MATH 131B: Homework #3

Professor Dave Penneys

Assignment: 21, 23, 25, 27, 30

Eric Chuu
UID: 604406828

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Problem 21

(Lebesgue number lemma). Suppose that (X, d) is a sequentially compact metric space. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X. Show that there exists a constant $\delta > 0$ which satisfies the following property: for every $x \in X$, there is an $\alpha \in I$ such that $B_\delta(x) \subseteq U_\alpha$.

Solution

Suppose no such δ exists. Then there exists $x \in X$ such that for all $\alpha \in I$, $B_\delta(x) \subset U_\alpha$ for no δ. Let $\{x_n\}$ be a sequence in X. Since X is sequentially compact, there exists a subsequence $\{x_{n_k}\}$ that converges to $x \in X$. It suffices to show that for some $\delta > 0$ and $\alpha \in I$, $B_\delta(x) \subset U_\alpha$.

Problem 23

(a) Show that sequential compactness implies compactness. (Start with an open cover, use Lebesgue number lemma and total boundedness)

Solution
Suppose that (X, d) is a sequentially compact metric space, and let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X. Then by the Lebesgue number lemma, there exists a $\delta > 0$ such that for every $x \in X$, there is an $\alpha \in I$ such that $B_\delta(x) \subseteq U_\alpha$. Note that taking the union of the δ-balls for each $x \in X$ give us an open cover of X: $X \subset \cup_{x \in X} B_\delta(x)$. X is sequentially compact, so X is totally bounded, so from the open cover of δ-balls, we can find $\delta_1, \delta_2, \ldots, \delta_N$ such that $X \subset B_{\delta_1}(x_1) \cup B_{\delta_2}(x_2) \cup \ldots \cup B_{\delta_N}(x_N)$, which is a finite subcover of X, so X is compact.

(b) Show that a compact metric space is sequentially compact.

Solution
To prove this, we first prove the following lemma: X is sequentially compact if and only if every sequence of X has a at least one limit point.

Proof: Let X be sequentially compact. Then every sequence in X has a subsequence that converges in X, so (by Proposition 1.4.5.) the limit of this convergent subsequence is a limit point of the sequence, and it follows that every sequence in X has at least one limit point. Conversely, suppose that X satisfies the property that every sequence of X has at least one limit point. Let $\{x_n\}$ be a sequence in X. It suffices to find a subsequence $\{x_{n_k}\}$ that converges in X. There are two cases to consider. Case 1: There are finitely many unique points in X and infinitely many recurring points. Then we can pick our subsequence to be a constant sequence after finitely many terms, which converges. Case 2: There are infinitely many distinct points in X. By hypothesis, $\{x_n\}$ has a limit point, say x_0, and by proposition 1.4.5, this is equivalent to having a subsequence $\{x_{n_k}\}$ that converges to x_0, and we've shown that for an arbitrary sequence in X, we can find a convergent subsequence. Thus X is sequentially compact.

Let X be a compact metric space, and suppose that X is not sequentially compact. Then there exists a sequence $\{x_n\} \subset X$ that has no limit points. Equivalently, $\{x_n\}$ has no convergent subsequence. Then for every $x \in X$ there exists $\epsilon > 0$ such that $B_\epsilon(x)$ contains x for at most finitely many elements of X. Then if we enumerate the finite number of points in $B_\epsilon(x)$: $x_1, x_2, \ldots, x_N$, then we can take $\delta := \min\{d(x, x_1), d(x, x_2), \ldots, d(x, x_N)\}$. Then $x_i \notin B_\delta(x)$ for $1 \leq i \leq N$, which means the ball contains the single point x. We can form such a ball for every point in X, and by taking the union of these balls around every $x \in X$, we form an open cover that cannot be made finite because if we take out any of the balls, then we fail to cover one of the points of X. Hence, we have contradict compactness of X, and the contradiction establishes the desired result. $\square$

Problem 25

Suppose that (X, d) is a compact metric space and $(K_n)_{n\in\mathbb{N}}$ is a sequence of nonempty nested compact subsets of X, i.e., the K_n's are compact subsets such that $K_n \supset K_{n+1} \neq \phi$ for all $n \in \mathbb{N}$. Show that $\cap_{n=1}^{\infty} K_n \neq \phi$. Then give a different proof of the problem using sequential compactness instead.

Solution

(1) Suppose that $\cap_{n=1}^{\infty} K_n = \phi$. Since each K_i for $1 \leq i \leq n$ is compact, then K_i is closed. Since $\cap_{n=1}^{\infty} K_n$ is an infinite intersection of closed sets, it is closed, and $(\cap_{n=1}^{\infty} K_n)^c = \cup_{n=1}^{\infty} K_n^c$ is open and covers X. By compactness of X, there exists $n_1, \ldots n_N$ such that $K_{n_1} \cup \ldots \cup K_{n_N} = X$. Taking the complement again, we get $K_{n_1} \cap \ldots \cap K_{n_N} = \phi$. Suppose that $K_{n_N}^c$ is the largest open set of the cover. Then we have that the empty intersection implies that $K_{n_N} = \phi$, which contradicts our the hypothesis that $K_n \neq \phi$ for all $n \in \mathbb{N}$, and we conclude that $\cap_{n=1}^{\infty} K_n \neq \phi$ $\square$

(2) We prove the result using sequential compactness instead. Let $K := \cap_{n=1}^{\infty} K_n$. Since each K_n is compact, then each K_n is also closed. Arbitrary intersection of closed sets are closed, so K is closed as well, and since it is a closed subset of each of the K_n's, each of which are compact, it follows that K is compact. We construct a sequence so that for each term of the sequence, we choose $x_n \in K_n$. Then the sequence $\{x_n\} \subset K_1$ since the K_n's are nested, i.e., $K_1 \supset K_2 \ldots$, and by sequential compactness of K_1, there exists a convergent subsequence $\{x_{n_k}\}$ that converges in $x_0 \in K_1$. We want to show that $x_0 \in K_n$ for all $n \in \mathbb{N}$. Let $j > 0$. Then it suffices to show that $x_0 \in K_j$. By construction of the original sequence, if we pick k large enough so that $n_k \geq j$, then $x_{n_k} \in K_j$, so all but finitely many terms are also in K_j. Since every term of the sequence after x_{n_k} is in K_j since $K_j \supset K_{j+1}$, and since K_j is closed, $x_0 \in K_j$, and we conclude that $x_0 \in \cap_{n=1}^{\infty} K_n$, and hence the intersection is nonempty. $\square$

Problem 27

Suppose $f : (X, d_X) \to (Y, d_Y)$. Prove the following two conditions are equivalent:

(1) f is continuous: for all $x_0 \in X$ and every $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(x, x_0) < \delta$ implies $d_Y(f(x), f(x_0)) < \epsilon$.

(2) f is topologically continuous, i.e., for every open $V \subseteq Y, f^{-1}(V) = \{x \in X | f(x) \in V\}$ is open in X.

Solution

Show (1) $\implies$ (2).

Suppose f is continuous as described by (1). We want to show that for every open $V \subseteq Y$, $f^{-1}(V) \subseteq X$ is open. Let V be open in Y and $f(x_0) \in V$. Then there exists $\epsilon > 0$ such that $B_\epsilon(f(x_0)) \subset V$, so for all $f(x) \in B_\epsilon(f(x_0)), d_Y(f(x_0), f(x)) < \epsilon$. By hypothesis, there exists $\delta > 0$ such that $d_X(x_0, x) < \delta$, so $x \in B_\delta(x_0)$. Set $f^{-1}(V) := B_\delta(x_0)$, which is open in X, and $f(B_\delta(x_0) \subseteq B_\epsilon(f(x_0)) \subset V$. Thus, we've shown that f is topologically continuous.

Show (2) $\implies$ (1).

Suppose f is topologically continuous. Then we want to show that for every $x_0 \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $d_X(x, x_0) < \delta$ implies $d_Y(f(x), f(x_0)) < \epsilon$. Let $\epsilon > 0$, $f(x_0) \in Y$. Set $V := B_\epsilon(f(x_0))$. Then $V \subseteq Y$ is open, and by hypothesis, $f^{-1}(V) \subseteq X$ is open. Then for all $x \in f^{-1}(V)$, there exists $\delta > 0$ such that $d_X(x, x_0) < \delta$, so $x \in B_\delta(x_0) \subseteq f^{-1}(V)$, and we conclude that $x \in B_\delta(x_0) \subset V \implies f(x) \in V = B_\epsilon(f(x_0))$, so (1) follows. $\square$

Problem 30

A function $f : (X, d) \to (X, d)$ is called a contraction if there is a $0 \leq c < 1$ such that $d(f(x), f(y)) \leq c \cdot d(x, y)$ for all $x, y \in X$.

(a) Show that a contraction is continuous.

Solution
Let $\epsilon > 0$, $x \in X$. Then we want to show there exists $\delta > 0$ such that $d(f(x), f(y)) < \epsilon$ whenever $d(x, y) < \delta$. Let $c \in (0, 1]$. Then let $\delta := \epsilon/c$. Then when $d(x, y) < \delta$, we have

$$d(f(x), f(y)) \leq c \cdot d(x, y) < c \cdot \epsilon/c = \epsilon,$$

so we conclude that a contraction is continuous. $\square$

(b) Pick a point $x_0 \in X$ and define the sequence $\{x_n\}$ inductively by $x_1 = f(x_0)$, and $x_{n+1} = f(x_n)$ for all $n \geq 1$, so that $x_n = f^n(x_0)$. Prove that $\{x_n\}$ is Cauchy.

(c) Suppose (X, d_X) is complete, so that the Cauchy sequence obtained in (b) converges to $x \in X$. Show that $f(x) = x$.

(d) Prove that if x, y are fixed points of f, then $x = y$.

Solution