma^\lambda\gamma^\sigma)] + B\\
&= (A)\mathrm{Tr}[(\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda - \gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\lambda\gamma^\sigma - \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda + \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\lambda\gamma^\sigma)] + B\\
&= (A)\mathrm{Tr}[(\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda - \gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma(-\gamma^\sigma\gamma^\lambda) - \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda + \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma(-\gamma^\sigma\gamma^\lambda))] + B\\
&= (A)\mathrm{Tr}[(\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda + \gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda - \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda - \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda))] + B\\
&= (A)\mathrm{Tr}[2\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda - 2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda] + B && (102)\\
&= (A)\mathrm{Tr}[2\gamma^\kappa(-\gamma^\nu\gamma^\mu)\gamma^\gamma\gamma^\sigma\gamma^\lambda - 2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda] + B && (103)\\
&= (A)\mathrm{Tr}[-2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda - 2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda] + B && (104)\\
&= (A)\mathrm{Tr}[-4\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda] + B && (105)\\
&= -\frac{1}{4}(p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[-4\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda] + B && (106)\\
&= (p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[\gamma^\kappa\gamma^\gamma] + B \text{ (Using a similar identity as shown in eq. 91)} && (107)\\
&= 4(p_1)_\kappa(p_2)_\gamma g^{\kappa\gamma} + B && (108)\\
&= 4p_1p_2 + m^2Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] && (109)\\
&= 4p_1p_2 + m^24(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) && (110)
\end{aligned}
$$

Branching Ratios

II The branching Ratio of a Pion

In order to calculate the Branching ratio of a charged pion, one must be familiar with the Feynman Rules of calculating amplitudes and the trace identities.

We must start with recognizing that the pion decay is a charged, weak, interaction which arises from the fact that a pion is made of quarks, and the decay is mediated by a massive W boson.

A diagram of the decay may been seen below:

Picture courtesy of Qora.com

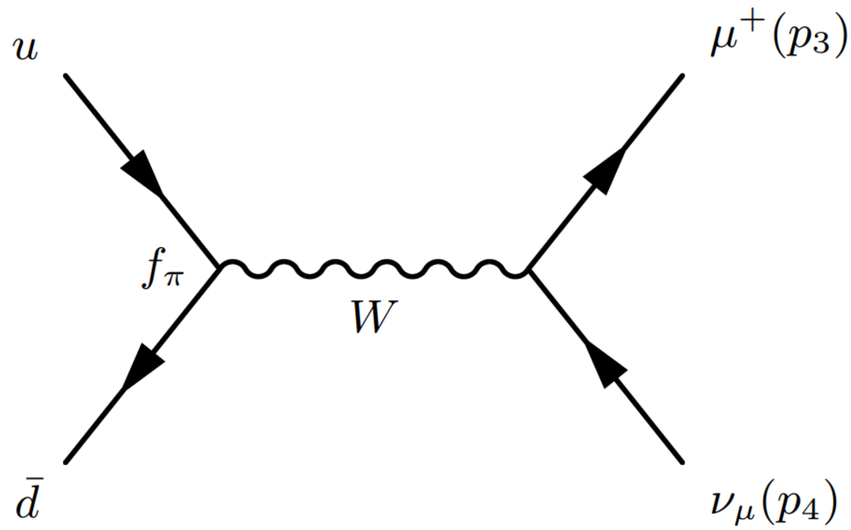

Figure 1: Pion Decay

Where the up and ani-down quark are the pion and the muon and muon neutrino are on the right. However, the muon and the muon neutrino could be the electron and electron neutrino.

The formula given to describe the branching ratio of the decay is given by [1]:

$$\Gamma \;=\; \frac{S|\mathbf{p}|}{8\pi\hbar m_1^2 c}|\mathcal{M}|^2 \tag{111}$$

Where $|\mathbf{p}|$ is the outgoing momentum, S is the product of statistical factors (in our case it will be equal to 1), and $\mathcal{M}$ is the Feynman amplitude. In our notation, we will use the Natural Units so our expression becomes:

$$\Gamma \;=\; \frac{S|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2 \tag{112}$$

The next step is to determine the Feynman Amplitude which can be done in a few steps In the charged weak lepton decays, we have different notation for the vertices and propagators.

(a) For each vertex add a factor of $\frac{-ig_w}{2\sqrt{2}}(\gamma^\nu(1-\gamma^5))$ where $g_w = \sqrt{4\pi\alpha_w}$

(b) For each propagator we add a factor of $\frac{-ig_{\mu\nu}-\frac{q_\mu q_\nu}{m^2}}{q^2-m^2}$ where m is the mass of the boson. In our case, $m_w \gg\; q$ so the expression simplifies to $\frac{ig_{\mu\nu}}{m_w^2}$

From these rules and Figure 1 we are able to calculate the value of $\mathcal{M}$

$$-i\mathcal{M} \;=\; [\overline{u}(3)(\frac{-ig_w}{2\sqrt{2}}(\gamma^\nu(1-\gamma^5))v(2)][\frac{ig_{\mu\nu}}{m_w^2}][\frac{-ig_w}{2\sqrt{2}}F^\mu] \tag{113}$$

We have a factor of -i with the $\mathcal{M}$ so that we obtain the real part of the expression and F^μ is the form factor of the coupling of the pion to the W boson. F^μ has the form of $f_\pi p^\mu$

$$-i\mathcal{M} = \frac{-ig_w^2}{8m_w^2}[\overline{u}(3)(\gamma^\mu(1-\gamma^5))v(2)]F^\mu \quad (114)$$

$$\mathcal{M} = \frac{g_w^2}{8m_w^2}[\overline{u}(3)(\gamma^\mu(1-\gamma^5))v(2)]F^\mu \quad (115)$$

In order to square the amplitude we do the following:

$$\langle|\mathcal{M}^2|\rangle = (\frac{g_w^2}{8m_w^2}f_\pi)^2 Tr[(\overline{u}(3)(\gamma^\mu(1-\gamma^5))v(2)p_\mu(\overline{v}(2)(\gamma^\nu(1-\gamma^5)u(3))]p_\nu \quad (116)$$

$$\langle|\mathcal{M}^2|\rangle = (\frac{g_w^2}{8m_w^2}f_\pi)^2 p_\mu p_\nu Tr[(u(3)\overline{u}(3)(\gamma^\mu(1-\gamma^5))v(2)\overline{v}(2)(\gamma^\nu(1-\gamma^5))] \quad (117)$$

$$\langle|\mathcal{M}^2|\rangle = (\frac{g_w^2}{8m_w^2}f_\pi)^2 p_\mu p_\nu Tr[((\not{p}_3+m_l)(\gamma^\mu(1-\gamma^5))\not{p}_2(\gamma^\nu(1-\gamma^5))] \quad (118)$$

$$\langle|\mathcal{M}^2|\rangle = (\frac{g_w^2}{8m_w^2}f_\pi)^2 p_\mu p_\nu (2Tr[\not{p}_3\gamma^\mu\not{p}_2\gamma^\nu] - 2Tr[\not{p}_3\gamma^\mu\not{p}_2\gamma^\nu\gamma^5]) \quad (119)$$

$$\langle|\mathcal{M}^2|\rangle = (\frac{g_w^2}{8m_w^2}f_\pi)^2 p_\mu p_\nu (8[p_3^\mu p_2^\nu + p_2^\mu p_3^\nu - (p_3\cdot p_2)g^{\mu\nu}] + 8i\epsilon^{\mu\lambda\nu\sigma}p_{3\lambda}p_{2\sigma} \quad (120)$$

Summing over the spins gives us:

$$\langle|\mathcal{M}^2|\rangle = (\frac{g_w^2}{8m_w^2}f_\pi)^2[2(p_1\cdot p_2)(p_1\cdot p_3) - p^2(p_2\cdot p_3)] \quad (121)$$

Since $p = p_2 + p_3$, we can simplify the equation further

$$\langle|\mathcal{M}^2|\rangle = (\frac{g_w^2}{2m_w^2}f_\pi)^2[m_l^2(m_\pi^2 - m_l^2)] \quad (122)$$

In this way we are able to calculate the branching ratio of a pion, since we know have the Feyneman Amplitude we simply return to equation (2)

$$\Gamma = \frac{f_\pi^2}{\pi m_\pi^3}(\frac{g_w}{4m_w})^4 m_l^2(m_\pi^2 - m_l^2)^2 \quad (123)$$

Expanding on this idea, we are able to graph $\frac{\Gamma_l}{\Gamma_\pi}$
Since the constants at the beginning of the equation cancel, we are able to show that:

$$\frac{\Gamma_l}{\Gamma_\pi} = \frac{m_l^2(m_\pi^2 - m_l^2)^2}{m_e^2(m_\pi^2 - m_e^2)^2 + m_e^2(m_\pi^2 - m_e^2)^2} \quad (124)$$

Using the equation:

$$\Gamma_\pi = \frac{f_\pi^2}{\pi m_\pi^3}(\frac{g_w}{4m_w})^4 m_e^2(m_\pi^2 - m_e^2)^2 + m_e^2(m_\pi^2 - m_e^2)^2 \quad (124)$$

Thus the graph of the equation is: Using the values from PDG [2] we can do calculations with the branching ratios.

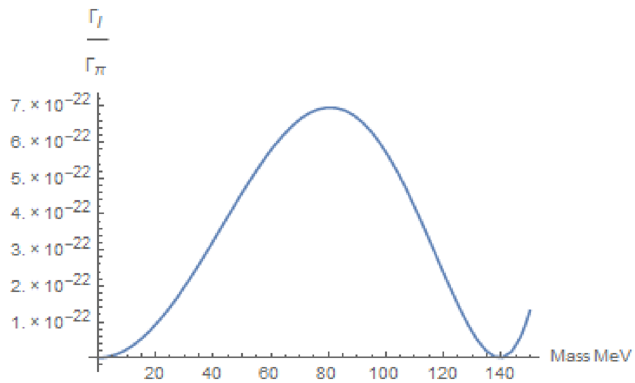

Figure 2: $\frac{\Gamma_l}{\Gamma_\pi}$

Observables	e	μ	π
τ	6.6×10^{28} yr	$2.1969811(22) \times 10^{-}6$ s	$2.6033(5) \times 10^{-}8$ s
Mass(MeV)	0.5109989461(31)	105.6583745(24)	139.57061(24)

If we would like to calculate the ratio of the $\pi^- \to e^- + \overline{\nu}_e$ and $\pi^- \to \mu^- + \overline{\nu}_\mu$ we simply do the following [2]:

$$\frac{\Gamma_e}{\Gamma_\mu} = \frac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2} \tag{125}$$

$$\frac{\Gamma_e}{\Gamma_\mu} = 1.28334(73) \times 10^{-4} \tag{126}$$

References

[1] Griffiths, D. (2014). Introduction to elementary particles. Weinheim: Wiley-VCH Verlag

[2] K. A. Olive *et al.* [Particle Data Group], Chin. Phys. C **38**, 090001 (2014). doi:10.1088/1674-1137/38/9/090001