École polytechnique fédérale de Lausanne

Master Thesis

About: convolution of $\mathbb{Q}_\ell$-sheaves on the affine line over a field of characteristic zero

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Ciao Pa'.
Questa è finita
un po'
come quella di Reggie al Garden.

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Contents

Chapter 1

Introduction

1.1 What is this master thesis about?

The core of this master thesis is presented in chapter 2 and chapter 3 of this manuscript. These two chapters have a different nature and represent the two objectives of my thesis, respectively.

The first and personal aim was to enter in the realm of étale cohomology.
Since the first time I read the "étale" on a blackboard, I felt that the mysterious word had to be understood.
Unfortunately the étale cohomology theory relies on a formalism made of sophisticated algebraic tools and, before this thesis, most of them were unknown to me.
During these months my approach to fight the lack of background was based on: (1) read [Sza09] to understand *la raison d'être* of the theory, (2) read [Fu11] to learn the theorems and definitions, (3) read parts of SGA to understand the nasty details, (4) feel stupid.
Chapter 2 is an exposition of what I've learned doing (1) (2) and (3).

The second, more concrete, aim of this master thesis was to understand and formulate a question regarding a notion of convolution that can be defined on sheaves.
The *additive convolution* of sheaves on $\mathbb{A}^1_k$, is a convolution operation defined on the level of the derived category of sheaves. When restricted to a simpler and appropriate subcategory (namely the category $\mathbf{Perv}_0(k)$, definition 3.2.2.1) this convolution produce non trivial sheaves and the problem of "explicitly compute" the convolution is an approachable problem. When $k = \mathbb{C}$ and the sheaves are usual topological sheaves, this computation is explicitly done in [FJ18, chapter 2].
Part of this master thesis was to understand, define and try to solve the analogue question in étale cohomology, and with k a field of characteristic zero.
Right now only the first two tasks are done and they are the content of chapter 3.

1.2 Detailed overture

The rest of chapter 1

After the overture, chapter 1 is devoted to prepare the reader for this reading. I give the prerequisites and I explain my conventions and notation.

Chapter 2

Chapter 2 is a collection of results in étale cohomology. It does not contain proofs, but references are given for each statement. The aim is mainly to give both a blueprint and an exposition of the theory I assume in chapter 3.

The first section is devoted to define the fundamental group of a scheme. The fundamental properties are given at the end of the section.

The second section is about étale sheaves, sites and contains the definition of the étale cohomology groups.

The the third section is an introduction to the *six functors formalism*. This section is an attempt to define and describe the six main tools to treat sheaves. I personally think that this formalism can be used as black box. Therefore I've tried to: (1) hide as much details as I could, (2) favor a logical order rather than a mathematical-chronological order. Because of (1), apologizes in advance if this section is not helpful. Because of (2), apologizes in advance if there are loops or redundant theorems. There is a risk that my attempt turned out merely in a copy of very personal, hopefully good, lecture notes.

The first part of the fourth section is a collection of results in étale cohomology, whose proofs, in some sense, are not related to étale cohomology. The second part of the section combines these results with the theory of the third and I compute the étale cohomology, with coefficients in $\mathbb{Z}/\ell^n\mathbb{Z}$, of an arbitrary smooth curve over a separably closed field of characteristic $p \neq \ell$, where ℓ is a prime number. The section ends with a couple of more results in étale cohomology

The last and fifth section explains the passage to the ℓ-adic formalism. There will be no discussions about the trace formula or other important results. This section ends with the fundamental equivalence of categories between lisse constructible sheaves and continuous representations of the fundamental group.

Chapter 3

Chapter 3 is about the additive convolution of constructible $\mathbb{Q}_\ell$-sheaves on $\mathbb{A}^1_k$ with k a field of characteristic zero.

The first section explains how to compute the ℓ-adic cohomology of constructible $\mathbb{Q}_\ell$-sheaves on $\mathbb{A}^1_k$.

The first part of the second section contains the definition and some properties of the additive convolution. This convolution is defined on the derived category of constructible $\mathbb{Q}_\ell$-sheaves, denoted $D^b_c(\mathbb{A}^1_k, \mathbb{Q}_\ell)$. The convolution of two objects in $D^b_c(\mathbb{A}^1_k, \mathbb{Q}_\ell)$ is not necessarily an object of $D^b_c(\mathbb{A}^1_k, \mathbb{Q}_\ell)$.
The second part of the section is dedicated to the definition of some subcategories of $D^b_c(\mathbb{A}^1_k, \mathbb{Q}_\ell)$, that are more stable convolution-wise, namely $\mathbf{Perv}(k)$ and $\mathbf{Perv}_0(k)$. The sections ends with

the proof that $\mathbf{Perv}_0(k)$ is stable under the convolution operation.[1]

The third section is about a subcategory of $\mathbf{Perv}_0(k)$, denoted $\mathbf{Perv}_{00}(k)$. This category is also stable under convolution. The objects in this category are lisse $\mathbb{Q}_\ell$-sheaves on $\mathbb{A}^1_k \setminus \{0\} := \mathtt{Spec}(k[T^{\pm 1}])$. The section contains the computation of the étale fundamental group of $\mathbb{A}^1_k \setminus \{0\}$, hence a description of the objects in $\mathbf{Perv}_{00}(k)$ in terms of continuous representations of $\pi_1^{\text{ét}}(\mathbb{A}^1_k \setminus \{0\})$.

The fourth section is very short and merely contains a formulation of the main questions about convolution.

The fifth and last section contains the computation of a convolution in $\mathbf{Perv}_{00}(k)$ and shows that the results are non trivial and interesting.

Appendix

The appendix is divided in two parts, A and B.
Appendix A contains a blue print of the formalism of derived categories.
Appendix B consists of: (1) a reference for the Riemann-Hurwitz formula for functions fields in one variable together with an example, (2) the abstract part of the direct and inverse image functors. The example in (1) is used in the proof of lemma 3.3.2.1, but it is not related to the rest of the manuscript. The reason why (2) is part of the appendix is explained above definition 2.3.1.1.

1.3 Assumptions, notation, conventions and some definitions

1.3.1 Assumptions on the reader

I assume that the reader is comfortable with the basic language of homological algebra, category theory and algebraic geometry. For instance, to read this text, it could be enough have a quick reading of [Vak] parts I, II, III and chapter 18.

1.3.2 Conventions for separable and algebraic extensions

Given a field k I denote by k^{sep} an implicit choice of a separable closure of k. Similarly I denote by k^{alg} an implicit choice of an algebraic closure of k. Since these choices are not canonical I remark now that, in every definition, statement, argument, etc. containing the symbols k^{sep} and k^{alg}, a choice is being made and that every other symbols in such definition, statement argument, etc. may depend on that choice.

If a field k has characteristic 0 (or k a perfect field) then $k^{\text{alg}} = k^{\text{sep}}$. When applying to such k a general result involving the symbol k^{sep} I will write k^{alg} without making the trivial remark that k^{sep} and k^{alg} are the same.

[1] I also wanted to talk about the fact that $\mathbf{Perv}_0(k)$ has the structure of a Tannakian neutral category, but for k an arbitrary fields, the proofs are not complete. For an introduction on Tannakian categories cf. [Sza09] chapter 6 and for the case $k = \mathbb{C}$ cf. [FJ18] chapter 2).

1.3.3 Notation and conventions on schemes

A *ring* is a commutative ring with unity.

All schemes are supposed to be noetherian.

The symbol ς is always used to refer to the structure morphism of a scheme over a field. If X is a scheme over a field k, I implicitly denote the morphism $X \to k$ by ς.

If X is a scheme I will systematically use the symbol $\mathcal{O}_X$ to denote its structure sheaf.

If $\mathfrak{p}$ is a point of a scheme X, $\kappa(\mathfrak{p})$ is the residue field at $\mathfrak{p}$. The scheme associated to the point $\mathfrak{p}$ is $\mathtt{Spec}(\kappa(\mathfrak{p}))$ and the morphism

$$x : \mathtt{Spec}(\kappa(\mathfrak{p})) \to X, \tag{1.3.1}$$

"is" the point $\mathfrak{p}$.

I've tried to systematically use gothic fonts for points of a scheme, and I've tried to reserve the use of latin letters for morphisms like (1.3.1). If x and $\mathfrak{p}$ are like above, I will write $x = \mathfrak{p}$. If $x = \mathfrak{p}$, by $\overline{x}$ I always mean, the *geometric* point $\mathtt{Spec}(\kappa(x)^{\mathrm{sep}}) \to X$ above x.

Let k be a field and let $\mathtt{Spec}(k) \to S$ and $X \to S$ be morphisms of schemes. Instead of writing $X \times_S \mathtt{Spec}(k)$ I will often write $X \otimes_S k$ or $X \otimes k$.

1.3.4 Notation and conventions on categories

I have made an *index of categories* that contains all the symbols that were used to denote precise categories without introducing the notation. In this index there may be categories that have not been used. This index is placed after the appendix.

A *functor* always mean a covariant functor. If $\mathbf{C}$ is a category, denote by $\mathbf{C}^{\mathrm{opp}}$ its opposite category.

"A category $\mathbf{C}$", if not stated otherwise, always means "a small category $\mathbf{C}$".

Functors between abelian categories are implicitly supposed additive, even if not stated explicitly.

Given a category $\mathbf{C}$ and a category $\mathbf{D}$ I denote by $\mathbf{Hom}(\mathbf{C}, \mathbf{D})$ the category of functors between them:

$\mathbf{Hom}(\mathbf{C}, \mathbf{D}) :=$

- Objects of $\mathbf{Hom}(\mathbf{C}, \mathbf{D})$: functors from $\mathbf{C}$ to $\mathbf{D}$
- Morphisms of $\mathbf{Hom}(\mathbf{C}, \mathbf{D})$: given two functors $\mathscr{F}, \mathscr{G} : \mathbf{C} \to \mathbf{D}$ a morphism between them (or a *natural transformation*) is a collection of morphism $\{f_X \in \mathrm{Hom}_{\mathbf{D}}(\mathscr{F}(X), \mathscr{G}(X))\}_X$, where X run over the objects of $\mathbf{C}$, such that for any morphism $X_1 \xrightarrow{\phi} X_2$, $\mathscr{G}(\phi) \circ f_{X_1} = f_{X_2} \circ \mathscr{F}(\phi)$

There is also the category of contravariant functors from **C** to **D**. It will be denoted by $\mathbf{PS_D}(\mathbf{C})$ and I will use the words *contravariant functor from* **C** *to* **D** and *presheaf on* **C** *with values in* **D** as synonym.

Notice that for any functor (resp. presheaf) $\mathscr{F}$, the automorphisms of $\mathscr{F}$ form group $\mathrm{Aut}_{\mathbf{Hom(C,D)}}(\mathscr{F})$ (resp. $\mathrm{Aut}_{\mathbf{PS_D(C)}}(\mathscr{F})$). This group will be denoted by $\mathrm{Aut}(\mathscr{F})$.

1.3.5 Sheaves and complexes of sheaves

For a sheaf on a scheme X I always use curly letters like $\mathscr{F}, \mathscr{G}, \mathscr{L}...$
For complexes of sheaves on X and for objects in the derived category of sheaves I always use capital letters like $F, G, L...$

1.3.6 Over and Under categories

Let **C** be any category. For any object X in **C** I define the categories $\mathbf{Over}(X)$ and $\mathbf{Under}(X)$. These are non-standard notions/notation.

$$\mathbf{Over}(X) := \begin{cases} \bullet \text{ Objects of } \mathbf{Over}(X)\text{: morphisms in } \mathbf{C} \text{ of the form } Y \to X \text{ for some object } Y \text{ in } \mathbf{C}. \text{ I say that } Y \text{ is an object over } X \text{ with structure morphism } Y \to X. \\ \bullet \text{ Morphisms of } \mathbf{Over}(X)\text{: } \mathrm{Hom}_{\mathbf{Over}(X)}(S_1, S_2) \text{ consists in morphisms in } \mathrm{Hom}_{\mathbf{C}}(Y_1, Y_2) \text{ commuting with the structure morphisms} \end{cases}$$

$$\mathbf{Under}(X) := \begin{cases} \bullet \text{ Objects of } \mathbf{Under}(X)\text{: morphisms in } \mathbf{C} \text{ of the form } X \to Y \text{ for some object } Y \text{ in } \mathbf{C}. \text{ I say that } Y \text{ is an object under } X \text{ with structure morphism } X \to Y. \\ \bullet \text{ Morphisms of } \mathbf{Under}(X)\text{: } \mathrm{Hom}_{\mathbf{Under}(X)}(Y_1, Y_2) \text{ consists in morphisms in } \mathrm{Hom}_{\mathbf{C}}(Y_1, Y_2) \text{ commuting with the structure morphism} \end{cases}$$

Let $\phi : Y \to X$ be a morphism in C. Using the over and under, the *neighborhood category* of ϕ could be defined as the category

$$\mathbf{Nbh}(\phi) := \begin{cases} \bullet \text{ Objects of } \mathbf{Nbh}(\phi)\text{: commutative diagrams of the form } \begin{array}{ccc} Y & \longrightarrow & U \\ & \searrow & \downarrow \\ & & X \end{array} \\ \bullet \text{ Morphisms of } \mathbf{Nbh}(\phi)\text{: } \mathrm{Hom}_{\mathbf{Nbh}(\phi)}(U_1, U_2) \text{ consists in morphisms in } \mathrm{Hom}_{\mathbf{Under}(Y)}(U_1, U_2) \cap \mathrm{Hom}_{\mathbf{Over}(X)}(U_1, U_2) \end{cases}$$

Many constructions and definitions are faster to state and a bit more intuitive if one uses (full sub-categories of) over and under categories.

Example 1.3.6.1. The categories $\mathbf{Alg}_R$ and $\mathbf{Under}(R)$ are tautologically the same.

Example 1.3.6.2. Let k be a field. The category $\mathbf{Under}(k)$ (in the category of fields) can be considered as the category of field extension of k. Indeed each object of $\mathbf{Under}(k)$ is given by an inclusion of fields $k \hookrightarrow L$.

Example 1.3.6.3. Let S be a scheme. The objects of $\mathbf{Over}(S)$ are called S-schemes. (If $S = \mathtt{Spec}(R)$ one says that objects of $\mathbf{Over}(S)$ are R-schemes).
The category of schemes admits fiber products, therefore given an S-scheme Z one can define the *base change functor*

$$\begin{aligned} \mathbf{Over}(Z \to S) : \mathbf{Over}(S) &\longrightarrow \mathbf{Over}(Z) \\ X &\longmapsto X \times_S Z, \end{aligned}$$

where the structure morphism of $X \times_S Z$ is given by the projection to Z.

Example 1.3.6.4. Let X be a connected, locally simply connected, topological space. Define the category $\mathbf{Cov}(X)$ as the full subcategory of $\mathbf{Over}(X)$ whose objects are covering spaces $Y \to X$ of X. Let $x : \{x\} \to X$ the inclusion of some point x in X. The category **Top** has fiber products and the base change functor $\mathbf{Over}(x) : Y \to Y \times_X \{x\}$ sends a covering space $Y \xrightarrow{\phi} X$ to $\phi^{-1}(x) \to x$, itself a covering space of x. The functor $\mathbf{Over}(x)$ restricts to a functor $\mathbf{Cov}(X) \to \mathbf{Cov}(x)$. Denote it by $\varpi(x)$. The automorphism group $\mathrm{Aut}(\varpi(x))$ of the functor $\varpi(x)$ is canonically isomorphic to $\pi_1(X, x)$.

Chapter 2

Quick and dirty introduction to ℓ-adic cohomology

ATTENTION: in this chapter the notation introduced in 1.3.6 is heavily used.

2.1 Fundamental groups of schemes

The fundamental group of a connected, locally simply connected, topological space X “classifies” the covering spaces of X.
The absolute Galois group of a field k “classifies” the separable extensions of k.
These two notions, that at first sight seem different, are in some sense the same. To see this one has to look at them from a category-theory point of view. This is exactly what was done by Grothendieck in [Gro71, V] where he gave the axioms of a Galois category, i.e. a category with a structure that allows to define a *fundamental group* that behaves just like the ones in the topological and Galois setting.

Let X be a scheme. The category $\mathbf{Over}(X)$ has a subcategory, namely $\mathbf{Over}_{\text{ét}}(X)$, which is a Galois category. The formation of its fundamental group is “coherent” when “compared” with the fundamental group of a topological space or with the Galois group of a field and, in some sense, is a patching of these two.

2.1.1 Étale morphisms

Definiton 2.1.1.1 (Flat modules). Let R be a ring. An object M in $\mathbf{Mod}_R$ is *flat* if the functor

$$\begin{aligned} \mathbf{Mod}_R &\longrightarrow \mathbf{Mod}_R \\ N &\longmapsto N \otimes_R M \end{aligned}$$

is exact.

In practice, to check flatness, one has:

Lemma 2.1.1.1 ([Eis13], Proposition 6.1). A R-module M is flat if and only if, for all finitely generated ideals $I \subset R$, the multiplication morphism

$$I \otimes_R M \longrightarrow M,$$

given by $r \otimes m \mapsto rm$, is injective.

Definiton 2.1.1.2 (Flat morphism). Let $X \xrightarrow{\phi} Y$ a morphism of schemes. One says that ϕ is flat at $x \in X$ if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,\phi(x)}$-module.

Definiton 2.1.1.3 (Unramified morphisms, [Sta20, Tag 02G3]). A morphism of schemes $X \xrightarrow{\phi} Y$ is *unramified at* $\mathfrak{p} \in X$ if it is of finite type and if any of the following conditions is satisfied.

1. Let $\mathfrak{q} = \phi(\mathfrak{p})$. Then $\kappa(\mathfrak{q}) \to \kappa(\mathfrak{p})$ is a finite separable extension and $\mathfrak{m}_\mathfrak{p} = \mathfrak{m}_\mathfrak{q}\mathcal{O}_{X,\mathfrak{p}}$, where $\mathfrak{m}_\mathfrak{p}$ is the maximal ideal of $\mathcal{O}_{X,\mathfrak{p}}$ and $\mathfrak{q}$ is the maximal ideal of $\mathcal{O}_{Y,\mathfrak{q}}$;
2. $(\Omega_{X/Y})_x = 0$ (where $\Omega_{X/Y}$ is the sheaf of relative differentials, cf. [Sza09] 5.1.26 or[Vak] chapter 21 for a detailed explanation);
3. the diagonal map $X \xrightarrow{\Delta} X \times_Y X$ is an open immersion around x (i.e. there is an open set $U \subset X$ such that $\Delta|_U$ is an open immersion).

If ϕ ramifies at x, one says that $\phi(x)$ is a *branch point* of ϕ.

Example 2.1.1.1. Consider the morphism

$$\phi : \mathrm{Spec}(\mathbb{Q}[T^{1/n}]) \to \mathrm{Spec}(\mathbb{Q}[T])$$

induced by the canonical inclusion $\mathbb{Q}[T] \hookrightarrow \mathbb{Q}[T^{1/n}]$. The point $\mathfrak{p} = (T) \in \mathrm{Spec}(\mathbb{Q}[T])$ has residue field

$$\kappa(\mathfrak{p}) = \mathbb{Q}[T]_\mathfrak{p}/\mathfrak{p}\mathbb{Q}[T]_\mathfrak{p} = \mathbb{Q}.$$

The fiber at $\mathfrak{p}$ is given by $\mathrm{Spec}(\mathbb{Q}[T^{1/n}] \otimes_{\mathbb{Q}[T]} \mathbb{Q}) \simeq \mathrm{Spec}(\mathbb{Q}[T]/T^n)$, thus ϕ has a branch point at $\mathfrak{p}$. Now consider the point $\mathfrak{q} = (T^2+1) \in \mathrm{Spec}(\mathbb{Q}[T])$, with residue field $\mathbb{Q}[i]$. The fiber at $\mathfrak{q}$ is therefore $\mathrm{Spec}(\mathbb{Q}[i,T]/(T^{2n}+1))$, hence $\mathfrak{q}$ is not a branch point of ϕ.

Definiton 2.1.1.4 (Étale morphisms). A morphism of schemes $X \to Y$ is *étale* at $x \in X$ if it is

(i) flat at x;

(ii) unramified at x.

If ϕ is étale at each x, ϕ is *étale*.

Definiton 2.1.1.5. Denote by $\mathbf{Over}_{\text{ét}}(X)$ the full subcategory of $\mathbf{Over}(X)$ whose objects are étale (cf. paragraph 1.3.6).

Proposition 2.1.1.1 ([Gro67], 17.3.3).

(i) Open immersions are étale

(ii) The composition of two étale morphisms is étale.

(iii) Morphisms in $\mathbf{Over}_{\text{ét}}(X)$ are étale.

(iv) If $\phi : X \to X'$ and $\psi : Y \to Y'$ are two étale morphisms in $\mathbf{Over}(S)$, then $\phi \times_S \psi : X \times_S Y \to X' \times_S Y'$ is étale.

Corollary 2.1.1.1 ([AGV71], VII.1.4). Let $X \xrightarrow{\phi} Y$ be a morphism of schemes. The base change functor defined in example 1.3.6.3 restricts to a functor $\mathbf{Over}_{\text{ét}}(Y) \to \mathbf{Over}_{\text{ét}}(X)$. Denote it by $\mathbf{Over}_{\text{ét}}(\phi)$.

Finite étale covers

Definiton 2.1.1.6. A finite *étale cover* of a scheme X is a surjective finite étale morphism $Y \to X$. Denote by $\mathbf{Cov}_{\text{ét}}(X)$ the full subcategory of $\mathbf{Over}_{\text{ét}}(X)$ whose objects are finite étale covers. An finite étale cover $E \to X$ is *trivial* if $\mathrm{Aut}_{\mathbf{Over}_{\text{ét}}X}(E)$ is the trivial group.

Example 2.1.1.2. A morphism of affine schemes $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ over $\mathbb{C}$ is given by a polynomial map $f = f(T) \in \mathbb{C}[T]$. If $f \notin \mathbb{C}$ the induced morphism, $\mathtt{Spec}(f)$, is necessarily surjective. With a little abuse of notation, write f instead of $\mathtt{Spec}(f)$.
A point $\mathfrak{p} = (T - z) \in \mathbb{A}^1_{\mathbb{C}}$ maps to $f(\mathfrak{p}) = (T - f(z)) \in \mathbb{A}^1_{\mathbb{C}}$. If f has degree n the fiber at $\mathfrak{p} = (T - z) \in \mathbb{A}^1_{\mathbb{C}}$ contains at most n points. The set underlying such fiber is computed solving $f(T) - z = 0$. The polynomial $f(T) - z$ has n distinct roots if and only if it has no roots in common with $f'(T)$. In particular if

$$w \in C_f = \text{critical points of } f := \left\{\text{roots of } f'(\mathrm{T})\right\},$$

then f ramifies at $(T - w)$, while it is unramified at all other points.
Let k be a field of characteristic zero. A k-morphism $\mathbb{A}^1_k \to \mathbb{A}^1_k$ is induced by a k-algebras morphism $f : k[T] \to k[T]$, thus it is completely described by the polynomial $f(T)$. Consider the diagram

$$\begin{array}{ccc} \mathbb{A}^1_{k^{\mathrm{alg}}} & \longrightarrow & \mathbb{A}^1_k \\ \big\downarrow f & & \big\downarrow f \\ \mathbb{A}^1_{k^{\mathrm{alg}}} & \longrightarrow & \mathbb{A}^1_k \end{array}$$ where the horizontal maps are $\mathtt{Spec}(k[X] \hookrightarrow k^{\mathrm{alg}}[X])$.

Since the horizontal maps are unramified, the right vertical map ramifies if and only if the left vertical arrow does. The ramification point in $\mathbb{A}^1_{k^{\mathrm{alg}}}$ are precisely the points $(T - w)$ such that $w \in C_f$. The ramification points in $\mathbb{A}^1_k$ are the ones below each of the $(T - w)$.
Since every morphism $\mathbb{A}^1_k \to \mathbb{A}^1_k$ is flat, the above gives a complete description of the étale endomorphisms of $\mathbb{A}^1_k$, when k has characteristic zero.

Classifying finite étale covers

This paragraph contains some general results that will be used to describe some fundamental groups. Before stating the first recall, not in full generality, what is the normalization of a scheme X.

Let X be an integral normal scheme and let K be its function field, that is the residue field of the generic point. Consider a finite separable extension $K \hookrightarrow L$. The sheaf on X given by:

$$\mathcal{O}_X^{(L)} : U \longmapsto \text{ integral closure of } \mathcal{O}_X(U) \text{ in } L,$$

for every open set of U, is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Recall that for every quasi-coherent sheaf $\mathcal{M}$, the ringed space $\mathtt{Spec}(\mathcal{M})$ defined in the obvious way is a scheme.
The scheme $X^{(L)} := \mathtt{Spec}(\mathcal{O}_X^{(L)})$ is called the *normalization* of X in L and X is *unramified in L* if n is unramified.

Theorem 2.1.1.1 ([Len08] Theorem 6.17 or [Fu11] Proposition 3.3.6). Let X be an integral normal scheme and let K be its function field. There is a bijection

$$\left\{ \begin{array}{c} \text{finite separable extensions} \\ K \hookrightarrow L \text{ s.t. } X^{(L)} \text{ is unramified} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{finite, connected} \\ \text{étale covers of } X \end{array} \right\}$$

given by $L \longmapsto X^{(L)}$ and $K(Y) \longleftarrow Y$ (here $K(Y)$ is the function field of a scheme Y).

In particular:

Corollary 2.1.1.2. Let k be a field. If $Y \to \mathtt{Spec}(k)$ is a finite étale cover, then $Y = \mathtt{Spec}(A)$, where A is a finite étale algebra, i.e. $A = \prod_{i=1}^n L_i$, where each L_i is a finite separable field extension $k \hookrightarrow L_i$.

Proposition 2.1.1.2 ([Fu11], Lemma 7.3.13). Let $(R, \mathfrak{m})$ be a noetherian henselian local ring. Let $X \overset{\phi}{\to} \mathtt{Spec}(R)$ a proper morphism. There is an equivalence of categories induced by the functor

$$\begin{aligned} \mathbf{Cov}_{\text{ét}}(X) &\xrightarrow{\sim} \mathbf{Cov}_{\text{ét}}(X \otimes \kappa(\mathfrak{m})) \\ E &\longmapsto E \times_X (X \otimes \kappa(\mathfrak{m})). \end{aligned}$$

Remark 2.1.1.1 (cf. [Sza09], discussion after 5.7.13). The "protagonist" of last proposition is a scheme proper over a henselian local ring. This may seem a very peculiar "protagonist", although to understand such schemes is in general a very good idea. Here's one reason.
Let k be a separably closed field. There exists a DVR $(R, \mathfrak{m})$ such that $\kappa(\mathfrak{m}) \simeq k$ and such that $\text{char}(\mathtt{Frac}(R)) = 0$. Taking the completion one may assume that R is complete. Complete local rings are henselian.

Let C be an integral proper normal curve over k. There exists a scheme $\mathscr{C}$ and a proper morphism $\mathscr{C} \to \mathtt{Spec}(R)$ such that $\mathscr{C}_m \simeq C$ and with $\mathscr{C}_\eta$ smooth over $\mathtt{Frac}(R)$ (here $m = \mathfrak{m}$ and $\eta = (0)$ are the special and generic point of $\mathtt{Spec}(R)$, while $\mathscr{C}_m$ and $\mathscr{C}_\eta$ are the corresponding fibers).

2.1.2 Fundamental groups of schemes

Denote by $\mathtt{f}$ the forgetful functor $\mathtt{f} : \mathbf{Sch} \to \mathbf{Set}$ mapping a scheme X to the set underlying the topological space of X.

Definiton 2.1.2.1 (Étale fundamental group). Let $x : \mathtt{Spec}(\Omega) \to X$ be a geometric point of X. Restricting the base change functor $\mathbf{Over}(x)$ (example 1.3.6.3) to $\mathbf{Cov}_{\text{ét}}(X)$ and composing it with $\mathtt{f}$, defines a functor

$$\begin{aligned} \varpi(x) : \mathbf{Cov}_{\text{ét}}(X) &\longrightarrow \mathbf{Set} \\ E &\longmapsto \mathtt{f}(E \otimes_X \Omega). \end{aligned}$$

The (étale) fundamental group of X at x is defined as the group $\mathrm{Aut}(\varpi(x))$ and it is denoted by $\pi_1^{\text{ét}}(X,x)$. An element of $\pi_1^{\text{ét}}(X,x)$ is, by definition, *the homotopy class of a loop in X based at* x. If $X = \mathtt{Spec}(R)$ I will sometimes write $\pi_1^{\text{ét}}(R,x)$. When unnecessary I'll omit x and simply write $\pi_1^{\text{ét}}(X)$.

Remark 2.1.2.1. The need of the forgetful functor in the definition of $\varpi(x)$ is because, for the theory to actually work, the automorphism group has to be on the level of sets.

Remark 2.1.2.2. The formation of the fundamental group is clearly functorial. If $X \xrightarrow{\phi} Y$ is a morphism of schemes, there is a natural morphism $\varphi : \pi_1^{\text{ét}}(X,x) \to \pi_1^{\text{ét}}(Y,y)$, where y is a geometric point of Y and x is a geometric point of the fiber at y.

A constructive description of $\pi_1^{\text{ét}}(X,x)$

Definiton 2.1.2.2 (Galois (étale) covers). A connected finite étale cover $E \to X$ of X is a Galois cover if $\mathrm{Aut}_{\mathbf{Over}_{\text{ét}}(X)}(E)$ acts transitively on the *geometric fibers*.

Lemma 2.1.2.1 ([Sza09],Proposition 5.3.9). For all connected étale covers $E \to G$, there is a morphism $\phi : G \to E$ such that the composition $G \to E \to X$ is Galois cover and such that all Galois covers $G' \to X$, any morphism $G' \to E$ factors through G. The commutative diagram below is a mnemonic of this lemma.

$$\begin{tikzcd} G \arrow[r, dashed, "\exists\phi"] \arrow[dr, dashed, "\mathrm{Gal}"'] & E \arrow[d, "\text{Ét}"] & G' \arrow[l, "\forall"'] \arrow[dl, "\mathrm{Gal}"] \arrow[ll, dashed, bend right, "\exists"'] \\ & X & \end{tikzcd}$$

Definiton 2.1.2.3. Let E and G be like in proposition 2.1.2.1. One says that G is a *Galois closure* of E (it is not unique).

Proposition 2.1.2.1 ([Sza09, pages 174-176]). The Galois covers of a scheme X form a full subcategory of $\mathbf{Cov}_{\text{ét}}(X)$. Choose a geometric point x in X and for each Galois cover E_α chose a geometric point e_α above x. If there exists a X-morphism $E_\beta \to E_\alpha$ of Galois covers, there is also a X-morphism $E_\beta \stackrel{\phi_{\alpha\beta}}{\to} E_\alpha$ such that $\phi_{\alpha\beta}(e_\beta) = e_\alpha$. Furthermore, if exists, $\phi_{\alpha\beta}$ is the unique X-morphism with such property. The collection of Galois covers $\{E_\alpha\}$ together with the collection of morphisms $\phi_{\alpha\beta}$ just defined form an inverse system. There is a natural isomorphism of functors

$$\varpi(x) \simeq \varinjlim_{\alpha} \operatorname{Hom}(E_\alpha, -)$$

that induce an isomorphism

$$\pi_1^{\text{ét}}(X,x) \simeq \varprojlim_{\alpha} \operatorname{Aut}_{\mathbf{Over}_{\text{ét}}(X)}(E_\alpha)$$

Since each $\operatorname{Aut}_{\mathbf{Over}_{\text{ét}}(X)}(E_\alpha)$ is a finite group, the étale fundamental group is a profinite group.

Definiton 2.1.2.4. Let x and y be two geometric points of X. An isomorphism $\varpi(x) \stackrel{\sim}{\to} \varpi(y)$ is called an *étale path* from x to y. In the literature étale paths are often called *chemins*.

First examples of fundamental groups

Example 2.1.2.1 (Fundamental group of a point, [Fu11] Proposition 3.2.14). Let k be a field and let Ω a separably closed field containing k^{sep}. This defines a geometric point x of $\mathtt{Spec}(k)$, namely

$$x : \mathtt{Spec}(\Omega) \to \mathtt{Spec}(k^{\text{sep}}) \to \mathtt{Spec}(k).$$

As a consequence of corollary 2.1.1.2, there is a canonical isomorphism

$$\pi_1^{\text{ét}}(k,x) \simeq \operatorname{Gal}(k^{\text{sep}}|k). \tag{2.1.1}$$

Example 2.1.2.2 (Fundamental group of $\mathbb{P}^1_k$). If $k(T) \hookrightarrow L$ is unramified everywhere, by elementary methods, $L = k'(T)$ for some finite separable extension $k \hookrightarrow k'$. Hence $\pi_1^{\text{ét}}(\mathbb{P}^1_k) \simeq \pi_1^{\text{ét}}(k)$.

Example 2.1.2.3 (Fundamental group of $\mathbb{A}^1_k$). The function field of $\mathbb{A}^1_k = \mathtt{Spec}(k[T])$ is $k(T)$. Étale covers of $\mathbb{A}^1_k$ are classified by theorem 2.1.1.1.

If k has characteristic zero, to understand the étale covers of $\mathbb{A}^1_k$, one has to classify all finite extensions $k(T) \hookrightarrow L$ that are either unramified everywhere either ramified only at the place $\mathfrak{p}_{1/T}$ (cf. Appendix).
But, if $k(T) \hookrightarrow L$ ramifies at $\mathfrak{p}_{1/T}$, example 4.2.1.1 implies that $L = k'(T)$ for some separable extension $k \hookrightarrow k'$.
Hence, if $\mathrm{char}(k) = 0$, all connected étale covers of $\mathbb{A}^1_k$ are trivial and $\pi_1^{\text{ét}}(k) \simeq \pi_1^{\text{ét}}(\mathbb{A}^1_k)$.
In positive characteristic this is not true. For instance the morphism of $\mathbb{F}_p$-algebras $\mathbb{F}_p[T] \to \mathbb{F}_p[T]$ given by $T \mapsto T^p$ is a non trivial étale cover of $\mathbb{A}^1_{\mathbb{F}_p}$

Example 2.1.2.4. For A any finite ring $\pi_1^{\text{ét}}(A) \simeq \widehat{\mathbb{Z}} :=$ profinite completion[1] of $\mathbb{Z}$.

For more details on these examples cf. [Len08, Chap. 6].

Some properties of the fundamental group

The main feature of fundamental groups is:

Theorem 2.1.2.1 ([Sza09], Theorem 5.4.2). Same notation as above. The functor $\varpi(x)$ induce an equivalence of categories

$$\mathbf{Cov}_{\text{ét}}(X) \xrightarrow{\sim} \pi_1^{\text{ét}}(X,x)\text{-}\mathbf{Setf}$$

by sending each étale cover E to the $\pi_1^{\text{ét}}(X,x)$-set underlying $\varpi(z)(E)$.
Here $\pi_1^{\text{ét}}(X,x)$-**Setf** is the category of finite sets equipped with a continuous $\pi_1^{\text{ét}}(X,x)$ action.

Proposition 2.1.2.2 ([Fu11, 3.3.4]). Let $X \xrightarrow{\phi} Y$ be a morphism of schemes and let y a geometric point of Y and x a geometric point of X above y. In other words consider a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(\Omega') \arrow[r, "x"] \arrow[d] & X \arrow[d, "\phi"] \\ \mathrm{Spec}(\Omega) \arrow[r, "y"] & Y \end{tikzcd}$$

with Ω and Ω' separably closed fields.

There is a natural morphism $\varphi : \pi_1^{\text{ét}}(X,x) \to \pi_1^{\text{ét}}(Y,y)$.

(i) For φ to be surjective it is sufficient and necessary that the base change functor **Over**(ϕ), when restricted to a functor $\mathbf{Cov}_{\text{ét}}(Y) \to \mathbf{Cov}_{\text{ét}}(X)$, sends connected schemes to connected schemes.

(ii) For φ to be injective it is sufficient and necessary that for every connected object E in $\mathbf{Cov}_{\text{ét}}(X)$, there exists an object P in $\mathbf{Cov}_{\text{ét}}(X)$ and a connected component P_0 of **Over**$(\phi)(P)$ such that

$$P_0 \hookrightarrow \mathbf{Over}(\phi)(P) = P \times_Y X \to X$$

factors as $P_0 \xrightarrow{\lambda} E \to X$ for an X-morphism λ.

[1] Recall that the *profinite completion* of a group is the inverse limit of its finite quotients.

Proposition 2.1.2.3 ([Sza09, 5.6.6]). Let k be an algebraically closed field and let X and Y be connected k-schemes. There is a canonical isomorphism

$$\pi_1^{\text{ét}}(X \times_k Y, (x, y)) \simeq \pi_1^{\text{ét}}(X, x) \times \pi_1^{\text{ét}}(Y, y) \tag{2.1.2}$$

for any couple of geometric points $x : \mathtt{Spec}(k) \to X$ and $y : \mathtt{Spec}(k) \to Y$. The geometric point (x, y) is the geometric point given by the spectrum of the tensor product of the residue fields of x and y.

Proposition 2.1.2.4 ([Sza09, 5.6.1] or [Fu11, 3.2.11]). Let k be a field, let X be a finite connected k-scheme and suppose $X \otimes_k k^{\text{sep}}$ is also connected. For a geometric point $x : \mathtt{Spec}(\Omega) \to X \otimes_k k^{\text{sep}}$, denote also by x its images in X and $\mathtt{Spec}(k)$. There is an exact sequence

$$1 \to \pi_1^{\text{ét}}(X \otimes_k k^{\text{sep}}, x) \longrightarrow \pi_1^{\text{ét}}(X, x) \longrightarrow \pi_1^{\text{ét}}(k, x) \to 1.$$

Remark 2.1.2.3 (cf. with remark 2.1.1.1). If k is replaced with an integral scheme X and if the structure morphism $X \xrightarrow{\varsigma} S$ is flat, proper and with integral geometric fibers, then proposition 2.1.2.4 can be partially generalized. More precisely let $X \xrightarrow{\varsigma} S$ be as above, let $s = \mathfrak{p}$ be a point on S and let x be a geometric point above $\bar{s}$. Then

$$\pi_1^{\text{ét}}(X \otimes \kappa(\mathfrak{p})^{\text{sep}}, x) \to \pi_1^{\text{ét}}(X, x) \to \pi_1^{\text{ét}}(S, \bar{s}) \to 1$$

is exact (cf. [Sza09, 5.6.4]).

Proposition 2.1.2.5 ([Sza09, 4.7.3]). Apply Proposition 2.1.2.4 to a curve over k and suppose that the image of x in X is a point $\mathfrak{p}$ with $\kappa(\mathfrak{p}) = k$. The exact sequence of proposition 2.1.2.4 splits and $\pi_1^{\text{ét}}(X, x)$ is a semi-direct product of $\pi_1^{\text{ét}}(k, x)$ and $\pi_1^{\text{ét}}(X \otimes k^{\text{sep}}, x)$.

Definiton 2.1.2.5. Given a variety X over $\mathbb{C}$, there is a canonical way to obtain an *analytic space* X^{an}. A precise construction and definition is given in [Har13] appendix B. If X is an algebraic, possibly punctured, curve in $\mathbb{P}^n_{\mathbb{C}}$ given by a polynomial, then X^{an} is the set of closed points of X endowed with the subspace topology coming from the usual topology on $\mathbb{P}^n_{\mathbb{C}}$.

Theorem 2.1.2.2. Let X be a $\mathbb{C}$-scheme and let $x : \mathtt{Spec}(\mathbb{C}) \to X$. Denote also be x the image of x in X. Then there is an isomorphism

$$\widehat{\pi_1(X^{\text{an}}, x)} \simeq \pi_1^{\text{ét}}(X, x)$$

where the group on the left hand side is the profinite completion of the classic fundamental group $\pi_1(X^{\text{an}}, x)$.

Let p be a prime number. The *profinite prime-to-p completion* of a group, is the inverse limit of the finite quotients with order prime to p. For a profinite group, like $\pi_1^{\text{ét}}(X, x)$, the maximal prime-to-p quotient coincide with the prime-to-p completion.

Theorem 2.1.2.3 ([Sza09, 4.9.1 and 5.7.14]). Let k be an algebraically closed field of characteristic $p \geqslant 0$ and let X be an integral proper normal curve of genus[2] g over k. Let S be a finite set of points of C and denote by X the complement of S in C. Let x be a geometric point of X.

(i) The group $\pi_1^{\text{ét}}(X, x)$ is topologically finitely generated.

(ii) The maximal prime-to-p quotient of $\pi_1^{\text{ét}}(X, x)$ is isomorphic to the profinite prime-to-p completion of the group with presentation

$$\langle a_1, b_1, \ldots, a_n, b_n, \gamma_1, \ldots, \gamma_m \mid [a_1, b_1] \cdots [a_n, b_n] \gamma_1 \cdots \gamma_n = 1 \rangle,$$

where n is the number of points in S.

Notice that if $p = 0$, then the "prime-to-p" conditions are trivial and (ii) becomes

(ii.0) $\pi_1^{\text{ét}}(X, x)$ is isomorphic to the profinite completion of the group with presentation

$$\langle a_1, b_1, \ldots, a_n, b_n, \gamma_1, \ldots, \gamma_m \mid [a_1, b_1] \cdots [a_n, b_n] \gamma_1 \cdots \gamma_n = 1 \rangle,$$

where n is the number of points in S.

2.2 Étale sheaves

2.2.1 Étale sheaves

Definiton 2.2.1.1. Let X be a scheme. Let U be an object in $\mathbf{Over}_{\text{ét}}(X)$. A *covering* (or cover) *of U in* $\mathbf{Over}_{\text{ét}}(X)$ is a set of étale morphisms $\{U_i \xrightarrow{\phi} U\}_i$ in $\mathbf{Over}_{\text{ét}}(X)$ such that $\bigcup \phi_i(U_i) = U$. An *étale* sheaf on X with values in a category $\mathbf{C}$ is an object $\mathscr{F}$ in $\mathbf{PS}_{\mathbf{C}}(\mathbf{Over}_{\text{ét}}(X))$ such that for all coverings $\{U_i \to U\}_{i \in I}$ in $\mathbf{Over}_{\text{ét}}(X)$, the diagram

$$\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \times_U U_j) \tag{2.2.1}$$

(with the obvious restriction morphisms) is exact.
The elements of $\mathscr{F}(X)$ are the *global sections* of $\mathscr{F}$.

All étale sheaves, when no confusion is possible, will be simply called sheaves. Usual sheaves, as structure sheaves, will be called *usual topological sheaves* or *topological sheaves*. The objects of $\mathbf{PS}_{\mathbf{C}}(X) := \mathbf{PS}_{\mathbf{C}}(\mathbf{Over}_{\text{ét}}(X))$ are referred as *presheaves on X*. Sheaves form a full subcategory of $\mathbf{PS}_{\mathbf{C}}(X)$. Denote it by $\mathbf{Sh}_{\mathbf{C}}(X)$.

Many of features of sheaves are independent of the target category $\mathbf{C}$ and some of the results in this manuscript hold for sheaves with values in a wide range of categories. This said I will only focus on étale sheaves with values in **Set**, **Ab**, or $\mathbf{Mod}_R$ for some commutative ring R. Furthermore assume that R is a finite ring. If not specified otherwise, a *sheaf*, in this chapter, will always mean an *étale sheaf with values in* **Set**, **Ab**, *or* $\mathbf{Mod}_R$ and R will always denote a finite commutative ring with unity.

[2]The arithmetic genus of a scheme X is 1-$\chi(X, \mathscr{O}_X)$, where $\chi(X, \mathscr{O}_X)$ is the Euler characteristic of X. This is defined as the alternating sum of the dimension of the Čech cohomology of $\mathscr{O}_X$. See [Vak, 18.4].

Remark 2.2.1.1. If $\mathbf{C} \in \{\mathbf{Ab}, \mathbf{Mod}_R\}$, the categories $\mathbf{PS_C}(X)$ and $\mathbf{Sh_C}(X)$ are abelian.

Grothendieck Topologies

The category $\mathbf{Over}_{\text{ét}}(X)$ together with the notion of covers defined above, is the *étale site* of X. To look at étale site of a scheme X is in some sense the same thing as to look at the scheme X endowed with a richer "topology", namely the étale topology. This notion of topology is not the same as the usual notion of "open and closed set topology".

Definiton 2.2.1.2. A *Grothendieck's topology* on a category $\mathbf{C}$ (with fiber products) is a family $\tau_{\mathbf{C}}$ whose elements, called *coverings*, are collection of morphisms $\{U_i \to U\}_i$ in $\mathbf{C}$, such that:

1. for any isomorphism $U \xrightarrow{\sim} V$ in $\mathbf{C}$, $\{U \xrightarrow{\sim} V\} \in \tau_{\mathbf{C}}$.
2. If $\{U_i \xrightarrow{\phi_i} U\}_i \in \tau_{\mathbf{C}}$ and if $V \xrightarrow{\phi} U$ is a morphism in $\mathbb{C}$, then $\{\mathbf{Over}(\phi_i)(V) \to V)\}_i \in \tau_{\mathbf{C}}$.
3. $\tau_{\mathbf{C}}$ is stable under *refinements*, i.e. given $\{U_i \to U\}_i \in \tau_{\mathbf{C}}$ and a system of $\{U_{j,i} \to U_i\}_j \in \tau_{\mathbf{C}}$, then $\{U_{j,i} \to U\}_{i,j} \in \tau_{\mathbf{C}}$.

A category endowed with a Grothendieck's topology is called a *site*. The sequence (2.2.1) provides a notion of sheaf definable in any kind of site.

The *étale site* of X is the category $\mathbf{Over}_{\text{ét}}(X)$ together with the Grothendieck's topology given by definition 2.2.1.1 [3].

Before étale cohomology the usual way to study the cohomology of a scheme was to study its, now called, *Zariski* site. Using the formalism of Grothendieck's topologies, the Zariski site of X is the category

$$\mathbf{Over}_{\text{open}}(X) := \text{ full subcategory of } \mathbf{Over}(X) \text{ whose objects are open immersions}$$

together with the topology given by

$$\{U_i \hookrightarrow U\}_i \text{ is a covering if and only if } \bigcup U_i = U.$$

But the Zariski site "is to coarse" and, in general, the cohomology groups in the Zariski site do not give satisfying answers.

Proposition 2.2.1.1 ([Sta20, Tag 04DY]). If $X \xrightarrow{\phi} Y$ is a morphism of schemes such that for all morphism $Z \to Y$,

$$X \times_Y Z \to Z \text{ is a homemorphism of the underlying top. spaces,} \tag{2.2.2}$$

then the étale site of X is *isomorphic* to the étale site of X. This means that the étale base change functor defines an equivalence of categories $\mathbf{Over}_{\text{ét}}(\phi) : \mathbf{Over}_{\text{ét}}(Y) \xrightarrow{\sim} \mathbf{Over}_{\text{ét}}(X)$, a bijection $\tau_{\mathbf{Over}_{\text{ét}}(Y)} \xrightarrow{\simeq} \tau_{\mathbf{Over}_{\text{ét}}(X)}$ and an isomorphism

$$\mathbf{Over}_{\text{ét}}(\phi) : \mathbf{Over}_{\text{ét}}(\phi)(U_i \times_Y E) \xrightarrow{\sim} \mathbf{Over}_{\text{ét}}(\phi)(U_i) \times_X \mathbf{Over}_{\text{ét}}(\phi)(E).$$

[3] This is sometimes called the *small étale site* of X, to not be confused with the *big étale site of* X, that is the category $\mathbf{Over}(X)$ endowed with the same notion of *étale coverings*.

Definiton 2.2.1.3. A morphsim $X \xrightarrow{\phi}$ that satisfies (2.2.2) is called a universal homeomorphism of schemes.

Sheafification and examples

Not all presheaves on X are sheaves, although there is a canonical way to "cook" a sheaf out of a presheaf. As for usual sheaves, the process is called *sheafification*.

Proposition 2.2.1.2 ([Fu11, 5.2.1]). There is a canonical functor $\# : \mathbf{PS_C}(X) \to \mathbf{Sh_C}(X)$ such that for any sheaf $\mathscr{F}$ and for any presheaf $\mathscr{P}$

$$\mathrm{Hom}_{\mathbf{Sh_C}(X)}(\#\mathscr{P}, \mathscr{F}) \simeq \mathrm{Hom}_{\mathbf{PS_C}(X)}(\mathscr{P}, \mathscr{F}).$$

Hence $\#$ is *left-adjoint* to the inclusion $\mathbf{PS_C}(X) \hookrightarrow \mathbf{Sh_C}(X)$.

I won't go through the details of the construction of the functor $\#$, it is not useful for the purpose of this exposition. The construction is similar to the shafification of usual sheaves (cf. [Sta20, Tag 007X]).

Definiton 2.2.1.4. Let A be an object in some category $\mathbf{C}$. Let $\mathscr{P}$ be the *constant* presheaf on X, i.e. the sheaf given by $\mathscr{P}(E) = A$ for every étale X-schemes E. The sheaf $\#\mathscr{P}$ is the *constant sheaf on* X. Denote it by $\underline{A}_X$.

Remark 2.2.1.2. If E in $\mathbf{Over}_{\text{ét}}(X)$ is connected, then $\underline{A}_X(E) = A$.

Definiton 2.2.1.5. Let X be a scheme and let Y be a X-scheme. One can show that the presheaf of sets on X given by

$$E \longmapsto \mathrm{Hom}_{\mathbf{Over}(X)}(E, Y)$$

is a sheaf. Denote by $\widetilde{Y}$ this sheaf. It is called *the sheaf on* X *represented by* Y.

There is a functor $\mathbf{Over}_{\text{ét}}(X) \to \mathbf{Ring}$ that plays the role of the structure sheaf of X. This functor, denoted by $\mathscr{O}_X^{\text{ét}}$ is called the *étale structure sheaf* of X and is defined by

$$\begin{aligned} \mathscr{O}_X^{\text{ét}} : \mathbf{Over}_{\text{ét}}(X) &\longrightarrow \mathbf{Ring} \\ E &\longmapsto \mathscr{O}_E(E) \end{aligned}$$

where $\mathscr{O}_E$ is the usual structure sheaf of E. Another important functor is the *multiplicative* *group* functor. It is often denoted by $\mathbb{G}_{m,X}$ and it is defined by

$$\begin{aligned}\mathbb{G}_{m,X} : \mathbf{Over}_{\text{ét}}(X) &\longrightarrow \mathbf{Grp} \\ E &\longmapsto (\mathscr{O}_E(E))^{\times}\end{aligned}$$

By descent theory (cf. [Fu11] 1.6.2) it is possible to show that both $\mathscr{O}_X^{\text{ét}}$ and $\mathbb{G}_{m,X}$ are sheaves. The same technique shows that the sheaf in definition 2.2.1.5 is indeed a sheaf.

Étale cohomology

Let X be a scheme. For simplicity denote by $\mathbf{Sh}_R(X)$ the category $\mathbf{Sh}_{\mathbf{Mod}_R}(X)$. For any object E in $\mathbf{Over}_{\text{ét}}(X)$ define the functor

$$\begin{aligned}\Gamma(E,-) : \mathbf{Sh}_R(X) &\longrightarrow \mathbf{Mod}_R \\ \mathscr{F} &\longmapsto \mathscr{F}(E)\end{aligned}$$

The functor $\Gamma(E,-)$ is left exact but not right exact, and the derived functors $R^p\Gamma(E,-)$ may be may be non-trivial. Set the notation

$$H^p_{\text{ét}}(E,\mathscr{F}) := R^p\Gamma(E,\mathscr{F}).$$

Definiton 2.2.1.6. The module $H^p_{\text{ét}}(X,\mathscr{F})$ is called the p^{th}-*cohomology* of X "with coefficients in $\mathscr{F}$" or simply the *étale cohomology of* $\mathscr{F}$.

One of the aim of this chapter is to give a sketch of the proof of the following theorem.

Theorem 2.2.1.1. Let C be proper smooth connected curve of genus g over a separably closed field k of characteristic $p \geqslant 0$. Let S be a finite set of closed points on C and denote by X the open subscheme $C \setminus S$. If $\ell \neq p$ is a prime number, then

$$\mathrm{H}^i_{\text{ét}}(X, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}_X) = \begin{cases} \mathbb{Z}/\ell^n\mathbb{Z} & \text{if } i = 1 \\ (\mathbb{Z}/\ell^n\mathbb{Z})^{2g+\alpha} & \text{if } i = 2 \\ (\mathbb{Z}/\ell^n\mathbb{Z})^{\beta} & \text{if } i = 3 \\ 0 & \text{otherwise} \end{cases}$$

where if $\#S = 0$, $\alpha = 0$ and $\beta = 1$, while if $\#S > 0$, then $\alpha = \#S - 1$ and $\beta = 0$.

2.3 Operations on sheaves

In this section assume that each sheaf is a sheaf of R-modules with R a finite ring.

2.3.1 Direct and inverse image: the functors ϕ_*, ϕ^*

Informally, any morphism of schemes $X \xrightarrow{\phi} Y$ gives canonical ways to "move" sheaves on X to sheaves on Y and vice-versa. More formally any morphism $X \xrightarrow{\phi} Y$ defines two canonical functors $\phi_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ and $\phi^* : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$. The precise and general construction of these functors is important, but is not necessary to proceed the reading. As announced in the introduction, part of the next definition was moved to the appendix.

Definiton 2.3.1.1. Same notation as in the appendix, section 4.2.2. Let $X \xrightarrow{\phi} Y$ a morphism of schemes. Define

$$\phi_* := \phi_{\mathbf{PS}}|_{\mathbf{Sh}(X)} \qquad\qquad \phi^* := \# \circ \phi^{\mathbf{PS}}\Big|_{\mathbf{Sh}(Y)}$$

This gives two functors $\phi_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ and $\phi^* : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$. They are called the *direct image* and *inverse image* functors of ϕ.
Concretely the functor ϕ_* sends a sheaf $\mathscr{F}$ (on X) to the sheafification of the presheaf

$$E \longmapsto \mathscr{F}(E \times_Y X) \text{ for all } E \text{ in } \mathbf{Over}_{\text{ét}}(Y).$$

Informally, and ignoring sheafification, the sheaf $\phi^*(\mathscr{G})$ (for $\mathscr{G}$ a sheaf on Y) sends an étale scheme $E \to X$ to the limit

$$\varinjlim \mathscr{G}(E'),$$

where E' run along a system of étale Y-schemes such that $\mathbf{Over}(\phi)(E')$ is an étale E-scheme. This description gives the intuition of the analogy with the usual topological sheaves.

Remark 2.3.1.1. Sometimes, in the literature, the inverse image functor is denote by ϕ^{-1}. The symbol ϕ^* is preferred for the *inverse image of sheaves of $\mathscr{O}$-modules*.

Remark 2.3.1.2. The inverse image of a constant sheaf is constant.

As the reader may guess the functors ϕ_* and ϕ^* are adjoints: for any couple of sheaves $\mathscr{G}$ on Y and $\mathscr{F}$ on X there is an isomorphism

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(\phi^*\mathscr{G}, \mathscr{F}) \simeq \mathrm{Hom}_{\mathbf{Sh}(Y)}(\mathscr{G}, \phi_*\mathscr{F}). \tag{2.3.1}$$

The formalism provided by ϕ_* and ϕ^* is very rich and is in some sense the main tool to define and study sheaves.
For instance, for any couple of morphism $X \xrightarrow{\phi} Y$ and $Y \xrightarrow{\psi} X$ and for any constant sheaf $\underline{A}_X$, the sheaf $\psi^*\phi_*\underline{A}_X$ could be arbitrary interesting.

Remark 2.3.1.3. If $E \xrightarrow{\phi} X$ is étale, then $H^p_{\text{ét}}(E, \mathscr{F}) = H^p_{\text{ét}}(E, \phi^*\mathscr{F})$. This is a good reason to set the following notation: if $\mathscr{F}$ is a sheaf on X and $E \xrightarrow{\phi} X$ is an étale X-scheme, denote by $\mathscr{F}_E$ the sheaf $\phi^*\mathscr{F}$. This is a very natural notation since the sheaf $\phi^*\mathscr{F}$ "is" the restriction of $\mathscr{F}$ to E.

Lemma 2.3.1.1 ([Fu11, 5.2.5 and]). Let $X \xrightarrow{\phi} Y$ be a morphism of schemes.

(i) ϕ^* is exact, ϕ_* is left-exact.

(ii) if $Y \xrightarrow{\psi} Z$ is another morphism of schemes, then $(\psi\circ\phi)^* = \phi^*\circ\psi^*$ and $(\psi\circ\phi)_* = \psi_*\circ\phi_*$.

(iii) For any presheaf $\mathscr{P}$ on a scheme Y and for any morphism $X \xrightarrow{\phi} Y$, $\phi^* \circ \# = \# \circ \phi^*$.

Stalks of étale sheaves and étale neighborhoods

Definiton 2.3.1.2 (Stalk at geometric point). Let $x : \mathtt{Spec}(\Omega) \to X$ be a geometric point of a scheme X and let $\mathscr{F}$ be a sheaf on X. The *stalk of $\mathscr{F}$ at x* is defined as

$$\mathscr{F}_x := x^*\mathscr{F}(\mathtt{Spec}(\Omega)).$$

Remark 2.3.1.4. By lemma 2.3.1.1.(iii) it is not necessary to understand the sheafification of a presheaf in order to compute stalks.

In topology, the stalk at x of a usual sheaf $\mathscr{F}$ is defined as the direct limit

$$\varinjlim_{U \ni x} \mathscr{F}(U),$$

where U runs over the family of open *neighborhoods* of x. The definition of stalk in the étale framework is essentially the same thing. This is seen by replacing the notion of usual open neighborhoods with a more “arrow-friendly” definition.

Definiton 2.3.1.3 (Étale neighborhood). Let $x : \mathtt{Spec}(\Omega) \to X$ be a geometric point of a scheme X. An étale neighborhood of X is a commutative diagram

$$\begin{tikzcd} \mathtt{Spec}(\Omega) \arrow[r] \arrow[dr, "x"'] & U \arrow[d] \\ & X \end{tikzcd} \quad \text{with } U \to X \text{ étale.}$$

Étale neighborhoods of x form a category (cf. section 1.3.6) denoted by $\mathbf{Nbh}_{\text{ét}}(x)$. Unfolding the definition of the functor x^*, it is possible to check that

$$\mathscr{F}_x \simeq \varinjlim_{U \text{ in } \mathbf{Nbh}_{\text{ét}}(x)} \mathscr{F}(U). \tag{2.3.2}$$

A remarkable, for someone tautological, consequence of (2.3.2), is that, by construction and by definition of the strict henselization, there is an identification

$$\mathscr{F}_x = \mathscr{F}(\mathscr{O}^{sh}_{X,\mathfrak{p}}),$$

where $x = \mathfrak{p}$ and where $\mathscr{O}^{sh}_{X,\mathfrak{p}}$ is a strict henselization of $\mathscr{O}_{X,\mathfrak{p}}$. For a definition of the *henselization* and *strict henselization* of a local ring, see [Sta20, Tag 0BSK].

Notice that the stalk at x of $\mathscr{O}^{\text{ét}}_X$ coincides with $\mathscr{O}^{sh}_{X,\mathfrak{p}}$.

Definiton 2.3.1.4. In the literature, the scheme $\mathtt{Spec}(\mathscr{O}^{sh}_{X,\mathfrak{p}})$ is called the *strict localization* of X at x. In some sense it is the smallest étale neighborhood of x.

A morphism of sheaves induce a morphism on the stalks. It is possible to deduce a lot of things about morphisms of sheaves by looking at the morphisms induced on the stalks.

Proposition 2.3.1.1 ([Fu11], proposition 5.3.3). A sequence of sheaves on X of the form

$$\mathscr{F} \to \mathscr{G} \to \mathscr{H}$$

is exact if and only if

$$\mathscr{F}_x \to \mathscr{G}_x \to \mathscr{H}_x,$$

is exact for all geometric point x of X.

Base change to k^{sep}

Let X be any k-scheme. A separable closure k^{sep} of k defines a geometric point of $\mathtt{Spec}(k)$, namely $\mathtt{Spec}(k \hookrightarrow k^{\mathrm{sep}})$. The base change functor $\mathbf{Over}(\mathtt{Spec}(k \hookrightarrow k^{\mathrm{sep}}))$ sends the scheme X to the scheme $X \times_{\mathtt{Spec}(k)} \mathtt{Spec}(k^{\mathrm{sep}})$ that I denote by $X \otimes k^{\mathrm{sep}}$ (as indicated in the introduction). The scheme $X \otimes k^{\mathrm{sep}}$ is a k^{sep}-scheme and it is simpler to study than the scheme X.

Remark 2.3.1.5. If Ω is a separable closed field containing k^{sep} then the canonical morphism $X \otimes \Omega \xrightarrow{\phi} X \otimes k^{\mathrm{sep}}$ is a universal homeomorphism. By proposition 2.2.1.1, the étale sites of $X \otimes \Omega$ and $X \otimes k^{\mathrm{sep}}$ are isomorphic. In particular

$$\mathrm{H}^p_{\text{ét}}(X \otimes \Omega, \phi^*\mathscr{F}) = \mathrm{H}^p_{\text{ét}}(X \otimes k^{\mathrm{sep}}, \mathscr{F}).$$

for any sheaf $\mathscr{F}$ on $X \otimes k^{\mathrm{sep}}$.

Remark 2.3.1.6. Since every geometric point $\mathtt{Spec}(\Omega) \to X$ factors as $\mathtt{Spec}(\Omega) \to X \otimes k^{\mathrm{sep}} \to X$ (for a suitable choice of k^{sep}). By (ii) in lemma 2.3.1.1, it is possible to assume that k is separably closed.

Stalks of $\phi_*\mathscr{F}$ and of $\phi^*\mathscr{F}$

Lemma 2.3.1.2 ([Fu11, 5.3]). Let $X \xrightarrow{\phi} Y$ be a morphism of schemes.

(i) For a geometric point $x : \mathtt{Spec}(\Omega) :\to X$ denote by $\phi(x)$ the geometric point given by $\phi \circ x$. For any $\mathscr{F}$ on Y there is an isomorphism

$$(\phi^*\mathscr{F})_x \simeq \mathscr{F}_{\phi(x)} \tag{2.3.3}$$

(ii) Suppose ϕ is finite and let y be a geometric point of Y. For any sheaf $\mathscr{F}$ on X there is an isomorphism

$$(\phi_*\mathscr{F})_y \simeq \bigoplus_{x \in X \otimes \kappa(y)} \mathscr{F}_{\overline{x}} \tag{2.3.4}$$

(iii) If ϕ is an either open or closed immersion, then $\phi^*\phi_*\mathscr{F} \simeq \mathscr{F}$ for any sheaf $\mathscr{F}$ on X.

2.3.2 Direct image with proper support: the functor $\phi_!$

Definiton 2.3.2.1. Let $\mathscr{F}$ be a sheaf on a scheme X. The support of $\mathscr{F}$, denoted $\mathrm{supp}(\mathscr{F})$ set of points $S \subset X$ such that $\mathscr{F}_{\overline{s}} = 0$.
Let $U \to X$ be an étale X-scheme and let $f \in \mathscr{F}(U)$. The support of f, denoted $\mathrm{supp}(f)$ is the closed subset $U \setminus U_0$ where U_0 is the largest open subset in U such that f vanish on U_0.

Remark 2.3.2.1. Notation as above. For any $V \xrightarrow{\phi} U$ étale over X, $\mathrm{supp}(\phi^* f) = \phi^{-1}(\mathrm{supp}(f))$.

Definiton 2.3.2.2. Let $X \xrightarrow{\phi} Y$ be a separated morphism of finite type and consider a sheaf $\mathscr{F}$ on X. The *direct image with proper support* of a sheaf $\mathscr{F}$ on X, with respect to ϕ, is a sheaf on Y denoted $\phi_!\mathscr{F}$ and given by

$$\phi_!\mathscr{F} : U \longmapsto \left\{s \in \phi_*\mathscr{F}(U) \text{ s.t. } \mathrm{supp}(s) \to U \text{ is proper.}\right\}$$

(Attention: if ϕ is not separated, this is not a sheaf.)
This gives a functor

$$\phi_! : \mathbf{Sh}(X) \to \mathbf{Sh}(Y).$$

Remark 2.3.2.2. If ϕ is a proper morphism $\phi_* = \phi_!$

For any couple of sheaves $\mathscr{G}$ on Y and $\mathscr{F}$ on X there is an isomorphism

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(\mathscr{F}, \phi^*\mathscr{G}) \simeq \mathrm{Hom}_{\mathbf{Sh}(Y)}(\phi_!\mathscr{F}, \mathscr{G}). \tag{2.3.5}$$

Remark 2.3.2.3. Compare (2.3.1) and (2.3.5): $\phi_!$ is the left adjoint of ϕ^*, while ϕ_* is the right adjoint of ϕ^*.

Lemma 2.3.2.1 ([Fu11, 5.5.1]). Let $X \xrightarrow{\phi} Y$ be any morphism of schemes

(i) If $Y \xrightarrow{\psi} Z$ is another morphism of schemes, then $\psi_! \circ \phi_! = (\psi \circ \phi)_!$.

(ii) Let y be a geometric point of Y and let $\mathscr{F}$ be a sheaf on X. Then

$$(\phi_!\mathscr{F})_y = \bigoplus_{x \in X \otimes \kappa(y)} \mathscr{F}_{\overline{x}}$$

(iii) The functor $\phi_!$ is exact and faithful.

2.3.3 Immersions and gluing

Let X be a scheme and let X_c be a closed subscheme of X. Let X_o be the complement of X_c and denote by i and j the closed immersions

$$X_c \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} X_0.$$

The functors i^*, j_* and $j_!$ are incredibly powerful tools to study sheaves. Some of these results will be useful in chapter 3.

Lemma 2.3.3.1 ([Fu11], proposition 5.4.2). Let i and j be as above.

(i) $i_*, i^*, j_!$ and j^* are exact, j_* is left exact, i_* is fully faithful.

(ii) $i^*i_* \simeq \mathrm{id}$, $j^*j_! \simeq j^*j_* \simeq \mathrm{id}$ and $i^*j_! \simeq 0$, $j^*i_* \simeq 0$.

(iii) There is an exact sequence

$$0 \to j_!j^*\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow i_*i^*\mathscr{F} \to 0,$$

induced by the adjunction morphisms. The exact sequence if furthermore functorial in the sense that for any morphism of sheaves $\mathscr{F} \to \mathscr{G}$ on X we obtain a morphisms of exact sequences.

Proposition 2.3.3.1 ([Fu11, 5.4.3]). The data of a sheaf $\mathscr{F}$ on X is the same as the data of:

(a) A sheaf $\mathscr{F}_c$ on X_c

(b) A sheaf $\mathscr{F}_o$ on X_o

(c) A morphism of sheaves $\sigma : \mathscr{F}_c \to i^*j_*\mathscr{F}_o$.

This induces am equivalence of categories between sheaves on X and the category of tuples $(\mathscr{F}_c, \mathscr{F}_0, \sigma)$ with morphisms defined in the obvious way.

To *glue the data* $(\mathscr{F}_c, \mathscr{F}_o, \sigma)$, take the pull-back $\mathscr{G}$ of the Cartesian diagram

$$\begin{tikzcd} \mathscr{G} \arrow[r] \arrow[d] & i_*\mathscr{F}_c \arrow[d, "i_*(\sigma)"] \\ j_*\mathscr{F}_o \arrow[r, "\mathtt{adj}"] & i_*i^*j_*\mathscr{F}_o \end{tikzcd} \tag{2.3.6}$$

($\mathtt{adj}$ denotes the adjunction induced by $\mathrm{id} \to i_*i^*$).

2.3.4 Higher direct image: the functors $R^p\phi_*$

I now implement the formalism of derived categories. See Appendix A for a blue print and [Vak, Chap. 23] for a gentle introduction.

Denote by $D(X,R)$ the derived category $D(\mathbf{Sh}_{\mathbf{Mod}_R}(X))$. It is a triangulated category and the category $\mathbf{Sh}_{\mathbf{Mod}_R}(X)$ can be realized as a full subcategory of $D(X,R)$. This is done by sending a sheaf $\mathscr{F}$ to the complex $\mathscr{F}[0]$ concentrated in degree 0:

$$\mathscr{F}[0] = \cdots \to 0 \to \mathscr{F} \to 0 \to \cdots$$

For $X \xrightarrow{\phi} Y$ a morphim of functors ϕ^* is an exact functor, while ϕ_* is only left exact. Thus one can study the functor $R\phi_* : D(X,R) \to D(Y,R)$.

For a definition of the categories $D^+(X,R)$, $D^-(X,R)$ and $D^b(X,R)$ cf. definition 4.1.2.4.

Remark 2.3.4.1. If k is separably closed and if X is a k-scheme with structure morphism ς, then, for any sheaf $\mathscr{F}$ on X,

$$R^p\varsigma_*\mathscr{F}(\mathtt{Spec}(k)) = \mathrm{H}^p_{\text{ét}}(X,\mathscr{F})$$

.

Lemma 2.3.4.1 ([Fu11, 5.6.11]). Let ϕ be as above and let $Y \xrightarrow{\psi} Z$ be another morphism of schemes.

(i) If $\mathscr{F}$ is a sheaf on Y, then $R^p\phi_*\mathscr{F}$ is the sheafification of the presheaf

$$E \longmapsto \mathrm{H}^p_{\text{ét}}(E \times_Y X, \mathscr{F})$$

(ii) There are spectral sequences

$$E_2^{p,q} = \mathrm{H}^p_{\text{ét}}(Y, R^q\phi_*\mathscr{F}) \Longrightarrow \mathrm{H}^{p+q}_{\text{ét}}(X,\mathscr{F}) \tag{2.3.7}$$

$$E_2^{p,q} = (R^p\psi_* \circ R^q\phi_*)\mathscr{F} \Longrightarrow R^{p+q}(\psi \circ \phi)_*\mathscr{F} \tag{2.3.8}$$

Proposition 2.3.4.1 ([Fu11, 5.7.4]). Let $X \xrightarrow{\phi} Y$ be a finite morphism and let $\mathscr{F}$ be a sheaf on X. Then

$$R^p\phi_*\mathscr{F} = 0 \qquad \text{for all } p \geqslant 1$$
$$\mathrm{H}^p_{\text{ét}}(Y, \phi_*\mathscr{F}) \simeq \mathrm{H}^p_{\text{ét}}(X,\mathscr{F}) \qquad \text{for all } p.$$

2.3.5 Base change theorems

Consider a Cartesian square

$$\begin{tikzcd} Y \times_S X \arrow[r, "\widetilde{\psi}"] \arrow[d, "\widetilde{\phi}"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "\psi"'] & S \end{tikzcd} \tag{2.3.9}$$

in the category of schemes. Composing the adjuction morphism $\mathrm{id} \to \widetilde{\psi}_*\widetilde{\psi}^*$ with $R^p\phi_*$, gives

$$R^p\phi_* \longrightarrow R^p\phi_* \circ (\widetilde{\psi}_* \circ \widetilde{\psi}^*)$$

By (2.3.8) there are morphisms

$$R^p\phi_* \circ \widetilde{\psi}_* \circ \widetilde{\psi}^* = R^p\phi_* \circ R^0\widetilde{\psi}_* \circ \widetilde{\psi}^* \longrightarrow R^p(\phi \circ \widetilde{\psi})_* \circ \widetilde{\psi}^*$$
$$R^p(\psi \circ \widetilde{\phi})_* \circ \widetilde{\psi}^* \longrightarrow \psi_* \circ R^p\widetilde{\phi}_* \circ \widetilde{\psi}^* = R^0\psi_* \circ R^p\widetilde{\phi}_* \circ \widetilde{\psi}^*$$

Since diagram (2.3.9) commutes, one obtains a morphism

$$R^p\phi_* \longrightarrow R^p\phi_* \circ \widetilde{\psi}_* \circ \widetilde{\psi}^* \longrightarrow \psi_* \circ R^p\widetilde{\phi}_* \circ \widetilde{\psi}^*.$$

Finally composing with ψ^* gives

$$\psi^* \circ R^p\phi_* \longrightarrow R^p\widetilde{\phi}_* \circ \widetilde{\psi}^* \tag{2.3.10}$$

This defines a morphism also on the level of derived categories.

Theorem 2.3.5.1 (Proper base change, [Fu11, 7.3.1]). If ϕ is proper, then (2.3.10) an isomorphism on on the category $D^+(X,R)$.

Theorem 2.3.5.2 (Smooth base change, [Fu11, 7.7.2]). If ψ is quasi-compact and quasi-separated and ϕ is smooth, then (2.3.10) induces an isomrophism in the category $D^+(X,\mathbb{Z}/n\mathbb{Z})$.

2.3.6 Hom and tensor products

Denote by $\underline{\mathrm{Hom}}(\mathscr{F},\mathscr{G})$ the sheafification of the presheaf given by:

$$E \longmapsto \mathrm{Hom}(\mathscr{F}(E),\mathscr{G}(E))$$

The sheaf $\mathscr{F} \otimes \mathscr{G}$ is defined as the sheafification of the presheaf

$$E \longmapsto \mathscr{F}(E) \otimes_R \mathscr{G}(E).$$

This defines two functors

$$\underline{\mathrm{Hom}} : \mathbf{Sh}(X) \times \mathbf{Sh}(X)^{\mathrm{opp}} \longrightarrow \mathbf{Sh}(X)$$

and

$$\otimes : \mathbf{Sh}(X) \times \mathbf{Sh}(X)^{\mathrm{opp}} \longrightarrow \mathbf{Sh}(X)$$

This functors can be derived and deriving them leads to definition of

$$R\underline{\mathrm{Hom}} : D(X,R) \times D^+(X,R) \longrightarrow D(X,R)$$

$$\otimes_R^{\mathbf{L}} : D(X,R) \times D^b_{tf}(X,R) \longrightarrow D(X,R).$$

Some restriction was made to assure the representability of the functors $R\underline{\mathrm{Hom}}$ and $\otimes_R^{\mathbf{L}}$ (cf. definition 4.1.4.1). Here the category $D_{tf}(X,R)$ is the full subcategory whose objects are complexes of sheaves F such that there exists an integer n such that $H^{-i}(F \otimes \underline{A}_X[0]) = 0$ for any $i > n$ and for any constant sheaf $\underline{A}_X$.
In practice, there is no need to worry about these conditions (cf. theorem 2.3.9.2 below).

Proposition 2.3.6.1. Let E, F and G be objects in $D(X,R)$, $D^b_{tf}(X,R)$ and $D^+(X,R)$ respectively. There is an isomorphism (of sets)

$$\mathrm{Hom}_{D(X,R)}(E, R\underline{\mathrm{Hom}}(F,G)) \simeq \mathrm{Hom}_{D(X,R)}(E \otimes^{\mathbf{L}}_R F, G)$$

given by

$$\phi \longmapsto \left(E \otimes^{\mathbf{L}}_R F \overset{\phi\otimes^{\mathbf{L}}_R \mathrm{id}}{\to} R\underline{\mathrm{Hom}}(F,G) \otimes^{\mathbf{L}}_R F \overset{\text{evaluation}}{\to} G \right)$$

2.3.7 Cohomology with support: the functor $R\phi_!$

Relative cohomology groups

Let S be a closed subscheme of X and denote its immersion by i. The functor

$$\Gamma_S(X,-) : \mathscr{F} \longmapsto \left\{ s \in \mathscr{F}(X) \text{ s.t. } \mathtt{supp}(s) \subset S \right\}$$

from $\mathbf{Sh}_R(X)$ to $\mathbf{Mod}_R$ is left exact but not right exact. Denote by $R\Gamma_S(X,-)$ its right derived functor. Set the notation

$$\mathrm{H}^p_{\text{ét},S}(X,\mathscr{F}) := R^p\Gamma_Z(S,-).$$

The functor $R\phi_!$

(cf. [Fu11, 7.4])

Let $X \overset{\phi}{\to} Y$ be a morphism of schemes. Thanks to the formalism of derived category it is possible to define "correctly" the higher image with proper support, denoted by $R\phi_!$. The notation is misleading since $R\phi_!$ is not the derived version of $\phi_!$: the latter is an exact functor (cf. lemma 2.3.3.1).
For $R\phi_!$ to be well defined, there is a mild condition on ϕ. More precisely ϕ has to be a compatctifiable S-morphsim for some scheme S. Without going through the details of the definition, if $Y = S$, and if ϕ is separated and of finite type, then the condition is satisfied.
This mild condition on ϕ assure a factorization

$$X \overset{j}{\hookrightarrow} \overline{X} \overset{\overline{\phi}}{\to} Y$$

with j an open immersion and with $\overline{\phi}$ proper (and S-compactifiable). One can show that the functor

$$R\phi_! := R\overline{\phi}_* \circ j_!$$

is well defined, i.e. it does not depend on the factorization $\phi = \overline{\phi} \circ j$.

If $\phi = \varsigma$ is the structure morphism of a k-scheme X, with k separably closed, the étale cohomology *with proper support*, is defined as

$$\mathrm{H}^p_{\text{ét},c}(X,\mathscr{F}) := R^p\varsigma_!\mathscr{F}(\mathtt{Spec}(k)).$$

for any F in $D^+(X,R)$.

2.3.8 Poincaré Duality and relative purity: the functor $R\phi^!$

The functor $R\phi^!$

The functor $R\phi_!$ admits a right adjoint, denoted $R\phi^!$. I won't give the general construction of this functor. The original reference is [AGV71] XVIII.3.1.
When $\phi = i$ is an open immersion, the functor $Ri^!$ can be described in a simple way.

Lemma 2.3.8.1 ([Fu11, 5.4.1]). Let X be a scheme. Let $i : X_c \to X$ be a closed immersion and denote by $j : X_o \to X$ the complementary open immersion, where $X_o = X \setminus X_c$.
A sheaf $\mathscr{F}$ on X is of the form $i_*\mathscr{G}$ for some sheaf $\mathscr{G}$ on X_c if and only if $j^*\mathscr{F} = 0$.

Let i and j be as in lemma 2.3.8.1. For a sheaf $\mathscr{F}$ on X, let $\mathscr{K}$ be the kernel of $\mathscr{F} \to j_*j^*\mathscr{F}$. By lemma 2.3.3.1, $j^*\mathscr{K}$ is the kernel of $j^*\mathscr{F} \to j^*j_*j^*\mathscr{F} = j^*\mathscr{F}$, which is an isomorphism. Hence $j^*\mathscr{K} = 0$ and by lemma 2.3.8.1, there is a sheaf $i^!\mathscr{F}$ such that $\mathscr{K} = i_*i^!\mathscr{F}$. This define a functor $i^!$ that is left exact. Its right derived functor coincide with $Ri^!$. In general this construction is not valid when ϕ is not a closed immersion.

Under particular circumstances there is a practical description of $R\phi^!$. For $n \in \mathbb{N}_{>0}$ let $[n] : \mathbb{G}_{m,X} \to \mathbb{G}_{m,X}$ be the morphism of sheaves locally defined by

$$\begin{aligned}[n] : \mathscr{O}_E(E)^\times &\longrightarrow \mathscr{O}_E(E)^\times \\ f &\longmapsto f^n.\end{aligned}$$

Denote by $\boldsymbol{\mu}_{n,X}$ the sheaf kernel of $[n]$.

Definiton 2.3.8.1. Let X be a scheme. For each integer d and positive integer n, set the notation:

$$\underline{\mathbb{Z}/n\mathbb{Z}}_X(d) := \begin{cases} \boldsymbol{\mu}_{n,X}^{\otimes d} & \text{if } d \geqslant 0 \\ \underline{\mathrm{Hom}}(\boldsymbol{\mu}_{n,X}^{\otimes -d}, \underline{\mathbb{Z}/n\mathbb{Z}}_X) & \text{if } d < 0 \end{cases}$$

$$\mathscr{F}(n,d) := \mathscr{F} \otimes \underline{\mathbb{Z}/n\mathbb{Z}}_X(d).$$

If n is understood, simply write $\mathscr{F}(d)$. This definition induces a similar definition for derived sheaves.

Remark 2.3.8.1. The sheaf $\boldsymbol{\mu}_{n,X}$ is locally constant. If X is a scheme over a strict local ring (eg. a separably closed field), then $\boldsymbol{\mu}_{n,X} \simeq \underline{\mathbb{Z}/n\mathbb{Z}}_X$ (cf. [Fu11, 7.2.2], 7.2.2).

Theorem 2.3.8.1 ([Fu11, 8.5.2]). If $X \xrightarrow{\phi} Y$ is a smooth, separable morphism of finite type which is furthermore pure of dimension d, then there is an isomorphism

$$R\phi^!F \simeq \phi^*F(n,d)[2d]$$

for any object in $D^+(Y, \mathbb{Z}/n\mathbb{Z})$.

Lemma 2.3.8.2 ([Fu11, 5.6.11]).

(i) There is a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^p_{\text{ét},Z}(X, \mathscr{F}) \longrightarrow \mathrm{H}^p_{\text{ét}}(X, \mathscr{F}) \longrightarrow \mathrm{H}^p_{\text{ét}}(X \setminus Z, \mathscr{F}) \longrightarrow \cdots$$

(ii) There is a spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(Y, R^q i^! \mathscr{F}) \Longrightarrow \mathrm{H}^{p+q}_Z(X, \mathscr{F}).$$

Duality and purity

Theorem 2.3.8.2. Let $X \xrightarrow{\phi} Y$ be an S-compactifiable morphism. For any object F in $D^-(X, R)$ and G in $D^+(X, R)$ there are isomorphisms

$$R\phi_* R\underline{\mathrm{Hom}}(F, R\phi^! G) \simeq R\underline{\mathrm{Hom}}(R\phi_! F, G)$$

$$R\underline{\mathrm{Hom}}(\phi^* F, R\phi^! G) \simeq R\phi^! R\underline{\mathrm{Hom}}(F, L)$$

Corollary 2.3.8.1. Let k be a separably closed field. If $X \xrightarrow{\phi} \mathtt{Spec}(k)$ is a smooth, separable morphism of finite type which is furthermore pure of dimension d, then for any locally constant sheaf F of $\mathbb{Z}/n$-modules there is an isomorphism

$$\mathrm{H}^{2d-p}_{\text{ét}}(X, R\underline{\mathrm{Hom}}(\mathscr{F}, \underline{\mathbb{Z}/n\mathbb{Z}}_X(d))) \simeq \mathrm{Hom}_{\mathbf{Mod}_{\mathbb{Z}/n\mathbb{Z}}}(\mathrm{H}^p_{\text{ét},c}(X, \mathscr{F}), \mathbb{Z}/n\mathbb{Z}))$$

This is the étale cohomology version of Poincaré duality.

Corollary 2.3.8.2. Let k be a separably closed field. Let X_c be a closed subscheme of X, X_o its complement, i and j the immersions of these schemes in X and let $X \xrightarrow{\varsigma} k$ the structure morphism of X. These objects sit in a diagram

$$\begin{tikzcd} X_c \arrow[r, hook, "i"] & X \arrow[d, "\varsigma"] & X_o \arrow[l, hook', "j"'] \\ & \mathtt{Spec}(k) & \end{tikzcd}$$

Suppose that X_c is pure of codimension d and that both X and X_c are smooth and of finite type. Then

$$R^p i^! \mathscr{F} \simeq \begin{cases} i^* \mathscr{F}(-d) & \text{if } p = 2d \\ 0 & \text{otherwise} \end{cases}$$

$$R^p j_* j^* \mathscr{F} \simeq \begin{cases} \mathscr{F} & \text{if } p = 0 \\ i_* i^* \mathscr{F}(-d) & \text{if } p = 2d-1 \\ 0 & \text{otherwise} \end{cases}$$

$$R^p \varsigma_* \mathscr{F} \simeq R^p (\varsigma \circ j)_* (j^* \mathscr{F}) \quad \text{if } 0 \leqslant p \leqslant 2d-2$$

Remark 2.3.8.2. Corollaries 2.3.8.1 and 2.3.8.2 hold in greater generality. The original reference is [AGV71], XVIII.3.2 and XIX.3.2.

2.3.9 Constructible sheaves

For a scheme X the category of $\mathbf{Sh}(X)$ is quite vast and the category $D^b(X,R)$ is even bigger. To study all kinds of sheaves at the same time is pretty much impossible and it would not necessarily lead to interesting and useful results. At the end of the day one wants to study the étale cohomology with coefficients in a sheaf $\mathscr{F}$ that is not too exotic.
For this one could try to focus on constant sheaves, but the full subcategory of constant sheaves is not very stable: for instance direct images of constant sheaves are not constant.
Thus one should aim to find a full subcategory of $D(X,R)$ which is stable under the operations defined in the previous section.

Definiton 2.3.9.1. A sheaf $\mathscr{F}$ on X is locally constant if there exists a covering $\{U_i \to X\}$, in the sense of definition 2.2.1.1, such that each $\mathscr{F}_{U_i}$ is constant.

Proposition 2.3.9.1 ([Fu11] proposition 5.8.1). A sheaf of sets $\mathscr{F}$ on X with finite stalks is locally constant if and only if $\mathscr{F} = \widetilde{E}$ for some finite étale cover of $E \to X$.
In particular (cf. theorem 2.1.2.1) the category $\mathbf{Sh}_{\mathbf{Setf}}(X)$ is equivalent to the category $\pi_1^{\text{ét}}(X,x)$-**Setf**.

Proposition 2.3.9.2. Let $X \xrightarrow{\phi} Y$ be any morphism of schemes and let $\mathscr{F}$ be a locally constant sheaf on Y. Then $\phi^* \mathscr{F}$ is a locally constant sheaf on X.

Definiton 2.3.9.2 (Constructible sheaves). Let $\mathscr{F}$ is *constructible* if there is a finite decomposition

$$X = \bigcup_{i=1}^{n} X_i \text{ with each } X_i \text{ locally closed,}$$

such that $\mathscr{F}_{X_i}$ is locally constant and with finite dimensional stalks.
For sheaves of sets or abelian groups the definition of constructible is the same but one replaces "finite dimensional" with "finite".

Definiton 2.3.9.3. Constructible sheaves define a full subcategory of $\mathbf{Sh}_R(X)$. Denote its derived category by $D_c(X,R)$. The full subcategory of $D^b(X,R)$ whose objects F satisfy

$$H^i(F) \text{ is a constructible object of } \mathbf{Sh}_{R,c}(X) \text{ for all } i,$$

coincide with $D^b_c(X,R)$.

Fix some regular scheme S of dimension $\leqslant 1$. Set let R be a finite ring such that $nR = 0$ for some integer n invertible in S.

Theorem 2.3.9.1. Let $X \xrightarrow{\phi} Y$ be an S-morphism of finite type. The functors $R\phi_*, \phi^*, R\phi_!, R\phi^!$ have finite cohomological dimension, i.e. there exists a positive integer n such that $R^N\phi_* = \phi^* = R^N\phi_! = R^N\phi^! = 0$ for all $N > n$.
Furthermore $R\phi_*$ and $R\phi_!$ map $D^b_c(X,A)$ to $D^b_c(Y,A)$, while ϕ^* and $R\phi^!$ map $D^b_c(Y,R)$ to $D^b_c(X,R)$.

Theorem 2.3.9.2. For any scheme X, the functor $-\otimes^{\mathbf{L}}_R-$ maps $D^b_c(X,R)\times D^b(X,R)$ to $D^b_c(X,R)$. For a S-scheme X the functor $R\underline{\mathrm{Hom}}(-,-)$ maps $D^b_c(X,R)\times D^b(X,R)$ to $D^b_c(X,R)$.

Definiton 2.3.9.4. Let X be a k-scheme. For F an object in D^b_c define

$$\mathbb{D}F := R\underline{\mathrm{Hom}}(F, R\varsigma^!\mathbb{Z}_\ell).$$

This gives a functor $\mathbb{D} : D^b_c(X,R) \to D^b_c(X,R)$, called *dualizing functor*.

Theorem 2.3.9.3. The functor $\mathbb{D}\circ\mathbb{D}$ is an isomorphism: for all objects F in $D^b_c(X,R)$,

$$F \simeq \mathbb{D}\circ\mathbb{D}(F).$$

Theorem 2.3.9.4 ([Fu11] 9.3.5.). Let X,Y,Z,W be k-schemes of finite type. Consider two k-morphisms $X \xrightarrow{\phi} Y$ and $W \xrightarrow{\psi} Z$ as well as two (derived) sheaves F in $D^b_{ctf}(X,R)$ and G in $D^b_{ctf}(W,R)$. Then

$$R\phi_*F \boxtimes R\psi_*G \simeq R\phi\times\psi_*(F\boxtimes G).$$

Here the map $\phi\times\psi$ denotes the obvious map $X\times_k W \to Y\times_k Z$.

Here $D^b_{ctf}(-,R)$ is the full subcategory of $D(-,R)$ whose objects are both in $D^b_c(-,R)$ and $D^b_{tf}(-,R)$. The latter category was defined at page 29.

2.4 Computing étale cohomology

2.4.1 First results

Relation with Zariski-cohomology

Any étale sheaf on X defines a sheaf on the Zariski site of X. Denote by ι_* the functor that sends étale sheaves to sheaves on the Zariski site. This functor is left exact and there is a spectral sequence

$$E_2^{p,q} = \mathrm{H}^p_{\mathrm{Zar}}(X, R^q\iota_*\mathscr{F}) \Longrightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathscr{F})$$

Here $\mathrm{H}^p_{\mathrm{Zar}}(X,-)$ is the right derived functor of $\Gamma(X,-) : \mathscr{F} \mapsto \mathscr{F}(X)$ that sends sheaves on the the Zariski site of X to their global sections.

Proposition 2.4.1.1. There are isomorphisms

$$\mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_{m,X}) \simeq \mathrm{H}^1_{\mathrm{Zar}}(X, \iota_*\mathbb{G}_{m,X}) \simeq \mathrm{Pic}(X) \tag{2.4.1}$$

where the second equality is well known. Here $\mathrm{Pic}(X)$ is the group whose elements are invertible $\mathscr{O}_X$-modules and whose composition law s given by the tensor product. If X is integral this can be identified with the class group of Cartier divisors on X.

Remark 2.4.1.1. The degree map $\mathtt{deg}$ that sends a divisor to its degree is well defined on $\mathrm{Pic}(X)$. Thus there is a map $\mathtt{deg} : \mathrm{Pic}(X) \to \mathbb{Z}$ which is furthermore surjective. The kernel of this map is denoted by $\mathrm{Pic}^0(X)$. It can be shown that $\mathrm{Pic}^0(X)$ has the structure of a k-abelian variety of dimension g, where g is the genus of X (see [Poo17] 5.7.1).
It is a standard result in the theory of abelian varieties that, over a separably closed field k of characteristic $p \neq \ell$, for an abelian variety A of dimension g, the kernel of the isogeny $[\ell^n] : A \to A$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$.

Galois cohomology

Galois cohomology is a special case of étale cohomology.

Proposition 2.4.1.2 ([Fu11, 5.7.8]). Let k be a field, let $x = \mathtt{Spec}(k \hookrightarrow k^{\mathrm{sep}})$ and let $\mathscr{F}$ be a sheaf on $\mathtt{Spec}(k)$. Then

$$\mathrm{H}^p_{\text{ét}}(\mathtt{Spec}(k^{\mathrm{sep}}), \mathscr{F}) = \mathrm{H}^p_{\mathbf{Grp}}(\mathrm{Gal}(k^{\mathrm{sep}}|k), \mathscr{F}_x),$$

where on the right hand side is the p^{th}-group cohomology of $\mathrm{Gal}(k^{\mathrm{sep}}|k)$ with coefficients in $\mathscr{F}_x$.

Remark 2.4.1.2. For k algebraically closed, it is possible to check that $\mathrm{Pic}(\mathtt{Spec}(k)) = 0$. This implies

$$\mathrm{H}^1_{\mathbf{Grp}}(\mathrm{Gal}(k^{\mathrm{sep}}|k), (k^{\mathrm{sep}})^\times) \simeq \mathrm{H}^1_{\text{ét}}(\mathtt{Spec}(k), \mathbb{G}_{m,\mathtt{Spec}(k)}) = 0.$$

This result is known as Hilbert's theorem 90.

Proposition 2.4.1.3 ([Fu11], 4.5.9). A field k is *quasi-algebraically closed* if the vanishing set of an arbitrary, non zero, homogeneous polynomial in $k[X_1,\dots,X_n]$ has a non trivial solution in k^n. If k is quasi algebraically closed, then

$$\mathrm{H}^p_{\mathbf{Grp}}(\mathrm{Gal}(k^{\mathrm{sep}}|k,(k^{\mathrm{sep}})^\times)) = 0 \text{ for all } p > 0.$$

Remark 2.4.1.3. It is a theorem of Tsen (cf [Fu11] 4.5.8) that, if k is an algebraically closed field, any extension $k \hookrightarrow L$ of transcendence degree $\geqslant 1$ is quasi-algebraically closed.

For the next two theorems the original reference is [AGV71] IX.3.

Theorem 2.4.1.1 (Kummer Theory). If n is invertible[4] on X there is a short exact sequence

$$0 \to \boldsymbol{\mu}_{n,X} \longrightarrow \mathbb{G}_{m,X} \xrightarrow{[n]} \mathbb{G}_{m,X} \to 0.$$

Theorem 2.4.1.2 (Artin-Schreier Theory). Let X be a scheme such that $p \cdot \mathscr{O}_X(X) = 0$ for some prime number p. There is a short exact sequence

$$0 \to \underline{\mathbb{Z}/p\mathbb{Z}}_X \longrightarrow \mathscr{O}_X^{\text{ét}} \xrightarrow{[p]-\mathrm{id}} \mathscr{O}_X^{\text{ét}} \to 0$$

Definiton 2.4.1.1. Let X be a noetherian scheme. A sheaf $\mathscr{F}$ on X is called a *skyscraper* sheaf if $\mathscr{F}_{\overline{x}} = 0$ for all non-closed points $\mathfrak{p} = x$.

Lemma 2.4.1.1 ([Fu11, 7.2.4]).

(i) A sheaf is skyscraper if and only if

$$\mathscr{F} \simeq \bigoplus_{\mathfrak{p} \text{ closed point in } X} i_{\mathfrak{p}*} i_{\mathfrak{p}}^* \mathscr{F}$$

where $i_{\mathfrak{p}} : \mathtt{Spec}(\kappa(\mathfrak{p})) \to X$ is the incusion of $\mathfrak{p}$

(ii) Let k be a separably closed field. Let X be a k-scheme and $\mathscr{F}$ a skyscraper sheaf on X. Then

$$H^i_{\text{ét}}(X,\mathscr{F}) = 0 \text{ for all } i \geqslant 1.$$

(iii) If $\mathscr{F}$ and $\mathscr{G}$ are sheaves on X with $\mathscr{F}$ skyscraper, then $\mathscr{F} \boxtimes \mathscr{G}$ is skyscraper.

Proposition 2.4.1.4. Let C be an integral regular curve and denote by η its generic point and by j the canonical inclusion $\mathtt{Spec}(\kappa(\eta)) \to C$. For a closed point $\mathfrak{p}$ in C denote by $i_{\mathfrak{p}}$ the inclusion $\mathtt{Spec}(\kappa(\mathfrak{p})) \to C$. There is an exact sequence

$$0 \to \mathbb{G}_{m,X} \longrightarrow j_* \mathbb{G}_{m,\mathtt{Spec}(\kappa(\eta))} \longrightarrow \bigoplus_{\mathfrak{p} \text{ closed point in } C} i_{\mathfrak{p}*} \underline{\mathbb{Z}}_{\mathtt{Spec}(\kappa(\mathfrak{p}))} \to 0,$$

[4] i.e. $n \in (O_X^{\text{ét}}(E))^\times$ for all E in $\mathbf{Over}_{\text{ét}}(X)$

where the third arrow is given by the the following morphism.
For E connected and étale over C denote by η_E its generic point and by $\nu_{\mathfrak{p}}(E)$ the discrete valuation $\kappa(\eta_E) \to \mathbb{Z}$ at $\mathfrak{p} \in V$. Then set

$$\begin{aligned} j_*\mathbb{G}_{m,\mathtt{Spec}(\kappa(\eta))}(E) = \kappa(\eta_E) &\longrightarrow \bigoplus_{\mathfrak{p} \text{ closed point in } E} \mathbb{Z} \simeq \left(\bigoplus_{\mathfrak{p} \text{ closed point in } C} i_{\mathfrak{p}*}\underline{\mathbb{Z}}_{\mathtt{Spec}(\kappa(\mathfrak{p}))} \right)(E) \\ a &\longmapsto \nu_{\mathfrak{p}}(E)(a) \end{aligned}$$

2.4.2 Comparison theorem

(cf. with definition 2.1.2.5)

The functor $X \mapsto X^{\mathrm{an}}$ and this functor induces a functor $\mathscr{F} \mapsto \mathscr{F}^{\mathrm{an}}$ from $\mathbf{Sh}(X)$ to $\mathbf{Sh}(X^{\mathrm{an}}) :=$ {category of usual topological sheaves on X^{an}}.

Theorem 2.4.2.1. If $\mathscr{F}$ is a locally constant sheaf on X with finite stalks, then

$$\mathrm{H}^p_{\text{ét}}(X, \mathscr{F}) \simeq \mathrm{H}^p(X^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}}),$$

where the right hand side is the usual cohomology.

Remark 2.4.2.1. Combining remark 2.3.1.5 and theorem 2.4.2.1, one obtains an easy way to study the étale cohomology of schemes over an algebraically closed field of characteristic 0.

2.4.3 Cohomology of curves

Assuming the above theorems as black boxes, I give an argument that implies theorem 2.2.1.1. Since some of the above theorems are very powerful my argument will be a total overkill.
Let me recall the statement. Let C be proper smooth connected curve of genus g over a separably closed field k of characteristic $p \geqslant 0$. Let S be a finite set of closed points on C and denote by X the open subscheme $C \setminus S$. If $\ell \neq p$ is a prime number, then

$$\mathrm{H}^i_{\text{ét}}(X, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}_X) = \begin{cases} \mathbb{Z}/\ell^n\mathbb{Z} & \text{if } i = 1 \\ (\mathbb{Z}/\ell^n\mathbb{Z})^{2g+\alpha} & \text{if } i = 2 \\ (\mathbb{Z}/\ell^n\mathbb{Z})^{\beta} & \text{if } i = 3 \\ 0 & \text{otherwise} \end{cases}$$

where if $\#S = 0$, $\alpha = 0$ and $\beta = 1$, while if $\#S > 0$, then $\alpha = \#S - 1$ and $\beta = 0$.

The proper case

First I treat the case $S = \emptyset$.
Remark 2.3.8.1 states that $\mathrm{H}^p_{\text{ét}}(C, \mathbb{Z}/n\mathbb{Z}) \simeq \mathrm{H}^p_{\text{ét}}(C, \boldsymbol{\mu}_{n,C})$, hence Kummer theory can be used to compute $\mathrm{H}^p_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z})$.
The exact sequence of proposition 2.4.1.1 induce the long exact sequence

$$\cdots \mathrm{H}^p_{\text{ét}}(X, \boldsymbol{\mu}_{n,C}) \longrightarrow \mathrm{H}^p_{\text{ét}}(X, \mathbb{G}_{k,C}) \xrightarrow{[n]} \mathrm{H}^p_{\text{ét}}(X, \mathbb{G}_{k,C}) \longrightarrow \cdots$$

There are isomorphisms

$$\begin{aligned} \mathrm{H}^0_{\text{ét}}(X, \mathbb{G}_{m,X}) &= k^\times && \text{(clear)} \\ \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_{m,X}) &= \mathrm{Pic}(X) && (2.4.1), \end{aligned}$$

and there are left to compute the others $\mathrm{H}^p_{\text{ét}}(X, \mathbb{G}_{k,X})$ for $p > 1$.

For this, by the sequence in proposition 2.4.1.4 it is enough to compute

$$\mathrm{H}^p_{\text{ét}}(C, j_*\mathbb{G}_{m,\mathtt{Spec}(\kappa(\eta))}) \quad \text{and} \quad \mathrm{H}^p_{\text{ét}}(C, \bigoplus_{\mathfrak{p} \text{ closed point in } C} i_{\mathfrak{p}*}\underline{\mathbb{Z}}_{\mathtt{Spec}(\kappa(\mathfrak{p}))}).$$

The computation of the second one is easy since, by lemma 2.4.1.1, the p^{th}-cohomology vanish for $p > 0$. For the first one, claim that

$$\mathrm{H}^p_{\text{ét}}(C, j_*\mathbb{G}_{m,\mathtt{Spec}(\kappa(\eta))}) = \mathrm{H}^p_{\text{ét}}(\mathtt{Spec}(\kappa(\eta)), \mathbb{G}_{m,\mathtt{Spec}(\kappa(\eta))}).$$

Assuming the claim, by proposition 2.4.1.3, for all $p > 0$

$$\mathrm{H}^p_{\text{ét}}(C, j_*\mathbb{G}_{m,\mathtt{Spec}(\kappa(\eta))}) = 0$$

For the claim, remark 2.4.1.3 implies

$$(R^p j_*\mathbb{G}_{m,\mathtt{Spec}(\kappa(\eta))})_x = \mathrm{H}^p_{\text{ét}}(\mathtt{Spec}(\underbrace{\kappa(\eta) \otimes \mathcal{O}^{sh}_{C,\mathfrak{p}}}_{\substack{\text{since } X \text{ is a curve} \\ \text{this is an over field of } \kappa(\text{ét}) \\ \text{of tr.deg. } \leqslant 1}}), \mathbb{G}_{m,\mathtt{Spec}(\kappa(\eta))}) = 0$$

which itself implies that the spectral sequence 2.3.8 collapse. This proves the claim.

Hence $\mathrm{H}^p_{\text{ét}}(C, j_*\mathbb{G}_{m,\mathtt{Spec}(\kappa(\eta))}) = \mathrm{H}^p_{\text{ét}}(C, \bigoplus_{\mathfrak{p} \text{ closed point in } C} i_{\mathfrak{p}*}\underline{\mathbb{Z}}_{\mathtt{Spec}(\kappa(\mathfrak{p}))}) = 0$ for $p > 1$, which means that $\mathrm{H}^p_{\text{ét}}(X, \mathbb{G}_{k,X})$ vanish for $p > 1$.
The long exact sequence of proposition 2.4.1.1 becomes

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^0_{\text{ét}}(X, \boldsymbol{\mu}_{n,X}) \arrow[r] & k^\times \arrow[r, "{[n]}"] & k^\times \arrow[dll] \\ & \mathrm{H}^1(X, \boldsymbol{\mu}_{n,X}) \arrow[r] & \mathrm{Pic}(X) \arrow[r, "{[n]}"] & \mathrm{Pic}(X) \arrow[dll] \\ & \mathrm{H}^2(X, \boldsymbol{\mu}_{n,X}) \arrow[r] & 0 \end{tikzcd} \tag{2.4.2}$$

From (2.4.2) and remark 2.4.1.1 it is a simple to verify that.

$$\mathrm{H}^p_{\text{ét}}(C, \mathbb{Z}/\ell^n\mathbb{Z}) = \begin{cases} \mathbb{Z}/\ell^n\mathbb{Z} & \text{if } p = 1 \\ (\mathbb{Z}/\ell^n\mathbb{Z})^{2g} & \text{if } p = 2 \\ \mathbb{Z}/\ell^n\mathbb{Z} & \text{if } p = 3 \\ 0 & \text{otherwise} \end{cases} \tag{2.4.3}$$

The affine case

By lemma 2.3.8.2.(i) there is a long exact sequence

$$\cdots \mathrm{H}^p_{\text{ét},S}(C, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}_C) \longrightarrow \mathrm{H}^p_{\text{ét}}(C, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}_C) \longrightarrow \mathrm{H}^p_{\text{ét}}(X, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}_C) \longrightarrow \cdots \tag{2.4.4}$$

By lemma 2.3.8.2.(ii) and corollary 2.3.8.2,

$$\mathrm{H}^p_{\text{ét},S}(C, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}_C) \simeq \begin{cases} \mathrm{Hom}(\boldsymbol{\mu}_{\ell^n}(k), (\mathbb{Z}/\ell^n\mathbb{Z}))^{\#S} & \text{if } p = 2 \\ 0 & \text{otherwise} \end{cases}$$

Since the endomorphism group of a cyclic group is isomorphic to the group itself,

$$\mathrm{H}^2_{\text{ét},S}(C, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}_C) = (\mathbb{Z}/\ell^n\mathbb{Z})^{\#S}$$

Combining this fact, (2.4.4) and (2.4.3), it is immediate that

$$\mathrm{H}^p_{\text{ét}}(X, \mathbb{Z}/\ell^n\mathbb{Z}) = \begin{cases} \mathbb{Z}/\ell^n\mathbb{Z} & \text{if } p = 1 \\ (\mathbb{Z}/\ell^n\mathbb{Z})^{2g+\#S-1} & \text{if } p = 2 \\ 0 & \text{otherwise} \end{cases} .$$

Thus theorem 2.2.1.1 is proved.

Artin's vanishing theorem

The vanishing of $\mathrm{H}^2_{\text{ét}}(X, \underline{\mathbb{Z}/\ell^n}_C) = 0$ is a particular case of a general theorem proved by Artin (cf. [AGV71] XIV 3.2) that will be used in next chapter.

Theorem 2.4.3.1. Let k be a separably closed field. Let $\mathscr{F}$ be a sheaf on a k-scheme X. If X is affine then $\mathrm{H}^p_{\text{ét}}(X, \mathscr{F}) = 0$ for all $p > \dim(X)$.

2.5 The formalism of ℓ-adic sheaves and $\mathbb{Q}_\ell$-sheaves

Let X a scheme and let ℓ be a prime number invertible on X.
If in the previous section the finite ring R is replaced with a field of characteristic 0, many theorems would not hold. For instance, for a curve C over a field k it is possible to define $\mathrm{H}^p_{\text{ét}}(X, \mathbb{Q})$ in the naive way, i.e. by considering étale sheaves of $\mathbb{Q}$-vectors spaces. Unfortunately doing this would not lead to the desired answer, which is theorem 2.2.1.1 with $\mathbb{Z}/\ell^n\mathbb{Z}$ replaced with $\mathbb{Q}$.
The formalism of ℓ-adic and $\mathbb{Q}_\ell$-sheaves is the solution to obtain a cohomology theory with non-torsion coefficients.

In this section ℓ is always a prime number invertible on X. Recall that

$$\mathbb{Z}_\ell := \varprojlim_n \mathbb{Z}/\ell^n\mathbb{Z}$$

and that $\mathbb{Q}_\ell := \mathtt{Frac}(\mathbb{Z}_\ell)$ is a field of characteristic 0.

Definiton 2.5.0.1. Let ℓ be a prime number. Let $\mathscr{F}$ be a sheaf of $\mathbb{Z}_\ell$-modules on a scheme X. Then $\mathscr{F}$ is *ℓ-torsion* if

$$\mathscr{F} = \bigcup_{m\geqslant 0} \mathrm{Ker}(\ell^m),$$

where $\ell^m \in \mathrm{End}(\mathscr{F})$ is the obvious map.

Remark 2.5.0.1. All sheaves of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules are ℓ-torsion.

Definiton 2.5.0.2 (Constructible and lisse ℓ-adic sheaves).
A constructible ℓ-adic sheaf $\mathscr{F}$ on a scheme X is an inverse system $\{\mathscr{F}_m, f_m\}_{m\in\mathbb{N}}$ of ℓ-torsion sheaves

$$\cdots \leftarrow \mathscr{F}_m \xleftarrow{f_{m+1}} \mathscr{F}_{m+1} \leftarrow \cdots$$

such that

0) Each sheaf $\mathscr{F}_m$ is constructible;

i) $\ell^{m+1}\mathscr{F}_m = 0$ for all $m \in \mathbb{N}$;

ii) $\mathscr{F}_m/\ell^m\mathscr{F}_m \simeq \mathscr{F}_{m-1}$.

A constructible ℓ-adic sheaf $\mathscr{F}$ is *lisse* if each $\mathscr{F}_m$ is locally constant.
Denote by $\mathbf{Sh}_{\mathbb{Z}_\ell}(X)$ the category of constructible ℓ-adic sheaves and by $\mathbf{LSh}_{\mathbb{Z}_\ell}$ the category of lisse ℓ-adic sheaves. Morphisms are defined in the usual way.

Remark 2.5.0.2 ([Fu11] 10.1.6). It is possible to replace replace condition 0) with:

0') $\mathscr{F}_0$ is constructible.

More precisely there exists a decomposition $X = \bigcup_j X_j$, with each X_j locally closed, such that each $\mathscr{F}_{i,X_j}$ is locally constant.

Definiton 2.5.0.3. For a ℓ-adic sheaf $\mathscr{F}$ on X, set

$$\mathrm{H}^p_{\text{ét}}(X,\mathscr{F}) := \varprojlim_m \mathrm{H}^p_{\text{ét}}(X,\mathscr{F}_m)$$

Remark 2.5.0.3. Since I've omitted the "passage to limit" discussion for the étale cohomology, let me remark that the formation of $\mathrm{H}^p_{\text{ét}}$ does not commute with inverse limits and that $R^p\Gamma(X,\underline{\mathbb{Z}_\ell}_X) \neq \varprojlim_m \mathrm{H}^p_{\text{ét}}(X,\mathbb{Z}/\ell^m\mathbb{Z})$.

Definiton 2.5.0.4. Let X be a scheme. The *bounded derived category of ℓ-adic (constructible) sheaves*, that denoted by $D^b_c(X,\mathbb{Z}_\ell)$ is the category whose objects are inverse systems $F = (F_m, f_m)_{m\in\mathbb{N}}$ with

$$F_0 \in D^b_c(X,\mathbb{Z}/\ell\mathbb{Z}) \qquad F_m \in D^-(X,\mathbb{Z}/\ell^{m+1}\mathbb{Z}) \qquad f_m : F_{m+1} \to F_{m+1} \underset{\mathbb{Z}/\ell^{m+2}\mathbb{Z}}{\otimes^{\mathbf{L}}} \mathbb{Z}/\ell^{m+1}\mathbb{Z} \simeq F_m,$$

and whose morphisms between two object $F = (F_m, f_m)$ and $G = (G_m, g_m)$ are family of morphisms

$$\{F_m \xrightarrow{\phi_m} G_m \in \mathrm{Hom}_{D(X,\mathbb{Z}/\ell^{m+1}\mathbb{Z})}\}_{m\in\mathbb{N}}$$

compatible with f_m and g_m in the sense that

$$\phi_m \circ f_m = g_m \circ (\phi_{m+1} \underset{\mathbb{Z}/\ell^{m+2}\mathbb{Z}}{\otimes^{\mathbf{L}}} \mathrm{id}) \;\; \text{“} = g_m \circ \phi_{m+1} \text{”}.$$

Proposition 2.5.0.1. Let F be an object of $D_c^b(X, \mathbb{Z}_\ell)$. For all i, the inverse system

$$(H^i(F_m), H^i(f_m))_{m\in\mathbb{N}}$$

is a constructible ℓ-adic sheaf.

Definiton 2.5.0.5. The category of constructible $\mathbb{Q}_\ell$-sheaves, that denoted by $\mathbf{Sh}_{\mathbb{Q}_\ell}(X)$, is defined as follows: the object are the same as the objects in $\mathbf{Sh}_{\mathbb{Z}_\ell}(X)$, while the morphisms are given by

$$\mathrm{Hom}_{\mathbf{Sh}_{\mathbb{Q}_\ell}}(-,-) := \mathrm{Hom}_{\mathbf{Sh}_{\mathbb{Z}_\ell}}(-,-) \otimes \mathbb{Q}_\ell.$$

Then it is natural to define the category $D_c^b(X, \mathbb{Q}_\ell)$. The objects are the same as the objects of $D_c^b(X, \mathbb{Z}_\ell)$ and the morphism are given by

$$\mathrm{Hom}_{D_c^b(X,\mathbb{Q}_\ell)}(-,-) := \mathrm{Hom}_{D_c^b(X,\mathbb{Z}_\ell)}(-,-) \otimes \mathbb{Q}_\ell.$$

Remark 2.5.0.4. Last definition, suggests how to define the functor $R\underline{\mathrm{Hom}}_{\mathbf{Sh}_{\mathbb{Q}_\ell}}$.

Definiton 2.5.0.6. For $\mathscr{F}$ a $\mathbb{Q}_\ell$-sheaf on X, the p^{th} cohomology of $\mathscr{F}$ is defined as

$$\mathrm{H}^p_{\text{ét}}(\mathscr{F}, \mathbb{Q}_\ell) := \mathrm{H}^p_{\text{ét}}(\mathscr{F}, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell.$$

Proposition 2.5.0.2. The operations on the categories $D_c^b(-, \mathbb{Z}/\ell^n\mathbb{Z})$, given by the functors $R\phi_*, \phi^*, R\phi_!, R\phi^!, R\underline{\mathrm{Hom}}, \otimes^{\mathbf{L}}$, are compatible with the formation of $D_c^b(-, \mathbb{Z}_\ell)$ and $D_c^b(-, \mathbb{Q}_\ell)$. Thus the categories $D_c^b(-, \mathbb{Z}_\ell)$ and $D_c^b(-, \mathbb{Q}_\ell)$ come equipped with a *six functors formalism*. The dualizing functor of definition 2.3.9.4 has an obvious analog in $D_c^b(-, \mathbb{Z}_\ell)$ and $D_c^b(-, \mathbb{Q}_\ell)$. Theorem 2.3.9.3 is also valid.

Remark 2.5.0.5. Kunneth's formula, theorem 2.3.9.4 also holds in $D_c^b(-, \mathbb{Q}_\ell)$. The tf condition, tensoring with $\mathbb{Q}_\ell$, becomes trivial.

Theorem 2.5.0.1 ([Fu11] 10.1.23). Let X be a connected scheme and let x a geometric point on X. There are two equivalence of categories induced by the functors

$$\begin{aligned} \mathbf{LSh}_{\mathbb{Z}_\ell}(X) &\xrightarrow{\sim} \mathbf{Repf}_{\mathbb{Z}_\ell}(\pi_1^{\text{ét}}(X,x)) \\ \mathscr{F} &\longmapsto \mathscr{F}_x \end{aligned} \qquad \begin{aligned} \mathbf{LSh}_{\mathbb{Q}_\ell}(X) &\xrightarrow{\sim} \mathbf{Repf}_{\mathbb{Q}_\ell}(\pi_1^{\text{ét}}(X,x)) \\ \mathscr{F} &\longmapsto \mathscr{F}_x \otimes \mathbb{Q}_\ell \end{aligned}$$

where, for any ring Λ, $\mathbf{Repf}_\Lambda(\pi_1^{\text{ét}}(X,x))$ denotes the category of finitely generated Λ-modules equipped with a continuous R-linear action of $\pi_1^{\text{ét}}(X,x)$.

Definiton 2.5.0.7. The continuous representation $\mathscr{F}_x \otimes \mathbb{Q}_\ell$ is called the *monodromy representation* of the $\mathbb{Q}_\ell$ sheaf $\mathscr{F}$.
Form now on, for $x : \mathtt{Spec}(\Omega) \to X$ a geometric point on X and $\mathscr{F}$ a $\mathbb{Q}_\ell$, write

$$\mathscr{F}_x := x^*\mathscr{F}(\mathtt{Spec}(\Omega)) \otimes \mathbb{Q}_\ell.$$

Remark 2.5.0.6. Let $\mathbb{Q}_\ell \hookrightarrow K$ be a finite algebraic extension and denote by $\mathscr{O}_K$ its ring of integer (i.e. the integral closure of $\mathbb{Z}_\ell$ in K). Then $\mathscr{O}_K$ is also a complete DVR, just as $\mathbb{Z}_\ell$. Denote by λ a generator of the maximal ideal of $\mathscr{O}_K$. Interchange the notation of the whole section following the following rules:

$$\ell \rightsquigarrow \lambda \qquad\qquad \mathbb{Z}_\ell \rightsquigarrow \mathscr{O}_K \qquad\qquad \mathbb{Q}_\ell \rightsquigarrow K,$$

all definitions make sense and all propositions hold.
Choose an algebraic closure $\mathbb{Q}_\ell \to \overline{\mathbb{Q}_\ell}$. One can define a category $D_c^b(X,\overline{\mathbb{Q}_\ell})$ as a limit of the categories $D_c^b(X,K)$ that we define via the rules above. The six functors formalism is again available. The proof of theorem 2.5.0.1 is compatible with the formation of $D_c^b(X,\overline{\mathbb{Q}_\ell})$ in the sense that there is an equivalence of categories

$$\mathbf{LSh}_{\overline{\mathbb{Q}_\ell}}(X) \xrightarrow{\sim} \mathbf{Repf}_{\overline{\mathbb{Q}_\ell}}(\pi_1(X,\overline{x})) := \varprojlim \mathbf{Repf}_K(\pi_1(X,\overline{x})).$$

(An object in the category of the right-hand-side is a morphism $\pi_1(X,\overline{x}) \to \mathrm{GL}(V)$ for some finite dimensional K-vector space, with K an arbitrary finite extension $\mathbb{Q}_\ell \hookrightarrow K$.)

Chapter 3

Convolution

(cf. [FJ18, Chapter 2])
In this chapter ℓ denotes a prime number.

3.1 Constructible sheaves on $\mathbb{A}^1_k$ and their cohomology

Recall that the word "sheaf", unless specified otherwise, always mean "étale sheaf". Also recall the notation for points and residue fields given in *Notation and conventions on schemes* (cf. introduction).

3.1.1 Constructible sheaves on $\mathbb{A}^1_k$

Let k be a field of arbitrary characteristic.
The affine line $\mathbb{A}^1_k$ is the scheme $\mathtt{Spec}(k[T])$.
A decomposotion of $\mathbb{A}^1_k$ in locally closed subschemes is the same as a decomposition

$$\mathbb{A}^1_k = X \sqcup S,$$

where $S := (\mathfrak{s}_1, \dots, \mathfrak{s}_n) \subset \mathbb{A}^1_k$ is a finite set of closed points and $X := \mathbb{A}^1_k \setminus S$.

Proposition 2.3.3.1 states that the data of a sheaf on $\mathbb{A}^1_k$ is equivalent to:

(a) A sheaf $\mathscr{V}$ on S;

(b) A locally constant sheaf $\mathscr{F}$ on X;

(c) A morphism $\sigma : \mathscr{V} \to i^* j_* \mathscr{F}$ of sheaves on S.

This equivalence of data induce an equivalence of categories. Morphims in the category of tuples $(\mathscr{V}, \mathscr{F}, \sigma)$ are defined in the obvious way.

This equivalence of categories is compatible with the ℓ-adic formalism and holds for ℓ-adic sheaves and $\mathbb{Q}_\ell$-sheaves.

Let x be a geometric point of X and consider a $\mathbb{Q}_\ell$-sheaf on $\mathbb{A}^1_k$, constructible with respect to the decomposition $X \sqcup S$. By theorem 2.5.0.1, it is possible to replace (a) (b) and (c) with the data:

(a') For each $\mathfrak{s} \in S$, a $\mathbb{Q}_\ell$-vector space $V_\mathfrak{s}$ together with a continuous linear $\mathrm{Gal}(\kappa(\mathfrak{s})^{\mathrm{sep}}|\kappa(\mathfrak{s}))$-action;

(b') A continuous morphism $\pi_1(X, x) \to \mathrm{GL}(\mathscr{F}_x \otimes \mathbb{Q}_\ell)$ for some geometric point x of X;

(c') For each $\mathfrak{s} \in S$ a $\mathrm{Gal}(\kappa(\mathfrak{s})^{\mathrm{alg}}|\kappa(\mathfrak{s}))$-morphism $V_\mathfrak{s} \to (i^* j_* \mathscr{F})_s$ of $\mathbb{Q}_\ell$-vector spaces, where $s : \mathtt{Spec}(\kappa(\mathfrak{s})^{\mathrm{alg}}) \to S$.

Remark 3.1.1.1. Let $\mathfrak{s} \in S \subset \mathbb{A}^1_k$ and let $s : \mathtt{Spec}(\kappa(\mathfrak{s})^{\mathrm{alg}}) \to \mathbb{A}^1_k$ (with the same choice of algebraic closure as in (a')). By construction $V_\mathfrak{s} = \mathscr{V}_s$.

3.1.2 Tiny ramification theory excursus

The henselization of the local ring of $\mathbb{A}^1_k$ at $\mathfrak{s}$, denoted $\mathscr{O}^h_{\mathbb{A}^1_k,\mathfrak{s}}$, plays the role of a little neighborhood of $\mathfrak{s}$. Denote by $K(\mathfrak{s})$ its fraction field. Choose a strict henselization of $\mathscr{O}^{sh}_{\mathbb{A}^1_k,\mathfrak{s}}$. The closed point of $\mathscr{O}^{sh}_{\mathbb{A}^1_k,\mathfrak{s}}$ is a geometric point above $\mathfrak{s}$. Denote by s the inclusion morphism of this point and by $\kappa(s)$ its residue field. Also, denote the fraction field of $\mathscr{O}^{sh}_{\mathbb{A}^1_k,\mathfrak{s}}$ by $K(s)$. The generic geometric point

$$\eta_s : \mathtt{Spec}(K(s)^{\mathrm{sep}}) \to \mathtt{Spec}(K(\mathfrak{s})) \to \mathtt{Spec}(\mathscr{O}^h_{\mathbb{A}^1_k,\mathfrak{s}}) \tag{3.1.1}$$

defines a geometric point of X. Via this immersion η "is" as a point of X nearby s.
The morphism (3.1.1) induces a morphism of fundamental groups

$$\mathrm{Gal}(K(s)^{\mathrm{sep}}|K(\mathfrak{s})) = \pi_1^{\text{ét}}(\mathtt{Spec}(K(\mathfrak{s})), \eta_s) \longrightarrow \pi_1^{\text{ét}}(\mathtt{Spec}(\mathscr{O}^h_{\mathbb{A}^1_k,\mathfrak{s}}), \eta_s)) \simeq \mathrm{Gal}(\kappa(s)|\kappa(\mathfrak{s}))$$

where the last isomorphism is proposition 2.1.1.2. In particular, is possible to replace (c') with

(c") For each $\mathfrak{s} \in S$ a $\mathrm{Gal}(K(s)^{\mathrm{sep}}|K(\mathfrak{s}))$-morphism $V_\mathfrak{s} \to (i^* j_* \mathscr{F})_s$ of $\mathbb{Q}_\ell$-vector spaces where $s : \mathtt{Spec}(\kappa(s)) \to S$.

Let $j_0 : \mathtt{Spec}(\mathscr{O}^h_{\mathbb{A}^1_k}) \to \mathbb{A}^1_k$ be the canonical inclusion so that $(i^* j_* \mathscr{F})_s = (j_{0*}\mathscr{F})_s$. One obtains a *specialization morphism*

$$\alpha_s : (i^* j_* \mathscr{F})_s = (j_{0*}\mathscr{F})_s \to (j_{0*}\mathscr{F})_{\eta_s} = \mathscr{F}_{\eta_s}.$$

which is injective. The group $\mathrm{Gal}(K(s)^{\mathrm{sep}}|K(\mathfrak{s}))$ acts on both the source and the target. One can check that α_s is $\mathrm{Gal}(K(s)^{\mathrm{sep}}|K(\mathfrak{s}))$-invariant. Then (c") is the same as:

(c"') For each $\mathfrak{s} \in S$ a $\mathrm{Gal}(K(s)^{\mathrm{sep}}|K(\mathfrak{s}))$-morphism $\sigma_s : V_\mathfrak{s} \to \mathscr{F}_{\eta_s}$ of $\mathbb{Q}_\ell$-vector spaces where $s : \mathtt{Spec}(\kappa(s)) \to S$.

Let x be the geometric point chosen in point (b') and set the notation

$$\mathbf{Paths}^{\text{ét}}_X(s, x) := \{\text{étale paths from } \eta_s \text{ to } x \text{ in } X\}.$$

Étale paths were defined in definition 2.1.2.4. The group $\pi_1^{\text{ét}}(X, x)$ acts on $\mathbf{Paths}^{\text{ét}}_X(s, x)$ in the obvious way. Since X is connected the action is transitive. An element $\gamma \in \mathbf{Paths}^{\text{ét}}_X(s, x)$ defines a morphism of vector spaces $\mathscr{F}_{\eta_s} \to \mathscr{F}_x = V$ which extends to a morphims $V_s \xrightarrow{\gamma} V$.

The group $\mathrm{Gal}(K(s)^{\mathrm{sep}}|K(\mathfrak{s}))$ also acts on $\mathbf{Paths}^{\text{ét}}_X(s,x)$: the action is the one induced by the morphism

$$\pi_1^{\text{ét}}(\mathtt{Spec}(K(\mathfrak{s}),\eta_s) \to \pi_1^{\text{ét}}(X,\eta_s),$$

itself induced by the inclusion $\mathtt{Spec}(K(\mathfrak{s})) \to X$. These facts shows that (c''') is the same as:

(c'''') A collection of maps of sets $\mathbf{p}_S := \{p_s\}_{s\in S}$ of the form:

$$\begin{aligned} p_s : \mathbf{Paths}^{\text{ét}}_X(s,x) &\longrightarrow \mathrm{Hom}_{\mathbf{Vec}}(V_s,V) \\ \gamma &\longmapsto \gamma\circ\alpha_s\circ\sigma_s \end{aligned}$$

and such that each p_s is both $\pi_1^{\text{ét}}(X,x)$ and $\mathrm{Gal}(K(s)^{\mathrm{sep}}|K(\mathfrak{s}))$-invariant.

3.1.3 Cohomology of constructible sheaves on $\mathbb{A}^1_k$

Let $\mathscr{G}$ be a constructible $\mathbb{Q}_\ell$-sheaf on $\mathbb{A}^1_k$. This subsection propose a strategy to compute $\mathrm{H}^p_{\text{ét}}(\mathbb{A}^1_{k^{\mathrm{sep}}},\mathscr{G})$.
Without loss of generality assume that $k = k^{\mathrm{sep}}$. In such case the data defining a constructible $\mathbb{Q}_\ell$-sheaf on $\mathbb{A}^1_k$ becomes simpler: forget about the action of $\mathrm{Gal}(K(s)^{\mathrm{sep}}|K(\mathfrak{s}))$.

Proposition 3.1.3.1. Let k be a separably closed field and let S be a finite collection of points of $\mathbb{A}^1_k$. A constructible sheaf on $\mathbb{A}^1_k$ with singularities in S is completely determined by the following data:

(A) A set of $\mathbb{Q}_\ell$-vector spaces $\{V_s\}_{s\in S}$.

(B) A $\mathbb{Q}_\ell$-vector space V together with a continuous representation $\varrho : \pi_1^{\text{ét}}(\mathbb{A}^1_k \setminus S, x) \to V$, where x is a geometric point of $\mathbb{A}^1_k \setminus S$.

(C) Maps $\left\{\varrho_s : \mathbf{Paths}^{\text{ét}}_X(s,x) \to \mathrm{Hom}_{\mathbf{Vec}_{\mathbb{Q}_\ell}}(V_s,V)\right\}_{s\in S}$ such that

$$\varrho_s(g\gamma) = \varrho(g)\varrho_s(\gamma)$$

for any $\gamma \in \mathbf{Paths}(s,x)$ and for any $g \in \pi_1^{\text{ét}}(X,x)$.

Proposition 3.1.3.1 justifies next definitions.

Definiton 3.1.3.1. Let G be a group, $\mathbf{P}_S := \{\mathbf{P}_s\}_{s\in S}$ a finite collection of G-sets. A *representation* $(V,\varrho;\mathscr{V},\mathbf{p}_S)$ *of the couple* $(G,\mathbf{P}_S)$, or a $(G,\mathbf{P}_S)$*-representation*, is the data of

(A') A set of $\mathbb{Q}_\ell$-vector spaces $\mathscr{V} = \{V_s\}_{s\in S}$;

(B') A $\mathbb{Q}_\ell$ vector space and a continuous representation $\varrho : G \to \mathrm{GL}(V)$;

(C') A set of maps

$$\mathbf{p}_S = \{p_s : \mathbf{P}_s \to \mathrm{Hom}_{\mathbf{Vec}}(V_s,V)\}_{s\in S}$$

such that

$$p_s(g\gamma) = \varrho(g)p_s(\gamma)$$

for all s, for all $\gamma \in \mathbf{P}_s$, for all $g \in G$.

Remark 3.1.3.1. Proposition 3.1.3.1 tautologically implies that a constructible $\mathbb{Q}_\ell$-sheaf on $\mathbb{A}^1_k$ can be seen as a $(G, \mathbf{P}_S)$-representation for an appropriate choice of $(G, \mathbf{P}_S)$. The choice of $(G, \mathbf{P}_S)$ is not unique and depends on the choice of a decomposition $\mathbb{A}^1_k = X \sqcup S$ and of a geometric point x of X.

Remark 3.1.3.2. Let $\mathscr{F}$ be a $\mathbb{Q}_\ell$-sheaf on $\mathbb{A}^1_k$, constructible with respect to a decomposition $X \sqcup S$. A global section $\mathscr{F}$, i.e. an element of $\mathscr{F}(\mathbb{A}^1_k)$, is given, in terms of the data (A), (B), (C) of proposition 3.1.3.1, by a tuple of elements $(v, \{v_s\}_{s \in S})$ such that $\gamma_s(v_s) = v$ for all $s \in S$ and for all $\gamma_s \in \mathbf{Paths}^{\text{ét}}_X(s; x)$.

Definiton 3.1.3.2. Let $(V, \varrho; \mathscr{V}, \mathbf{p}_S)$ and $(W, \theta; \mathscr{W}, \mathbf{q}_S)$ two $(G, \mathbf{P}_S)$-representations. A morphism $(V, \varrho; \mathscr{V}, \mathbf{p}_S) \to (W, \theta; \mathscr{W}, \mathbf{q}_S)$ is a set of morphisms of vector spaces

$$\{\phi : V \to W\} \cup \{\phi_s : V_s \to W_s\}_{s \in S}$$

such that

$$\phi \circ \varrho(g) = \theta(g) \circ \phi \text{ and } \phi \circ p_s(\gamma) = q_s(\gamma) \circ \phi_s.$$

If it brings no confusion, omit the maps p_s and simply write γv_s instead of $p_s(\gamma)v_s$.

Denote by $\mathbf{Rep}_{(G,\mathbf{P}_S)}$ the category defined by definitions (3.1.3.1 and 3.1.3.2). This category is easily seen to be abelian and $\mathbb{Q}_\ell$-linear. The functor

$$\mathrm{Hom}_{\mathbf{Rep}_{(G,\mathbf{P}_S)}}(\mathbb{Q}_\ell, -) : \mathbf{Rep}_{(G,\mathbf{P}_S)} \longrightarrow \mathbf{Vec}_{\mathbb{Q}_\ell}$$

where $\mathbb{Q}_\ell$ is the constant $(G, \mathbf{P}_S)$-representation, i.e. $V = V_s = \mathbb{Q}_\ell$ and $\varrho(g) = p_s(\gamma) = \mathrm{id}$, plays the role of the invariants-functor in group cohomology and is left-exact.

Proposition 3.1.3.2. Let $(V, \varrho; \mathscr{V}, \mathbf{p}_S)$ be a $(G, \mathbf{P}_S)$-representation. For every p there is an isomorphism

$$R^p\mathrm{Hom}_{\mathbf{Rep}_{(G,\mathbf{P}_S)}}(\mathbb{Q}_\ell, (V, \varrho; \mathscr{V}, \mathbf{p}_S)) \simeq H^p(C^\bullet),$$

where $C^\bullet$ is the complex

$$C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \cdots$$

with entries

$$\begin{aligned} C^0 := V \oplus \bigoplus_s V_s, \text{ and } C^p =& \mathrm{Hom}_{\mathbf{Set}}(G^p \amalg \left(\coprod_s G^{p-1} \times \mathbf{P}_s\right), V), \\ &= \mathrm{Hom}_{\mathbf{Set}}(G^p, V) \times \prod_s \mathrm{Hom}_{\mathbf{Set}}(G^{p-1} \times \mathbf{P}_s, V) \end{aligned}$$

and differentials

$$\begin{aligned} d^0(v, \{v_s\}) : g &\mapsto v - gv \quad \text{ and } \quad d^p\phi : (g_1 \dots, g_p, \xi_{p+1}) \mapsto g_1\phi(g_2, \dots, \xi_{p+1}) \\ p_s &\mapsto v - p_s v_s, \qquad\qquad + \sum_{i=1}^{p}(-1)^i \phi(g_1 \dots, g_i g_{i+1}, \dots, \xi_{p+1}) \\ &\qquad\qquad\qquad + (-1)^{p+1}\phi(g_1, \dots, g_n). \end{aligned}$$

Here $\xi_{p+1} \in G \cup \bigcup_s \mathbf{P}_s$ and $\phi \in C^p$.

Proof. One can apply the same proof of lemma 2.2.4 in [FJ18]. I omit the proof. $\square$

Remark 3.1.3.3. For later write down explicitly $\mathrm{Ker}(d^1)$ and $\mathrm{Im}(d^0)$:

$$\mathrm{Ker}(d^1) = \{\phi : G \to V \text{ s.t. } \phi(gh) = \phi(g) + g\phi(h)\} \times \prod_s \{\phi_s : \mathbf{P}_s \to V \text{ s.t. } \phi_s(g\gamma) = \phi(g) + g\phi_s(\gamma).\}$$

$$\mathrm{Im}(d^0) = \left\{ (\phi : G \to V, \{\phi_s : \mathbf{P}_s \to V\}_s) \text{ s.t. } \begin{array}{cc} \phi(g) = v - gv & \phi_s(\gamma) = v - \gamma v_s \\ \text{for some } v \in V & \text{for some } v_s \in V_s \end{array} \right\}$$

3.1.4 Cohomology of constructible sheaves on $\mathbb{A}^1_k$

Theorem 3.1.4.1. Let k be a separably closed field. Let $\mathscr{F}$ be a constructible sheaf on $\mathbb{A}^1_k$ and let $C^\bullet$ be the complex computing the cohomology of the $(G, \mathbf{P}_S)$-representation associated to $\mathscr{F}$. Then the cohomology groups $H^p(C^\bullet)$ coincide with the groups $H^p_{\text{ét}}(\mathbb{A}^1_k, \mathscr{F})$.

Proof. Let $\mathbb{A}^1_k = X \sqcup S$ be a decomposition such that $\mathscr{F}$ is lisse on X, let x be a geometric point of X and set $\mathbf{P}_S := \{\mathbf{Paths}^{\text{ét}}_X(s,x)\}_{s\in S}$. Then proposition 3.1.3.1 induce an an equivalence of categories that sits in a commutative diagram

$$\begin{tikzcd}
\left\{ \begin{array}{l} \text{full subcategory of} \\ \mathbf{Sh}_{\mathbb{Q}_\ell}(\mathbb{A}^1_k) \text{ whose} \\ \text{objects are lisse on } X \end{array} \right\} \arrow[rr, "\sim"] \arrow[dr, "\mathrm{Hom}_{\mathbf{Sh}_{\mathbb{Q}_\ell}(\mathbb{A}^1_k)}(\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k}, -)"'] & & \mathbf{Rep}_{(\pi_1(X,x), \mathbf{P}_S)} \arrow[dl, "\mathrm{Hom}_{\mathbf{Rep}_{(\pi_1(X,x),\mathbf{P}_S)}}(\mathbb{Q}_\ell, -)"] \\
& \mathbf{Vec}_{\mathbb{Q}_\ell} &
\end{tikzcd}$$

The functor $\Gamma(\mathbb{A}^1_k, -)$, defined at page 22 coincide with the functor $\mathrm{Hom}_{\mathbf{Sh}_{\mathbb{Q}_\ell}(\mathbb{A}^1_k)}(\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k}, -)$ and their derived functors agree. $\square$

Lemma 3.1.4.1. Let k be an algebraically closed field of characteristic 0. Let $C^\bullet$ be the complex associated to a $\mathbb{Q}_\ell$-sheaf $\mathscr{G}$ on $\mathbb{A}^1_k$, supposed constructible with respect to a decomposition $X \sqcup S$. The complex $C^0 \xrightarrow{d^0} \mathrm{Ker}(d^1)$ is isomorphic to the complex

$$\begin{aligned} V \oplus \bigoplus_s V_s &\longrightarrow \bigoplus_s V \\ (v, \{v\}_s) &\longmapsto (v - \gamma_s v_s) \end{aligned} \tag{3.1.2}$$

for some set of paths $\{\gamma_s \in \mathbf{Paths}^{\text{ét}}_X(s,x)\}_{s\in S}$.

Proof. For each $s \in S$ choose a path each $\gamma_s \in \mathbf{Paths}^{\text{ét}}_X(s;x)$. The action of $\pi_1^{\text{ét}}(X,x)$ on each set $\mathbf{Paths}^{\text{ét}}_X(s,x)$ is transitive and by the orbit-stabilizer theorem $\mathbf{Paths}^{\text{ét}}_X(s,x)$ is identified with the coset $\pi_1^{\text{ét}}(X,x)/G_s$, where G_s is the stabilizer of γ_s. By theorem 2.1.2.3 $\pi_1^{\text{ét}}(X,x)$ is the profinite completion of the group given by the presentation

$$\left\langle g_s | \prod_s g_s = 1 \right\rangle$$

and without loss of generality it is possible to identify G_s with $\langle g_s \rangle$.
Remark 3.1.3.3 implies that ϕ is completely described by the values $\phi_s(g_s)$, but also that

$$\phi(g_s) = \phi_s(\underbrace{g_s\gamma_s}_{=\gamma_s}) - g_s\phi(\gamma_s).$$

In particular

$$\begin{aligned} \mathrm{Ker}(d^1) &\longrightarrow \bigoplus_s V \\ (\phi, \{\phi_s\}) &\longmapsto \phi_s(\gamma_s) \end{aligned} \tag{3.1.3}$$

is injective. It is left as an exercise to check the surjectivity. □

Remark 3.1.4.1. Let $\mathscr{F}$ be a constructible $\mathbb{Q}_\ell$-sheaf on $\mathbb{A}_k^1$ and let $C^\bullet = (C^i, d^i)$ the the complex given by proposition 3.1.3.2 for a suitable $(G, \mathbf{P}_S)$-representation. Combining remark 3.1.3.2 and lemma 3.1.4.1 it is possible to explicitly describe the condition $R\varsigma_* S\mathscr{F} = 0$.
As a consequence of theorem 2.4.3.1, $\mathrm{H}^p_{\text{ét}}(\mathbb{A}_k^1, \mathscr{F})$ vanish for $p > 1$. Hence to explicit the condition $R\varsigma_*\mathscr{F} = 0$ it is enough to unfold the conditions

$$\mathrm{H}^0_{\text{ét}}(\mathbb{A}_k^1, \mathscr{F}) = \mathrm{Ker}(d^0) = 0 \quad \text{and} \quad \mathrm{H}^1_{\text{ét}}(\mathbb{A}_k^1, \mathscr{F}) = \frac{\mathrm{Ker}(d^1)}{\mathrm{Im}(d^0)} = 0.$$

By lemma 3.1.4.1, $\mathrm{Ker}(d^0) = 0$ holds if and only if (3.1.2) is injective while $\mathrm{Ker}(d^1)/\mathrm{Im}(d_0) = 0$ if and only if (3.1.2) is surjective. This is equivalent to say that

$$\text{all the } \gamma_s : V_s \to V \text{ are injective,} \qquad \bigcap_{s\in S} \gamma_s V_s = 0 \quad \text{and} \quad \sum \dim(V/\gamma_s V_s) = \dim(V), \tag{3.1.4}$$

where γ_s are chosen as in the proof of lemma 3.1.4.1.

Proposition 3.1.4.1. Let k be an algebraically closed field of characteristic 0. Let $\mathscr{F}$ and $\mathscr{G}$ be constructible sheaves on $\mathbb{A}_k^1$. Suppose $\mathscr{F}$ has no global sections and $\mathscr{G}$ has no global section with finite support, i.e,

$$\text{if } f \in \mathscr{G}(\mathbb{A}_k^1) \text{ and } |\mathrm{supp}(f)| < \infty, \text{ then } f = 0. \tag{3.1.5}$$

Then $\mathscr{F} \otimes \mathscr{G}$ has no global sections.

Remark 3.1.4.2 (cf. with remarks 3.1.3.2 and 3.1.4.1). Condition (3.1.5) can be made explicit in terms of $(G, \mathbf{P}_S)$-representation. The global sections with finite support are the elements represented by tuples of the form $(0, \{v_s\}_{s\in S})$ with $v_s \neq 0$ for at least one s and with $\gamma_s(v_s) = 0$. To say that there are no such elements is to say that $\gamma_s : V_s \to V$ is injective for each s.

Proof. Choose S large enough so that S contains the singularities of both $\mathscr{F}$ and $\mathscr{G}$ and denote by X the complement of S in $\mathbb{A}_k^1$. Choose a geometric point x of X. Then one can look at $\mathscr{F}$ and $\mathscr{G}$ as $(G, \mathbf{P}_S)$-representations with $G = \pi_1^{\text{ét}}(X, x)$ and $\mathbf{P}_S = \{\mathbf{Paths}_X^{\text{ét}}(s, x)\}_{s \in S}$.
Write $(V, \varrho; \mathscr{V}, \mathbf{p}_S)$ and $(W, \theta; \mathscr{W}, \mathbf{q}_S)$ for the representations of $\mathscr{F}$ and $\mathscr{G}$.
The representation of $\mathscr{F} \otimes \mathscr{G}$ is given by the tuple $(V \otimes W, \varrho \otimes \theta, \mathscr{V} \otimes \mathscr{W}, \mathbf{p}_s \otimes \mathbf{q}_s)$, where $\mathscr{V} \otimes \mathscr{W} = \{V_s \otimes W_s\}_{s \in S}$ and where $\mathbf{p}_S \otimes \mathbf{q}_S = \{p_s \otimes q_s\}_{s \in S}$. [1]
To conclude the proof combine remarks 3.1.4.1 and 3.1.4.2. □

3.2 Additive convolution of constructible sheaves on $\mathbb{A}_k^1$

3.2.1 Definition and stalks

Definition of the convolution

The scheme $\mathbb{A}_k^1$ comes equipped with the group structure coming from the additive group $(k, +)$. We denote by $\mathtt{sum} : \mathbb{A}_k^2 \to \mathbb{A}_k^1$ the morphism of schemes given by "+". Explicitly $\mathtt{sum}$ is the morphism of k-schemes induced by

$$\begin{tikzcd} k \arrow[r, hook] \arrow[dr, hook] & k[T_1, T_2] \\ & k[T] \arrow[u, "T \mapsto T_1 + T_2"'] \end{tikzcd}$$

Given two sheaves $\mathscr{F}$ and $\mathscr{G}$, let

$$\mathscr{F} \boxtimes \mathscr{G} := \mathtt{pr}_1^* \mathscr{G} \otimes \mathtt{pr}_2^* \mathscr{F} \text{ in } \mathbf{Sh}(\mathbb{A}_k^2),$$

where $\mathtt{pr}_1, \mathtt{pr}_2 : \mathbb{A}_k^2 \to \mathbb{A}_k^1$ denote the projection maps to the first and second coordinate, respectively.
To obtain back a sheaf on $\mathbb{A}_k^1$, "push" $\mathtt{sum}$ and consider the sheaf

$$\mathtt{sum}_*(\mathscr{F} \boxtimes \mathscr{G}) \text{ in } \mathbf{Sh}(\mathbb{A}_k^1).$$

This sheaf it is essentially never interesting. To obtain a more rich theory pass to the derived category and consider the derived push forward

$$R\mathtt{sum}_*(F \boxtimes^{\mathbf{L}} G) \text{ in } D_b(\mathbb{A}_k^1)$$

where F and G are complexes of sheaves and where

$$F \boxtimes^{\mathbf{L}} G := \mathtt{pr}_1^*(F) \otimes^{\mathbf{L}} \mathtt{pr}_2^*(G).$$

Definiton 3.2.1.1 (Convolution). Given two objects F and G in $D_c^b(\mathbb{A}_k^1)$ define their convolution to be

$$F * G := R\mathtt{sum}_*(F \boxtimes^{\mathbf{L}} G, \mathbb{Q}_\ell).$$

Remark 3.2.1.1. The superscript $^{\mathbf{L}}$ is actually superfluous: for any scheme Y, and for any $\mathbb{Q}_\ell$-sheaf $\mathscr{F}$ on Y, the functor $\mathscr{F} \otimes - : \mathbf{Sh}_{\mathbb{Q}_\ell}(Y) \to \mathbf{Sh}_{\mathbb{Q}_\ell}(Y)$ is exact. Thus, for any object F in $D_c^b(\mathbb{A}_k^1, \mathbb{Q}_\ell)$, the functor $F \boxtimes - : D_c^b(\mathbb{A}_k^1, \mathbb{Q}_\ell) \to D_c^b(\mathbb{A}_k^2, \mathbb{Q}_\ell)$ is well defined.

[1] here $(p_s \otimes q_s)(\gamma)(v_s \otimes w_s) := p_s(\gamma)v_s \otimes q_s(\gamma)w_s \in V \otimes W$.

For simplicity, call *sheaves* the objects of $D^b_c(\mathbb{A}^1_k, \mathbb{Q}_\ell)$. This should bring no confusion, cf. *Sheaves and complexes of sheaves* at page 9. A sheaf L is lisse if $H^p(L)$ is lisse for every p.

Lemma 3.2.1.1. Let E, F, G be sheaves on $\mathbb{A}^1_k$. One has a natural isomorphisms

$$F * G = G * F \tag{3.2.1}$$

$$E * (F * G) \simeq R\mathtt{sum}^{(3)}_*(E \boxtimes F \boxtimes G) \simeq (E * F) * G. \tag{3.2.2}$$

where $\mathtt{sum}^{(3)} : \mathbb{A}^3_k \to \mathbb{A}^1_k$ is the morphism induced by $T \mapsto T_1 + T_2 + T_3$.

Sketch of the proof. This is a straightforward consequence of the commutativity and associativity of "+".

Stalk of the convolution

Let k be a field of characteristic 0. Recall that the symbol ς systematically stands for the structure morphism of a k-scheme.

Lemma 3.2.1.2. Let $z : \mathtt{Spec}(\Omega) \to \mathbb{A}^1_k$ be a geometric point of $\mathbb{A}_k$ and denote by $\mathbb{A}^1(z)$ the scheme given by the Cartesian diagram

$$\begin{tikzcd} \mathbb{A}(z) \arrow[r, "\iota"] \arrow[d, "\varsigma"'] & \mathbb{A}^2_k \arrow[d, "\mathtt{sum}"] \\ \mathtt{Spec}(\Omega) \arrow[r, "z"] & \mathbb{A}^1_k \end{tikzcd} \tag{3.2.3}$$

Let F and G be sheaves on $\mathbb{A}^1_k$. There is a natural quasi-isomorphism

$$z^*(F * G) \simeq R\varsigma_* \iota^*(F \boxtimes G) \tag{3.2.4}$$

of complexes of $\mathbb{Q}_\ell$-sheaves on a point.
In particular if $\mathscr{F}$ and $\mathscr{G}$ are constructible $\mathbb{Q}_\ell$ sheaves on $\mathbb{A}^1_k$, then

$$H^p(\mathscr{F}[0] * \mathscr{G}[0])_z = \mathrm{H}^p_{\text{ét}}(\mathbb{A}^1_k(z), \mathscr{F} \boxtimes \mathscr{G}).$$

Remark 3.2.1.2. The proof of this lemma uses theorem 2.4.2.1 and a topological argument. The idea of the proof should generalize to arbitrary characteristic and for arbitrary, i.e. non-geometric, points. Unfortunately, at this time, there is a step that I don't know how to justify.

Proof. By remark 2.3.1.6, assume that k is algebraically closed.
By theorem 2.4.2.1 and its following remark, it is enough to work the case $k = \mathbb{C}$ and it is possible to treat F and G as objects in the derived category of topological sheaves of $\mathbb{Q}_\ell$-vector spaces on $\mathbb{C}$.
For simplicity assume that $F = \mathscr{F}[0]$ and $G = \mathscr{G}[0]$ for some topological sheaves on $\mathbb{C}$. By a *dévissage* argument this case is enough. Let $\mathscr{C} = \mathscr{F} \boxtimes \mathscr{G}$ and denote by $S_{\mathscr{F}}$ and $S_{\mathscr{G}}$ the singularities of $\mathscr{F}$ and $\mathscr{G}$, so that they are locally constant on $\mathbb{C} \setminus S_{\mathscr{F}}$ and $\mathbb{C} \setminus S_{\mathscr{G}}$. Notice that

(i) $\mathscr{C}$ is locally constant outside $S := S_{\mathscr{F}} \times \mathbb{C} \cup \mathbb{C} \times S_{\mathscr{G}}$;

(ii) the morphism $\mathtt{sum}|_S : S \to \mathbb{C}$ is proper;

(iii) the morphism $\mathtt{sum} : \mathbb{C}^2 \to \mathbb{C}$ is a fiber bundle.

Then (3.2.4) follows from [Nor, lemma 2.7] or [Nor02, 1.3A]: it states that, if (i), (ii) and (iii) are verified, then

$$R^p \varsigma_* \iota^* \mathscr{C}(z) \simeq (R^p \mathtt{sum}_* \mathscr{C})_z$$

holds.

□

3.2.2 The category $\mathbf{Perv}_0$ for ℓ-adic sheaves

Definiton 3.2.2.1. Let k be an algebraically closed field of characteristic 0. Denote by $\mathbf{Perv}(k)$ the full subcategory of $D^b_c(\mathbb{A}^1_k, \mathbb{Q}_\ell)$ whose objects are complexes of sheaves F such that

(P.i) $H^p(F) = 0$ for all $p \neq -1, 0$.

(P.ii) $H^0(F)$ is a skyscraper sheaf.

(P.iii) if $H^{-1}(F)$ has no non-zero global sections with finite support (see 3.1.5).

For k non-algebraically closed denote by $\mathbf{Perv}(k)$ the full subcategory of $D^b_c(\mathbb{A}^1_k, \mathbb{Q}_\ell)$ whose objects F pull-back to sheaves in $\mathbf{Perv}(k^{\mathrm{alg}})$.

Definiton 3.2.2.2. Define $\mathbf{Perv}_0(k)$ as the full subcategory of $\mathbf{Perv}(k)$ whose objects are sheaves F such that $R\varsigma_* F = 0$.

Lemma 3.2.2.1. A sheaf F in $\mathbf{Perv}(k)$, is in $\mathbf{Perv}_0(k)$ if and only if $F = \mathscr{F}[1]$ for some constructible $\mathbb{Q}_\ell$-sheaf $\mathscr{F}$ on $\mathbb{A}^1_k$ such that $\mathrm{H}^p_{\text{ét}}(\mathbb{A}^1_k, \mathscr{F}) = 0$ for all p.

Proof. The *if* part is left as an exercise. For the *only if* let F a sheaf in $\mathbf{Perv}(k)$ with $R\varsigma_* F = 0$. There is the spectral sequence (apply (4.1.2) with $\Phi = \varsigma$)

$$E_2^{p,q} = R^p \varsigma_*(H^q(F)) \Longrightarrow R\varsigma_*^{p+q}(F)$$

and since F is a sheaf in $\mathbf{Perv}(k)$, the combination of lemma 2.4.1.1 and theorem 2.4.3.1 shows

$$R^0 \varsigma_*(H^{-1}(F)) = R^{-1} \varsigma_*(F)$$

$$R^1 \varsigma_*(H^{-1}(F)) \oplus R^0 \varsigma_*(H^0(F)) = R^0 \varsigma_*(F).$$

By hypothesis $R^p \varsigma_*(F) = 0$ for all p. This implies that all the objects in the identities above are 0. Using theorem 2.4.3.1 again, $R\varsigma_*(H^{-1}(F)) = 0$.
It would be enough to show that F and $H^{-1}(F)[1]$ are quasi isomorphic.
For this invoke the distinguished triangle,

$$H^{-1}(F)[1] \longrightarrow F \longrightarrow H^0(F)[0] \longrightarrow H^{-1}(F)[2].$$

which exists since $H^i(F)=0$ for $i \neq 0.-1$.
It is enough to show that $H^{-1}(F)$ vanish. This is true since $H^{-1}(F)$ is a skyscraper $\mathbb{Q}_\ell$-sheaf with $R^0\varsigma_* H^{-1}(F)=0$. □

Theorem 3.2.2.1. Let F be a sheaf in $\mathbf{Perv}_0(k)$ and let G be a sheaf in $\mathbf{Perv}(k)$.

(i) $F * G$ is also in $\mathbf{Perv}_0$

(ii) $(F * j_! j^* \underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k})[1] \simeq F$, were j is the inclusion $\mathbb{A}^1_k \setminus \{0\} \hookrightarrow \mathbb{A}^1_k$.

Proof.

(i) Assume $k = k^{\mathrm{alg}}$.
The sheaf $F \boxtimes G$ is concentrated in cohomological degree -1 and -2.
As a consequence of 2.2.1.1 the sheaf $F * G$ is concentrated in cohomological degree -2, -1 and 0.
Thus it is enough to show that $H^{-2}(F*G)$ and $H^0(F*G)$ vanish. For this it is enough to show that the stalks of these sheaves at each geometric point vanish (see proposition 2.3.1.1).
By lemma 3.2.1.2 $H^p(F*B)_z = R^q\varsigma_*(F \boxtimes G|_{\mathbb{A}^1_k(z)})$ (where $F \boxtimes G|_{\mathbb{A}^1_k(z)}$ is concentrated in degree -2 and -1). The spectral sequence

$$E_2^{p,q} = R^p\varsigma_* \circ H^q \Longrightarrow R^{p+q}\varsigma_*,$$

says that for each sheaf S supported in degree -2 and -1 one has

$$R^0\varsigma_*(S) = R^1\varsigma_* \circ H^{-1}(S) \quad \text{and} \quad R^{-2}\varsigma_*(S) = R^0\varsigma_* \circ H^{-2}(S).$$

If $S = F \boxtimes G$,

$$H^{-2}(S) = H^{-1}(\mathtt{pr}_1^* F) \otimes H^{-1}(\mathtt{pr}_2^* G) \qquad H^{-1}(S) = H^0(\mathtt{pr}_1^* F) \otimes H^{-1}(\mathtt{pr}_2^* G) \oplus H^{-1}(\mathtt{pr}_1^* F) \otimes H^0(\mathtt{pr}_2^* G).$$

Since F is concentrated in degree -1, so is $\mathtt{pr}_1^* F$, therefore $H^{-1}(S) = H^{-1}(\mathtt{pr}_1^* F) \otimes H^0(\mathtt{pr}_2^* G)$. Thus $H^{-1}(S)$ is a skyscraper sheaf and $R^1\varsigma_* \circ H^{-1}(S) = H^1_{\text{ét}}(\mathbb{A}^1_k, H^{-1}(S))$ vanishes by lemma 2.4.1.1.
Finally $R^0\varsigma_* \circ H^{-2}(S)$ vanishes by proposition 3.1.4.1.

(ii) Let $i : \mathtt{Spec}(\kappa((T))) \to \mathbb{A}^1_k$ be the closed immersion of $0 = (T)$ in $\mathbb{A}^1_k = \mathtt{Spec}(k[T])$. By looking at the stalks, it is easy to see that for any sheaf G in $\mathbf{Perv}(k)$ on has

$$R\mathtt{sum}_*(G \boxtimes i_* i^* \underline{\mathbb{Q}_\ell}) = G.$$

Consider the exact sequence given by lemma 2.3.1.2,

$$0 \to j_! j^* \underline{\mathbb{Q}_\ell} \longrightarrow \underline{\mathbb{Q}_\ell} \longrightarrow i_* i^* \underline{\mathbb{Q}_\ell} \to 0,$$

where j is the complement immersion of i. It induces an exact triangle

$$R\mathtt{sum}_*(F \boxtimes j_! j^* \underline{\mathbb{Q}_\ell}) \to R\mathtt{sum}_*(F \boxtimes \underline{\mathbb{Q}_\ell}) \to R\mathtt{sum}_*(F \boxtimes i_* i^* \underline{\mathbb{Q}_\ell}) \to R\mathtt{sum}_*(F \boxtimes j_! j^* \underline{\mathbb{Q}_\ell})[1]$$

that can be rewritten as

$$R\mathtt{sum}_*(F \boxtimes j_! j^* \underline{\mathbb{Q}_\ell}) \to R\mathtt{sum}_*(F \boxtimes \underline{\mathbb{Q}_\ell}) \to F \to (F * j_! j^* \underline{\mathbb{Q}_\ell})[1].$$

One can check (for instance looking at the stalks) that $R\mathtt{sum}_*(F \boxtimes \mathbb{Q}_\ell) = H_{\text{ét}}(\mathbb{A}^1_k, F)$ and by hypothesis this is 0. Therefore

$$0 \to F \to (F * j_! j^* \underline{\mathbb{Q}_\ell})[1]$$

is exact, hence the result.

□

3.3 Monodromic Objects

3.3.1 The category $\mathbf{Perv}_{00}$

Let k be a subextension of $\mathbb{Q} \hookrightarrow \mathbb{C}$ and denote by u the canonical morphism $\mathbb{A}^1_{k^{\text{alg}}} \to \mathbb{A}^1_k$.

Definiton 3.3.1.1. Let $\mathscr{F}$ be a lisse $\mathbb{Q}_\ell$-sheaf on $\mathbb{A}^1_k \setminus \{0\}$. The *geometric monodromy* of $\mathscr{F}$ is the the monodromy representation of $u^*\mathscr{F}$ given by theorem 2.5.0.1.
Define the category $\mathbf{Perv}_{00}(k)$ as the category full subcategory of $\mathbf{LSh}_{\mathbb{Q}_\ell}(\mathbb{A}^1_k \setminus \{0\})$ whose objects have finite order geometric monodromy.

Let $j : \mathbb{A}^1_k \setminus \{0\} \hookrightarrow \mathbb{A}^1_k$ be the canonical inclusion. The functor $\mathscr{F} \mapsto j_!\mathscr{F}[1]$ is a fully faithful additive exact functor from $\mathbf{Perv}_{00}(k)$ to $\mathbf{Perv}_0(k)$.

3.3.2 Lisse sheaves on $\mathbb{A}^1_k \setminus \{0\}$

Recall that k is a field of characteristic 0.

Lemma 3.3.2.1. The inclusion $k[T^{\pm 1}] \hookrightarrow k((T))$ induces a morphism of schemes $\mathtt{Spec}(k((T))) \to \mathbb{A}^1_k \setminus \{0\}$. We have an equivalence of categories induced by the (base change) functor

$$\begin{aligned} \mathbf{Cov}_{\text{ét}}(\mathbb{A}^1_k \setminus \{0\}) &\xrightarrow{\sim} \mathbf{Cov}_{\text{ét}}(\mathtt{Spec}(k((T)))) \\ E &\longmapsto E \times_{\mathbb{A}^1_k \setminus \{0\}} \mathtt{Spec}(k((T))) \end{aligned}$$

Proof. It is enough to check the equivalence on connected covers. Recall theorem 2.1.1.1. A finite connected étale cover of $\mathbb{A}^1_k \setminus \{0\}$ is a finite morphism $E \to \mathbb{P}^1_k$ with two branch points, namely ∞ and 0, and with E connected. By example 4.2.1.1, E has genus zero, hence $E = \mathbb{P}^1_{k'}$ for some field extension k' of k. Applying a suitable automorphism, choose the ramification point of $\mathbb{P}^1_{k'} \to \mathbb{P}^1_k$ to be ∞ and 0. Then, example 2.1.1.2 implies that a finite connected étale cover of $\mathbb{A}^1_k \setminus \{0\}$ is uniquely described by a k-morphism

$$\begin{aligned} k[T] &\longrightarrow k'[T^{\pm 1}] \\ T &\longmapsto T^n. \end{aligned}$$

A finite connected étale cover of $\mathtt{Spec}(k((T)))$ is given by a finite extension of $k((T))$. By [Eis13], Corollary 13.15, these are given by k-morphisms

$$\begin{aligned} k((T)) &\longrightarrow k'((T)) \\ T &\longmapsto T^n, \end{aligned}$$

with k' a finite extension of k. Hence the equivalence. $\square$

Corollary 3.3.2.1. Let $\eta : \mathtt{Spec}(\Omega) \to \mathtt{Spec}(k((T)))$ be a geometric point and denote also by η the composition

$$\mathtt{Spec}(\Omega) \xrightarrow{\eta} \mathtt{Spec}(k((T))) \to \mathbb{A}^1_k \setminus \{0\},$$

seen as a geometric point of $\mathbb{A}^1_k$. Then there is a canonical isomorphism

$$\mathrm{Gal}(k((T))^{\mathrm{alg}}|k((T))) \overset{(2.1.1)}{\simeq} \pi_1^{\text{ét}}(k((T)),\eta) \xrightarrow{\sim} \pi_1^{\text{ét}}(\mathbb{A}^1_k \setminus \{0\},\eta)$$

induced by the inclusion $\mathtt{Spec}(k((T))) \to \mathbb{A}^1_k \setminus \{0\}$.

Let $x : \mathtt{Spec}(k^{\mathrm{alg}}) \to \mathbb{A}^1_k \setminus \{0\}$ be a geometric point and let η be as in corollary 3.3.2.1. By Proposition 2.1.2.4 there are exact sequences

$$0 \to \pi_1(\mathbb{A}^1_{k^{\mathrm{alg}}} \setminus \{0\},x) \longrightarrow \pi_1(\mathbb{A}^1_k \setminus \{0\},x) \longrightarrow \pi_1^{\text{ét}}(k,x) \to 0. \tag{3.3.1}$$

and

$$0 \to \pi_1(\mathbb{A}^1_{k^{\mathrm{alg}}} \setminus \{0\},\eta) \longrightarrow \pi_1(\mathbb{A}^1_k \setminus \{0\},\eta) \longrightarrow \pi_1(k,\eta) \overset{(2.1.1)}{\simeq} \mathrm{Gal}(k^{\mathrm{alg}}|k) \to 0 \tag{3.3.2}$$

Choosing an étale path in $\mathbb{A}^1_k \setminus \{0\}$ from η to x defines an isomorphism between (3.3.1) and (3.3.2). By theorem 2.1.2.3 this path is unique, hence the isomorphism is canonical.
By Corollary 3.3.2.1 there is also a canonical isomorphism $\pi_1(\mathbb{A}^1_k\setminus\{0\},y) \simeq \pi_1(\mathtt{Spec}(k((T))),y)$ that transforms (3.3.2) in the exact sequence

$$1 \to \mathrm{Gal}(k((T))^{\mathrm{alg}}|k^{\mathrm{alg}}((T))) \longrightarrow \mathrm{Gal}(k((T))^{\mathrm{alg}}|k((T))) \longrightarrow \mathrm{Gal}(k^{\mathrm{alg}}|k) \to 1, \tag{3.3.3}$$

The sequence (3.3.3) splits since there is a retraction

$$\begin{tikzcd} \mathrm{Gal}(k((T))^{\mathrm{alg}}|k((T))) \arrow[r] & \mathrm{Gal}(k^{\mathrm{alg}}|k) \arrow[l, bend right, dashed, "\epsilon"'] \end{tikzcd}$$

given by a choice of an extension of $\sigma \in \mathrm{Gal}(k^{\mathrm{alg}}|k)$ to $k((T))^{\mathrm{alg}}$. Next paragraph explains how construct such extension.

Existence and choices for ϵ

(a) The action of $\mathrm{Gal}(k^{\mathrm{alg}}|k)$ on k^{alg} extends to an action on $k^{\mathrm{alg}}((T))$, by acting on the coefficients of each Laurent series. This action is trivial when restricted to $k((T))$.

(b) Let $n \geqslant 1$ an integer and let $T^{1/n}$ be a formal symbol satisfying the algebraic relation $(T^{1/n})^n = T$. It is possible to extend the action of $\mathrm{Gal}(k^{\mathrm{alg}}|k)$ on $k((T))$ to an action on $k^{\mathrm{alg}}((T^{1/n}))$ in different ways, one for each $\xi \in \boldsymbol{\mu}_n(k^{\mathrm{alg}})$: just set $\sigma(T^{1/n}) = \xi T^{1/n}$ for all $\sigma \in \mathrm{Gal}(k^{\mathrm{alg}}|k)$.

(c) The proof of lemma 3.3.2.1 implies that all finite extensions $k^{\mathrm{alg}}((T)) \hookrightarrow L$ are uniquely determined by the degree. In particular if $[L : k^{\mathrm{alg}}((T))] = n$, then $L = k^{\mathrm{alg}}((T^{1/n}))$ and therefore one would like to write

$$k((T))^{\mathrm{alg}} = \bigcup_n k^{\mathrm{alg}}((T^{1/n})). \tag{3.3.4}$$

(d) (3.3.4) is a bit imprecise. Make the following choice: if n and m are positive integers such that $n|m$ and $m = q \cdot n$, force $(T^{1/m})^q = T^{1/n}$. Under this convention (3.3.4) is precise.

(e) To extend the action defined in (b) to an action on $k((T))^{\mathrm{alg}}$ use (3.3.4): for each topological generator of $\boldsymbol{\mu}(k^{\mathrm{alg}})$ one obtains a way to retract $\mathrm{Gal}(k^{\mathrm{alg}}|k)$ to $\mathrm{Gal}(k((T))^{\mathrm{alg}}|k((T)))$.

Theorem 3.3.2.1. For any geometric point $x : \mathtt{Spec}(\Omega) \to \mathbb{A}^1_k$ point there is a non-canonical isomorphism

$$\pi_1^{\text{ét}}(\mathbb{A}^1_k \setminus \{0\}, x) \simeq \boldsymbol{\mu}(k^{\mathrm{alg}}) \times \mathrm{Gal}(k^{\mathrm{alg}}|k),$$

that depends on the choice of a retraction.

Proof. By steps (a)—(e) in the above recipe, one identifies $\pi_1^{\text{ét}}(\mathbb{A}^1_k \setminus \{0\}, x)$ with the semi-direct product

$$\mathrm{Gal}(k((T))^{\mathrm{alg}}|k^{\mathrm{alg}}((T))) \rtimes \epsilon(\mathrm{Gal}(k^{\mathrm{alg}}|k)),$$

The actions of $\epsilon(\mathrm{Gal}(k^{\mathrm{alg}}|k))$ and $\mathrm{Gal}(k((T))^{\mathrm{alg}}|k^{\mathrm{alg}}((T)))$ on $k((T))^{\mathrm{alg}}$ commute, hence the action of $\epsilon(\mathrm{Gal}(k^{\mathrm{alg}}|k))$ on $\mathrm{Gal}(k((T))^{\mathrm{alg}}|k^{\mathrm{alg}}((T)))$ is trivial and the semi-direct product is a product.

Finally by point (c) (and (d)) of the recipe one has

$$\mathrm{Gal}(k((T))^{\mathrm{alg}}|k^{\mathrm{alg}}((T))) = \varprojlim_n \mathrm{Gal}(k^{\mathrm{alg}}((T^{1/n}))|k^{\mathrm{alg}}((T))) = \varprojlim_n \boldsymbol{\mu}_n(k^{\mathrm{alg}}) =: \boldsymbol{\mu}(k^{\mathrm{alg}}).$$

□

Corollary 3.3.2.2. The data of a lisse sheaf $\mathscr{L}$ on $\mathbb{A}^1_k \setminus \{0\}$ is the same data of two continuous representations:

$$\rho_{\mathrm{geom}} : \widehat{\mathbb{Z}} \longrightarrow \mathrm{GL}(\mathscr{L}_{\overline{x}} \otimes \mathbb{Q}_\ell) \quad \text{and} \quad \rho_{\mathrm{Gal}} : \mathrm{Gal}(k^{\mathrm{alg}}|k) \longrightarrow \mathrm{GL}(\mathscr{L}_{\overline{x}} \otimes \mathbb{Q}_\ell)$$

Proof. By point (e) of the recipe, the choice of a retraction consists in the choice of a topological generator of $\boldsymbol{\mu}(k^{\mathrm{alg}})$. This choice defines an isomorphism $\boldsymbol{\mu}_n(k^{\mathrm{alg}}) \simeq \widehat{\mathbb{Z}}$. □

3.4 Formulation of the “convolution problems”

Problem

Let k be a field of characteristic 0. Given two sheaves F and G in $\mathbf{Perv}_0(k)$ describe the convolution $F * G$ in terms of a suitable $(G, \mathbf{P}_S)$-representation.

Baby Problem

Let k be a field of characteristic 0. Given some two sheaves $\mathscr{F}$ and $\mathscr{G}$ in $\mathbf{Perv}_{00}(k)$ describe the sheaf

$$\mathscr{F} \star \mathscr{G} := H^{-1}(j_!\mathscr{F}[1] * j_!\mathscr{G}[1]).$$

Big problem

Define the category $\mathbf{Perv}_0(k)$ for k a field of characteristic $p > 0$ and study the convolution of two objects in $D_b^c(\mathbb{A}_l^1, \mathbb{Q}_\ell)$ for $\ell \neq p$.

Big big problem

Define the category $\mathbf{Perv}_0(k)$ for k a field of characteristic $p > 0$ and study the convolution of two objects in $D_b^c(\mathbb{A}_l^1, \mathbb{Q}_\ell)$ for $\ell = p$.

3.5 The “baby problem”

3.5.1 Computation of one example

Let $\mathscr{L}$ be the object of $\mathbf{Perv}_{00}(k)$ defined by

$$\rho_{\text{geom}} : 1_{\widehat{\mathbb{Z}}} \mapsto -1 \in \mathbb{Q}_\ell^\times \text{ and } \rho_{\text{Gal}} : \sigma \mapsto 1 \in \mathbb{Q}_\ell^\times.$$

I will compute the representation associated to the sheaf $\mathscr{F}[1] * \mathscr{F}[1]$.

Lemma 3.5.1.1. Let $j : \mathbb{A}_k^1 \setminus \{0\} \hookrightarrow \mathbb{A}_k^1$ be the canonical inclusion. The sheaf $j_!\mathscr{L}$, with $\mathscr{L}$ defined as above, is the sheaf $\phi_*\phi^*\underline{\mathbb{Q}_\ell}_{\mathbb{A}_k^1}/\underline{\mathbb{Q}_\ell}_{\mathbb{A}_k^1}$ where $\phi : \mathbb{A}_k^1 \to \mathbb{A}_k^1$ is the morphism induced by

$$\begin{aligned} k[T] &\longrightarrow k[T] \\ T &\longmapsto T^2. \end{aligned}$$

In other words $j_!\mathscr{L}$ sits in the exact sequence

$$0 \to \underline{\mathbb{Q}_\ell}_{\mathbb{A}_k^1} \longrightarrow \phi_*\phi^*\underline{\mathbb{Q}_\ell}_{\mathbb{A}_k^1} \longrightarrow j_!\mathscr{L} \to 0,$$

where the second arrow is the adjunction morphism.

Proof. Choose a geometric point x of $\mathbb{A}_k^1 \setminus \{0\}$. The stalks at x of $\underline{\mathbb{Q}_\ell}_{\mathbb{A}_k^1}$ and $\phi_*\underline{\mathbb{Q}_\ell}_{\mathbb{A}_k^1}$ are given by

$$(\underline{\mathbb{Q}_\ell}_{\mathbb{A}_k^1})_x = \mathbb{Q}_\ell \quad \text{and} \quad (\phi_*\underline{\mathbb{Q}_\ell}_{\mathbb{A}_k^1})_x \overset{(2.3.4)}{=} \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell.$$

The sheaf $\phi^*\underline{\mathbb{Q}_\ell}_{\mathbb{A}_k^1}$ is still the constant sheaf $\underline{\mathbb{Q}_\ell}_{\mathbb{A}_k^1}$ and the adjunction morphism

$$\underline{\mathbb{Q}_\ell}_{\mathbb{A}_k^1} \longrightarrow \phi_*\phi^*\underline{\mathbb{Q}_\ell}_{\mathbb{A}_k^1}$$

induces a morphism on the stalks $\mathbb{Q}_\ell \to \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell$. By construction of the adjunction this morphism is given by sending $1 \in \mathbb{Q}_\ell$ to $\binom{1}{1} \in \mathbb{Q}_\ell^2$ and then projecting it to its first and second coordinate. Thus the stalk $\mathbb{Q}_\ell \oplus \mathbb{Q}_\ell$ is identified with $\langle\binom{1}{0}\rangle \oplus \langle\binom{0}{1}\rangle$

The monodromy of $\phi_*\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k}$ is given by swapping the coordinates, while the monodromy of $\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k}$ is trivial. Therefore the quotient monodromy, the monodromy of $\phi_*\phi^*\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k}/\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k}$, is given by inverting (multiplication by -1) the only coordinate of a vector in $(\mathbb{Q}_\ell \oplus \mathbb{Q}_\ell)/\mathbb{Q}_\ell$. This is indeed the monodromy of $j_!\mathscr{L}$, hence $\mathscr{L}$ and $\phi_*\phi^*\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k}/\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k}$ have the same monodromy, hence they are the same sheaf. □

Lemma 3.5.1.2. Denote by $\phi \boxplus \phi$ the morphism $\mathbb{A}^2_k \to \mathbb{A}^1_k \setminus \{0\}$ of affine schemes induced by

$$\begin{aligned} k[T] &\longrightarrow k[T_1, T_2] \\ T &\longmapsto T_1^2 + T_2^2. \end{aligned}$$

Then $\mathscr{L} \star \mathscr{L} = R^1(\phi \boxplus \phi)_*(\underline{\mathbb{Q}_\ell}_{\mathbb{A}^2_k})$.

Proof. This is an immediate consequence of lemma 3.5.1.1 and Kunneth's formula (theorem 2.3.9.4): there is an exact sequence of sheaves on $\mathbb{A}^1_k$

$$0 \to \underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k} \longrightarrow \phi_*\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k} \longrightarrow j_!\mathscr{L} \to 0,$$

that induce the two distinguished triangles

$$R\mathrm{sum}_*(j_!\mathscr{L} \boxtimes \underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k}) \to R\mathrm{sum}_*(j_!\mathscr{L} \boxtimes R\phi_*\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k}) \to R\mathrm{sum}_*(j_!\mathscr{L} \boxtimes j_!\mathscr{L}) \to R\mathrm{sum}_*(j_!\mathscr{L} \boxtimes \underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k})[1]$$

$$R\mathrm{sum}_*(R\phi_*\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k} \boxtimes \underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k}) \to R\mathrm{sum}_*(R\phi_*\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k} \boxtimes R\phi_*\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k}) \to R\mathrm{sum}_*(R\phi_*\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k} \boxtimes j_!\mathscr{L}) \to R\mathrm{sum}_*(R\phi_*\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k} \boxtimes j_!\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k})[1]$$

By inspection the first and last entries of this two sequences vanish. Therefore there is an isomorphism

$$R\mathrm{sum}_*(R\phi_*\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k} \boxtimes R\phi_*\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k}) \simeq R\mathrm{sum}_*(j_!\mathscr{L} \boxtimes j_!\mathscr{L}). \tag{3.5.1}$$

By Kunneth's formula the left hand side of (3.5.1) is

$$R\mathrm{sum}_* \circ R(\phi \times \phi)_*(\underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k} \boxtimes \underline{\mathbb{Q}_\ell}_{\mathbb{A}^1_k}) = R(\phi \boxplus \phi)_*(\underline{\mathbb{Q}_\ell}_{\mathbb{A}^2_k}).$$

This implies the result since, for any two object $\mathscr{F}$ and $\mathscr{G}$ of $\mathbf{Perv}_{00}(k)$,

$$\begin{aligned} \mathscr{F} \star \mathscr{G} &= H^{-1}R\mathrm{sum}_*(j_!\mathscr{F}[1] \boxtimes j_!\mathscr{G}[1]) \\ &= H^{-1}(R\mathrm{sum}(j_!\mathscr{F} \boxtimes j_!\mathscr{G})[2]) \\ &= R^1\mathrm{sum}(j_!\mathscr{F} \boxtimes j_!\mathscr{G}). \end{aligned}$$

□

Lemma 3.5.1.3. Let z be a geometric point of $\mathbb{A}^1_k$. There is an isomorphism

$$(R^p(\phi \boxplus \phi)_*(\underline{\mathbb{Q}_\ell}_{\mathbb{A}^2_k}))_z = \mathrm{H}^p_{\text{ét}}(X(z), \underline{\mathbb{Q}_\ell}_{X(z)}),$$

where $X(z)$ is given by the Cartesian diagram

$$\begin{array}{ccc} X(z) & \longrightarrow & \mathbb{A}^2_k \\ \downarrow & & \downarrow{\scriptstyle \phi\boxplus\phi} \\ \mathrm{Spec}(\Omega) & \xrightarrow{z} & \mathbb{A}^1_k \end{array} \tag{3.5.2}$$

Proof. The proof of lemma 3.2.1.2 generalizes to this situation without effort. □

Lemma 3.5.1.4. Let $X(z)$ be as in the Cartesian diagram (3.5.2). If the image of z is in $\mathbb{A}^1_k \setminus \{0\}$, then $X(z) \simeq \mathbb{A}^1_\Omega \setminus \{0\}$.

Proof. The isomorphism is the one induced by

$$\begin{aligned} \Omega[T^{\pm 1}] &\longrightarrow \Omega[T_1, T_2]/(T_1^2 + T_2^2 - z) \simeq \Omega[(T_1 - iT_2)^{\pm 1}] \\ T &\longmapsto T_1 - iT_2. \end{aligned}$$

□

Corollary 3.5.1.1. Let $z : \mathtt{Spec}(k^{\mathrm{alg}}) \to \mathbb{A}^1_k \setminus \{0\}$ be a geometric point. By theorem 2.2.1.1

$$\begin{aligned} (\mathscr{L} * \mathscr{L})_z = H^1_{\text{ét}}(\mathbb{A}^1_{k^{\mathrm{alg}}} \setminus \{0\}, \underline{\mathbb{Q}_\ell}) &= (\varprojlim_n H^1_{\text{ét}}(\mathbb{A}^1_{k^{\mathrm{alg}}} \setminus \{0\}, \mathbb{Z}/\ell^n\mathbb{Z})) \otimes \mathbb{Q}_\ell \\ &= (\varprojlim_n \boldsymbol{\mu}_{\ell^n}(k^{\mathrm{alg}})) \otimes \mathbb{Q}_\ell =: \mathbb{Q}_\ell(1) \end{aligned}$$

These isomorphisms are isomorphisms in the category $\mathbf{Perv}_{00}(k)$ in the sense that the action of $\pi_1^{\text{ét}}(\mathbb{A}^1_k \setminus \{0\}, z)$ on the left hand side is the same as the action on the right hand side. Thus $\mathscr{L} \star \mathscr{L}$ is an object in $\mathbf{Perv}_{00}(k)$ described by the representations

$$\rho_{\mathrm{geom}} : 1_{\widehat{\mathbb{Z}}} \longmapsto 1 \in \mathbb{Q}_\ell^\times \quad \text{and} \quad \rho_{\mathrm{Gal}} : \sigma \longmapsto \sigma|_{\boldsymbol{\mu}(k^{\mathrm{alg}})} \otimes \mathrm{id}$$

Chapter 4

Appendix

4.1 Appendix A

(cf. [Fu11, Chapter 6] and [Vak, Chapter 1 and 24])

4.1.1 Resolutions

The frame of this subsection is an arbitrary category. The symbol $\hookrightarrow$ denotes a monomorphisms while $\twoheadrightarrow$ denotes an epimorphism.

Injective and projective objects

An object I is injective if every diagram $Y \hookleftarrow X \to I$ defines a (non-necessarily unique) morphism $Y \dashrightarrow I$ such that

$$X \hookrightarrow Y \dashrightarrow I = X \to I$$

An object P is projective if every diagram $P \to Y \twoheadleftarrow X$ defines a (non-necessarily unique) morphism $P \dashrightarrow X$ such that

$$P \dashrightarrow X \twoheadrightarrow Y = P \to Y$$

Injective and projective resolutions

An injective resolution for X is an exact sequence

$$0 \to X \to I_0 \to I_1 \to I_2 \to \cdots$$

where the objects I_i are injective.
A projective resolution for X is an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to X \to 0$$

where the objects P_i are projective.

Enough injective/surjective objects

A category has enough injective (resp. projective) objects if for every object X there is an injective object I (resp. a projective object P) and a morphism $X \hookrightarrow I$ (resp. $P \twoheadrightarrow X$).

Lemma 4.1.1.1. If a category has enough injective (resp. projective) objects, then every object X has an injective (resp. projective) resolution.

Lemma 4.1.1.2. Let X be a scheme. The category of étale sheaves on X, with values in $\mathbf{C} \in \{\mathbf{Ab}, \mathbf{Mod}_R\}$ has enough injective objects.

4.1.2 Localization of categories

Definiton 4.1.2.1 (Localized Categories). Consider a category $\mathbf{C}$ and let S be a collection of morphisms in $\mathbf{C}$. Suppose that the following axioms are satisfied:

(a) for all $\phi, \psi \in S$, if $\psi \circ \phi$ exists, then $\psi \circ \phi \in S$;

(b) for all morphisms $\phi, \psi : X \to Y$ in $\mathbf{C}$:

$$\exists \theta \in S \text{ s.t. } \theta \circ \phi = \theta \circ \psi \quad \text{iff} \quad \exists \theta \in S \text{ s.t. } \phi \circ \theta = \psi \circ \theta;$$

(c) given red (resp. blue) morphisms as in the diagram below (defined in $\mathbf{C}$) we can find blue (resp. red) arrows so that

$$\begin{tikzcd} X \arrow[r, red] \arrow[d, red, "\phi\in S"'] & Y \arrow[d, blue, "\psi\in S"] \\ Z \arrow[r, blue] & W \end{tikzcd} \qquad \text{commutes.}$$

Define a new category, denoted by $S^{-1}\mathbf{C}$, whose objects are the same as the objects of $\mathbf{C}$ while the morphisms are defined by

$$\mathrm{Hom}_{S^{-1}\mathbf{C}}(X, Y) := \varinjlim_{X' \text{ in } \mathbf{Over}_S(X)^{\mathrm{opp}}} \mathrm{Hom}(X', Y),$$

where $\mathbf{Over}_S(X)$ is the full subcategory of $\mathbf{Over}(X)$ whose objects $X' \to X$ are also in S.

A set S that satisfies (a), (b) and (c) above, is a *multiplicative family* and $S^{-1}\mathbf{C}$ is the *localization at S* of $\mathbf{C}$.

The composition of morhpisms in $S^{-1}\mathbf{C}$ is described as follows.
An element $\phi \in \mathrm{Hom}_{S^{-1}X}(X, Y)$ can be represented by a diagram

$$X \leftsquigarrow X' \to Y,$$

where the squig left-arrow is in S.
Given two morphisms $\phi \in \mathrm{Hom}_{S^{-1}\mathbf{C}}(X, Y)$ and $\psi \in \mathrm{Hom}_{S^{-1}\mathbf{C}}(Y, Z)$ represented by

$$\phi = X \leftsquigarrow X' \to Y \text{ and } \psi = Y \leftsquigarrow Y' \to Z,$$

obtain the diagram $X \to Y \leftsquigarrow Y'$, and complete it, using axiom (c), in a square

$$
\begin{tikzcd}
 & X'' \arrow[dl, squiggly] \arrow[dr] & \\
X' \arrow[dr] & & Y' \arrow[dl, squiggly] \\
 & Y &
\end{tikzcd} .
$$

Then define $\psi \circ \phi \in \mathrm{Hom}_{S^{-1}\mathbf{C}}(X,Z)$ to be the morphism represented by

$$X \leftsquigarrow X' \leftsquigarrow X'' \to Y' \to Z.$$

Lemma 4.1.2.1.

(i) The canonical functor $\mathbf{C} \to S^{-1}\mathbf{C}$ sends morphisms in S to isomorphisms in $S^{-1}\mathbf{C}$.

(ii) Let $\mathbf{D}$ be another, arbitrary, category. If $\mathbf{C} \to \mathbf{D}$ is a functor that sends morphisms in S to isomorphisms, then $\mathbf{C} \to \mathbf{D}$ factors through $\mathbf{C} \to S^{-1}\mathbf{C}$ in a unique way.

Definiton 4.1.2.2. Let $\mathbf{C}$ be an additive category. Denote by $K(\mathbf{C})$, is the category whose objects are complexes

$$C^\bullet = (C^i, d^i)_i = \quad \cdots \longrightarrow C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \xrightarrow{d^{i+1}} \cdots$$

and whose morphism are homotopy classes of morphisms of complexes. Call this category the *category of complexes in* $\mathbf{C}$ and set the notation

$$H^i(C^\bullet) := \ i^{\text{th}}\text{-cohomology of the complex } C^\bullet.$$

Definiton 4.1.2.3. Let $\mathbf{C}$ be an abelian category. The set

$$S := \left\{F^\bullet \to G^\bullet \text{ in } K(\mathbf{C}) \text{ s.t. } H^i(F^\bullet) \simeq H^i(G^\bullet) \text{ for all } i\right\}$$

is a multiplicative family in $K(\mathbf{C})$. An element in S is often called a *quism* (short for *quasi-isomorphism*). I systematically use squiggly arrows for quisms.
Write $D(\mathbf{C})$ for the category $S^{-1}\mathbf{K}(\mathbf{C})$. It is called the *derived category of* $\mathbf{C}$.

Definiton 4.1.2.4 (Bounded complexes and their derived categories). A complex $C = (C^i, d^i)$ is said to be *bounded below* (resp. *above*) if $C^n = 0$ for an integer n small (resp. big) enough. A complex is *bounded* if it is bounded below and above.
Denote by $K^+(\mathbf{C})$ (resp. $K^-(\mathbf{C})$ and $K^b(\mathbf{C})$) the full subcategories of $K(\mathbf{C})$ whose objects are complexes bounded below (resp. bounded above and bounded). Similarly denote by $D^+(\mathbf{C})$, $D^-(\mathbf{C})$ and $D^b(\mathbf{C})$ the corresponding derived categories.

4.1.3 Triangulated structure on $D(\mathbf{C})$

Triangulated categories

Let $\mathbf{C}$ be an additive category equipped with a functor

$$
\begin{aligned}
[1] : \mathbf{C} &\longrightarrow \mathbf{C} \\
X &\longmapsto X[1],
\end{aligned}
$$

and with a collection Δ of diagrams, with arrows in $\mathbf{C}$, of the form

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1].$$

The elements of Δ are the *distinguished triangles* of $\mathbf{C}$. The tuple $(\mathbf{C},[1],\Delta)$ is a *triangulated category* if it satisfies the following axioms.

(TR1) a) Any morphism $X \to Y$ extends to a distinguished triangle.

b) $X \xrightarrow{\mathrm{id}} X \xrightarrow{0} 0 \xrightarrow{0} 0$ is a distinguished triangle.

c) The set of distinguished triangles is closed under isomorphisms.
(A morphism of distinguished triangles, say from $X \to Y \to Z \to X[1]$ to $X' \to Y' \to Z' \to X'[1]$ its a triplet (α, β, γ) of morphisms in $\mathbf{C}$ such that
$\begin{tikzcd} X \arrow[r] \arrow[d, "\alpha"] & Y \arrow[r] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, "\gamma"] & X[1] \arrow[d, "\alpha[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$ commutes.)

(TR2) $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \xrightarrow{\theta} X[1]$ is a distinguished triangle if and only if $Y \xrightarrow{\psi} Z \xrightarrow{\theta} X[1] \xrightarrow{-\phi[1]} Y[1]$ is.

(TR3) Any commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] & X[1] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

whose rows are distinguished triangles, extends to a morphism of distinguished triangles.

(TR4) Given a diagram as below, where the rows are distinguished triangles, it is possible to find f and g such that (ϕ, id, g) is a morphism of distinguished triangles and such that the red arrows defines a distinguished triangle.

$$\begin{tikzcd} Y \arrow[r, "\psi"] & Z \arrow[r] & X' \arrow[r] & Y[1] \\ X \arrow[u, "\phi"] \arrow[r, "\psi\circ\phi"] & Z \arrow[u, "\mathrm{id}"] \arrow[r] & Y' \arrow[u, dashed, "g"] \arrow[r] & X[1] \arrow[u, "\phi[1]"] \\ X \arrow[r, "\phi"] & Y \arrow[r] & Z' \arrow[u, dashed, "f"] \arrow[r] & X[1] \end{tikzcd}$$

Triangulated structure on K(C)

Consider the functor

$$[1] : K(\mathbf{C}) \longrightarrow K(\mathbf{C})$$

that sends a complex $F^\bullet = (F^i, f^i)_i$ to the complex

$$F^\bullet[1] := (F^{i+1}, -f^{i+1})_i,$$

and a morphism $u = (u_i)_i \in \mathrm{Hom}_{\mathbf{K(C)}}(F, G)$ to the morphism

$$u[1] := (u_{i+1})_i.$$

Consider two complexes $F^\bullet = (F^i, f^i)_i$ and $G^\bullet = (G^i, g^i)_i$ in $K(\mathbf{C})$. Given a homotopy class of morphisms $F^\bullet \xrightarrow{u} G^\bullet$, define the complex

$$F^\bullet[1] \oplus_u G^\bullet := (F^{i+1} \oplus G^i, -f^{i+1} \oplus u_{i+1} + g^i)_i \tag{4.1.1}$$

known as the *mapping cone* of u. This is often denoted by $C(u)$. This gives a diagram

$$F^\bullet \xrightarrow{u} G^\bullet \to F^\bullet[1] \oplus_u G^\bullet \to F^\bullet[1],$$

where $G^\bullet \to F^\bullet[1] \oplus_u G^\bullet$ is the canonical inclusion and $F^\bullet[1] \oplus_u G^\bullet \to F^\bullet[1]$ is the canonical projection.

Lemma 4.1.3.1. Denote by Δ the set of diagrams in $\mathbf{K}(\mathbf{C})$ isomorphic to diagrams of the form of (4.1.1). The triplet $(\mathbf{K}(\mathbf{C}), [1], \Delta)$ defines a triangulated category.

Triangulated structure on $D(\mathbf{C})$

Lemma 4.1.3.2. If $F^\bullet \overset{q}{\rightsquigarrow} F'^\bullet$ is a quism, so is $q[1]$. If $G^\bullet \overset{q'}{\rightsquigarrow} G'^\bullet$ is another quism that sits in a commutative diagram

$$\begin{tikzcd} F^\bullet \arrow[r] \arrow[d, squiggly] & G^\bullet \arrow[d, squiggly] \\ F'^\bullet \arrow[r] & G'^\bullet \end{tikzcd}$$

then the induced map $C(q) \rightsquigarrow C(q')$ is also a quism.

Corollary 4.1.3.1. Let Δ be as in lemma 4.1.3.1 and let $D(\Delta)$ set of diagrams in $\mathbf{K}(\mathbf{C})$ quasi isomorphic to a diagram in Δ. The triplet $(D(\mathbf{C}), [1], D(\Delta))$ defines a triangulated category.

Remark 4.1.3.1. Similarly $D^+(\mathbf{C})$, $D^-(\mathbf{C})$ and $D^b(\mathbf{C})$ are triangulated categories.

4.1.4 Derived functors

In this subsection "?" denotes a symbol in $\{b, +, -\}$. Let X and be a scheme, $\mathbf{C}$ some abelian category and let $\Phi : K^?(\mathbf{C}) \to K(\mathbf{D})$ be a functor.
For a complex $F^\bullet$ in $K(\mathbf{C})$, denote by $\mathbf{Under}_S(F^\bullet)$ the full subcategory of $\mathbf{Under}(F^\bullet)$ whose objects are quisms, hence the category whose objects are of the form $F^\bullet \rightsquigarrow F'^\bullet$ for some complex $F'^\bullet$.
Define $\Phi'_{F^\bullet} : D(\mathbf{D}) \to \mathbf{Set}$ to be the functor that sends an object $G^\bullet$ in $D(\mathbf{D})$ to

$$\Phi'_{F^\bullet}(G^\bullet) := \varinjlim_{F'^\bullet \text{ in } \mathbf{Under}_S(F^\bullet)} \mathrm{Hom}_{D(\mathbf{D})}(G^\bullet, \Phi(F'^\bullet)).$$

This gives a map $\Phi' : F^\bullet \mapsto \Phi'_{F^\bullet}$ that can be made into a functor.
More precisely, if $F_1^\bullet \to F_2^\bullet$ is a morphism in $K(\mathbf{C})$, define a canonical morphisms of functors $\Phi'_{F_1^\bullet} \to \Phi'_{F_2^\bullet}$ as follows. For any quism $F_1^\bullet \rightsquigarrow F_1'^\bullet$, by axiom (c), there is a commutative diagram

$$\begin{tikzcd} F_1^\bullet \arrow[r] \arrow[d, rightsquigarrow] & F_2^\bullet \arrow[d, rightsquigarrow] \\ F_1'^\bullet \arrow[r] & F_2'^\bullet \end{tikzcd},$$

hence there is a map $\mathrm{Hom}_{D(\mathbf{D})}(G^\bullet, \Phi(F_1'^\bullet)) \longrightarrow \mathrm{Hom}_{D(\mathbf{D})}(F^\bullet, \Phi(F_2'^\bullet))$ induced by $F_1^\bullet \to F_2^\bullet$ and $F_1^\bullet \rightsquigarrow F_1'^\bullet$. Denote this map by $\phi_{F_1^\bullet \to F_2^\bullet}(F_1'^\bullet)$ and define $\Phi'_{F_1^\bullet} \to \Phi'_{F_2^\bullet}$ to be

$$\varinjlim_{F^{\bullet\prime} \text{ in } \mathbf{Under}_S(F^\bullet)} \phi_{F_1^\bullet \to F_2^\bullet}(F_1'^\bullet).$$

This defines a functor $\Phi' : K^?(C) \to \mathbf{Hom}(D(\mathbf{D})^{\mathrm{opp}}, \mathbf{Set})$ that sends $F^\bullet$ to $\Phi_F'^\bullet$.

Lemma 4.1.4.1. The functor Φ' sends quisms to isomorphisms. In particular it induces a canonical functor

$$R\Phi : D^?(\mathbf{C}) \to \mathbf{Hom}(D(\mathbf{D})^{\mathrm{opp}}, \mathbf{Set}),$$

that is called the *right derived functor* of Φ.

Definiton 4.1.4.1. Say that $R\Phi$ is *representable* (or *defined everywhere*) if $R\Phi(F^\bullet)$ is representable for every $F^\bullet$ in $D(\mathbf{D})$. If $R\Phi$ is representable, then it essentially takes values in $D(\mathbf{D})$. For $R\Phi$ representable, set the notation

$$R^i\Phi(F) := H^i(R\Phi(F)).$$

4.1.5 Some features and properties of the derived categories formalism

Lemma 4.1.5.1. The functor

$$\mathscr{F} \longmapsto \mathscr{F}[0] : \quad \cdots \to 0 \to \mathscr{F} \to 0 \to \cdots \ ,$$

(that sends a sheaf $\mathscr{F}$ to the complex, concentrated in degree 0, with non-zero entry $\mathscr{F}$) is fully faithful.
(In particular it defines an equivalence between the full subcategory of $D(X)$, whose objects are those complexes F such that $H^i(F) = 0$ for all $i \neq 0$.)

Proposition 4.1.5.1. Let $\mathbf{C}$ be an abelian category. For every exact sequence

$$0 \to X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \to 0$$

where X, X, Z are complexes in $K(\mathbf{C})$, there is:

(i) a canonical long exact sequence

$$\cdots \longrightarrow H^{i-1}(Z) \xrightarrow{\delta_{i-1}} H^i(X) \xrightarrow{\phi_i} H^i(Y) \xrightarrow{\psi_i} H^i(Z) \xrightarrow{\delta_i} H^{i+1}(X) \longrightarrow \cdots ;$$

(ii) a distinguished triangle in $D(\mathbf{C})$

$$X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \longrightarrow X[1].$$

Proposition 4.1.5.2. Let $\Phi : \mathbf{C}_1 \to \mathbf{C}_2$ be a left exact functor such that $R\Phi$ has finite cohomological dimension. Then we have the two spectral sequences.

$$E_2^{p,q} = R^p\Phi(H^q(F)) \Longrightarrow R^{p+q}\Phi(F) \tag{4.1.2}$$

4.2 Appendix B

4.2.1 Riemann-Hurwitz for function fields in one variable

(cf. [Che51])

Let k be any field. A *function field in one variable* over k is a finite extension $k(T) \hookrightarrow K$, where T is transcendental over k.
Let k and ℓ be two fields. Consider two function fields in one variable $k(T) \hookrightarrow K$ and $\ell(T) \hookrightarrow L$. Assume that $K \hookrightarrow L$ and that $k = \ell \cap K$.

Definiton 4.2.1.1 (Ramification index). Let $\mathfrak{P}$ be a place of L. The discrete valuation $v_{\mathfrak{P}} : L \setminus \{0\} \to \mathbb{Z}$ can be restricted to $K \setminus \{0\}$. The image of $v_{\mathfrak{P}}\big|_{K\setminus 0}$ is a subgroup of $\mathbb{Z}$, hence a group of the form $n\mathbb{Z}$ for some integer $n \geqslant 1$. Define such n as the ramification index of $\mathfrak{P}$. Denote it by $\mathtt{ram}_K(\mathfrak{P})$.

Theorem 4.2.1.1 ([Che51] IV.1 theorem 1). If $\mathfrak{p}$ is a place of K, there exists at least a place $\mathfrak{P}$ of L above $\mathfrak{p}$. Furthermore the number of possible such places is finite, say $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$ and if $K \hookrightarrow L$ is algebraic

$$[L : K] = \sum_{i=1}^{n} d(\mathfrak{P}_i) \cdot \mathtt{ram}_K(\mathfrak{P}_i)$$

Definiton 4.2.1.2 (Differential exponent). Let $K \hookrightarrow L$ as above. Let $\mathfrak{P}$ be a place in L and $\mathfrak{p}$ the place in K below $\mathfrak{P}$. Denote by $\mathcal{O}_{\mathfrak{p}} \subset K$ the DVR of $\mathfrak{p}$. Denote by $K_{\mathfrak{p}}$ the fraction field of the $\mathfrak{p}$-completion of $\mathcal{O}_{\mathfrak{p}}$[1] and similarly define $L_{\mathfrak{P}}$. Denote by $m(\mathfrak{P})$ the largest integer with the following property:

$$\forall y \in L \text{ s.t. } v_{\mathfrak{P}}(y) + m(\mathfrak{P}) \geqslant 0, \text{ then } v_{\mathfrak{p}}(\mathrm{Tr}^{\mathfrak{P}}(y)) \geqslant 0.$$

where $\mathrm{Tr}^{\mathfrak{P}}(y)$ is the trace of y with respect to the field extension $K_{\mathfrak{p}} \hookrightarrow L_{\mathfrak{P}}$.

Theorem 4.2.1.2 ([Che51] IV theorem 7). Suppose that $K \to L$ is finite and separable. Then $m(\mathfrak{P}) \geqslant \mathtt{ram}_K(\mathfrak{P}) - 1$ and equality holds if and only if (i) and (ii) below hold.

(i) $\kappa(\mathfrak{p}) \hookrightarrow \kappa(\mathfrak{P})$ is separable.

(ii) $\mathrm{char}(L) \nmid \mathtt{ram}_K(\mathfrak{P})$.

---

[1] That is, the inverse limit $\varprojlim_n \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n\mathcal{O}_{\mathfrak{p}}$

The definition of the *genus* of a function field in one variable is given in [Che51] II.1.

Theorem 4.2.1.3 ([Che51] VI.2 corollary 2 of theorem 2). Let $K \hookrightarrow L$ be finite and separable and let g_K and g_L the genera of K and L. Assume furthermore that the field of constants of L is k. Then

$$2g_L - 2 = [L:K]\cdot(2g_K - 2) + \sum_{\mathfrak{P}} m(\mathfrak{P})\cdot[\kappa(\mathfrak{P}):k]$$

Example 4.2.1.1. Let $K = k(T)$ and let L a separable extension with field of constant k. Then Riemann-Hurwitz says that

$$2g_L - 2 = [L:K]\cdot(-2) + \sum_{\mathfrak{P}}(\mathtt{ram}_K(\mathfrak{P}) - 1)\cdot[\kappa(\mathfrak{p}):k].$$

(i) For any place $\mathfrak{p} \neq 1/T$ in K, suppose that $\mathtt{ram}_K(\mathfrak{P}) = 1$ for all primes above $\mathfrak{p}$, so that

$$2g_L - 2 = -[L:K] - \sum_{\mathfrak{P}\text{ above }1/T}[\kappa\mathfrak{P}:k].$$

Since $g_L \geqslant 0$ the left hand side is $\geqslant -2$ while the right hand side is $\leqslant -2$. Hence $[L:K] = 1$ and $L = k(T)$.

(ii) For any place $\mathfrak{p} \neq T, 1/T$ in K, suppose that $\mathtt{ram}_K(\mathfrak{P}) = 1$ for all primes above $\mathfrak{p}$, so that

$$2g_L - 2 = -\left(\sum_{\mathfrak{P}\text{ above }T}[\kappa(\mathfrak{P}):k] + \sum_{\mathfrak{P}\text{ above }1/T}[\kappa\mathfrak{P}:k]\right)$$

Since $g_L \in \mathbb{N}$ we obtain

$$1 - \frac{1}{2}\left(\sum_{\mathfrak{P}\text{ above }T}[\kappa(\mathfrak{P}):k] + \sum_{\mathfrak{P}\text{ above }1/T}[\kappa\mathfrak{P}:k]\right) \in \mathbb{N},$$

which implies

$$\sum_{\mathfrak{P}\text{ above }T}[\kappa(\mathfrak{P}):k] + \sum_{\mathfrak{P}\text{ above }1/T}[\kappa(\mathfrak{P}):k] = 2 \quad\text{and}\quad g_L = 0.$$

Therefore we see that:

(a) there is only one place, say $\mathfrak{P}_T$, above T;

(b) there is only one place, say $\mathfrak{P}_{1/T}$, above $1/T$;

(c) $[\kappa(\mathfrak{P}_T):k] = [\kappa(\mathfrak{P}_{1/T}):k] = 1$

4.2.2 Construction of ϕ_* and ϕ^*

In this section I define the functors $\phi_{\mathbf{PS}}$ and $\phi^{\mathbf{PS}}$ used in definition 2.3.1.1

Definiton 4.2.2.1. Let

$$\Phi : \mathbf{C}_1 \to \mathbf{C}_2$$

be a functor and denote by $\mathbf{PS}(\mathbf{C}_i)$ $(i = 1, 2)$ the category of presheaves on $\mathbf{C}_i$ with values in some category. Define the functors

$$\Phi^{\mathbf{PS}} : \mathbf{PS}(\mathbf{C}_2) \to \mathbf{PS}(\mathbf{C}_1) \qquad \text{and} \qquad \Phi_{\mathbf{PS}} : \mathbf{PS}(\mathbf{C}_1) \to \mathbf{PS}(\mathbf{C}_2)$$

by

$$\Phi^{\mathbf{PS}}(\mathscr{P}) : X \mapsto \mathscr{P}(\Phi(X)) \qquad \text{and} \qquad \Phi_{\mathbf{PS}}(\mathscr{P}) : X \longmapsto \varinjlim_{X' \text{ in } \mathbf{Under}_{\Phi}(X)^{\mathrm{opp}}} \mathscr{P}(X').$$

Here the category $\mathbf{Under}_{\Phi}(X)$ is the category whose objects are objects X' in $\mathbf{C_1}$ together with a structure morphism $X \to \Phi(X')$ in $\mathbf{C}_2$. Morphisms are defined in the obvious way, i.e.*so that everything commutes*.

Remark 4.2.2.1. There is a natural isomorphism

$$\mathrm{Hom}(\Phi_{\mathbf{PS}}(\mathscr{P}_1), \mathscr{P}_2) \simeq \mathrm{Hom}(\mathscr{P}_1, \Phi^{\mathbf{PS}}(\mathscr{P}_2)).$$

Definiton 4.2.2.2. Let $X \xrightarrow{\phi} Y$ be a morphism of schemes. Recall the definition of $\mathbf{OVer}_{\text{ét}}(\phi)$ (see corollary 2.1.1.1). Set the notation

$$\phi_{\mathbf{PS}} := \mathbf{Over}_{\text{ét}}(\phi)^{\mathbf{PS}} : \mathbf{PS}(X) \to \mathbf{PS}(Y), \qquad \phi^{\mathbf{PS}} := \mathbf{Over}_{\text{ét}}(\phi)_{\mathbf{PS}} : \mathbf{PS}(Y) \to \mathbf{PS}(X).$$

Explicitly for $E \to Y$ étale, one has

$$\phi_{\mathbf{PS}}(\mathscr{P}) : E \longmapsto \mathscr{P}(E \times_Y X)$$

while for $E \to X$ étale, one has

$$\phi^{\mathbf{PS}}(\mathscr{P}) : E \longmapsto \varinjlim_{U \text{ in } \mathbf{Under}_{\mathbf{Over}_{\text{ét}}(\phi)}(E)^{\mathrm{opp}}} \mathscr{P}(U') = \left(\coprod_{U \text{ in } \mathbf{Under}_{\mathbf{Over}_{\text{ét}}(\phi)}(E)^{\mathrm{opp}}} \mathscr{P}(U) \right) / \sim$$

where $\sim$ is the relation defined as: for a morphism $U \to V$ in $\mathbf{Under}_{\mathbf{Over}_{\text{ét}}(\phi)}(E)$, $f \in \mathscr{P}(U)$ and $g \in \mathscr{P}(V)$ one has

$$f \sim g$$

if and only if

$$\mathscr{P}(V) \to \mathscr{P}(U)$$

maps g to f.

index of categories

$$\textbf{Set} := \begin{cases} \bullet \text{ Objects of } \textbf{Set}\text{: Sets} \\ \bullet \text{ Morphisms of } \textbf{Set}\text{: Functions} \end{cases}$$

$$\textbf{Setf} := \begin{cases} \bullet \text{ Objects of } \textbf{Setf}\text{: Finite Sets} \\ \bullet \text{ Morphisms of } \textbf{Setf}\text{: Functions} \end{cases}$$

$$\textbf{Group} := \begin{cases} \bullet \text{ Objects of } \textbf{Group}\text{: Groups} \\ \bullet \text{ Morphisms of } \textbf{Group}\text{: Morphisms of groups} \end{cases}$$

$$\textbf{Ab} := \begin{cases} \bullet \text{ Objects of } \textbf{Ab}\text{: Abelian groups} \\ \bullet \text{ Morphisms of } \textbf{Ab}\text{: Morphisms of groups} \end{cases}$$

$$\textbf{Ring} := \begin{cases} \bullet \text{ Objects of } \textbf{Ring}\text{: Commutative rings with unity} \\ \bullet \text{ Morphisms of } \textbf{Ring}\text{: Morphisms of rings sending the unity to the unity} \end{cases}$$

$$\textbf{Field} := \begin{cases} \bullet \text{ Objects of } \textbf{Field}\text{: fields} \\ \bullet \text{ Morphisms of } \textbf{Field}\text{: morphisms of rings} \end{cases}$$

Let k be a field and let R be a ring.

$$\textbf{Vec}_k := \begin{cases} \bullet \text{ Objects of } \textbf{Vec}_k\text{: Vectors spaces over } k \\ \bullet \text{ Morphisms of } \textbf{Vec}_k\text{: } k\text{-linear maps} \end{cases}$$

$$\textbf{Mod}_R := \begin{cases} \bullet \text{ Objects of } \textbf{Mod}_R\text{: Modules over } R \\ \bullet \text{ Morphisms of } \textbf{Mod}_R\text{: } R\text{-linear maps} \end{cases}$$

$$\textbf{Alg}_R := \begin{cases} \bullet \text{ Objects of } \textbf{Alg}_R\text{: Morphism of rings of the form } R \to S \text{ (for some object } S \text{ in } \textbf{Ring}\text{). In such case we also say that } S \text{ is an object of } \textbf{Alg}_R \text{ with } \textit{structure morphism } R \to S. \\ \bullet \text{ Morphisms of } \textbf{Alg}_R\text{: morphisms of rings commuting with the structure morphism} \end{cases}$$

$$\textbf{Vecf}_k := \begin{cases} \bullet \text{ Objects of } \textbf{Vecf}_k\text{: finite dimensional vector spaces over } k \\ \bullet \text{ Morphisms of } \textbf{Vecf}_k\text{: morphisms of } k\text{-vector spaces} \end{cases}$$

$$\textbf{Modf}_R := \begin{cases} \bullet \text{ Objects of } \textbf{Modf}_R\text{: modules of finite type over } R \\ \bullet \text{ Morphisms of } \textbf{Modf}_R\text{: morphisms of } R\text{-modules} \end{cases}$$

$$\mathbf{Algf}_R := \begin{cases} \bullet \text{ Objects of } \mathbf{Algf}_R\text{: finitely generated } R\text{-algebras} \\ \bullet \text{ Morphisms of } \mathbf{Algf}_R\text{: morphims of algebras} \end{cases}$$

$$\mathbf{Sch} := \begin{cases} \bullet \text{ Objects of } \mathbf{Sch}\text{: schemes} \\ \bullet \text{ Morphisms of } \mathbf{Sch}\text{: morphisms of schemes} \end{cases}$$

$$\mathbf{AffSch} := \begin{cases} \bullet \text{ Objects of } \mathbf{AffSch}\text{: affine schemes} \\ \bullet \text{ Morphisms of } \mathbf{AffSch}\text{: morphisms of schemes} \end{cases}$$

Let X be a scheme.

$$\mathbf{Sh}_{\mathbf{R}}(X) := \begin{cases} \bullet \text{ Objects of } \mathbf{Sh}_{\mathbf{R}}(X)\text{: étale sheaves of } R\text{-modules on } X \text{ (see definition 2.2.1.1)} \\ \bullet \text{ Morphisms of } \mathbf{Sh}_{\mathbf{R}}(X)\text{: morphism of sheaves, i.e. morphisms of contravariant functors} \end{cases}$$

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