

Practice with Spinors

I Algebra with Dirac γ matrices

1. $(\bar{u}_1 \gamma^\mu u_2)^* = ?$

Note: $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$

$(\bar{u}_1 \gamma^\mu u_2)$ is a 1×1 matrix. Therefore, its complex conjugate is the same as its Hermitian conjugate, i.e. if we call $L^\mu = (\bar{u}_1 \gamma^\mu u_2)$, then $(L^\mu)^* = (L^\mu)^\dagger$. We can then express this quantity as follows:

$$\begin{aligned}
 L^\mu &= \bar{u}_1 \gamma^\mu u_2, \\
 \Rightarrow (L^\mu)^* &= (L^\mu)^\dagger, \\
 &= (\bar{u}_1 \gamma^\mu u_2)^\dagger, \\
 &= ((u_1)^\dagger \gamma^0 \gamma^\mu u_2)^\dagger \quad \text{using } (A \dots Z)^\dagger = Z^\dagger \dots A^\dagger, \\
 &= (u_2^\dagger) (\gamma^\mu)^\dagger (\gamma^0)^\dagger (u_1) \\
 &= (u_2^\dagger) \gamma^0 \gamma^\mu \gamma^0 \gamma^0 (u_1) \\
 &= (u_2^\dagger) \gamma^0 \gamma^\mu (u_1) \\
 &= \bar{u}_2 \gamma^\mu (u_1)
 \end{aligned}$$

Therefore $(\bar{u}_1 \gamma^\mu u_2)^* = \bar{u}_2 \gamma^\mu u_1$

2. $(\bar{u}_1 \gamma^\mu \gamma^5 u_2)^*$ is also a 1×1 Matrix so the same reasoning applies as above in 1. Note: $(\gamma^5)^\dagger = \gamma^5$

We define: R^μ as $\bar{u}_1 \gamma^\mu \gamma^5 u_2$ thus:

$$\begin{aligned}
 \Rightarrow (R^\mu)^* &= (R^\mu)^\dagger \\
 &= (\bar{u}_1 \gamma^\mu \gamma^5 u_2)^\dagger \\
 &= ((u_1)^\dagger \gamma^0 \gamma^\mu \gamma^5 u_2)^\dagger \\
 &= (u_2^\dagger) (\gamma^5)^\dagger (\gamma^\mu)^\dagger (\gamma^0)^\dagger (u_1) \\
 &= (u_2^\dagger) \gamma^5 \gamma^0 \gamma^\mu \gamma^0 \gamma^0 u_1 \\
 &= (u_2^\dagger) \gamma^5 \gamma^0 \gamma^\mu (1) u_1 \\
 &= -(u_2^\dagger) \gamma^0 \gamma^5 \gamma^\mu u_1 \\
 &= -\bar{u}_2 \gamma^5 \gamma^\mu u_1 \\
 &= \bar{u}_2 \gamma^\mu \gamma^5 u_1
 \end{aligned}$$

Therefore $(\bar{u}_1 \gamma^\mu \gamma^5 u_2)^* = \bar{u}_2 \gamma^\mu \gamma^5 u_1$

3. $(\bar{u}_1 u_2)^* = ?$

We let: $P = \bar{u}_1 u_2$

$$\Rightarrow (P)^* = (P)^\dagger,$$

$$\begin{aligned}
&= (\bar{u}_1 u_2)^\dagger, \\
&= ((u_1)^\dagger \gamma^0 u_2)^\dagger \\
&= (u_2)^\dagger (\gamma^0)^\dagger (u_1) \\
&= (u_2)^\dagger \gamma^0 (u_1) \\
&= \bar{u}_2 (u_1)
\end{aligned}$$

Therefore $(\bar{u}_1 u_2)^* = \bar{u}_2 u_1$

4. By the same reasoning as shown above it can be shown that $(\bar{u}_1 \gamma^5 u_2)^* = \bar{u}_2 \gamma^5 u_1$
 If we let $T = \bar{u}_1 \gamma^5 u_2$ then:

$$\begin{aligned}
\Rightarrow (T)^* &= (T)^\dagger, \\
&= (\bar{u}_1 \gamma^5 u_2)^\dagger, \\
&= ((u_1)^\dagger \gamma^0 \gamma^5 u_2)^\dagger \\
&= (u_2)^\dagger (\gamma^5)^\dagger (\gamma^0)^\dagger (u_1) \\
&= (u_2)^\dagger (-\gamma^5) \gamma^0 (u_1) \\
&= (u_2)^\dagger \gamma^0 \gamma^5 (u_1) \\
&= \bar{u}_2 \gamma^5 (u_1)
\end{aligned}$$

Therefore $(\bar{u}_1 \gamma^5 u_2)^* = \bar{u}_2 \gamma^5 u_1$

5. While the above identities could be shown to be trivial, the identity: $(\bar{u}_1 \sigma^{\mu\nu} u_2)^* = \bar{u}_2 \sigma^{\mu\nu} u_1$ is more difficult to solve

The identity: $(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$ is needed

$$\begin{aligned}
(\sigma^{\mu\nu})^\dagger &= \left(\frac{i}{2}[\gamma^\mu, \gamma^\nu]\right)^\dagger \\
&= (u_2)^\dagger (\sigma^{\mu\nu})^\dagger (\gamma^0)^\dagger u_1 \\
&= \frac{i}{2}([\gamma^\mu, \gamma^\nu])^\dagger \\
&= \frac{i}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)^\dagger \\
&= \frac{i}{2}((\gamma^\mu)^\dagger (\gamma^\nu)^\dagger - (\gamma^\nu)^\dagger (\gamma^\mu)^\dagger) \\
&= \frac{i}{2}(\gamma^0 \gamma^\mu \gamma^0 \gamma^0 \gamma^\nu \gamma^0 - \gamma^0 \gamma^\nu \gamma^0 \gamma^0 \gamma^\mu \gamma^0) \\
&= \frac{i}{2}(\gamma^0 \gamma^\mu \gamma^\nu \gamma^0 - \gamma^0 \gamma^\nu \gamma^\mu \gamma^0) \\
&= \frac{i}{2}((-1)^2 \gamma^\mu \gamma^\nu - (-1)^2 \gamma^\nu \gamma^\mu) \\
&= \frac{i}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \\
&= \sigma^{\mu\nu}
\end{aligned} \tag{-34}$$

After showing $(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$ is true it is trivial to show $(\bar{u}_1 \sigma^{\mu\nu} u_2)^* = \bar{u}_2 \sigma^{\mu\nu} u_1$

We let $B^{\mu\nu} = \bar{u}_1 \sigma^{\mu\nu} u_2$

$$\Rightarrow (B^{\mu\nu})^* = (B^{\mu\nu})^\dagger,$$

$$\begin{aligned}
&= (\bar{u}_1 \sigma^{\mu\nu} u_2)^\dagger, \\
&= ((u_1)^\dagger \gamma^0 \sigma^{\mu\nu} u_2)^\dagger \\
&= (u_2^\dagger (\sigma^{\mu\nu})^\dagger (\gamma^0)^\dagger (u_1)) \\
&= (u_2^\dagger) \sigma^{\mu\nu} \gamma^0 (u_1) \\
&= (u_2^\dagger) (\gamma^0) \sigma^{\mu\nu} (u_1) \\
&= \bar{u}_2 \sigma^{\mu\nu} (u_1)
\end{aligned}$$

II Squaring Expressions

$$\begin{aligned}
6. \quad |\bar{u}_1 \gamma^\mu u_2|^2 &= (\bar{u}_1 \gamma^\mu u_2) (\bar{u}_1 \gamma^\mu u_2)^* \\
|\bar{u}_1 \gamma^\mu u_2|^2 &= (\bar{u}_1 \gamma^\mu u_2) (\bar{u}_2 \gamma^\mu u_1)
\end{aligned}$$

I know that both of the expressions in the parentheses are 1x1 matrices but I fail to see how it can be simplified anymore.

Or is this the way to proceed: $|\bar{u}_1 \gamma^\mu u_2|^2 = |\bar{u}_1|^2 |\gamma^\mu|^2 |u_2|^2$

$$\begin{aligned}
|\bar{u}_1 \gamma^\mu u_2|^2 &= (\bar{u}_1) (\bar{u}_1)^* \gamma^\mu (\gamma^\mu)^* (u_2) (u_2)^* \\
|\bar{u}_1 \gamma^\mu u_2|^2 &= (\bar{u}_1) (\bar{u}_1)^* \gamma^\mu \gamma^\mu (u_2) (u_2)^* \\
|\bar{u}_1 \gamma^\mu u_2|^2 &= (\bar{u}_1) \bar{u}_1^* (u_2) (u_2)^*
\end{aligned}$$