# MATH 131B: Homework #2

*Professor Dave Penneys*
Assignment: 10, 12, 13, 15, 17

**Eric Chuu**
UID: 604406828

June 30, 2016

## Problem 10

Let $(X, d)$ be a metric space.

**(a)** Let $E$ be a subset of $X$. Then $E$ is open if and only if $E = \text{int}(E)$.

**Solution**
Suppose that $E$ is open. To show $E = \text{int}(E)$, we show inclusion in both directions. Let $x \in E$. Since $E$ is open, then $\partial(E) \cap E = \phi$. This means $x \notin \partial E$. Then there exists $\epsilon > 0$ such that $B_\epsilon(x) \cap \partial E = \phi$. Then $x$ can either be in the interior or exterior of $E$. Suppose $x \in \text{ext}(E)$, then there exists $\delta > 0$ such that $B_\delta(x) \cap E = \phi$, which contradicts $x \in E$, so $x \in \text{int}(E)$, and so $E \subset \text{int}(E)$. For the reverse inclusion, since every $x \in \text{int}(E)$ has an $\epsilon > 0$ such that $B_\epsilon(x) \subset E$, then $\text{int}(E) \subset E$, so we have $E = \text{int}(E)$.

Conversely, suppose that $E = \text{int}(E)$. Since for all $x \in \partial E$, $x$ is not in the exterior nor the interior, then $\partial E \cap E = \phi$, so $E$ is open. $\square$

**(b)** Let $E$ be a subset of $X$. Then $E$ is closed if and only if $E$ contains all its adherent points.

**Solution**
Suppose $E$ is closed. Then it contains all of its boundary points, i.e., $\partial E \subset E$. Let $x$ be adherent to $E$. Then for all $r > 0, B_r(x) \cap E \neq 0$, so $x \notin \text{ext(E)}$, so $x \in \text{Int}(E) \cup \partial E$, so $x \in E$.

Conversely, suppose $E$ contains all its adherent points. We show that $E$ is closed, $\partial E \subset E$. Let $x \in \partial E$. $x \notin \text{ext}(E)$, so for all $r > 0$, $B_r(x) \cap E \neq \phi$, so $x$ is adherent to $E$, and since $E$ contains all of its adherent points, we've shown that $\partial E \subset E$. $\square$

**(c)** For any $x_0 \in X$ and $r > 0$, then the ball $B_r(x_0)$ is an open set. The set $\{x \in X : d(x, x_0) \leq r\}$ is a closed set.

**Solution**
To show that $B_r(x_0)$ is an open set, we show that for any $x \in B_r(x_0)$, there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset B_r(x_0)$. Let $s := r - d(x_0, x)$. Then let $y \in B_s(x)$. Then

$$d(x_0, y) \leq d(x_0, x) + d(x, y) < d(x_0, x) + s = d(x_0, x) + r - d(x_0, x) = r,$$

so $d(x_0, y) < r$, and we've shown that $B_s(x) \subset B_r(x_0)$, so $B_r(x_0)$ is open.

Let $K := \{x \in X : d(x, x_0) \leq r\}$. To show $K$ is closed, it suffices to show that $K^c = \{x \in X : d(x, x_0) > r\}$ is open, that is, for all $x \in K^c$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset K^c$. Let $x \in K^c, x_0 \in K$. Then let $s := d(x, x_0) - r$. Note this is greater than 0 since $x \notin K$. Let $y \in B_s(x)$. Then,

$$\begin{aligned} &d(x, x_0) \leq d(x, y) + d(y, x_0) < s + d(y, x_0) = d(x, x_0) - r + d(y, x_0) \\ \implies &r < d(y, x_0), \end{aligned}$$

which implies that $B_s(x_0) \subset K^c$, so $K^c$ is open $\Leftrightarrow$ $K$ is closed. $\square$

**(d)** Any singleton set $\{x_0\}$, where $x_0 \in X$, is automatically closed.

**Solution**
By part (b), to show that $\{x_0\}$ is closed, it suffices to show that $\{x_0\}$ contains all of its adherent points. Let $x$ be adherent to $\{x_0\}$. Then for all $r > 0$, $B_r(x) \cap \{x_0\} \neq \phi$, so $d(x, x_0) < r$ for all $r > 0$. Since $r$ is arbitrary $d(x, x_0) \to 0$, so $x = x_0$, so $\{x_0\}$ contains all of its adherent points, and $\{x_0\}$ is closed. $\square$

**(e)** If $E$ is a subset of $X$, then $E$ is open if and only if the complement $E^c := \{x \in X : x \notin E\}$ is closed.

**Solution**
Suppose that $E$ is open. In order to show that $E^c$ is closed, it suffices to show that it contains all of its adherent points. Let $x$ be adherent to $E^c$. Then for all $\epsilon > 0$, $B_\epsilon(x) \cap E^c \neq \phi$. Then it follows that there is no such $\epsilon > 0$ such that $B_\epsilon(x) \subset E$, so $x \notin \text{Int}(E)$. Since $E = \text{Int}(E)$ by openness of $E$, so $x \notin E$, so $x \in E^c$, so $E^c$ is closed.

Conversely, suppose that $E^c$ is closed. To show that $E$ is open, it suffices to show that for all $x \in E$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset E$. Let $x \in E$. Since $E^c$ closed, it contains all of its adherent points, so $x$ is not adherent to $E^c$. Then there exists $\epsilon > 0$ such that $B_\epsilon(x) \cap E^c = \phi$, so for this $\epsilon$, $B_\epsilon(x) \subset E$. Thus, $E$ is open. $\square$

**(f)** If $E_1, \dots, E_n$ is a finite collection of open sets in $X$, then $E_1 \cap E_2 \dots \cap E_n$ is also open. If $F_1, \dots, F_n$ is a finite collection of closed sets in $X$, then $F_1 \cup \dots \cup F_n$ is also closed.

**Solution**
In order to show that $\cap_{i=1}^n E_i$ is open, it suffices to show for any $x \in \cap_{i=1}^n E_i$, there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset \cap_{i=1}^n E_i$. Let $x \in \cap_{i=1}^n E_i$. Then $x \in E_i$ for every $1 \leq i \leq n$. Since each of these $E_i$ is open, there exists $\epsilon_i > 0$ such that $B_{\epsilon_i}(x) \subset E_i$. Then choose $\epsilon := \min\{\epsilon_1, \dots \epsilon_n\}$. Note we can take the minimum because this collection of $\epsilon_i$'s forms a finite set. Then $B_\epsilon(x) \subset E_i$ for $1 \leq i \leq n$, which implies that $B_\epsilon(x) \subset \cap_{i=1}^n E_i$, and it follows that $\cap_{i=1}^n E_i$ is open. From part (e), we know that $\cap_{i=1}^n E_i$ is open $\Leftrightarrow$ $(\cap_{i=1}^n E_i)^c$ is closed. Applying De Morgan's law,

$$\left( \bigcap_{i=1}^n E_i \right)^c = \bigcup_{i=1}^n E_i^c,$$

where each $E_i^c$ is closed since $E_i$ is open for $1 \leq i \leq n$. Thus, if we take $F_1 = E_1^c, F_2 = E_2^c, \dots, F_n = E_n^c$, then we find that finite unions of closed sets are closed. $\square$

**(g)** If $\{E_\alpha\}_{\alpha\in I}$ is a collection of open sets in $X$, then $\cup_{\alpha\in I}E_\alpha$ is open. If $\{F_\alpha\}_{\alpha\in I}$ is a collection of closed sets in $X$, then $\cap_{\alpha\in I}F_\alpha$ is closed.

**Solution**
To show that the arbitrary union of open sets is open, we show that for any $x \in \cup_{\alpha\in I}E_\alpha$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset \cup_{\alpha\in I}E_\alpha$. Let $x \in E_\alpha$ for one of the open sets in the union. Since $E_\alpha$ is open, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset E_\alpha \subset \cup_{\alpha\in I}E_\alpha$, so it follows that $\cup_{\alpha\in I}E_\alpha$ is open. $\cup_{\alpha\in I}E_\alpha$ open $\Leftrightarrow (\cup_{\alpha\in I}E_\alpha)^c$ is closed. Applying De Morgan's law,

$$\left(\bigcup_{\alpha\in I} E_\alpha\right)^c = \bigcap_{\alpha\in I} E_\alpha^c,$$

where each $E_\alpha^c$ is closed since each $E_\alpha$ is open. Then it follows that arbitrary intersections of closed sets are closed. □

**(h)** If $E$ is any subset of $X$, then $\mathrm{Int}(E)$ is the largest open set which is contained in $E$; in other words $\mathrm{int}(E)$ is open, and given any other open set $V \subset E$, we have $V \subset \mathrm{int}(E)$. Similarly, $\overline{E}$ is the smallest closed set which contains $E$; in other words, $\overline{E}$ is closed, and given any other closed set $K \supset E, K \supset \overline{E}$.

**Solution**
We first show that $\mathrm{Int}(E)$ is open. Since $\mathrm{Int}(E)$ is defined as the set of interior points of $E$, for all $x \in \mathrm{Int}(E)$, there exists $r > 0$ such that $B_r(x) \subset E$. To show that $\mathrm{Int}(E)$ is open, we show that for all $y \in \mathrm{Int}(E)$, there exists $s > 0$ such that $B_s(y) \subset \mathrm{Int}(E)$. Let $y \in B_r(x)$. Then let $s := r - d(x, y)$. Then let $z \in B_s(y)$, and the triangle inequality yields

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + s = d(x, y) + r - d(x, y) = r,$$

so $B_s(y) \subset B_r(x) \subset E$, so we've shown that for an arbitrary point in the interior of $E$, there exists a ball around that point whose points are also in the interior of $E$, so $B_r(x) \subset \mathrm{Int}(E)$, and $\mathrm{Int}(E)$ is open.

Let $V$ be open in $E$. Then for all $x \in V$, there exists $r > 0$ such that $B_r(x) \subset V \subset E$, so $x \in \mathrm{Int}(E)$, and we conclude that $V \subset \mathrm{Int}(E)$.

We first show that $\overline{E}$ is closed. Let $x$ be adherent to $\overline{E}$. Then it suffices to show that $x \in \overline{E}$, that is, for all $r > 0, B_r(x) \cap E \neq \phi$. Since $x$ is adherent to $\overline{E}$, then for all $r > 0$, $B_{r/2}(x) \cap \overline{E} \neq \phi$. Let $y \in B_{r/2}(x) \cap \overline{E}$. Then $y \in \overline{E}$, the set containing all points adherent to $E$, which implies that $B_{r/2}(y) \cap E \neq \phi$. Let $z \in B_{r/2}(y) \cap E$. Then,

$$d(x, z) \leq d(x, y) + d(y, z) < \frac{r}{2} + \frac{r}{2} = r,$$

so $z \in B_r(x)$, and since $z \in E, z \in B_r(x) \cap E$, and we've shown that $B_r(x) \cap E \neq \phi$, so $\overline{E}$ is closed.

Suppose $K$ is closed and $K \supset E$. We want to show that $\overline{E} \subset K$, that for all $x \in \overline{E}$, $x \in K$. This is equivalent to showing that if $x \notin K$, then $x \notin \overline{E}$. Let $x \in K^c$. Since $K$ is closed, then $K^c$ is open, so there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset K^c$. Then $B_\epsilon(x) \cap K = \phi$, which means that $x$ is not adherent to $K$. Since $E \subset K$, then it follows that $B_\epsilon(x) \cap E = \phi$, so $x$ is not adherent to $E$, so $x \notin \overline{E}$. Thus, we've shown that if $x \notin K$, then $x \notin \overline{E}$, or equivalently, $\overline{E} \subset K$. □

# Problem 12

Let $(X, d)$ be a nonempty metric space.

(a) Fix $x_0 \in X$. Suppose we have a convergent sequence $y_n \to y_0$ in $(X, d)$. Show that $d(x_0, y_n) \to d(x_0, y_0)$ in $(\mathbb{R}, |\cdot|)$.

(b) Fix $x_0, y_0 \in X$. Suppose we have two convergent sequences $x_m \to x_0$ and $y_n \to y_0$ in $(X, d)$. Show that $d(x_m, y_m) \to d(x_0, y_0)$ in $(\mathbb{R}, |\cdot|)$, i.e., show that for every $\epsilon > 0$, there is an $N > 0$ such that $m, n > N$ implies $|d(x_m, y_n) - d(x_0, y_0)| < \epsilon$.

**Solution**

(a) Let $\epsilon > 0$. Since $y \to y_0$ in $(X, d)$, there exists $N > 0$ such that for all $n > N, d(y_n, y_0) < \epsilon$. We want to show that for all $n > N$, $|d(x_0, y_n) - d(x_0, y_0)| < \epsilon$. By the triangle inequality,

$$d(x_0, y_n) \leq d(x_0, y_0) + d(y_0, y_n),$$

so for all $n > N$,

$$\begin{aligned} |d(x_0, y_n) - d(x_0, y_0)| &\leq |d(x_0, y_0) + d(y_0, y_n) - d(x_0, y_0)| \\ &\leq |d(y_0, y_n)| = d(y_0, y_n) < \epsilon. \end{aligned}$$

Note we can safely take off the absolute value because the distance is non-negative, so we've shown that $d(x_0, y_n) \to d(x_0, y_0)$ in $(\mathbb{R}, |\cdot|)$. $\square$

(b) Let $\epsilon > 0$. Since $x_m \to x_0$ in $(X, d)$, then there exists $N_1 > 0$ such that for all $m > N_1$, $d(x_m, x_0) < \epsilon/2$. Similarly, since $y_n \to y_0$ in $(X, d)$, then there exists $N_2 > 0$ such that for all $n > N_2$, $d(y_n, y_0) < \epsilon/2$. We want to show that there exists $N$ such that for all $m, n > N$, $|d(x_m, y_n) - d(x_0, y_0)| < \epsilon$. Take $N = \max\{N_1, N_2\}$. Then applying the triangle inequality twice, we have that for all $m, n < N$

$$\begin{aligned} |d(x_m, y_n) - d(x_0, y_0)| &\leq |d(x_m, x_0) + d(x_0, y_0) + d(y_0, y_n) - d(x_0, y_0)| \\ &= |d(x_m, x_0) + d(y_0, y_n)| \\ &\leq |d(x_m, x_0)| + |d(y_0, y_n)| \\ &= d(x_m, x_0) + d(y_0, y_n) \\ &< \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

so $d(x_m, y_m) \to d(x_0, y_0)$ in $(\mathbb{R}, |\cdot|)$. $\square$

# Problem 13

Prove the following conditions are equivalent for a subset $S$ of a metric space $(X, d)$.

**(a)** For every $x \in X$, there is an $R > 0$ such that $S \subset B_R(x)$.

**(b)** There exists an $x \in X$ and an $R > 0$ such that $S \subset B_R(x)$.

**(c)** $\operatorname{diam}(S) = \sup\{d(x, y) | x, y \in S\} < \infty$.

We say $S$ is bounded if it satisfies one and thus all of the conditions above.

**Solution**

We show that (a) $\implies$ (b).
By (a), we have that for all $x \in X$, there exists $R > 0$ such that $S \subset B_R(x)$, so we pick $x \in X$. Then it follows directly from (a) that there exists $R > 0$ such that $S \subset B_R(x)$.

We show (b) $\implies$ (a).
Suppose that there exists $x \in X$ and $R > 0$ such that $S \subset B_R(x)$. Let $y \in X$, and set $R' := R + d(x, y)$. Let $z \in S$. Then

$$d(y, z) \leq d(y, x) + d(x, z) < d(y, x) + R = R'$$

so $z \in B_{R'}(y)$, so we've shown that for all $y \in X$, there exists an $R'$ such that $S \subset B_{R'}(y)$, so (a) holds.

We show that (b) $\implies$ (c).
By (b), there exists $x \in X$ and $R > 0$ such that $S \subset B_R(x)$. Then for any $s \in S$, $s \in B_R(x)$, so $d(x, s) < R$. Let $s, t \in S$. Then

$$d(s, t) \leq d(s, x) + d(x, t) < R + R = 2R.$$

Then $2R$ is an upper bound for $d(x, y)$ for all $x, y \in S$, so $\operatorname{diam}(S) = \sup\{d(x, y) | x, y \in S\} \leq 2R < \infty$.

We show (c) $\implies$ (b).
We're given that $\operatorname{diam}(S) = \sup\{d(x, y) | x, y \in S\} < \infty$. Let $x \in S$. Then take $R := \operatorname{diam}(S) + 1$. Let $y \in S$. Then $d(x, y) \leq \operatorname{diam}(S) < R$, $S \subset B_R(x)$. $\square$

# Problem 15

A subset $S \subset X$ is called dense if for every nonempty open $U \subset X, S \cap U \neq \phi$.

**(a)** Show that $S$ is dense in $X$ if and only if for every $\epsilon > 0$ and every $x \in X$, $B_\epsilon(x) \cap S \neq \phi$.

**(b)** Show that $S$ is dense in $X$ if and only if $X = \overline{S}$.

**Solution**

(a) Suppose $S$ is dense in $X$. Then we show that for every $\epsilon > 0$ and every $x \in X$, $B_\epsilon(x) \cap S \neq \phi$. Let $\epsilon > 0, x \in X$. Then consider the open ball $B_\epsilon(x)$ is a nonempty open subset of $X$, and since $S$ is dense in $X$, $B_\epsilon(x) \cap S \neq \phi$, which is what we wanted to show.

Conversely, suppose that for every $\epsilon > 0$ and $x \in X$, $B_\epsilon(x) \cap S \neq \phi$. let $U$ be an open subset of $X$. Let $x \in U$. Then there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. By hypothesis, $B_\epsilon(x) \cap S \neq \phi$, and since $B_\epsilon(x) \subset U$, then it follows that $U \cap S \neq \phi$, so $S$ is dense in $X$. □

(b) Suppose $S$ is dense in $X$. We want to show that $X = \overline{S}$. Let $x \in X$, $\epsilon > 0$. Then $B_\epsilon(x)$ is open in $X$. By hypothesis, $B_\epsilon \cap S \neq \phi$, so $x$ is adherent to $S$, so $x \in \overline{S}$, and $X \subset \overline{S}$. For the reverse inclusion, let $x \in \overline{S}$. Then $x$ is adherent to $S$, and for all $r > 0$, $B_r(x) \cap S \neq \phi$. Since $x \in B_r(x)$, $x \in X$, so $\overline{S} \subset X$, and it follows that $X = \overline{S}$.

Conversely, suppose $X = \overline{S}$. Then we want to show that for every nonempty open $U \subset X$, $U \cap S \neq \phi$. Let $U$ be open in $X$, and let $x \in U$. Then there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. Since $x \in X = \overline{S}$, then $x$ is adherent to $S$, so $B_\epsilon(x) \cap S \neq \phi$, so it follows that $U \cap S \neq \phi$, and we are done. □

# Problem 17

Find a Cauchy sequence in a metric space $(X, d)$ which does not converge. You must prove that your sequence is Cauchy, and you must prove that it does not converge.

**Solution**

Let $(X, d)$ be a metric space, where $X = (0, 1]$. Then let $\{x_n\}$ be a sequence in $X$, where $x_n = \frac{1}{n+1}$ for $n \geq 1$. We claim that $\{x_n\}$ is Cauchy. Let $\epsilon > 0$. Then by the Archimedean Property, there exists $N > 0$ such that $\frac{1}{N} < \epsilon/2$. Then we can inductively construct a sequence such that for $x_1 \in B_1(0), x_2 \in B_{1/2}(0), \ldots, x_n \in B_{1/n}(0)$, so we have that for all $n, m > N$, $\mathrm{x}_n \in B_{\epsilon/2}(0)$ and $x_m \in B_{\epsilon/2}(0)$ that is $d(x_n, x_m) \leq d(x_n, 0) + d(0, x_m) < \epsilon/2 + \epsilon/2 = \epsilon$, so $\{x_n\}$ is Cauchy. However, as $n \to \infty$, the sequence approaches 0, but $0 \notin X$, so $\{x_n\}$ does not converge in $(X, d)$. □