MATH 128A: Homework #3

Professor John Strain
Assignment: 1-7

Eric Chuu

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Problem 1 (BF 3.3.11)

(a) Show both (1) and (2) interpolate the data

$$P(x) = 3 - 2(x+1) + 0(x+1)x + (x+1)x(x-1) \tag{1}$$
$$Q(x) = -1 + 4(x+2) - 3(x+2)(x+1) + (x+2)(x+1)x \tag{2}$$

(b) Why does part (a) not violate the uniqueness property of interpolating polynomials.

Solution
(a) Using equation (1) and substituting the x values in, we get

$$\begin{aligned}
P(-2) &= 3 - 2(-2+1) + (-2+1)(-2)(-2-1) = 3 + 2 - 6 = -1 \\
P(-1) &= 3 - 2(-1+1) + (-1+1)(-1)(-1-1) = 3 \\
P(0) &= 3 - 2(-0+1) + 0 = 1 \\
P(1) &= 3 - 2(1+1) + 1 + 1)(1-1) = 3 - 4 = -1 \\
P(2) &= 3 - 2(2+1) + (2+1)2(2-1) = 3 - 6 + 6 = 3
\end{aligned}$$

Using equation (2), we get

$$\begin{aligned}
Q(-2) &= -1 + 4(-2+2) - 3(-2+2) + (-2+2) = -1 \\
Q(-1) &= -1 + 4(-1+2) = -1 + 4 = 3 \\
Q(0) &= -1 + 4(0+2) - 3(2) = -1 + 8 - 6 = 1 \\
Q(1) &= -1 + 4(2+1) - 3(1-2)(1+1) + 3(2)1 = -1 \\
Q(2) &= -1 + 0 - 0 + 0 = -1
\end{aligned}$$

Clearly, both P and Q interpolate the data.
(b) If we expand both P and Q we see that

$$\begin{aligned}
P(x) &= 3 - 2x - 2 + x^3 - x^2 + x^2 - x \\
&= 1 - 3x + x^3 \\
Q(x) &= -1 + 4x + 8 - 3x^2 - 9x - 6 + x^3 + 3x^2 + 2x \\
&= 1 - 3x + x^3
\end{aligned}$$

so $P(x) = Q(x).P, Q$ are different representations of the same polynomial, so uniqueness of interpolating polynomials holds is not violated. □

Problem 2 (BF 3.4.5)

(a) Use the following values and five-digit rounding arithmetic to construct the Hermite interpolating polynomial to approximate $\sin 0.34$.

x	$\sin x$	$D_x \sin x = \cos x$
0.30	0.29552	0.95534
0.32	0.31457	0.94924
0.35	0.34290	0.93937

(b) Determine the error bound for the approximation in part (a), and compare it to the actual error.

(c) Add $\sin 0.33 = 0.32404$ and $\cos 0.33 = 0.94606$ to the data, and redo the calculations.

Solution
See following page for solution.

Problem 3 (BF 4.1.13)

Use the following data and the knowledge that the first five derivatives of f are bounded on $[1, 5]$ by $2, 3, 6, 12, 23$ respectively to approximate $f'(3)$ as accurately as possible. Find a bound for the error.

x	1	2	3	4	5
$f(x)$	2.4142	2.6734	2.8974	3.0976	3.2804

Solution
The formula for approximating $f'(x_0)$ is given by

$$f'(x_0) = \frac{1}{12h}\left[f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)\right]$$

We can approximate $f'(x_0)$ for $x_0 = 3, h = 1$ with

$$\begin{aligned} f'(3) &= \frac{1}{12} \cdot [f(3-2) - 8f(3-1) + 8f(3+1) - f(3+2)] \\ &= \frac{1}{12} \cdot [f(1) - 8f(2) + 8f(4) - f(5)] \\ &= \frac{1}{12} \cdot [2 \cdot 2.4142 - 8 \cdot 2.6734 + 8 \cdot 3.0976 - 3.23804] \\ &= 0.21062 \end{aligned}$$

We can bound the error with

$$\left|\frac{h^4}{30} f^{(5)}(\xi)\right| \leq \left|\frac{1}{30} \cdot 23\right| = 0.76667$$

where $\xi \in [1, 5]$. $\square$

Problem 4

For equidistatnt points $x_j = j, 0 \le j \le n, n$ even, let

$$\omega(x) = (x - x_0)(x_0 - x_1)\cdots(x - x_n) \tag{3}$$

Use Stirling's formula to estimate the ratio $\omega(1/2)/\omega(n/2+1/2)$ for large n. Explain the Runge phenomenon.

Solution
We first consider $\omega(x)$ for $x = 1/2$, and if we factor out $1/2$ from each of the terms in the product, we get

$$\begin{aligned} w(1/2) &= (1/2)(-1/2)(-3/2)\ldots(-(n-1/2)) \\ &= 2^{-(n+1)}\frac{(2n-1)!}{2^{n-1}(n-1)!} \\ &\approx O(2^n) \end{aligned}$$

If we consider the denominator of the ratio we are interested in, then we see that as $n \to \infty$, in $\omega(1/2+n/2)$, the $1/2$ is negligible, so it suffices to only analyze the $\omega(n/2)$ term:

$$\begin{aligned} \omega\left(\frac{n}{2}\right) &= \frac{n}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\cdots\left(\frac{n}{2}-n\right) \\ &= \left(\left(\frac{n}{2}\right)!\right)^2 \\ &\approx \left(\frac{n}{e}\right)^n \\ &= 2^{-n}\left(\frac{n}{e}\right)^n \end{aligned}$$

Note we used Sterling's formula in the approximation above. If we now consider the ratio, then

$$\frac{\omega\left(\frac{1}{2}\right)}{\left(\frac{n}{2}+\frac{1}{2}\right)} \approx \frac{O(2^n)}{2^{-n}\left(\frac{n}{e}\right)^n} \approx O(2^n)$$

This implies that $n \to \infty$ does not give convergence of this ratio, i.e., $\omega(x)$ near the middle of the interval differs from $\omega(x)$ near the ends of the interval by a multiple of 2^n. Runge's phenomenon is thus the issue that occurs when interpolating a high order polynomial on an interval with equidistant points. This tends to give large error and oscillation at the ends of the interval, even though the middle of the interval gives good behavior. $\square$

Problem 5

Interpolate Runge's function

$$f(x) = \frac{1}{1+x^2} \tag{4}$$

on the interval $[-5, 5]\}$ at
(a) equidistant points
(b) Chebyshev points
(c) geometrically distrubted points
with $x_0 = 0, x_{\pm 1} = \pm 5, x_{\pm(j+1)} = (x_{\pm j} + x_{\pm(j-1)})/2$. Use $n = 3, 5, 9, 17, 33$ points.

For each case:
(1) tabulate the maximum error over 1000 random points y_k with $|y_k| < 5$
(2) plot $\omega(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$
(3) identify subregions of $[-6, 6]$ where $|\omega(x)|$ is exceptionally large or small.

Solution
As expected, when interpolating at equidistant points for the function in (4), we see that at the ends of the interval, there is high error and oscillation, particularly when the degree of the interpolating polynomial is large. Chebyshev points partially solve this problem, and we can examine the errors by uncommenting the line that returns the error table. See the following page for the respective plots. □

Problem 6

Let p be a positive integer and

$$f(x) = 2^x \tag{5}$$

for $0 \leq x \leq 2$.

(a) Find a formula for the pth derivative $f^{(p)}(x)$.
(b) For $p = 0, 1, 2$, find a formula for the polynomial H_p of degree $2p + 1$ such that

$$H_p^{(k)}(x_j) = f^{(k)}(x_j)$$

for $0 \leq k \leq p, 0 \leq j \leq 1$, $x_0 = 0, x_1 = 2$.
(c) For general p prove that

$$|f(x) - H_p(x)| \leq \left(\frac{1}{p+1}\right)^{2p+2} \tag{6}$$

for $0 \leq x \leq 2$.
(d) Show that one step of Newton's method for solving

$$g(y) = x \ln 2 - \ln y = 0$$

starting from $y_0 = H_4(x)$ gives $y_1 = f(x) = 2^x$ to full double precision accuracy for $0 \leq x \leq 2$.

Solution
(a) We consider the first three derivatives of the function f, where $f(x) = 2^x$, for $0 \leq x < 2$.

$$\begin{aligned}
f(x) &= 2^x \\
f'(x) &= 2^x \ln(2) \\
f''(x) &= 2^x \ln(2) \cdot \ln(2) \\
f'''(x) &= 2^x \ln(2) \cdot \ln(2) \cdot \ln(2)
\end{aligned}$$

Generalizing for $p \in \mathbb{Z}^+$, we see that the pth derivative is given by

$$\boxed{f^{(p)}(x) = 2^x (\ln 2)^p} \tag{7}$$

Problem 7

For integer $k \geq 4$ let

$$p_k = k \sin\left(\frac{\pi}{k}\right) \qquad P_k = k \tan\left(\frac{\pi}{k}\right) \tag{8}$$

(a) Show that $p_4 = 2\sqrt{2}$ and $P_4 = 4$.
(b) Show that

$$P_{2k} = \frac{2p_k P_k}{p_k + P_k} \qquad p_{2k} = \sqrt{p_k P_{2k}} \tag{9}$$

for $k \geq 4$.
(c) Approximate π within 10^{-4} by computing p_k and P_k until $P_k - p_k < 10^{-4}$.
(d) Use Taylor series to show that

$$\pi = p_k + \sum_{j=1}^{\infty} q_j k^{-2j} \qquad \pi = P_k + \sum_{j=1}^{\infty} Q_j k^{-2j}$$

for some constants q_j and Q_j.
(e) Use extrapolation with $h = 1/k$ to approximate π within 10^{-12}.

Solution
(a)

$$\begin{aligned} p_4 &= 4\sin\left(\frac{\pi}{4}\right) = 4 \cdot \frac{\sqrt{2}}{2} = 2\sqrt{2} \\ P_4 &= 4\tan\left(\frac{\pi}{4}\right) = 4 \cdot 1 = 4 \end{aligned}$$

(b) We expand the left hand side of P_{2k}, and we see that by definition

$$P_{2k} = 2k \tan\left(\frac{\pi}{2k}\right) \tag{10}$$

If we consider the right hand side and recall the half angle formulas for $\sin x$ and $\cos x$, then we see that

$$\begin{aligned} \frac{2p_k P_k}{p_k + P_k} &= \frac{2k\sin\left(\frac{\pi}{k}\right) \cdot \tan\left(\frac{\pi}{k}\right)}{k\sin\left(\frac{\pi}{k}\right) + k\tan\left(\frac{\pi}{k}\right)} \\ &= \frac{2k\sin\left(\frac{\pi}{k}\right)\tan\left(\frac{\pi}{k}\right)}{\sin\left(\frac{\pi}{k}\right) + \tan\left(\frac{\pi}{k}\right)} \\ &= \frac{2k\sin\left(\frac{\pi}{k}\right)}{1 + \cos\left(\frac{\pi}{k}\right)} \\ &= \frac{2k \cdot 2\sin\left(\frac{\pi}{2k}\right)\cos\left(\frac{\pi}{2k}\right)}{1 + 2\cos^2\left(\frac{\pi}{2k}\right) - 1} \\ &= 2k\tan\left(\frac{\pi}{2k}\right), \end{aligned}$$

which is the left hand side as expressed in equation (10).

For the second equation in (9), we use the definition to express the left hand side as

$$p_{2k} = 2k \cdot \sin\left(\frac{\pi}{2k}\right) \tag{11}$$

Expanding the right hand side and using double angle formulas, we see that

$$\begin{aligned}
p_{2k} &= \sqrt{k \sin\left(\frac{\pi}{k}\right) \cdot 2k \tan\left(\frac{\pi}{2k}\right)} \\
&= 2k\sqrt{\frac{1}{2}\sin\left(\frac{\pi}{k}\right)\tan\left(\frac{\pi}{2k}\right)} \\
&= 2k\sqrt{\frac{1}{2}\cdot 2\sin\left(\frac{\pi}{2k}\right)\cos\left(\frac{\pi}{2k}\right)\cdot\frac{\sin\left(\frac{\pi}{2k}\right)}{\cos\left(\frac{\pi}{2k}\right)}} \\
&= 2k\sqrt{\sin^2\left(\frac{\pi}{2k}\right)} \\
&= 2k\sin\left(\frac{\pi}{2k}\right),
\end{aligned}$$

which coincides with our result in the left hand side as expressed in equation (11).

(c) The MATLAB code below approximates π within 10^{-4} according the specification.

```
% approximate pi within 10^(-4) using the recursive
% definitions derived in part (b).
% returns the approximations of both p_2k and P_2k
function [pi_1, pi_2] = appox_pi()
    TOL = 10^(-4);

    % initial conditions for recurrence relation
    p_k = 2 * sqrt(2);
    P_k = 4;

    while abs(p_k - P_k) > TOL
        P_2k = 2 * p_k * P_k / (p_k + P_k);
        p_2k = sqrt(p_k * P_2k);

        P_k = P_2k;
        p_k = p_2k;
    end

    pi_1 = p_k;
    pi_2 = P_k;

%% OUTPUT OF FUNCTION CALL:
>> [x,y] = approx_pi()
x =
   3.141572940367092
y =
   3.141632080703182
```

(d) We consider p_k first and use the Taylor series expansion for $\sin x$,

$$k \sin\left(\frac{\pi}{k}\right) = \sum_{j=0}^{\infty} (-1)^j \frac{1}{(2j+1)!} \cdot \pi^{2j+1} \frac{1}{k^{2j+1}} = \sum_{j=0}^{\infty} (-1)^j \frac{1}{(2j+1)!} \cdot \pi^{2j+1} \frac{1}{k^{2j}}$$

We can use this last equality and re-index by taking the 0th term out of the sum,

$$\begin{aligned} p_k + \sum_{j=1}^{\infty} q_j k^{-2j} &= \pi + \sum_{j=1}^{\infty} (-1)^j \frac{1}{(2j+1)!} \cdot \pi^{2j+1} \frac{1}{k^{2j}} + \sum_{j=1}^{\infty} q_j k^{-2j} \\ &= \pi + \sum_{j=1}^{\infty} \left((-1)^j \frac{\pi^{2j+1}}{(2j+1)!} + q_j \right) \cdot \frac{1}{k^{2j}} \\ &= \pi \end{aligned}$$

for

$$q_j := (-1)^{j+1} \frac{\pi^{2j+1}}{(2j+1)!},$$

so we've shown that for some constants, q_j, we have

$$\pi = p_k + \sum_{j=1}^{\infty} q_j k^{-2j}.$$

For the second equation, we consider the Taylor series expansion for $\tan x$,

$$k \tan\left(\frac{\pi}{k}\right) = k \cdot \sum_{j=0}^{\infty} t^j \left(\frac{\pi}{k}\right)^{2j+1} = \sum_{j=0}^{\infty} t^j \pi^{2j+1} \cdot \frac{1}{k^2 j}$$

where t_j are the coefficients of the Taylor series expansion for $\tan x$, whose values are not necessary for this calculation. Like before we take the 0th element out of the sum, and we evaluate

$$\begin{aligned} P_k + \sum_{j=1}^{\infty} &= \pi + \sum_{j=1}^{\infty} t^j \pi^{2j+1} \cdot \frac{1}{k^2 j} + \sum_{j=1}^{\infty} Q_j k^{-2j} \\ &= \pi + \sum_{j=1}^{\infty} \left(t^j \pi^{2j+1} + Q_j \right) \frac{1}{k^{2j}} \\ &= \pi \end{aligned}$$

for

$$Q_j := -t^j \pi^{2j+1},$$

and we have shown that for some constants Q_j, we have

$$\pi = P_k + \sum_{j=1}^{\infty} Q_j k^{-2j}$$

(e)

MATLAB Code for extrapolation to approximate π

```
% Richardson extrapolation to approximate pi within 10^(-12)
% using p_k and P_k as defined above
function M = richardson()
    x = 1;
    h = 0.25;
    TOL = 10^(-12);

    R(1, 1) = (f(x + h) - f(x - h))/(2*h);
    for i = 1:100 % iterate only 100 times, else diverge
        h = h / 2;

        % Store results in matrix form
        R(i + 1, 1) = abs((f(x + h) - f(x - h)) / (2 * h));

        for j = 1:i
            R(i + 1, j + 1) = abs((4^j * R(i + 1, j) - R(i, j)) / (4^j - 1));
        end

        if ( abs( R(i + 1, i + 1) - R(i, i) ) < TOL )
            break;
        end
    end

    M = R;

    % comment out function that's not being used
    % one for each p_k and P_k
    function p = f(k)
        % p = k * sin(pi/k); %% p_k
        p = k * tan(pi/k); %% P_k
```

Result from extrapolation:

The results of the extrapolation are on the attached sheet. The first matrix contains the results from using P_k, and the second matrix contains the results from using p_k. □