Rapid mixing of Glauber dynamics via spectral independence for all degrees

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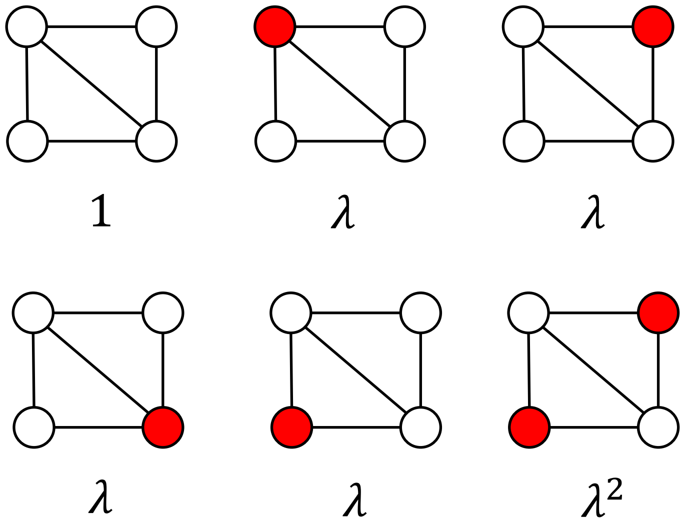
Hardcore model

- Graph $G = (V, E)$: n-vertex and max degree Δ;
- Fugacity parameter $\lambda \in \mathbb{R}_{\geq 0}$;
- Independent set $\Omega = \{S \subseteq V \mid S \text{ is an independent set}\}$;
- Gibbs distribution μ:

$$\forall S \in \Omega, \qquad \mu(S) = \frac{\lambda^{|S|}}{Z}, \qquad \text{where } Z = \sum_{I \in \Omega} \lambda^{|I|}.$$



Partition function
$Z = 1 + 4\lambda + \lambda^2$

Counting problem: calculate the partition function Z

- Exact counting: #P hard
- Approximate counting: $(1-\epsilon)Z \leq Z_{\text{out}} \leq (1+\epsilon)Z$

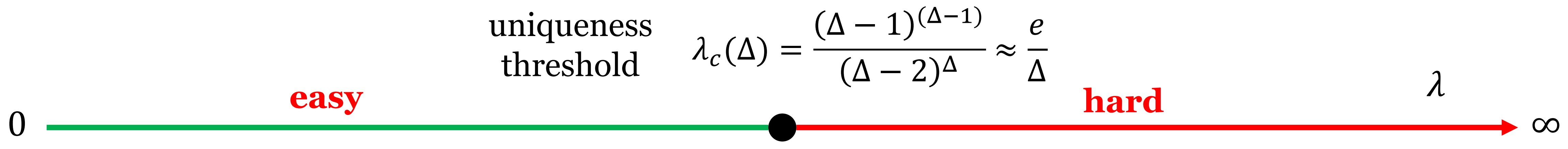
Sampling problem: draw (approximate) random independent set $S \sim \mu$

Physics

Machine Learning

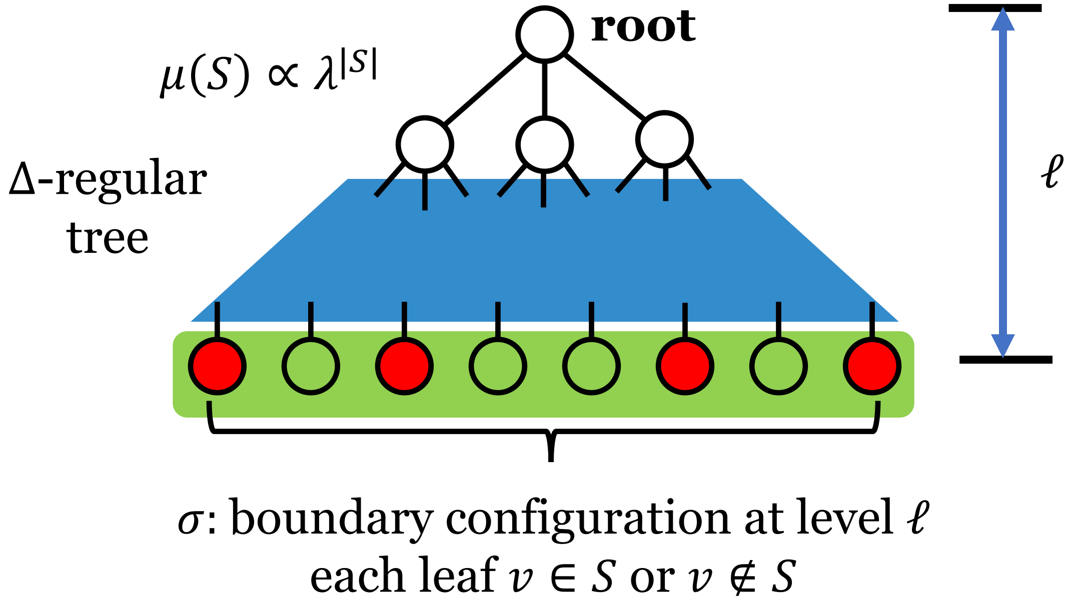
- Inference

Jerrum-Valiant-Vazirani Theorem [JVV86]: *Approximate counting* $\Longleftrightarrow$ *Sampling*



Computational phase transition

- $\lambda < \lambda_c$: poly-time ($n^{O(\log \Delta)}$) algorithm for sampling and approx. counting [Wei06]
- $\lambda > \lambda_c$: no poly-time algorithm unless $\boldsymbol{NP} = \boldsymbol{RP}$ [Sly10]



Uniqueness Threshold

$\Pr[\text{root} \in S \mid \sigma]$ is independent of σ if $\ell \to \infty$

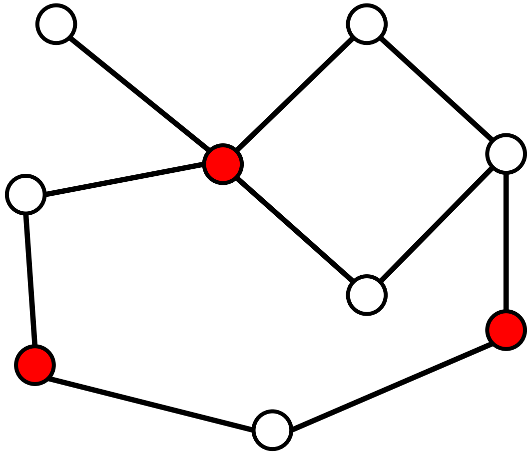
if $\lambda < \lambda_c(\Delta) = \frac{(\Delta-1)^{(\Delta-1)}}{(\Delta-2)^{\Delta}}$

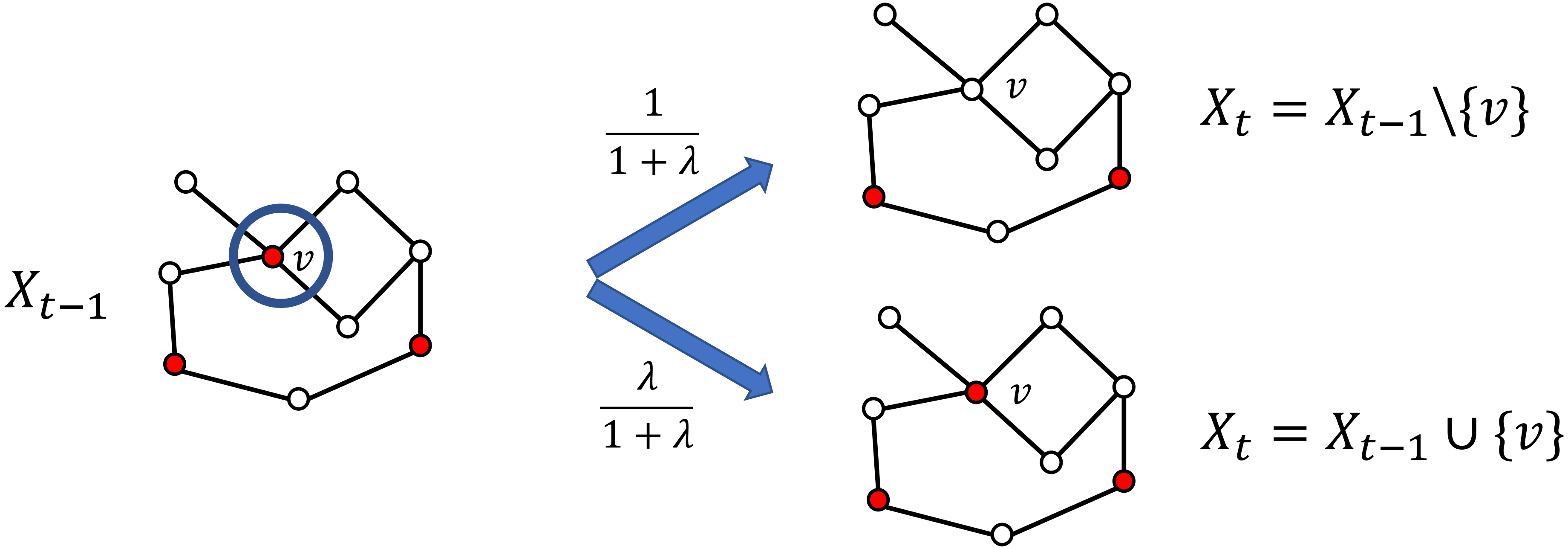
Glauber dynamics for hardcore model

Start from an arbitrary independent set X_0;

For t from 1 to T **do**

- Pick a vertex $v \in V$ uniformly at random;
- **If** $\Gamma_G(v) \cap X_{t-1} = \emptyset$, **then** $X_t = \begin{cases} X_{t-1} \setminus \{v\} & \text{with prob. } \frac{1}{1+\lambda}; \\ X_{t-1} \cup \{v\} & \text{with prob. } \frac{\lambda}{1+\lambda} \end{cases}$
- **If** $\Gamma_G(v) \cap X_{t-1} \neq \emptyset$, **then** $X_t = X_{t-1}$.





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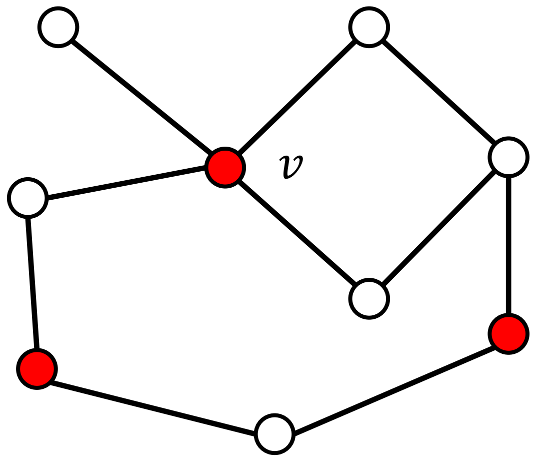
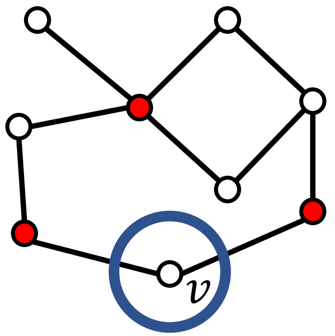
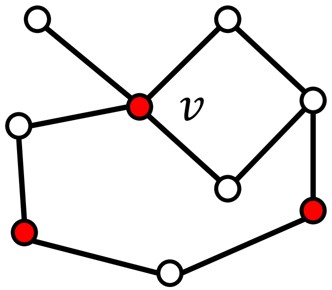


X_{t-1}





$X_t = X_{t-1}$

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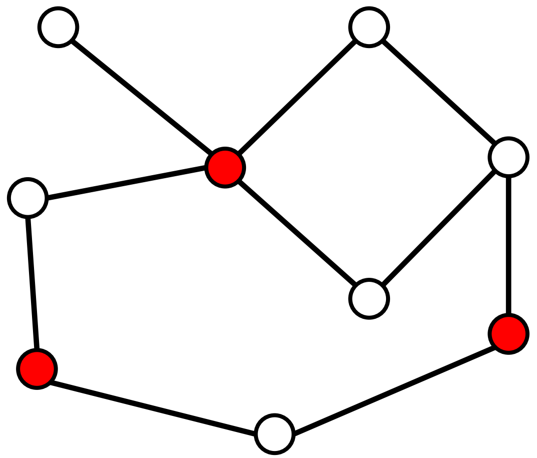

irreducible +aperiodic + reversible ➡ $X_t \sim \mu$ as $t \to \infty$

Question: How fast does the Glauber dynamics converge to μ

$$\textbf{mixing time}: T_{\text{mix}} = \max_{X_0} \min\{\, t \mid d_{TV}(X_t, \mu) \leq 0.001 \,\}$$

Total variation distance: $d_{TV}(X_t, \mu) = \frac{1}{2}\sum_{S \in \Omega} |\Pr[X_t = S] - \mu(S)|$.

Glauber dynamics for hardcore model

Start from an arbitrary independent set X_0;

For t from 1 to T **do**

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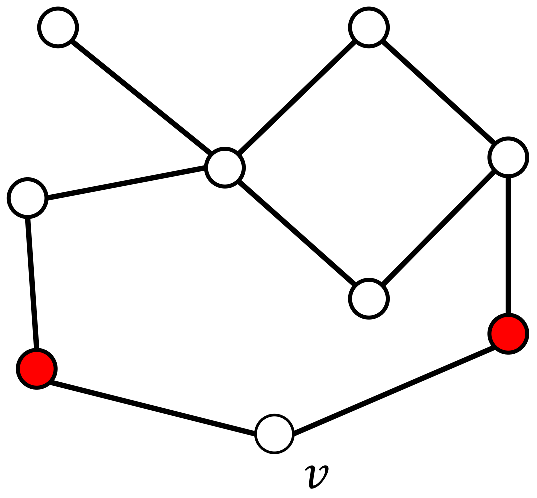

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Question: How fast does the Glauber dynamics converge to μ

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$$T_{\text{mix}}(\epsilon) = \max_{X_0} \min\{\, t \mid d_{TV}(X_t, \mu) \le \epsilon \,\}, \qquad T_{\text{mix}}(\epsilon) \le T_{\text{mix}} \log \frac{1}{\epsilon}$$

Glauber dynamics for hardcore model

Start from an arbitrary independent set X_0;

For t from 1 to T **do**

- Pick a vertex $v \in V$ uniformly at random;
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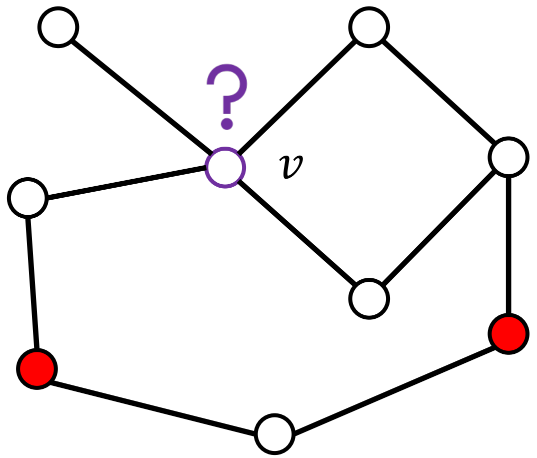


Conjecture: For hardcore model satisfying the ***uniqueness condition***

$$\lambda < \lambda_c(\Delta) = \frac{(\Delta - 1)^{(\Delta - 1)}}{(\Delta - 2)^{\Delta}} \approx \frac{e}{\Delta}$$

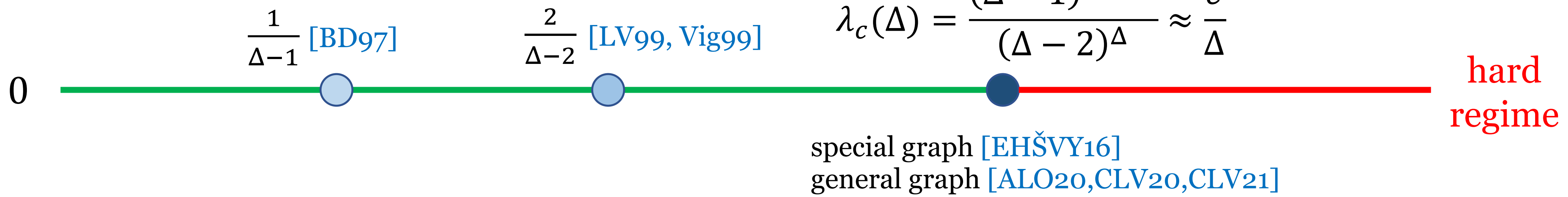
The Glauber dynamics is **rapid mixing** $T_{\text{mix}} = \text{poly}(n)$, where $n = |V|$.

Previous works

Following results holds for all $\delta \in (0,1)$

Work	Condition	Mixing Time
Bubley, Dyer *FOCS*'97	$\lambda \le (1-\delta)\frac{1}{\Delta-1}$	$O\left(\frac{1}{\delta} n \log n\right)$
Luby, Vigoda *RSA* 1999	$\lambda \le (1-\delta)\frac{2}{\Delta-2}$	$O\left(\frac{1}{\delta} n \log n\right)$
Efthymiou *et al FOCS*'16	$\lambda \le (1-\delta)\lambda_c(\Delta)$ $\Delta \ge \Delta_0(\delta)$, girth ≥ 7	$O\left(\frac{1}{\delta} n \log n\right)$
Anari, Liu, Oveis Gharan *FOCS*'20	$\lambda \le (1-\delta)\lambda_c(\Delta)$	$n^{\exp(O(1/\delta))}$
Chen, Liu, Vigoda *FOCS*'20	$\lambda \le (1-\delta)\lambda_c(\Delta)$	$n^{O(1/\delta)}$
Chen, Liu, Vigoda *STOC*'21	$\lambda \le (1-\delta)\lambda_c(\Delta)$	$\Delta^{O(\Delta^2/\delta)} n \log n$

$$\lambda_c(\Delta) = \frac{(\Delta-1)^{(\Delta-1)}}{(\Delta-2)^{\Delta}} \approx \frac{e}{\Delta}$$



Our results

Following results holds for all $\delta \in (0,1)$

Work	Condition	Mixing Time
Anari, Liu, Oveis Gharan *FOCS'20*	$\lambda \leq (1-\delta)\lambda_c(\Delta)$	$n^{\exp(O(1/\delta))}$
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Chen, Liu, Vigoda *STOC'21*	$\lambda \leq (1-\delta)\lambda_c(\Delta)$	$\Delta^{O(\Delta^2/\delta)} n \log n$
Our Result	$\boldsymbol{\lambda \leq (1-\delta)\lambda_c(\Delta)}$	$\boldsymbol{\exp\left(O\left(\frac{1}{\delta}\right)\right) \cdot n^2 \log n}$

FPT w.r.t. parameter δ

Theorem (hardcore model) [this work]

For any $\delta \in (0,1)$, any hardcore model satisfying $\lambda \leq (1-\delta)\lambda_c(\Delta)$

Glauber dynamics mixing time: $O_\delta(n^2 \log n)$.

General 2-spin systems

- Graph $G = (V, E)$, parameters β, γ, λ with $\beta \le \gamma$

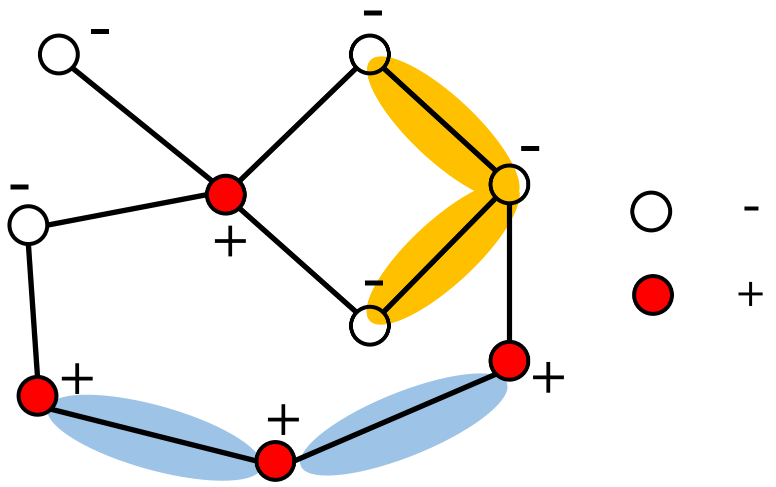
$$\boldsymbol{b} = \begin{array}{c} + \\ - \end{array}\begin{bmatrix} \lambda \\ 1 \end{bmatrix} \qquad \boldsymbol{A} = \begin{array}{cc} & \begin{array}{cc} + & - \end{array} \\ \begin{array}{c} + \\ - \end{array} & \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix} \end{array}$$

external field on vertex   interaction on edge

- Gibbs distribution μ : for any $\sigma \in \{-, +\}^V$

$$\mu(\sigma) \propto \prod_{v \in V} b(\sigma_v) \prod_{e=\{u,v\} \in E} A(\sigma_u, \sigma_v) = \lambda^{n_+(\sigma)} \beta^{m_+(\sigma)} \gamma^{m_-(\sigma)}$$

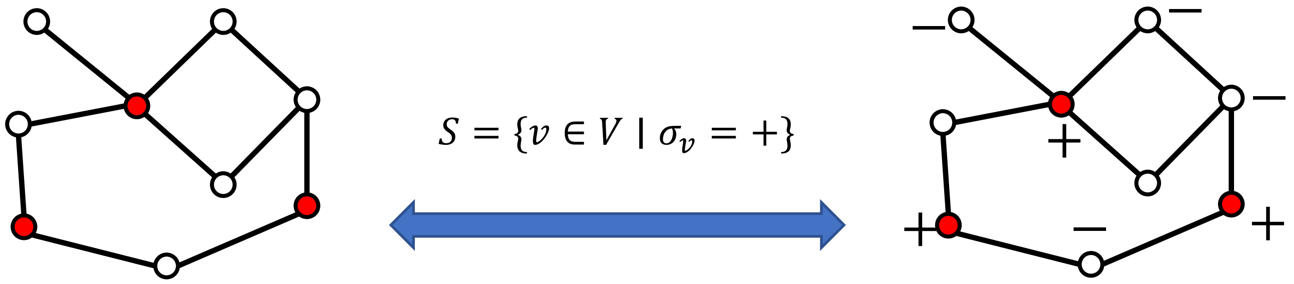
$$n_+(\sigma) = \#\{ v \in V \mid \sigma_v = + \}, m_\pm(\sigma) = \#\{\{u, v\} \in E \mid \sigma_u = \sigma_v = \pm\}$$



$$\mu(\sigma) \propto \lambda^4 \beta^2 \lambda^2$$

Hardcore model:

- $\beta = 0$ and $\gamma = 1$
- $\sigma_v = +$: v ***is in*** the independent set
- $\sigma_v = -$: v ***is not in*** the independent set



Independent set $S \subseteq V$ Vector $\sigma \in \{-, +\}^V$

- Graph $G = (V, E)$, parameters β, γ, λ with $\beta \leq \gamma$

$$\boldsymbol{b} = \begin{array}{c} + \\ - \end{array}\begin{bmatrix} \lambda \\ 1 \end{bmatrix} \qquad \boldsymbol{A} = \begin{array}{c|cc} & + & - \\ \hline + & \beta & 1 \\ - & 1 & \gamma \end{array}$$

external field on vertex — interaction on edge

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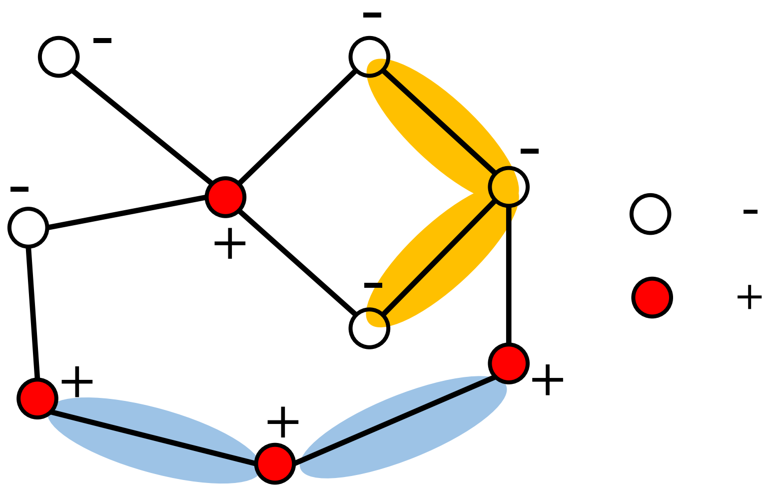


$\mu(\sigma) \propto \lambda^4 \beta^2 \lambda^2$

Glauber dynamics for distribution μ over $\{-,+\}^V$

Start from an arbitrary feasible configuration $X \in \{-,+\}^V$;

For t from 1 to T **do**

- Pick a vertex $v \in V$ uniformly at random;
- Resample $X(v) \sim \mu_v(\cdot \mid X_{V \setminus \{v\}})$.

Ising Model

- Graph $G = (V, E)$, parameters β, λ

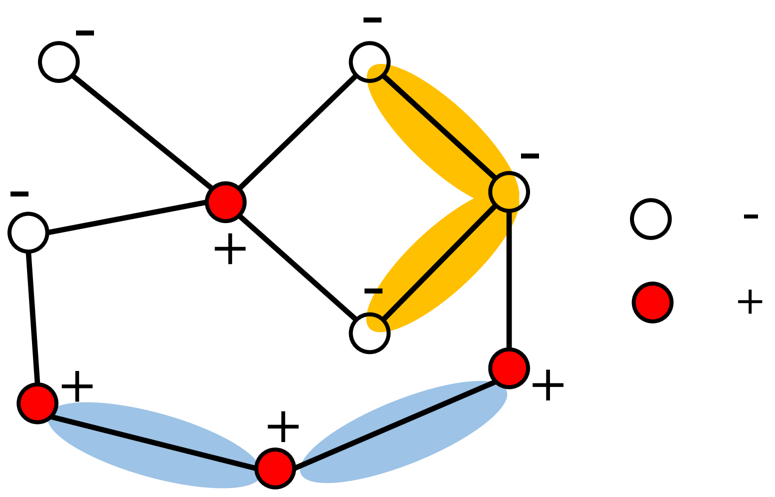
$$\boldsymbol{b} = \begin{array}{c} + \\ - \end{array} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \qquad \boldsymbol{A} = \begin{array}{c c} & \begin{array}{cc} + & - \end{array} \\ \begin{array}{c} + \\ - \end{array} & \begin{bmatrix} \color{red}{\beta} & 1 \\ 1 & \color{red}{\beta} \end{bmatrix} \end{array}$$

external field on vertex   interaction on edge

- Gibbs distribution μ : for any $\sigma \in \{-,+\}^V$

$$\mu(\sigma) \propto \prod_{v \in V} b(\sigma_v) \prod_{e=\{u,v\} \in E} A(\sigma_u, \sigma_v) = \lambda^{n_+(\sigma)} \color{red}{\beta}^{m(\sigma)}$$

$$n_+(\sigma) = \#\{\, v \in V \mid \sigma_v = +\,\}, m(\sigma) = \#\{\{u,v\} \in E \mid \sigma_u = \sigma_v\}$$



$$\mu(\sigma) \propto \lambda^4 \beta^4$$

- $\beta > 1$ ***ferromagnetic*** Ising model : each edge favors to be ***monochromatic***
- $\beta < 1$ ***anti-ferromagnetic*** Ising model : each edge favors to be ***non-monochromatic***

Ising Model

- Graph $G = (V, E)$, parameters β, λ

$$\boldsymbol{b} = \begin{array}{c c} + & \multirow{2}{*}{$\begin{bmatrix}\lambda \\ 1\end{bmatrix}$} \\ - & \end{array} \qquad \boldsymbol{A} = \begin{array}{c c c} & + & - \\ + & \beta & 1 \\ - & 1 & \beta \end{array}$$

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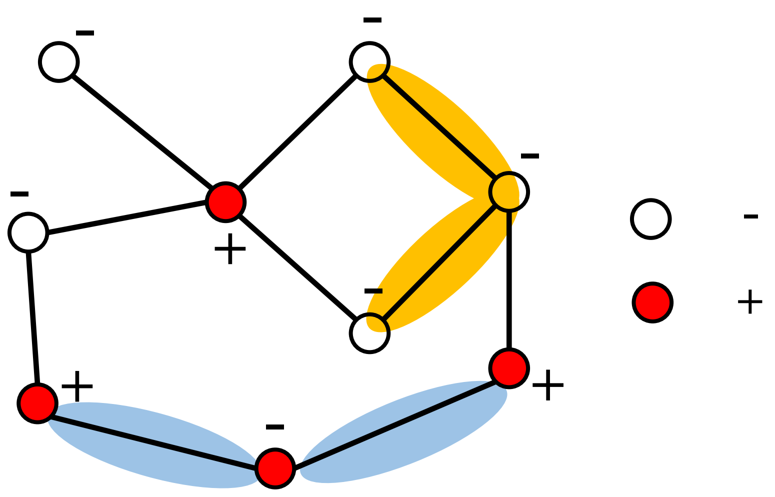
$$n_+(\sigma) = \#\{\, v \in V \mid \sigma_v = + \,\}, m_\pm(\sigma) = \#\{\{u,v\} \in E \mid \sigma_u = \sigma_v = \pm\}$$



$$\mu(\sigma) \propto \lambda^4 \beta^4$$

$$\beta_c(\Delta) = \frac{\Delta - 2}{\Delta} \qquad \hat{\beta}_c(\Delta) = \frac{\Delta}{\Delta - 2}$$

Hard Regime   0 — 1 — λ — ∞

No poly alg. unless NP=RP [SS12],[GŠV12]

Uniqueness Regime

Glauber dynamics slow mixing [GM07]
Other alg. works [JS93,GJ18,LSS17]

Our results

Work	Condition	Mixing Time
Mossel and Sly *Ann probab.* 2012	$1 \le \beta \le \frac{\Delta-\delta}{\Delta-2+\delta}$	$\exp\left(\Delta^{O(1/\delta)}\right) n \log n$
Chen, Liu, Vigoda *FOCS*'20	$\frac{\Delta-2+\delta}{\Delta-\delta} \le \beta \le \frac{\Delta-\delta}{\Delta-2+\delta}$	$n^{O(1/\delta)}$
Chen, Liu, Vigoda *STOC*'21	$\frac{\Delta-2+\delta}{\Delta-\delta} \le \beta \le \frac{\Delta-\delta}{\Delta-2+\delta}$	$\Delta^{O(1/\delta)} n \log n$
This work	$\boldsymbol{\frac{\Delta-2+\delta}{\Delta-\delta} \le \beta \le \frac{\Delta-\delta}{\Delta-2+\delta}}$	$\boldsymbol{\exp\left(O\left(\frac{1}{\delta}\right)\right) n^2}$

Theorem (Ising model) [this work]

For any $\delta \in (0,1)$, any Ising model (with par. β, λ) satisfying $\frac{\Delta-2+\delta}{\Delta-\delta} \le \beta \le \frac{\Delta-\delta}{\Delta-2+\delta}$

Glauber dynamics mixing time: $O_\delta(n^2)$

Theorem (hardcore model) [this work]

For any $\delta \in (0,1)$, any hardcore model satisfying $\lambda \leq (1-\delta)\lambda_c(\Delta)$

Glauber dynamics mixing time: $O_\delta(n^2 \log n)$.

Theorem (Ising model) [this work]

For any $\delta \in (0,1)$, any Ising model (with par. β, λ) satisfying $\frac{\Delta-2+\delta}{\Delta-\delta} \leq \beta \leq \frac{\Delta-\delta}{\Delta-2+\delta}$

Glauber dynamics mixing time $O_\delta(n^2)$.

Theorem (anti-ferro 2-spin system) [this work]

For any $\delta \in (0,1)$, any anti-ferro 2-spin system (with par. λ, β, γ and $\beta\gamma < 1$)

satisfying up-to-Δ uniqueness condition with gap δ:

Glauber dynamics mixing time: $C(\delta, \lambda, \beta, \gamma)n^3$

Mixing time, spectral gap and relaxation time

Distribution: μ over $\{-,+\}^V$ with support Ω;

Transition matrix of Glauber dynamics : $P: \Omega \times \Omega \to \mathbb{R}_{\geq 0}$

Eigenvalues : P has $|\Omega|$ real eigenvalues $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|\Omega|} \geq 0$

Spectral gap : $\lambda_{\text{gap}} = 1 - \lambda_2$

$$\boxed{\lambda_{\text{gap}} \geq \frac{1}{\text{poly}(n)}} \iff \boxed{T_{\text{mix}} \leq \text{poly}(n)}$$

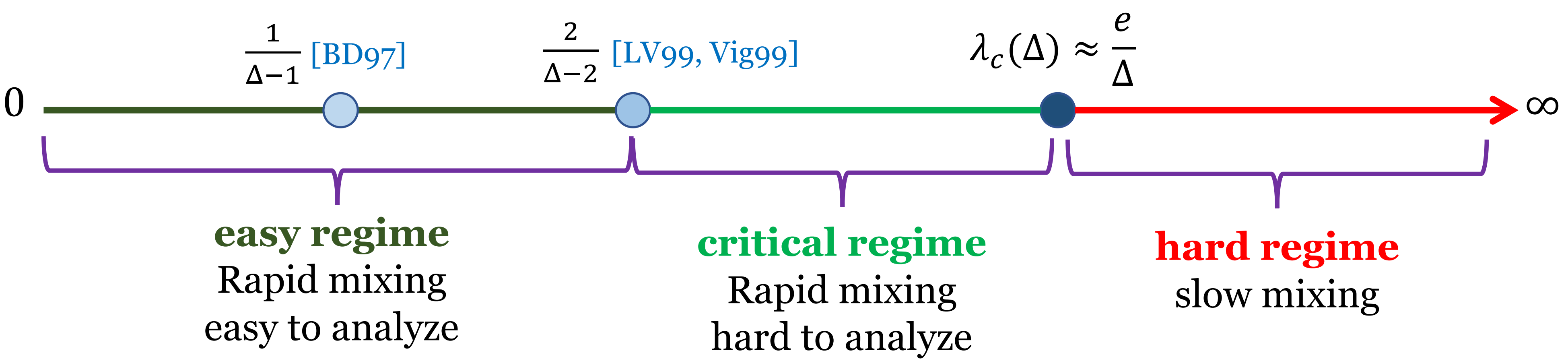
$$T_{\text{mix}} \leq O\left(\frac{1}{\lambda_{\text{gap}}} \log \frac{1}{\mu_{\min}}\right), \qquad \text{where } \mu_{\min} = \min_{\sigma \in \Omega} \mu(\sigma)$$

typically poly(n)

Relaxation time: $T_{\text{rel}} = \dfrac{1}{\lambda_{\text{gap}}}$

To prove the ***rapid mixing*** result, we only need to bound the ***relaxation time***.

Boosting lemma



$\boxed{\lambda \leq (1-\delta)\lambda_c(\Delta)}$ ⟹ $\boxed{\frac{\lambda}{25} \leq \frac{1}{2\Delta} \in \text{Easy regime}}$

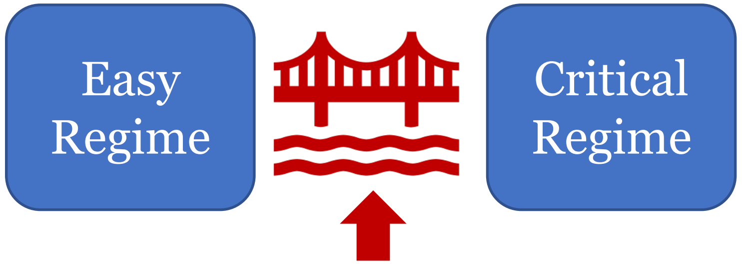
$P(\lambda), P\left(\frac{\lambda}{25}\right)$: Glauber dynamics for hardcore model with λ and $\lambda/25$

$t_{\mathrm{rel}}\left(P\left(\frac{\lambda}{25}\right)\right) = O(n)$ are already known [BD97]

Our contribution: *Boosting lemma*:

$$t_{\mathrm{rel}}(P(\lambda)) \leq \exp\left(O\left(\frac{1}{\delta}\right)\right) \cdot t_{\mathrm{rel}}\left(P\left(\frac{\lambda}{25}\right)\right) = \exp\left(O\left(\frac{1}{\delta}\right)\right) n$$

Boosting lemma



New Markov chain: ***Field Dynamics***

Field Dynamics (hardcore model)

Input: hardcore model on $G = (V, E)$ with fugacity $\lambda \in \mathbb{R}_{\geq 0}$, a parameter $\theta \in (0,1)$

Start from an arbitrary configuration $X \in \{-,+\}^V$ ($X_v = +$: v is in the independent set)

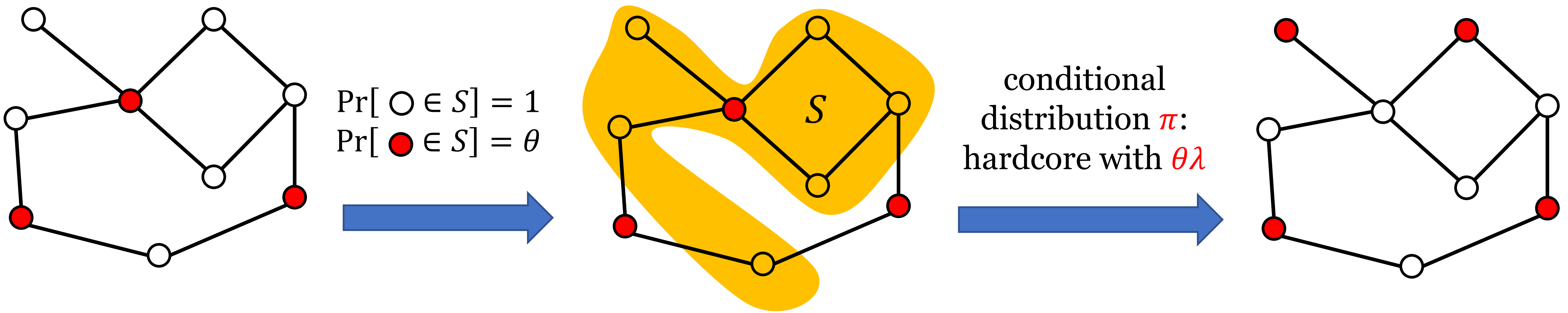
For each t from 1 to T **do**

- Construct $S \subseteq V$ be selecting each $v \in V$ independently with probability

$$p_v = \begin{cases} 1 & \text{if } X(v) = - \ \ (\text{i.e. } v \text{ is not in indepedent set }) \\ \theta & \text{if } X(v) = + \ \ (\text{i.e. } v \text{ is in independent set}) \end{cases}$$

- resample $X_t(S) \sim \pi_S(\cdot \mid X(V \backslash S))$

π: the Gibbs distribution of hardcore model with fugacity $\theta\lambda$



Field Dynamics (hardcore model)

Input: hardcore model on $G = (V, E)$ with fugacity $\lambda \in \mathbb{R}_{\geq 0}$, a parameter $\theta \in (0,1)$

Start from an arbitrary configuration $X \in \{-,+\}^V$ $\quad$ ($X_v = +$: v is in the independent set)

For each t from 1 to T **do**

- Construct $S \subseteq V$ be selecting each $v \in V$ independently with probability

$$p_v = \begin{cases} 1 & \text{if } X(v) = - \ \ (\text{i.e. } v \text{ is not in indepedent set }) \\ \theta & \text{if } X(v) = + \ \ (\text{i.e. } v \text{ is in independent set}) \end{cases}$$

- resample $X_t(S) \sim \pi_S(\cdot \mid X(V \setminus S))$

π: the Gibbs distribution of hardcore model with fugacity $\theta\lambda$

Proposition (Field Dynamics): for any $\lambda \in \mathbb{R}_{\geq 0}$, any $\theta \in (0,1)$

The Field Dynamics $P_{FD}(\lambda, \theta)$ is irreducible, aperiodic and reversible with respect to μ.

$\Longrightarrow$ $P_{FD}(\lambda, \theta)$ has the unique stationary distribution μ.

μ: the Gibbs distribution of hardcore model with fugacity λ

Critical Regime
$T_{\mathrm{rel}}(P(\lambda))$

Easy Regime
$T_{\mathrm{rel}}(P(\theta\lambda))$

Field Dynamics

Filed dynamics: comparison lemma: for any $\lambda \in \mathbb{R}_{\geq 0}$, any $\theta \in (0,1)$,

$$\underbrace{T_{\mathrm{rel}}\big(P(\lambda)\big)}_{\textit{Target relaxation time}} \leq \underbrace{T_{\mathrm{rel}}\big(P_{FD}(\lambda,\theta)\big)}_{\textit{Filed Dynamics relaxation time}} \cdot \boxed{\underbrace{T_{\mathrm{rel}}^{\mathrm{worst}}(P(\theta\lambda))}_{\textit{Easy regime relaxation time}}}$$

Relaxation time for π with **worst condition**

Filed dynamics: comparison lemma: for any $\lambda \in \mathbb{R}_{\geq 0}$, any $\theta \in (0,1)$,

$$T_{\mathrm{rel}}\big(P(\lambda)\big) \leq T_{\mathrm{rel}}\big(P_{FD}(\lambda,\theta)\big) \cdot \boxed{T_{\mathrm{rel}}^{\mathrm{worst}}(P(\theta\lambda))}$$

Target relaxation time — *Filed Dynamics relaxation time* — *Easy regime relaxation time*

Relaxation time for π with **worst condition**

π: the Gibbs distribution of hardcore model with fugacity $\theta\lambda$

for any $H \subseteq V$, any $\sigma \in \{-,+\}^H$

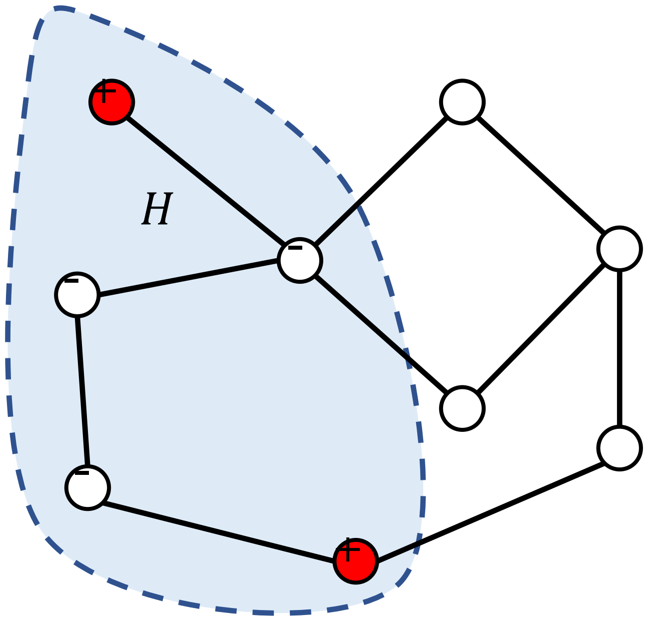
π^σ: π *conditional on* the configuration on Λ is fixed to σ

$P(\pi^\sigma)$: Glauber dynamics for π^σ

worst-pinning relaxation time

$$T_{\mathrm{rel}}^{\mathrm{worst}}(P(\theta\lambda)) = \max_{\Lambda \subseteq V, \sigma \in \{-,+\}^\Lambda} t_{\mathrm{rel}}(P(\pi^\sigma))$$

Hardcore:$\theta\lambda = \frac{\lambda}{25} \leq \frac{2}{\Delta} \in$ Easy regime , thus $T_{\mathrm{rel}}^{\max}(P(\theta\lambda)) = O(n)$



Critical Regime
$T_{\mathrm{rel}}(P(\lambda))$

Easy Regime
$T_{\mathrm{rel}}(P(\theta\lambda))$

Field Dynamics

Filed dynamics: comparison lemma: for any $\lambda \in \mathbb{R}_{\geq 0}$, any $\theta \in (0,1)$,

$$T_{\mathrm{rel}}\big(P(\lambda)\big) \leq T_{\mathrm{rel}}\big(P_{FD}(\lambda,\theta)\big) \cdot T_{\mathrm{rel}}^{\mathrm{worst}}(P(\theta\lambda))$$

Target relaxation time | *Filed Dynamics relaxation time* | *Easy regime relaxation time*

$\lambda < \lambda_c$
$\theta = \frac{1}{25}$

$\theta\lambda \in$ Easy Regime

Standard Analysis

$T_{\mathrm{rel}}^{\mathrm{worst}}\big(P(\theta\lambda)\big) = O(n)$

Critical Regime
$T_{\text{rel}}(P(\lambda))$

Easy Regime
$T_{\text{rel}}(P(\theta\lambda))$

Field Dynamics

Filed dynamics: comparison lemma: for any $\lambda \in \mathbb{R}_{\geq 0}$, any $\theta \in (0,1)$,

$$\underbrace{T_{\text{rel}}\big(P(\lambda)\big)}_{\text{Target relaxation time}} \leq \underbrace{\boxed{T_{\text{rel}}\big(P_{FD}(\lambda,\theta)\big)}}_{\text{Filed Dynamics relaxation time}} \cdot \underbrace{T_{\text{rel}}^{\text{worst}}(P(\theta\lambda))}_{\text{Easy regime relaxation time}}$$

Boosting Cost

Critical Regime
$T_{\mathrm{rel}}(P(\lambda))$

Easy Regime
$T_{\mathrm{rel}}(P(\theta\lambda))$

Field Dynamics

Filed dynamics: comparison lemma: for any $\lambda \in \mathbb{R}_{\geq 0}$, any $\theta \in (0,1)$,

$$\underbrace{T_{\mathrm{rel}}\big(P(\lambda)\big)}_{\text{Target relaxation time}} \leq \underbrace{T_{\mathrm{rel}}\big(P_{FD}(\lambda,\theta)\big)}_{\text{Filed Dynamics relaxation time}} \cdot \underbrace{T_{\mathrm{rel}}^{\mathrm{worst}}(P(\theta\lambda))}_{\text{Easy regime relaxation time}}$$

Proved by an involved calculation

Filed dynamics: mixing lemma: for any $\lambda \leq (1-\delta)\lambda_c(\Delta)$, any $\theta \in (0,1)$,

$$T_{\mathrm{rel}}\big(P_{FD}(\lambda,\theta)\big) \leq \left(\frac{2}{\theta}\right)^{O\left(\frac{1}{\delta}\right)}$$

Original distribution
μ over $\{-,+\}^V$

k-transformation

Transformed distribution
μ_k over $\{-,+\}^{V_k}$

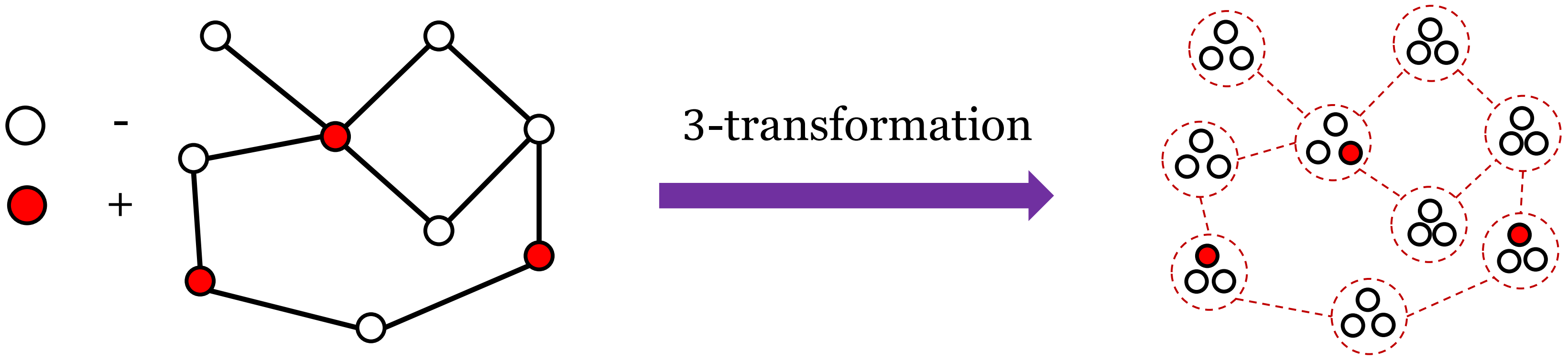
$V_k = \{v_1, v_2, \ldots, v_k \mid v \in V\}$

$X \sim \mu$

For each variable $v \in V$ **do**

- **If** $X(v) = -$, **then** $Y(v_i) = -$ for all $i \in [k]$;
- **If** $X(v) = +$, **then**
  - Sample $j \in \{1,2,\ldots,k\}$ uniformly at random;
  - $Y(v_j) = +$ and $Y(v_i) = -$ for all $i \in [k]\backslash\{j\}$;

$Y \sim \mu_k$



$X \sim \mu$

$Y \sim \mu_3$

Original distribution
μ over $\{-,+\}^V$

inverse
k-transformation

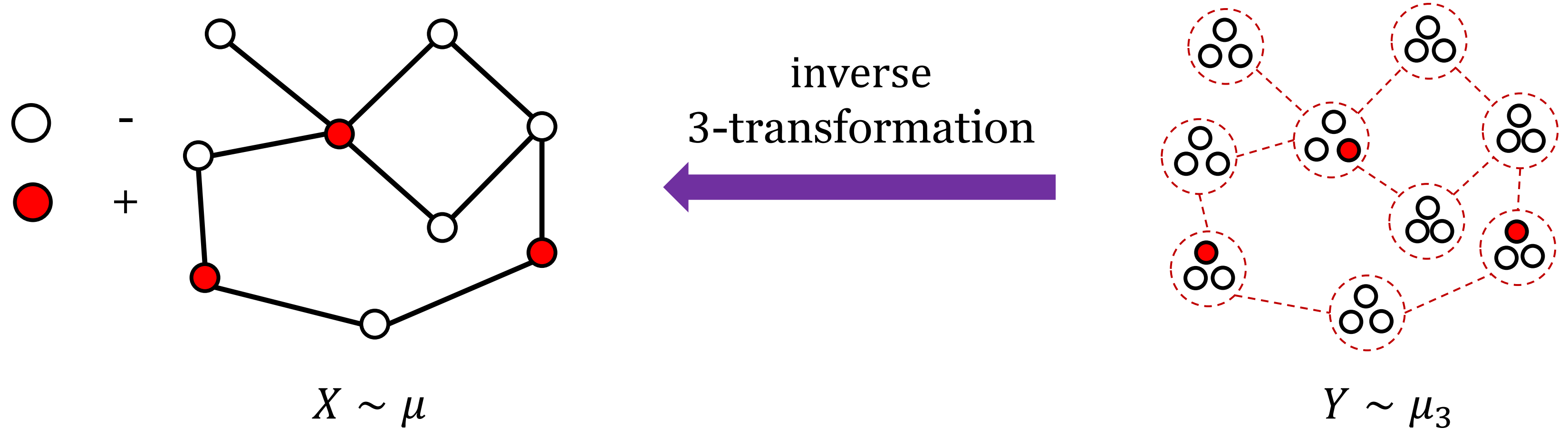
Transformed distribution
μ_k over $\{-,+\}^{V\times[k]}$
$V\times[k] = \{v_1, v_2, \dots, v_k \mid v \in V\}$

$X \sim \mu$

For each variable $v \in V$ **do**

- **If** $Y(v_i) = -$ for all $i \in [k]$, **then** $X(v) = -$;
- **If** $\exists j \in [k]$ s.t. $Y(v_j) = +$, **then** $X(v) = +$;

$Y \sim \mu_k$



fix $Y \in V\times[k]$, $X = \text{inverse}(Y)$ is uniquely fixed, denote $X = Y^\star$

Goal: prove the *filed dynamics on μ* is rapid mixing $t_{\mathrm{rel}}\left(P_{FD}(\lambda,\theta)\right) \le \left(\frac{2}{\theta}\right)^{O\left(\frac{1}{\delta}\right)}$

Approach: use *block dynamics on μ_k* to approximate the *filed dynamics on μ*

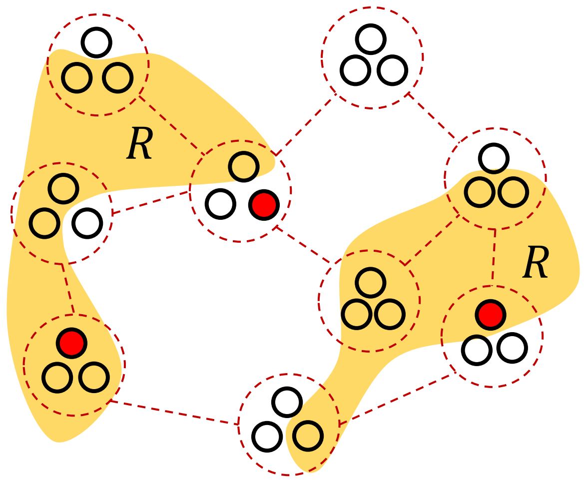
ℓ-Block dynamics for distribution μ_k over $\{-,+\}^{V_k}$

Parameter: integer $1 \le \ell \le nk$, where $n = |V|$

Start from an arbitrary feasible configuration $Y_0 \in \{-,+\}^{V_k}$;

For t from 1 to T **do**

- Sample $R \in \binom{V_k}{\ell}$ uniformly at random and let $Y_t(V_k \backslash R) = Y_{t-1}(V_k \backslash R)$;
- Sample $Y(R) \sim \mu_{k,R}(\cdot \mid Y(V \backslash R))$.



Goal: prove the *filed dynamics on μ* is rapid mixing $t_{\mathrm{rel}}\left(P_{FD}(\lambda,\theta)\right) \le \left(\frac{2}{\theta}\right)^{300/\delta}$

Approach: use *block dynamics on μ_k* to approximate the *filed dynamics on μ*

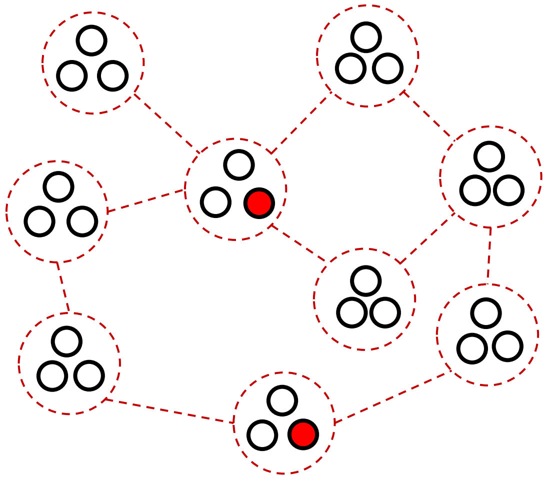
ℓ-Block dynamics for distribution μ_k over $\{-,+\}^{V_k}$

Parameter: integer $1 \le \ell \le nk$, where $n = |V|$

Start from an arbitrary feasible configuration $Y_0 \in \{-,+\}^{V_k}$;

For t from 1 to T **do**

- Sample $R \in \binom{V_k}{\ell}$ uniformly at random and let $Y_t(V_k \setminus R) = Y_{t-1}(V_k \setminus R)$;
- Sample $Y_t(R) \sim \mu_{k,R}(\cdot \mid Y_{t-1}(V \setminus R))$.



Glauber dynamics: $\ell = 1$

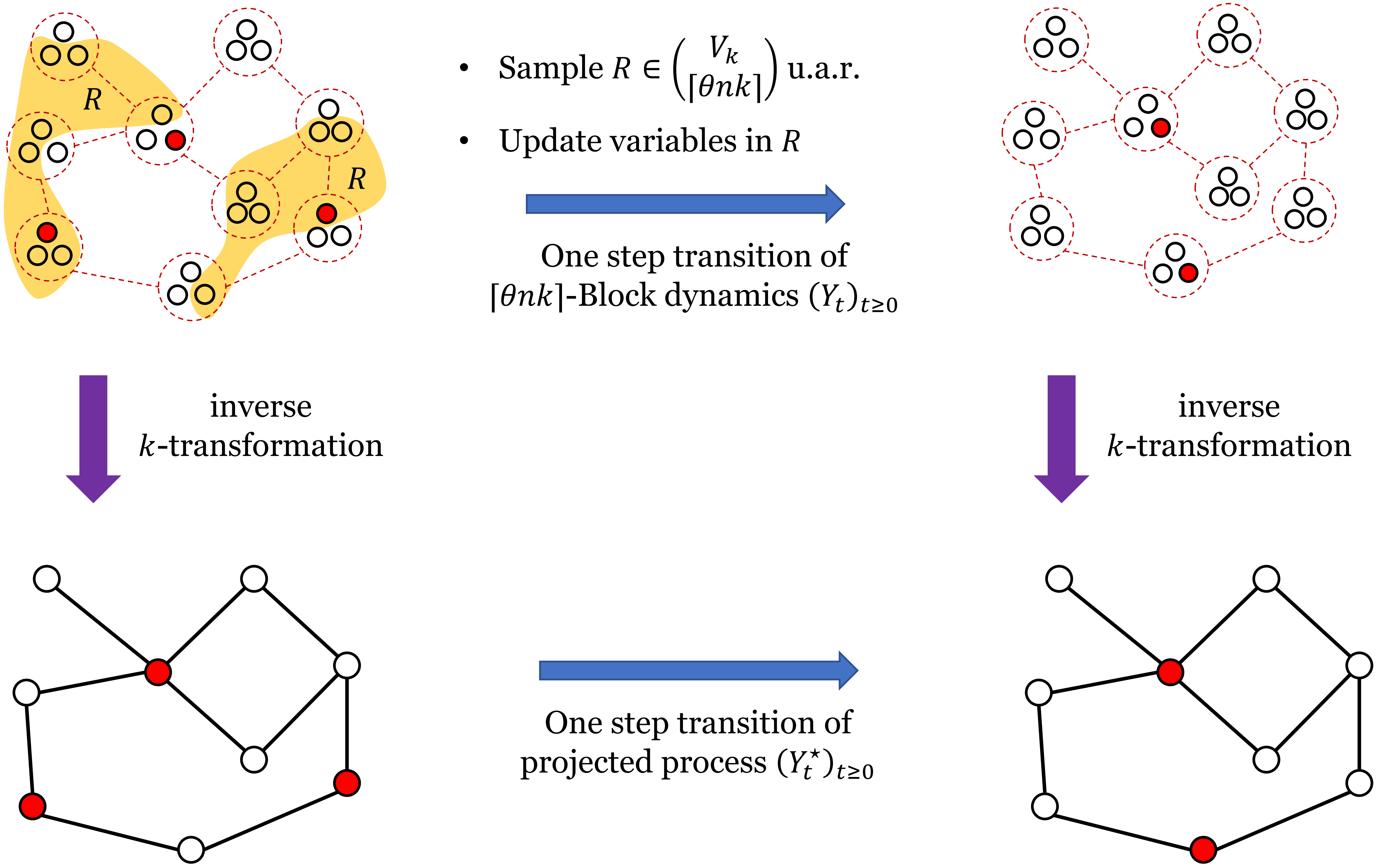
$[\theta kn]$-block dynamics on μ_k: $(Y_t)_{t\ge 0}$

Random walk over $\{-,+\}^{V_k}$

inverse k-transformation

projected process $(Y_t^\star)_{t\ge 0}$:

Random process over $\{-,+\}^V$



R
R
• Sample $R \in \binom{V_k}{\lceil \theta nk \rceil}$ u.a.r.
• Update variables in R
One step transition of $\lceil \theta nk \rceil$-Block dynamics $(Y_t)_{t\geq 0}$
inverse k-transformation
inverse k-transformation
One step transition of projected process $(Y_t^\star)_{t\geq 0}$

Goal: prove the *filed dynamics on μ* is rapid mixing $t_{\mathrm{rel}}(P_{FD}(\lambda,\theta)) \le \left(\frac{2}{\theta}\right)^{300/\delta}$

Approach: use *block dynamics on μ_k* to approximate the *filed dynamics on μ*

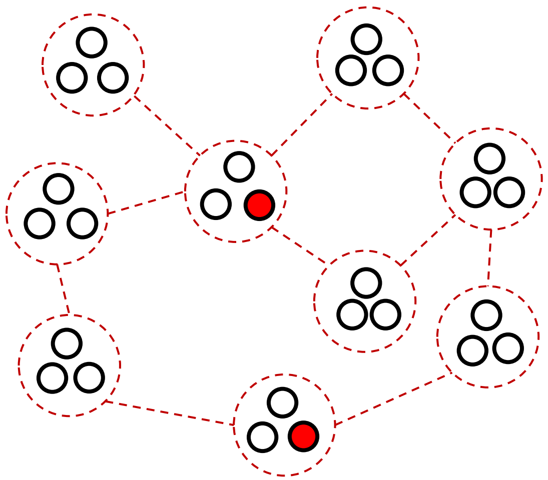
ℓ-Block dynamics for distribution μ_k over $\{-,+\}^{V_k}$

Parameter: integer $1 \le \ell \le nk$, where $n = |V|$

Start from an arbitrary feasible configuration $Y_0 \in \{-,+\}^{V_k}$;

For t from 1 to T **do**

- Sample $R \in \binom{V_k}{\ell}$ uniformly at random and let $Y_t(V_k \backslash R) = Y_{t-1}(V_k \backslash R)$;
- Sample $Y_t(R) \sim \mu_{k,R}(\cdot \mid Y_{t-1}(V \backslash R))$.



Glauber dynamics: $\ell = 1$

$[\theta kn]$-block dynamics on μ_k: $(Y_t)_{t\ge 0}$

Random walk over $\{-,+\}^{V_k}$

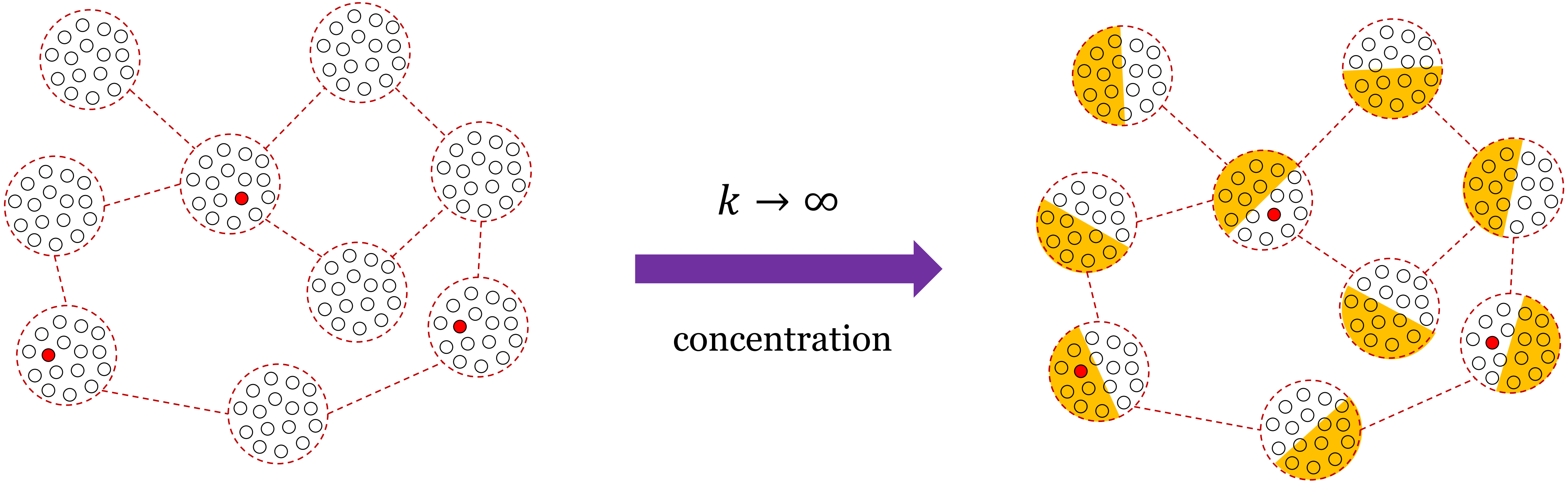
inverse
k-transformation

projected process $(Y_t^\star)_{t\ge 0}$:

Random process over $\{-,+\}^{V}$

Approximation lemma: For any $k \ge 1$, $(Y_t^\star)_{t\ge 0}$ is a Markov chain.

If $k \to \infty$, *Markov chain* $(Y_t^\star)_{t\ge 0}$ = *field dynamics* $P_{FD}(\lambda,\theta)$



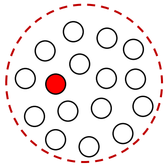
Pick $S \in \binom{V_k}{\lceil \theta n k \rceil}$ uniformly at random
S: θ fraction of vertices

For each $v \in V$,
Pick θk vertices in $\{v_1, v_2, .. v_k\}$ u.a.r.

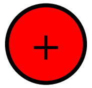
Block Dynamics when $k \to \infty$

Field Dynamics

one vertex with value +
Pr[● $\in R] \to \theta$

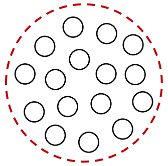


inverse
k-transformation

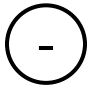


Selected into S with probability $p_v = \theta$

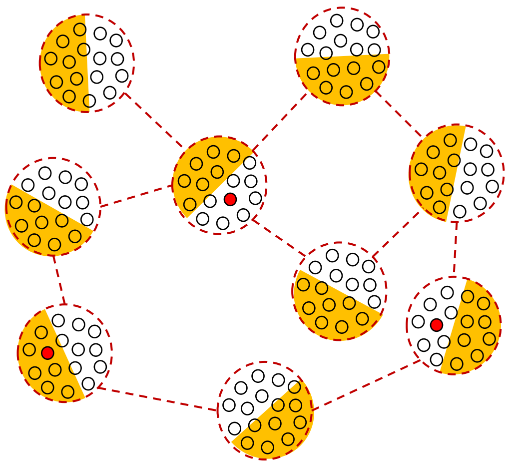
all vertices with value −



inverse
k-transformation



Selected into S with probability $p_v = 1$



Pr[all balls pick θ fraction of variables] $\to 1$

inverse
k-transformation

- Resample variables S from the conditional distribution w.r.t. π
- π: hardcore model **with fugacity $\theta\lambda$**

Filed dynamics: mixing lemma: for any $\lambda \le (1-\delta)\lambda_c(\Delta)$, any $\theta \in (0,1)$,

$$t_{\text{rel}}\left(P_{FD}(\lambda,\theta)\right) \le \left(\frac{2}{\theta}\right)^{\frac{300}{\delta}}$$

Goal

Approximation lemma: For any $k \ge 1$, $(Y_t^\star)_{t\ge 0}$ is a Markov chain.

If $k \to \infty$, *Markov chain* $(Y_t^\star)_{t\ge 0}$ approaches to *field dynamics* $P_{FD}(\lambda,\theta)$

$T_{\text{rel}}(\text{Block},\mu_k)$: relaxation time of $[\theta n k]$-block dynamics on μ_k

$$\boxed{T_{\text{rel}}\left(P_{\text{FD}}(\lambda,\theta)\right) \le \limsup_{k\to\infty} T_{\text{rel}}(\text{Block},\mu_k)}$$

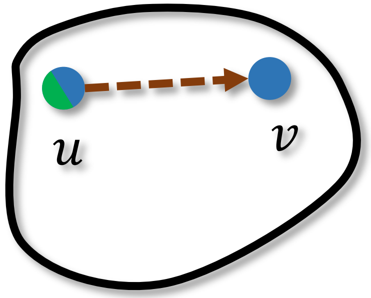
Block dynamics mixing lemma:

for any $\lambda \le (1-\delta)\lambda_c(\Delta)$, any $\theta \in (0,1)$, any sufficiently large k

$$T_{\text{rel}}(\text{Block},\mu_k) \le \left(\frac{2}{\theta}\right)^{O\left(\frac{1}{\delta}\right)}$$

Filed dynamics mixing lemma

Spectral independence

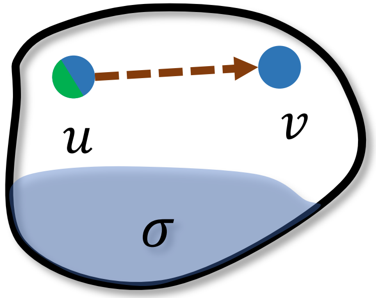


influence on v caused by a **disagreement** on u

μ: a distribution over $\Omega \subseteq \{-1,+1\}^V$

$|V|\times|V|$ **influence matrix** $\Psi \in \mathbb{R}^{V\times V}$ such that

$$\Psi(u,v) = \left|\Pr_{\mu}[\, v = + \mid u = + \,] - \Pr_{\mu}[\, v = + \mid u = - \,]\right|$$



influence matrix for **conditional distribution**

For any subset $S \subseteq V$, any feasible $\sigma \in \{-1,+1\}^{V\setminus S}$

μ_S^{σ} distribution on S conditional on σ

influence matrix $\Psi_S^{\sigma} \in \mathbb{R}^{S\times S}$ for **conditional distribution**

$$\Psi_S^{\sigma}(u,v) = \left|\Pr_{\mu_S^{\sigma}}[\, v = + \mid u = + \,] - \Pr_{\mu_S^{\sigma}}[\, v = + \mid u = - \,]\right|$$

Spectral independence [ALO20, FGYZ21, CGŠV21]

There is a constant $C > 0$ s.t.

for all conditional distribution μ_S^σ,

spectral radius of influence matrices $\rho(\Psi_S^\sigma) \le C$.

Mixing via spectral independence [Chen, Liu, Vigoda, 2021]

For any distribution π over $\{-1,+1\}^V$, where $n = |V|$

if π is $\boldsymbol{C}$ **spectral independence**,

then $\lceil \theta n \rceil$-block dynamics satisfies $T_{\mathrm{rel}}(\mathrm{Block}, \pi) \le \left(\frac{2}{\theta}\right)^{2C+3}$

if $\lambda \le (1-\delta)\lambda_c(\Delta)$, for all $k \ge 1$

μ_k is $O\left(\frac{1}{\delta}\right)$ spectrally independent.

$\Rightarrow$

Block dynamics mixing lemma

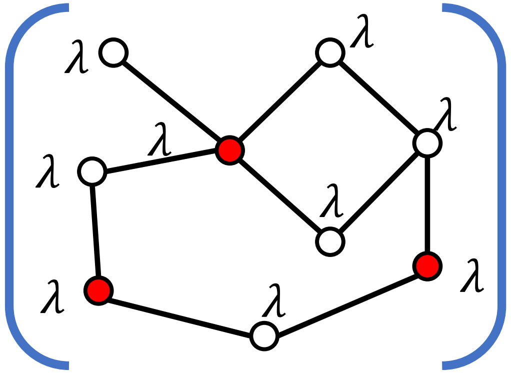
Complete spectral independence

Magnetizing joint distribution with local fields

Joint distribution μ over $\{-1,+1\}^V$, local fields $\phi = (\phi_v)_{v \in V} \in \mathbb{R}^V_{>0}$

$$\mu^{(\phi)}(\sigma) \propto \mu(\sigma) \prod_{v \in V : \sigma_v = +1} \phi_v$$

μ

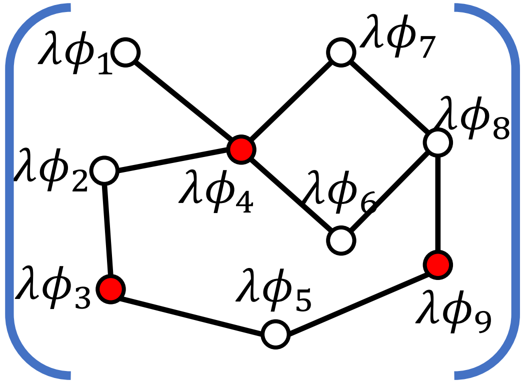


$\propto \lambda^3$

Hardcore model: $\mu(S) \propto \lambda^{|S|}$

magnetizing

$\mu^{(\phi)}$



$\propto \lambda^3 \phi_3 \phi_4 \phi_9$

Hardcore mode with local fields

$\mu^{(\phi)}(S) \propto \lambda^{|S|} \prod_{v \in S} \phi_v = \prod_{v \in S} \lambda \phi_v$

Complete Spectral independence [This work]

There is a constant $C > 0$ s.t.

for all local fields $\phi \in (0,1]^V$ (for all $v \in V$, $0 < \phi_v \leq 1$),

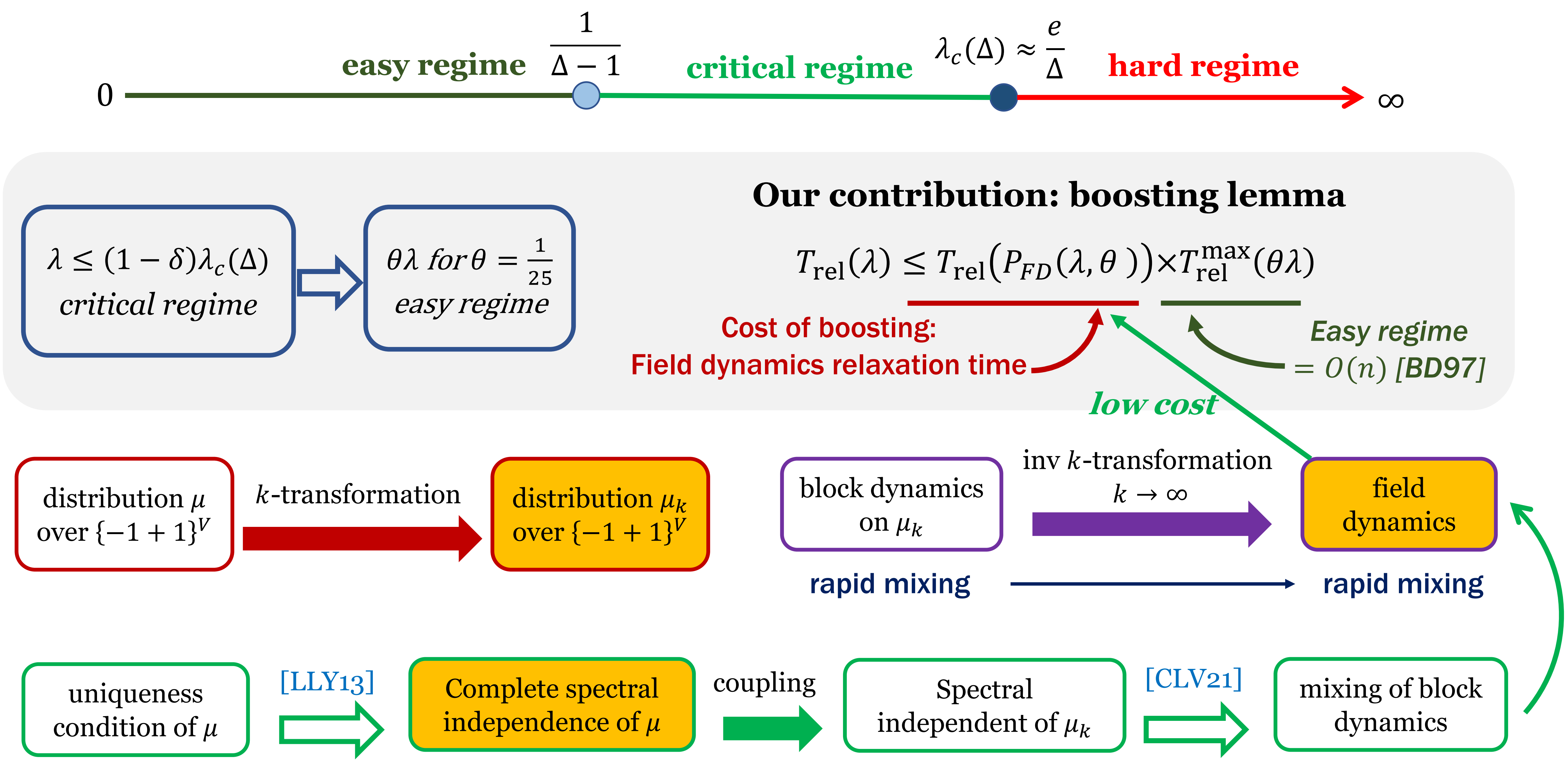
$\mu^{(\phi)}$ is *spectrally independent* with parameter C

Complete spectral independence implies spectral independence

μ is C-***completely*** spectrally independent $\xrightarrow{\textit{Coupling}}$ for **all** $\boldsymbol{k \geq 1}$, μ_k is $(C+2)$-spectrally independent

Establish complete spectral independence

Hardcore model μ satisfying ***uniqueness condition*** $\lambda \leq (1-\delta)\lambda_c(\Delta)$ $\xrightarrow{\textit{Correlation Decay}}$ μ is $O\left(\frac{1}{\delta}\right)$-***completely*** spectrally independent



Our technique works for **all completely spectrally independent** distributions over $\{-1,+1\}^V$

Open problems

- Prove the ***optimal*** $O(n \log n)$ mixing time
  - our technique is based on *spectral gap*:

$$T_{\text{mix}} \leq O\left(T_{\text{rel}} \log \frac{1}{\mu_{\min}}\right)$$

  - new technique for *modified log-Sobolev* constant?

Hardcore model:

- our result $T_{\text{mix}} = O(n^2 \log n)$
- $T_{\text{rel}} = O(n)$ **optimal!**
- $\log \frac{1}{\mu_{\min}} = O(n \log n)$ **tight!**

- Extend our technique to ***general distributions*** beyond the Boolean domain i.e. q-coloring
  - what is the *field dynamics* for general distributions?

- Improve the dependency on δ
  - our result for **Ising** and **hardcore** $C(\delta) = \exp(O(1/\delta))$;
  - our result for **general anti-ferro 2-spin** $C(\delta) = \left(\frac{1}{\delta}\right)^{O(1/\delta)}$;
  - improve the dependency to $C(\delta) = \text{poly}\left(\frac{1}{\delta}\right)$.

Thank you
Q&A