HW7 due 11:30a Mon Nov 28

Stability of discretization schemes

Consider the (CL) model

$$\dot{x} = Ax$$

and the (DL) model of the form

$$\bar{x}^+ = \bar{A}\bar{x}$$

that results from discretizing (CL) using the Forward Euler, Backward Euler, or Exact discretization scheme with step size $\delta > 0$.

Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; note that this is the matrix obtained by linearizing the pendulum model around the downward-pointing equilibrium.

a. Determine whether the (CL) system $\dot{x} = Ax$ is stable, unstable, or neutral.

b. Determine whether the (DL) system $\bar{x}^+ = \bar{A}\bar{x}$ is stable, unstable, or neutral, where $\bar{A}$ is obtained with $\delta = \frac{1}{2}$ via:

(i) Forward Euler: $\bar{x}^+ = \bar{x} + \delta A\bar{x}$;

(ii) Backward Euler: $\bar{x}^+ = \bar{x} + \delta A\bar{x}^+$;

(iii) Exact discretization: $\bar{x}^+ = e^{A\delta}\bar{x}$.

Asteroids linear system

This problem deals with the asteroids system nonlinear DE (N)

$$\dot{x} = f(x, u)$$

with equilibrium

$$f(x_0, u_0) = 0$$

and associated linear DE (L)

$$\dot{\xi} = A\xi + B\mu$$

obtained with $A = D_x f(x_0, u_0)$, $B = D_u f(x_0, u_0)$.

a. Simulate two "interesting" nonequilibrium trajectories for (L); you may wish to apply nonzero inputs. Show that the trajectories sum linearly, i.e. with $\xi_1, \xi_2 : [0, t] \to \mathbb{R}^n$ denoting the two trajectories generated by inputs $\mu_1, \mu_2 : [0, t] \to \mathbb{R}^m$, provide a plot that convinces the reader that the trajectory $\xi_1 + \xi_2$ is the same as that obtained by simulating from initial condition $\xi_1(0) + \xi_2(0)$ with input $\mu_1 + \mu_2$.

b. Repeat (a.) using (N); this time, the trajectories will not sum linearly. Use initial conditions $x_1(0) = x_0 + \xi_1(0)$, $x_2(0) = x_0 + \xi_2(0)$ and inputs $u_1 = u_0 + \mu_1$, $u_2 = u_0 + \mu_2$. Provide the analogous plot from (a.) to convince the reader that the trajectories do not sum linearly.

c. Assess stability of the linear system (L): for each eigenvalue of A, determine whether it corresponds to a stable, unstable, or neutral eigensystem, and discuss (with reference to the corresponding eigenvector) what directions in state space are governed by this eigensystem.

Project linear system

This problem deals with your project system nonlinear DE (N)

$$\dot{x} = f(x, u)$$

with equilibrium

$$f(x_0, u_0) = 0$$

and associated linear DE (L)

$$\dot{\xi} = A\xi + B\mu$$

obtained with $A = D_x f(x_0, u_0)$, $B = D_u f(x_0, u_0)$.

a. Simulate two "interesting" nonequilibrium trajectories for (L); you may wish to apply nonzero inputs. Show that the trajectories sum linearly, i.e. with $\xi_1, \xi_2 : [0, t] \to \mathbb{R}^n$ denoting the two trajectories generated by inputs $\mu_1, \mu_2 : [0, t] \to \mathbb{R}^m$, provide a plot that convinces the reader that the trajectory $\xi_1 + \xi_2$ is the same as that obtained by simulating from initial condition $\xi_1(0) + \xi_2(0)$ with input $\mu_1 + \mu_2$.

b. Repeat (a.) using (N); this time, the trajectories will not sum linearly. Use initial conditions $x_1(0) = x_0 + \xi_1(0)$, $x_2(0) = x_0 + \xi_2(0)$ and inputs $u_1 = u_0 + \mu_1$, $u_2 = u_0 + \mu_2$. Provide the analogous plot from (a.) to convince the reader that the trajectories do not sum linearly.

c. Assess stability of the linear system (L): for each eigenvalue of A, determine whether it corresponds to a stable, unstable, or neutral eigensystem, and discuss (with reference to the corresponding eigenvector) what directions in state space are governed by this eigensystem.