FORMAL GROUP LAWS IN ALGEBRAIC TOPOLOGY

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Formal Group Laws

Let's start with a classical result, an addition formula for an elliptic integral.

Theorem (Euler, 1756)**.** *If*

$$\int_0^x \frac{dt}{\sqrt{1-t^4}} + \int_0^y \frac{dt}{\sqrt{1-t^4}} = \int_0^z \frac{dt}{\sqrt{1-t^4}},$$

then

$$z = \frac{x\sqrt{1-y^4} + y\sqrt{1-x^4}}{1+x^2y^2}.$$

Proof. The general solution of the differential equation

$$\frac{dx}{\sqrt{1-x^4}} = \frac{dy}{\sqrt{1-y^4}}$$

is

$$x^2 + y^2 + c^2x^2y^2 = c^2 + 2xy\sqrt{1-c^4}.$$

□

We should compare this with a more familiar result.

Theorem. *If*

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} + \int_0^y \frac{dt}{\sqrt{1-t^2}} = \int_0^z \frac{dt}{\sqrt{1-t^2}},$$

then

$$z = x\sqrt{1-y^2} + y\sqrt{1-x^2}.$$

In other words

$$\sin^{-1} x + \sin^{-1} y = \sin^{-1}\bigl(x\sqrt{1-y^2} + y\sqrt{1-x^2}\bigr),$$

or

$$\sin(\theta+\varphi) = \sin\theta\cos\varphi + \sin\varphi\cos\theta.$$

These two expressions

$$\frac{x\sqrt{1-y^4} + y\sqrt{1-x^4}}{1+x^2y^2}$$

and

$$x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

are our first examples of *formal group laws* (FGLs). Think of each of them as a formal power series; we can use the binomial theorem to expand the square-roots:

$$\begin{aligned}\sqrt{1-x^2} &= 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \cdots \\ &= 1 - \sum_{n\geq 0} \frac{1}{2^{2n+1}} \frac{1}{n+1}\binom{2n}{n} x^{2n+2}.\end{aligned}$$

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The numbers $\dfrac{1}{n+1}\dbinom{2n}{n}$ are integers (they're Catalan numbers), so the series has all its coefficients in the ring $\mathbb{Z}[\frac{1}{2}]$. We say that the two FGLs are defined *over* this ring. More formally

Definition. A *formal group law* over a ring R (commutative with 1) is a power series $F(x,y) \in R[[x,y]]$ satisfying

$$F(x,y) = F(y,x), \qquad F(x,0) = x, \quad F\big(F(x,y),z\big) = F\big(x,F(y,z)\big).$$

Note that we have made commutativity part of the definition, but in fact this follows from mild assumptions on R (no torsion nilpotents), and we have left out the inverse axiom: there exists a series $i(x) \in R[[x]]$ such that $F\big(x,i(x)\big) = 0$, since this is a consequence of the other axioms (we can just solve for the coefficients recursively).

So the axioms look like the axioms for an abelian group. In fact if we imagine the multiplication function $(g,h) \mapsto gh$ in a group as given by a power series, then the group axioms give us the axioms for a formal group law. So a FGL is like a group without elements — the Grin without the Cheshire Cat. I should say that there is a higher level of abstraction. A coordinate-free formal group law defines a *formal group*.

Examples.

(1) The *additive* FGL $x+y$.
(2) The *multiplicative* FGL $x+y+xy$.

Both of these are visibly FGLs over $\mathbb{Z}$.

A FGL provides a *Hopf algebra* structure on the ring $R[[t]]$: as well as the usual multiplication

$$R[[t]] \otimes_R R[[t]] \to R[[t]]$$

the ring homomorphism given by

$$t \mapsto F(t \otimes 1, 1 \otimes t)$$

gives a *coproduct*

$$R[[t]] \to R[[t]] \widehat{\otimes}_R R[[t]] = R[[t \otimes 1, 1 \otimes t]].$$

Here we have to have a completed tensor product (with respect to the t-adic topology) to allow infinite series involving both variables.

Topology

A standard tool for topologists is the concept of a *cohomology theory*. For us (we are not being very fussy with the details here) a cohomology theory is a contravariant functor

$$E^* : \textbf{Top} \to \textbf{Graded Rings}$$

satisfying

(1) *homotopy invariance*: If $f \simeq g : X \to Y$ are two homotopic maps, then $f^* = g^* : E^*(Y) \to E^*(X)$.
(2) *Exactness*: If $A \subset X$ is a closed subspace, and X/A denotes X with A collapsed to a point, then

$$A \xrightarrow{i} X \xrightarrow{p} X/A$$

gives rise to a *long exact sequence*

$$\cdots \xleftarrow{p^*} E^{n+1}(X/A) \xleftarrow{\delta} E^n(A) \xleftarrow{i^*} E^n(X) \xleftarrow{p^*} E^n(X/A) \xleftarrow{\delta} \cdots.$$

$E^* = E^*(\text{point})$ is the *coefficient ring*. Each cohomology theory has a dual homology theory $E_*(X)$. Sometimes this is the more natural, geometric object.

Examples of cohomology theories.

(1) Ordinary cohomology $H^*(X)$, the coefficient ring is $\mathbb{Z}$. The homology $H_*(X)$ is built out of maps of simplices into X, subject to the relation of *homology*.

(2) K-theory $K^*(X)$ has coefficient ring $K^* = \mathbb{Z}[u, u^{-1}]$, where $u \in K^{-2}$. The ring $K^*(X)$ is built out of complex vector bundles on X.

(3) The theory $MU^*(X)$ is called *complex cobordism*. Its corresponding homology theory is built out of maps of manifolds (with complex structure) into X, subject to the relation of bordism.

This is a very rich theory. The coefficient ring is a polynomial ring in infinitely many generators

$$MU^* = \frac{\text{manifolds}}{\text{bordism}} \cong \mathbb{Z}[v_1, v_2, \dots],$$

where $v_i \in MU^{-2i}$. The simplest complex manifolds are the complex projective spaces

$$\mathbb{C}P^n = \frac{\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}}{\mathbf{z} \sim \lambda\mathbf{z} \text{ for all } \lambda \in \mathbb{C} \setminus \{0\}}.$$

The bordism class $[\mathbb{C}P^n] \in MU^{-2n}$ is non-zero and

$$MU^* \otimes \mathbb{Q} = \mathbb{Q}\big[[\mathbb{C}P^1], [\mathbb{C}P^2], \dots\big],$$

but

$$MU^* \supsetneqq \mathbb{Z}\big[[\mathbb{C}P^1], [\mathbb{C}P^2], \dots\big].$$

Why are we interested in cohomology theories? One answer: In order to get better invariants. The hope is (and experience provides some justification) that by choosing exactly the "right" theory we can solve geometric problems.

One example is the "Hopf invariant-one" problem: When does the n-sphere S^n have a continuous multiplication with an identity? Answer: for $n = 0$, 1, 3 or 7. This was solved in 1960 by Adams using ordinary cohomology (85 pages), and then in 1966 by Adams and Atiyah using K-theory (8 pages) — the "postcard proof".

Complex-oriented theories

We can include the complex projective spaces

$$\mathbb{C}P^1 \subset \mathbb{C}P^2 \subset \cdots \subset \mathbb{C}P^n \subset \mathbb{C}P^{n+1} \subset \cdots$$

and form the union $\mathbb{C}P^\infty$.

The first of these complex projective spaces $\mathbb{C}P^1$ is the 2-sphere (Riemann sphere). The exactness axiom means that we know its E-cohomology: $E^n(\mathbb{C}P^1) = E^n(S^2) \cong E^{n-2}$. (This is only the first of a whole range of spaces whose E-cohomology is easily computed for complex-oriented theories. The FGL which we are about to define is the key to many of these constructions.)

A cohomology theory $E^*(\_)$ is *complex-oriented* if there is an element $t \in E^2(\mathbb{C}P^\infty)$ which under the inclusion $\mathbb{C}P^1 \subset \mathbb{C}P^\infty$ corresponds to $1 \in E^0$.

All the examples above ($H^*(\_)$, $K^*(\_)$ and $MU^*(\_)$) are complex-oriented. An example which is not is the theory $MSO^*(\_)$ built out of *real* oriented manifolds modulo bordism.

A consequence of complex orientability is that $E^*(\mathbb{C}P^\infty) = E^*[[t]]$.

We could think of $\mathbb{C}P^\infty$ as

$$\frac{\text{complex polynomials}}{\text{scalar multiplication}} \cong \text{monic polynomials}.$$

Multiplication of polynomials provides a continuous product

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\mu} \mathbb{C}P^\infty$$

and hence a ring homomorphism

$$E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \xleftarrow{\mu^*} E^*(\mathbb{C}P^\infty).$$

The complex-orientability assumption also gives

$$E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[t_1, t_2]]$$

and μ^* is determined by the power series

$$\mu^*(t) = F^E(t_1, t_2) \in E^*[[t_1, t_2]].$$

Note that μ^* is continuous with respect to the t-adic topology. The product μ has an identity (constant polynomials) and is associative, hence $F^E(t_1, t_2)$ is a FGL over the ring E^*.

This observation was first made explicitly by Quillen in 1969; Boardman and Novikov were also studying μ^* at the same time.

Examples.

(1) $F^H(t_1, t_2) = t_1 + t_2$, the additive FGL.
(2) $F^K(t_1, t_2) = t_1 + t_2 + ut_1t_2$, a graded version of the multiplicative FGL.

Universal examples

It is rather easy to see that there is a *universal* FGL. This is a ring L and a FGL

$$F^L(x, y) = x + y + \sum_{i,j>0} a_{ij} x^i y^j \in L[[x, y]]$$

such that for any FGL $F(x, y)$ over a ring R, there is a ring homomorphism $\varphi : L \to R$ such that $\varphi F^L = F$, i.e.,

$$F(x, y) = x + y + \sum_{i,j>0} \varphi(a_{ij}) x^i y^j \in R[[x, y]].$$

To form L, we just take the polynomial ring $\mathbb{Z}[a_{ij} : i, j > 0]$ and impose the relations given by commutativity and associativity. Thus $a_{ij} = a_{ji}$; the first non-trivial relation is $2a_{11}a_{12} = 2a_{22} - 3a_{13}$.

Theorem (Lazard, 1950s)**.** $L = \mathbb{Z}[v_1, v_2, \dots]$.

Theorem (Quillen, 1969)**.** *$MU^* = L$ and $F^{MU} = F^L$.*

Every complex-oriented cohomology theory comes equipped with a natural transformation $MU^*(\_) \to E^*(\_)$. On coefficient rings this gives the map $L = MU^* \to E^*$ which classifies the FGL F^E for E-cohomology.

This correspondence of universal objects implies a close connection, which we still don't fully understand, between FGLs and complex-oriented cohomology theories.

Logarithms

The second FGL we wrote down was $\sin\big(\sin^{-1}(x) + \sin^{-1}(y)\big)$. Can we always write FGLs in such a simple form, i.e., write

$$F(x, y) = g\big(f(x) + f(y)\big), \quad \text{where } f = g^{-1}?$$

Here $f(x) = x + a_1x^2 + a_2x^3 + \cdots$.

Rewrite this as $f\big(F(x, y)\big) = f(x) + f(y)$ and differentiate with respect to y:

$$\frac{\partial F}{\partial y} f'\big(F(x, y)\big) = f'(y).$$

Now let $y = 0$:

$$\left.\frac{\partial F}{\partial y}\right|_{y=0} f'(x) = f'(0) = 1 \qquad [\text{recall } F(x,0) = x]$$

so that

$$f(x) = \int \frac{dx}{\left.\frac{\partial F}{\partial y}\right|_{y=0}}.$$

Examples.

(1) For the multiplicative FGL $F(x,y) = x + y + xy$,

$$\frac{\partial F}{\partial y} = 1 + x,$$

so that

$$f(x) = \int \frac{dx}{1+x} = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots,$$

and

$$g(x) = e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

(2) For the FGL $F(x,y) = x\sqrt{1-y^2} + y\sqrt{1-x^2}$,

$$\frac{\partial F}{\partial y} = -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}, \qquad \left.\frac{\partial F}{\partial y}\right|_{y=0} = \sqrt{1-x^2},$$

and

$$f(x) = \sin^{-1}(x).$$

(3) For the Euler FGL

$$F(x,y) = \frac{x\sqrt{1-y^4} + y\sqrt{1-x^4}}{1+x^2y^2},$$

$$\left.\frac{\partial F}{\partial y}\right|_{y=0} = \sqrt{1-x^4},$$

and

$$f(x) = \int \frac{dx}{\sqrt{1-x^4}}.$$

The process works, as long as the ring R over which the FGL is defined is torsion-free, but the series

$$f(x) = \text{the log series} \qquad \text{and} \qquad g(x) = f^{-1}(x) = \text{the exp series}$$

are power series over $R \otimes \mathbb{Q}$. The names "log" and "exp" come from the multiplicative case.

In fact, since

$$f'(x) = \frac{1}{\left.\frac{\partial F}{\partial y}\right|_{y=0}} \in R[[x]],$$

$$f(x) = \sum_{n\geq 1} c_{n-1}\frac{x^n}{n}, \quad \text{with } c_{n-1} \in R, \text{ and } c_0 = 1,$$

and

$$g(x) = \sum_{n\geq 1} e_{n-1}\frac{x^n}{n!}, \quad \text{with } e_{n-1} \in R, \text{ and } e_0 = 1.$$

In the universal case $F^{MU}(x,y)$ we have $c_n = [\mathbb{C}P^n] \in MU^{-2n}$.

Lagrange inversion gives a formula for e_n in terms of the c_k in which the coefficient of $c_1^{k_1} c_2^{k_2} \cdots c_s^{k_s}$ is

$$(-1)^{k_1+k_2+\cdots+k_s} \binom{n+k_1+k_2+\cdots+k_s}{2k_1, 3k_2, \ldots, (s+1)k_s} \frac{(2k_1)!}{2^{k_1}k_1!} \frac{(3k_2)!}{3^{k_2}k_2!} \cdots \frac{\big((s+1)k_s\big)!}{(s+1)^{k_s}k_s!}$$

This leads to *Kummer congruences* for the coefficients of the exp series of a FGL, the first of which is

$$e_{n+p-1} \equiv e_n e_{p-1} \mod p \quad \text{in } R,$$

for any prime p.

Landweber Exactness

Which FGLs arise from cohomology theories? All of them? There is no strong evidence for this guess, but there are still no counterexamples. We will look at one approach to constructing a cohomology theory with a given FGL.

Suppose $F(x,y) \in R[[x,y]]$ is a FGL defined over a ring R. By universality there is a homomorphism $\varphi : L \to R$ such that $\varphi F^L(x,y) = F(x,y)$. Quillen's theorem tells us that $MU^* = L$ so φ gives R the structure of an MU^*-module. We define

$$E_F^*(X) = R \otimes_{MU^*} MU^*(X).$$

This is intended to be a cohomology theory whose coefficient ring is R and whose FGL is F.

But this doesn't work: tensoring can destroy exact sequences, so the exactness axiom for $E_F^*(\_)$ fails, in general.

There are cases where the construction does give a cohomology theory.

Theorem (Conner-Floyd, 1966)**.**

$$K^*(X) = K^* \otimes_{MU^*} MU^*(X).$$

This result says that in some sense K-theory comes directly from the multiplicative FGL.

In 1976, Landweber gave necessary and sufficient criteria for $E_F^*(\_)$ to be a cohomology theory, i.e., to satisfy the exactness axiom:

If n is a natural number, the *n-series* of a FGL F is defined as

$$[1]_F(x) = x, \quad [2]_F(x) = F(x,x), \quad [3]_F(x) = F\big(F(x,x),x\big), \quad \ldots,$$
$$[n]_F(x) = F\big([n-1]_F(x),x\big).$$

If f and g are the log and exp series, then $[n]_F(x) = g\big(nf(x)\big)$.

Now pick a prime p and write

$$[p]_F(x) = px + \cdots + u_1x^p + \cdots + u_2x^{p^2} + \cdots + u_jx^{p^j} + \cdots.$$

Landweber's criteria say that, for all primes p, the sequence

$$p, u_1, u_2, \ldots, u_j, \ldots$$

is *regular* in R, that is

$$\text{multiplication by } u_j \text{ on } R/(p, u_1, u_2, \ldots, u_{j-1}) \text{ is injective.}$$

Note that this includes the case $u_0 = p$, so that R is torsion-free.

Consequence. FGLs satisfying these criteria give rise to (new) cohomology theories.

Examples.

(1) Take the graded multiplicative FGL
$$x + y + uxy = \frac{(1+ux)(1+uy)-1}{u}$$
over the ring $\mathbb{Z}[u, u^{-1}]$. Its p-series is
$$[p](x) = \frac{(1+ux)^p - 1}{u} = px + \binom{p}{2}ux^2 + \cdots + u^{p-1}x^p,$$
so $u_1 = u^{p-1}$, and $u_j = 0$ for $j > 1$.

The ring $\mathbb{Z}[u, u^{-1}]$ is certainly torsion-free, and multiplication by u^{p-1} on $\mathbb{Z}/p\mathbb{Z}[u, u^{-1}]$ is injective. In fact it is bijective, so that all the higher quotients are zero. This makes Landweber's criteria hold; we get K-theory. This is the result of Conner and Floyd cited above.

(2) (Landweber, Ravenel, Stong, 1986) Modify the Euler FGL with which we began, to obtain
$$F^{\mathrm{Ell}}(x,y) = \frac{x\sqrt{1-2\delta y^2+\varepsilon y^4} + y\sqrt{1-2\delta x^2+\varepsilon x^4}}{1-\varepsilon x^2y^2},$$
whose log series is the elliptic integral
$$\int \frac{dx}{\sqrt{1-2\delta x^2+\varepsilon x^4}} = \sum_{n\geq 0} \varepsilon^{n/2} P_n\big(\tfrac{\delta}{\sqrt{\varepsilon}}\big)\frac{x^{2n+1}}{2n+1},$$
where $P_n(z)$ is the Legendre polynomial. This FGL is defined over the ring $\mathbb{Z}[\frac{1}{2}, \delta, \varepsilon]$. This ring should be thought of as a ring of modular forms, in which δ and ε are particular Eisenstein series. That the Landweber criteria are satisfied (when the ring is extended by inverting the discriminant $\Delta = \delta^2 - \varepsilon$) can be thought of either as expressing certain congruences amongst the Legendre polynomials, or as reflecting a fact about FGLs arising from elliptic curves.

The cohomology theory $\mathrm{Ell}^*(\_)$ which we obtain is *elliptic cohomology*. The weakness in this approach is that $\mathrm{Ell}^*(X)$ is defined algebraically. We still do not have a geometric definition (analogous to the way K-theory is given by vector bundles) though there are many hints that such a description does exist. The absence of a geometric way of thinking of elliptic cohomology is not necessarily an obstacle to doing calculations. For example recent work of Gerd Laures detects classes in the homotopy groups of spheres by using results on the q-expansions of modular forms. But the lack of geometry does limit our understanding.

(3) The same can be done for the FGL with logarithm
$$\int \frac{dx}{\sqrt[p-1]{1+a_1x^{p-1}+\cdots+a_px^{p(p-1)}}},$$
where p is prime. This is recent work of Hopkins and Mahowald. Here cohomology theories are associated to curves of higher genus.

Examples over $\mathbb{Z}$

FGLs over the integers are classified. Transform the log series into a Dirichlet series (by a Mellin transform):
$$f(x) = \sum_{n\geq 1} c_{n-1}\frac{x^n}{n} \longmapsto \sum_{n\geq 1}\frac{c_{n-1}}{n^s}.$$

Then there is an Euler product

$$\sum_{n\geq 1} \frac{c_{n-1}}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{b_{1,p}}{p^s} - \cdots - \frac{p^{k-1} b_{k,p}}{p^{ks}} - \cdots\right)^{-1},$$

where $b_{k,p} \in \mathbb{Z}$.

Consider the cases

$$b_{1,p} = \pm(1+p), \qquad b_{2,p} = -1, \qquad b_{k,p} = 0 \quad (k > 2).$$

We can make a choice of sign for each prime.

Landweber's criteria are satisfied, giving us uncountably many distinct FGLs thus uncountably many cohomology theories. These are "fake" K-theories.

One example gives us genuine K-theory. Taking $b_{1,p} = 1 + p$ for all p, gives

$$f(x) = \sum_{n\geq 1} \sigma(n) \frac{x^n}{n},$$

where $\sigma(n)$ is the sum of the divisors of n. This is the multiplicative FGL in disguise, and so gives rise to K-theory. The identity

$$e^{f(x)} = \sum_{n\geq 0} p(n) x^n,$$

where $p(n)$ is the partition function shows how to change coordinates to get the multiplicative FGL.

Out of all these cohomology theories, only one of them has a geometric definition. However they all detect exactly the same homotopy classes.

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