Towards derandomising Markov chain Monte Carlo

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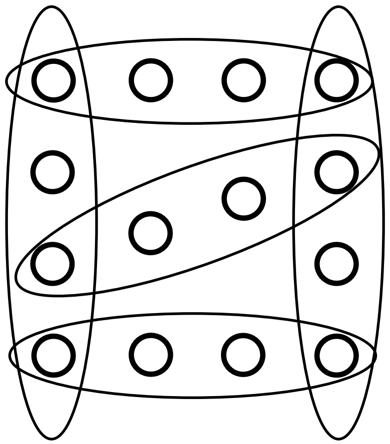
Feb 24th, 2023

Hypergraph independent set

Hypergraph $H = (V, \mathcal{E})$

- k-uniform: each hyperedge contains k vertices
- Δ-max degree: each vertex belongs to $\leq \Delta$ hyperedges



4-uniform with max degree 2

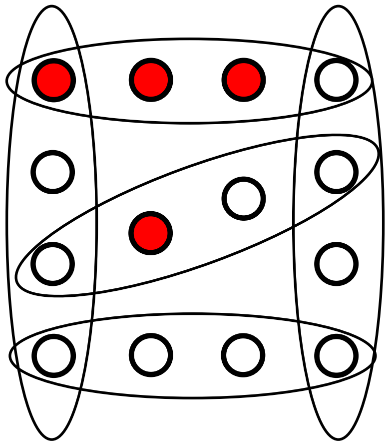
Independent set $S \subseteq V$ in hypergraph H

- For any $e \in \mathcal{E}$, $e \nsubseteq S$

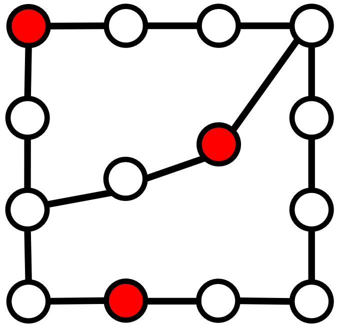
equivalent definition $v \in S$ iff $\sigma(v) = 1$

Independent set $\sigma \in \{0,1\}^V$ in hypergraph H

- For any $e \in \mathcal{E}$, there exists $v \in e$ such that $\sigma(v) = 0$



Example: an independent set in 4-unifrom hypergraph



Example: an independent set in graph ($k = 2$)

Counting independent sets

Input: a hypergraph $H = (V, \mathcal{E})$

- k uniform and max degree Δ $(k, \Delta = O(1))$
- number of vertices n

Output: the total number of independent sets in H

Hardness of exact counting

Counting independent sets is $\#P$ complete

- Hardness result holds even if $k = 2$ and $\Delta = 3$ [Greenhill 2000]

Approximate counting

Approximate counting problem

Input: a hypergraph $H = (V, \mathcal{E})$

- k uniform and max degree Δ ($k, \Delta = O(1)$)
- number of vertices n

an error bound $0 < \epsilon < 1$

Output: a number $\hat{Z}$ such that

$$(1-\epsilon)Z \leq \hat{Z} \leq (1+\epsilon)Z$$

Z: the total number of independent sets in H

Approximate counting algorithms

FPTAS (fully polynomial time approximation scheme)

Deterministic algorithm that solves the problem in time $\mathrm{poly}\left(n, \frac{1}{\epsilon}\right)$

FPRAS (fully polynomial time randomised approximation scheme)

Randomised algorithm that ***solves*** the problem in time $\mathrm{poly}\left(n, \frac{1}{\epsilon}\right)$

output a random number $\boldsymbol{\hat{Z}}$ such that

$$\Pr\left[(1-\epsilon)Z \le \hat{Z} \le (1+\epsilon)Z\right] \ge \frac{2}{3}$$

Graph case ($k = 2$): computational phase transitions

Algorithm	Complexity
FPTAS in time $\left(\frac{n}{\epsilon}\right)^{O(\log \Delta)}$ [Weitz05]	The approximate counting is
FPRAS in time $\tilde{O}\left(\frac{n^2}{\epsilon^2}\right)$ [CLV21, ŠVV07]	**NP-Hard**
$\Delta \leq 5$	$\Delta \geq 6$

General hypergraphs ($k > 2$)

Work	Regime	Time
[BGGGŠ 16]	$k \geq \Delta \geq 200$	$\left(\frac{n}{\epsilon}\right)^{O(\log(k\Delta))}$
[Moitra 19]	$k \gtrsim 60 \log \Delta$	$\left(\frac{n}{\epsilon}\right)^{\mathrm{poly}(k\Delta)}$
[JPV 21]	$k \gtrsim 7 \log \Delta$	
[HWY 22]	$k \gtrsim 5 \log \Delta$	

FPTAS (deterministic algorithm)

Work	Regime	Time
[BDK06]	$k \geq \Delta + 2$	$\tilde{O}\left(\frac{n^2}{\epsilon^2}\right)$
[BDK08]		
[HSZ19]	$k \gtrsim 2 \log \Delta$	
[QWZ22]		

FPRAS (randomised algorithm)

Hardness: the approximate counting is **NP-Hard** if $k \leq 2 \log \Delta - C$ [BGGGŠ 16]

Our result [F., Guo, Wang, Wang, Yin, 22]: there is a **FPTAS** if $k \gtrsim 2 \log \Delta$

Theorem [this work] Let $k \geq 2$ and $\Delta \geq 2$ be two constants s.t.

$$k \geq 2\log\Delta + 4\log k + O(1).$$

There is a ***deterministic algorithm*** such that

- **Input:** a k-uniform hypergraph with n vertices and max degree Δ, an error bound ϵ
- **Output:** an $(1 \pm \epsilon)$-approximation to the *number of independent sets*
- **Running time:** $(n/\epsilon)^{O(\Delta^2 k^4)}$

Hardness: NP-Hard if $k \leq 2\log\Delta - C$ [BGGGŠ 16]

Linear hypergraph: two hyperedges share at most 1 vertex

Better condition: Let $\delta > 0$ be a constant. Constants $k \geq \frac{25(1+\delta)^2}{\delta^2}$ and $\Delta \geq 2$ satisfy

$$k \geq (1+\delta)\log\Delta + 3(1+\delta)\log k + O(1)$$

Running time: $(n/\epsilon)^{\mathrm{poly}(\Delta k/\delta)}$

Hardness for linear hypergraph: NP-Hard if $k \leq \log\Delta - C$ [QW 22]

Work	Regime
[JPV 21]	$k \gtrsim 7 \log \Delta$
[HWY 22]	$k \gtrsim 5 \log \Delta$

FPTAS (deterministic algorithm)

Work	Regime
[HSZ 21]	$k \gtrsim 2 \log \Delta$
[QWZ 22]	

FPRAS (randomised algorithm)

Q: Why there is a **gap** between the regimes for FPTAS and FPRAS ?

A: Previous FPTASes and FPRASes are based on very **different** techniques.

Techniques for FPTAS:

- Dynamic programming on computation tree [BGGGŠ 16]
- Linear programming [Moitra 19] [JPV 21]
- Derandomisation of a **marginal recursive sampler** [HWY 22, AJ 21]

Techniques for FPRAS:

- **MCMC sampling algorithm** & **reduction from counting to sampling** [HSZ 22, QWZ 22]

The sampling problem

- **Input:** a hypergraph $H = (V, \mathcal{E})$

$$\Omega = \{X \in \{0,1\}^V \mid X \text{ } is \text{ an independent set in } H\}$$

μ: the ***uniform distribution*** over Ω

$$\forall X \in \Omega, \qquad \mu(X) = \frac{1}{Z} = \frac{1}{|\Omega|}$$

- **Output:** a random sample $X \sim \mu$.

The approximate sampling problem

- **Input:** a hypergraph $H = (V, \mathcal{E})$ specifying the uniform distribution μ

an error bound $\epsilon > 0$

- **Output:** a random sample $X \in \{0,1\}^V$ such that

total variation distance $d_{TV}(X, \mu) \leq \epsilon$

Counting-to-sampling reduction

Fix the independent set $\emptyset$, denote it by $\mathbf{0}$

$$\mu(\mathbf{0}) = \frac{1}{Z}$$

approximate Z $\Longleftrightarrow$ approximate $\mu(\mathbf{0})$

Fix an arbitrary ordering of vertices $V = \{v(1), v(2), \ldots, v(n)\}$. By ***chain rule***

$$\mu(\mathbf{0}) = \mu_{v(1)}(0) \times \mu_{v(2)}(\,0 \mid v(1) = 0\,) \times \cdots \times \mu_{v(n)}(\,0 \mid \forall j < n, v(j) = 0\,)$$

$$= \prod_{i=1}^{n} \mu_{v(i)}(0 \mid \forall j < i, v(j) = 0)$$

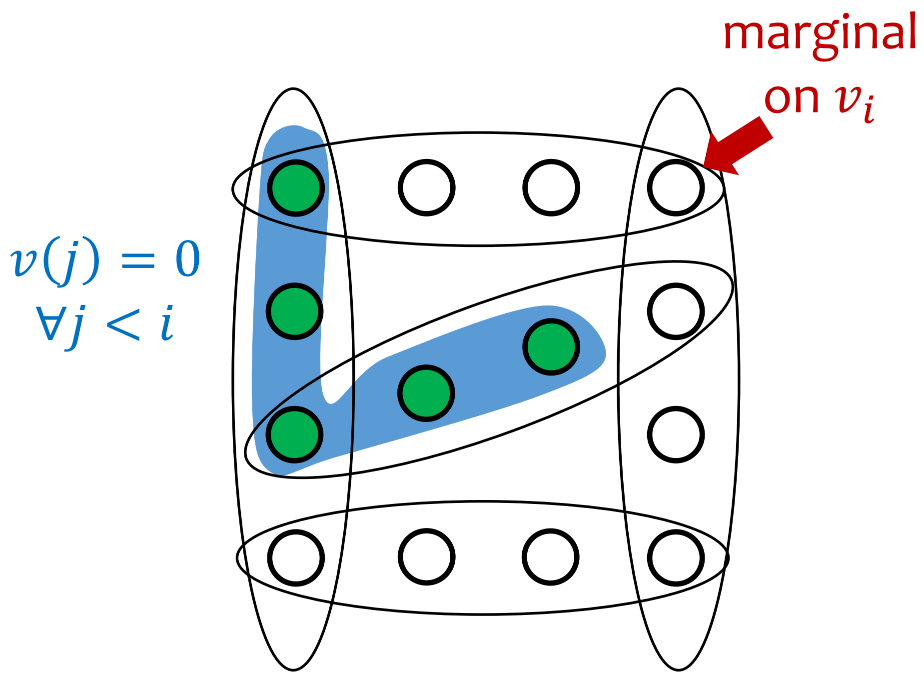
Conditional Marginal distribution $\boldsymbol{\mu_{v(i)}(0 \mid \forall j < i, v(j) = 0)}$

given $X \sim \mu$, conditional on $X_{v(j)} = 0$ for all $j < i$, the prob of $X_{v(i)} = 0$

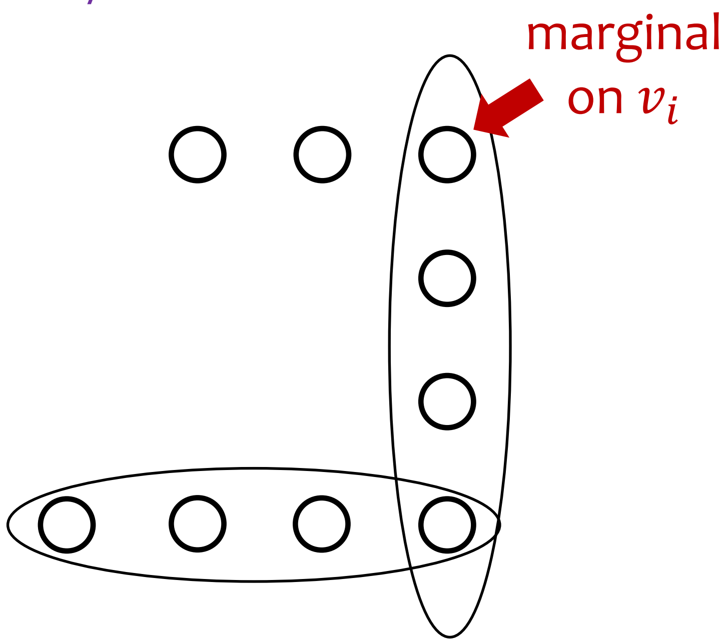
Counting-to-sampling reduction

approximate Z $\Leftrightarrow$ approximate $\mu(\mathbf{0})$ $\Rightarrow$ approximate $\mu_{v(i)}(\,0 \mid \forall j < i, v(j) = 0\,)$

with error ϵ/n



Self-reduction
[JVV86]



Conditional marginal distribution
$\mu_{v(i)}(\,0 \mid \forall j < i, v(j) = 0\,)$

Marginal distribution $\pi_{v(i)}$
π: uniform distribution in another hypergraph

Counting-to-sampling reduction

approximate Z $\Leftrightarrow$ approximate $\mu(\mathbf{0})$ $\Rightarrow$ approximate $\mu_v(0)$ with error ϵ/n

Solve approximate counting problem via sampling algorithm

The sampling algorithm $\mathcal{S}(H, \epsilon)$

- Input: a hypergraph H and error bound ϵ
- Output: a random sample X satisfying $d_{TV}(X, \mu) \leq \epsilon$

The algorithm for estimating $\mu_v(\mathbf{0})$

- Run $\mathcal{S}\left(H, \frac{\epsilon}{2n}\right)$ independently $N = \text{poly}\left(\frac{n}{\epsilon}\right)$ times to get $X^{(1)}, X^{(2)}, \ldots, X^{(N)}$
- Compute the value $\hat{m} = \frac{\text{number of } i \text{ with } X_v^{(i)}=0}{N}$

MCMC for hypergraph independent sets

Systematic scan for hypergraph independent sets

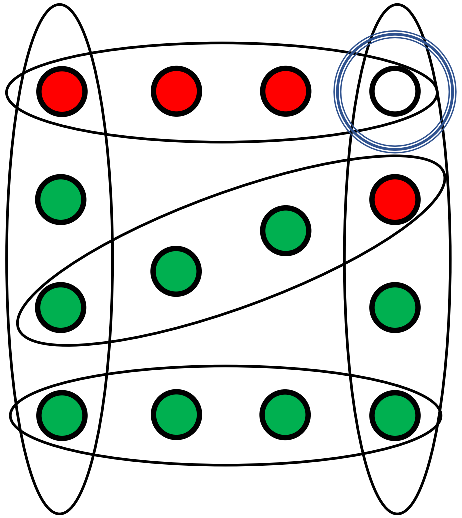
Input: a hypergraph $H = (V, E)$, each $v \in V$ has a unique label in $\{0,1,2,\ldots,n-1\}$

Start from an arbitrary independent set $X \in \{0,1\}^V$

For each t from 1 to T

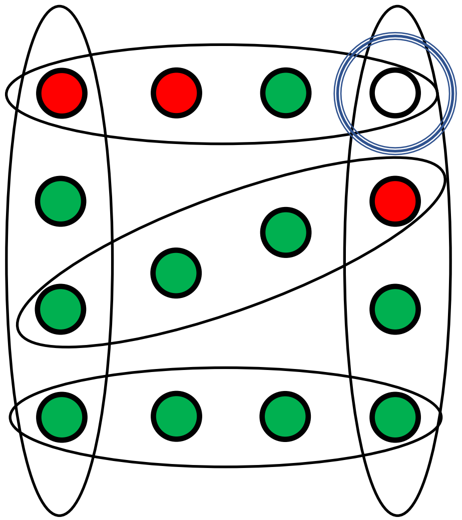
- Pick the vertex $v \in V$ with label$(v) = t \bmod n$
- Update $X(v) \sim \mu_v(\cdot \mid X(V \backslash \{v\}))$

Output X



update

$X_v \leftarrow 0$



update

$X_v \leftarrow r$

$r \in \{0,1\}$ is a uniform random bit

MCMC for hypergraph independent sets

Systematic scan for hypergraph independent sets

Input: a hypergraph $H = (V, E)$, each $v \in V$ has a unique label in $\{0,1,2,\ldots,n-1\}$

Start from an arbitrary independent set $X \in \{0,1\}^V$

For each t from 1 to T

- Pick the vertex $v \in V$ with label $t \bmod n$
- Update $X(v) \sim \mu_v(\cdot \mid X(V \setminus \{v\}))$

Output X

Mixing time of systematic scan [HSZ16, JPV21, HSW21]

Systematic scan $(X_t)_{t=0}^{T}$: $X_t \in \{0,1\}$ random independent set after the t-th update

$k \gtrsim 2 \log \Delta$ $\Rightarrow$ $d_{TV}(X_T, \mu) \le \mathrm{poly}\left(\frac{\epsilon}{n}\right)$, where $T = O\left(n \log \frac{n}{\epsilon}\right)$

Approximate counting via MCMC algorithm

approximate Z ⟷ approximate $\mu(\mathbf{0})$ → approximate $\mu_v(0)$ with error ϵ/n

solve the problem

Run $O\left(n\log\frac{n}{\epsilon}\right)$-step **systematic scan** to generate $N = \mathrm{poly}\left(\frac{n}{\epsilon}\right)$ independent samples

$$X^{(1)}, X^{(2)}, \dots, X^{(N)} \in \{0,1\}^V$$

Full configuration $X \in \{0,1\}^V$

Compute the fraction

$$\hat{m} = \frac{\text{number of } i \text{ with } X_v^{(i)} = 0}{N}$$

Only use one bit $X_v \in \{0,1\}$

Previous results [HSZ16, JPV21, HSW21]: there is a **FPRAS** if $k \gtrsim 2\log\Delta$

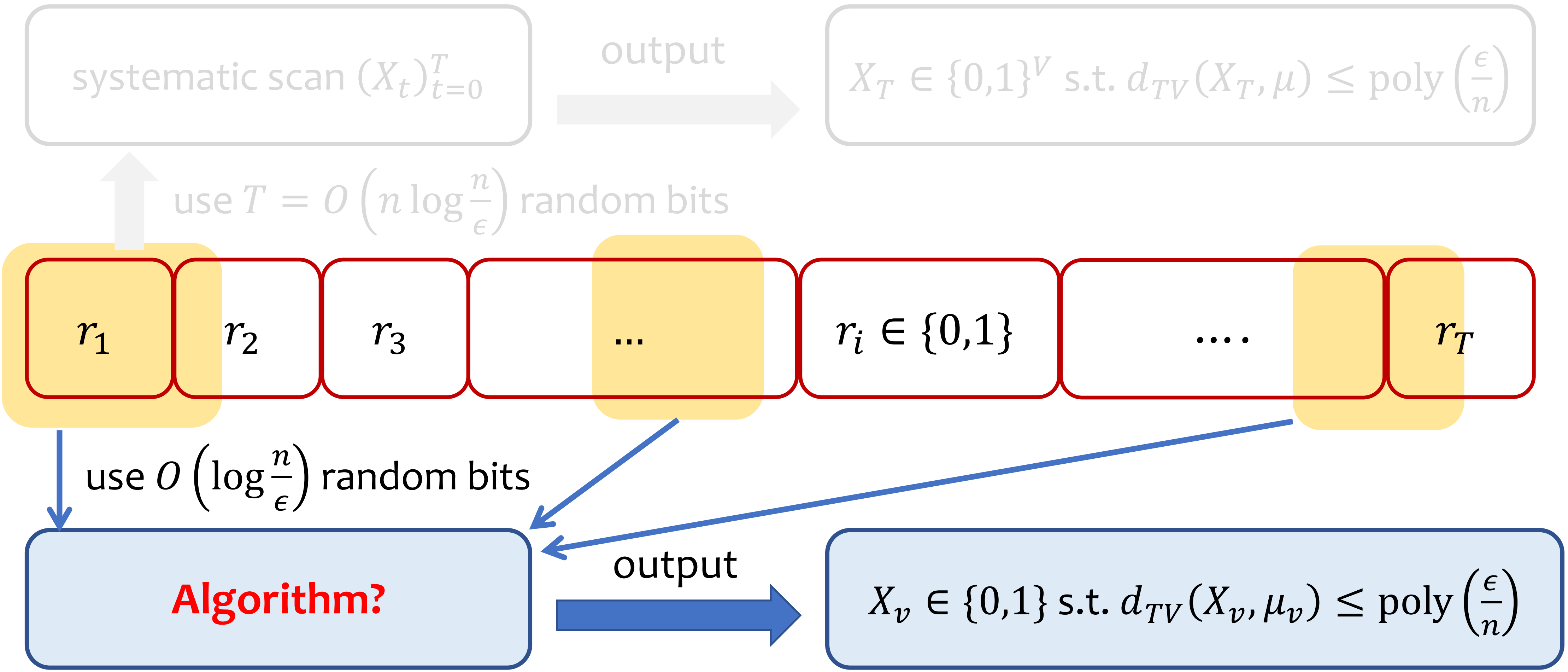
Our idea: Derandomising MCMC

systematic scan $(X_t)_{t=0}^{T}$ $\xrightarrow{\text{output}}$ $X_T \in \{0,1\}^V$ s.t. $d_{TV}(X_T, \mu) \leq \mathrm{poly}\left(\frac{\epsilon}{n}\right)$

use $T = O\left(n \log \frac{n}{\epsilon}\right)$ random bits

r_1	r_2	r_3	…	$r_i \in \{0,1\}$	….	r_T

Our idea: Derandomising MCMC



Enumerate ***all possible choices*** of $O\left(\log \frac{n}{\epsilon}\right)$ random bits $\Rightarrow$ ***Deterministically*** approximate $\mu_v(0)$ with error $\mathrm{poly}\left(\frac{\epsilon}{n}\right)$

Our results: log-time sampling via MCMC

Theorem [this work] Let constants $k \geq 2, \Delta \geq 2$ satisfying $k \gtrsim 2 \log \Delta$

There is a sampling algorithm such that

Input: a k-uniform *hypergraph* max degree Δ, a *vertex* v, an *error bound* ϵ

Output: a random sample $X_v \in \{0,1\}$ with

$$d_{TV}(X_v, \mu_v) \leq \epsilon$$

Running time & number of random bits used by alg.: $\mathrm{poly}(\Delta k) \log \frac{n}{\epsilon}$

Straightforward derandomisation

- The algorithm uses $\mathrm{poly}(\Delta k) \log \frac{n}{\epsilon}$ random bits
- Enumerate all $\left(\frac{n}{\epsilon}\right)^{\mathrm{poly}(\Delta k)}$ possible assignments for random bits
- Deterministically compute $\Pr[X_v = 0] \in (1 \pm 2\epsilon)\mu_v(0)$ (as $\mu_v(0) \geq 1/2$)

Our results: log-time sampling via MCMC

Theorem [this work] Let constants $k \geq 2, \Delta \geq 2$ satisfying $k \gtrsim 2 \log \Delta$

There is a sampling algorithm such that

Input: a k-uniform *hypergraph* max degree Δ, a *vertex* v, an *error bound* ϵ

Output: a random sample $X_v \in \{0,1\}$ with

$$d_{TV}(X_v, \mu_v) \leq \epsilon$$

Running time & number of random bits used by alg.: $\mathrm{poly}(\Delta k) \log \frac{n}{\epsilon}$

***Result on linear* hypergraphs** [this work]

- Let constants $k \geq 2, \Delta \geq 2, \delta > 0$ satisfying $k \gtrsim (1+\delta) \log \Delta$ and $k \geq k_0(\delta)$
- Running time & number of random bits used by alg.

$$\mathrm{poly}\left(\frac{\Delta k}{\delta}\right) \log \frac{n}{\epsilon}$$

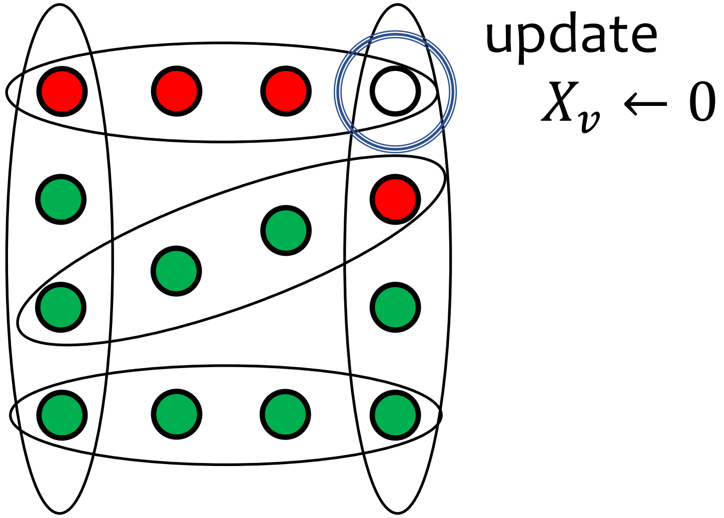
Systematic scan for hypergraph independent sets

Start from an arbitrary independent set $X \in \{0,1\}^V$

For each t from 1 to T

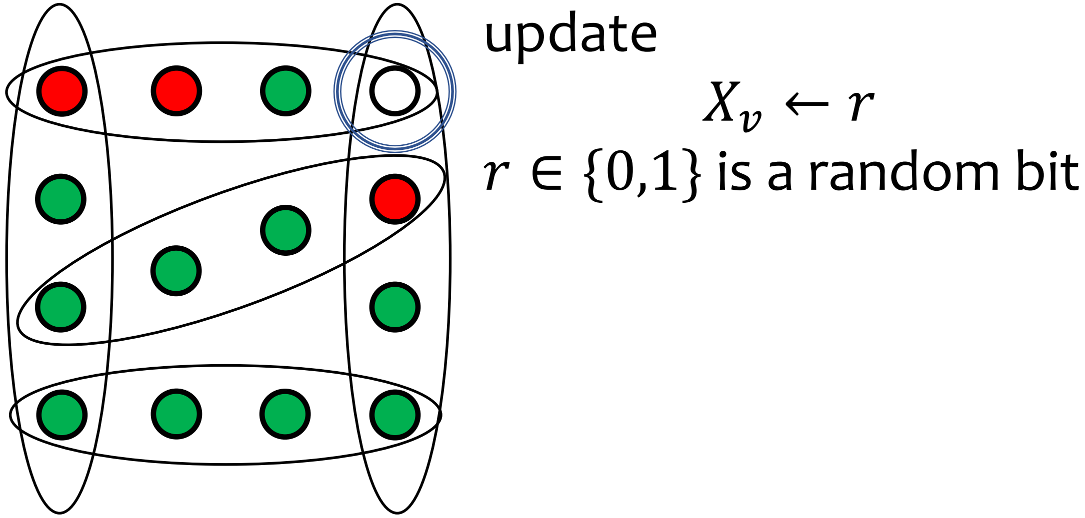
- Pick the vertex $v \in V$ with label $t \bmod n$
- Update $X(v) \sim \mu_v(\cdot \mid X(V \setminus \{v\}))$

Output X



Blocked: X_v is updated to 0

$\exists e \ni v$ s.t. $\forall u \in e \setminus \{v\}, X_u = 1$



Unblocked: X_v is updated to 0 w.p. $1/2$

$\forall e \ni v, \exists u \in e \setminus \{v\}$ s.t. $X_u = 0$

Systematic scan for hypergraph independent sets

Start from an arbitrary independent set $X \in \{0,1\}^V$

For each t from 1 to T

- Pick the vertex $v \in V$ with label $t \bmod n$
- Update $X(v) \sim \mu_v(\cdot \mid X(V \setminus \{v\}))$

Output X

The t-th transition step of systematic scan

Sample a random bit $r_t \in \{0,1\}$ uniformly at random;

If X is in the ***blocked*** case ($\forall e$ with $v \in e$, $\exists u \in e \setminus \{v\}$ s.t. $X_u = 0$)

$X_v \leftarrow 0$;

If X is in the ***unblocked*** case ($\exists e$ with $v \in e$, $\forall u \in e \setminus \{v\}$ s.t. $X_u = 1$)

$X_v \leftarrow r_t$

$$\boxed{r_t = 0} \xrightarrow{\text{in both cases}} \boxed{X_v \leftarrow 0}$$

Systematic scan for hypergraph independent sets

Start from an arbitrary independent set $X \in \{0,1\}^V$

For each t from 1 to T

- Pick the vertex $v \in V$ with label $t \bmod n$
- Update $X(v) \sim \mu_v(\cdot \mid X(V \setminus \{v\}))$

Output X

The *t*-th transition step of systematic scan

Sample a random bit $r_t \in \{0,1\}$ uniformly at random;

If X is in the ***blocked*** case ($\forall e$ with $v \in e$, $\exists u \in e \setminus \{v\}$ s.t. $X_u = 0$)

$X_v \leftarrow 0;$

If X is in the ***unblocked*** case ($\exists e$ with $v \in e$, $\forall u \in e \setminus \{v\}$ s.t. $X_u = 1$)

$X_v \leftarrow r_{\mathrm{t}}$

If $r_t = 0$ (with probability 1/2)

decide X_v ***immediately*** (*no need* to distinguish blocked or unblocked cases) systematic scan $(X_t)_{t=0}^{T}$ $\xrightarrow{\text{our goal}}$ output $X_T(v) \in \{0,1\}$

- For any vertex $v \in V$, any time $0 \le t \le T$,

$$S(v,t) = \{1 \le j \le t \mid \text{vertex } v \text{ is picked in } j^{th} \text{ step, i.e. label}(v) = j \bmod n\}$$

- ***Previous*** update time for $v \in V$ ***up to time*** t:

$$\mathrm{Pred}(v,t) = \begin{cases} \max\{j \mid j \in S(v,t)\} & \text{if } S(v,t) \neq \emptyset \\ 0 & \text{if } S(v,t) = \emptyset \end{cases}$$

Our goal: output the value of $X_T(v) = X_{\mathrm{Pred}(v,T)}(v) \in \{0,1\}$

compute the value of v after the last time that v is updated

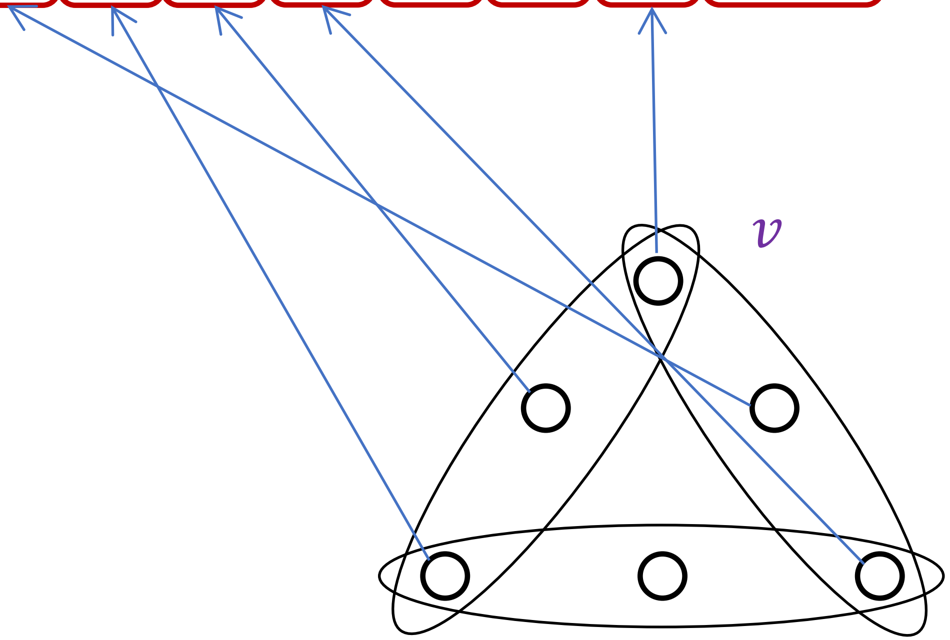
Resolve(v, t)

- **Input:** $v \in V$ and $1 \le t \le T$ such that v is picked at the time t
- **Output:** the random value $X_t(v) \in \{0,1\}$

Random bits for simulating Markov chain up to time *T*

$t = 19$

r_1	r_2	r_3	r_4	r_5	r_5	r_6	r_8	r_9	r_{10}	r_{11}	r_{12}	r_{13}	r_{14}	r_{15}	r_{16}	r_{17}	r_{18}	r_{19}	r_{20}



reveal the random bit r_t;

if $r_t = 0$, **return** 0;

else //distinguish blocked or unblocked cases

reveal $r_{\mathrm{pred}(w,t)}$ for ***all neighbours*** w;

Resolve(v, t)

- **Input:** $v \in V$ and $1 \le t \le T$ such that v is picked at the time t
- **Output:** the random value $X_t(v) \in \{0,1\}$

Random bits for simulating Markov chain up to time *T*

r_1	r_2	r_3	r_4	r_5	r_5	r_6	r_8	r_9	r_{10}	r_{11}	r_{12}	r_{13}	r_{14}	r_{15}	r_{16}	r_{17}	r_{18}	r_{19}	$r_{T=20}$

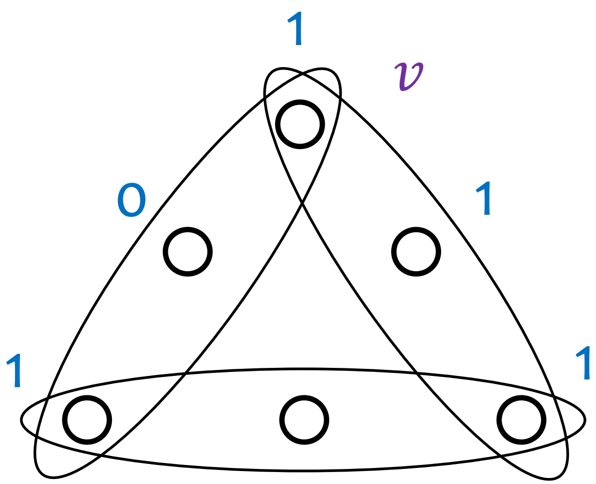
reveal the random bit r_t;

if $r_t = 0$, **return** 0;

else //distinguish blocked or unblocked cases

reveal $r_{\mathrm{pred}(w,t)}$ for ***all neighbours*** w;

for each hyperedge e incident to v **do**



Resolve(v, t)

- **Input:** $v \in V$ and $1 \leq t \leq T$ such that v is picked at the time t
- **Output:** the random value $X_t(v) \in \{0,1\}$

Random bits for simulating Markov chain up to time *T*

r_1	r_2	r_3	r_4	r_5	r_5	r_6	r_8	r_9	r_{10}	r_{11}	r_{12}	r_{13}	r_{14}	r_{15}	r_{16}	r_{17}	r_{18}	r_{19}	$r_{T=20}$

reveal the random bit r_t;

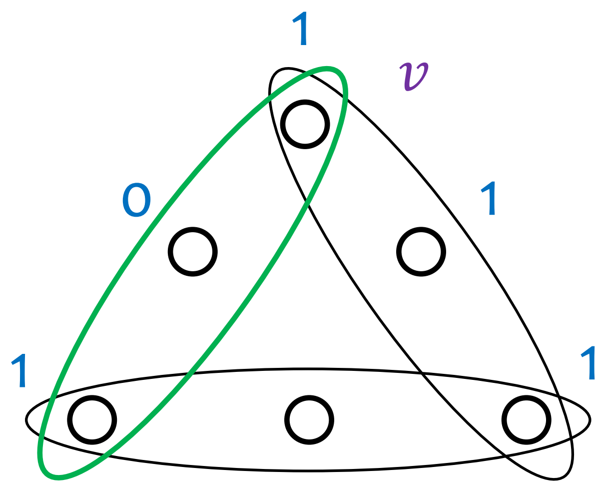
if $r_t = 0$, **return** 0;

else //distinguish blocked or unblocked cases

reveal $r_{\text{pred}(w,t)}$ for ***all neighbours*** w;

for each hyperedge e incident to v **do**

$\begin{cases} \exists w \in e\backslash\{v\} \text{ s.t. } r_{\text{pred}(w,t)} = 0 \Longrightarrow X_t(w) = 0 \quad \textbf{good edge} \\ \end{cases}$



Resolve(v, t)

- **Input:** $v \in V$ and $1 \le t \le T$ such that v is picked at the time t
- **Output:** the random value $X_t(v) \in \{0,1\}$

Random bits for simulating Markov chain up to time *T*

r_1	r_2	r_3	r_4	r_5	r_5	r_6	r_8	r_9	r_{10}	r_{11}	r_{12}	r_{13}	r_{14}	r_{15}	r_{16}	r_{17}	r_{18}	r_{19}	$r_{T=20}$

reveal the random bit r_t;

if $r_t = 0$, **return** 0;

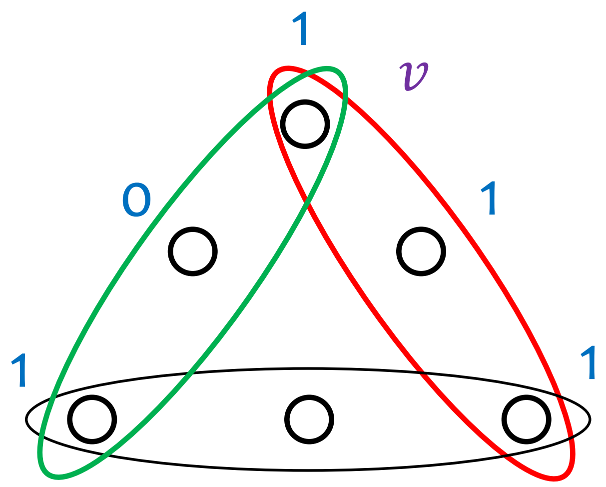
else //distinguish blocked or unblocked cases

reveal $r_{\mathrm{pred}(w,t)}$ for ***all neighbours*** w;

for each hyperedge e incident to v **do**

$\exists w \in e \backslash \{v\}$ s.t. $r_{\mathrm{pred}(w,t)} = 0 \Longrightarrow X_t(w) = 0$ **good edge**

$\forall w \in e \backslash \{v\}, r_{\mathrm{pred}(w,t)} = 1$ **bad edge (cannot decide $X_t(w)$)**



Resolve(v, t)

- **Input:** $v \in V$ and $1 \le t \le T$ such that v is picked at the time t
- **Output:** the random value $X_t(v) \in \{0,1\}$

Random bits for simulating Markov chain up to time *T*

r_1	r_2	r_3	r_4	r_5	r_5	r_6	r_8	r_9	r_{10}	r_{11}	r_{12}	r_{13}	r_{14}	r_{15}	r_{16}	r_{17}	r_{18}	r_{19}	$r_{T=20}$

reveal random bits towards the past

reveal the random bit r_t;

if $r_t = 0$, **return** 0;

else //distinguish blocked or unblocked cases

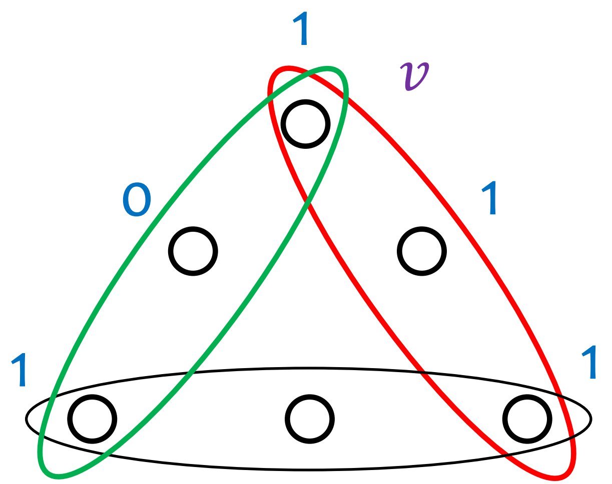
reveal $r_{\text{pred}(w,t)}$ for ***all neighbours*** w;

for each hyperedge e incident to v **do**

- **if** $\forall w \in e \backslash \{v\}, r_{\text{pred}(w,t)} = 1$ //bad edge
  - **If all $w \in e\backslash\{v\}$, $\mathbf{Resolve}(w, \mathbf{pred}(w, t)) = 1$**

    return 0; //blocked case

return 1; //unlocked case



reveal the random bit r_t;

if $r_t = 0$, **return** 0;

else //distinguish blocked or unblocked cases

reveal $r_{\mathrm{pred}(w,t)}$ for ***all neighbours*** w;

for each hyperedge e incident to v **do**

- **if** $\forall w \in e \backslash \{v\}, r_{\mathrm{pred}(w,t)} = 1$ //bad edge
    - If all $w \in e \backslash \{v\}, \mathrm{Resolve}(w, \mathrm{pred}(w,t)) = 1$

      return 0; //blocked case

return 1; //unlocked case

Run Resolve recursively ***only if***

$\mathcal{B}$: $\forall w \in e \backslash \{v\}, r_{\mathrm{pred}(w,t)} = 1$

$$\Pr[\mathcal{B}] = \left(\frac{1}{2}\right)^{k-1}$$

An informal analysis of the branching process

- Vertex v has $\leq \Delta$ incident hyperedges and each hyperedge has k vertices

$$\mathbb{E}[\#\text{recursive calls}] \leq \Delta k \left(\frac{1}{2}\right)^{k-1} < 1 \implies k \gtrsim \log \Delta$$

- However, hyperedges share vertices $\Longrightarrow$ ***dependency*** of recursive calls

$k \gtrsim 2 \log \Delta$ $\rightarrow$ w.h.p. $\#\{\text{resolve instances}\} = O_{\Delta,k}(\log n)$

pay extra factor to overcome the dependency reveal the random bit r_t;

if $r_t = 0$, **return** 0;

else //distinguish blocked or unblocked cases

reveal $r_{\mathrm{pred}(w,t)}$ for ***all neighbours*** w;

for each hyperedge e incident to v **do**

- **if** $\forall w \in e \backslash \{v\}, r_{\mathrm{pred}(w,t)} = 1$ //bad edge
  - If all $w \in e \backslash \{v\}, \mathrm{Resolve}(w, \mathrm{pred}(w,t)) = 1$

    return 0; //blocked case

return 1; //unlocked case

Run Resolve recursively ***only if***

$\mathcal{B}$: $\forall w \in e \backslash \{v\}, r_{\mathrm{pred}(w,t)} = 1$

$$\Pr[\mathcal{B}] = \left(\frac{1}{2}\right)^{k-1}$$

Better bound for linear hypergraphs (informal analysis)

- Linear hypergraph: two hyperedges share at most 1 vertex

$\Longrightarrow$ ***dependency*** of recursive calls is ***much weaker*** than the general case

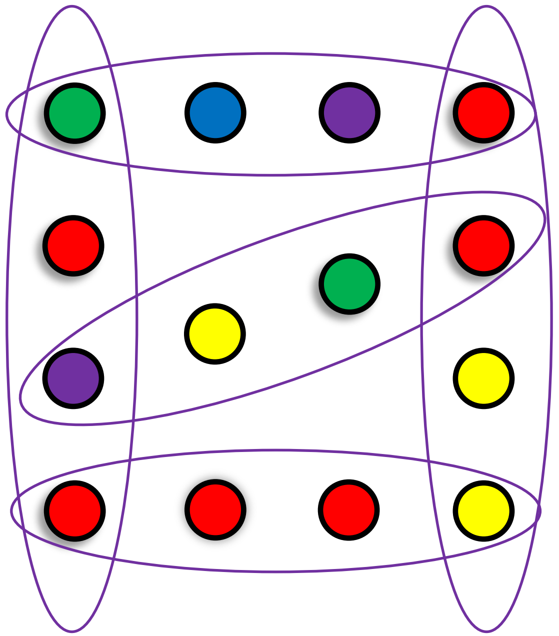
$k \gtrsim (1+\delta)\log\Delta$ ➡ w.h.p. #{resolve instances}$= O_{\delta,\Delta,k}(\log n)$

pay extra δ-factor to overcome the dependency, where $\delta > 0$ is an ***arbitrary*** constant

Hypergraph colouring

Instance

- a k-uniform hypergraph with max degree Δ
- colour set $[q] = \{1,2,\ldots,q\}$



Hypergraph colouring: $X \in [q]^V$ s.t.

- no hyperedge is monochromatic

  $(\forall e \in \mathcal{E}, \ \ |\{X_v \mid v \in e\}| \geq 2)$

Lovász Local lemma and algorithmic LLL

- find a hypergraph colouring when $q \gtrsim \Delta^{1/k}$ $(q \geq C_k \Delta^{1/(k-1)})$

Sampling Lovász Local lemma

- ***sampling*** / ***approx. counting*** hypergraph colourings in the local lemma regime

Previous results for approximate counting

Work	Regime	Algorithm Type
[Frieze, Anastos, 17]	Linear & $q \gtrsim \max\{\log n, \Delta^{1/k}\}$	Randomised
[Guo, Liao, Lu, Zhang, 19]	$q \gtrsim \Delta^{16/k}$	Deterministic
[Jain, Pham, Vuong, 21]	$q \gtrsim \Delta^{7/k}$	

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[Feng, He, Yin, 21]	$q \gtrsim \Delta^{9/k}$	Randomised
[Jain, Pham, Vuong, 21]	$q \gtrsim \Delta^{3/k}$	
[Feng, Guo, Wang, 22]	Linear & $q \gtrsim \Delta^{(2+\delta)/k}$	

MCMC on projected distribution

Lower Bound [Galanis, Guo, Wang, 21]

- NP-Hard for $q \lesssim \Delta^{2/k}$ (general hypergraph) and $q \lesssim \Delta^{1/k}$ (linear hypergraph)

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Counting-to-sampling reduction

Instance: hypergraph $H = (V, \mathcal{E})$ and colour set $[q]$

Colouring: $\Omega \subseteq [q]^V$ set of all proper colourings

$$Z = |\Omega|$$

Distribution: μ the uniform distribution over all proper colourings

$$\forall X \in \Omega, \qquad \mu(X) = \frac{1}{Z}$$

Approximate counting to sampling reduction

approximate Z $\Rightarrow$ approximate μ_e within TV distance poly $\left(\frac{\epsilon}{\Delta n}\right)$

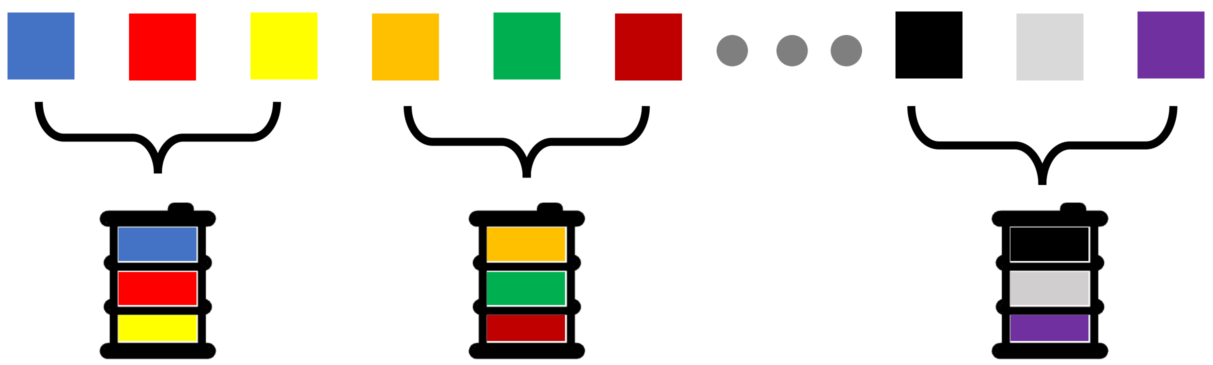
$e \in \mathcal{E}$ is a hyperedge

μ_e over $[q]^e$: Marginal distribution on k vertices

MCMC on projected distribution

- The systematic scan on μ **does not work** (connectivity issue)
- Use systematic scan on ***projected distribution***

the set of q colours



Bucket #1 Bucket #2 Bucket # s

the set of s buckets, $s = q^c$ for $c < 1$

Balanced projection scheme

$$h: [q] \to [s]$$

for any $j \in [s]$,

$$|h^{-1}(j)| \in \frac{q}{s} \pm 1$$

Projected distribution π over $[s]^V$ [Feng, He, Yin, 21]

$$Y = \{h(X_v)\}_{v \in V} \sim \pi \text{ if } X \sim \mu$$

Systematic scan for projected distribution

Start from a uniform random $Y \in [s]^V$;

For each t from 1 to T

- Pick the vertex $v \in V$ with label $t \bmod n$
- Update $Y(v) \sim \pi_v(\cdot \mid Y(V \setminus \{v\}))$ conditional marginal distribution induced by π

Output Y

Local uniformity in the local lemma regime

For any $v \in V$, any condition $\sigma \in [q]^{V \setminus \{v\}}$

$$\forall j \in [s], \qquad \mu_v(j \mid \sigma) \in \left(1 \pm \frac{1}{s}\right)\frac{|h^{-1}(j)|}{q} \approx \left(1 \pm \frac{1}{s}\right)\frac{1}{s}$$

given any condition, the marginal on v is close to a uniform distribution

⬇

Mixing in the local lemma regime: If $T = O\left(n \log \frac{n}{\epsilon}\right)$ **then** $d_{TV}(\pi, X_T) \leq \frac{\epsilon}{n^2}$

Log-time sampling (informal)

$$\mathrm{Resolve}(v, t)$$

- **Input:** $v \in V$ and $1 \le t \le T$ such that v is picked at the time t
- **Output:** the random value $Y_t(v) \in [s]$

Local uniformity $\Longrightarrow$ Marginal lower bound

For any $v \in V$, any condition $\sigma \in [q]^{V \setminus \{v\}}$

$$\forall j \in [s], \qquad \mu_v(j \mid \sigma) \gtrsim \left(1 - \frac{1}{s}\right)\frac{1}{s}$$

By **marginal lower bound**, even if $Y_t(V \setminus \{v\})$ is **unknown**, we can decide $Y_t(v)$ w.p.

$$p_{LB} \approx \sum_{j \in [s]} \left(1 - \frac{1}{s}\right)\frac{1}{s} = 1 - \frac{1}{s} \approx 1 - \Delta^{-\Omega(1/k)}$$

Resolve(v, t) (informal description)

- Reveal the randomness used in t-th step
- **With probability** $p_{LB} = 1 - 1/s$
  - Determine the value of $X_t(v)$ and **return.**
- **With probability** $1 - p_{LB}$ get ***enough information*** to determine $\mu_v(\cdot \mid Y_t(V \backslash \{v\}))$
  - Reveal other randomness & call resolve recursively if necessary

How to sample from μ_e using a partial sample from projected distribution π ?

sample Y_M for some $M \subseteq V$ → sample X_e conditional on Y_M

How to sample Y_M?

Resolve(v, t) returns $X_t(v) \in [s]$ $\xrightarrow{\textbf{\textit{Generalise}}}$ Resolve(M, t) returns $X_t(M) \in [s]^M$

Summary

Fully Poly-time ***deterministic*** approximate counting algorithms:

✓ Hypergraph independent sets

- general case $k \gtrsim 2\log\Delta$
- linear case $k \gtrsim (1+\delta)\log\Delta$
- almost match the **hardness conditions**

✓ Hypergraph colourings

- general case $q \gtrsim \Delta^{3/k}$
- linear case $q \gtrsim \Delta^{(2+\delta)/k}$
- match the conditions of **current best randomised algorithms**

Technique

MCMC → Log-time sampling algorithm for marginal distributions → FPTAS for approx. counting

Open problems

Close the ***gap*** for ***hypergraph colouring***

- general case $q \gtrsim \Delta^{3/k}$ v.s. $q \lesssim \Delta^{2/k}$
- linear case $q \gtrsim \Delta^{(2+\delta)/k}$ v.s. $q \lesssim \Delta^{1/k}$

Thank You
Q&A

Faster algorithm for ***deterministic*** approximate counting

- FPRAS running time $\tilde{O}(n^2/\epsilon^2)$
- FPTAS running time $n^{\mathrm{poly}(k\Delta)}$
- Question: $f(k\Delta)n^C$ running time (can we use pseudorandom generator?)

Sublinear time sampling (*related to local access to huge random objects* [BRY ITCS2020])

- **Input:** distribution μ over $[q]^V$ and $v \in V$
- **Output:** *a sample* or *an approximate sample* from μ_v
- For which μ, it can be solved in sublinear time (say $n^{1-\epsilon}$ time or even $O(\log n)$ time)