Practice with Spinors

I Algebra with Dirac γ matrices

Notation:

1. $S = \overline{u}u$
2. $P = \overline{u}\gamma^5 u$
3. $V^\mu = \overline{u}\gamma^\mu u$
4. $A^\mu = \overline{u}\gamma^\mu\gamma^5 u$
5. $T^{\mu\nu} = \overline{u}\sigma^{\mu\nu} u$

1. $(\overline{u}_1\gamma^\mu u_2)^* \ = ?$

   Note: $(\gamma^0)^\dagger \ = \ \gamma^0$ and $(\gamma^\mu)^\dagger \ = \ \gamma^0\gamma^\mu\gamma^0$

   $(\overline{u}_1\gamma^\mu u_2)$ is a 1×1 matrix. Therefore, its complex conjugate is the same as its Hermitian conjugate, i.e. if we call $L^\mu = (\overline{u}_1\gamma^\mu u_2)$, then $(L^\mu)^* = (L^\mu)^\dagger$. We can then express this quantity as follows:

$$\begin{aligned}
L^\mu &= \overline{u}_1\gamma^\mu u_2 \ , \\
\Rightarrow (L^\mu)^* &= (L^\mu)^\dagger \ , \\
&= (\overline{u}_1\gamma^\mu u_2)^\dagger \ , \\
&= ((u_1)^\dagger\gamma^0\gamma^\mu u_2)^\dagger \quad \text{using } (A\ldots Z)^\dagger = Z^\dagger\ldots A^\dagger \ , \\
&= (u_2^\dagger)(\gamma^\mu)^\dagger(\gamma^0)^\dagger(u_1) \\
&= (u_2^\dagger)\gamma^0\gamma^\mu\gamma^0\gamma^0(u_1) \\
&= (u_2^\dagger)\gamma^0\gamma^\mu(u_1) \\
&= \overline{u}_2\gamma^\mu(u_1) \qquad (1)
\end{aligned}$$

   Therefore $(\overline{u}_1\gamma^\mu u_2)^* \ = \ \overline{u}_2\gamma^\mu u_1$. To solve for $|L^\mu|^2$ we simply use $|L^\mu|^2 \ = \ \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1]$.

   Question: Do not erase! Answer it.

   $L^\mu \ = \ \overline{u}_1\gamma^\mu u_2$ clearly has one Lorentz index μ. But, $|L^\mu|^2 \ = \ \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1]$ can be written as some other quantity $\mathcal{L}^{\mu\nu}$, i.e. it has two Lorentz indices μ and ν. Why?

   Note: $\mathrm{Tr}[\gamma^\mu\gamma^\nu] \ = \ 4g^{\mu\nu}$, $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$, The trace over the product of an odd number of gamma matrices is zero.

$$\begin{aligned}
|L^\mu|^2 &= \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1] \\
&= \mathrm{Tr}[\overline{u}_1\gamma^\mu(\not{p}_2 + m)\gamma^\nu u_1] \\
&= \mathrm{Tr}[u_1\overline{u}_1\gamma^\mu(\not{p}_2 + m)\gamma^\nu]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Tr}[(\not{p}_1 + m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu] \\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] + m[\mathrm{Tr}(\gamma^\mu\not{p}_1\gamma^\nu) + \mathrm{Tr}(\gamma^\mu\gamma^\nu\not{p}_2)] + m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] \\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] + m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] \\
&= \mathrm{Tr}[(p_1)_\lambda\gamma^\lambda\gamma^\mu(p_2)_\sigma\gamma^\sigma\gamma^\nu] + 4m^2g^{\mu\nu} \\
&= (p_1)_\lambda(p_2)_\sigma\mathrm{Tr}[\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu] + 4m^2g^{\mu\nu} \\
&= (p_1)_\lambda(p_2)_\sigma 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) + 4m^2g^{\mu\nu} \\
&= 4[p_1^\mu p_2^\nu - g^{\mu\nu}(p_1\cdot p_2) + p_2^\mu p_1^\nu] + 4m^2g^{\mu\nu} \\
&= 4[p_1^\mu p_2^\nu - 4g^{\mu\nu}(p_1\cdot p_2 + m^2) + p_2^\mu p_1^\nu] \qquad (2)
\end{aligned}
$$

2. $(\overline{u}_1\gamma^\mu\gamma^5u_2)^*$ is also a 1×1 Matrix so the same reasoning applies as above in 1. Note: $(\gamma^5)^\dagger = \gamma^5$ We define: R^μ as $\overline{u}_1\gamma^\mu\gamma^5u_2$ thus:

$$
\begin{aligned}
(R^\mu)^* &= (R^\mu)^\dagger \\
&= (\overline{u}_1\gamma^\mu\gamma^5u_2)^\dagger \\
&= ((u_1)^\dagger\gamma^0\gamma^\mu\gamma^5u_2)^\dagger \\
&= (u_2^\dagger)(\gamma^5)^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger(u_1) \\
&= (u_2^\dagger)\gamma^5\gamma^0\gamma^\mu\gamma^0\gamma^0u_1 \\
&= (u_2^\dagger)\gamma^5\gamma^0\gamma^\mu(1)u_1 \\
&= -(u_2^\dagger)\gamma^0\gamma^5\gamma^\mu u_1 \\
&= -\overline{u}_2\gamma^5\gamma^\mu u_1 \\
&= \overline{u}_2\gamma^\mu\gamma^5u_1 \qquad (3)
\end{aligned}
$$

Therefore $(\overline{u}_1\gamma^\mu\gamma^5u_2)^* \;=\; \overline{u}_2\gamma^\mu\gamma^5u_1$

We also are able to calculate $|R^\mu|^2$

$$
\begin{aligned}
|R^\mu|^2 &= \mathrm{Tr}[(\overline{u}_1\gamma^\mu\gamma^5u_2)(\overline{u}_2\gamma^\nu\gamma^5u_1)] \\
&= \mathrm{Tr}[\overline{u}_1\gamma^\mu\gamma^5(\not{p}_2 + m)\gamma^\nu\gamma^5u_1] \\
&= \mathrm{Tr}[u_1\overline{u}_1\gamma^\mu\gamma^5(\not{p}_2 + m)\gamma^\nu\gamma^5] \\
&= \mathrm{Tr}[(\not{p}_1 + m)\gamma^\mu\gamma^5(\not{p}_2 + m)\gamma^\nu\gamma^5] \\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5 + m(\not{p}_1\gamma^\mu\gamma^5\gamma^\nu\gamma^5 + \gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5) + m^2(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)] \\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5 + m^2(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)] \\
&= \mathrm{Tr}[(p_1)_\lambda\gamma^\lambda\gamma^\mu\gamma^5(p_2)_\sigma\gamma^\sigma\gamma^\nu\gamma^5] + m^2\mathrm{Tr}[\gamma^\mu\gamma^5\gamma^\nu\gamma^5] \\
&= (p_1)_\lambda(p_2)_\sigma\mathrm{Tr}[\gamma^\lambda\gamma^\mu\gamma^5\gamma^\sigma\gamma^\nu\gamma^5] - m^2\mathrm{Tr}[\gamma^\mu\gamma^5\gamma^5\gamma^\nu] \\
&= (p_1)_\lambda(p_2)_\sigma\mathrm{Tr}[\gamma^\lambda\gamma^\mu\gamma^5\gamma^5\gamma^\sigma\gamma^\nu] - m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] \\
&= (p_1)_\lambda(p_2)_\sigma\mathrm{Tr}[\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu] - m^2(g^{\mu\nu}) \\
&= (p_1)_\lambda(p_2)_\sigma 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) - 4m^2g^{\mu\nu} \\
&= 4[p_1^\mu p_2^\nu - g^{\mu\nu}(p_1\cdot p_2) + p_2^\mu p_1^\nu] - 4m^2g^{\mu\nu} \\
&= 4[p_1^\mu p_2^\nu - 4g^{\mu\nu}(p_1\cdot p_2 - m^2) + p_2^\mu p_1^\nu] \qquad (4)
\end{aligned}
$$

3. $(\overline{u}_1 u_2)^* \;=\; ?$ We let:$P \;=\; \overline{u}_1 u_2$

$$\begin{aligned}
(P)^* &= (P)^\dagger \\
&= (\overline{u}_1 u_2)^\dagger \;, \\
&= ((u_1)^\dagger \gamma^0 u_2)^\dagger \\
&= (u_2)^\dagger (\gamma^0)^\dagger (u_1) \\
&= (u_2)^\dagger \gamma^0 (u_1) \\
&= \overline{u}_2 (u_1) \qquad (5)
\end{aligned}$$

Therefore$(\overline{u}_1 u_2)^* \;=\; \overline{u}_2 u_1$. In order to find $|P|^2$ we simply do the following:

$$\begin{aligned}
|P|^2 &= \mathrm{Tr}[\overline{u}_1 u_2 \overline{u}_2 u_1] \qquad (6) \\
&= \mathrm{Tr}[(\not{p}_1 + m)(\not{p}_2 + m)] \\
&= \mathrm{Tr}[\not{p}_1 \not{p}_2 + m(\not{p}_1 + \not{p}_2) + m^2] \\
&= \mathrm{Tr}[\not{p}_1 \not{p}_2] + \mathrm{Tr}[m(\not{p}_1 + \not{p}_2)] + \mathrm{Tr}[m^2] \\
&= \mathrm{Tr}[\not{p}_1 \not{p}_2] + m(\mathrm{Tr}[\not{p}_1] + \mathrm{Tr}[\not{p}_2]) + m^2 \mathrm{Tr}[1] \\
&= \mathrm{Tr}[\not{p}_1 \not{p}_2] + 4m^2 \\
&= 4(p_1 \cdot p_2) + 4m^2 \qquad (7)
\end{aligned}$$

4. By the same reasoning as shown above it can be shown that $(\overline{u}_1 \gamma^5 u_2)^* \;=\; \overline{u}_2 \gamma^5 u_1$
   If we let $T \;=\; \overline{u}_1 \gamma^5 u_2$ then:

$$\begin{aligned}
\Rightarrow (T)^* &= (T)^\dagger \qquad (8) \\
&= (\overline{u}_1 \gamma^5 u_2)^\dagger \;, \\
&= ((u_1)^\dagger \gamma^0 \gamma^5 u_2)^\dagger \\
&= (u_2)^\dagger (\gamma^5)^\dagger (\gamma^0)^\dagger (u_1) \\
&= (u_2)^\dagger (-\gamma^5) \gamma^0 (u_1) \\
&= (u_2)^\dagger \gamma^0 \gamma^5 (u_1) \\
&= \overline{u}_2 \gamma^5 (u_1) \qquad (9)
\end{aligned}$$

   Therefore $(\overline{u}_1 \gamma^5 u_2)^* \;=\; \overline{u}_2 \gamma^5 u_1$

   In order to square T we do the following:

$$\begin{aligned}
|T|^2 &= \mathrm{Tr}[\overline{u}_1 \gamma^5 u_2 \overline{u}_2 \gamma^5 u_1] \qquad (10) \\
&= \mathrm{Tr}[u_1 \overline{u}_1 \gamma^5 \not{p}_2 \gamma^5] \\
&= \mathrm{Tr}[\not{p}_1 \gamma^5 \not{p}_2 \gamma^5] \\
&= \mathrm{Tr}[(p_1)_\mu \gamma^\mu \gamma^5 (p_2)_\nu \gamma^\nu \gamma^5] \\
&= (p_1)_\mu (p_2)_\nu \mathrm{Tr}[\gamma^\mu \gamma^5 \gamma^\nu \gamma^5] \\
&= -(p_1)_\mu (p_2)_\nu \mathrm{Tr}[\gamma^\mu \gamma^5 \gamma^5 \gamma \nu] \\
&= -(p_1)_\mu (p_2)_\nu \mathrm{Tr}[\gamma^\mu \gamma^\nu]
\end{aligned}$$

$$\begin{aligned} &= -(p_1)_\mu (p_2)_\nu (4g^{\mu\nu}) \\ &= -4(p_1)(p_2) \end{aligned} \tag{11}$$

5. While the above identities could be shown to be trivial, the identity: $(\overline{u}_1\sigma^{\mu\nu}u_2)^* = \overline{u}_2\sigma^{\mu\nu}u_1$ is more difficult to solve. The identity:$(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$ is needed

$$(\sigma^{\mu\nu})^\dagger = (\frac{i}{2}[\gamma^\mu, \gamma^\nu])^\dagger \tag{12}$$

$$\begin{aligned} &= (u_2)^\dagger(\sigma^{\mu\nu})^\dagger(\gamma^0)^\dagger u_1 \\ &= \frac{i}{2}([\gamma^\mu, \gamma^\nu])^\dagger \\ &= \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)^\dagger \\ &= \frac{i}{2}((\gamma^\mu)^\dagger(\gamma^\nu)^\dagger - (\gamma^\nu)^\dagger(\gamma^\mu)^\dagger) \\ &= \frac{i}{2}(\gamma^0\gamma^\mu\gamma^0\gamma^0\gamma^\nu\gamma^0 - \gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^\mu\gamma^0) \\ &= \frac{i}{2}(\gamma^0\gamma^\mu\gamma^\nu\gamma^0 - \gamma^0\gamma^\nu\gamma^\mu\gamma^0) \\ &= \frac{i}{2}((-1)^2\gamma^\mu\gamma^\nu - (-1)^2\gamma^\nu\gamma^\mu) \\ &= \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \\ &= \sigma^{\mu\nu} \end{aligned} \tag{13}$$

After showing $(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$ is true it is trivial to show $(\overline{u}_1\sigma^{\mu\nu}u_2)^* = \overline{u}_2\sigma^{\mu\nu}u_1$. We let $B^{\mu\nu} = \overline{u}_1\sigma^{\mu\nu}u_2$

$$(B^{\mu\nu})^* = (B^{\mu\nu})^\dagger \ , \tag{14}$$

$$\begin{aligned} &= (\overline{u}_1\sigma^{\mu\nu}u_2)^\dagger \\ &= ((u_1)^\dagger\gamma^0\sigma^{\mu\nu}u_2)^\dagger \\ &= (u_2^\dagger)(\sigma^{\mu\nu})^\dagger(\gamma^0)^\dagger(u_1) \\ &= (u_2^\dagger)\sigma^{\mu\nu}\gamma^0(u_1) \\ &= (u_2^\dagger)(\gamma^0)\sigma^{\mu\nu}(u_1) \\ &= \overline{u}_2\sigma^{\mu\nu}(u_1) \end{aligned} \tag{15}$$

An interesting thing to note is that the expression $\overline{u}\sigma^{\mu\nu}\gamma^5 u$ is not an independent quantity. Since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ it follows that the product of $\sigma^{\mu\nu}$ and γ^5 can be simplified to an expression with only 2 γ matrices which has been defined as a pseudoscalar. For example, let $\mu = 0$ and $\nu = 1$:

$$\overline{u}\sigma^{01}\gamma^5 u = \overline{u}\sigma^{01}(i\gamma^0\gamma^1\gamma^2\gamma^3)u \tag{16}$$

$$= \overline{u}((\frac{i}{2})(\gamma^0\gamma^1 - \gamma^1\gamma^0))(i\gamma^0\gamma^1\gamma^2\gamma^3)u$$

$$
\begin{aligned}
&= \overline{u}(\frac{i}{2})[\gamma^0\gamma^1(i\gamma^0\gamma^1\gamma^2\gamma^3) - \gamma^1\gamma^0(i\gamma^0\gamma^1\gamma^2\gamma^3)]u \\
&= \overline{u}(\frac{-1}{2})[\gamma^0\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 - \gamma^1\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3)]u \\
&= \overline{u}(\frac{-1}{2})[-\gamma^2\gamma^3 - \gamma^2\gamma^3)]u \\
&= \overline{u}(\frac{-1}{2})[-2\gamma^2\gamma^3]u \\
&= 2\overline{u}\gamma^2\gamma^3 u \qquad (17)
\end{aligned}
$$

This expression (because it contains two gamma matrices) is a pseudoscalar. Any values of μ and ν can be shown to be similar to this because of the communal and identity properties of the gamma matrices.

In order to find the value of $|B^{\mu\nu}|^2$ one needs to do the following:

$$
\begin{aligned}
|B^{\mu\nu}|^2 &= \mathrm{Tr}[\overline{u}_1\sigma^{\mu\nu}u_2\overline{u}_2\sigma^{\sigma\lambda}u_1] \\
&= \mathrm{Tr}[\overline{u}_1\sigma^{\mu\nu}\not{p}_2\sigma^{\sigma\lambda}u_1] \\
&= \mathrm{Tr}[\not{p}_1\sigma^{\mu\nu}\not{p}_2\sigma^{\sigma\lambda}] \\
&= \mathrm{Tr}[p_\kappa\gamma^\kappa\sigma^{\mu\nu}p_\gamma\gamma^\gamma\sigma^{\sigma\lambda}] \\
&= p_\kappa p_\gamma\mathrm{Tr}[\gamma^\kappa\sigma^{\mu\nu}\gamma^\gamma\sigma^{\sigma\lambda}] \\
&= -\frac{1}{2}p_\kappa p_\gamma\mathrm{Tr}[\gamma^\kappa(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma^\gamma(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma)] \\
&= -\frac{1}{2}p_\kappa p_\gamma\mathrm{Tr}[2\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda - 2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda] \\
&= -\frac{1}{2}p_\kappa p_\gamma\mathrm{Tr}[4\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda] \\
&= -2p_\kappa p_\gamma\mathrm{Tr}[\gamma^\kappa\gamma^\gamma] \text{ Using a similar identity as shown in eq. 23} \\
&= -8p_\kappa p_\gamma g^{\kappa\gamma} \qquad (18)
\end{aligned}
$$