Field dynamics: a new tool to boost mixing results

Weiming Feng
University of Edinburgh

Joint work with: Xiaoyu Chen (Nanjing University)
Yitong Yin (Nanjing University)
Xinyuan Zhang (Nanjing University)

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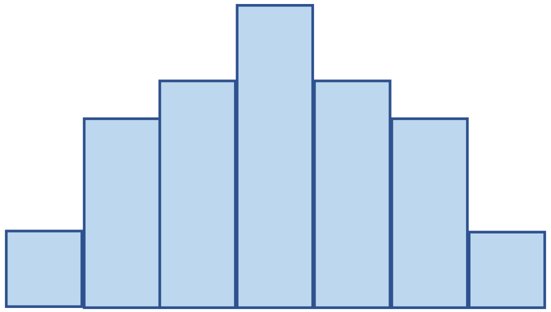
Sampling, counting and phase transition

Boolean variables set V, weight function $w: \{-,+\}^V \to \mathbb{R}_{\geq 0}$

joint distribution μ:

$$\forall X = (X_v)_{v \in V} \in \{-,+\}^V, \qquad \mu(X) \propto w(X)$$

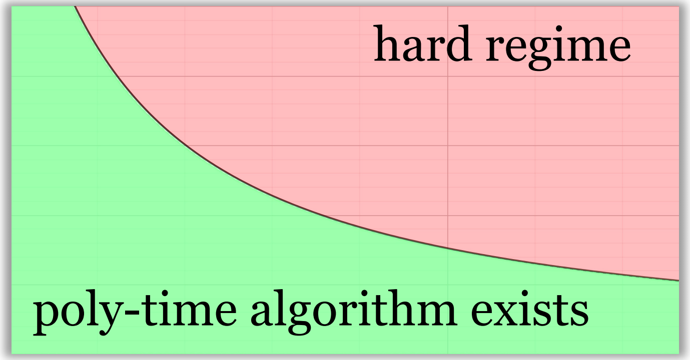


Sampling problem

Draw (approximate) random samples from distribution μ

Computational phase transition

computational complexity of sampling problem
changes sharply around some parameters of μ

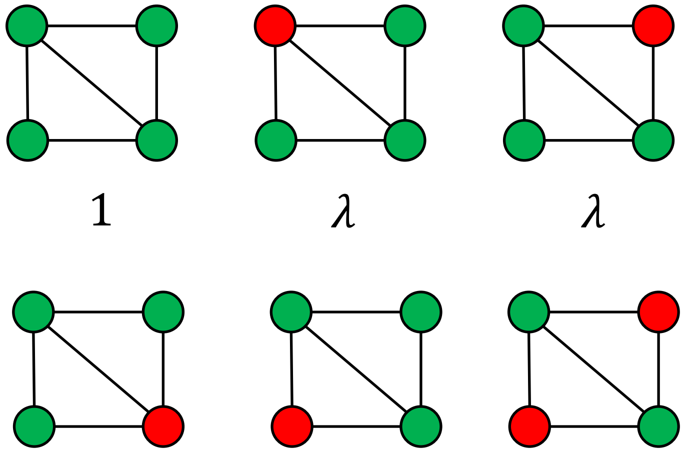


Hardcore model

- Graph $G = (V, E)$: n-vertex and max degree Δ;
- Fugacity parameter $\lambda \in \mathbb{R}_{\geq 0}$;
- Configuration $X \in \{-, +\}^V$
  - $X_v = +$: vertex v is **occupied**
  - $X_v = -$: vertex v is **unoccupied**
- $X \in \Omega$ if occupied vertices form an independent set
- Gibbs distribution μ:

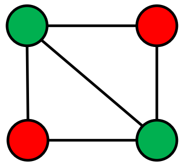
$$\forall X \in \Omega, \qquad \mu(X) \propto w(X) = \lambda^{|X|_+}.$$

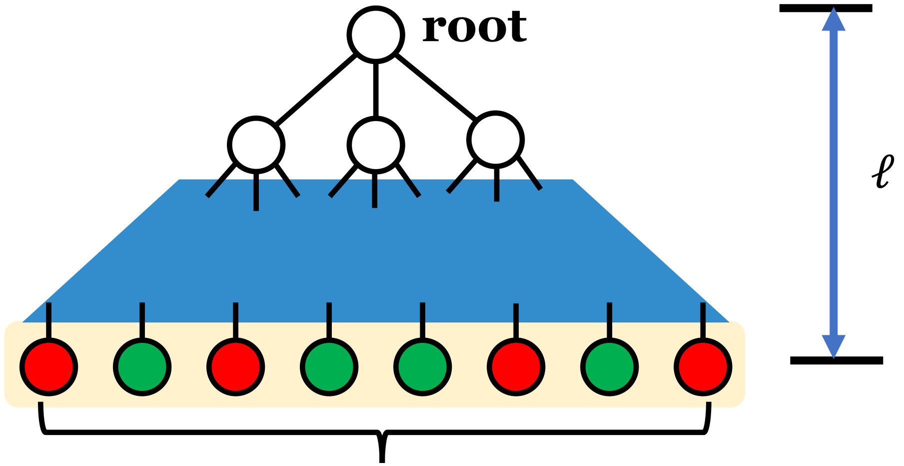
$|X|_+ =$ *number of occupied vertices* $(X_v = +)$



1 $\quad\lambda\quad$ λ

λ $\quad\lambda\quad$ λ^2

$\mu\Bigg($  $\Bigg) = \dfrac{\lambda^2}{1 + 4\lambda + \lambda^2}$



σ: boundary configuration at level ℓ

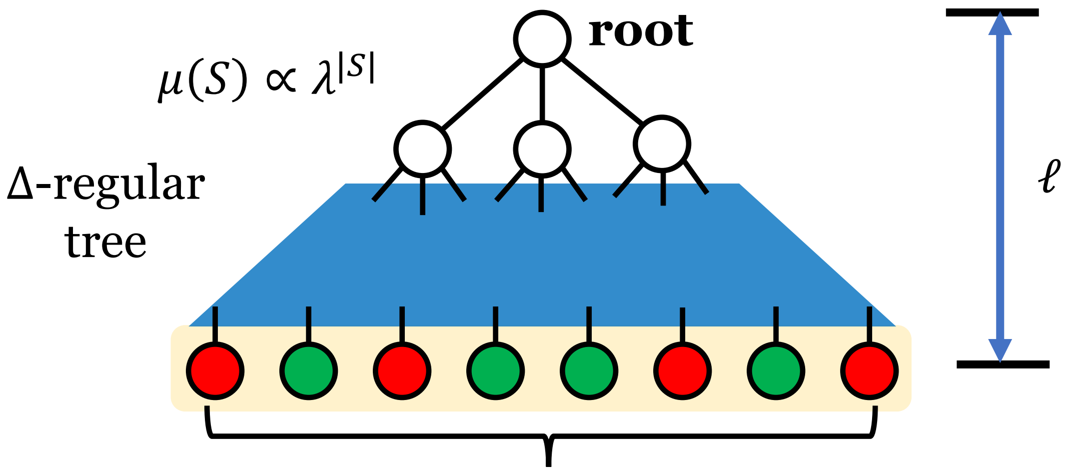
each leaf $\sigma_v \in +$ or $\sigma_v = -$

conditional probability

$$P_{\text{root}}^{\sigma} = \Pr[X(\text{root}) = + \mid \sigma]$$

Computational phase transition

- $\lambda < \lambda_c$: poly-time algorithm for sampling [Weitz06]
- $\lambda > \lambda_c$: no poly-time algorithm unless $\boldsymbol{NP} = \boldsymbol{RP}$ [Sly10]



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each leaf $\sigma_v \in +$ or $\sigma_v = -$

Uniqueness Threshold

$\Pr[X(\text{root}) = + \mid \sigma]$ is independent of σ if $\ell \to \infty$

$$\text{iff } \lambda \le \lambda_c(\Delta) = \frac{(\Delta-1)^{(\Delta-1)}}{(\Delta-2)^{\Delta}} \approx \frac{e}{\Delta}$$

Δ: maximum degree

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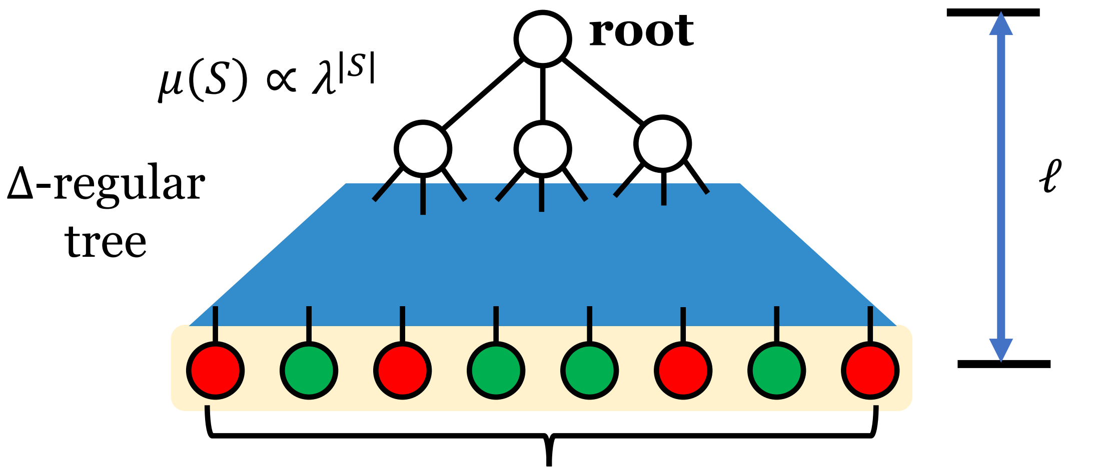

σ: boundary configuration at level ℓ
each leaf $v \in S$ or $v \notin S$

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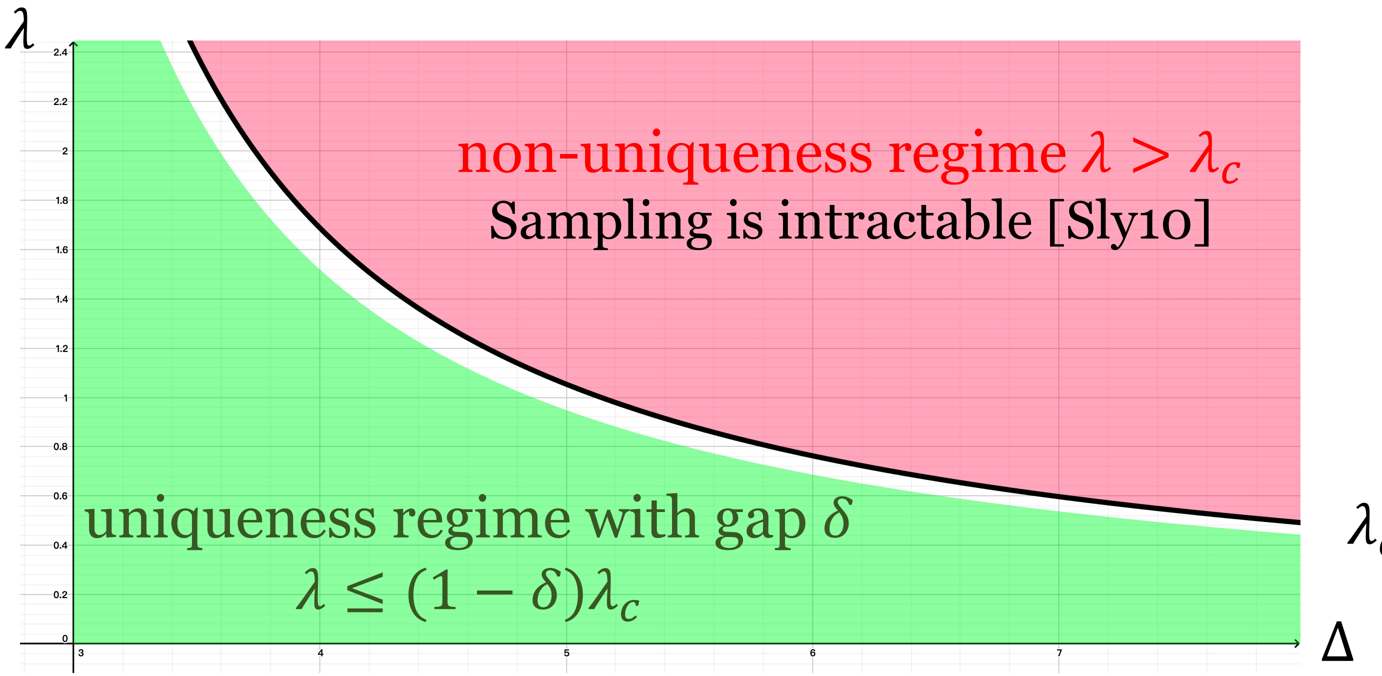
Δ: maximum degree

Computational phase transition

- bounded degree $\Delta = O(1)$
- δ in the exponent of n

- $\lambda \le (1-\delta)\lambda_c$: $n^{O\left(\frac{\log \Delta}{\delta}\right)}$-time algorithms for sampling (via approx. counting) [Weitz06]
- $\lambda > \lambda_c$: no poly-time algorithm unless $\boldsymbol{NP} = \boldsymbol{RP}$ [Sly10]



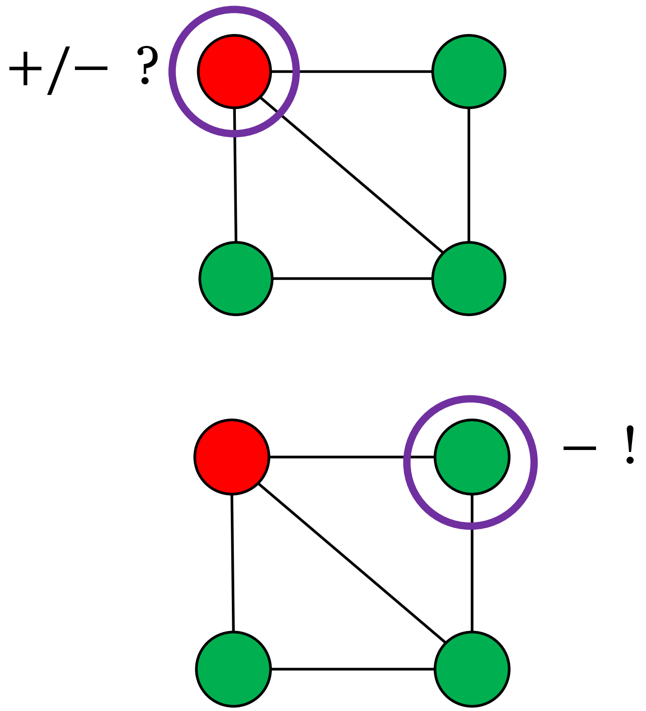
$$\lambda_c(\Delta) = \frac{(\Delta-1)^{(\Delta-1)}}{(\Delta-2)^{\Delta}}$$

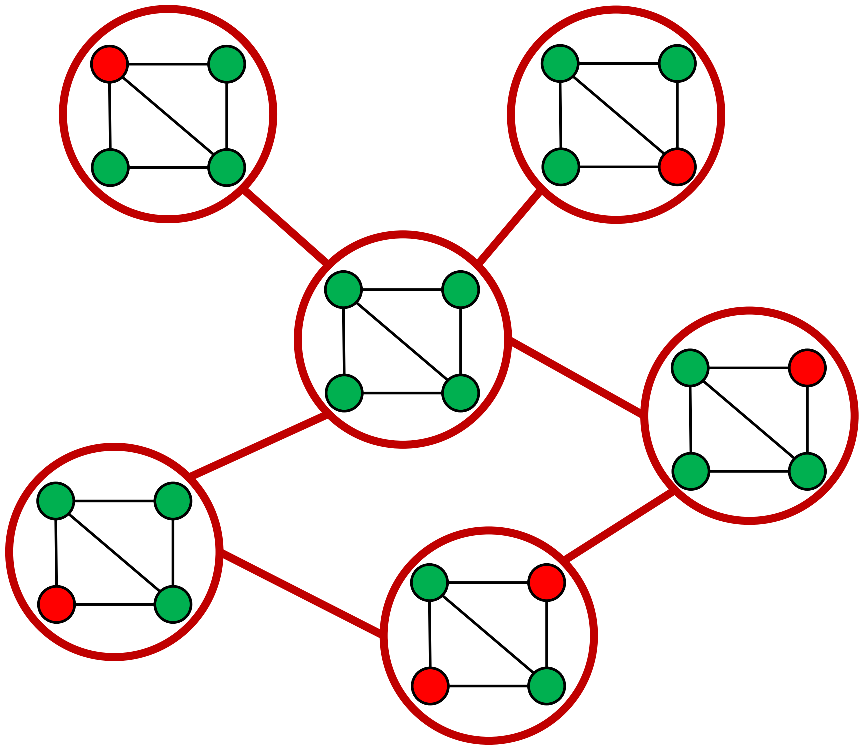
Problem : *fixed parameter trackable* sampling algorithm for hardcore model

Let $\delta > 0$ be an *arbitrary gap*. For any hardcore model with $\lambda \le (1-\delta)\lambda_c(\Delta)$,

can we sample from Gibbs distribution in time $C(\delta) \cdot \mathrm{poly}(n)$?

Glauber dynamics for hardcore model





Glauber dynamics for hardcore model

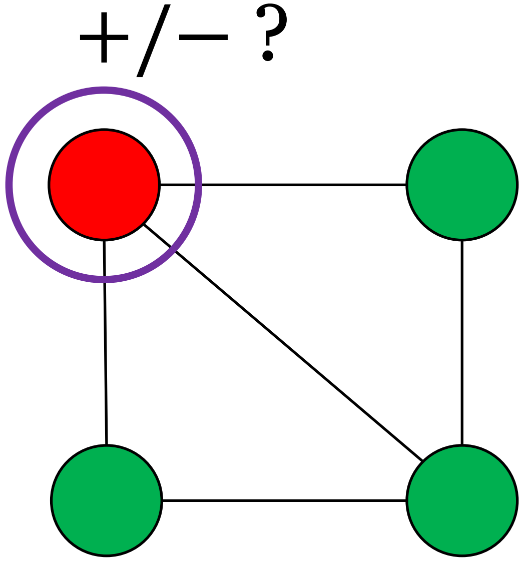
Start from an arbitrary independent set X;

For each transition step **do**

- Pick a vertex v uniformly at random;
- **If** $X_u = -$ for all neighbors u **then**

$$X_v = \begin{cases} + & \text{w.p.} \quad \lambda/(1+\lambda) \\ - & \text{w.p.} \quad 1/(1+\lambda) \end{cases}$$

- **Else** $X_v \leftarrow -$



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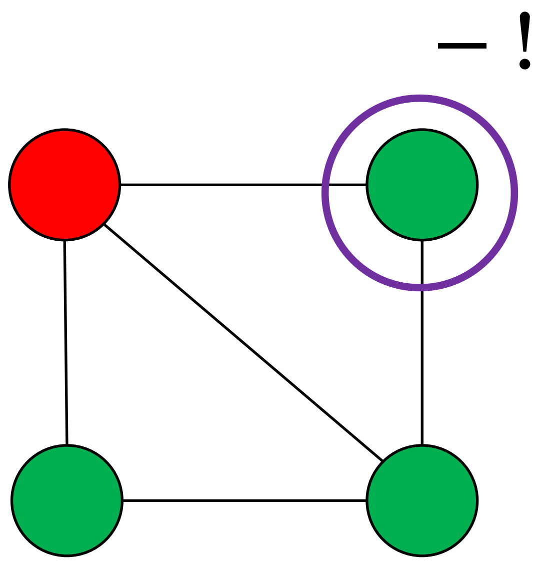

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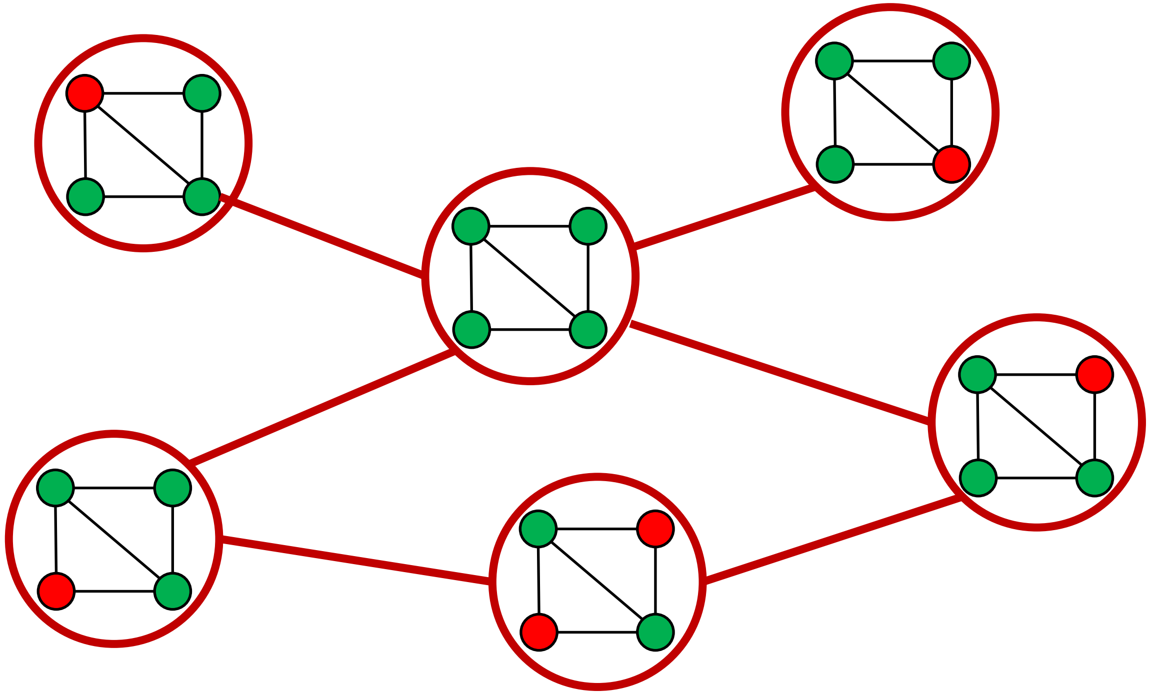

Mixing time: $T_{\mathrm{mix}} = \max_{X_0 \in \Omega} \min\{\, t \mid d_{TV}(X_t, \mu) \leq 0.001 \,\}$,

$d_{TV}(X_t, \mu)$: the *total variation distance* between X_t and μ.

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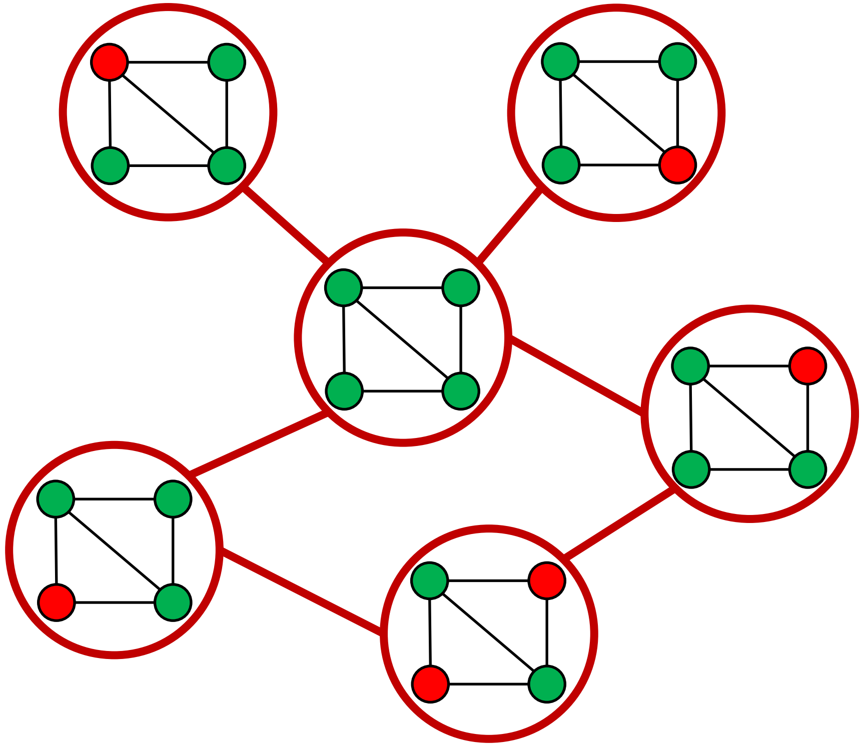

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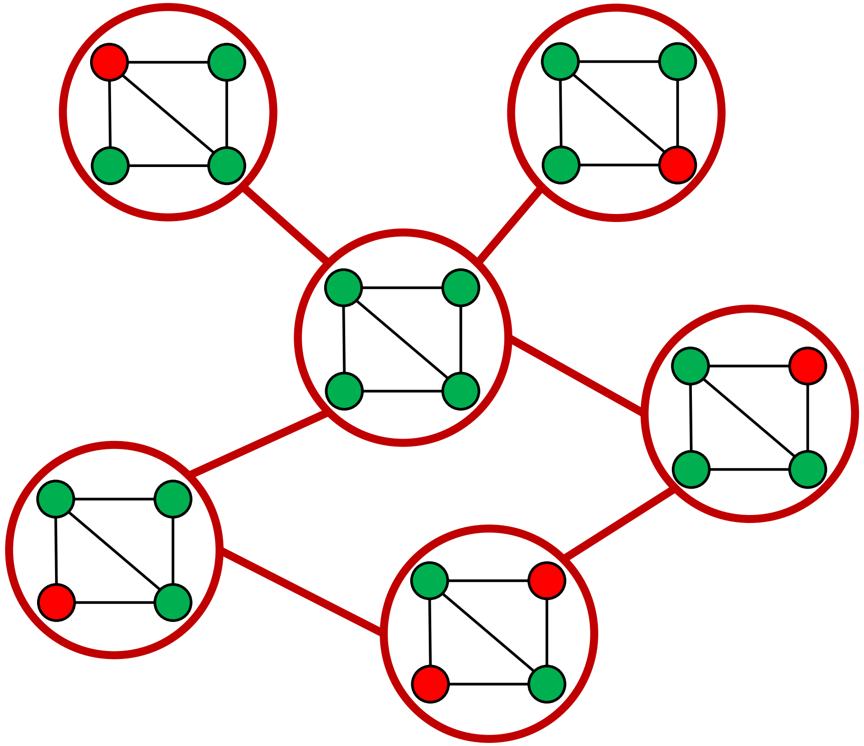

Mixing time (continuous version) T_{mix}

$$T_{\text{mix}} = \max_{X_0 \in \Omega} \min\{\, t \mid d_{TV}(X_N, \mu) \le 0.001, \text{where } N \sim \text{Poisson}(t) \,\}$$

Previous works

Work	Condition	Mixing Time
Dobrushin 1970	$\lambda \le \frac{1-\delta}{\Delta - 1}$	$O\left(\frac{1}{\delta} n \log n\right)$

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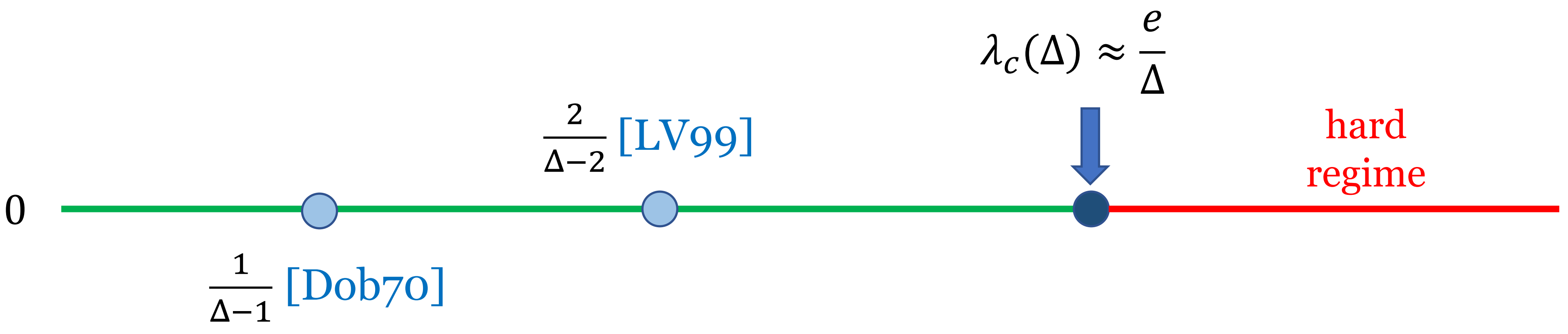

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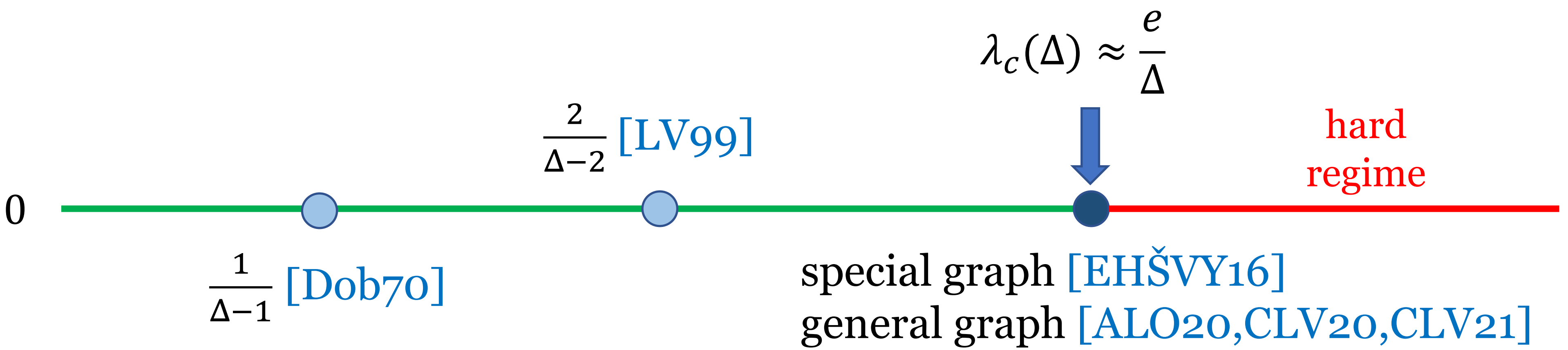

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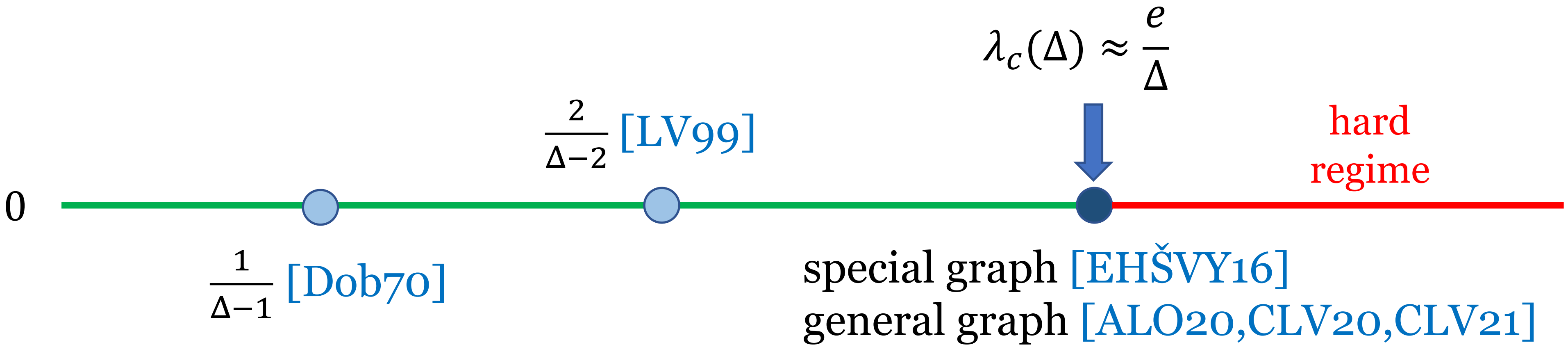

Mixing time of Glauber dynamics when $\lambda \le (1-\delta)\lambda_c$

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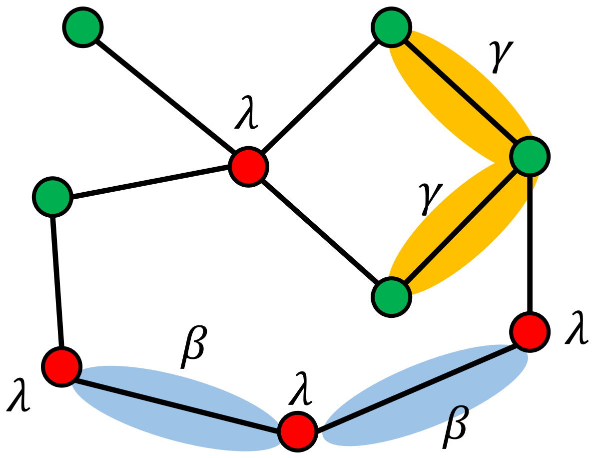
Chen, Eldan, 2022	$e^{O(1/\delta)} n \log n$	Localization Scheme

- $\Omega(n \log n)$ mixing time lower bound (Hayes, Sinclair, 2005)



$\mu(\sigma) \propto \lambda^4 \beta^2 \gamma^2$

Anti-ferro two-spin systems

- Hardcore model
- Ising model
- …

Joint distribution defined by external fields and local interactions

Anti-ferro two-spin systems [Chen, Feng, Yin, Zhang, 2021 & 2022]

For anti-ferro two-spin system that is up-to-Δ unique,

- $O(n^3)$ mixing time
- $O(n \log n)$ mixing time if $\beta, \gamma < 1$ or G is regular

Applications for Ising model:

$O(n \log n)$ for anti-ferro *Ising models* in the uniqueness regime.

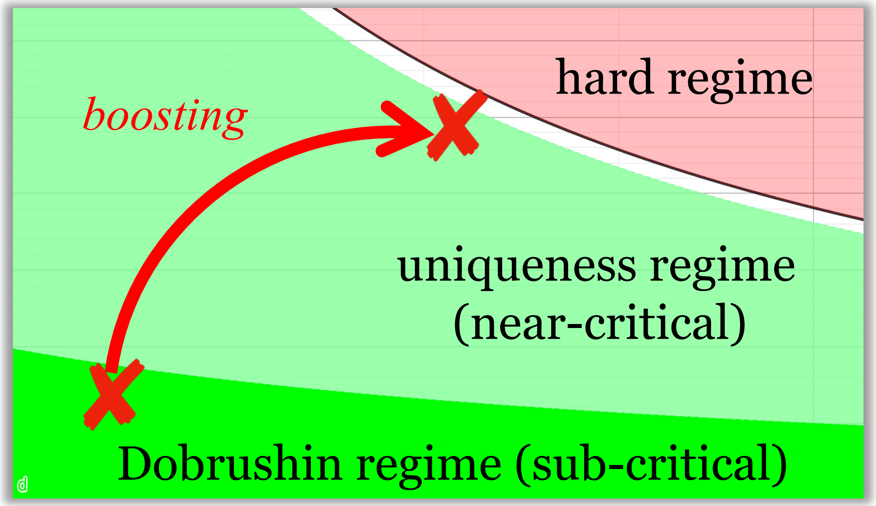
Hardcore model in uniqueness regime

If λ is ***close*** to $\lambda_c(\Delta)$, e.g., $\lambda = 0.999\lambda_c$ (**near-critical**)

analyzing mixing time is ***hard***

- If λ is ***far-away*** from $\lambda_c(\Delta)$, e.g., $\lambda \leq 0.1\lambda_c$ (**sub-critical**)

analyzing mixing time is ***easy***



General technical results (informal)

Boosting mixing results from **sub-critical regime** to **near-critical regime**

for general distributions with certain conditions.

General technical results

- Distributions satisfying *complete spectral independence*

  Boost **spectral gap** (Poincaré constant) → **polynomial** mixing time

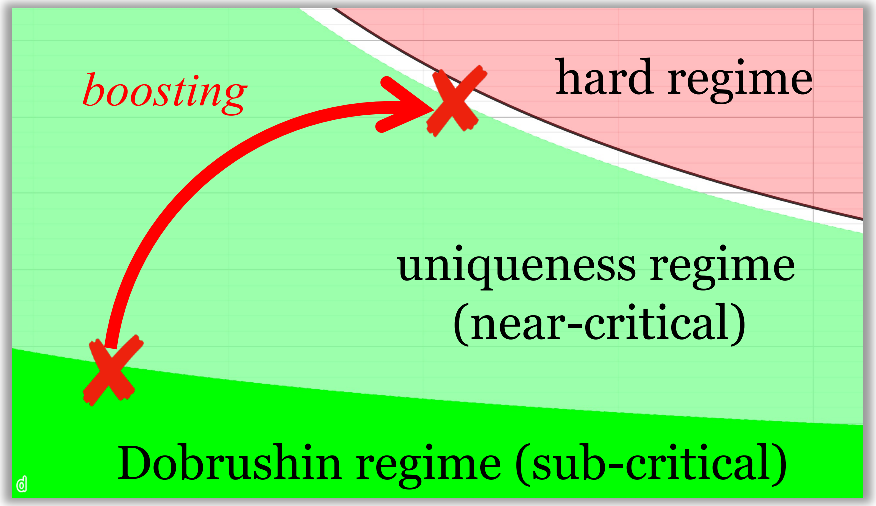
- Distributions satisfying *complete spectral independence* + *marginal ratio bound*

  Boost **modified log-Sobolev constant** → **optimal** $O(n \log n)$ mixing time

Application to Hardcore model

uniqueness condition → complete spectral independence

uniqueness condition → marginal ratio bound



General technical result (I)

Boost the ***spectral gap***

for ***completely spectrally independence distributions***

⇨ $O(n^2 \log n)$ mixing time for hardcore model

Markov chain analysis background

Transition matrix of Glauber dynamics for $\boldsymbol{\mu}$: $P: \Omega \times \Omega \to \mathbb{R}_{\geq 0}$

Reversibility: $\mu(x)P(x,y) = \mu(y)P(y,x)$;

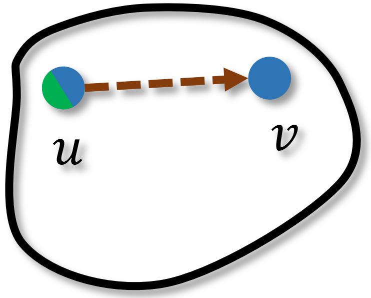
Eigenvalues : P has $|\Omega|$ non-negative real eigenvalues

$$1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|\Omega|} \geq 0$$

Spectral gap (Poincaré constant) $\boldsymbol{\lambda_{\mathrm{gap}}(P) = 1 - \lambda_2}$

$$T_{\mathrm{mix}}(P) = O\left(\frac{1}{\lambda_{\mathrm{gap}}} \log \frac{1}{\mu_{\min}}\right), \qquad \mu_{\min} = \min_{\sigma \in \Omega} \mu(\sigma)$$

Influence matrix and spectral independence



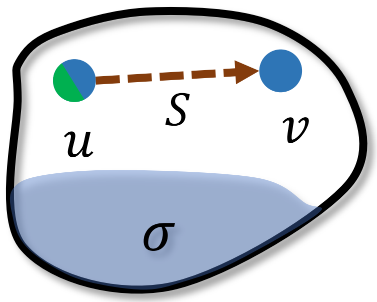
influence on v caused by a **disagreement** on u

μ: a distribution over $\Omega \subseteq \{-1,+1\}^V$

$|V|\times|V|$ **influence matrix** $\Psi \in \mathbb{R}^{V\times V}$ such that

$$\Psi(u,v) = \left|\Pr_{\mu}[\, v = + \mid u = + \,] - \Pr_{\mu}[\, v = + \mid u = - \,]\right|$$

Influence matrix and spectral independence



Influence from u to v
for **conditional distribution**

For any subset $S \subseteq V$, any feasible $\sigma \in \{-1,+1\}^{V\setminus S}$

μ_S^σ distribution on S conditional on σ

influence matrix $\Psi_S^\sigma \in \mathbb{R}^{S\times S}$ for **conditional distribution**

$$\Psi_S^\sigma(u,v) = \left|\Pr_{\mu_S^\sigma}[\, v=+ \mid u=+\,] - \Pr_{\mu_S^\sigma}[\, v=+ \mid u=-\,]\right|$$

Spectral independence (SI) [ALO20, CGŠV21, FGYZ21]

There is a constant $C>0$ s.t. for all conditional distribution μ_S^σ,

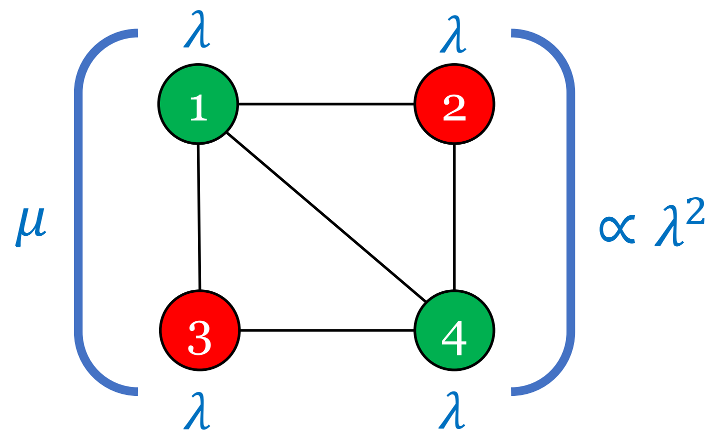
spectral radius of influence matrices $\rho(\Psi_S^\sigma) \le C$.

Complete spectral independence

Magnetizing joint distribution with local fields

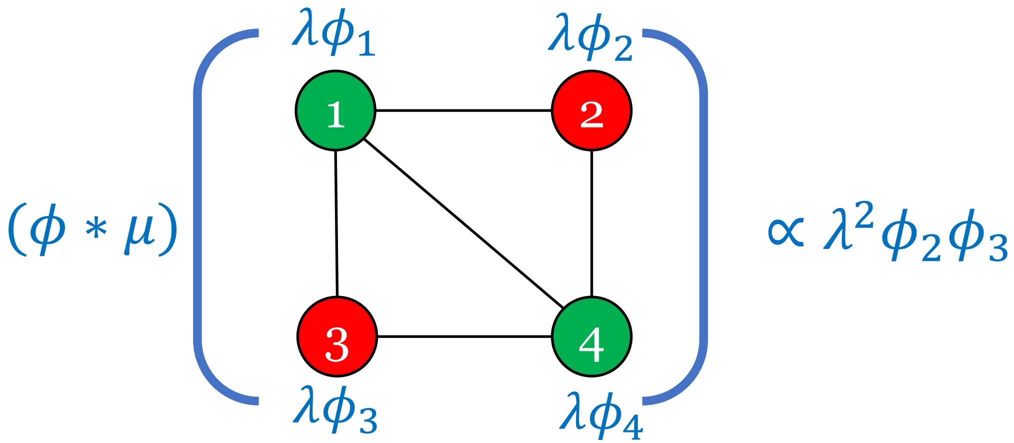
Joint distribution μ over $\{-,+\}^V$, local fields $\phi = (\phi_v)_{v\in V} \in \mathbb{R}^V_{>0}$

$$(\boldsymbol{\phi} * \mu)(\sigma) \propto \mu(\sigma) \prod_{v\in V:\sigma_v=+} \phi_v$$



Hardcore model: $\mu(S) \propto \lambda^{|S|}$

magnetizing



Hardcore mode with local fields

$\mu^{(\phi)}(S) \propto \lambda^{|S|} \prod_{v\in S} \phi_v = \prod_{v\in S} \lambda\phi_v$

Complete spectral independence

Complete Spectral independence [Chen, F., Yin, Zhang, 2021]

There are constant $C > 0$ s.t.

for all local fields $\phi \in (0,1]^V$ (for all $v \in V$, $0 < \phi_v \leq 1$),

$(\boldsymbol{\phi} * \mu)$ is *spectrally independent* with parameter C

Example: hardcore model (G, λ) is ***completely spectrally independent*** if

any hardcore models $(G, (\lambda_v)_{v \in V})$ with $\lambda_v \leq \lambda$

are ***spectrally independent***

Boosting result of spectral gap [Chen, F., Yin, Zhang, 2021]

If μ is C-completely spectrally independent, for any $\theta \in (0,1)$

$$\lambda_{\text{gap}}^{\text{GD}}(\mu) \geq \theta^{O(C)} \cdot \lambda_{\text{gapmin}}^{\text{GD}}(\boldsymbol{\theta} * \mu), \qquad \boldsymbol{\theta}_v = \theta \text{ for all } v \in V$$

$\lambda_{\text{mingap}}^{\text{GD}}(\boldsymbol{\theta} * \mu)$: *minimum spectral gap of Glauber dynamics for all conditional distributions induced by* $\boldsymbol{\theta} * \mu$.

Boosting result of spectral gap [Chen, F., Yin, Zhang, 2021]

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Near-Critical Regime

Boosting with cost $O(1)$

Impose local fields → Easy Regime

Boosting result of spectral gap [Chen, F., Yin, Zhang, 2021]

If μ is C-completely spectrally independent, for any $\theta \in (0,1)$

$$\lambda_{\text{gap}}^{\text{GD}}(\mu) \geq \theta^{O(C)} \cdot \lambda_{\text{gap}}^{\min}(\boldsymbol{\theta} * \mu), \qquad \boldsymbol{\theta}_v = \theta \text{ for all } v \in V$$

Application: polynomial mixing of hardcore model

$\lambda \leq (1-\delta)\lambda_c(\Delta)$

correction decay

[Weitz06,LLY13, ALO20 CLV20]

$O\left(\frac{1}{\delta}\right)$-completely SI

$\theta = 1/50$

$\theta\lambda \leq \frac{1}{2\Delta} \ll \lambda_c$

Dobrushin condition

path coupling [BD97]

coupling v.s. spectral gap [Chen98]

$\lambda_{\text{mingap}}^{\text{GD}}(\boldsymbol{\theta} * \mu) \geq \frac{1}{2n}$

$\lambda_{\text{gap}}^{\text{GD}}(\mu) = \Omega(1/n)$

$T_{\text{mix}} = O(n^2 \log n)$

Proof of boosting results

New Markov chain: *field dynamics*

Field Dynamics

Input: a distribution μ over $\{-1,+1\}^V$, a parameter $\theta \in (0,1)$

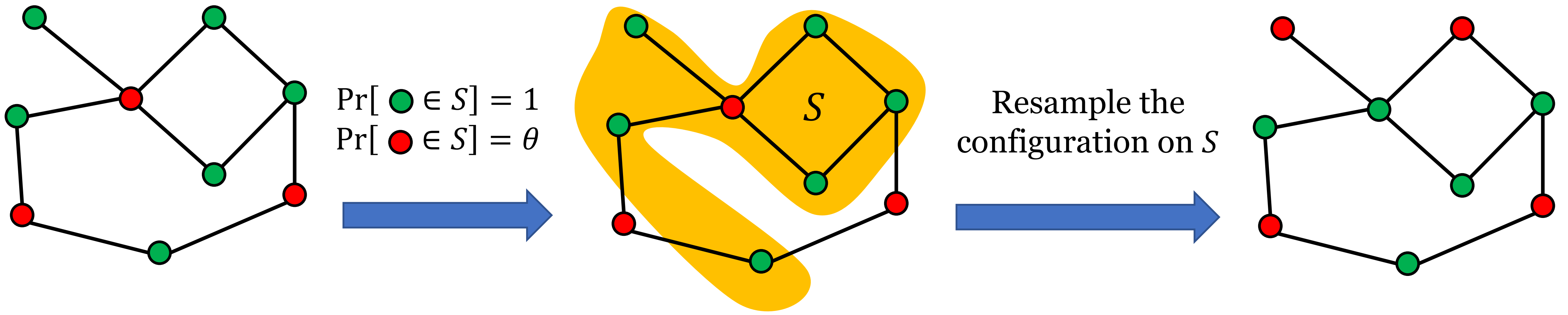
Start from an arbitrary feasible configuration $X \in \{-,+\}^V$

For each t from 1 to T **do**

- Construct $S \subseteq V$ be selecting each $v \in V$ independently with probability

$$p_v = \begin{cases} 1 & \text{if } X_v = - \\ \theta & \text{if } X_v = + \end{cases}$$

- Resample $X_S \sim (\boldsymbol{\theta} * \mu)_S(\cdot \mid X_{V \setminus S})$ conditional distribution induced from $(\boldsymbol{\theta} * \mu)$



Field Dynamics

Input: a distribution μ over $\{-1,+1\}^V$, a parameter $\theta \in (0,1)$

Start from an arbitrary feasible configuration $X \in \{-,+\}^V$

For each transition step $X \to X'$

- Construct $S \subseteq V$ be selecting each $v \in V$ independently with probability
$$p_v = \begin{cases} 1 & \text{if } X_v = - \\ \theta & \text{if } X_v = + \end{cases}$$
- Resample $X_S \sim (\boldsymbol{\theta} * \mu)_S(\cdot \mid X_{V \setminus S})$ conditional distribution induced from $(\boldsymbol{\theta} * \mu)$

Proposition (Field Dynamics): for any $\theta \in (0,1)$

The Field Dynamics $P_{FD}(\theta)$ is irreducible, aperiodic and reversible with respect to μ.

$P_{FD}(\theta)$ has the unique stationary distribution μ.

Comparison lemma of spectral gap

For any distribution μ over $\{-,+\}^V$

$$\lambda_{\text{gap}}^{GD}(\mu) \geq \lambda_{\text{gap}}^{\text{Field}}(\mu,\theta) \cdot \lambda_{\text{mingap}}^{\text{GD}}(\boldsymbol{\theta} * \mu), \qquad \boldsymbol{\theta}_v = \theta \text{ for all } v \in V$$

Spectral gap of field dynamics

If μ is C-completely spectrally independent, for any $\theta \in (0,1)$

$$\lambda_{\text{gap}}^{\text{Field}}(\mu,\theta) \geq \theta^{O(C)}$$

Comparison lemma + Spectral gap ➡ Boosting spectral gap

$$\lambda_{\text{gap}}^{\text{GD}}(\mu) \geq \theta^{O(C)} \cdot \lambda_{\text{mingap}}^{\text{GD}}(\boldsymbol{\theta} * \mu)$$

Comparison lemma of spectral gap

Proved by a calculation

For any distribution μ over $\{-,+\}^V$

$$\lambda_{\text{gap}}^{GD}(\mu) \geq \lambda_{\text{gap}}^{\text{Field}}(\mu,\theta) \cdot \lambda_{\text{mingap}}^{\text{GD}}(\boldsymbol{\theta} * \mu), \qquad \boldsymbol{\theta}_v = \theta \text{ for all } v \in V$$

Stationary distribution is μ $\longleftarrow$ **Field Dynamics** $\longrightarrow$ Transition uses $(\boldsymbol{\theta} * \mu)$

Comparison lemma of spectral gap

For any distribution μ over $\{-,+\}^V$

$$\lambda_{\text{gap}}^{GD}(\mu) \geq \lambda_{\text{gap}}^{\text{Field}}(\mu,\theta) \cdot \lambda_{\text{mingap}}^{\text{GD}}(\boldsymbol{\theta} * \mu), \qquad \boldsymbol{\theta}_v = \theta \text{ for all } v \in V$$

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Comparison lemma + Spectral gap ⟶ Boosting spectral gap

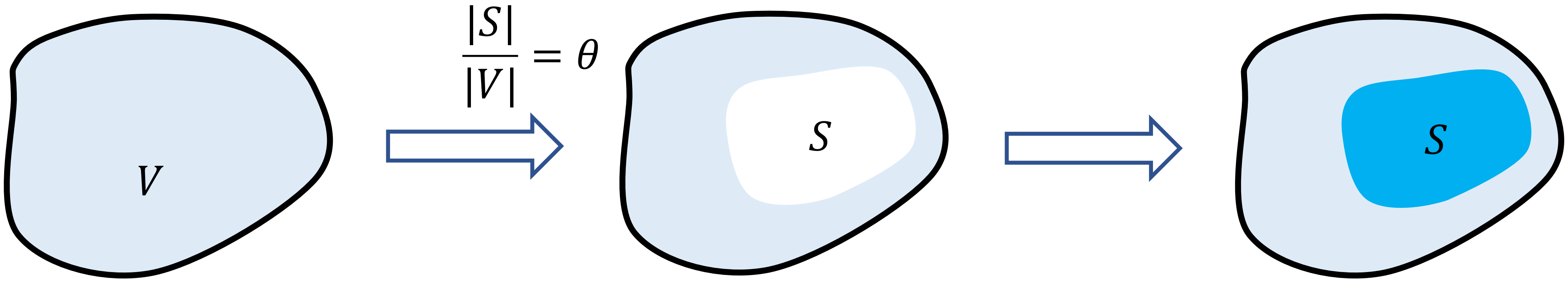
$$\lambda_{\text{gap}}^{\text{GD}}(\mu) \geq \theta^{O(C)} \cdot \lambda_{\text{mingap}}^{\text{GD}}(\boldsymbol{\theta} * \mu)$$

Block dynamics

Standard Markov chain: θ-fractional block dynamics on μ

Transition step: given configuration $X \in \{-,+\}^V$

- pick θ fraction of variables $S \subseteq V$ uniformly at random *Non-Adaptive*
- resample $X_S \sim \mu_S(\cdot \mid X_{\overline{S}})$ *Conditional distribution induced by μ*



$$\boxed{\theta = \frac{1}{|V|}} \Longrightarrow \boxed{\text{Glauber dynamics}}$$

Step-1: k-transformation

Transform μ into a sequence $(\mu_k)_{k\geq 1}$ (μ_k over $\{-1,+1\}^{V_k}$)

Step-2: Connect field dynamics on μ to the block dynamics on μ_k when $k \to \infty$

spectral gap of ***block dynamics*** *on* μ_k when $k \to \infty$

$\Rightarrow$ *spectral gap of* ***field dynamics***

Step-3: analyze θ-fractional ***block dynamics*** on μ_k

μ is completely SI $\xrightarrow{\text{all large } k}$ Spectral gap lower bound of the block dynamics on μ_k

μ over $\{-,+\}^V$
integer $k \geq 1$

k-transformation $\Rightarrow$

μ_k over $\{-,+\}^{V_k}$
$V_k = \{u_1, u_2, \ldots, u_k \mid u \in V\}$

$X \sim \mu$ $\Rightarrow$

For each variable $u \in V$ **do**

- **If** $X(u) = -$, **then** $Y(u_i) = -$ for all $i \in [k]$;
- **If** $X(u) = +$, **then**
  - Sample $j \in \{1,2,..,k\}$ uniformly at random;
  - $Y(u_j) = +$ and $Y(u_i) = -$ for all $i \in [k] \backslash \{j\}$;

$\Rightarrow$ $Y \sim \mu_k$

$X \sim \mu$ $\qquad$ $Y \sim \mu_3$

Original distribution
μ over $\{-,+\}^V$

k-transformation

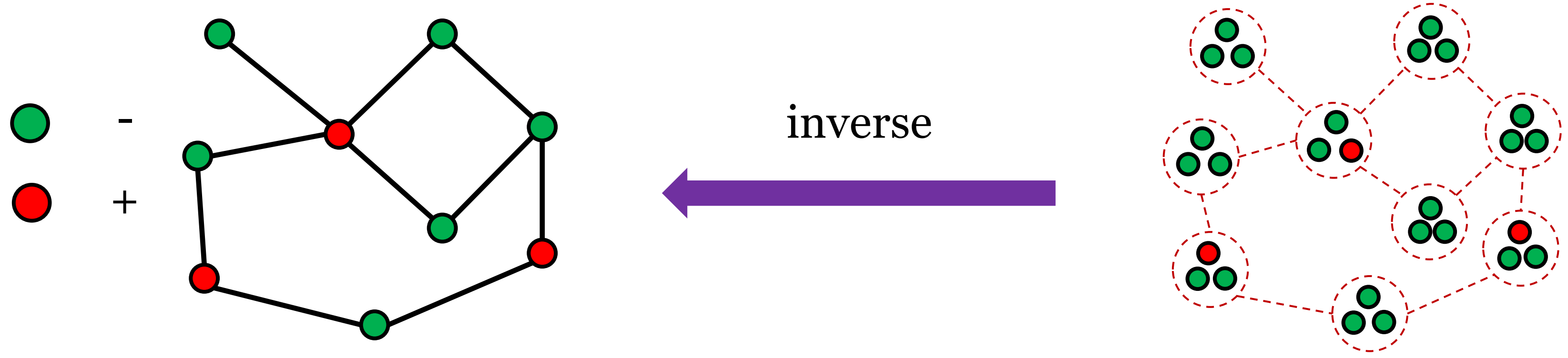
Transformed distribution
μ_k over $\{-,+\}^{V_k}$
$V_k = \{u_1, u_2, \ldots, u_k \mid u \in V\}$

$X \sim \mu$

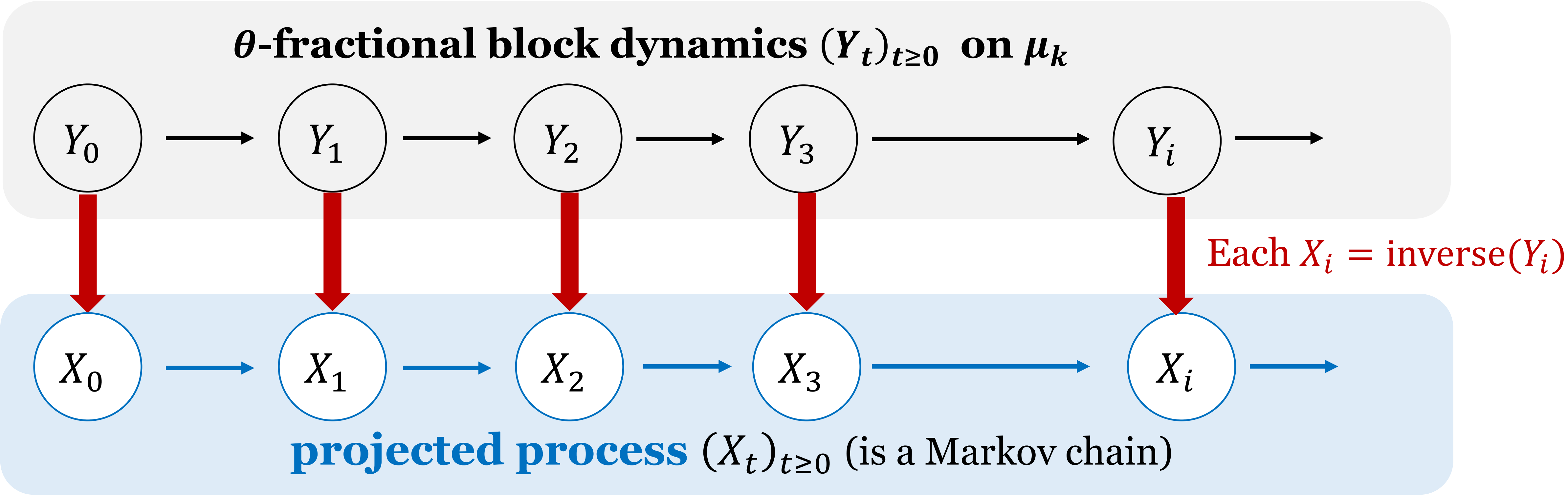
inverse
k-transformation

$Y \sim \mu_k$

$X = \text{inverse}(Y)$ is uniquely fixed by Y



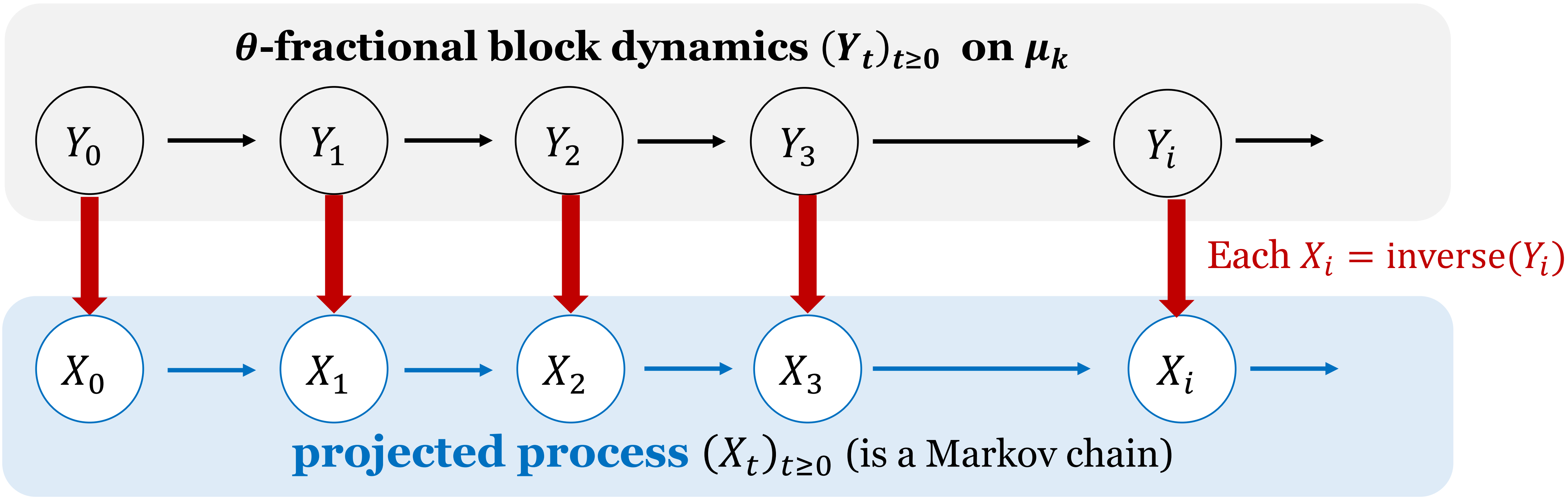
$X \sim \mu$ $\qquad$ $Y \sim \mu_3$



Theorem [Chen, F., Yin, Zhang, 2021]

Field dynamics is the projected process when $k \to \infty$

$$\forall \epsilon > 0, \exists K_0 \text{ s.t. } \forall k \geq K_0$$
$$\forall \sigma, \tau \in \{-,+\}^V, \left| P^{FD}(\sigma,\tau) - P^{\mathrm{Proj}}_{\mu_k}(\sigma,\tau) \right| \leq \epsilon$$



Theorem [Chen, F., Yin, Zhang, 2021]

Field dynamics is the projected process when $k \to \infty$

Block dynamics spectral gap $\Rightarrow$ Field dynamics spectral gap

Analysis of block dynamics

- k-transformation ***preserve*** spectral independence (SI)

C-Complete SI of μ $\xrightarrow{\text{all large } k}$ **$10C$-SI of μ_k**

- Spectral gap of the block [CLV21]

For ***C*-SI** distribution μ_k, **θ-fractional block dynamics**

$$\lambda_{gap}^{\theta-BD}(\mu_k) \geq \left(\frac{\theta}{2}\right)^{2C+2}$$

General technical result (II)

Boost the **modified log-Sobolev constant**

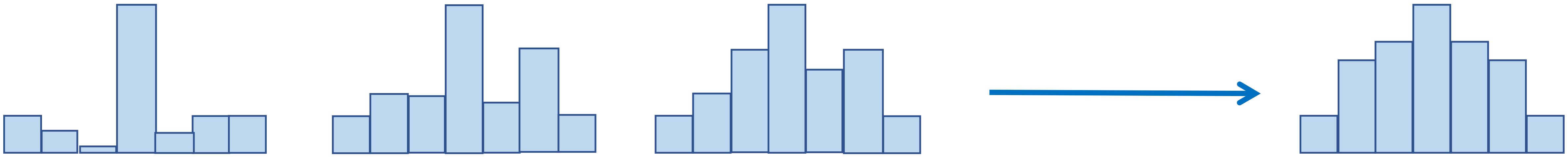
for ***completely spectrally independence distributions***

with marginal ratio bounds

$\Rightarrow$ $O(n \log n)$ mixing time for hardcore model

Why does the spectral gap lower bound imply rapid mixing?

P: Transition matrix of the Glauber dynamics for distribution μ



ν $\qquad$ νP $\qquad$ νP^2 $\qquad$ $\nu P^\infty = \mu$

Spectral gap captures the **decay rate of the χ^2-divergence** for Glauber dynamics

$$D_{\chi^2}(\nu P \parallel \mu) \leq (1 - \lambda_{\text{gap}}) D_{\chi^2}(\nu, \mu)$$

$$D_{\chi^2}(\nu \parallel \mu) = \sum_{\sigma} \frac{\nu^2(\sigma)}{\mu(\sigma)} - 1$$

Modified log-Sobolev constant ρ_{mls}

$$D_{KL}(\nu \parallel \mu) = \sum_{\sigma} \nu(\sigma) \log \frac{\nu(\sigma)}{\mu(\sigma)}$$

captures the **decay rate of the KL-divergence** for Glauber dynamics

Continuous-time Glauber dynamics

$$D_{KL}(\nu_t \parallel \mu) \leq \exp(-2\rho_{\mathrm{mls}}) D_{KL}(\nu_0 \parallel \mu)$$

→

Mixing time

$$T_{\mathrm{mix}} = O\left(\frac{1}{\rho_{\mathrm{mls}}} \log\log \frac{1}{\mu_{\min}}\right)$$

Formal Definition (don't need to know)

$$\rho_{\mathrm{mls}}(P) = \inf\left\{ \frac{\mathcal{E}_P(f, \log f)}{\mathrm{Ent}_\mu(f)} \,\middle|\, \mathrm{Ent}_\mu(f) \neq 0, f: \Omega \to \mathbb{R}_+ \right\}$$

- $\mathcal{E}_P(f, \log f) = \frac{1}{2} \sum_{xy \in \Omega} \mu(x) P(x,y) \big(f_x - f_y\big)\big(\log f_x - \log f_y\big)$
- $\mathrm{Ent}_\mu(f) = \sum_{x \in \Omega} \mu(x) f_x \log f_x + (\sum_{x \in \Omega} \mu(x) f_x) \log(\sum_{x \in \Omega} \mu(x) f_x)$

(C, ϵ)-Complete SI

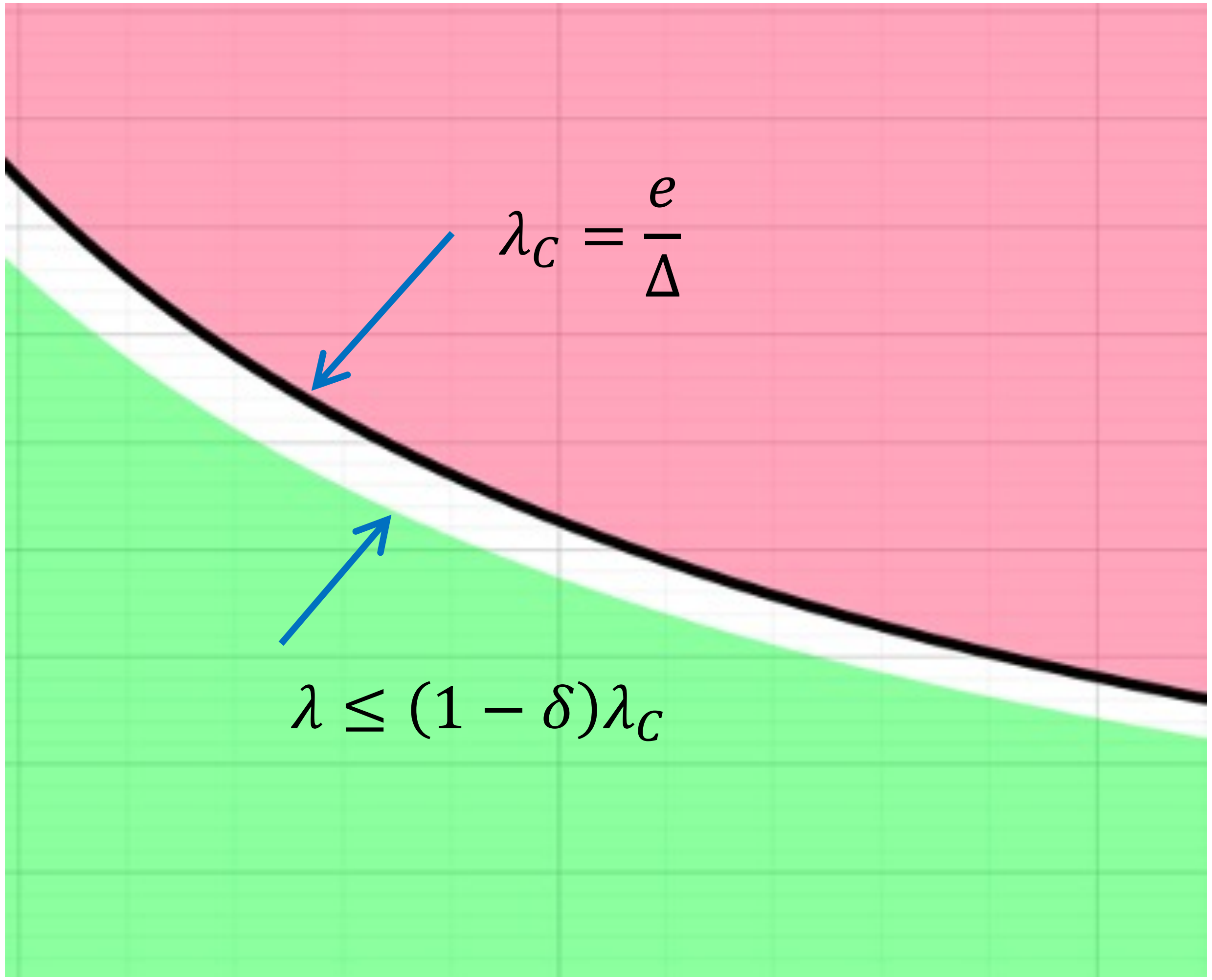
There are constants $C > 0$ and $\epsilon \geq 0$ s.t.

for all local fields $\phi \in (0,1+\epsilon]^V$

$(\forall,\ v \in V, 0 < \phi_v \leq 1+\epsilon)$,

$(\boldsymbol{\phi} * \mu)$ is *SI* with parameter C

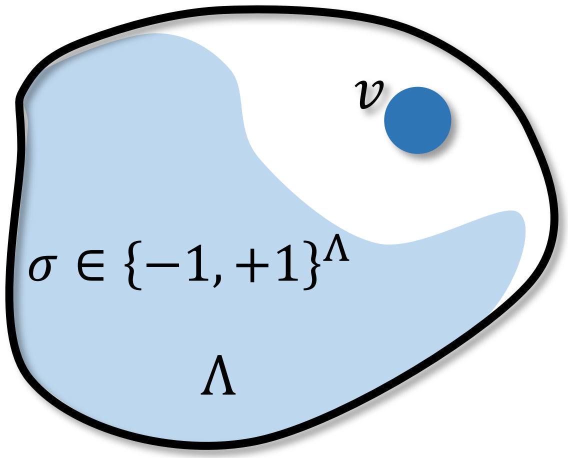
$$\boxed{\epsilon = \frac{\delta}{10}} \Rightarrow \boxed{\left(O\left(\frac{1}{\delta}\right), \epsilon\right)\text{-CSI}}$$



Marginal stability [Chen, F., Yin, Zhang, 2022]

For any pinning $\sigma \in \{-,+\}^\Lambda$ and $v \notin \Lambda$, let

marginal ratio $$R_v^\sigma = \frac{\mu_v^\sigma(+)}{\mu_v^\sigma(-)},$$



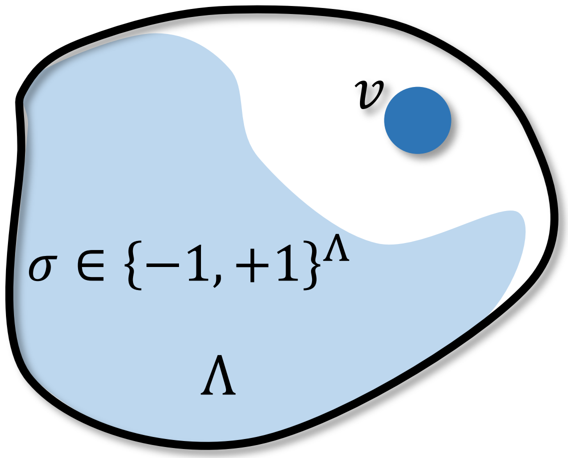
marginal ratio R_v^σ

Marginal stability [Chen, F., Yin, Zhang, 2022]

For any pinning $\sigma \in \{-,+\}^{\Lambda}$ and $v \notin \Lambda$, let

marginal ratio $$R_v^{\sigma} = \frac{\mu_v^{\sigma}(+)}{\mu_v^{\sigma}(-)},$$

- Bound on the ratio $R_v^{\sigma} \leq \zeta$



marginal ratio R_v^{σ}

Marginal stability [Chen, F., Yin, Zhang, 2022]

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- Stability of the ratio $R_v^{\sigma} \le \zeta R_v^{\sigma_S}$

marginal ratio R_v^{σ} marginal ratio $R_v^{\sigma_S}$

Marginal stability [Chen, F., Yin, Zhang, 2022]

For any pinning $\sigma \in \{-,+\}^\Lambda$ and $v \notin \Lambda$, let

marginal ratio $$R_v^\sigma = \frac{\mu_v^\sigma(+)}{\mu_v^\sigma(-)},$$

- Bound on the ratio $R_v^\sigma \le \zeta$
- Stability of the ratio $R_v^\sigma \le \zeta R_v^{\sigma s}$

Complete Marginal stability [Chen, F., Yin, Zhang, 2022]

$(\boldsymbol{\phi} * \mu)$ is marginally stable for all $\boldsymbol{\phi} \in (0,1]^V$

Main result: boosting modified log-Sobolev constant

Boosting result of modified log-Sobolev constant [Chen, F., Yin, Zhang, 2022]

If μ is (C,ϵ)-completely spectrally independent

and ζ-completely marginally stable,

then for any $\theta \in (0,1)$

$$\rho_{\text{mls}}^{\text{GD}}(\mu) \geq f(\theta, C, \epsilon, \zeta) \cdot \rho_{\text{minmls}}^{\text{GD}}(\boldsymbol{\theta} * \mu), \qquad \boldsymbol{\theta}_v = \theta \text{ for all } v \in V$$

Boosting modified log-Sobolev constant
with cost $O(1)$

optimal mixing time bound

Marginal stability condition：

strong: guarantee the modified log-Sobolev constant bound

mild: verifiable for general 2-spin systems

Application: Optimal mixing of hardcore model

$\lambda \le (1-\delta)\lambda_c(\Delta)$

correlation decay
marginal recursion
[Weitz06,LLY13, ALO20 CLV20]

Complete SI & complete marginal stable with

- $C = O(1/\delta)$
- $\epsilon = \Theta(1/\delta)$
- $\eta = O(1)$

$\theta = 1/50$

$\theta\lambda \le \frac{1}{2\Delta} \ll \lambda_c$

Ricci curvature
[EHMT17]

$\rho^{\mathrm{GD}}_{\mathrm{minmls}}(\boldsymbol{\theta} * \mu) \ge \frac{1}{4n}$

$\rho^{\mathrm{GD}}_{\mathrm{mls}}(\mu) = \Omega(1/n)$

$T_{\mathrm{mix}} = O(n \log n)$

Boosting Spectral Gap

Field dynamics: Mixing lemma

Complete Spectral independence

$\Rightarrow$ Spectral gap of field dynamics
$\lambda_{\text{gap}}^{\text{Field}}(\mu, \theta) = \Omega(1)$

Field dynamics: Comparison lemma

$$\lambda_{\text{gap}}^{GD}(\mu) \geq \lambda_{\text{gap}}^{\text{Field}}(\mu, \theta) \cdot \lambda_{\text{mingap}}^{\text{GD}}(\boldsymbol{\theta} * \mu)$$

Boosting Spectral Gap

Field dynamics: Mixing lemma

Complete Spectral independence

$\Rightarrow$

field dynamics has some
χ^2-divergence decay with
rate $\alpha(\mu,\theta) = \lambda_{\text{gap}}^{\text{Field}}(\mu,\theta) = \Omega(1)$

Field dynamics: Comparison lemma

$$\lambda_{\text{gap}}^{GD}(\mu) \geq \alpha(\mu,\theta) \cdot \lambda_{\text{mingap}}^{\text{GD}}(\boldsymbol{\theta} * \mu)$$

Boosting Spectral Gap

Field dynamics: Mixing lemma

Complete Spectral independence

field dynamics has some χ^2-divergence decay with rate $\alpha(\mu,\theta)=\lambda_{\text{gap}}^{\text{Field}}(\mu,\theta)=\Omega(1)$

Field dynamics: Comparison lemma

$$\lambda_{\text{gap}}^{GD}(\mu)\geq \alpha(\mu,\theta)\cdot\lambda_{\text{mingap}}^{\text{GD}}(\boldsymbol{\theta}*\mu)$$

Boosting MLS constant

Field dynamics: Mixing lemma

Complete Spectral independence
Marginal stability

field dynamics has some KL-divergence decay with rate $\beta(\mu,\theta)=\Omega(1)$

Field dynamics: Comparison lemma

$$\rho_{\text{mls}}^{GD}(\mu)\geq \beta(\mu,\theta)\cdot\rho_{\text{minmls}}^{\text{GD}}(\boldsymbol{\theta}*\mu)$$

Boosting Spectral Gap

Field dynamics: Mixing lemma

Complete Spectral independence

field dynamics has some χ^2-divergence decay with rate $\alpha(\mu,\theta) = \lambda_{\text{gap}}^{\text{Field}}(\mu,\theta) = \Omega(1)$

Field dynamics: Comparison lemma

$$\lambda_{\text{gap}}^{GD}(\mu) \geq \alpha(\mu,\theta) \cdot \lambda_{\text{mingap}}^{\text{GD}}(\boldsymbol{\theta} * \mu)$$

Boosting MLS constant

Field dynamics: Mixing lemma

Complete Spectral independence
Marginal stability

field dynamics has some KL-divergence decay with rate $\beta(\mu,\theta) = \Omega(1)$

main contribution

Field dynamics: Comparison lemma

$$\rho_{\text{mls}}^{GD}(\mu) \geq \beta(\mu,\theta) \cdot \rho_{\text{minmls}}^{\text{GD}}(\boldsymbol{\theta} * \mu)$$

Summary

- New Markov chain: field dynamics
  Boost mixing results for Glauber dynamics
- Applications
  Optimal $O(n \log n)$ mixing for hardcore / Ising model in the uniqueness regime

Thank you!

Open problem

- Optimal mixing for all anti-ferro two spin systems in the uniqueness regime
- *General distributions* beyond the Boolean domain i.e., q-coloring
- More applications of the field dynamics [algorithmic application AJKPV21]
- Other version of field dynamics?