Problem 10. (§6.4, #30) If λ_{max} is the largest eigenvalue of a symmetric matrix A, no diagonal entry can be larger tha λ_{max}. What is the first entry a_{11} of $A = Q\Lambda Q^T$? Show why $a_{11} \le \lambda_{max}$.

Solution. **(12 points)** Set $\mathbf{e}_1 = (1, 0, 0, \ldots)^T$ and $\mathbf{v} = Q^{\mathrm{T}}\mathbf{e}_1 = (v_1, \ldots, v_n)$. Then,

$$a_{11} = \mathbf{e}_1^{\mathrm{T}} A \mathbf{e}_1 = \mathbf{e}_1^{\mathrm{T}} Q\Lambda Q^{\mathrm{T}} \mathbf{e}_1 = (Q^{\mathrm{T}}\mathbf{e}_1)^{\mathrm{T}} \Lambda (Q^{\mathrm{T}}\mathbf{e}_1) = \mathbf{v}^{\mathrm{T}}\Lambda\mathbf{v} = \sum_{i=1}^{n} \lambda_i v_i^2.$$

Since Q^{T} is orthogonal,

$$\|\mathbf{v}\| = \|Q^{\mathrm{T}}\mathbf{e}_1\| = \|\mathbf{e}_1\| = 1$$

and so

$$a_{11} \le \lambda_{max} \sum_{i=1}^{n} v_i^2 = \lambda_{max}\|\mathbf{v}\|^2 = \lambda_{max}.$$

When Q is square, $Q^{\mathrm{T}}Q = I$ means that $Q^{\mathrm{T}} = Q^{-1}$: transpose = inverse.

To repeat: $Q^{\mathrm{T}}Q = I$ even when Q is rectangular. In that case Q^{T} is only an inverse from the left. For square matrices we also have $QQ^{\mathrm{T}} = I$, so Q^{T} is the two-sided inverse of Q. The rows of a square Q are orthonormal like the columns. ***The inverse is the transpose***. In this square case we call Q an ***orthogonal matrix***.[1]

If Q has orthonormal columns ($Q^{\mathrm{T}}Q = I$), it leaves lengths unchanged:

Same length $\quad \|Qx\| = \|x\|$ **for every vector x.** (3)

Q also preserves dot products: $(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}}Q^{\mathrm{T}}Qy = x^{\mathrm{T}}y$. Just use $Q^{\mathrm{T}}Q = I$!

Proof $\|Qx\|^2$ equals $\|x\|^2$ because $(Qx)^{\mathrm{T}}(Qx) = x^{\mathrm{T}}Q^{\mathrm{T}}Qx = x^{\mathrm{T}}Ix = x^{\mathrm{T}}x$.
总是无法区分 Q is square 和 Q is rectangular