Stability of Discretization Schemes

a. $A = [0\ \ 1\ ;\ -1\ \ 0]$

Stability: finding eigenvalues gives that $\lambda = \pm\, i$. This means that the system is **marginally stable**.

b. $\delta = ½$.

i. Forward Euler:

$x^+ = x + \delta(Ax)$ → $x^+ = x(I + \delta A)$ → $\bar{A} = (I + \delta A) = [1\ \ ½\ ;\ -½\ \ 1]$

Finding the eigenvalues of this matrix gives $\lambda = 1 \pm ½i$. This means the system is **unstable** because $|\lambda| > 1$.

ii. Backward Euler:

$x^+ = x + \delta(Ax^+)$ → $x^+ - \delta Ax^+ = x$ → $x^+(I - \delta A) = x$
$x^+ = (I - \delta A)^{-1}$ → $\bar{A} = (I - \delta A)^{-1} = [1\ \ -½\ ;\ ½\ \ 1]^{-1}$

Finding the eigenvalues of this matrix gives $\lambda = .8 \pm .4i$. This means the system is **stable** because $|\lambda| < 1$.

iii. Exact Discretization: By definition of exact discretization:

$\bar{A} = e^{\delta A}$ → $\bar{A} = [0.878\ \ 0.479\ ;\ -0.479\ \ 0.878]$

Finding the eigenvalues of this matrix gives $\lambda = 0.878 \pm 0.479i$. This means the system is **marginally stable** because $|\lambda| = 1$.