MATH 131AH: Homework #6

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Assignment: 36-43
February 24, 2016

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Problem 1

Prove that $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence when $s_n = \sum_{k=1}^{n} \frac{1}{k!}$.

Solution

We first show inductively that $n! \geq 2^{n-1}$ for $n \in N$.

Base Case: $n = 1$. Then $1! = 1 \geq 2^0 = 1$, so the base case is satisfied.

Inductive Step: Suppose the inequality holds for n. We must show it is true for $n+1$, that is $(n+1)! = 2^n$. Evaluating the LHS, we get $(n+1)! = (n+1)n!$. By the inductive hypothesis, we know that $(n+1)n! \geq (n+1)2^{n-1}$. Since $n \geq 1$ for $n \in \mathbb{N}$, then $(n+1) \geq 2$, so $(n+1)! = (n+1)n! \geq (n+1)2^{n-1} \geq 2 \cdot 2^{n-1} = 2^n$. Having shown the inequality holds for $n+1$, by induction it holds for all $n \in N$.

Since $n \geq 2^{n-1}$, we can bound s_n above, so we have

$$\sum_{k=1}^{n} \frac{1}{k!} \leq \sum_{k=1}^{n} \frac{1}{2^{n-1}} = t_n.$$

If we can show that t_n converges, then by the comparison test, s_n also converges. It is clear that t_n is a geometric series,

$$\sum_{k=1}^{n} \frac{1}{2^{n-1}} = \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{n-1},$$

and since $\frac{1}{2} < 1$, then t_n converges by Theorem 3.26. Thus, s_n converges, and since convergent sequences are Cauchy sequences, it follows that s_n is also a Cauchy sequence. ■

Problem 2

Determine which of the following sequences are Cauchy sequences:

$$\text{(i) } a_n = \frac{(-1)^n + n^2}{2n^2+1}$$
$$\text{(ii) } a_n = \left(1 + \frac{1}{\sqrt{n}}\right)^n$$
$$\text{(iii) } a_n = 2^{(-1)^n n}$$

Solution
(i) We can write a_n as a sum of two terms, as shown by

$$a_n = \frac{(-1)^n + n^2}{2n^2+1} = \frac{(-1)^n}{2n^2+1} + \frac{n^2}{2n^2+1}$$

Note that $(-1)^n$ is bounded, so $\frac{(-1)^n}{2n^2+1} \leq \frac{B}{2n^2+1}$, so when n is large, this term goes to 0. We can evaluate the other term of the sum

$$\frac{n^2}{2n^2+1} = \frac{1}{2+1/n^2}$$

which converges to $\frac{1}{2}$ when n is large. Thus, a_n converges $\frac{1}{2}$. Since a_n is convergent, it is also Cauchy. ■

(ii) We can write a_n as

$$a_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}\cdot\sqrt{n}}$$

Taking the limit as n approaches infinity, we see that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}\cdot\sqrt{n}} = \lim_{n\to\infty} e^{\sqrt{n}}$$

Since this is an unbounded sequence, a_n does not converge, so it is not Cauchy. ■

(iii) For n even, we can write $a_n = 2^n$, which is monotonically increasing but unbounded, so it diverges. For n odd, we can write $a_n = \frac{1}{2^n}$, which is bounded, monotonically decreasing, and converges to 0. A sequence converges to $L \in \mathbb{R}$ if and only if all of its subsequences converge to L, so it is clear that a_n does not converge, so it is not Cauchy. ■

Problem 3

What are lim sup a_n ad lim inf a_n for the three sequences in the previous problem?

Solution
(i) Since the limit exists, and $\lim_{n\to\infty} a_n = \frac{1}{2}$, so lim inf $a_n =$ lim sup $a_n = \frac{1}{2}$.

(ii) The only subsequential limit of $\{a_n\}$ is $+\infty$, so for this sequence, the lim sup $a_n =$ lim inf $a_n = +\infty$.

(iii) For the sequence $\{a_n\}$, the set of subsequential limits consists of $\{0, \infty\}$, so lim sup $a_n = +\infty$ and lim inf $a_n = 0$. ■

Problem 4

Suppose that $\sum_{k=1}^{\infty}(a_k+b_k)$ is convergent. What can you say about $\sum_{k=1}^{\infty} c_k$ when $c_{2k}=b_k$ and $c_{2k-1}=a_k$?

Solution
We can take $\{a_n\}=1$ and $\{b_n\}=-1$. Then

$$\sum_{n=1}^{\infty}(a_k+b_k)=(1+(-1))+(1+(-1)+...$$

which converges since every partial sum is equal to 0. However, if we use the same sequences to evaluate the series $\sum c_k$, then

$$\begin{aligned}\sum_{k=1}^{\infty} c_k &= a_1+b_1+a_2+b_2+... \\ &= 1+(-1)+1+(-1)+... \\ &= (1+(-1)+1)+(1+(-1)+1)+... \\ &= 1+1+1+...\end{aligned}$$

which is a divergent series. Therefore, we see that addition when determining convergence of infinite series is not necessarily commutative. In this case, when the elements of the sum that is initially convergent are rearranged, the series can become divergent. ■

Problem 5

Prove $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for any $a > 0$.

Solution

Let $\{s_n\} = \frac{a^n}{n!}$ for any $a > 0$. Then $a^n > 0$, so it follows that that $s_n > 0$ for all n. Since a is fixed, then for some N, $n \geq N$ results in $n \geq a$, so $\frac{a}{n+1} < 1$. Then

$$s_{n+1} = \frac{a^{n+1}}{(n+1)!} = \frac{a}{n+1} \cdot \frac{a^n}{n!} < s_n.$$

Thus, we've shown that for $n \geq N$, the sequence $s_{n+1} < s_n$, so $\{s_n\}$ is monotonically decreasing. Since it is also bounded, it converges to some number L, so $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a^n}{n!} = L$. Taking the limit, we see

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \left(\frac{a}{n+1} \cdot \frac{a^n}{n!} \right) = \lim_{n\to\infty} \frac{a}{n+1} \cdot \lim_{n\to\infty} \frac{a_n}{n!} = 0 \cdot L = 0.$$

Hence, L = 0, and we've shown that for any $a > 0$, $\lim_{n\to\infty} \frac{a_n}{n!}$ converges to 0. ■

Problem 6

(a) Take $c > 0$ and choose $a_1 > \sqrt{c}$. Then define $a_{n+1} = \frac{1}{2}(a_n + \frac{c}{a_n})$ for $n \in \mathbb{N}$. Prove that a_n is monotonic decreasing and $\lim a_n = \sqrt{c}$.

Solution
We first prove an inequality that will later help show that $\{a_n\}$ is bounded below. Observe that for $x, y \in R$, $(x-y)^2 \geq 0$ implies that $x^2 - 2xy + y^2 \geq 0$, which implies that

$$\frac{x^2 + y^2}{2} \geq xy \tag{1}$$

where equality holds if and only if $x = y$.

We claim that $\{a_n\}$ is bounded below by $\sqrt{c}$ for all n, that is, $a_n \geq \sqrt{c}$ for all n. We prove this by induction.

Base Case: $n = 1$. $a_1 > \sqrt{c}$ holds by hypothesis.

Inductive Step: Suppose the inequality holds for n. We must show that it holds for $n+1$. Consider

$$a_{n+1} = \frac{1}{2}(a_n + \frac{c}{a_n}) = \frac{a + \frac{c}{a_n}}{2} \geq \sqrt{a_n} \cdot \frac{\sqrt{c}}{\sqrt{a_n}} = \sqrt{c}$$

The inequality before the last equality holds by using the inequality in (1) above. Having shown the original inequality holds for $n+1$, we can conclude that $a_n \geq \sqrt{c}$ holds for all n.

We know show that $\{a_n\}$ is monotonic decreasing. Consider:

$$\begin{aligned} a_n - a_{n+1} &= a_n - \frac{1}{2}\left(a_n + \frac{c}{a_n}\right) \\ &= \frac{1}{2}a_n - \frac{1}{2}\frac{c}{a_n} = \frac{1}{2}\left(\frac{a_n^2 - c}{a_n}\right) \geq 0 \end{aligned}$$

The last inequality holds since we we've shown that $a_n \geq \sqrt{c}$ for all n. Thus $\{a_n\}$ is monotonic decreasing. Since $\{a_n\}$ is both bounded and monotonic decreasing, it converges. Let $\lim_{n\to\infty} a_n = L$.

We now show that $\lim_{n\to\infty} a_n = L = \sqrt{c}$.

$$\begin{aligned} &\lim a_{n+1} = \lim \frac{1}{2}\left(a_n + \frac{c}{a_n}\right) \\ &\Rightarrow L = \frac{1}{2}\left(L + \frac{c}{L}\right) \\ &\Rightarrow L = \frac{1}{2}\left(\frac{L^2 + c}{L}\right) \\ &\Rightarrow L^2 = c \\ &\Rightarrow L = \sqrt{c} \end{aligned}$$

which is exactly what we wanted. Limits are unique, so $\lim a_n = \sqrt{c}$. ■

(b) The sequence $\{a_n\}$ can converge to $\sqrt{c}$ very rapidly: let $r_n = a_n - \sqrt{c}$. Show

$$r_{n+1} = \frac{r_n^2}{2a_n} < \frac{r_n^2}{2\sqrt{c}}.$$

Hence, setting $A = 2\sqrt{c}$, you have $r_{n+1} \leq A(\frac{r_1}{A})^{2^n}$. If $r_1 < A$, that goes to zero very as a function of n as $n \to \infty$.

Solution
We can write r_{n+1} as follows

$$\begin{aligned} r_{n+1} &= a_{n+1} - \sqrt{c} \\ &= \frac{1}{2}\left(a_n + \frac{c}{a_n}\right) - \sqrt{c} \\ &= \frac{a_n}{2} + \frac{c}{2a_n} - \sqrt{c} \\ &= \frac{a_n^2 - 2a_n\sqrt{c} + c}{2_a n} = \frac{(a_n - \sqrt{c})^2}{2_a n} = \frac{r_n^2}{2a_n} < \frac{r_n^2}{2\sqrt{c}} \end{aligned}$$

The last equality follows from the inequality $a_n \geq \sqrt{c}$ for all n. This is exactly what we are trying to prove. If we let $A = 2\sqrt{c}$, then we see that

$$r_{n+1} < \frac{r_n^2}{A} < \frac{(r_{n-1}/A)^2}{A} = A\left(\frac{r_{n-1}}{A}\right)^{2^2} < \cdots < A\left(\frac{r_1}{A}\right)^{2^n}$$

which is the inequality desired. ∎

Problem 7

Suppose (χ, d) and $(\tilde{\chi}, \tilde{d})$ are metric spaces and f is a continuous mapping of (χ, d) into $\tilde{\chi}, \tilde{d}$. Show that for every subset S of χ, $f(\overline{S})$ is contained in $\overline{f(S)}$ and sometimes $f(\overline{S})$ can be strictly smaller than $\overline{f(S)}$.

Solution

We're given that f is a continuous mapping, $f : (\chi, d) \to (\tilde{\chi}, \tilde{d})$. Let $x \in \overline{S}$. Then we want to show that $f(x) \in \overline{f(S)}$, which would imply that $f(\overline{S}) \subseteq \overline{f(S)}$. For some $r > 0$, let $B_r(f(x))$ be a ball around $f(x)$. Since f is continuous, then $f^{-1}(B_r(f(x)))$ is a ball around x. Since $x \in \overline{S}$, there exists a point $p \in S$ such that $p \in f^{-1}(B_r(f(x)))$, which implies that $f(p) \in f^{-1}(B_r(f(x))) \cap f(S)$. Therefore $f(x) \in \overline{f(S)}$.

Consider $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \frac{x}{1+x^2}$. Let $S = \overline{S} = [1, \infty)$. Then $f(S) = f(\overline{S}) = (0, \frac{1}{2}]$, whereas the closure, $\overline{f(S)} = [0, \frac{1}{2}]$, so in this case the containment is strict, and $f(\overline{S}) \subset \overline{f(S)}$. ■

Problem 8

Define $f : \mathbb{R} \to \mathbb{R}$ as follows: if x is irrational or zero, define $f(x) = 0$, if x is rational and not zero write it as p/q, where $p, z \in \mathbb{Z}$ have no common factors, and define $f(x) = 1/q$. Where is f continuous? Where is it not continuous?

Solution

Let $x \in \mathbb{Q} \setminus \{0\}$ and we choose an $\epsilon < 1/q$. Since irrationals are dense in $\mathbb{R}$, we can find an irrational y such that $|x - y| < \delta$, but this implies that $|f(x) - f(y)| = |1/q - 0| = 1/q < \epsilon$. Since we have found an ϵ that does not have a corresponding δ such that $|x - y| < \epsilon$ implies $|f(x) - f(y)| < \delta$, we have shown that f is not continuous at $\mathbb{Q} \setminus \{0\}$.

Now consider the set of irrational numbers, from which we choose a number z. Then we choose $n \in N$ such that $1/n < \epsilon$. If we progressively pick $\delta > 0$ so that balls of radius δ exclude all rationals with denominator less than our chosen n, then after a finite number of steps will have picked a δ that satisfies:

$$|f(x) - f(z)| = |f(x) - 0| = |f(x)| < 1/n < \epsilon$$

Thus, we have shown that f is continuous at all the irrational numbers. ■