A Markov chain approach to the sampling Lovász local lemma

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Base on joint works with:

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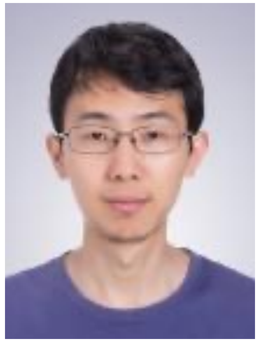

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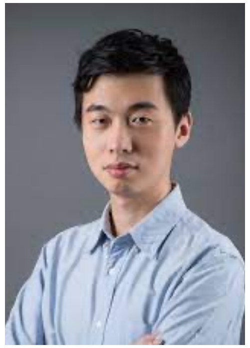

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Conjunctive normal form (CNF)

- **Instance**: a formula $\Phi = (V, C)$, for example

$$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge \boxed{(x_3 \vee \neg x_4 \vee \neg x_5)}$$ clause

$V = \{x_1, x_2, x_3, x_4, x_5\}$: set of Boolean variables; C: set of clauses.

- **SAT solutions**: an assignment of variables in V s.t. $\mathbf{\Phi = true}$.

- **Computational tasks**:
  - **Decision**: Does SAT solution exist?

    NP-Complete problem [Cook 1971, Levin 1973]
  - **Counting**: How many SAT solutions?

    #P-Complete problem [Valiant 1979].

(k,d)-CNF formula $\Phi = (V, C)$

$$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge \underbrace{(x_3 \vee \neg x_4 \vee \neg x_5)}_{\text{clause size } k}$$

variable degree d

Lovász local lemma (LLL) [Erdős and Lovász, 1975]

SAT solution exists if $k \gtrsim \log d$ ($k \geq \log d + \log k + C$)

Algorithmic Lovász local lemma (ALLL) [Moser and Tardos, 2009]

Find a SAT solution when $k \gtrsim \log d$

Sampling Lovász local lemma (SLLL)

Sample a SAT solution uniformly at random if $k = \Omega(\log d)$

Counting Lovász local lemma (CLLL)

Approximately count number of SAT solutions if $k = \Omega(\log d)$

Sampling & counting k-SAT solutions

- **Input:** a (k, d)-CNF formula $\Phi = (V, C)$ with $|V| = n$, an error bound $\epsilon > 0$.

- **Almost uniform sampling:** random solution $\boldsymbol{X} \in \{\text{true}, \text{false}\}^V$ s.t.

  the *total variation distance* $d_{TV}(\boldsymbol{X}, \mu) \le \epsilon$,

  μ: the uniform distribution of all SAT solutions

- **Approximate counting:** estimate the number of SAT solutions, output

$$(1-\epsilon)Z \le \hat{\boldsymbol{Z}} \le (1+\epsilon)Z,$$

  Z = the number of SAT solutions

almost uniform sampling

self-reduction

[Jerrum, Valiant, Vazirani 1986]

$\Longleftrightarrow$

approximate counting

Work	**Regime**	**Running time or lower bound**	**Technique**
HSZ19	Monotone CNF[1] $k \gtrsim 2\log d$	$\mathrm{poly}(dk)n\log n$	Markov chain Monte Carlo (MCMC)

[1] *Monotone CNF: all variables appear* ***positively*** $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_4 \vee x_5) \wedge (x_3 \vee x_4 \vee x_6)$.

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Moitra'17	$k \gtrsim 60\log d$	$n^{\mathrm{poly}(dk)}$	linear programming

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BGGGŠ15	$k \leq 2\log d - C$	**NP-hard**	-

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Open Problem

fast sampling algorithm when $k \gtrsim 2\log d$

Our Results [F., Guo, Yin and Zhang, 2020 & F., He and Yin, 2021]

For any (k,d)-CNF formula satisfying $k \gtrsim 13 \log d$,

- ***sampling algorithm (main algorithm)***
  draw almost uniform random SAT solution in time $\tilde{O}(d^2 k^3 n^{1.001})$;

- ***counting algorithm (by reduction)***
  count SAT solutions approximately in time $\tilde{O}(d^3 k^3 n^{2.001})$.

Further Improvements on our algorithm

- Better analysis [Jain, Pham and Vuong, 2021]: improve to $k \gtrsim 5.741 \log d$.
- Better accuracy [He, Sun and Wu, 2021]: perfect sampling via CTFP

State-of-the-art [He, Wang and Yin, 2022]

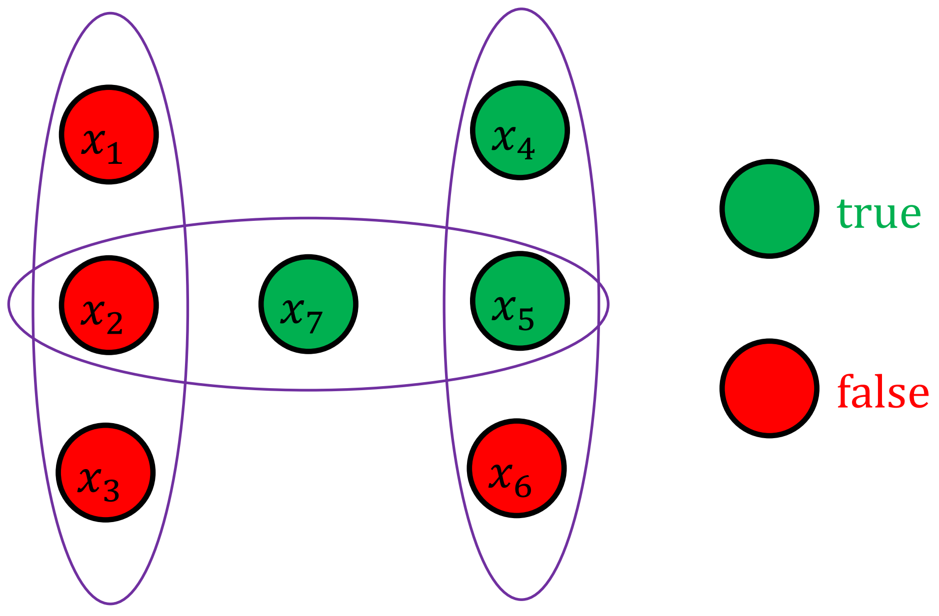
- $k \gtrsim 5 \log d$ (non-MCMC sampling algorithm)

Glauber dynamics (Gibbs sampling)

Start from an arbitrary solution $Y \in \{T, F\}^V$;

For each t from 1 to T **do**

- Pick $v \in V$ uniformly at random;
- Resample $Y_v \sim (\cdot \mid Y_{V \setminus v})$;



$(x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee x_7 \vee x_5) \wedge (x_4 \vee \neg x_5 \vee x_6)$

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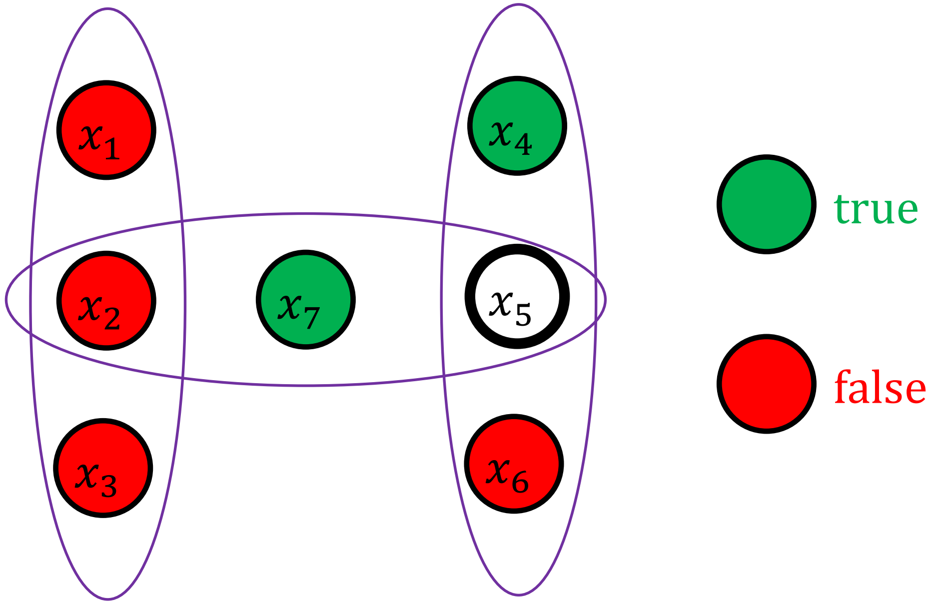

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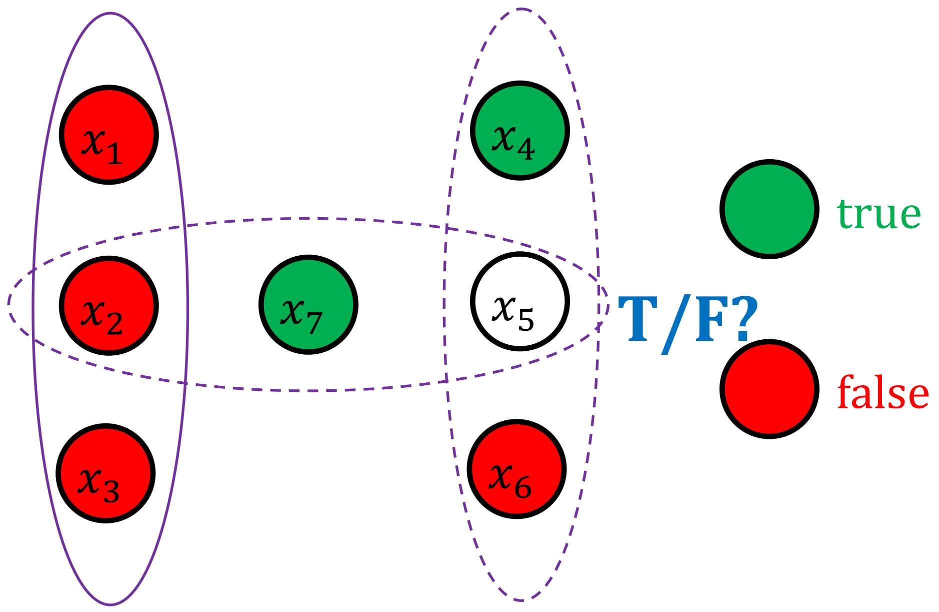

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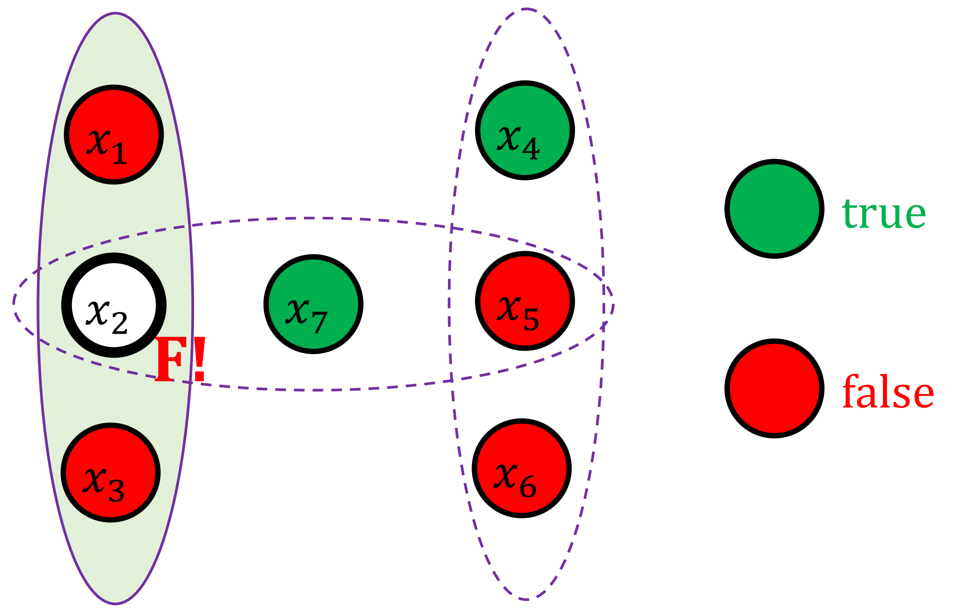

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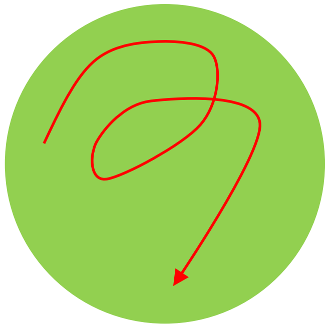
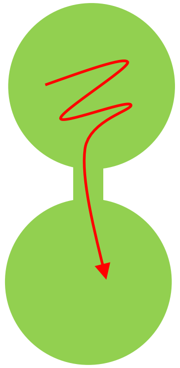
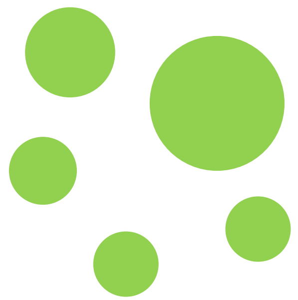
Glauber dynamics: *random walk* over solution space via *local update.*



rapid mixing



slow mixing



not mixing

Connectivity barrier (toy example)

- (k,d)-CNF formula $\Phi = (V, C)$ with $V = \{x_1, x_2, \dots x_k\}$:

$$\Phi = C_1 \wedge C_2 \wedge \cdots \wedge C_k.$$

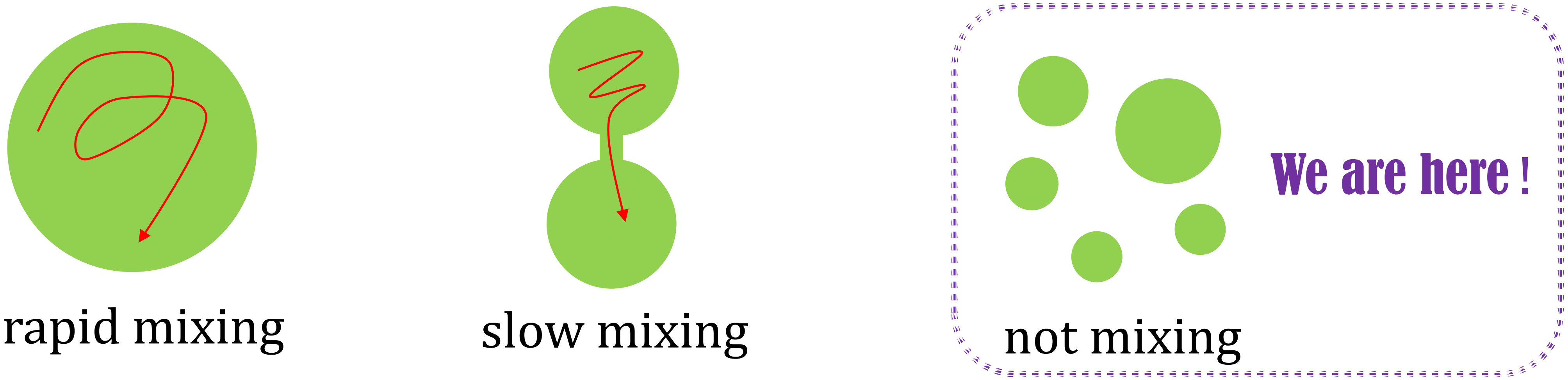
$$C_1 = (\neg x_1 \vee x_2 \vee x_3 \vee \cdots \vee x_k) \text{ forbids } 100\dots0$$

$$C_2 = (x_1 \vee \neg x_2 \vee x_3 \vee \cdots \vee x_k) \text{ forbids } 010\dots0$$

$$C_k = (x_1 \vee x_2 \vee x_3 \vee \cdots \vee \neg x_k) \text{ forbids } 000\dots1$$

- Any assignment $X \in \{0,1\}^V$ with $\|X\|_1 = 1$ is infeasible.
- All false solution **0** is **disconnected** with others.

Glauber dynamics: *random walk* over solution space via *local update.*



Question for SLLL

Can we obtain ***fast*** sampling algorithm
when the solution space is ***disconnected***?

Our technique: projection

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Projecting from a high dimension to a lower dimension to improve connectivity

Construct a *good subset* of variables $S \subseteq V$

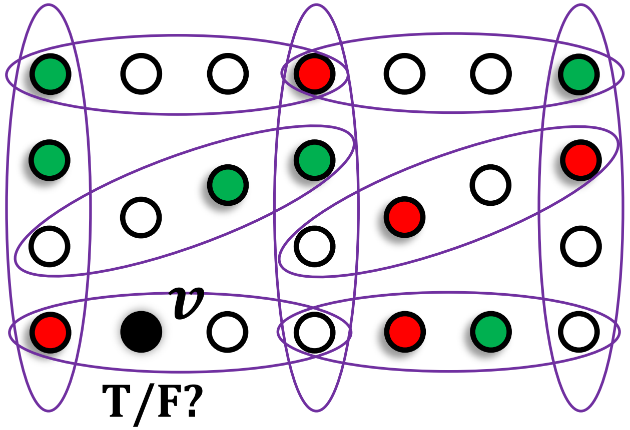
Run *Glauber dynamics* on *projected distribution* μ_S to draw sample $X \sim \mu_S$

Start from a uniform random $X \in \{\text{true,false}\}^S$;

For each t from 1 to T

- Pick a variable $v \in S$ uniformly at random;
- Resample $X_v \sim \mu_v(\cdot \mid X_{S \setminus v})$;

Return X;



Draw sample $Y \sim \mu_{V \setminus S}(\cdot \mid X)$ from the *conditional distribution*

There exists an ***efficiently constructible subset*** $S \subseteq V$ such that:

- the Glauber dynamics on μ_S is ***rapidly mixing***,
- the Glauber dynamics on μ_S can be ***implemented efficiently*** (draw $X_v \sim \boxed{\mu_v(\cdot \mid X_{S\setminus v})}$),
- sampling assignment for $V \setminus S$ can be ***implemented efficiently*** (draw $Y \sim \boxed{\mu_{V\setminus S}(\cdot \mid X)}$).

computing exact distr. can be #P-hard

Construct a *good subset* of variables $S \subseteq V$

Run *Glauber dynamics* on *projected distribution* μ_S to draw sample $X \sim \mu_S$

Start from a uniform random $X \in \{\text{true,false}\}^S$;

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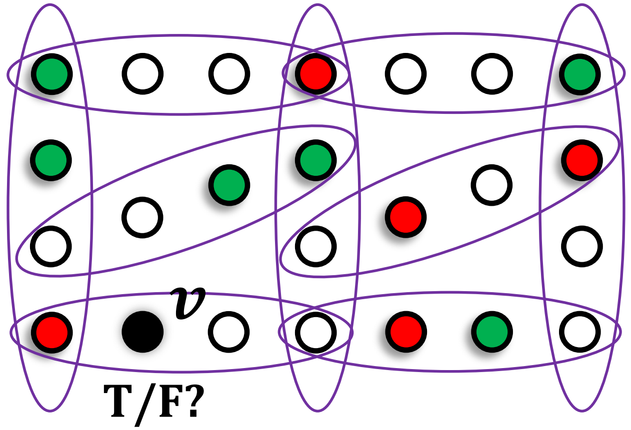


Draw sample $Y \sim \mu_{V \setminus S}(\cdot \mid X)$ from the *conditional distribution*

Our Tasks:

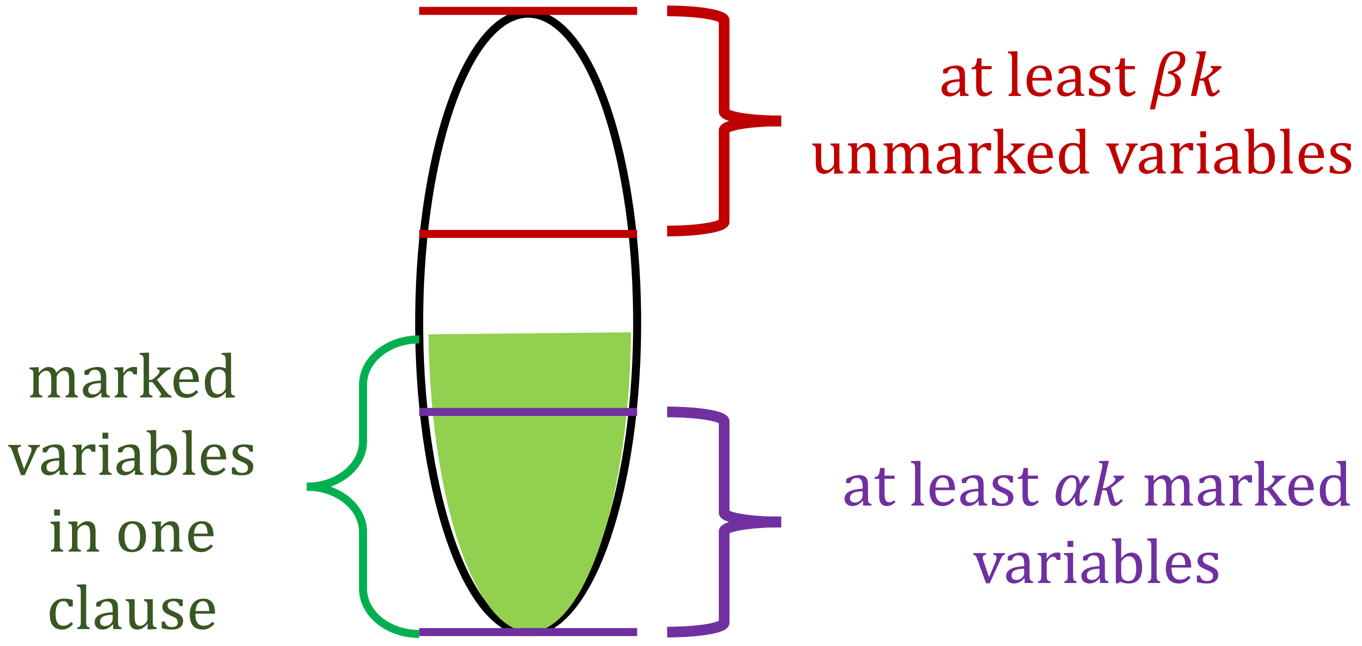
- Construct such a ***good subset*** $S \subseteq V$.
- Show that the Glauber dynamics on μ_S is ***rapidly mixing***.
- Given assignment on S, draw samples ***efficiently*** from the conditional distribution.

Mark variables [Moitra' 17]

Mark a set of variables $S \subseteq V$ such that

- each clause contains ***at least*** $\alpha k = \Theta(k)$ marked variables;
- each clause contains ***at least*** $\beta k = \Theta(k)$ unmarked variables;

where $0 < \alpha, \beta < 1$ are two constant parameters with $\alpha + \beta < 1$



Marked set $S \subseteq V$
is constructed by
algorithmic LLL
(Moser-Tardos)

The rapid mixing of Glauber dynamics on μ_S

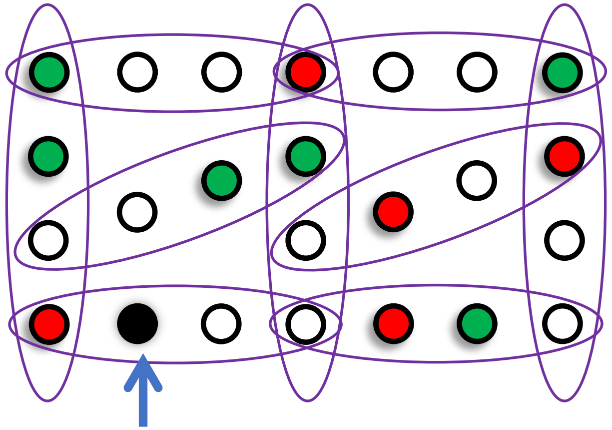
Each clause has **at least** βk **unmarked variables**

by LLL

Key Property: local uniformity

For any $v \in S$, any $X_{S \setminus v} \in \{T, F\}^{S \setminus v}$,

$$\forall c \in \{T, F\}, \qquad \mu_v(\, c \mid X_{S \setminus v}\,) = \frac{1}{2}\left(1 + \frac{1}{\mathrm{poly}(dk)}\right)$$



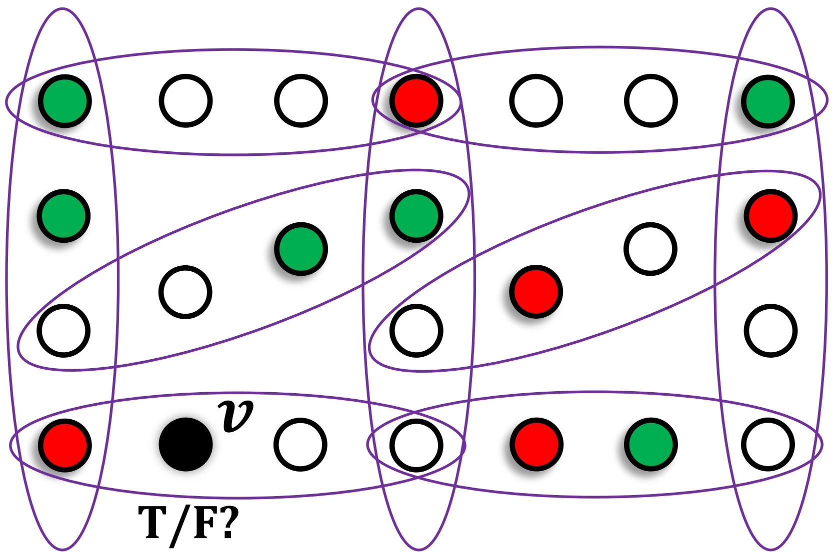
Pr[T]≈ 0.5 and Pr[F]≈ 0.5

- The Glauber dynamics is **connected** ($\mu_S(X_S) > 0$ for all $X_S \in \{T, F\}^S$)
- The Glauber dynamics **mixes** in $O(n \log n)$ steps
  - path coupling analysis [F., Guo, Yin and Zhang, 2020] [F., He and Yin, 2021]
  - information percolation analysis [Jain, Pham and Vuong, 2021]

Implementation of the algorithm

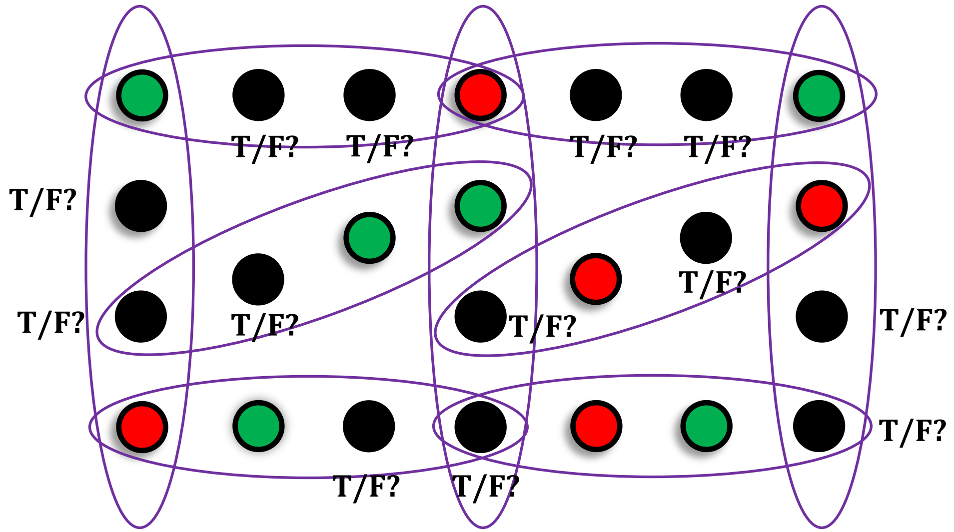
Transition of Glauber dynamics

resample $X_v \sim \mu_v(\cdot \mid X_{S\setminus v})$



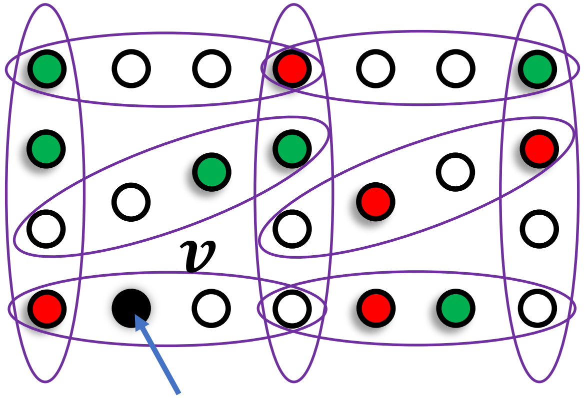
Sample unmarked variable

sample $Y \sim \mu_{V\setminus S}(\cdot \mid X)$



Challenge: computing the ***exact*** conditional distributions can be **#P-hard**.

Solution: draw ***approximate*** samples via rejection sampling

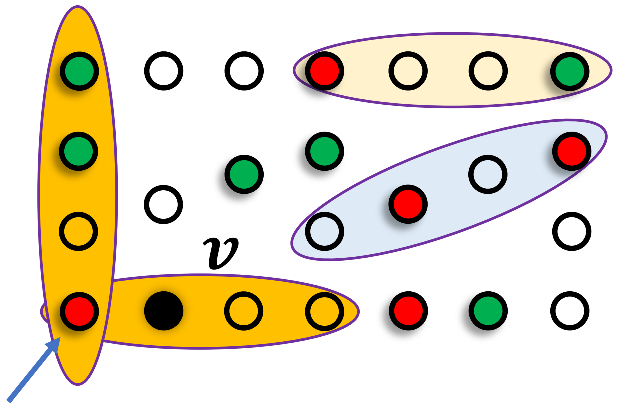


resample X_v from $\mu_v(\cdot \mid X_{S\setminus v})$

remove satisfied clauses

$(x_1 \vee x_2 \vee \neg x_3 \vee \neg x_4)$

$x_1 = \mathrm{T}$ or $x_4 = \mathrm{F}$ ✓



C: connected component containing v

each clause has
at least αk **marked variables**

local uniformity
each marked variables takes
an almost uniform random value

remove
each clause
with prob.
$\approx 1 - \left(\frac{1}{2}\right)^{\alpha k}$

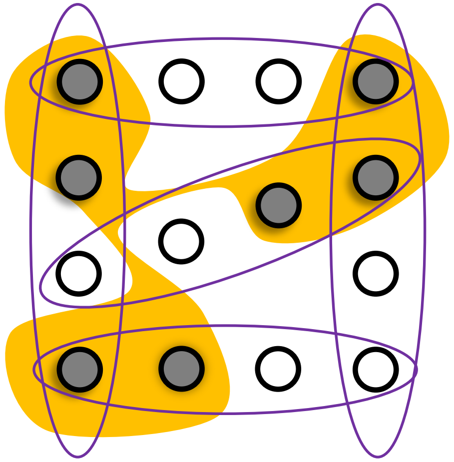
with high probability
component size is
$O(\log n)$

rejection sampling
on component

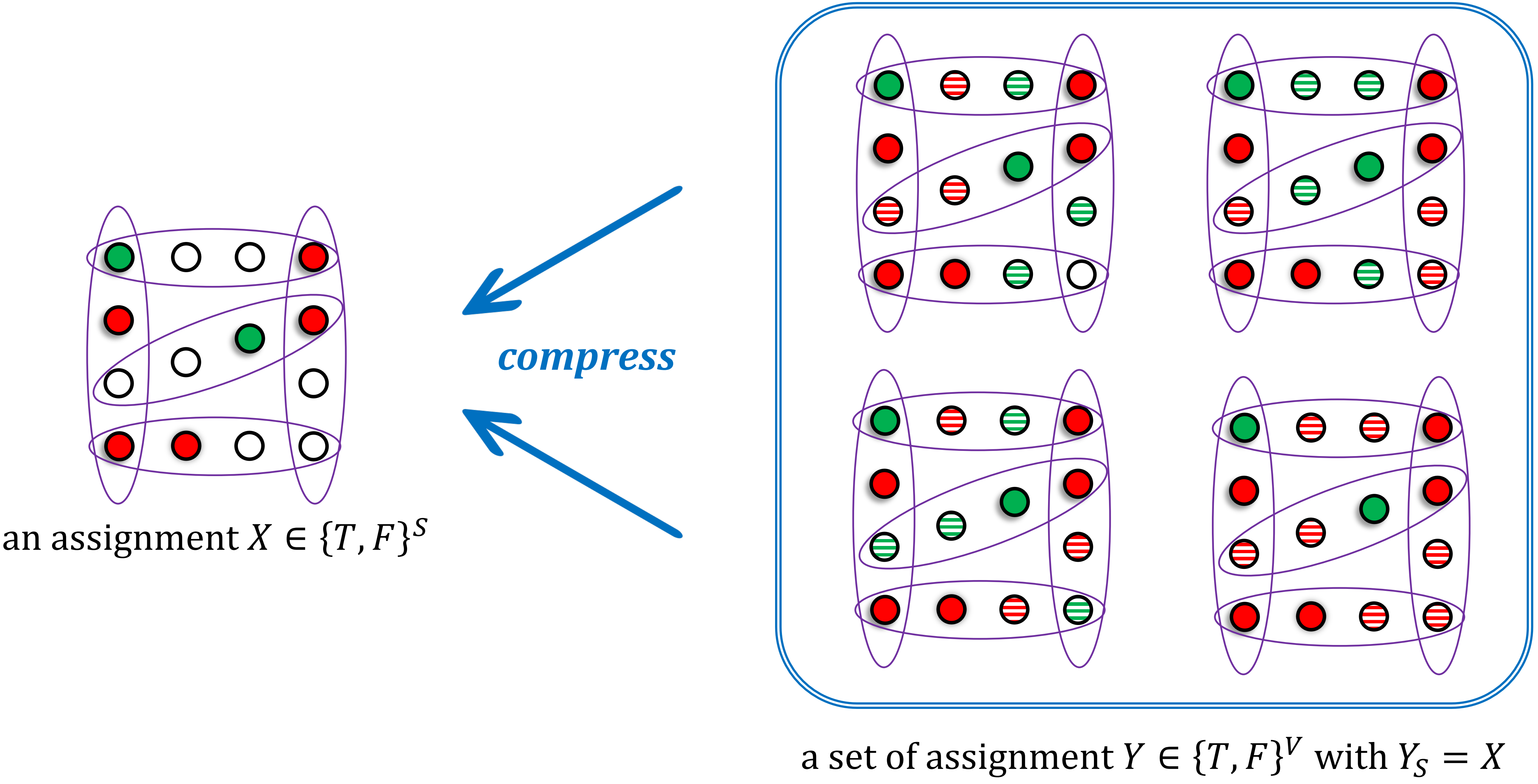
Input: a k-CNF formula $\Phi = (V, E)$ with maximum degree d, an error bound $\epsilon > 0$.

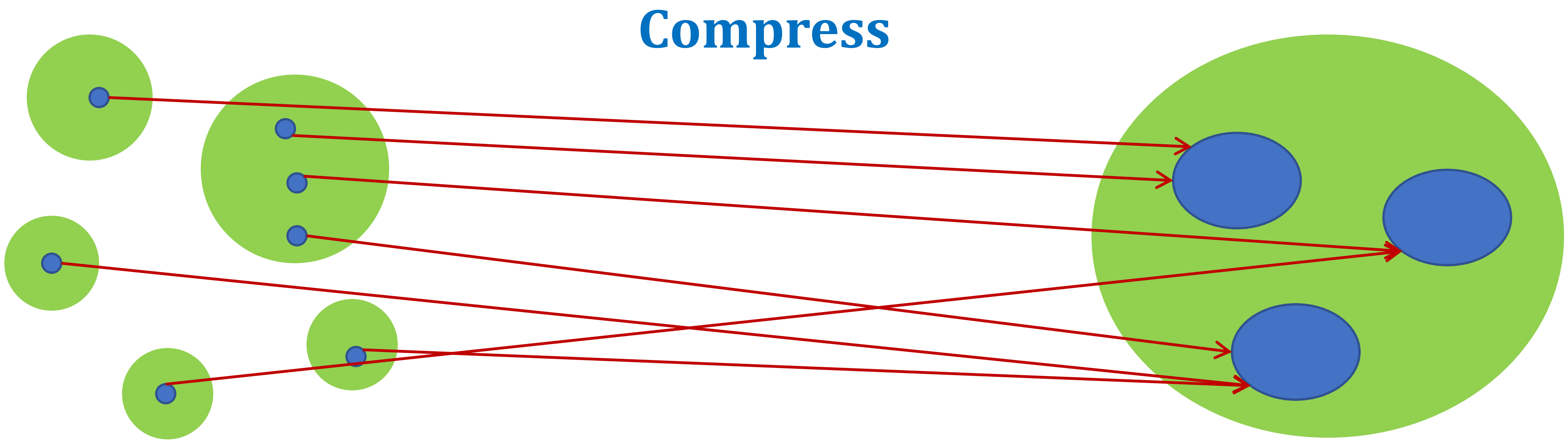
Output: a random sample $\sigma \in \{\text{true}, \text{false}\}^V$ s.t. $d_{TV}(\sigma, \mu) \leq \epsilon$.

1. Run ***Moser-Tardos*** algorithm to construct marked set $S \subseteq V$;
2. Run $\boxed{\textit{\textbf{Glauber dynamics}} \text{ on } \mu_S}$ for $O\left(n \log \frac{n}{\epsilon}\right)$ steps to sample $X \sim \mu_S$;
   (implemented using rejection sampling)
3. Run ***rejection sampling*** to draw $Y \sim \mu_{V \setminus S}(\cdot \mid X)$;
4. Return $X \cup Y$.



set of marked variables $S \subseteq V$


compress
an assignment $X \in \{T, F\}^S$
a set of assignment $Y \in \{T, F\}^V$ with $Y_S = X$

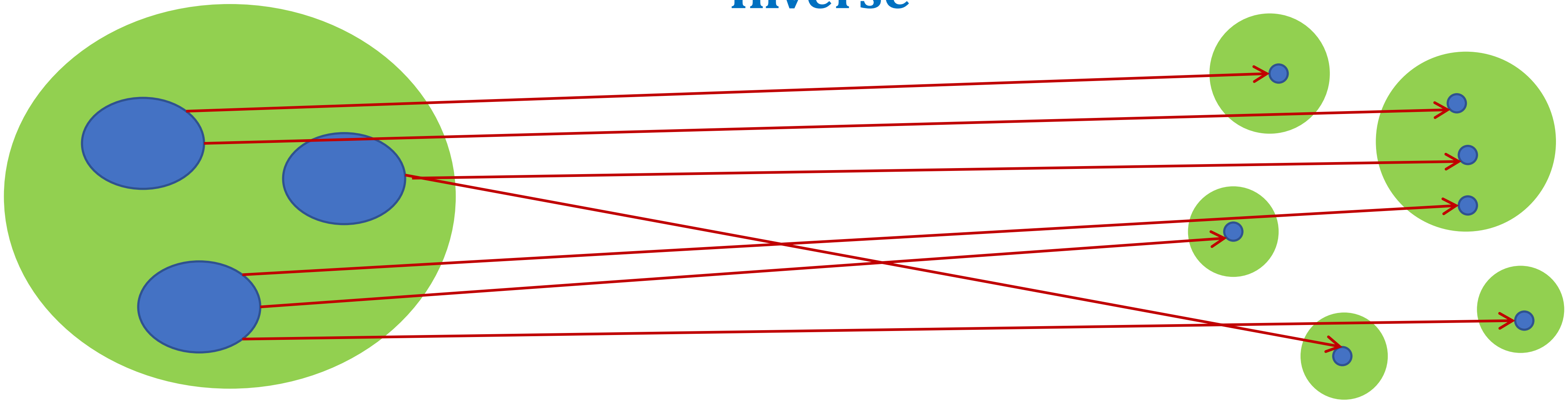


CNF solution space: disconnected

compressed space: connected

Rapid mixing of Glauber dynamics

Inverse



compressed state

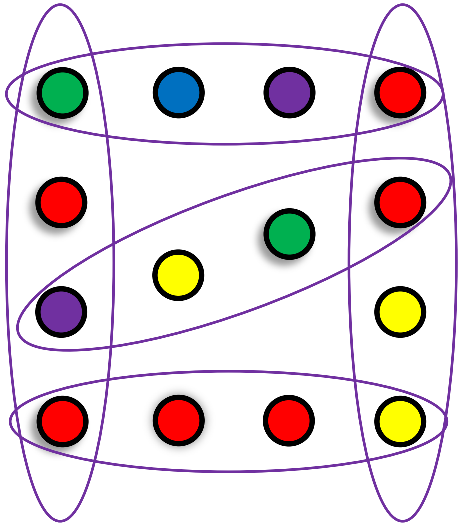
random CNF solution

Fast implementation of algorithm

Hypergraph colouring

Instance: a k-uniform hypergraph $H(V,E)$ with max degree d and q colours

- each hyperedge contains k vertices
- each vertex belongs to $\leq d$ hyperedges



Hypergraph colouring: $X \in [q]^V$ assign $v \in V$ a colour X_v

- no hyperedge is monochromatic

Lovász Local lemma and algorithmic LLL

- find a hypergraph colouring when $q \gtrsim d^{1/k}$ ($q \geq C_k d^{1/(k-1)}$)

Sampling Lovász Local lemma

- Sample a uniform hypergraph colouring in the local lemma regime

Work	Regime	Running time or lower bound	Technique
FA17	Linear hypergraph[1] $q \gtrsim \max\{\log n, d^{1/k}\}$	$O(n \log n)$	Markov chain Monte Carlo (MCMC)

[1] *Linear hypergraph: for all distinct hyperedge* $e_1, e_2 \in E, |e_1 \cap e_2| \leq 1$

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FA17	Linear hypergraph[1] $q \gtrsim \max\{\log n, d^{1/k}\}$	$O(n \log n)$	Markov chain Monte Carlo (MCMC)
GLLZ17	$q \gtrsim d^{16/k}$	$n^{\mathrm{poly}(dk\log q)}$	linear programming

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GLLZ17	$q \gtrsim d^{16/k}$	$n^{\mathrm{poly}(dk\log q)}$	linear programming
GGW22	$q \lesssim d^{2/k}$	**NP-hard**	-
GGW22	Linear hypergraph $q \lesssim d^{1/k}$	**NP-hard**	

[1] *Linear hypergraph: for all distinct hyperedge* $e_1, e_2 \in E, |e_1 \cap e_2| \leq 1$

Open Problem

fast sampling algorithm when $q \gtrsim d^{2/k}$ (general) and $q \gtrsim d^{1/k}$ (linear)

Results obtained by MCMC with compression

MCMC with compression [F., He and Yin, 2021]

- $\tilde{O}(\text{poly}(dk) \cdot n^{1.001})$ running time if $q \gtrsim d^{9/k}$

Improved analysis on general hypergraph [Jain, Pham and Vuong, 2021]

- $\tilde{O}(\text{poly}(dk) \cdot n^{1.001})$ running time if $q \gtrsim d^{3/k}$

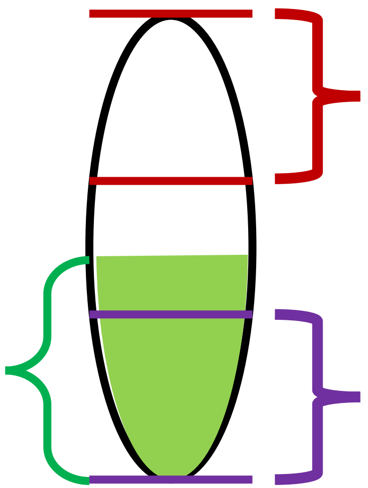
Improved analysis on linear hypergraph [F., Guo and Wang, 2022]

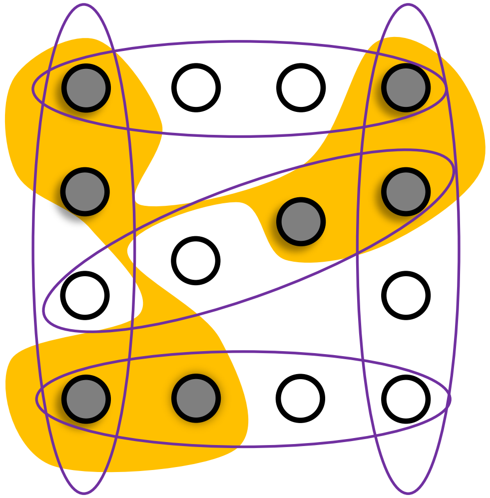
- $\tilde{O}(\text{poly}(dk) \cdot n^{1.001})$ running time if $q \gtrsim d^{(2+\delta)/k}$ for any constant $\delta > 0$

Perfect sampling via CFTP [He, Sun and Wu, 2021]

- $\tilde{O}(\text{poly}(dk) \cdot n)$ expected running time if $q \gtrsim d^{3/k}$

Mark/unmarked paradigm





set of marked variables

By **Lovász Local lemma,**

such marked set exists if

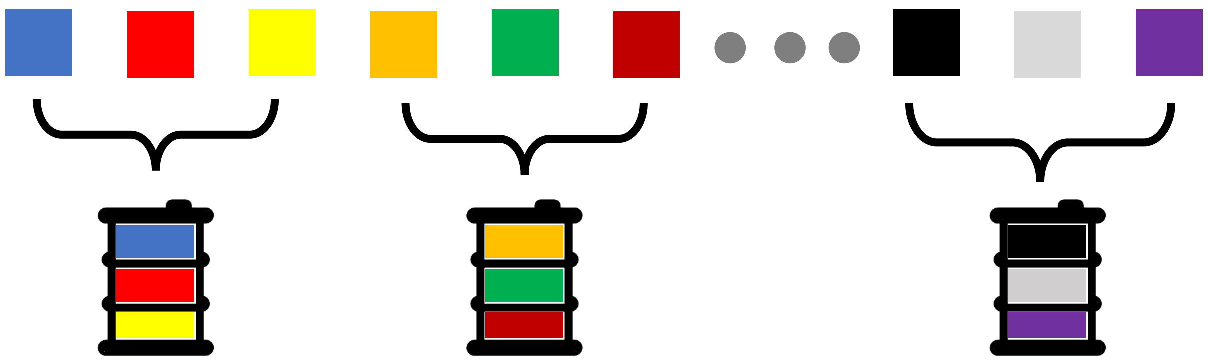
$k = \Omega(\log d)$

Local lemma regime:

- CNF: $k = \Omega(\log d)$
- Hypergraph colouring: $q = \Delta^{\Omega(1/k)}$

State compression for hypergraph colouring

the set of q colours



Bucket #1 Bucket #2 Bucket # s

the set of s buckets

Balanced projection scheme

$$h: [q] \to [s]$$

for any $j \in [s]$,

$$|h^{-1}(j)| = \frac{q}{s} \pm 1$$

$s = q^{\gamma}$ for constant $0 < \gamma < 1$

Projection of colouring

for any hypergraph colouring $X \in [q]^V$,

$$Y = h(X) \text{ s.t. } Y_v = h(X_v) \text{ for all } v \in V$$

Compression: ***different*** colouring $X \in [q]^V$ may be mapped to the ***same*** $Y \in [s]^V$

Distribution $\pi = \pi_h$ over the ***compressed space*** $[s]^V$

$$h(X) \sim \pi \text{ if } X \sim \mu,$$

μ is the ***uniform*** distribution over all hypergraph colourings

Sampling algorithm for hypergraph colourings

1. Choose a proper $s = q^\gamma$ to define the balanced projection scheme h;
2. Run ***Glauber dynamics*** on π_h for $O(n \log n)$ steps to sample $Y \sim \pi_h$; *(implemented using rejection sampling)*
3. Run ***rejection sampling*** to draw $X \in h^{-1}(Y)$ uniformly at random;
4. Return X.

Deterministic approximate counting

almost uniform sampling

standard reduction
[Bezáková, Štefankovič, Vazirani, Vigoda, 2008]
[Jerrum, Valiant, Vazirani 1986]

randomised approximate counting

Randomised counting: ***with probability at least*** $\mathbf{2/3}$, output $\hat{Z}$ satisfying

$$(1-\epsilon)Z \leq \hat{Z} \leq (1+\epsilon)Z,$$

Z total number of solutions (say total number of hypergraph colourings)

Deterministic counting: output $\hat{Z}$ satisfying

$$(1-\epsilon)Z \leq \hat{Z} \leq (1+\epsilon)Z$$

?

Deterministic approximate counting for hypergraph colourings

Work	Regime	Running time	Technique
GLLZ17	$q \gtrsim d^{14/k}$	$n^{\mathrm{poly}(dk \log q)}$	linear programming
JPV21	$q \gtrsim d^{7/k}$	$n^{\mathrm{poly}(dk \log q)}$	linear programming
HWY22	$q \gtrsim d^{5/k}$	$n^{\mathrm{poly}(dk \log q)}$	derandomisation

MCMC & Compression: sampling full colouring $X \in [q]^V$ in $O(n \log n)$ transition steps

⬇ Coupling towards the past [F., Guo, Wang, Wang and Yin, 2022]

Sample from marginal distribution μ_S for a small subset $S \subseteq V$ in $O(\log n)$ step

⬇ derandomisation

$n^{\mathrm{poly}(dk \log q)}$-time deterministic approximate counting if $q \gtrsim d^{3/k}$

Open problems

CNF formula

NP-Hard	?	Poly-Time Algorithm
$k \lesssim 2 \log d$		$k \gtrsim 5 \log d$

my guess

Hypergraph colouring

NP-Hard	?	Poly-Time Algorithm
$q \lesssim \Delta^{2/k}$		$q \lesssim \Delta^{3/k}$

my guess

Thank you! Q&A