Bootstrapping Big Data

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The Setting

Observe data $X_1, \ldots, X_n$

Form an estimate $\hat{\theta}_n = \theta(X_1, \ldots, X_n)$
(e.g., θ could be a classifier)

Want to compute an assessment ξ of the quality of $\hat{\theta}_n$
(e.g., ξ could compute a confidence region)

Our Goal

A procedure for quantifying estimator quality which is

accurate
automatic
scalable

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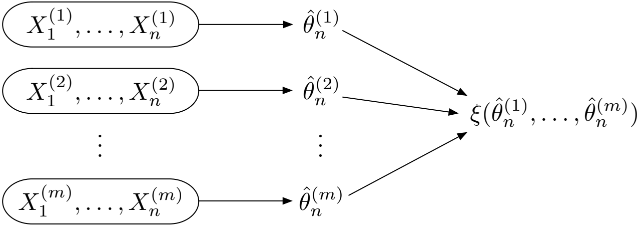

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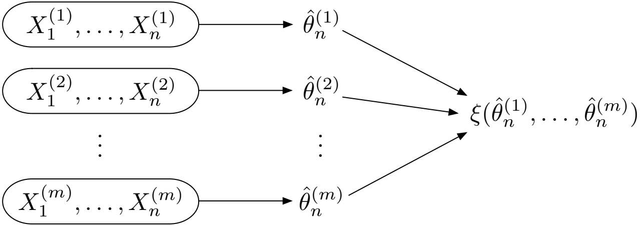

But, we only observe *one* dataset of size n

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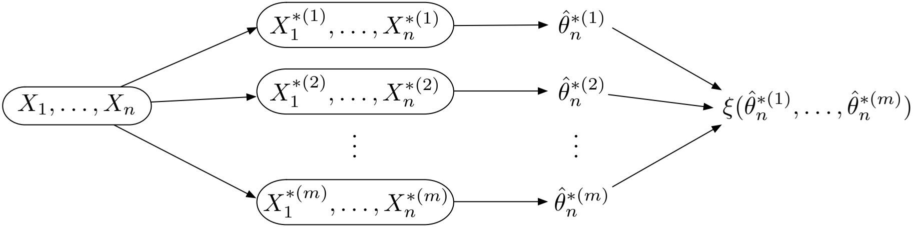

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  - Use weighted representation of resampled datasets to avoid physical data replication.
  - Example: If original dataset has size 1 TB, then expect resample to have size $\sim$ 632 GB.

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Suppose that the original dataset has size 1 TB. The bootstrap does the following:

```
for i ← 1 to 300
      resample ∼ 632 GB of data
      compute θ on resample
compute ξ based on the resampled θ's
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Disadvantages

- Must repeatedly compute θ on $\sim$ 63% of the data
- For big data, difficult to parallelize across different computations of θ (though θ could perhaps be parallelized internally)

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Issues

- Accuracy sensitive to choice of $b(n)$.
- Still fairly automatic, though analytical correction introduces some dependency on internals of θ.

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- Similar results obtained with Normal and StudentT data generating distributions, as well as if estimate a misspecified model.

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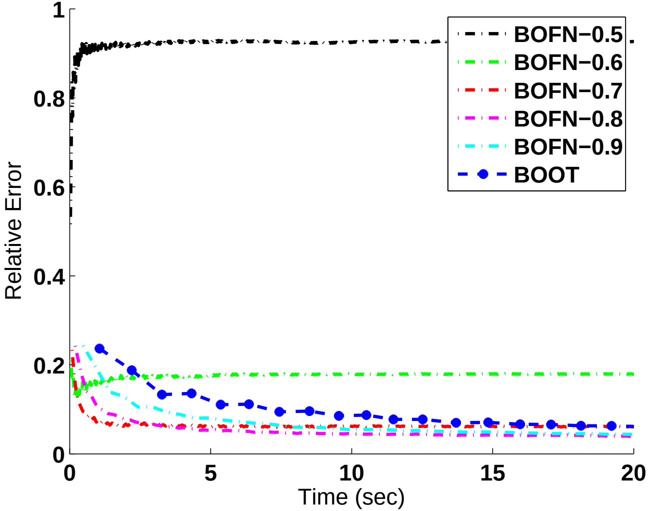

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3. We now have one estimate of ξ per subsample. Output their average as our final estimate of ξ for $\hat{\theta}_n$.

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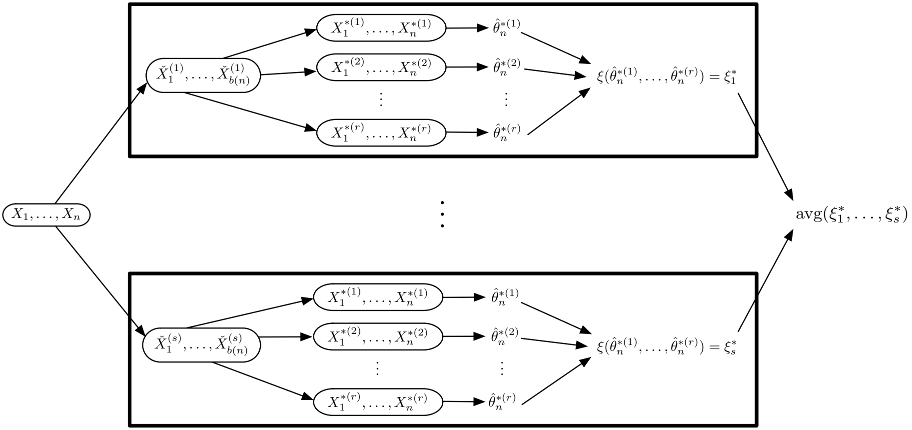

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  - (in contrast, bootstrap resamples have size $\sim$ 632 GB)

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- Easy to parallelize across different computations of θ (in addition to parallelizing θ internally).

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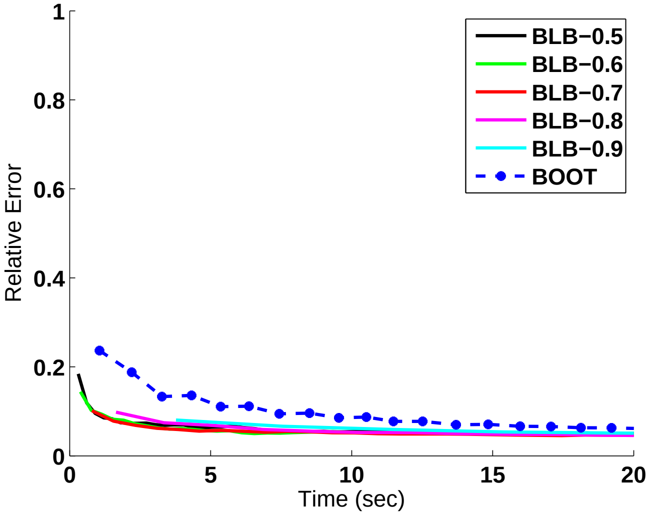

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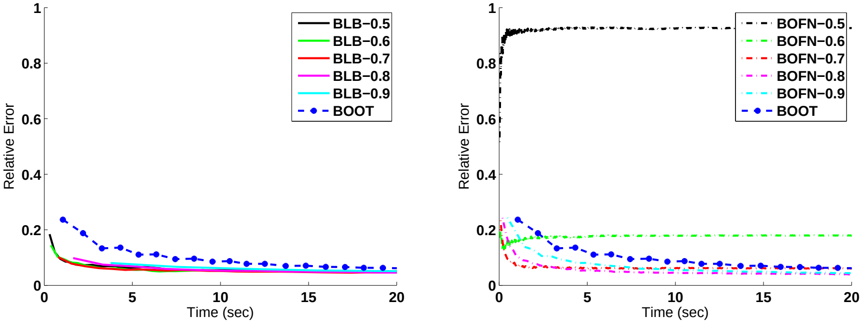

BLB: Theoretical Results

BLB shares the bootstrap's favorable statistical properties (consistency & higher-order correctness)

under the same conditions that have been used in prior analysis of the bootstrap