Practice with Spinors

I Algebra with Dirac γ matrices

Notation:

1. $S = \overline{u}u$
2. $P = \overline{u}\gamma^5 u$
3. $V^\mu = \overline{u}\gamma^\mu u$
4. $A^\mu = \overline{u}\gamma^\mu\gamma^5 u$
5. $T^{\mu\nu} = \overline{u}\sigma^{\mu\nu}u$

1. $(\overline{u}_1\gamma^\mu u_2)^* \ = ?$

   Note: $(\gamma^0)^\dagger \ = \ \gamma^0$ and $(\gamma^\mu)^\dagger \ = \ \gamma^0\gamma^\mu\gamma^0$

   $(\overline{u}_1\gamma^\mu u_2)$ is a 1×1 matrix. Therefore, its complex conjugate is the same as its Hermitian conjugate, i.e. if we call $L^\mu = (\overline{u}_1\gamma^\mu u_2)$, then $(L^\mu)^* = (L^\mu)^\dagger$. We can then express this quantity as follows:

$$\begin{aligned} L^\mu &= \overline{u}_1\gamma^\mu u_2 \ , \\ \Rightarrow (L^\mu)^* &= (L^\mu)^\dagger \ , \\ &= (\overline{u}_1\gamma^\mu u_2)^\dagger \ , \\ &= ((u_1)^\dagger\gamma^0\gamma^\mu u_2)^\dagger \quad \text{using } (A\ldots Z)^\dagger = Z^\dagger\ldots A^\dagger \ , \\ &= (u_2^\dagger)(\gamma^\mu)^\dagger(\gamma^0)^\dagger(u_1) \\ &= (u_2^\dagger)\gamma^0\gamma^\mu\gamma^0\gamma^0(u_1) \\ &= (u_2^\dagger)\gamma^0\gamma^\mu(u_1) \\ &= \overline{u}_2\gamma^\mu(u_1) \end{aligned} \tag{1}$$

   Therefore $(\overline{u}_1\gamma^\mu u_2)^* \ = \ \overline{u}_2\gamma^\mu u_1$. To solve for $|L^\mu|^2$ we simply use $|L^\mu|^2 \ = \ \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1]$.

   Question: Do not erase! Answer it.

   $L^\mu \ = \ \overline{u}_1\gamma^\mu u_2$ clearly has one Lorentz index μ. But, $|L^\mu|^2 \ = \ \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1]$ can be written as some other quantity $\mathcal{L}^{\mu\nu}$, i.e. it has two Lorentz indices μ and ν. Why? The reason is because you cannot have two of the same indices written together in the same line, if so they cannot be contracted due to index notation rules.
   But what's wrong with $\mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma_\mu u_1]$? Why increase the number of indices and not decrease it?

   If you have a term such as $\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma_\mu u_1$ you need to state explicitly that the indices are not summed because that is the convention with an expression such as $\gamma^\mu\gamma_\mu$.

That is not correct. $\gamma^\mu\gamma_\mu = \gamma^\mu\gamma^\nu g_{\mu\nu} = \gamma^0\gamma^0 - \gamma^1\gamma^1 - \gamma^2\gamma^2 - \gamma^3\gamma^3$. This is just an expression. The question is why is the left-hand side not the same as the right-hand side?

Also with the upper and lowering of matrices that implies a change of basis since you cannot arbitrarily raise/lower an index. Nothing wrong with raising/lowering indices: see above.

Why is $|L^\mu|^2 \neq \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma_\mu u_1]$?
Why is $|L^\mu|^2 = \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1]$?

You start with one index μ. When you square why do you get two indices and not a sum over two of the same index?
When you square you have two indices because you must increase the number of components. The number of components when you square should go as n^2 not simply n. When you have one index, you restrict the number of components, because you have 4 components, and not 16.
Note: $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$, $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$, The trace over the product of an odd number of gamma matrices is zero.

$$\begin{aligned}
|L^\mu|^2 &= \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1] \\
&= \mathrm{Tr}[\overline{u}_1\gamma^\mu(\not{p}_2 + m)\gamma^\nu u_1] \\
&= \mathrm{Tr}[u_1\overline{u}_1\gamma^\mu(\not{p}_2 + m)\gamma^\nu] \\
&= \mathrm{Tr}[(\not{p}_1 + m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu] \\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] + m[\mathrm{Tr}(\gamma^\mu\not{p}_1\gamma^\nu) + \mathrm{Tr}(\gamma^\mu\gamma^\nu\not{p}_2)] + m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] \\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] + m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] \\
&= \mathrm{Tr}[(p_1)_\lambda\gamma^\lambda\gamma^\mu(p_2)_\sigma\gamma^\sigma\gamma^\nu] + 4m^2g^{\mu\nu} \\
&= (p_1)_\lambda(p_2)_\sigma\mathrm{Tr}[\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu] + 4m^2g^{\mu\nu} \\
&= (p_1)_\lambda(p_2)_\sigma 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) + 4m^2g^{\mu\nu} \\
&= 4[p_1^\mu p_2^\nu - g^{\mu\nu}(p_1\cdot p_2) + p_2^\mu p_1^\nu] + 4m^2g^{\mu\nu} \\
&= 4[p_1^\mu p_2^\nu - 4g^{\mu\nu}(p_1\cdot p_2 + m^2) + p_2^\mu p_1^\nu]
\end{aligned} \tag{2}$$

2. $(\overline{u}_1\gamma^\mu\gamma^5u_2)^*$ is also a 1×1 Matrix so the same reasoning applies as above in 1. Note: $(\gamma^5)^\dagger = \gamma^5$ We define: R^μ as $\overline{u}_1\gamma^\mu\gamma^5u_2$ thus:

$$\begin{aligned}
(R^\mu)^* &= (R^\mu)^\dagger \\
&= (\overline{u}_1\gamma^\mu\gamma^5u_2)^\dagger \\
&= ((u_1)^\dagger\gamma^0\gamma^\mu\gamma^5u_2)^\dagger \\
&= (u_2^\dagger)(\gamma^5)^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger(u_1) \\
&= (u_2^\dagger)\gamma^5\gamma^0\gamma^\mu\gamma^0\gamma^0u_1 \\
&= (u_2^\dagger)\gamma^5\gamma^0\gamma^\mu(1)u_1 \\
&= -(u_2^\dagger)\gamma^0\gamma^5\gamma^\mu u_1 \\
&= -\overline{u}_2\gamma^5\gamma^\mu u_1
\end{aligned}$$

$$= \quad \overline{u}_2\gamma^\mu\gamma^5 u_1 \tag{3}$$

Therefore $(\overline{u}_1\gamma^\mu\gamma^5 u_2)^* \;=\; \overline{u}_2\gamma^\mu\gamma^5 u_1$

We also are able to calculate $|R^\mu|^2$

$$\begin{aligned}
|R^\mu|^2 &= \mathrm{Tr}[(\overline{u}_1\gamma^\mu\gamma^5 u_2)(\overline{u}_2\gamma^\nu\gamma^5 u_1)] \\
&= \mathrm{Tr}[\overline{u}_1\gamma^\mu\gamma^5(\not{p}_2+m)\gamma^\nu\gamma^5 u_1] \\
&= \mathrm{Tr}[u_1\overline{u}_1\gamma^\mu\gamma^5(\not{p}_2+m)\gamma^\nu\gamma^5] \\
&= \mathrm{Tr}[(\not{p}_1+m)\gamma^\mu\gamma^5(\not{p}_2+m)\gamma^\nu\gamma^5] \\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5 + m(\not{p}_1\gamma^\mu\gamma^5\gamma^\nu\gamma^5 + \gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5) + m^2(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)] \\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5 + m^2(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)] \\
&= \mathrm{Tr}[(p_1)_\lambda\gamma^\lambda\gamma^\mu\gamma^5(p_2)_\sigma\gamma^\sigma\gamma^\nu\gamma^5] + m^2\mathrm{Tr}[\gamma^\mu\gamma^5\gamma^\nu\gamma^5] \\
&= (p_1)_\lambda(p_2)_\sigma\mathrm{Tr}[\gamma^\lambda\gamma^\mu\gamma^5\gamma^\sigma\gamma^\nu\gamma^5] - m^2\mathrm{Tr}[\gamma^\mu\gamma^5\gamma^5\gamma^\nu] \\
&= (p_1)_\lambda(p_2)_\sigma\mathrm{Tr}[\gamma^\lambda\gamma^\mu\gamma^5\gamma^5\gamma^\sigma\gamma^\nu] - m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] \\
&= (p_1)_\lambda(p_2)_\sigma\mathrm{Tr}[\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu] - m^2(g^{\mu\nu}) \\
&= (p_1)_\lambda(p_2)_\sigma 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) - 4m^2 g^{\mu\nu} \\
&= 4[p_1^\mu p_2^\nu - g^{\mu\nu}(p_1\cdot p_2) + p_2^\mu p_1^\nu] - 4m^2 g^{\mu\nu} \\
&= 4[p_1^\mu p_2^\nu - 4g^{\mu\nu}(p_1\cdot p_2 - m^2) + p_2^\mu p_1^\nu]
\end{aligned} \tag{4}$$

3. $(\overline{u}_1 u_2)^* \;=\; ?$ We let:$P \;=\; \overline{u}_1 u_2$

$$\begin{aligned}
(P)^* &= (P)^\dagger \\
&= (\overline{u}_1 u_2)^\dagger\ , \\
&= ((u_1)^\dagger\gamma^0 u_2)^\dagger \\
&= (u_2)^\dagger(\gamma^0)^\dagger(u_1) \\
&= (u_2)^\dagger\gamma^0(u_1) \\
&= \overline{u}_2(u_1)
\end{aligned} \tag{5}$$

Therefore$(\overline{u}_1 u_2)^* \;=\; \overline{u}_2 u_1$. In order to find $|P|^2$ we simply do the following:

$$|P|^2 = \mathrm{Tr}[\overline{u}_1 u_2\overline{u}_2 u_1] \tag{6}$$

$$\begin{aligned}
&= \mathrm{Tr}[(\not{p}_1+m)(\not{p}_2+m)] \\
&= \mathrm{Tr}[\not{p}_1\not{p}_2 + m(\not{p}_1+\not{p}_2) + m^2] \\
&= \mathrm{Tr}[\not{p}_1\not{p}_2] + \mathrm{Tr}[m(\not{p}_1+\not{p}_2)] + \mathrm{Tr}[m^2] \\
&= \mathrm{Tr}[\not{p}_1\not{p}_2] + m(\mathrm{Tr}[\not{p}_1] + \mathrm{Tr}[\not{p}_2]) + m^2\mathrm{Tr}[1] \\
&= \mathrm{Tr}[\not{p}_1\not{p}_2] + 4m^2 \\
&= 4(p_1\cdot p_2) + 4m^2
\end{aligned} \tag{7}$$

4. By the same reasoning as shown above it can be shown that $(\overline{u}_1\gamma^5 u_2)^* \;=\; \overline{u}_2\gamma^5 u_1$
If we let $T \;=\; \overline{u}_1\gamma^5 u_2$ then:

$$\Rightarrow (T)^* \;=\; (T)^\dagger \tag{8}$$

$$
\begin{aligned}
&= (\overline{u}_1\gamma^5 u_2)^\dagger\ , \\
&= ((u_1)^\dagger\gamma^0\gamma^5 u_2)^\dagger \\
&= (u_2)^\dagger(\gamma^5)^\dagger(\gamma^0)^\dagger(u_1) \\
&= (u_2)^\dagger(-\gamma^5)\gamma^0(u_1) \\
&= (u_2)^\dagger\gamma^0\gamma^5(u_1) \\
&= \overline{u}_2\gamma^5(u_1) \qquad (9)
\end{aligned}
$$

Therefore $(\overline{u}_1\gamma^5 u_2)^* \ = \ \overline{u}_2\gamma^5 u_1$

In order to square T we do the following:

$$
\begin{aligned}
|T|^2 &= \mathrm{Tr}[\overline{u}_1\gamma^5 u_2\overline{u}_2\gamma^5 u_1] \qquad (10) \\
&= \mathrm{Tr}[u_1\overline{u}_1\gamma^5 \not{p}_2\gamma^5] \\
&= \mathrm{Tr}[\not{p}_1\gamma^5\not{p}_2\gamma^5] \\
&= \mathrm{Tr}[(p_1)_\mu\gamma^\mu\gamma^5(p_2)_\nu\gamma^\nu\gamma^5] \\
&= (p_1)_\mu(p_2)_\nu\mathrm{Tr}[\gamma^\mu\gamma^5\gamma^\nu\gamma^5] \\
&= -(p_1)_\mu(p_2)_\nu\mathrm{Tr}[\gamma^\mu\gamma^5\gamma^5\gamma\nu] \\
&= -(p_1)_\mu(p_2)_\nu\mathrm{Tr}[\gamma^\mu\gamma^\nu] \\
&= -(p_1)_\mu(p_2)_\nu(4g^{\mu\nu}) \\
&= -4(p_1)(p_2)
\end{aligned}
$$

$$(11)$$

5. While the above identities could be shown to be trivial, the identity: $(\overline{u}_1\sigma^{\mu\nu}u_2)^* \ = \ \overline{u}_2\sigma^{\mu\nu}u_1$ is more difficult to solve. The identity:$(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$ is needed

$$
\begin{aligned}
(\sigma^{\mu\nu})^\dagger &= (\frac{i}{2}[\gamma^\mu,\gamma^\nu])^\dagger \qquad (12) \\
&= (u_2)^\dagger(\sigma^{\mu\nu})^\dagger(\gamma^0)^\dagger u_1 \\
&= \frac{i}{2}([\gamma^\mu,\gamma^\nu])^\dagger \\
&= \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)^\dagger \\
&= \frac{i}{2}((\gamma^\mu)^\dagger(\gamma^\nu)^\dagger - (\gamma^\nu)^\dagger(\gamma^\mu)^\dagger) \\
&= \frac{i}{2}(\gamma^0\gamma^\mu\gamma^0\gamma^0\gamma^\nu\gamma^0 - \gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^\mu\gamma^0) \\
&= \frac{i}{2}(\gamma^0\gamma^\mu\gamma^\nu\gamma^0 - \gamma^0\gamma^\nu\gamma^\mu\gamma^0) \\
&= \frac{i}{2}((-1)^2\gamma^\mu\gamma^\nu - (-1)^2\gamma^\nu\gamma^\mu) \\
&= \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \\
&= \sigma^{\mu\nu} \qquad (13)
\end{aligned}
$$

After showing $(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$ is true it is trivial to show $(\overline{u}_1\sigma^{\mu\nu}u_2)^* = \overline{u}_2\sigma^{\mu\nu}u_1$. We let $B^{\mu\nu} = \overline{u}_1\sigma^{\mu\nu}u_2$

$$
\begin{aligned}
(B^{\mu\nu})^* &= (B^{\mu\nu})^\dagger \ , \qquad (14)\\
&= (\overline{u}_1\sigma^{\mu\nu}u_2)^\dagger\\
&= ((u_1)^\dagger\gamma^0\sigma^{\mu\nu}u_2)^\dagger\\
&= (u_2^\dagger)(\sigma^{\mu\nu})^\dagger(\gamma^0)^\dagger(u_1)\\
&= (u_2^\dagger)\sigma^{\mu\nu}\gamma^0(u_1)\\
&= (u_2^\dagger)(\gamma^0)\sigma^{\mu\nu}(u_1)\\
&= \overline{u}_2\sigma^{\mu\nu}(u_1) \qquad (15)
\end{aligned}
$$

An interesting thing to note is that the expression $\overline{u}\sigma^{\mu\nu}\gamma^5 u$ is not an independent quantity. Since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ it follows that the product of $\sigma^{\mu\nu}$ and γ^5 can be simplified to an expression with only 2 γ matrices which has been defined as a pseudoscalar. For example, let $\mu = 0$ and $\nu = 1$:

$$
\begin{aligned}
\overline{u}\sigma^{01}\gamma^5 u &= \overline{u}\sigma^{01}(i\gamma^0\gamma^1\gamma^2\gamma^3)u \qquad (16)\\
&= \overline{u}((\frac{i}{2})(\gamma^0\gamma^1 - \gamma^1\gamma^0))(i\gamma^0\gamma^1\gamma^2\gamma^3)u\\
&= \overline{u}(\frac{i}{2})[\gamma^0\gamma^1(i\gamma^0\gamma^1\gamma^2\gamma^3) - \gamma^1\gamma^0(i\gamma^0\gamma^1\gamma^2\gamma^3)]u\\
&= \overline{u}(\frac{-1}{2})[\gamma^0\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 - \gamma^1\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3)]u\\
&= \overline{u}(\frac{-1}{2})[-\gamma^2\gamma^3 - \gamma^2\gamma^3)]u\\
&= \overline{u}(\frac{-1}{2})[-2\gamma^2\gamma^3]u\\
&= 2\overline{u}\gamma^2\gamma^3 u \qquad (17)
\end{aligned}
$$

This expression (because it contains two gamma matrices) is a pseudoscalar. Any values of μ and ν can be shown to be similar to this because of the communal and identity properties of the gamma matrices.

In order to find the value of $|B^{\mu\nu}|^2$ one needs to do the following:

$$
\begin{aligned}
|B^{\mu\nu}|^2 &= \mathrm{Tr}[\overline{u}_1\sigma^{\mu\nu}u_2\overline{u}_2\sigma^{\sigma\lambda}u_1]\\
&= \mathrm{Tr}[\overline{u}_1\sigma^{\mu\nu}\not{p}_2\sigma^{\sigma\lambda}u_1]\\
&= \mathrm{Tr}[\not{p}_1\sigma^{\mu\nu}\not{p}_2\sigma^{\sigma\lambda}]\\
&= \mathrm{Tr}[p_\kappa\gamma^\kappa\sigma^{\mu\nu}p_\gamma\gamma^\gamma\sigma^{\sigma\lambda}]\\
&= p_\kappa p_\gamma \mathrm{Tr}[\gamma^\kappa\sigma^{\mu\nu}\gamma^\gamma\sigma^{\sigma\lambda}]\\
&= -\frac{1}{2}p_\kappa p_\gamma \mathrm{Tr}[\gamma^\kappa(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma^\gamma(\gamma^\sigma\gamma^\lambda - \gamma^\lambda\gamma^\sigma)]\\
&= -\frac{1}{2}p_\kappa p_\gamma \mathrm{Tr}[2\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda - 2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda]
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}p_\kappa p_\gamma \mathrm{Tr}[4\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda] \\
&= -2p_\kappa p_\gamma \mathrm{Tr}[\gamma^\kappa\gamma^\gamma] \text{ Using a similar identity as shown in eq. 23} \\
&= -8p_\kappa p_\gamma g^{\kappa\gamma} \qquad (18)
\end{aligned}
$$