MATH 131B: Homework #4

Professor Dave Penneys

Assignment: 34, 37, 39, 40; Optional: 35

Eric Chuu

UID: 604406828

May 3, 2016

Problem 34

Let $(X, d_X), (Y, d_Y)$ be metric spaces.

(a) Show that the map $\pi_X : X \times Y \to X$ by $(x, y) \mapsto x$ is continuous.

(b) Prove that π_X is open, i.e., if $U \subseteq X \times Y$ is open, then $\pi_X(U) \subseteq X$ is open.

(c) Is π_X closed? That is, if $F \subseteq X \times Y$ is closed, then is $\pi_X(F) \subseteq X$ closed?

Solution

(a) We will show that the map $\pi_X : X \times Y \to X$ is (topologically) continuous by showing that all open sets in X have pre-images that are open in $X \times Y$. We first prove a few lemmas that help with the proof:

Lemma 1: $B_r(x) = \prod_{i=1}^n B_r^i(x_i)$, $x = (x_1, x_2, \ldots, x_n)$, in X is the product of open balls $B_r^1(x_1), B_r^2(x_2), \ldots, B_r^n(x_n)$, where $B_r^i(x_i)$ is the open ball of radius $r > 0$ centered about $x_i \in X_i$.

Proof: Consider $y \in B_r(x) \Leftrightarrow \max\{d_i(x_i, y) : 1 \leq i \leq n\} < r \Leftrightarrow d_i(x_i, y) < r \Leftrightarrow y \in B_r^i(x_i)$ for $1 \leq i \leq n \Leftrightarrow y \in B_r^1(x_1) \times B_r^2(x_2) \times \ldots \times B_r^n(x_n)$, and we are done.

Lemma 2: If $U_i \in X_i, 1 \leq i \leq n$, are open subsets of X_i, then $\prod_{i=1}^n U_i$ is open in $X = X_1 \times \ldots X_n$.

Proof: Let $x = (x_1, \ldots, x_n) \in \prod_{i=1}^n U_i$. Then there exist positive $r_1, \ldots, r_n$ such that $B_{r_i}(x_i) \subset U_i, 1 \leq i \leq n$. Take $r := \min\{r_1, \ldots, r_n\}$. Then $B_r^i(x_i) \subset U_i$, $1 \leq i \leq n$, and by Lemma 1, we can conclude that $B_r^1(x_1) \times \ldots \times B_r^n(x_n) \subset \prod_{i=1} U_i$, so $\prod_{i=1}^n U_i$ is open in X.

We now consider the map $\pi_X : X \times Y \to X$. Consider an open ball around $x \in X$, $B_r(x) \subset X$. Then we can write the pre-image of this as

$$\pi_X^{-1}(B_r(x)) = B_r(x) \times Y \tag{1}$$

since $B_r(x)$ is a subset of one of the metric spaces in the product space $X \times Y$. Let $U \subseteq X$ be open. We will show that $\pi_X^{-1}(U)$ is open in $X \times Y$. Note that U can be written as the union of open balls around points in U, $U = \bigcup_{x \in U} B_{r_x}(x)$, where each radius r_x depends on the point $x \in U$. Then

$$\pi_X^{-1}(U) = \pi_X^{-1}\big(\bigcup_{x \in U} B_{r_x}(x)\big) = \bigcup_{x \in U} \big(\pi_X^{-1}(B_{r_x}(x))\big)$$

We can apply the equality in (1) to get

$$\bigcup_{x\in U}\big(\pi_X^{-1}(B_{r_x}(x))\big)=\bigcup_{x\in U}\big(B_{r_x}(x)\times Y\big)=\big(\bigcup_{x\in U}B_{r_x}(x)\big)\times Y=U\times Y.$$

Since U is open in X and Y is open in Y, then applying Lemma 2, we see that $U\times Y$ is open in $X\times Y$, so the map π_X is continuous. $\square$

(b) Let $U\in X\times Y$ be open. We will show that $\pi_X(U)$ is open in X. U can be written as a union of open balls about points in U, $U=\bigcup_{z\in U}B_{r_z}(z)$, and applying Lemma 1 from above, we know that this can be that opens balls in $U\subset X\times Y$ can be written as: $B_{r_z}(z)=B_r(x)\times B_r(y)$, the product of open balls in X and Y, respectively. Applying the map π_X, we get

$$\pi_X(U)=\pi_X\big(\bigcup_{z\in U}B_{r_z}(z)\big)=\bigcup_{z\in U}\pi_X(B_{r_z}(z))=\bigcup_{x\in G\cap X}B_r(x),$$

which is an arbitrary union of open sets in X, hence open in X. $\square$

(c) Consider the map $\pi_X:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Then the set $F=\{(x,y):xy\geq 1,x>0\}$ is closed in $\mathbb{R}\times\mathbb{R}$, as it it contains all of its boundary points. However, $\pi_X(F)=(0,\infty)$, which is not closed in $\mathbb{R}$ since 0 is adherent to $(0,\infty)$, but $0\notin(0,\infty)$, and we conclude that the map π_X is not a closed map. $\square$

Problem 35

Let X,Y be metric spaces and let $\pi_X:X\times Y\to X$ be the map defined by $(x,y)\mapsto x$ as in Problem 34. Assume Y is compact. Show that π_X is closed.

Solution

We first show the following lemmas.

(a) Show that for every $x\in X$, the set $\{x\}\times Y$ is compact.

Proof Let $(x_n,y_n)\subset\{x\}\times Y$ be a sequence. Since the only element in $\{x\}$ is x, then $x_n=x$ for all n, so we have $x_n\to x$. By assumption Y is compact $\Leftrightarrow$ sequentially compact, so $(y_n)\subset Y$ has a convergent subsequence that converges to $y\in Y$. Hence, we've shown that for a sequence $(x_n,y_n)\subset\{x\}\times Y$, we can find a subsequence $(x_{n_k},y_{n_k})\to(x,y)\in\{x\}\times Y$, so $\{x\}\times Y$ is sequentially compact $\Leftrightarrow$ compact.

(b) Suppose $F\subset X\times Y$ is closed, $x\in X$ such that $(\{x\}\times Y)\cap F=\phi$. Show there is an $\epsilon>0$ such that $(B_\epsilon(x)\times Y)\cap F=\phi$.

Proof Suppose for contradiction that $B_\epsilon(x)\cap F\neq\phi$ for every $\epsilon>0$. Let $(x_0,y_0)\in(B_\epsilon(x)\times Y)\cap F$. Then let (x_n,y_n) be a sequence in $B_\epsilon(x)\times Y$. Since Y is compact $\Leftrightarrow$ sequentially compact, then we can pass y_n to a subsequence $(y_{n_k})\subset Y$ that converges to $y\in Y$. Since $x_n\in B_\epsilon(x)$ for every $\epsilon>0$, then $x_n\to x$, and $x_{n_k}\to x$. Then, we have that $(x_{n_k},y_{n_k})\to(x,y)$. Since $(x_n,y_n)\subset F$, and F is closed in $X\times Y$, then $(x,y)\in F$, but this contradicts our assumption that $(\{x\}\times Y)\cap F=\phi$. Thus, we conclude that there does exists $\epsilon>0$ such that $(B_\epsilon(x)\times Y)\cap F=\phi$. $\square$

We now show that π_X is closed. Let $F \subset X \times Y$ be closed. We want to show that $\pi_X(F)$ is closed. It suffices to show that $(X - \pi_X(F))$ is open, that is for any $x \in (X - \pi_X(F))$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \cap \pi_X(F) = \phi$. Let $x \in (X - \pi_X(F))$. Note that this means that $(x, y) \notin F$ for all $y \in Y$ since $x \notin \pi_X(F)$. In other words, this means that $(\{x\} \times Y) \cap F = \phi$. Applying part (b) above, we know there exists $\epsilon > 0$ such that

$$(B_\epsilon(x) \times Y) \cap F = \phi \tag{2}$$

Applying π_X, we get $\pi_X(B_\epsilon(x) \times Y) = B_\epsilon(x) \subset X$, and by the the empty intersection in (2) above, we see that $B_\epsilon(x) \cap \pi_X(F) = \phi$, so $B_\epsilon(x) \subset (X - \pi_X(F))$, so $(X - \pi_X(F))$ is open $\Leftrightarrow \pi_X(F)$ closed, so the map π_X is closed. $\square$

Problem 37

Suppose $f : (X, d_X) \to (Y, d_Y)$.

(a) Prove that if f is continuous, then the graph of f is closed in $X \times Y$.

Solution
The graph of f is graph$(f) = \{(x, f(x)) : x \in X\}$. Let $(x_n, f(x_n))$ be a sequence in graph(f) that converges to (x_0, y_0). To show graph(f) is closed, we need to show that $(x_0, y_0) \in$ graph(f), that is $(x_0, y_0) = (x_0, f(x_0))$. Since $(x_n, f(x_n)) \to (x_0, y_0)$, we have component-wise convergence, so $x_n \to x_0$ and $f(x_n) \to y_0$. By continuity of f, $x_n \to x_0$ implies $f(x_n) \to f(x_0)$. Since limits are unique, we conclude that $f(x_0) = y_0$, so we have $(x_0, y_0) = (x_0, f(x_0))$, and $(x_0, y_0) \in$ graph(f), so graph(f) is closed in $X \times Y$. $\square$

(b) Find metric spaces $(X, d_X), (Y, d_Y)$ and a function $f : X \to Y$ which is not continuous such that the graph of f is closed in $X \times Y$.

Solution
Consider the metric space $(\mathbb{R}, |\cdot|)$, and the function $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \frac{1}{x}$ if $x \neq 0$, and $f(0) = 0$. Then clearly, f is not continuous at 0. If we consider the graph of f, however, graph$(f) = \{(x, f(x)) : x \in \mathbb{R}\}$, then we see that it is closed in $\mathbb{R} \times \mathbb{R}$, as it contains all of its boundary points. In particular, $f(0) = 0$, which is a limit point as $x \to \infty$ or $x \to -\infty$. $\square$

(c) Suppose now that (Y, d_Y) is compact. Prove that if the graph of $f : X \to Y$ is closed, then f is continuous.

Solution
Let $x_0 \in X$. Then $f(x_0) \in Y$. Let V be the open ball around $f(x_0) \in Y$. Then $(Y - V)$ is closed, and since graph(f) is closed in $X \times Y$, then graph$(f) \cap (X \times (Y - V))$ is closed in $X \times Y$. By Problem 35, we also know that $\pi_X(\text{graph}(f) \cap (X \times (Y - V)))$ is closed in X, so $X - \pi_X(\text{graph}(f) \cap (X \times (Y - V))) =: U$ is open in X. Since $f(x_0) \in V$, then $f(x_0) \notin Y - V$, so $(x_0, f(x_0)) \notin X \times (Y - V)$, and $x \notin \pi_X(\text{graph}(f) \cap (X \times (Y - V)))$, so $x_0 \in U$. We are done if we can show that for arbitrary $x \in U, f(x) \in V$. Let $x \in U$. Then we can follow similar logic as above: $(x, f(x)) \notin (X \times (Y - V)) \implies f(x) \notin Y - V \implies f(x) \in V \implies f(U) \subset V$, so f is continuous at x_0, which was arbitrary, so we conclude that f is continuous. $\square$

Problem 39

Let (X, d) be a metric space and $(E_{\alpha\alpha\in I})$ be a collection of connected subsets of X. Suppose $\bigcap_{\alpha\in I} E_\alpha \neq \phi$. Show that $\bigcup_{\alpha\in I} E_\alpha$ is connected.

Solution
Let $X := \bigcup_{\alpha\in I} E_\alpha$. Since X is connected if and only if every continuous two valued function on X is constant, then it suffices to show that $f : X \to \{0,1\}$ is constant. Since each E_α is connected, then $f|_{E_\alpha} : X \to \{0,1\}$ is constant. Since $\bigcap_{\alpha\in I} E_\alpha \neq \phi$, let $x_0 \in \bigcap_{\alpha\in I} E_\alpha$. Then $x_0 \in E_\alpha$ for all $\alpha \in I$, hence in the union, X. By connectedness of each of the E_α's, we can say without loss of generality that $f(x_0) = 0$, hence constant. Since $E_\alpha \subset X$ for all $\alpha \in I$, then the continuous function $f : X \to \{0,1\}$ is also constant, so $X = \bigcup_{\alpha\in I} E_\alpha$ is connected. $\square$

Problem 40

Let (X, d_X) be a metric space, and let $E \subseteq X$.

(1) Show that if E is connected, then $\overline{E}$ is connected.

Solution
Suppose E is connected. Then all continuous two-valued functions on E are constant. Suppose without loss of generality that for all $x \in E$, $f(x) = 0$. Then it suffices to show that for any $x \in \overline{E}$, $f(x) = 0$ as well. Let $x_0 \in \overline{E}$. Then x_0 is adherent to E, so there exists a sequence $(x_n) \subset E$ that converges to x_0. Then we take the limit of $f(x_n)$, and by continuity of f on E, we can write

$$\lim_{n\to\infty} f(x_n) = f\big(\lim_{n\to\infty} x_n\big) = f(x_0).$$

Since $f(x_n) = 0$ for all n, then we conclude that $f(x_0) = 0$, so the continuous two valued function $f : \overline{E} \to \{0,1\}$ is in fact constant, so $\overline{E}$ is connected. $\square$

(2) Is the converse true?

Solution
Let $E \subset (\mathbb{R}^2, d_2)$ be the union of the open balls of radius 1 about $x_1 := (-1,0)$ and $x_2 := (1,0)$, $E = B_1(x_1) \cup B_1(x_2)$. Then if we consider the closure of E, then we see that $\overline{E}$ is connected, as $\overline{E}$ is the union of two closed balls in $\mathbb{R}^2$, which cannot be written as the union of two open, nonempty sets. However, by construction, E is disconnected, so $\overline{E}$ being connected does not imply that E is connected. $\square$

(3) Is it true that if E is connected, then $\text{int}(E)$ connected?

Solution
Let $E \subset (\mathbb{R}^2, d_2)$ be the union of the closed balls of radius 1 about the points $x_1 := (-1,0)$ and $x_2 := (1,0)$. Then E is connected, as it cannot be written as the union of two open, nonempty sets. However, if we consider $\text{int}(E)$, which consists of the union of the two open balls of radius 1 about x_1 and x_2, then we see that the interior is exactly the union of two open, nonempty sets, which implies that $\text{int}(E)$ is disconnected. Thus, E being connected does not imply that $\text{int}(E)$ is connected. $\square$