Outline for Continuity

We covered most of Chapter 4 in Rudin, and added one topic. Here is an outline of what we did:

1. The definition of $\lim_{p \to q} f(p)$ when $q \in \overline{E}$ for a function $f : E \subset \chi \to \mathcal{Y}$, when (χ, d_χ) and $(\mathcal{Y}, d_\mathcal{Y})$ are metric spaces, and the equivalence of the ϵ, δ and sequential definitions of that limit (Theorem 4.2 in Rudin).

2. The definition of continuity at a point for functions between metric spaces. As Rudin points out in the proof to Theorem 4.4, the equivalence of ϵ, δ and sequential limits reduces the continuity of sums, products and quotients of continuous functions, to the limit theorems for sequences. I did not discuss continuity of compositions of continuous functions (Theorem 4.7) because it seemed obvious and boring.

3. The equivalence of the continuity of f mapping (χ, d_χ) into $(\mathcal{Y}, d_\mathcal{Y})$ to $f^{-1}(\mathcal{O})$ is open for every open set $\mathcal{O} \subset \mathcal{Y}$ (Theorem 4.8). Note that by this time we were only talking about functions whose domains were (whole) metric spaces. This is natural since if in (χ, d) the function f has domain $E \subset \chi$, then it is defined on the whole metric space (E, d).

4. If $(\mathcal{Y}, d_\mathcal{Y})$ is $(\mathbb{R}^k, d_2)$, then $f : \chi \to \mathbb{R}^k$ is continuous at p, if and only if each component f_j is continuous at p – here $f = (f_1, f_2, .., f_k)$. This is Theorem 4.10, and you should know it, but I did not discuss it in class.

5. If f is continuous, and (χ, d_χ) is compact, then $f(\chi)$ is compact in $(\mathcal{Y}, d_\mathcal{Y})$ (Theorem 4.14). Using that, you should know how to prove that a real-valued function on a compact metric space is bounded, and takes the values $\sup\{f(p) : p \in \chi\}$ and $\inf\{f(p) : p \in \chi\}$, i.e. "a continuous function on a compact metric space takes maximum and minimum values."

6. If $f : \chi \to \mathcal{Y}$ is continuous and one-to-one, and (χ, d_χ) is compact, then f^{-1} is continuous as a function from $(f(\chi), d_\mathcal{Y})$ to (χ, d_χ). This is a relatively minor theorem, but its proof is a good exercise in set theory!

7. The definition of uniform continuity, and the theorem that continuous functions on compact metric spaces are automatically uniformly continuous (Theorem 4.19). After this Rudin does several examples, these are good, but I skipped most of them. Read them before the final.

8. The connectedness version of 5. If f is continuous, and χ is connected, then $f(\chi)$ is connected (Theorem 4.22). Using that, you should know how to prove that the range of a real-valued function on a connected metric space is an interval (Theorem 2.3). At the end of "Continuity and Connectedness", I left Chapter 4, but I did do two more results:

9. Suppose that $f : (\chi, d) \to (\mathcal{Y}, d_\mathcal{Y})$ is uniformly continuous. If $\{p_n\}_{n=1}^\infty$ is a Cauchy sequence in χ, then $\{f(p_n)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{Y}$. This is Exercise 11 on page 99, and its proof is easy.

10. Assume f is uniformly continuous on $E \subset \chi$ with values in $\mathcal{Y}$ and $(\mathcal{Y}, d_\mathcal{Y})$ is a

complete metric space. If $\chi = \overline{E} = E \cup Bdy(E)$, i.e."E is dense in χ", then there is a continuous (actually uniformly continuous) function on χ such that $\tilde{f}(p) = f(p)$ when $p \in E$. This is Exercise 13 on page 100 in Rudin. The proof suggested there does not use 9. I gave the following proof using 9.

Proof: Given $q \in \chi$, either $q \in E$ and $f(q)$ is defined, or $q \in Bdy(E)$ and there is a sequence $\{p_n\}_{n=1}^{\infty} \subset E$ such that $\lim_{n\to\infty} p_n = q$. So by 9. we have $\{f(p_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in χ which converges to a point $\tilde{f}(p)$ in $\mathcal{Y}$ because $(\mathcal{Y}, d_{\mathcal{Y}})$ is complete. To show that this defines $\tilde{f}$ on χ you need to show that, if $\{p'_n\}_{n=1}^{\infty} \subset E$ is any *other* sequence converging to q, then $\lim_{n\to\infty} f(p'_n) = \lim_{n\to\infty} f(p_n)$. This follows easily since given $\epsilon > 0$ there is a $\delta > 0$ such that $d_{\mathcal{Y}}(f(p), f(p')) < \epsilon$ when $d_{\chi}(p, p') < \delta$ for all $p, p' \in E$. Since $\{p_n\}$ and $\{p'_n\}$ are both converging to q, there is an N such that $d_{\chi}(p_n, p'_n) < \delta$ when $n > N$. So $d_{\mathcal{Y}}(f(p_n), f(p'_n)) < \epsilon$ when $n > N$. That leads easily to $\lim_{n\to\infty} f(p'_n) = \lim_{n\to\infty} f(p'_n)$ (why?).

Now comes the harder part. We need to show that $\tilde{f}$ is continuous on χ. We have two consequences of the triangle inequality: for any $q, q' \in \chi$ and $p, p' \in E$

$$d_{\mathcal{Y}}(\tilde{f}(q), \tilde{f}(q')) \le d_{\mathcal{Y}}(\tilde{f}(q), f(p)) + d_{\mathcal{Y}}(f(p), f(p')) + d_{\mathcal{Y}}(f(p'), \tilde{f}(q')) \text{ and} \tag{1}$$

$$d_{\chi}(p, p') \le d_{\chi}(p, q) + d_{\chi}(q, q') + d_{\chi}(q', p') \tag{2}$$

Given $\epsilon > 0$, choose $\delta > 0$ so that $d_{\mathcal{Y}}(f(p), f(p')) < \epsilon/3$ when $d_{\chi}(p, p') < \delta$. Then we will be able to make $d_{\mathcal{Y}}(\tilde{f}(q), \tilde{f}(q')) < \epsilon$ when $d_{\chi}(q, q') < \delta/3$ by choosing p and p' correctly. Since we have a sequence $\{p_n\}_{n=1}^{\infty} \subset E$ such that $\lim_{n\to\infty} p_n = q$ and $\lim_{n\to\infty} f(p_n) = \tilde{f}(q)$, we can choose one of the p_n's so that $d_{\mathcal{Y}}(\tilde{f}(q), f(p_n)) < \epsilon/3$ and $d_{\chi}(p_n, q) < \delta/3$. Use that p_n for p. In the same way we can find a $p' \in E$ so that $d_{\mathcal{Y}}(\tilde{f}(q'), f(p')) < \epsilon/3$ and $d_{\chi}(q', p') < \delta/3$. With those choices of p and p' the inequalities (1) and (2) show that $d_{\mathcal{Y}}(\tilde{f}(q), \tilde{f}(q')) < \epsilon$ when $d_{\chi}(q, q') < \delta/3$. Note that this says $\tilde{f}$ is uniformly continuous on χ.