Practice with Spinors

I Algebra with Dirac γ matrices

Notation:

1. $S = \overline{u}u$
2. $P = \overline{u}\gamma^5 u$
3. $V^\mu = \overline{u}\gamma^\mu u$
4. $A^\mu = \overline{u}\gamma^\mu\gamma^5 u$
5. $T^{\mu\nu} = \overline{u}\sigma^{\mu\nu}u$

Please follow the above naming convention in the rest of the notes. Make appropriate modifications.

Anything marked in red has a mistake in it. Please work on it and fix it.

1. $(\overline{u}_1\gamma^\mu u_2)^* \ = ?$

   Note: $(\gamma^0)^\dagger \ = \ \gamma^0$ and $(\gamma^\mu)^\dagger \ = \ \gamma^0\gamma^\mu\gamma^0$

   $(\overline{u}_1\gamma^\mu u_2)$ is a 1×1 matrix. Therefore, its complex conjugate is the same as its Hermitian conjugate, i.e. if we call $V^\mu = (\overline{u}_1\gamma^\mu u_2)$, then $(V^\mu)^* = (V^\mu)^\dagger$. We can then express this quantity as follows:

$$
\begin{aligned}
V^\mu &= \overline{u}_1\gamma^\mu u_2 \ , && (1)\\
\Rightarrow (V^\mu)^* &= (V^\mu)^\dagger \ , && (2)\\
&= (\overline{u}_1\gamma^\mu u_2)^\dagger \ , && (3)\\
&= ((u_1)^\dagger\gamma^0\gamma^\mu u_2)^\dagger \quad \text{using } (A\ldots Z)^\dagger = Z^\dagger\ldots A^\dagger \ , && (4)\\
&= (u_2^\dagger)(\gamma^\mu)^\dagger(\gamma^0)^\dagger(u_1) && (5)\\
&= (u_2^\dagger)\gamma^0\gamma^\mu\gamma^0\gamma^0(u_1) && (6)\\
&= (u_2^\dagger)\gamma^0\gamma^\mu(u_1) && (7)\\
&= \overline{u}_2\gamma^\mu(u_1) && (8)
\end{aligned}
$$

   Therefore $(\overline{u}_1\gamma^\mu u_2)^* \ = \ \overline{u}_2\gamma^\mu u_1$. To solve for $|V^\mu|^2$ we simply use $|V^\mu|^2 \ = \ \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1]$.

   Question:
   Why is $|V^\mu|^2 \ \neq \ \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma_\mu u_1]$?
   Why is $|V^\mu|^2 \ = \ \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1]$?

   You start with one index μ. When you square why do you get two indices and not a sum over two of the same index?
   When you square you have two indices because you must increase the number of components. The number of components when you square should go as n^2 not simply

n. When you have one index, you restrict the number of components, because you have 4 components and not 16.

Note: $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$, $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$, The trace over the product of an odd number of gamma matrices is zero.

$$
\begin{aligned}
|V^\mu|^2 &= \mathrm{Tr}[\overline{u}_1\gamma^\mu u_2\overline{u}_2\gamma^\nu u_1] && (9)\\
&= \mathrm{Tr}[\overline{u}_1\gamma^\mu(\not{p}_2 + m)\gamma^\nu u_1] && (10)\\
&= \mathrm{Tr}[u_1\overline{u}_1\gamma^\mu(\not{p}_2 + m)\gamma^\nu] && (11)\\
&= \mathrm{Tr}[(\not{p}_1 + m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu] && (12)\\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] + m[\mathrm{Tr}(\gamma^\mu\not{p}_1\gamma^\nu) + \mathrm{Tr}(\gamma^\mu\gamma^\nu\not{p}_2)] + m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] && (13)\\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] + m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] && (14)\\
&= \mathrm{Tr}[(p_1)_\lambda\gamma^\lambda\gamma^\mu(p_2)_\sigma\gamma^\sigma\gamma^\nu] + 4m^2g^{\mu\nu} && (15)\\
&= (p_1)_\lambda(p_2)_\sigma\mathrm{Tr}[\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu] + 4m^2g^{\mu\nu} && (16)\\
&= (p_1)_\lambda(p_2)_\sigma 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) + 4m^2g^{\mu\nu} && (17)\\
&= 4[p_1^\mu p_2^\nu - g^{\mu\nu}(p_1\cdot p_2) + p_2^\mu p_1^\nu] + 4m^2g^{\mu\nu} && (18)
\end{aligned}
$$

2. $(\overline{u}_1\gamma^\mu\gamma^5u_2)^*$ is also a 1×1 Matrix so the same reasoning applies as above in 1. Note: $(\gamma^5)^\dagger = \gamma^5$ We define: A^μ as $\overline{u}_1\gamma^\mu\gamma^5u_2$ thus:

$$
\begin{aligned}
(A^\mu)^* &= (A^\mu)^\dagger && (19)\\
&= (\overline{u}_1\gamma^\mu\gamma^5u_2)^\dagger && (20)\\
&= ((u_1)^\dagger\gamma^0\gamma^\mu\gamma^5u_2)^\dagger && (21)\\
&= (u_2^\dagger)(\gamma^5)^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger(u_1) && (22)\\
&= (u_2^\dagger)\gamma^5\gamma^0\gamma^\mu\gamma^0\gamma^0u_1 && (23)\\
&= (u_2^\dagger)\gamma^5\gamma^0\gamma^\mu(1)u_1 && (24)\\
&= -(u_2^\dagger)\gamma^0\gamma^5\gamma^\mu u_1 && (25)\\
&= -\overline{u}_2\gamma^5\gamma^\mu u_1 && (26)\\
&= \overline{u}_2\gamma^\mu\gamma^5u_1 && (27)
\end{aligned}
$$

Therefore $(\overline{u}_1\gamma^\mu\gamma^5u_2)^* = \overline{u}_2\gamma^\mu\gamma^5u_1$

We also are able to calculate $|A^\mu|^2$

$$
\begin{aligned}
|A^\mu|^2 &= \mathrm{Tr}[(\overline{u}_1\gamma^\mu\gamma^5u_2)(\overline{u}_2\gamma^\nu\gamma^5u_1)] && (28)\\
&= \mathrm{Tr}[\overline{u}_1\gamma^\mu\gamma^5(\not{p}_2 + m)\gamma^\nu\gamma^5u_1] && (29)\\
&= \mathrm{Tr}[u_1\overline{u}_1\gamma^\mu\gamma^5(\not{p}_2 + m)\gamma^\nu\gamma^5] && (30)\\
&= \mathrm{Tr}[(\not{p}_1 + m)\gamma^\mu\gamma^5(\not{p}_2 + m)\gamma^\nu\gamma^5] && (31)\\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5 + m(\not{p}_1\gamma^\mu\gamma^5\gamma^\nu\gamma^5 + \gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5) + m^2(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)] && (32)\\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5 + m(\not{p}_1\gamma^\mu(-\gamma^5\gamma^5)\gamma^\nu + \not{p}_2\gamma^\mu(-\gamma^5\gamma^5)\gamma^\nu) + m^2(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5 - m(\not{p}_1\gamma^\mu\gamma^\nu + \not{p}_2\gamma^\mu\gamma^\nu) + m^2(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)] && (33)\\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5] - mTr[\not{p}_1\gamma^\mu\gamma^\nu] - mTr[\not{p}_2\gamma^\mu\gamma^\nu] + m^2(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)] && (34)\\
&= \mathrm{Tr}[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\nu\gamma^5 + m^2(\gamma^\mu\gamma^5\gamma^\nu\gamma^5)] && (35)\\
&= \mathrm{Tr}[(p_1)_\lambda\gamma^\lambda\gamma^\mu\gamma^5(p_2)_\sigma\gamma^\sigma\gamma^\nu\gamma^5] + m^2\mathrm{Tr}[\gamma^\mu\gamma^5\gamma^\nu\gamma^5] && (36)\\
&= (p_1)_\lambda(p_2)_\sigma\mathrm{Tr}[\gamma^\lambda\gamma^\mu\gamma^5\gamma^\sigma\gamma^\nu\gamma^5] - m^2\mathrm{Tr}[\gamma^\mu\gamma^5\gamma^5\gamma^\nu] && (37)\\
&= (p_1)_\lambda(p_2)_\sigma\mathrm{Tr}[\gamma^\lambda\gamma^\mu\gamma^5\gamma^5\gamma^\sigma\gamma^\nu] - m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] && (38)\\
&= (p_1)_\lambda(p_2)_\sigma\mathrm{Tr}[\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\nu] - m^2(g^{\mu\nu}) && (39)\\
&= (p_1)_\lambda(p_2)_\sigma 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) - 4m^2g^{\mu\nu} && (40)\\
&= 4[p_1^\mu p_2^\nu - g^{\mu\nu}(p_1\cdot p_2) + p_2^\mu p_1^\nu] - 4m^2g^{\mu\nu} && (41)
\end{aligned}
$$

3. $(\overline{u}_1u_2)^* \ = \ ?$ We let $S \ = \ \overline{u}_1u_2$

$$
\begin{aligned}
(S)^* &= (S)^\dagger && (42)\\
&= (\overline{u}_1u_2)^\dagger\ , && (43)\\
&= ((u_1)^\dagger\gamma^0u_2)^\dagger && (44)\\
&= (u_2)^\dagger(\gamma^0)^\dagger(u_1) && (45)\\
&= (u_2)^\dagger\gamma^0(u_1) && (46)\\
&= \overline{u}_2(u_1) && (47)
\end{aligned}
$$

Therefore$(\overline{u}_1u_2)^* \ = \ \overline{u}_2u_1$. In order to find $|S|^2$ we simply do the following:

$$
\begin{aligned}
|S|^2 &= \mathrm{Tr}[\overline{u}_1u_2\overline{u}_2u_1] && (48)\\
&= \mathrm{Tr}[(\not{p}_1+m)(\not{p}_2+m)] && (49)\\
&= \mathrm{Tr}[\not{p}_1\not{p}_2 + m(\not{p}_1+\not{p}_2) + m^2] && (50)\\
&= \mathrm{Tr}[\not{p}_1\not{p}_2] + \mathrm{Tr}[m(\not{p}_1+\not{p}_2)] + \mathrm{Tr}[m^2] && (51)\\
&= \mathrm{Tr}[\not{p}_1\not{p}_2] + m(\mathrm{Tr}[\not{p}_1] + \mathrm{Tr}[\not{p}_2]) + m^2\mathrm{Tr}[1] && (52)\\
&= \mathrm{Tr}[\not{p}_1\not{p}_2] + 4m^2 && (53)\\
&= 4(p_1\cdot p_2) + 4m^2 && (54)
\end{aligned}
$$

4. By the same reasoning as shown above it can be shown that $(\overline{u}_1\gamma^5u_2)^* \ = \ \overline{u}_2\gamma^5u_1$
If we let $P \ = \ \overline{u}_1\gamma^5u_2$ then:

$$
\begin{aligned}
(P)^* &= (P)^\dagger && (55)\\
&= (\overline{u}_1\gamma^5u_2)^\dagger\ , && (56)\\
&= ((u_1)^\dagger\gamma^0\gamma^5u_2)^\dagger && (57)\\
&= (u_2)^\dagger(\gamma^5)^\dagger(\gamma^0)^\dagger(u_1) && (58)\\
&= (u_2)^\dagger(\gamma^5)\gamma^0(u_1) && (59)\\
&= -(u_2)^\dagger\gamma^0\gamma^5(u_1) && (60)\\
&= -\overline{u}_2\gamma^5(u_1) && (61)
\end{aligned}
$$

Therefore $(\overline{u}_1\gamma^5 u_2)^* \;=\; -\,\overline{u}_2\gamma^5 u_1$

In order to square P we do the following:

$$
\begin{aligned}
|P|^2 &= \mathrm{Tr}[\overline{u}_1\gamma^5 u_2(-\overline{u}_2\gamma^5 u_1)] && (62)\\
&= \mathrm{Tr}[u_1\overline{u}_1\gamma^5(-\not{p}_2 - m)\gamma^5] && (63)\\
&= \mathrm{Tr}[(\not{p}_1 + m)\gamma^5(-\not{p}_2 - m)\gamma^5] && (64)\\
&= \mathrm{Tr}[((p_1)_\mu\gamma^\mu + m)\gamma^5((-p_2)_\nu\gamma^\nu - m)\gamma^5] && (65)\\
&= \mathrm{Tr}[((p_1)_\mu\gamma^\mu\gamma^5 + m\gamma^5)((-p_2)_\nu\gamma^\nu\gamma^5 - m\gamma^5)] && (66)\\
&= \mathrm{Tr}[(p_1)_\mu\gamma^\mu\gamma^5(-p_2)_\nu\gamma^\nu\gamma^5 + m\gamma^5(-p_2)_\nu\gamma^\nu\gamma^5 - m\gamma^5(p_1)_\mu\gamma^\mu\gamma^5 - \gamma^5\gamma^5 m^2)]\\
&= \mathrm{Tr}[(p_1)_\mu\gamma^\mu\gamma^5(-p_2)_\nu\gamma^\nu\gamma^5] + Tr[m\gamma^5(-\gamma^5)(-p_2)_\nu\gamma^\nu] - Tr[m\gamma^5(-\gamma^5)(p_1)_\mu\gamma^\mu] - Tr[m^2]\\
&= \mathrm{Tr}[(p_1)_\mu\gamma^\mu\gamma^5(-p_2)_\nu\gamma^\nu\gamma^5] - Tr[m(-p_2)_\nu\gamma^\nu] + Tr[m(p_1)_\mu\gamma^\mu] - 4m^2 && (67)\\
&= \mathrm{Tr}[(p_1)_\mu\gamma^\mu\gamma^5(-p_2)_\nu\gamma^\nu\gamma^5] - 4m^2 && (68)\\
&= (p_1)_\mu(-p_2)_\nu\mathrm{Tr}[\gamma^\mu\gamma^5\gamma^\nu\gamma^5] - 4m^2 && (69)\\
&= (p_1)_\mu(-p_2)_\nu\mathrm{Tr}[\gamma^\mu\gamma^5(-\gamma^5\gamma^\nu)] - 4m^2 && (70)\\
&= (p_1)_\mu(-p_2)_\nu(-\mathrm{Tr}[\gamma^\mu\gamma^\nu]) - 4m^2 && (71)\\
&= (p_1)_\mu(-p_2)_\nu(-4g^{\mu\nu}) - 4m^2 && (72)\\
&= 4(p_1)(p_2) - 4m^2 && (73)
\end{aligned}
$$

5. While the above identities could be shown to be trivial, the identity: $(\overline{u}_1\sigma^{\mu\nu}u_2)^* \;=\; \overline{u}_2\sigma^{\nu\mu}u_1$ is more difficult to solve. The identity:$(\sigma^{\mu\nu})^\dagger = \sigma^{\mu\nu}$ is needed

$$
\begin{aligned}
(\sigma^{\mu\nu})^\dagger &= (\frac{i}{2}[\gamma^\mu,\gamma^\nu])^\dagger && (74)\\
&= -\frac{i}{2}([\gamma^\mu,\gamma^\nu])^\dagger && (75)\\
&= -\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)^\dagger && (76)\\
&= -\frac{i}{2}((\gamma^\nu)^\dagger(\gamma^\mu)^\dagger - (\gamma^\mu)^\dagger(\gamma^\nu)^\dagger) && (77)\\
&= -\frac{i}{2}(\gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^\mu\gamma^0 - \gamma^0\gamma^\mu\gamma^0\gamma^0\gamma^\nu\gamma^0) && (78)\\
&= -\frac{i}{2}(\gamma^0\gamma^\nu\gamma^\mu\gamma^0 - \gamma^0\gamma^\mu\gamma^\nu\gamma^0) && (79)\\
&= -\frac{i}{2}((-1)^2\gamma^\nu\gamma^\mu - (-1)^2\gamma^\mu\gamma^\nu) && (80)\\
&= -\frac{i}{2}(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu) && (81)\\
&= -\sigma^{\nu\mu} && (82)
\end{aligned}
$$

After showing $(\sigma^{\mu\nu})^\dagger = \sigma^{\nu\mu}$ is true it is trivial to show $(\overline{u}_1\sigma^{\mu\nu}u_2)^* \;=\; -\,\overline{u}_2\sigma^{\nu\mu}u_1$.
We let $T^{\mu\nu} \;=\; \overline{u}_1\sigma^{\mu\nu}u_2$

$$
(T^{\mu\nu})^* \;=\; (T^{\mu\nu})^\dagger\ , \qquad (83)
$$

$$
\begin{aligned}
&= (\overline{u}_1\sigma^{\mu\nu}u_2)^\dagger && (84)\\
&= ((u_1)^\dagger\gamma^0\sigma^{\mu\nu}u_2)^\dagger && (85)\\
&= (u_2^\dagger)(\sigma^{\mu\nu})^\dagger(\gamma^0)^\dagger(u_1) && (86)\\
&= (u_2^\dagger)(-\sigma^{\nu\mu})\gamma^0(u_1) && (87)\\
&= (u_2^\dagger)(-\gamma^0)(-\sigma^{\nu\mu})(u_1) && (88)\\
&= \overline{u}_2\sigma^{\mu\nu}(u_1) && (89)
\end{aligned}
$$

In order to find the value of $|T^{\mu\nu}|^2$ one needs to find the value of $Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}]$

$$
\begin{aligned}
Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] &= Tr[\frac{i}{2}(\gamma^\sigma\gamma^\lambda-\gamma^\lambda\gamma^\sigma)\frac{i}{2}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)] && (90)\\
&= Tr[\frac{i}{2}(\gamma^\sigma\gamma^\lambda-\gamma^\lambda\gamma^\sigma)\frac{i}{2}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)] && (91)\\
&= -\frac{1}{4}Tr[(\gamma^\sigma\gamma^\lambda-\gamma^\lambda\gamma^\sigma)(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)] && (92)\\
&= -\frac{1}{4}Tr[\gamma^\sigma\gamma^\lambda\gamma^\mu\gamma^\nu]+\frac{1}{4}Tr[\gamma^\sigma\gamma^\lambda\gamma^\nu\gamma^\mu]+\frac{1}{4}Tr[\gamma^\lambda\gamma^\sigma\gamma^\mu\gamma^\nu]-\frac{1}{4}Tr[\gamma^\lambda\gamma^\sigma\gamma^\nu\gamma^\mu]
\end{aligned}
$$

Here we must label each of the traces individually:

$$
\begin{aligned}
A &= -\frac{1}{4}Tr[\gamma^\sigma\gamma^\lambda\gamma^\mu\gamma^\nu] && (93)\\
&= -(g^{\sigma\lambda}g^{\mu\nu}-g^{\sigma\mu}g^{\lambda\nu}+g^{\sigma\nu}g^{\lambda\mu}) && (94)
\end{aligned}
$$

$$
\begin{aligned}
B &= +\frac{1}{4}Tr[\gamma^\sigma\gamma^\lambda\gamma^\nu\gamma^\mu] && (95)\\
&= +(g^{\sigma\lambda}g^{\nu\mu}-g^{\sigma\nu}g^{\lambda\mu}+g^{\sigma\mu}g^{\lambda\nu}) && (96)
\end{aligned}
$$

$$
\begin{aligned}
C &= \frac{1}{4}Tr[\gamma^\lambda\gamma^\sigma\gamma^\mu\gamma^\nu] && (97)\\
&= (g^{\lambda\sigma}g^{\mu\nu}-g^{\lambda\mu}g^{\sigma\nu}+g^{\lambda\nu}g^{\sigma\mu}) && (98)
\end{aligned}
$$

$$
\begin{aligned}
D &= -\frac{1}{4}Tr[\gamma^\lambda\gamma^\sigma\gamma^\nu\gamma^\mu] && (99)\\
&= -(g^{\lambda\sigma}g^{\nu\mu}-g^{\lambda\nu}g^{\sigma\mu}+g^{\lambda\mu}g^{\sigma\nu}) && (100)
\end{aligned}
$$

$$
\begin{aligned}
Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] &= A+B++C+D && (101)\\
&= 2g^{\sigma\mu}g^{\lambda\nu}-2g^{\sigma\nu}g^{\lambda\mu}-2g^{\lambda\mu}g^{\sigma\nu}+2g^{\lambda\nu}g^{\sigma\mu}) && (102)
\end{aligned}
$$

In order to find the value of $|T^{\mu\nu}|^2$ one needs to do the following:

$$
|T^{\mu\nu}|^2 = \mathrm{Tr}[\overline{u}_1\sigma^{\mu\nu}u_2\overline{u}_2\sigma^{\sigma\lambda}u_1] \qquad (103)
$$

$$
\begin{aligned}
&= \mathrm{Tr}[\overline{u}_1\sigma^{\mu\nu}(\not{p}_2+m)\sigma^{\sigma\lambda}u_1] && (104)\\
&= \mathrm{Tr}[(\not{p}_1+m)\sigma^{\mu\nu}(\not{p}_2+m)\sigma^{\sigma\lambda}] && (105)\\
&= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa+m)\sigma^{\mu\nu}((p_2)_\gamma\gamma^\gamma+m)\sigma^{\sigma\lambda}] && (106)\\
&= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}+m\sigma^{\mu\nu})((p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}+m\sigma^{\sigma\lambda})] && (107)\\
&= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}+(p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}m\sigma^{\sigma\lambda}+(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}m\sigma^{\mu\nu}+m\sigma^{\sigma\lambda}m\sigma^{\mu\nu})] && (108)\\
&= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}]+Tr[(p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}m\sigma^{\sigma\lambda}+(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}m\sigma^{\mu\nu}]+Tr[m\sigma^{\sigma\lambda}m\sigma^{\mu\nu}]\\
&= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}]+Tr[m\sigma^{\sigma\lambda}m\sigma^{\mu\nu}] && (108)\\
&= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}]+m^2Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] \text{ Let B } = \mathrm{m}^2\mathrm{Tr}[\sigma^{\sigma\lambda}\sigma^{\mu\nu}]\\
&= \mathrm{Tr}[((p_1)_\kappa\gamma^\kappa\sigma^{\mu\nu}(p_2)_\gamma\gamma^\gamma\sigma^{\sigma\lambda}]+B\\
&= (p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[\gamma^\kappa\sigma^{\mu\nu}\gamma^\gamma\sigma^{\sigma\lambda}]+B\\
&= (p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[\gamma^\kappa(\frac{i}{2}((\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)\gamma^\gamma(\frac{i}{2}(\gamma^\sigma\gamma^\lambda-\gamma^\lambda\gamma^\sigma))]+B && (106)\\
&= -\frac{1}{4}(p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[\gamma^\kappa(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)\gamma^\gamma(\gamma^\sigma\gamma^\lambda-\gamma^\lambda\gamma^\sigma)]+B \text{ (Let A } = -\frac{1}{4}(\mathrm{p}_1)_\kappa(\mathrm{p}_2)_\gamma)\\
&= (A)\mathrm{Tr}[(\gamma^\kappa\gamma^\mu\gamma^\nu-\gamma^\kappa\gamma^\nu\gamma^\mu)(\gamma^\gamma\gamma^\sigma\gamma^\lambda-\gamma^\gamma\gamma^\lambda\gamma^\sigma)]+B\\
&= (A)\mathrm{Tr}[(\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda-\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\lambda\gamma^\sigma-\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda+\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\lambda\gamma^\sigma)]+B\\
&= (A)\mathrm{Tr}[(\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda-\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma(-\gamma^\sigma\gamma^\lambda)-\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda+\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma(-\gamma^\sigma\gamma^\lambda))]+B\\
&= (A)\mathrm{Tr}[(\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda+\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda-\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda-\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda))]+B\\
&= (A)\mathrm{Tr}[2\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda-2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda]+B && (102)\\
&= (A)\mathrm{Tr}[2\gamma^\kappa(-\gamma^\nu\gamma^\mu)\gamma^\gamma\gamma^\sigma\gamma^\lambda-2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda]+B && (103)\\
&= (A)\mathrm{Tr}[-2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda-2\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda]+B && (104)\\
&= (A)\mathrm{Tr}[-4\gamma^\kappa\gamma^\nu\gamma^\mu\gamma^\gamma\gamma^\sigma\gamma^\lambda]+B && (105)\\
&= -\frac{1}{4}(p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[-4\gamma^\kappa\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\sigma\gamma^\lambda]+B && (106)\\
&= (p_1)_\kappa(p_2)_\gamma\mathrm{Tr}[\gamma^\kappa\gamma^\gamma]+B \text{ (Using a similar identity as shown in eq. 91)} && (107)\\
&= 4(p_1)_\kappa(p_2)_\gamma g^{\kappa\gamma}+B && (108)\\
&= 4p_1p_2+m^2Tr[\sigma^{\sigma\lambda}\sigma^{\mu\nu}] && (109)\\
&= 4p_1p_2+m^24(g^{\mu\nu}g^{\lambda\sigma}-g^{\mu\lambda}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\lambda}) && (110)
\end{aligned}
$$