The R_K Puzzle and Analysis

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Abstract

The R_K is a B meson ratio puzzle that seems to provide substantial evidence of New Physics(NP) beyond the Standard Model (SM). Recent experimental results from the BaBar, Belle, and LHCb collaborations demonstrate intriguing results compared to SM predictions. The loops present in certain B meson decays are sensitive to the possibility of NP. This article seeks to constrain and analyze the Wilson Coefficients that are present in the branching ratio of such decays. We hope to provide a substantial analysis and demonstrate a indication of new research to come.

Contents

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1 Introduction

The Standard Model (SM) of particle physics works extremely well when calculating decay rates of subatomic particle and explaining different types of physical phenomena that occur at the subatomic level. Included in the SM are lists of rules that govern interactions and different conserved quantities that have aligned well with the observed experimental results. The SM spectrum includes 4 types of particles: gauge bosons, quarks, leptons, and the Higgs boson. The gauge bosons are particles with spin 1 that mediate the different forces. For instance, the W and Z bosons mediate the weak force. In Table 1 we summarize the physical properties of this type of particles.

Gauge Bosons			
Boson	Mass	Charge	Force
γ	0 MeV	0	Electromagnetic
Z	91.1876 ± 0.00214 GeV	0	Weak
$W^{\pm 1}$	80.385 ± 0.01596 GeV	± 1	Weak
Gluon	0 MeV	0	Strong

Table 1: Table of Gauge Bosons in the SM [2]

In addition to the gauge bosons, the SM contains other particles known as fermions. Fermions have half-integer spin and are comprised of quarks and leptons. There are 3 types of leptons (electron, muon and tau), each with its corresponding neutrino. Each fermion (f) has an antiparticle ($\bar{f}$), which has the same mass but opposite charge. These leptons can be seen in Table 2.

Leptons		
Type	Mass	Charge
e	0.51099 MeV	-1
μ	105.6583745(24) MeV	-1
τ	1776.86 ± 0.12 MeV	-1

Table 2: Table of Leptons in the SM [2]

The other types of fermions are quarks. Quarks differ from leptons because quarks participate in the strong force, while leptons do not. Quarks also have either a $\frac{2}{3}$ or $-\frac{1}{3}$ charge and can be seen below in Table 3. Finally, there is the recently discovered Higgs boson, a spinless scalar particle responsible for generating gauge boson and fermion masses in the SM.

In spite of its remarkable success in explaining subatomic phenomena, recently it has been shown that there is need for physics beyond the SM in order to explain experimental observations made by BaBar, Belle and LHCb collaborations, while studying decays of B mesons. In this article, we will study decays of the B^+ ($\bar{b}u$) and B_s ($\bar{b}s$) mesons. We will focus on a few interesting channels that have received quite a bit of attention in the literature.

Quarks		
Type	Mass	Charge
u	$2.2^{+0.6}_{-0.4}$ MeV	$\frac{2}{3}$
d	$4.7^{+0.5}_{-0.4}$ MeV	$-\frac{1}{3}$
s	96^{+8}_{-4} MeV	$-\frac{1}{3}$
c	1.28 ± 0.03 GeV	$\frac{2}{3}$
b	$4.18^{+0.04}_{-0.03}$ GeV	$-\frac{1}{3}$
t	173.1 ± 0.6 GeV	$\frac{2}{3}$

Table 3: Table of Quarks in the SM [2]

By exploring the properties and characteristics of quarks and leptons, our goal is to understand the properties of such new physics (NP).

The R_K and $R_{D^{(*)}}$ are two ratios of these B meson decays that show some puzzling experimental results. The R_K ratio is defined as $\mathcal{B}(B^+ \to K^+\mu^+\mu^-)/\mathcal{B}(B^+ \to K^+e^+e^-)$ The experimental value found by the LHCb Collaboration is [6]

$$R_K^{expt} = 0.745^{+0.090}_{-0.074} \text{ (stat) } \pm 0.036 \text{ (syst)}$$

Which differs from the SM prediction of $R_K^{expt} = 1 \pm 0.01$ differs from the standard model prediction by 2.6σ. [7] The next puzzle is the $R_{D^{(*)}}$ ratio which is defined as $\mathcal{B}(\bar{B} \to D^{(*)}\tau^-\overline{\nu}_\tau/\mathcal{B}(\bar{B} \to D^{(*)}l^-\overline{\nu}_l$ where ($l = \mu$ or e) this ratio has been measured by the LHCb Collaboration, BaBar, and Belle. The experimental values of R_D and R_{D^*}The ratios are as follows [8–10] :

$$R_D = 1.29 \pm 0.17 \text{ , } R_{D^*} = 1.28 \pm 0.09$$

R_D differs from the standard model by 1.7σ and the R_{D^*} differs by 3.1σ. We will focus on analyzing the possible NP contributions to the Wilson Coefficients (WC) arising from the effective Hamiltonian of the $b \to s\mu^+\mu^-$ transition. The complete Hamiltonian is as follows: [5].

$$H = -\frac{\alpha G_f}{\sqrt{2}\pi} V_{tb}V_{ts}^* \sum_{a=9,10} (C_a O_a + C'_a O'_a) \tag{1}$$

This article will take the Hamiltonian and use the WC found in the branching ratio of the $B_s \to \mu^+\mu^-$ decay to constrain the effects of NP parameters. The paper begins by first exploring and computing different observable and particulars of the $\pi^+ \to l^+\nu_l$ decay. (Where $l = \mu$ or e) This will provide us with a background in manipulating and understanding the theoretical expression for branching ratios and decays. It will also serve the reader as a guide to the format of the rest of the paper. In section 3 the WC from the branching ratio of the $B_s \to \mu^+\mu^-$ decay will be extracted and constrained with the experimental numbers. We will use the Python package Flavio [4] to generate our data and construct our plots. Flavio's packages focus on flavor physics and allows the user to find the contributions of different WC for varying observable. Finally in section 4 plots of the Branching ratio based upon the WC will be shown and fits to the data will be provided. We conclude the article in section 5.

2 The branching Ratio of a Charged Pion decay

In order to calculate the Branching ratio of a charged pion, we will use the Feynman Rules of calculating amplitudes and the trace identities.We start by with recognizing that the pion decay is a charged current interaction which arises from the fact that a pion is made of quarks, and the decay is mediated by a massive W boson. A diagram of the decay may be seen in Figure 1:

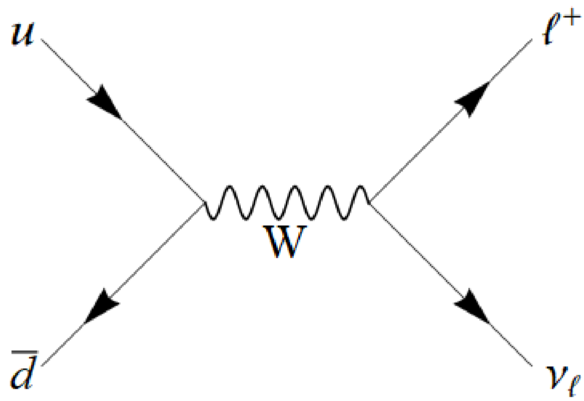

Figure 1: Pion Decay (where ℓ is μ or e)

Where the up and ani-down quark are the pion and the lepton and corresponding lepton neutrino are on the right.

The formula given to describe the branching ratio of this decay is given by [1]:

$$\Gamma = \frac{S|\mathbf{p}|}{8\pi\hbar m_1^2 c}|\mathcal{M}|^2 \quad (2)$$

Where $|\mathbf{p}|$ is the outgoing momentum, S is the product of statistical factors (in our case it will be equal to 1), m_1 is the mass of the pion, and $\mathcal{M}$ is the Feynman amplitude. In our notation, we will use the Natural Units so our expression becomes:

$$\Gamma = \frac{S|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2 \quad (3)$$

The next step is to determine the $\mathcal{M}$. In the charged current decay, we have different notation for the vertices and propagators.

1. For each vertex add a factor of $\frac{-ig_w}{2\sqrt{2}}(\gamma^\nu(1-\gamma^5))$ where $g_w = \sqrt{4\pi\alpha_w}$

2. For each propagator we add a factor of $\frac{-ig_{\mu\nu}-\frac{q_\mu q_\nu}{m^2}}{q^2-m^2}$ where m is the mass of the boson. In our case, $m_w \gg q$ so the expression simplifies to $\frac{ig_{\mu\nu}}{m_w^2}$

From these rules and Figure 1 we are able to calculate the value of $\mathcal{M}$

$$-i\mathcal{M} = \left[\overline{u}(3)\left(\frac{-ig_w}{2\sqrt{2}}(\gamma^\nu(1-\gamma^5)\right)v(2)\right]\left[\frac{ig_{\mu\nu}}{m_w^2}\right]\left[\frac{-ig_w}{2\sqrt{2}}F^\mu\right] \tag{4}$$

Where F^μ is the form factor of the coupling of the pion to the W boson. F^μ has the form of $f_\pi p^\mu$.

$$\mathcal{M} = \frac{g_w^2}{8m_w^2}\left[\overline{u}(3)(\gamma^\mu(1-\gamma^5))v(2)\right]F^\mu \tag{5}$$

In order to square the amplitude we do the following:

$$\langle|\mathcal{M}^2|\rangle = \left(\frac{g_w^2}{8m_w^2}f_\pi\right)^2 Tr[(\overline{u}(3)(\gamma^\mu(1-\gamma^5))v(2)p_\mu(\overline{v}(2)(\gamma^\nu(1-\gamma^5)u(3))]p_\nu \tag{6}$$

$$\langle|\mathcal{M}^2|\rangle = \left(\frac{g_w^2}{8m_w^2}f_\pi\right)^2 p_\mu p_\nu(8[p_3^\mu p_2^\nu + p_2^\mu p_3^\nu - (p_3\cdot p_2)g^{\mu\nu}] + 8i\epsilon^{\mu\lambda\nu\sigma}p_{3\lambda}p_{2\sigma} \tag{7}$$

Summing over the spins gives us:

$$\langle|\mathcal{M}^2|\rangle = 8\left(\frac{g_w^2}{8m_w^2}f_\pi\right)^2\left[2(p_1\cdot p_2)(p_1\cdot p_3) - p^2(p_2\cdot p_3)\right] \tag{8}$$

Since $p = p_2 + p_3$, we can simplify the equation further. For simplicity and consistency we will use the following notation: $p_1 = p_\pi, p_2 = p_l, p_3 = p_{\nu_l}$. We begin with showing the value of the 4-momentum squared:

$$\begin{aligned} p_1 &= (E, \vec{p}_1) && (9)\\ (p_1)^2 &= p_\mu p_\nu g_{\mu\nu} \\ (p_1)^2 &= p_1 p_1(1) + p_2 p_2(-1) + ... \\ (p_1)^2 &= E^2 - (\vec{p}_1)^2 \\ (p_1)^2 &= m_1^2 && (10)\end{aligned}$$

Similarly: $(p_\pi)^2 = m_\pi^2, (p_l)^2 = m_l^2, (p_{\nu_l})^2 = (m_{\nu_l})^2 = 0$. Using this we can further simplify Equation 8:

$$\frac{1}{2}[(m_\pi)^2 - (m_l)^2] = (p_2\cdot p_3) \tag{11}$$

$$\frac{1}{2}[(m_\pi)^2 - (m_l)^2] = (p_1\cdot p_3) \tag{12}$$

$$\frac{1}{2}[(m_\pi)^2 + (m_l)^2] = (p_1\cdot p_2) \tag{13}$$

Returning to Equation 8 we now have

$$\langle|\mathcal{M}^2|\rangle = 8\left(\frac{g_w^2}{8m_w^2}f_\pi\right)^2\left[\frac{1}{2}(m_l)^2((m_\pi)^2 - (m_l)^2)\right] \tag{14}$$

With the Feynman amplitude, we are able to complete the expression for the branching ratio of a pion. Returning to equation (3)

$$\Gamma = \frac{S|\mathbf{p}|}{8\pi m_1^2}|\mathcal{M}|^2 \tag{15}$$

$$\Gamma = \frac{S|\mathbf{p}|}{\pi m_\pi^2}\left(\frac{g_w^2}{8m_w^2}f_\pi\right)^2\left[\frac{1}{2}(m_l)^2((m_\pi)^2-(m_l)^2)\right] \tag{16}$$

We can simplify S because in this case S=1. We then need to find the value of $|\mathbf{p}|$ which is simply the outgoing momentum of the two particles.

$$|\mathbf{p}| = \frac{\sqrt{(m_1+m_2+m_3)(m_1-m_2-m_3)(m_1+m_2-m_3)(m_1-m_2+m_3)}}{2m_1} \tag{17}$$

$$|\mathbf{p}| = \frac{\sqrt{m_1^4+m_2^4-m_1^2m_2^2}}{2m_1} \tag{18}$$

(Because the neutrino is massless)

$$|\mathbf{p}| = \frac{\sqrt{(m_1^2-m_2^2)^2}}{2m_1} \tag{19}$$

For our case $m_1 = m_\pi$ and $m_2 = m_l$

$$|\mathbf{p}| = \frac{(m_\pi^2-m_l^2)}{2m_\pi} \tag{20}$$

Combining with Equation 14:

$$\Gamma = \frac{|\mathbf{p}|}{2\pi m_1^2}\left(\frac{g_w^2}{8m_w^2}f_\pi\right)^2\left[(m_l)^2((m_\pi)^2-(m_l)^2)\right] \tag{21}$$

$$\Gamma = \frac{1}{4\pi}\left(\frac{g_w^2 f_\pi}{8m_w^2}\right)^2\left(\frac{1}{m_\pi^3}\right)m_l^2(m_\pi^2-m_l^2)^2 \tag{22}$$

Because $\frac{g_w^2}{8m_w^2} = \frac{G_f}{\sqrt{2}}$ we are able to say:

$$\Gamma = \frac{1}{8\pi}(G_f f_\pi)^2\left(\frac{1}{m_\pi^3}\right)m_l^2(m_\pi^2-m_l^2)^2 \tag{23}$$

Expanding on this idea, we are able to graph $\frac{\Gamma_l}{\Gamma_\pi}$ by:

$$\Gamma = \frac{1}{8\pi}(G_f f_\pi)^2 m_\pi^3\left(\frac{m_l}{m_\pi}\right)^2\left(1-\left(\frac{m_l}{m_\pi}\right)^2\right)^2 \tag{24}$$

In order to find $\frac{\Gamma_l}{\Gamma_\pi}$ we need to divide the above expression by $\frac{1}{\tau_\pi}$ but normalize it with $\hbar$ so our final expression is:

$$\frac{\Gamma_l}{\Gamma_\pi} = \frac{1}{8\pi}(G_f f_\pi)^2 m_\pi^3\left(\frac{\tau_\pi}{\hbar}\right)\left(\frac{m_l}{m_\pi}\right)^2\left(1-\left(\frac{m_l}{m_\pi}\right)^2\right)^2 \tag{25}$$

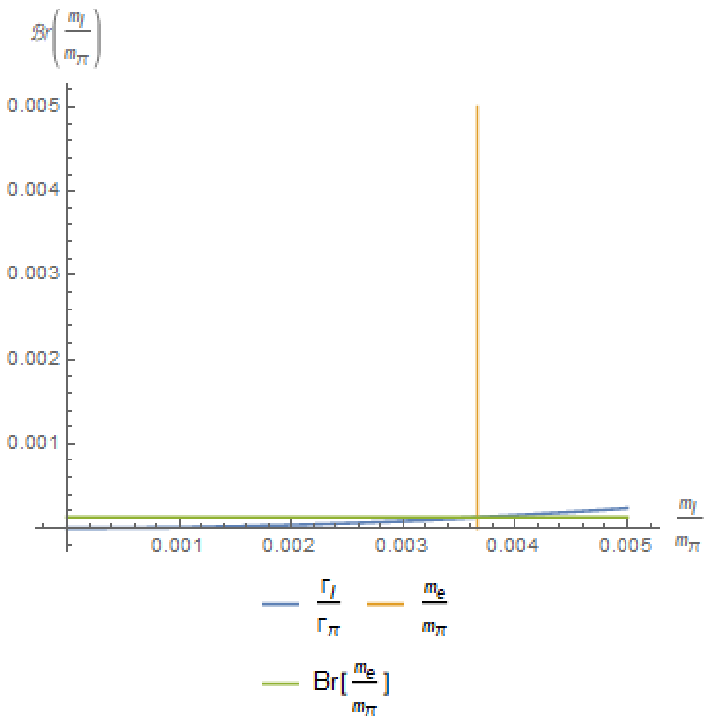

Figure 2: Graph of $\frac{\Gamma_{m_e}}{\Gamma_\pi}$ and the plots of $\frac{m_e}{m_\pi}$ and the value of $Br(\pi^- \to e^- + \nu_e)$

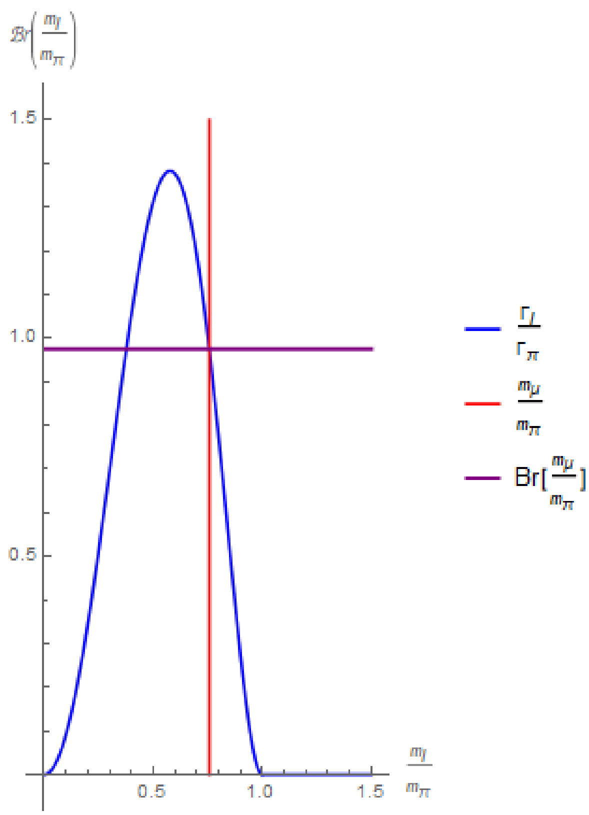

Figure 3: Graph of $\frac{\Gamma_{m_\mu}}{\Gamma_\pi}$ and the plots of $\frac{m_\mu}{m_\pi}$ and the value of $Br(\pi^- \to \mu^- + \nu_\mu)$

The graph of the Equation 25 (with $m_l = m_e$) is found in Figure 2. and the graph of Equation 25 (with $m_l = m_\pi$) is found in Figure 3. Using the values from PDG [2] we may compute different ratios with the branching ratios.

Observables	e	μ	π
τ (Lifetime)	6.6×10^{28} yr	$2.1969811(22) \times 10^-6$ s	$2.6033(5) \times 10^-8$ s
Mass(MeV)	0.5109989461(31)	105.6583745(24)	139.57061(24)

If we would like to calculate the ratio of the $\pi^- \to e^- + \nu_e$ and $\pi^- \to \mu^- + \nu_\mu$ we simply do the following:

$$\frac{\Gamma_e}{\Gamma_\mu} = \frac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2} \tag{26}$$

$$\frac{\Gamma_e}{\Gamma_\mu} = 1.28334(73) \times 10^{-4} \tag{27}$$

This is an interesting observation because the value of $\frac{\Gamma_e}{\Gamma_\mu}$ suggests that the probability of $\pi^- \to \mu^- + \nu_\mu$ is higher than $\pi^- \to e^- + \nu_e$. This is somewhat striking because the mass of a muon is greater than the mass of an electron, indicating that the pion does not decay into the lightest particle most frequently.

3 The Branching Ratio of the $B_s \to \mu^+\mu^-$ decay

Following the procedure outlined above and using the branching ratio found in [3] we are able to find an expression for the decay rate of the $B_s \to \mu^+\mu^-$ with some coefficients of new physics included. The decay rate is as follows:

$$\begin{aligned}\mathcal{B}(B_s \to \mu^+\mu^-) &= \frac{G_F^2\alpha_{em}^2 m_{B_s}^5 f_{B_s}^2 \tau_{B_s}}{64\pi^3}\sqrt{1 - \frac{4m_\mu^2}{m_{B_s}^2}}\left\{\left(1 - \frac{4m_\mu^2}{m_{B_s}^2}\right)\right. \\ &\left.\left|\zeta\frac{C_S - C_S^{'}}{m_b + m_s}\right|^2 + \left|\zeta\frac{C_P - C_P^{'}}{m_b + m_s} + \frac{2m_\mu}{m_{B_s}^2}[|V_{tb}V_{ts}^*|C_{10} + \zeta(C_A - C_A^{'})]\right|^2\right\}\end{aligned} \tag{28}$$

Where $\zeta \equiv (\frac{g_{NP}^2}{\Lambda^2})(\frac{\sqrt{2}}{4G_F})(\frac{4\pi}{\alpha_{em}})$

The Feynman Diagram of the decay can be seen below: Since we seek to find the constraints on the parameters $C_S, C_S^{'}, C_P, C_P^{'}, C_A$, and $C_A^{'}$ we set the equation equal to the branching ratio given in [4] and using the values for $C_{10}, |V_{tb}V_{ts}^*|, g_{NP}$, and Λ given in [3] thus we are able to find the constraints on the parameters. In order to solve for one parameter, we allow the other two to be equal to 0, this will simply our calculations.We will solve each coefficient by the upper and lower limits of the branching ratio then take the average by setting the expression with the unknown coefficient equal to the branching ratio, the upper

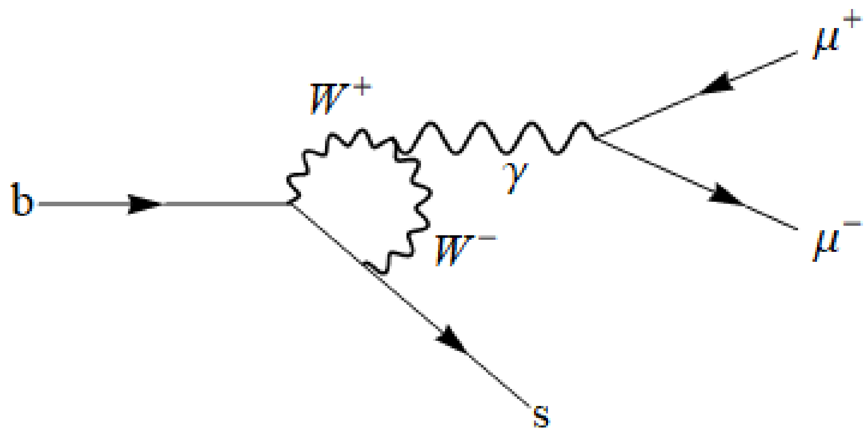

Figure 4: Graph of $B_s \to \mu^+\mu^-$

limit of the branching ratio, and its lower limit.

The values for the Branching ratio of $B_s \to \mu^+\mu^-$ decay is equal to $2.4^{+0.9}_{-0.7} \times 10^{-9}$ [2]

$$\begin{aligned} C_S &= \pm 1.6215i \times 10^{-4} \text{ For the lower BR value} \\ &= \pm 4.63981 \times 10^{-5} \text{ For the upper BR value} \end{aligned}$$

$$\begin{aligned} C_P &= 6.3832 \times 10^{-5} \text{ or } 4.1125 \times 10^{-4} \text{ For the lower BR value} \\ &= -4.482 \times 10^{-6} \text{ or } 4.7957 \times 10^{-4} \text{ For the upper BR value} \end{aligned}$$

$$\begin{aligned} C_A &= 2.0348 \times 10^{-3} \text{ or } 1.31095 \times 10^{-2} \text{ For the lower BR value} \\ &= 1.48237 \times 10^{-4} \text{ or } 1.5287 \times 10^{-2} \text{ For the upper BR value} \end{aligned}$$

Notes about the Wilson Coefficients:
The C_S coefficient is equal to $\pm(0.811i + 0.232) \times 10^{-4}$ (We are able to calculate the mean by a simply average).The C_P has two values the first value is equal to 2.967×10^{-5} and the second value is equal to 4.454×10^{-4}. The last coefficient C_A also has two values, the first one is equal to 1.089×10^{-3} and the second value is equal to 1.420×10^{-2}

After finding the Wilson Coefficients, we use the Python package Flavio [4] to compute the NP values of $B(B_s \to \mu^+\mu^-)$. We again only apply one Wilson Coefficient at a time to find the NP branching ratio:

C_S:

$$C_S = \pm(0.811i + 0.232) \times 10^{-4}$$
$$\text{Flavio Prediction: BR} = 3.610 \times 10^{-9}$$

C_P:

$$C_P = 2.967 \times 10^{-5}$$
$$\text{Flavio Prediction: } Br = 3.603 \times 10^{-9}$$
$$C_P = 4.454 \times 10^{-4}$$
$$\text{Flavio Prediction: } Br = 3.508 \times 10^{-9}$$

C_A:

$$C_A = 1.089 \times 10^{-3}$$
Flavio Prediction: $Br = 3.608 \times 10^{-9}$
$$C_A = 1.420 \times 10^{-2}$$
Flavio Prediction: $Br = 3.584 \times 10^{-9}$

4 Plots of $Br(WC)$

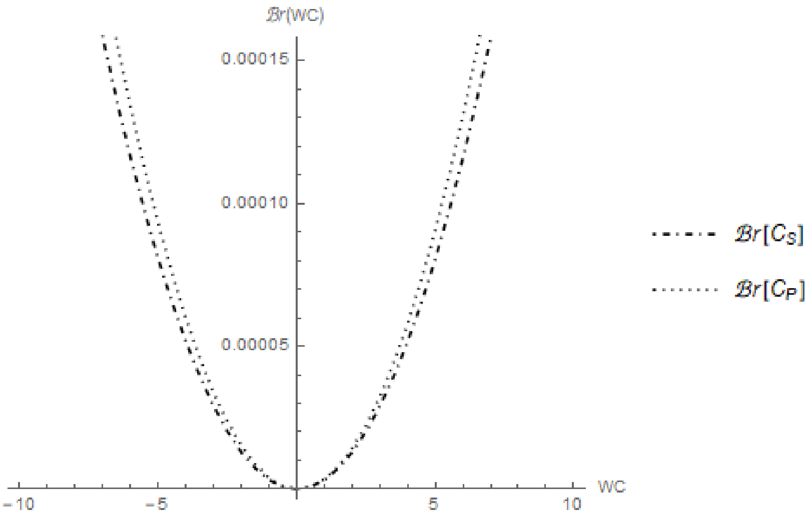

Figure 5: Graph of $Br(B_s \to \mu^+\mu^-)$ as functions of the C_P and C_S Wilson Coefficients

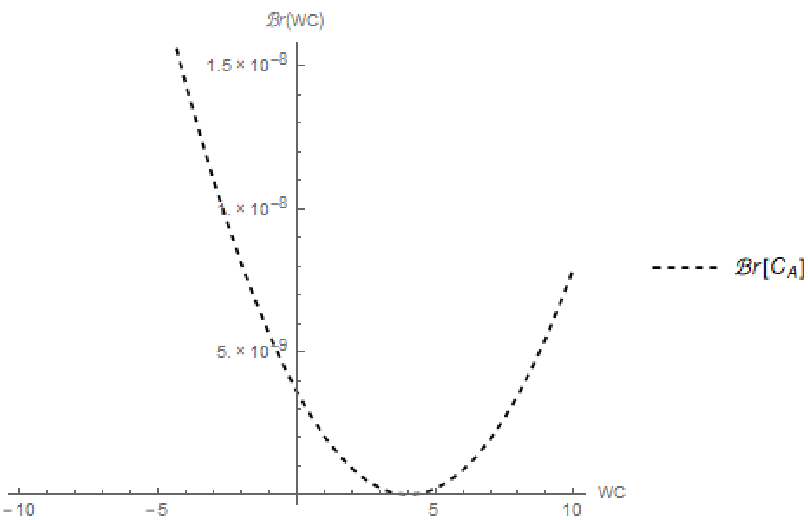

Figure 6: Graph of $Br(B_s \to \mu^+\mu^-)$ as functions of the C_A Wilson Coefficient

Figures 5 and 6 give an indication of how drastically the variation of a Wilson Coefficient could change the value of the Branching ratio.

For each of the plots we have conducted a fit to the lines in order to find the coefficient values to the graphs.
$Br(C_S) = 3.22046 \times 10^{-6}x^2$
$Br(C_P) = 3.57491 \times 10^{-6}x^2$
$Br(C_A) = 2.79479 \times 10^{-10}x^2$
Here, in Figure 7 the fits to the plots are as follows:

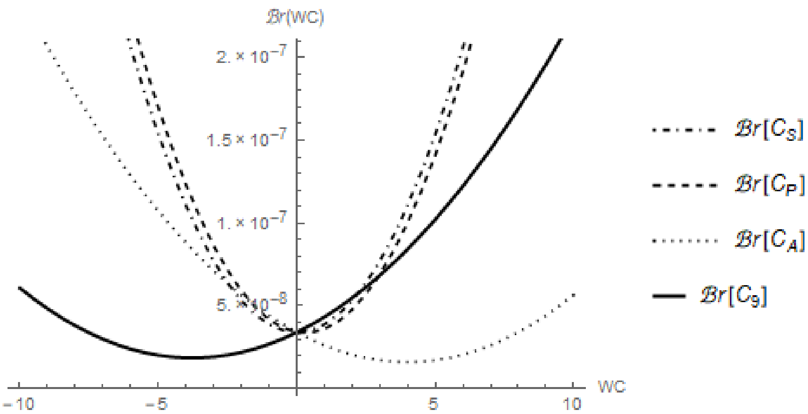

Figure 7: Graph of $Br(B \to K\mu^+\mu^-)$ as functions of the Wilson Coefficients

$Br(C_S) = 5.73539 \times 10^{-9}x^2$
$Br(C_P) = 5.81325 \times 10^{-9}x^2$
$Br(C_A) = 1.7963 \times 10^{-9}x^2$
$Br(C_9) = 1.75406 \times 10^{-9}x^2$

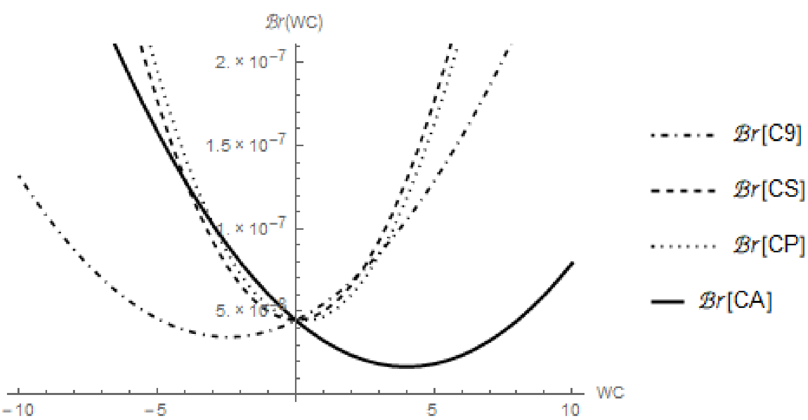

Figure 8: Graph of $Br(B \to K^*\mu^+\mu^-)$ as functions of the Wilson Coefficients

In Figure 8 the fits for the plots are as follows:
$Br(C_S) = 5.97644 \times 10^{-9}x^2$
$Br(C_P) = 6.056 \times 10^{-9}x^2$
$Br(C_A) = 2.45199 \times 10^{-9}x^2$

$Br(C_9) = 2.40283 \times 10^{-9} x^2$

5 Conclusion

The work done with the Wilson Coefficients gives new suggestions to the possibility of NP. In the future, the expectation is that work will bo done using a chi squared minimization technique to help limit and constrain the WC. The chi squared technique would use the observables that have been calculated experimentally from different collaborations. Although there have been several suggested solutions to the WC such as the ones found in [11] the solutions still raise questions. We hope in the future to provide a rigorous explanation that shows the existance of NP.

6 Acknowledgements

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7 Appendix

Practice with Spinors

Notation:

1. $S = \overline{u}u$
2. $P = \overline{u}\gamma^5 u$
3. $V^\mu = \overline{u}\gamma^\mu u$
4. $A^\mu = \overline{u}\gamma^\mu\gamma^5 u$
5. $T^{\mu\nu} = \overline{u}\sigma^{\mu\nu} u$

It was necessary to practice with spinor notation and the different mathematical techniques before starting this project. This appendix summarizes what I was able to do in that regard.

1. $(\overline{u}_1\gamma^\mu u_2)^* \;= ?$

   Note: $(\gamma^0)^\dagger \;=\; \gamma^0$ and $(\gamma^\mu)^\dagger \;=\; \gamma^0\gamma^\mu\gamma^0$

   $(\overline{u}_1\gamma^\mu u_2)$ is a 1×1 matrix. Therefore, its complex conjugate is the same as its Hermitian conjugate, i.e. if we call $V^\mu = (\overline{u}_1\gamma^\mu u_2)$, then $(V^\mu)^* = (V^\mu)^\dagger$. We can then express this quantity as follows:

$$
\begin{aligned}
V^\mu &= \overline{u}_1\gamma^\mu u_2 \;, \qquad (29)\\
\Rightarrow (V^\mu)^* &= (V^\mu)^\dagger \;,\\
&= (\overline{u}_1\gamma^\mu u_2)^\dagger \;,\\
&= ((u_1)^\dagger\gamma^0\gamma^\mu u_2)^\dagger \quad \text{using } (A\ldots Z)^\dagger = Z^\dagger\ldots A^\dagger \;,\\
&= (u_2^\dagger)(\gamma^\mu)^\dagger(\gamma^0)^\dagger(u_1)\\
&= (u_2^\dagger)\gamma^0\gamma^\mu\gamma^0\gamma^0(u_1)\\
&= (u_2^\dagger)\gamma^0\gamma^\mu(u_1)\\
&= \overline{u}_2\gamma^\mu(u_1) \qquad (30)
\end{aligned}
$$

   Therefore $(\overline{u}_1\gamma^\mu u_2)^* \;=\; \overline{u}_2\gamma^\mu u_1$. To solve for $|V^\mu|^2$ we simply use $|V^\mu|^2 \;=\; \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1]$.

   Note: $\mathrm{Tr}[\gamma^\mu\gamma^\nu] \;=\; 4g^{\mu\nu}$, $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$, The trace over the product of an odd number of gamma matrices is zero.

$$
\begin{aligned}
|V^\mu|^2 &= \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1] \qquad (31)\\
&= \mathrm{Tr}[\overline{u}_1\gamma^\mu(\not{p}_2 + m)\gamma^\nu u_1]\\
&= \mathrm{Tr}[u_1\overline{u}_1\gamma^\mu(\not{p}_2 + m)\gamma^\nu]\\
&= \mathrm{Tr}[(\not{p}_1 + m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu]\\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] + m[\mathrm{Tr}(\gamma^\mu\not{p}_1\gamma^\nu) + \mathrm{Tr}(\gamma^\mu\gamma^\nu\not{p}_2)] + m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu]\\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] + m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{Tr}[(p_1)_\lambda \gamma^\lambda \gamma^\mu (p_2)_\sigma \gamma^\sigma \gamma^\nu] + 4m^2 g^{\mu\nu} \\
&= (p_1)_\lambda (p_2)_\sigma \mathrm{Tr}[\gamma^\lambda \gamma^\mu \gamma^\sigma \gamma^\nu] + 4m^2 g^{\mu\nu} \\
&= (p_1)_\lambda (p_2)_\sigma 4(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda}) + 4m^2 g^{\mu\nu} \\
&= 4[p_1^\mu p_2^\nu - g^{\mu\nu}(p_1 \cdot p_2) + p_2^\mu p_1^\nu] + 4m^2 g^{\mu\nu} \qquad (32)
\end{aligned}
$$

2. $(\overline{u}_1 \gamma^\mu \gamma^5 u_2)^*$ is also a 1×1 Matrix so the same reasoning applies as above in 1. Note: $(\gamma^5)^\dagger = \gamma^5$ We define: A^μ as $\overline{u}_1 \gamma^\mu \gamma^5 u_2$ thus:

$$
\begin{aligned}
(A^\mu)^* &= (A^\mu)^\dagger \qquad (33) \\
&= (\overline{u}_1 \gamma^\mu \gamma^5 u_2)^\dagger \\
&= ((u_1)^\dagger \gamma^0 \gamma^\mu \gamma^5 u_2)^\dagger \\
&= (u_2^\dagger)(\gamma^5)^\dagger (\gamma^\mu)^\dagger (\gamma^0)^\dagger (u_1) \\
&= (u_2^\dagger)\gamma^5 \gamma^0 \gamma^\mu \gamma^0 \gamma^0 u_1 \\
&= (u_2^\dagger)\gamma^5 \gamma^0 \gamma^\mu (1) u_1 \\
&= -(u_2^\dagger)\gamma^0 \gamma^5 \gamma^\mu u_1 \\
&= -\overline{u}_2 \gamma^5 \gamma^\mu u_1 \\
&= \overline{u}_2 \gamma^\mu \gamma^5 u_1 \qquad (34)
\end{aligned}
$$

Therefore $(\overline{u}_1 \gamma^\mu \gamma^5 u_2)^* \;=\; \overline{u}_2 \gamma^\mu \gamma^5 u_1$

We also are able to calculate $|A^\mu|^2$

$$
\begin{aligned}
|A^\mu|^2 &= \mathrm{Tr}[(\overline{u}_1 \gamma^\mu \gamma^5 u_2)(\overline{u}_2 \gamma^\nu \gamma^5 u_1)] \qquad (35) \\
&= \mathrm{Tr}[\overline{u}_1 \gamma^\mu \gamma^5 (\not{p}_2 + m) \gamma^\nu \gamma^5 u_1] \\
&= \mathrm{Tr}[u_1 \overline{u}_1 \gamma^\mu \gamma^5 (\not{p}_2 + m) \gamma^\nu \gamma^5] \\
&= \mathrm{Tr}[(\not{p}_1 + m) \gamma^\mu \gamma^5 (\not{p}_2 + m) \gamma^\nu \gamma^5] \\
&= \mathrm{Tr}[\not{p}_1 \gamma^\mu \gamma^5 \not{p}_2 \gamma^\nu \gamma^5 + m(\not{p}_1 \gamma^\mu \gamma^5 \gamma^\nu \gamma^5 + \gamma^\mu \gamma^5 \not{p}_2 \gamma^\nu \gamma^5) + m^2(\gamma^\mu \gamma^5 \gamma^\nu \gamma^5)] \\
&= \mathrm{Tr}[\not{p}_1 \gamma^\mu \gamma^5 \not{p}_2 \gamma^\nu \gamma^5 + m(\not{p}_1 \gamma^\mu (-\gamma^5 \gamma^5) \gamma^\nu + \not{p}_2 \gamma^\mu (-\gamma^5 \gamma^5) \gamma^\nu) + m^2(\gamma^\mu \gamma^5 \gamma^\nu \gamma^5)] \\
&= \mathrm{Tr}[\not{p}_1 \gamma^\mu \gamma^5 \not{p}_2 \gamma^\nu \gamma^5 - m(\not{p}_1 \gamma^\mu \gamma^\nu + \not{p}_2 \gamma^\mu \gamma^\nu) + m^2(\gamma^\mu \gamma^5 \gamma^\nu \gamma^5)] \\
&= \mathrm{Tr}[\not{p}_1 \gamma^\mu \gamma^5 \not{p}_2 \gamma^\nu \gamma^5] - mTr[\not{p}_1 \gamma^\mu \gamma^\nu] - mTr[\not{p}_2 \gamma^\mu \gamma^\nu] + m^2(\gamma^\mu \gamma^5 \gamma^\nu \gamma^5)] \\
&= \mathrm{Tr}[\not{p}_1 \gamma^\mu \gamma^5 \not{p}_2 \gamma^\nu \gamma^5 + m^2(\gamma^\mu \gamma^5 \gamma^\nu \gamma^5)] \\
&= \mathrm{Tr}[(p_1)_\lambda \gamma^\lambda \gamma^\mu \gamma^5 (p_2)_\sigma \gamma^\sigma \gamma^\nu \gamma^5] + m^2 \mathrm{Tr}[\gamma^\mu \gamma^5 \gamma^\nu \gamma^5] \\
&= (p_1)_\lambda (p_2)_\sigma \mathrm{Tr}[\gamma^\lambda \gamma^\mu \gamma^5 \gamma^\sigma \gamma^\nu \gamma^5] - m^2 \mathrm{Tr}[\gamma^\mu \gamma^5 \gamma^5 \gamma^\nu] \\
&= (p_1)_\lambda (p_2)_\sigma \mathrm{Tr}[\gamma^\lambda \gamma^\mu \gamma^5 \gamma^5 \gamma^\sigma \gamma^\nu] - m^2 \mathrm{Tr}[\gamma^\mu \gamma^\nu] \\
&= (p_1)_\lambda (p_2)_\sigma \mathrm{Tr}[\gamma^\lambda \gamma^\mu \gamma^\sigma \gamma^\nu] - m^2(g^{\mu\nu}) \\
&= (p_1)_\lambda (p_2)_\sigma 4(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda}) - 4m^2 g^{\mu\nu} \\
&= 4[p_1^\mu p_2^\nu - g^{\mu\nu}(p_1 \cdot p_2) + p_2^\mu p_1^\nu] - 4m^2 g^{\mu\nu} \qquad (36)
\end{aligned}
$$

3. $(\overline{u}_1 u_2)^* \;=\;$? We let $S \;=\; \overline{u}_1 u_2$

$$
(S)^* = (S)^\dagger \qquad (37)
$$

$$
\begin{aligned}
&= (\overline{u}_1 u_2)^\dagger\ , \\
&= ((u_1)^\dagger \gamma^0 u_2)^\dagger \\
&= (u_2)^\dagger (\gamma^0)^\dagger (u_1) \\
&= (u_2)^\dagger \gamma^0 (u_1) \\
&= \overline{u}_2 (u_1) \qquad (38)
\end{aligned}
$$

Therefore$(\overline{u}_1 u_2)^* = \overline{u}_2 u_1$. In order to find $|S|^2$ we simply do the following:

$$
\begin{aligned}
|S|^2 &= \text{Tr}[\overline{u}_1 u_2 \overline{u}_2 u_1] \qquad (39) \\
&= \text{Tr}[(\not{p}_1 + m)(\not{p}_2 + m)] \\
&= \text{Tr}[\not{p}_1 \not{p}_2 + m(\not{p}_1 + \not{p}_2) + m^2] \\
&= \text{Tr}[\not{p}_1 \not{p}_2] + \text{Tr}[m(\not{p}_1 + \not{p}_2)] + \text{Tr}[m^2] \\
&= \text{Tr}[\not{p}_1 \not{p}_2] + m(\text{Tr}[\not{p}_1] + \text{Tr}[\not{p}_2]) + m^2 \text{Tr}[1] \\
&= \text{Tr}[\not{p}_1 \not{p}_2] + 4m^2 \\
&= 4(p_1 \cdot p_2) + 4m^2 \qquad (40)
\end{aligned}
$$

4. By the same reasoning as shown above it can be shown that $(\overline{u}_1 \gamma^5 u_2)^* = \overline{u}_2 \gamma^5 u_1$
   If we let $P = \overline{u}_1 \gamma^5 u_2$ then:

$$
\begin{aligned}
(P)^* &= (P)^\dagger \qquad (41) \\
&= (\overline{u}_1 \gamma^5 u_2)^\dagger\ , \\
&= ((u_1)^\dagger \gamma^0 \gamma^5 u_2)^\dagger \\
&= (u_2)^\dagger (\gamma^5)^\dagger (\gamma^0)^\dagger (u_1) \\
&= (u_2)^\dagger (\gamma^5) \gamma^0 (u_1) \\
&= -(u_2)^\dagger \gamma^0 \gamma^5 (u_1) \\
&= -\overline{u}_2 \gamma^5 (u_1) \qquad (42)
\end{aligned}
$$

   Therefore $(\overline{u}_1 \gamma^5 u_2)^* = -\overline{u}_2 \gamma^5 u_1$

   In order to square P we do the following:

$$
\begin{aligned}
|P|^2 &= \text{Tr}[\overline{u}_1 \gamma^5 u_2 (-\overline{u}_2 \gamma^5 u_1)] \qquad (43) \\
&= \text{Tr}[u_1 \overline{u}_1 \gamma^5 (-\not{p}_2 - m) \gamma^5] \\
&= \text{Tr}[(\not{p}_1 + m) \gamma^5 (-\not{p}_2 - m) \gamma^5] \\
&= \text{Tr}[((p_1)_\mu \gamma^\mu + m) \gamma^5 ((-p_2)_\nu \gamma^\nu - m) \gamma^5] \\
&= \text{Tr}[((p_1)_\mu \gamma^\mu \gamma^5 + m\gamma^5)((-p_2)_\nu \gamma^\nu \gamma^5 - m\gamma^5)] \\
&= \text{Tr}[(p_1)_\mu \gamma^\mu \gamma^5 (-p_2)_\nu \gamma^\nu \gamma^5 + m\gamma^5 (-p_2)_\nu \gamma^\nu \gamma^5 - m\gamma^5 (p_1)_\mu \gamma^\mu \gamma^5 - \gamma^5 \gamma^5 m^2)] \\
&= \text{Tr}[(p_1)_\mu \gamma^\mu \gamma^5 (-p_2)_\nu \gamma^\nu \gamma^5] + Tr[m\gamma^5 (-\gamma^5)(-p_2)_\nu \gamma^\nu] - Tr[m\gamma^5 (-\gamma^5)(p_1)_\mu \gamma^\mu] - Tr[m^2] \\
&= \text{Tr}[(p_1)_\mu \gamma^\mu \gamma^5 (-p_2)_\nu \gamma^\nu \gamma^5] - Tr[m(-p_2)_\nu \gamma^\nu] + Tr[m(p_1)_\mu \gamma^\mu] - 4m^2 \\
&= \text{Tr}[(p_1)_\mu \gamma^\mu \gamma^5 (-p_2)_\nu \gamma^\nu \gamma^5] - 4m^2 \\
&= (p_1)_\mu (-p_2)_\nu \text{Tr}[\gamma^\mu \gamma^5 \gamma^\nu \gamma^5] - 4m^2
\end{aligned}
$$

$$
\begin{aligned}
&= (p_1)_\mu(-p_2)_\nu \mathrm{Tr}[\gamma^\mu\gamma^5(-\gamma^5\gamma^\nu)] - 4m^2 \\
&= (p_1)_\mu(-p_2)_\nu(-\mathrm{Tr}[\gamma^\mu\gamma^\nu]) - 4m^2 \\
&= (p_1)_\mu(-p_2)_\nu(-4g^{\mu\nu}) - 4m^2 \\
&= 4(p_1)(p_2) - 4m^2 \qquad (44)
\end{aligned}
$$

5. While the above identities could be shown to be trivial, the identity: $(\overline{u}_1\sigma^{\mu\nu}u_2)^* \;=\; \overline{u}_2\sigma^{\nu\mu}u_1$ is more difficult to solve. The identity:$(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$ is needed

$$
\begin{aligned}
(\sigma^{\mu\nu})^\dagger &= (\frac{i}{2}[\gamma^\mu,\gamma^\nu])^\dagger \qquad (45)\\
&= -\frac{i}{2}([\gamma^\mu,\gamma^\nu])^\dagger \\
&= -\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)^\dagger \\
&= -\frac{i}{2}((\gamma^\nu)^\dagger(\gamma^\mu)^\dagger - (\gamma^\mu)^\dagger(\gamma^\nu)^\dagger) \\
&= -\frac{i}{2}(\gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^\mu\gamma^0 - \gamma^0\gamma^\mu\gamma^0\gamma^0\gamma^\nu\gamma^0) \\
&= -\frac{i}{2}(\gamma^0\gamma^\nu\gamma^\mu\gamma^0 - \gamma^0\gamma^\mu\gamma^\nu\gamma^0) \\
&= -\frac{i}{2}((-1)^2\gamma^\nu\gamma^\mu - (-1)^2\gamma^\mu\gamma^\nu) \\
&= -\frac{i}{2}(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu) \\
&= -\sigma^{\nu\mu} \qquad (46)
\end{aligned}
$$

After showing $(\sigma^{\mu\nu})^\dagger = -\sigma^{\nu\mu}$ is true it is trivial to show $(\overline{u}_1\sigma^{\mu\nu}u_2)^* \;=\; -\,\overline{u}_2\sigma^{\nu\mu}u_1$. We let $T^{\mu\nu} \;=\; \overline{u}_1\sigma^{\mu\nu}u_2$

$$
\begin{aligned}
(T^{\mu\nu})^* &= (T^{\mu\nu})^\dagger\ , \qquad (47)\\
&= (\overline{u}_1\sigma^{\mu\nu}u_2)^\dagger \\
&= ((u_1)^\dagger\gamma^0\sigma^{\mu\nu}u_2)^\dagger \\
&= (u_2^\dagger)(\sigma^{\mu\nu})^\dagger(\gamma^0)^\dagger(u_1) \\
&= (u_2^\dagger)(-\sigma^{\nu\mu})\gamma^0(u_1) \\
&= (u_2^\dagger)(-\gamma^0)(-\sigma^{\nu\mu})(u_1) \\
&= \overline{u}_2\sigma^{\mu\nu}(u_1) \qquad (48)
\end{aligned}
$$

In order to find the value of $|T^{\mu\nu}|^2$ one needs to find the value of $Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}]$

$$
\begin{aligned}
Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] &= Tr[\frac{i}{2}(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma)\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)] \qquad (49)\\
&= Tr[\frac{i}{2}(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma)\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)] \\
&= -\frac{1}{4}Tr[(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma)(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)]
\end{aligned}
$$

$$= -\frac{1}{4}Tr[\gamma^\sigma\gamma^\lambda\gamma^\mu\gamma^\nu] + \frac{1}{4}Tr[\gamma^\sigma\gamma^\lambda\gamma^\nu\gamma^\mu] + \frac{1}{4}Tr[\gamma^\lambda\gamma^\sigma\gamma^\mu\gamma^\nu] - \frac{1}{4}Tr[\gamma^\lambda\gamma^\sigma\gamma^\nu\gamma^\mu]$$

Here we must label each of the traces individually:

$$\begin{aligned} A &= -\frac{1}{4}Tr[\gamma^\sigma\gamma^\lambda\gamma^\mu\gamma^\nu] \\ &= -(g^{\sigma\lambda}g^{\mu\nu} - g^{\sigma\mu}g^{\lambda\nu} + g^{\sigma\nu}g^{\lambda\mu}) \end{aligned} \tag{50}$$

$$\begin{aligned} B &= +\frac{1}{4}Tr[\gamma^\sigma\gamma^\lambda\gamma^\nu\gamma^\mu] \\ &= +(g^{\sigma\lambda}g^{\nu\mu} - g^{\sigma\nu}g^{\lambda\mu} + g^{\sigma\mu}g^{\lambda\nu}) \end{aligned} \tag{51}$$

$$\begin{aligned} C &= \frac{1}{4}Tr[\gamma^\lambda\gamma^\sigma\gamma^\mu\gamma^\nu] \\ &= (g^{\lambda\sigma}g^{\mu\nu} - g^{\lambda\mu}g^{\sigma\nu} + g^{\lambda\nu}g^{\sigma\mu}) \end{aligned} \tag{52}$$

$$\begin{aligned} D &= -\frac{1}{4}Tr[\gamma^\lambda\gamma^\sigma\gamma^\nu\gamma^\mu] \\ &= -(g^{\lambda\sigma}g^{\nu\mu} - g^{\lambda\nu}g^{\sigma\mu} + g^{\lambda\mu}g^{\sigma\nu}) \end{aligned} \tag{53}$$

$$\begin{aligned} Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] &= A + B + +C + D \\ &= 2g^{\sigma\mu}g^{\lambda\nu} - 2g^{\sigma\nu}g^{\lambda\mu} - 2g^{\lambda\mu}g^{\sigma\nu} + 2g^{\lambda\nu}g^{\sigma\mu}) \end{aligned} \tag{54}$$

In order to find the value of $|T^{\mu\nu}|^2$ one needs to do the following:

$$\begin{aligned} |T^{\mu\nu}|^2 &= \mathrm{Tr}[\overline{u}_1\sigma^{\mu\nu}u_2\overline{u}_2\sigma^{\sigma\lambda}u_1] \qquad (55) \\ &= \mathrm{Tr}[\overline{u}_1\sigma^{\mu\nu}(\not{p}_2 + m)\sigma^{\sigma\lambda}u_1] \\ &= \mathrm{Tr}[(\not{p}_1 + m)\sigma^{\mu\nu}(\not{p}_2 + m)\sigma^{\sigma\lambda}] \\ &= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa + m)\sigma^{\mu\nu}((p_2)_\gamma\gamma^\gamma + m)\sigma^{\sigma\lambda}] \\ &= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu} + m\sigma^{\mu\nu})((p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda} + m\sigma^{\sigma\lambda})] \\ &= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda} + (p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}m\sigma^{\sigma\lambda} + (p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}m\sigma^{\mu\nu} + m\sigma^{\sigma\lambda}m\sigma^{\mu\nu})] \\ &= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}] + Tr[(p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}m\sigma^{\sigma\lambda} + (p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}m\sigma^{\mu\nu}] + Tr[m\sigma^{\sigma\lambda}m\sigma^{\mu\nu}] \\ &= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}] + Tr[m\sigma^{\sigma\lambda}m\sigma^{\mu\nu}] \\ &= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}] + m^2Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] \text{ Let B } = \mathrm{m}^2\mathrm{Tr}[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] \\ &= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}] + B \\ &= (p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[\gamma^\kappa\sigma^{\mu\nu}\gamma^\gamma\sigma^{\sigma\lambda}] + B \\ &= (p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[\gamma^\kappa(\frac{i}{2}((\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma^\gamma(\frac{i}{2}(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma))] + B \\ &= -\frac{1}{4}(p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[\gamma^\kappa(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma^\gamma(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma)] + B \text{ (Let A } = -\frac{1}{4}(\mathrm{p}_1)_\kappa(\mathrm{p}_2)_\gamma) \end{aligned}$$

$$
\begin{aligned}
&= (A)\mathrm{Tr}[(\gamma^\kappa\gamma^\mu\gamma^\nu - \gamma^\kappa\gamma^\nu\gamma^\mu)(\gamma^\gamma\gamma^\sigma\gamma^\lambda - \gamma^\gamma\gamma^\lambda\gamma^\sigma)] + B \\
&= (A)\mathrm{Tr}[(\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda - \gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\lambda\gamma^\sigma - \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda + \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\lambda\gamma^\sigma)] + B \\
&= (A)\mathrm{Tr}[(\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda - \gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma(-\gamma^\sigma\gamma^\lambda) - \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda + \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma(-\gamma^\sigma\gamma^\lambda))] + B \\
&= (A)\mathrm{Tr}[(\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda + \gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda - \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda - \gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda))] + B \\
&= (A)\mathrm{Tr}[2\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda - 2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda] + B \\
&= (A)\mathrm{Tr}[2\gamma^\kappa(-\gamma^\nu\gamma^\mu)\gamma^\gamma\gamma^\sigma\gamma^\lambda - 2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda] + B \\
&= (A)\mathrm{Tr}[-2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda - 2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda] + B \\
&= (A)\mathrm{Tr}[-4\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda] + B \\
&= -\frac{1}{4}(p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[-4\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda] + B \\
&= (p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[\gamma^\kappa\gamma^\gamma] + B \\
&= 4(p_1)_\kappa(p_2)_\gamma g^{\kappa\gamma} + B \\
&= 4p_1p_2 + m^2Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] \\
&= 4p_1p_2 + m^2 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) \qquad (56)
\end{aligned}
$$