Practice with Spinors

I Algebra with Dirac γ matrices

1. $(\overline{u}_1\gamma^\mu u_2)^* \;= ?$

   Note: $(\gamma^0)^\dagger \;=\; \gamma^0$ and $(\gamma^\mu)^\dagger \;=\; \gamma^0\gamma^\mu\gamma^0$

   $(\overline{u}_1\gamma^\mu u_2)$ is a 1×1 matrix. Therefore, its complex conjugate is the same as its Hermitian conjugate, i.e. if we call $L^\mu = (\overline{u}_1\gamma^\mu u_2)$, then $(L^\mu)^* = (L^\mu)^\dagger$. We can then express this quantity as follows:

$$\begin{aligned} L^\mu &= \overline{u}_1\gamma^\mu u_2 \;, \quad (1)\\ \Rightarrow (L^\mu)^* &= (L^\mu)^\dagger \;,\\ &= (\overline{u}_1\gamma^\mu u_2)^\dagger \;,\\ &= ((u_1)^\dagger\gamma^0\gamma^\mu u_2)^\dagger \quad \text{using } (A\ldots Z)^\dagger = Z^\dagger\ldots A^\dagger \;,\\ &= (u_2^\dagger)(\gamma^\mu)^\dagger(\gamma^0)^\dagger(u_1)\\ &= (u_2^\dagger)\gamma^0\gamma^\mu\gamma^0\gamma^0(u_1)\\ &= (u_2^\dagger)\gamma^0\gamma^\mu(u_1)\\ &= \overline{u}_2\gamma^\mu(u_1) \quad (2)\end{aligned}$$

   Therefore $(\overline{u}_1\gamma^\mu u_2)^* \;=\; \overline{u}_2\gamma^\mu u_1$. To solve for $(L^\mu)^2$ we simply use $(L^{\mu\nu})^2 = Tr[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1]$
   Note: $Tr[\gamma^\mu\gamma^\nu] = 0$, $Tr[\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda] = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$
   Correct: $\mathrm{Tr}[\gamma^\mu\gamma^\nu] \;=\; 4g^{\mu\nu}$, $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$

$$\begin{aligned} (L^{\mu\nu})^2 &= Tr[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1]\\ &= Tr[\overline{u}_1\gamma^\mu(\not{p}_2 + m)\gamma^\nu u_1]\\ &= Tr[u_1\overline{u}_1\gamma^\mu(\not{p}_2 + m)\gamma^\nu]\\ &= Tr[(\not{p}_1 + m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu](\text{Good Job!})\\ &= Tr[\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2] + m[Tr(\gamma^\mu\not{p}_1\gamma^\nu) + Tr[\gamma^\mu\gamma^\nu\not{p}_2)] + m^2Tr[\gamma^\mu\gamma^\nu]\\ &= Tr[\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2] + m[Tr(\gamma^\mu\not{p}_1\gamma^\nu) + Tr[\gamma^\mu\gamma^\nu\not{p}_2)] + m^2Tr[\gamma^\mu\gamma^\nu]\\ &= Tr[\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2] + m^2Tr[\gamma^\mu\gamma^\nu]\\ &= (p_1)_\lambda(p_2)_\sigma Tr[\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda]4m^2g^{\mu\nu}\\ &= (p_1)_\lambda(p_2)_\sigma 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})4m^2g^{\mu\nu}\\ &= 4[p_1^\mu p_2^\nu - g^{\mu\nu}(p_1\cdot p_2) + p_2^\mu p_1^\nu] + 4m^2g^{\mu\nu} \quad (3)\end{aligned}$$

2. $(\overline{u}_1\gamma^\mu\gamma^5 u_2)^*$ is also a 1×1 Matrix so the same reasoning applies as above in 1. Note: $(\gamma^5)^\dagger = \gamma^5$
   We define: R^μ as $\overline{u}_1\gamma^\mu\gamma^5 u_2$ thus:

$$\begin{aligned} \Rightarrow (R^\mu)^* &= (R^\mu)^\dagger\\ &= (\overline{u}_1\gamma^\mu\gamma^5 u_2)^\dagger\end{aligned}$$

$$
\begin{aligned}
&= ((u_1)^\dagger\gamma^0\gamma^\mu\gamma^5u_2)^\dagger \\
&= (u_2^\dagger)(\gamma^5)^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger(u_1) \\
&= (u_2^\dagger)\gamma^5\gamma^0\gamma^\mu\gamma^0\gamma^0u_1 \\
&= (u_2^\dagger)\gamma^5\gamma^0\gamma^\mu(1)u_1 \\
&= -(u_2^\dagger)\gamma^0\gamma^5\gamma^\mu u_1 \\
&= -\overline{u}_2\gamma^5\gamma^\mu u_1 \\
&= \overline{u}_2\gamma^\mu\gamma^5u_1
\end{aligned}
$$

Therefore $(\overline{u}_1\gamma^\mu\gamma^5u_2)^* \;=\; \overline{u}_2\gamma^\mu\gamma^5u_1$

3. $(\overline{u}_1u_2)^* \;=\; ?$
   We let:$P \;=\; \overline{u}_1u_2$

$$
\begin{aligned}
\Rightarrow (P)^* &= (P)^\dagger\ , \\
&= (\overline{u}_1u_2)^\dagger\ , \\
&= ((u_1)^\dagger\gamma^0u_2)^\dagger \\
&= (u_2)^\dagger(\gamma^0)^\dagger(u_1) \\
&= (u_2)^\dagger\gamma^0(u_1) \\
&= \overline{u}_2(u_1)
\end{aligned}
$$

   Therefore$(\overline{u}_1u_2)^* \;=\; \overline{u}_2u_1$

4. By the same reasoning as shown above it can be shown that $(\overline{u}_1\gamma^5u_2)^* \;=\; \overline{u}_2\gamma^5u_1$
   If we let $T \;=\; \overline{u}_1\gamma^5u_2$ then:

$$
\begin{aligned}
\Rightarrow (T)^* &= (T)^\dagger\ , \\
&= (\overline{u}_1\gamma^5u_2)^\dagger\ , \\
&= ((u_1)^\dagger\gamma^0\gamma^5u_2)^\dagger \\
&= (u_2)^\dagger(\gamma^5)^\dagger(\gamma^0)^\dagger(u_1) \\
&= (u_2)^\dagger(-\gamma^5)\gamma^0(u_1) \\
&= (u_2)^\dagger\gamma^0\gamma^5(u_1) \\
&= \overline{u}_2\gamma^5(u_1)
\end{aligned}
$$

   Therefore $(\overline{u}_1\gamma^5u_2)^* \;=\; \overline{u}_2\gamma^5u_1$

5. While the above identities could be shown to be trivial, the identity: $(\overline{u}_1\sigma^{\mu\nu}u_2)^* \;=\; \overline{u}_2\sigma^{\mu\nu}u_1$ is more difficult to solve
   The identity:$(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$ is needed

$$
\begin{aligned}
(\sigma^{\mu\nu})^\dagger &= (\frac{i}{2}[\gamma^\mu,\gamma^\nu])^\dagger \\
&= (u_2)^\dagger(\sigma^{\mu\nu})^\dagger(\gamma^0)^\dagger u_1
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{i}{2}([\gamma^\mu, \gamma^\nu])^\dagger \\
&= \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)^\dagger \\
&= \frac{i}{2}((\gamma^\mu)^\dagger(\gamma^\nu)^\dagger - (\gamma^\nu)^\dagger(\gamma^\mu)^\dagger) \\
&= \frac{i}{2}(\gamma^0\gamma^\mu\gamma^0\gamma^0\gamma^\nu\gamma^0 - \gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^\mu\gamma^0) \\
&= \frac{i}{2}(\gamma^0\gamma^\mu\gamma^\nu\gamma^0 - \gamma^0\gamma^\nu\gamma^\mu\gamma^0) \\
&= \frac{i}{2}((-1)^2\gamma^\mu\gamma^\nu - (-1)^2\gamma^\nu\gamma^\mu) \\
&= \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \\
&= \sigma^{\mu\nu} \qquad \text{(-31)}
\end{aligned}
$$

After showing $(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$ is true it is trivial to show $(\overline{u}_1\sigma^{\mu\nu}u_2)^* = \overline{u}_2\sigma^{\mu\nu}u_1$
We let $B^{\mu\nu} = \overline{u}_1\sigma^{\mu\nu}u_2$

$$
\begin{aligned}
\Rightarrow (B^{\mu\nu})^* &= (B^{\mu\nu})^\dagger \ , \\
&= (\overline{u}_1\sigma^{\mu\nu}u_2)^\dagger \ , \\
&= ((u_1)^\dagger\gamma^0\sigma^{\mu\nu}u_2)^\dagger \\
&= (u_2^\dagger)(\sigma^{\mu\nu})^\dagger(\gamma^0)^\dagger(u_1) \\
&= (u_2^\dagger)\sigma^{\mu\nu}\gamma^0(u_1) \\
&= (u_2^\dagger)(\gamma^0)\sigma^{\mu\nu}(u_1) \\
&= \overline{u}_2\sigma^{\mu\nu}(u_1)
\end{aligned}
$$

An interesting thing to note is that the expression $\overline{u}\sigma^{\mu\nu}\gamma^5 u$ is not an independent quantity. Since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ it follows that the product of $\sigma^{\mu\nu}$ and γ^5 can be simplified to an expression with only 2 γ matrices which has been defined as a pusedoscalar.
For example, let $\mu = 0$ and $\nu = 1$:

$$
\begin{aligned}
\overline{u}\sigma^{01}\gamma^5 u &= \overline{u}\sigma^{01}(i\gamma^0\gamma^1\gamma^2\gamma^3)u \\
&= \overline{u}((\frac{i}{2})(\gamma^0\gamma^1 - \gamma^1\gamma^0))(i\gamma^0\gamma^1\gamma^2\gamma^3)u \\
&= \overline{u}(\frac{i}{2})[\gamma^0\gamma^1(i\gamma^0\gamma^1\gamma^2\gamma^3) - \gamma^1\gamma^0(i\gamma^0\gamma^1\gamma^2\gamma^3)]u \\
&= \overline{u}(\frac{-1}{2})[\gamma^0\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 - \gamma^1\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3)]u \\
&= \overline{u}(\frac{-1}{2})[-\gamma^2\gamma^3 - \gamma^2\gamma^3)]u \\
&= \overline{u}(\frac{-1}{2})[-2\gamma^2\gamma^3]u \\
&= 2\overline{u}\gamma^2\gamma^3 u
\end{aligned}
$$

This expression (because it contains two gamma matrices) is a puesdoscalar. Any values of μ and ν can be shown to be similar to this because of the communal and indentity properties of the gamma matrices.

II Squaring Expressions

6. $\mid \overline{u}_1 \gamma^\mu u_2 \mid^2 \;=\; (\overline{u}_1 \gamma^\mu u_2)(\overline{u}_1 \gamma^\mu u_2)^*$

   $\mid \overline{u}_1 \gamma^\mu u_2 \mid^2 \;=\; (\overline{u}_1 \gamma^\mu u_2)(\overline{u}_2 \gamma^\mu u_1)$

   I know that both of the expressions in the parentheses are 1x1 matrices but I fail to see how it can be simplified anymore.

   Or is this the way to proceed: $\mid \overline{u}_1 \gamma^\mu u_2 \mid^2 \;=\; \mid \overline{u}_1 \mid^2 \mid \gamma^\mu \mid^2 \mid u_2 \mid^2$

   $\mid \overline{u}_1 \gamma^\mu u_2 \mid^2 \;=\; (\overline{u}_1)(\overline{u}_1)^* \gamma^\mu (\gamma^\mu)^* (u_2)(u_2)^*$

   $\mid \overline{u}_1 \gamma^\mu u_2 \mid^2 \;=\; (\overline{u}_1)(\overline{u}_1)^* \gamma^\mu \gamma^\mu (u_2)(u_2)^*$

   $\mid \overline{u}_1 \gamma^\mu u_2 \mid^2 \;=\; (\overline{u}_1)\overline{u}_1)^* (u_2)(u_2)^*$