Rapid mixing from spectral independence beyond the Boolean domain

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Glauber dynamics

Sampling from joint distribution

Set of variables V

Finite domain $[q] = \{1,2,\dots,q\}$ for $q \geq 2$

Joint distribution μ over $\Omega = \mathrm{supp}(\mu) \subseteq [q]^V$

Problem draw random samples from μ

Fundamental MCMC: Glauber dynamics

Start from an arbitrary feasible configuration $X \in \Omega$;

For each t from 1 to T **do**

- pick a variable $v \in V$ uniformly at random;
- resample $X_v \sim \mu_v(\cdot \mid X_{V\setminus\{v\}})$;

Return $\boldsymbol{X}$;

Example: proper q-coloring

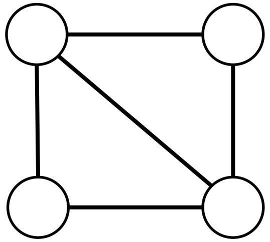
Uniform proper q-coloring

Undirected graph $G = (V, E)$

Finite set of colors $[q] = \{1, 2, \ldots, q\}$

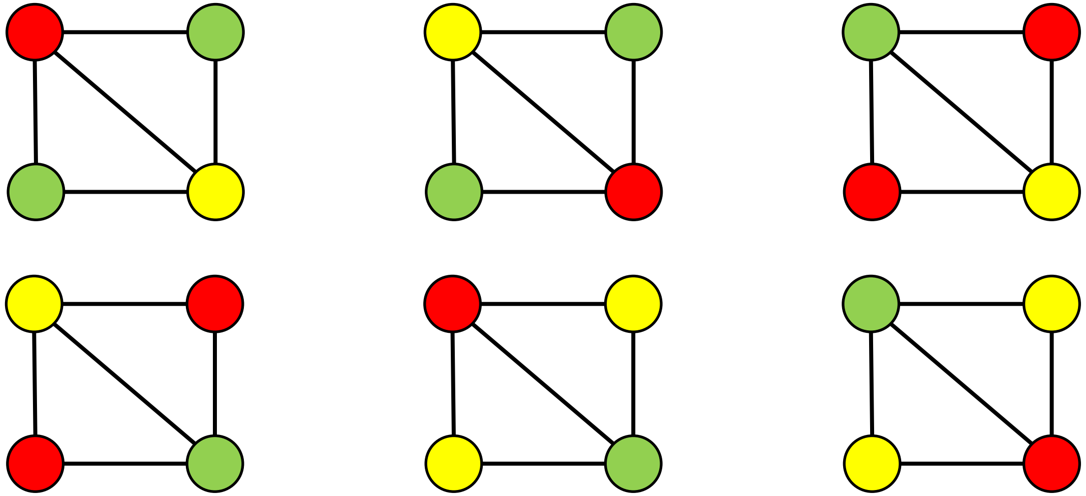
Gibbs distribution μ **uniform distribution** over Ω

$$\Omega = \{X \in [q]^V \mid X \text{ is a proper coloring}\}$$



graph $G = (V, E)$

colors $[q] = \{$  $\}$



Ω: set of all proper colors

Example: proper q-coloring

Uniform proper q-coloring

Undirected graph $G = (V, E)$

Finite set of colors $[q] = \{1,2,\ldots,q\}$

Gibbs distribution μ **uniform distribution** over $\Omega = \{X \in [q]^V \mid X \text{ is a proper coloring}\}$

Problem sample proper coloring u.a.r.

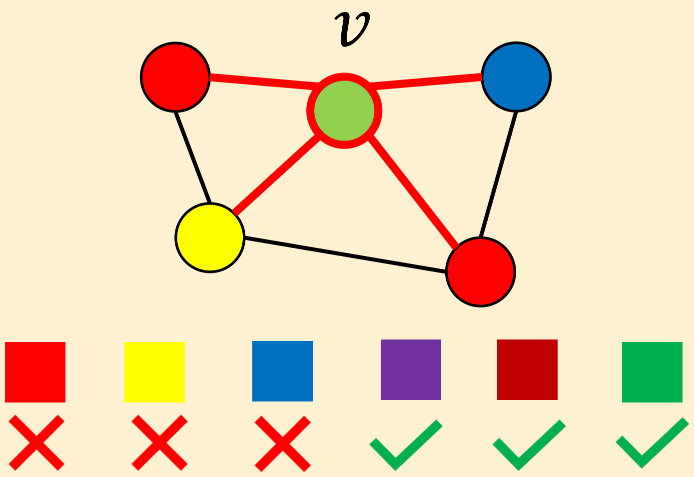
Glauber dynamics for proper q-coloring

Start from an arbitrary proper coloring $X \in \Omega$;

For each t from 1 to T **do**

- pick a vertex $v \in V$ uniformly at random;
- resample X_v from $[q] \backslash \{X_u \mid u \in \Gamma(\text{v})\}$ uniformly at random;

Return X;



Convergence

Glauber dynamics: Markov chain over Ω

Transition Matrix $P \in \mathbb{R}^{\Omega \times \Omega}$

Glauber dynamics is *reversible*

detailed balance equation with respect to μ

$$\forall X, Y \in \Omega, \mu(X)P(X,Y) = \mu(Y)P(Y,X)$$

$\Rightarrow$ ***Stationary distribution*** $\mu P = \mu$

move among any states with positive probability

Proposition (convergence)

If Glauber dynamics is **connected**, it converges to **unique** stationary distribution μ.

If $\boldsymbol{q \geq \Delta + 2}$, Glauber dynamics converges to uniform distribution over q-colorings.

Mixing time

How fast does the Glauber dynamics converge to stationary distribution μ?

Glauber dynamics $X_0, X_1, X_2, \ldots$ where each $X_i \in \Omega \subseteq [q]^V$

Mixing time

$$T_{\text{mix}} = \max_{X_0 \in \Omega} \min \left\{ t \,\middle|\, d_{TV}(X_t, \mu) \le \frac{1}{4e} \right\},$$

$d_{TV}(X_t, \mu)$: the **total variation distance** between X_t and μ.

The Glauber dynamics is **rapid mixing** if

$$T_{\text{mix}} = \text{Poly}(n) \qquad n = |V| = \#\{\text{variables}\}$$

✓ Sample from an exponential space $|\Omega| = \exp(O(n))$ within polynomial steps $T_{\text{mix}} = \text{poly}(n)$.

Open problems

Under what **condition** of the distribution $\boldsymbol{\mu}$

the Glauber dynamics for μ rapid mixing ?

Under what **relation** between $\boldsymbol{q}$ and max degree $\boldsymbol{\Delta}$

the Glauber dynamics for coloring rapid mixing ?

Previous works

Glauber dynamics for graph coloring

General graphs [Jer95,Vig00, SS97, CDMPP19]

current best result $q \geq \left(\frac{11}{6} - \epsilon_0\right)\Delta$ [CDMPP19]

Special graphs [DF01, Hay03,HV03,GMP05, HV06, Mol04, Hay13,DFHV13]

High-dimensional expansion (HDX)

Strongly log-concave distribution [ALOV19, CGM19]

Spectral independence with **Boolean domain $\{0,1\}^V$** [ALO20]

Spectral Independence → Mixing up to uniqueness for **hardcore model** [ALO20] / **anti-ferro 2-spin systems** [CLV20]

Previous works

Rapid mixing condition	Techniques
Dobrushin's condition [Dob70, Hay06]	**Coupling**
Path coupling contraction [BD97]	
Spin system on girds with strong spatial mixing [DSVW02]	
High-dimensional expansion (HDX)	**High-dimensional expansion (HDX)**
Strongly log-concave distribution [ALOV19, CGM19]	
Spectral independence with **Boolean domain $\{0,1\}^V$** [ALO20]	

Spectral Independence

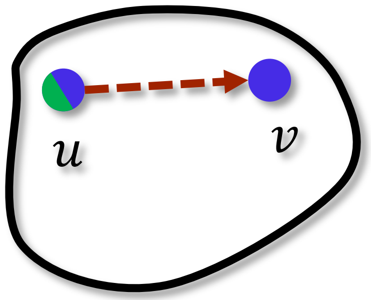
A ***tight mixing condition*** for ***hardcore model*** [ALO20,CLV20]

Our results

- A **spectral independence** condition for **general distribution.**
- **Rapid mixing** of Glauber dynamics from spectral independence.
  - combinatorial proof: **coupling;**
  - algebraic proof for Boolean variables [ALO20].
- Application: a new rapid mixing regime for **graph coloring.**
  - relate spectral independence with **correlation decay;**
  - a refined **recursive coupling** [GMP05] argument.

Our results

Result (I). A spectral independence condition beyond the Boolean domain.

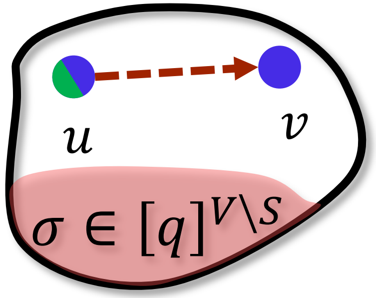


maximum influence on v caused by a **disagreement** on u

μ: a distribution over $\Omega \subseteq [q]^V$

$|V|\times|V|$ **influence matrix** $\Psi \in \mathbb{R}^{V\times V}$ with $\Psi(u,u)=0$ and

$$\Psi(u,v) = \max_{i,j\in[q]} d_{TV}\big(\mu_v(\cdot \mid u\leftarrow i), \mu_v(\cdot \mid u \leftarrow j)\big)$$



influence matrix for **conditional distribution**

For any subset $S \subseteq V$, any feasible $\sigma \in [q]^{V\setminus S}$

μ_S^σ distribution on S conditional on σ

influence matrix $\Psi_S^\sigma \in \mathbb{R}^{S\times S}$ for **conditional distribution**

$$\Psi_S^\sigma(u,v) = \max_{i,j\in[q]} d_{TV}\big(\mu_v^\sigma(\cdot \mid u\leftarrow i), \mu_v^\sigma(\cdot \mid u \leftarrow j)\big)$$

Our results

Result (I). **A spectral independence condition beyond the Boolean domain**

Spectral independence [This work]

There is a constant $C > 0$ such that

for all conditional distributions μ_S^σ,

spectral radius of influence matrices $\rho(\Psi_S^\sigma) \le C$.

Spectral independence for Boolean variables[Anari, Liu, Oveis Gharan 20]

Distribution over **Boolean domain** $\{0,1\}^V$

signed influence matrix: $I_S^\sigma(u,v) = \mu_v^\sigma(1 \mid u \leftarrow 1) - \mu_v^\sigma(1 \mid u \leftarrow 0)$.

Relation: $\Psi_S^\sigma(u,v) = |I_S^\sigma(u,v)|$.

Spectral independence: for all influence matrices, max eigenvalue $\lambda_{\max}(I_S^\sigma) \le C$.

Our results

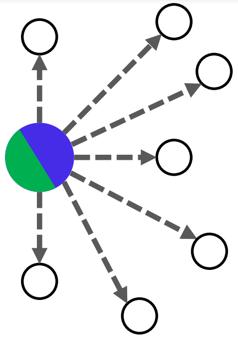
Result (II). **Rapid mixing** of Glauber dynamics from **spectral independence**

Theorem [This work]

μ is spectrally independent with constant C

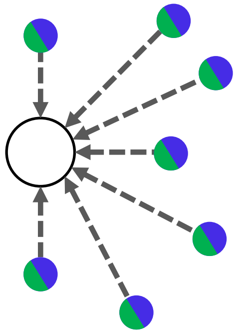
$$\Rightarrow T_{\text{mix}} = O\left(n^{1+2C} \log\left(\frac{1}{\mu_{\min}}\right)\right),$$

where $\mu_{\min} = \min_{X \in \Omega} \mu(X)$.



Bounded one-to-all influence

spectral radius $\leq \sum_{v \in S} \Psi_S^{\sigma}(u, v) \leq C$



Bounded all-to-one influence

spectral radius $\leq \sum_{u \in S} \Psi_S^{\sigma}(u, v) \leq C$

Spectral Independence $\rightarrow$ Rapid Mixing

Our results

Result (III). Rapid mixing for q-coloring on triangle-free graph with $q > 1.763\Delta$

Theorem [This work]

constant

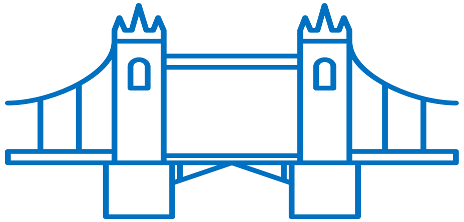
Triangle free graph and $q \geq (\alpha + \delta)\Delta$ where $\alpha \approx 1.763$ s.t. $\alpha = \exp(1/\alpha)$

$$\Rightarrow T_{\text{mix}} \leq n^{2+O(1/\delta)} \log q\,.$$

Work	Regime	Girth	Addition condition	Mixing time
[GMP05]	$q > \alpha\Delta$	≥ 4	$\Delta = O(1)$ + amenable graph	$O(n^2)$
[HV06]	$q \geq (\alpha + \delta)\Delta$	≥ 4	$\Delta = \Omega(\log n)$	$O(n \log n)$
[DFHV13]	$q \geq (\alpha + \delta)\Delta$	≥ 5	$\Delta \geq \Delta_0(\delta)$	$O(n \log n)$
This work	$q \geq (\alpha + \delta)\Delta$	≥ 4	--	$n^{2+O(1/\delta)} \log q$

Proof outline

Spectral Independence



Bridge: HDX

Rapid mixing of Glauber dynamics

Boolean domain [ALO20]

General domain [this work]

Rapid mixing of **local walks**

local-to-global [AL20]

Rapid mixing of **global walk**

Graph Coloring

Decay analysis [this work]

Based on recursive coupling [GMP05]

Spectral Independence

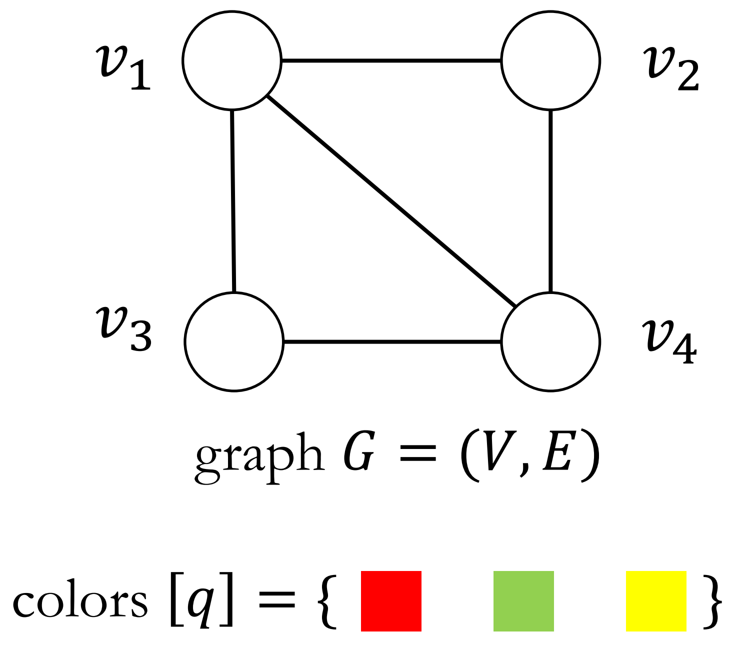
Rapid mixing of Glauber dynamics

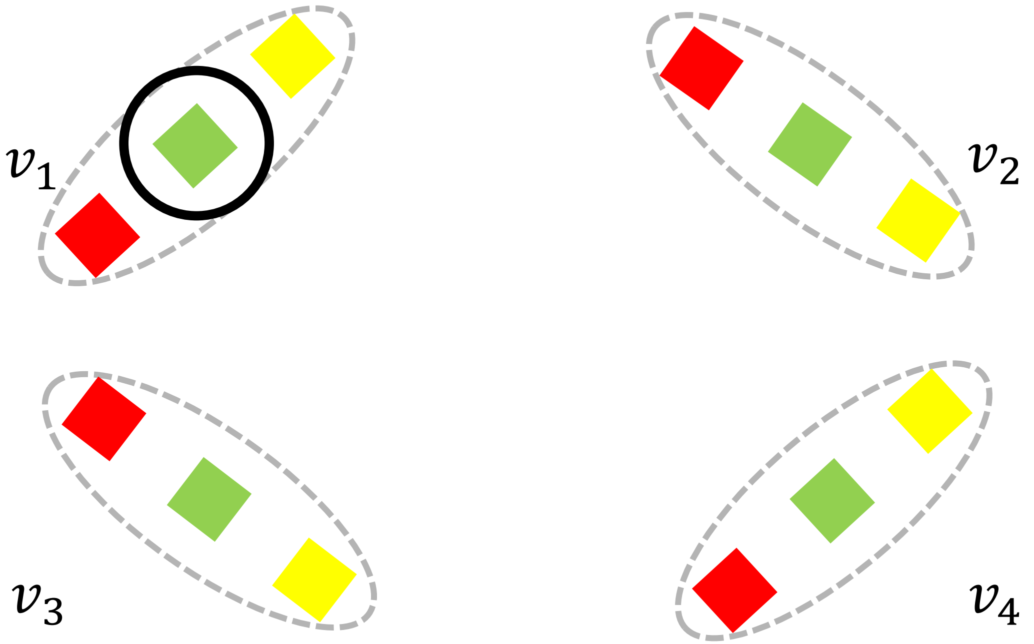
Lazy local random walk

State space $U = \{ (v,i) \mid v \in V, i \in [q] \}$

Current state $(v,i) \in U$. Transition $(v,i) \to (u,j)$

- pick a vertex $u \in V$ uniformly at random;
- sample $j \sim \mu_u(\cdot \mid v \leftarrow i)$.

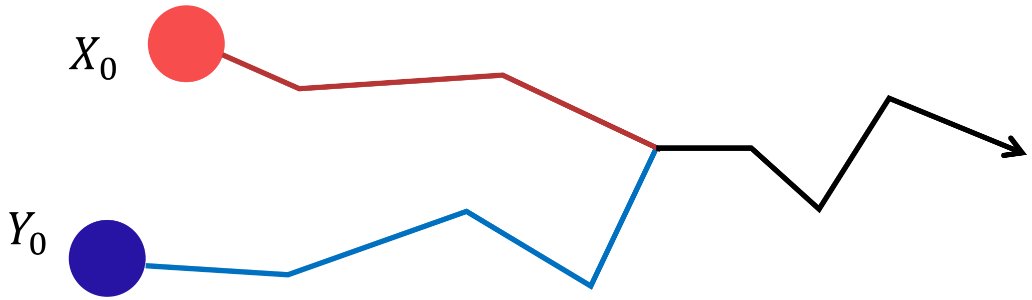


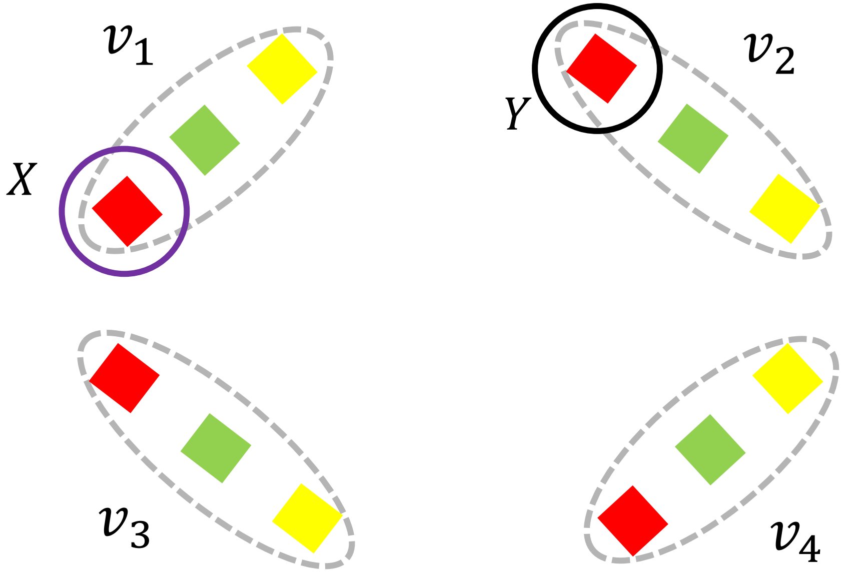


Our technique: coupling

Coupling $(\boldsymbol{X_t}, \boldsymbol{Y_t})_{t\geq 0}$ of local walk

- start from two states $X_0, Y_0 \in U$
- two chains $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ follow local walk





Coupling

Current state $X_t = (u, i)$ and $Y_t = (v, j)$

Next state $X_{t+1} = (u', i')$ and $Y_{t+1} = (v', j')$

- Pick the **same** $u' = v' \in V$ uniformly at random;
- Sample (i', j') from the **optimal coupling** between $\mu_{u'}(\cdot \mid u \leftarrow i)$ and $\mu_{v'}(\cdot \mid v \leftarrow j)$.

Observation: for any $t \geq 1$, X_t and Y_t must be on the **same vertex**.

$$X_t = (v, i) \text{ and } Y_t = (v, j) \text{ (same vertex, different color)}$$

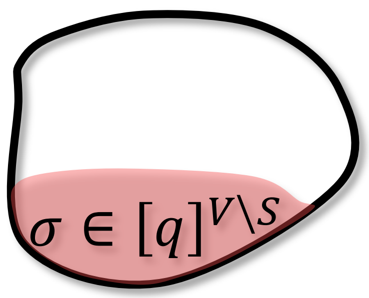
- Pick the same vertex $u \in V$ uniformly at random.
- Couple the colors on u **optimally**, the coupling fails with probability

$$d_{TV}\left(\mu_u(\cdot \mid v \leftarrow i), \mu_u(\cdot \mid v \leftarrow j)\right) \leq \Psi(v, u). \quad \text{(Influence } v \to u\text{)}$$

Spectral independence → **X, Y meet quickly in the coupling** → **local mixing** $\xrightarrow{\text{[Alev and Lau 20]}}$ **global mixing**

Remark (conditional distributions)

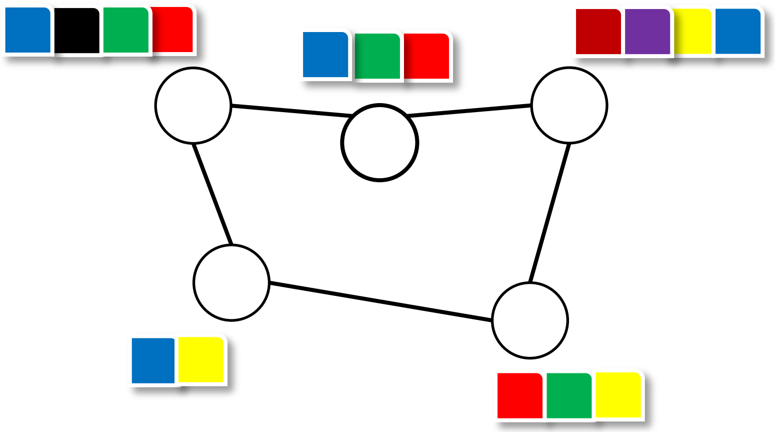
- [AL20] requires local mixing on all **conditional distributions**.
- Our coupling also works for all **conditional distributions**.



Spectral independence for coloring

List coloring instance

- graph $G = (V, E)$ with max degree Δ;
- each vertex $v \in V$ has a color list L_v.



Proper list coloring $\boldsymbol{X}$

- $X_v \in L_v$ for all $v \in V$;
- $X_u \neq X_v$ for all $\{u, v\} \in E$.

Gibbs distribution $\boldsymbol{\mu}$

- uniform distribution over all proper list colorings.

Theorem [this work]

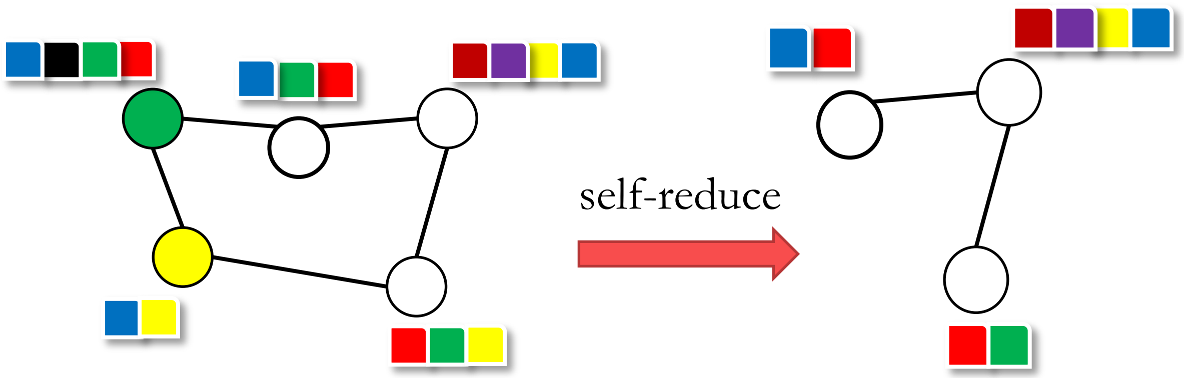
In triangle-free graph, if for all $v \in V$,

$$|L(v)| \geq (\alpha + \delta)\Delta \approx (1.763 + \delta)\Delta,$$

then under any pinning,

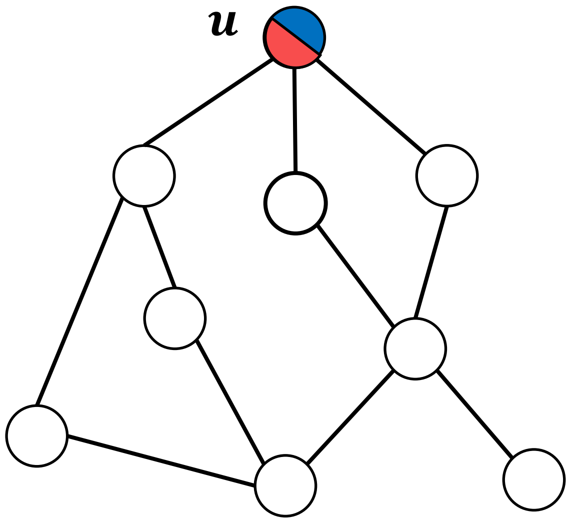
$$\text{one-to-all influence} = O\left(\frac{1}{\delta}\right),$$

μ is spectrally independent with $C = O\left(\frac{1}{\delta}\right)$.



closed under pinning

Recursive coupling



Influence from u to v

$$\mathrm{Inf}(u \to v) = \max_{i,j \in L(u)} d_{TV}\big(\mu_v(\cdot \mid u \leftarrow i),\mu_v(\cdot \mid u \leftarrow j)\big)$$

One-to-all influence

$$\sum_{v \in V \setminus \{u\}} \mathrm{Inf}(u \to v)$$

by **recursive coupling** **[Goldberg, Martin, Paterson 05]**

Proof sketch

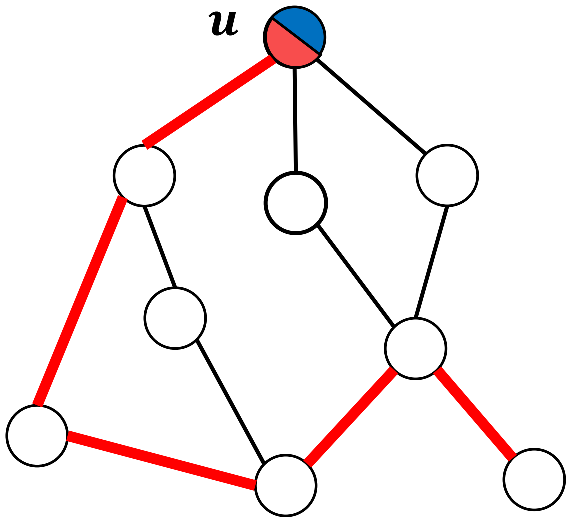
Construct a **coupling** (c_v, c_v') between $\mu_v(\cdot \mid u \leftarrow i)$ and $\mu_v(\cdot \mid u \leftarrow j)$

$$d_{TV}\big(\mu_v(\cdot \mid u \leftarrow i),\mu_v(\cdot \mid u \leftarrow j)\big) \le \Pr[c_v \neq c_v']$$

Bound **one-to-all influence** by **coupling inequality**

$$\sum_{v \in V \setminus \{u\}} \mathrm{Inf}(u \to v) \le \sum_{v \in V \setminus \{u\}} \Pr[c_v \neq c_v']$$

Recursive coupling



- Staring from the "**disagreement vertex**" u.
- Coupling vertex one by one in a "**DFS-manner**".
- If the coupling on v fails (i.e. $c_v \neq c'_v$)
  then **there is a path** $\mathcal{P}$ from u to v
  ALL vertices in $\mathcal{P}$ **FAIL** in coupling.

Bound **one-to-all influence** by **enumerating all the paths from u**

$$\sum_{u \neq v} \mathrm{Inf}(u \to v) \leq \sum_{\text{all paths } P \text{ from } u} \text{Influence alone the path } P \leq O\left(\frac{1}{\delta}\right)$$

Triangle-free
Many colors

→

Coupling succeeds
with high prob.

→

Bounded total influence

Independent work

arXiv.org > cs > arXiv:2007.08058

Computer Science > Data Structures and Algorithms

[Submitted on 16 Jul 2020]

Rapid Mixing for Colorings via Spectral Independence

Zongchen Chen, Andreas Galanis, Daniel Štefankovič, Eric Vigoda

arXiv.org > cs > arXiv:2007.08091

Computer Science > Data Structures and Algorithms

[Submitted on 16 Jul 2020]

Rapid mixing from spectral independence beyond the Boolean domain

Weiming Feng, Heng Guo, Yitong Yin, Chihao Zhang

Theorem [Chen, Galanis, Štefankovič, Vigoda 20]

The Glauber dynamics for coloring is rapid mixing if $q \geq \alpha\Delta + 1$

- **different definition** of spectral independence
- **different method** to prove spectral independence for coloring

Summary

- A definition a **spectral independence** for general distribution (generalize def. in [ALO20])
- **Rapid mixing** of **Glauber dynamics** from spectral independence
- Application: sampling uniform q-coloring on **triangle-free graph** when

$$q \geq (\alpha + \delta)\Delta \approx (1.763 + \delta)\Delta.$$

Future work

- Improve the $n^{O(C)}$ mixing time for **general distribution**.
- Improve the $n^{O(1/\delta)}$ mixing time for **spin systems** (including coloring)

  $O(n \log n)$ optimal mixing for spin systems with

  - spectral independence and $\Delta = O(1)$ [CLV20, arXiv:2011.02075].
- **Better condition** for spectral independence
  - example: prove spectral independence for graph coloring with **fewer colors**.

Thank you