More Metric Spaces Light

This continues the first set of notes, Metric Spaces Light. While I said that I would follow Rudin for the rest of pages 36 to 40, that is not quite true. Using "S closed $\Leftrightarrow$ $bdy(S) \subset S$" it is a little easier to see that compact sets are closed:

Proof: If K is compact, but not closed, there is $p \in Bdy(K)$ which is not in K. Consider the sets $\mathcal{U}_n = \{q \in \mathbb{R}^n : d(p,q) > 1/n\}$. These are open (that's an exercise), and $\cup_{n=1}^{\infty}\mathcal{U}_n = \mathbb{R}^n \backslash \{p\} \supset K$. However, since $p \in Bdy(K)$, for any N we have $B_{1/N}(p) \cap K \neq \emptyset$. So $\cup_{n=1}^{N}\mathcal{U}_n$ cannot cover K. Thus we have a contradiction to the assumption that K is (HB)-compact.

The proofs of Theorems 2.35 (Closed subsets of compact sets are compact) and 2.36 (Compact sets have the "finite intersection property") will be repeated in lecture exactly as they appear in Rudin. Theorem 2.37 is a slightly disguised version of "(HB)-compact sets are (BW)-compact" in Metric Spaces Light. At this point I would like to go directly to a version of Theorem 2.42.

Theorem. Closed bounded sets in $\mathbb{R}^n$ are (BW)-compact.

Proof. It is easy to show that closed subsets of compact sets are compact using the (BW) definition of compact (remember that we will eventually show that (HB)-compact is *equivalent* to (BW)-compact): Suppose that $C \subset K$ where C is closed and K is compact, and $\{p_n\}_{n=1}^{\infty} \subset C \subset K$. Since K is compact, there is $p_\infty \in K$ such that for every $r > 0$ we have $p_n \in B_r(p_\infty)$ for infinitely many n. This says p_∞ is either in $Int(C)$ or $Bdy(C)$, but, since C is closed, it says $p_\infty \in C$. So C is compact in the (BW)-sense.

Any closed bounded set in $\mathbb{R}^n$ is contained in one of the boxes, $B_R = \{x = (x_1, .., x_n) \in \mathbb{R}^n : |x_j| \leq R,\ j = 1, .., n\}$. So, using the first paragraph in this proof, it will be enough to show that the B_R's are (BW)-compact. Given $\{p_j\}_{j=1}^{\infty} \subset B_R$, divide B_R into 2^n boxes of side length R, taking either $0 \leq x_j \leq R$ or $-R \leq x_j \leq 0$, $j = 1, .., n$, in each box. One of those boxes must contain p_j for infinitely many j. Call it $B_R^{(1)}$. Now divide $B_R^{(1)}$ into 2^n boxes of side length $R/2$ the same way. One of those boxes must contain p_j for infinitely many j. Call it $B_R^{(2)}$. Continue, getting $B_R^{(k)}$ of side length $R/2^k$ containing p_j for infinitely many j for each $k \in \mathbb{N}$. Note that the boxes are nested: $B_R^{(k+1)} \subset B_R^{(k)}$.

I claim that there is a point $p_\infty \in \cap_{k=1}^{\infty} B_R^{(k)}$. This follows from Theorems 2.38 and 2.39 in Rudin. This is where it matters that we are in $\mathbb{R}^n$, and that bounded sets of real numbers have least upper bounds. In some metric spaces this step in the proof can fail and the result is false. More about this later. For the proofs of Theorems 2.38 and 2.39 I will just follow Rudin.

Given any $r > 0$, $B_r(p_\infty)$ will contain $B_R^{(k)}$ for k sufficiently large. How large? My computation is that, if we are using the metric d_2 – Euclidean distance – we need $\sqrt{n}R/2^k < r$ which holds for k sufficiently large. I will leave it as an exercise to find what we need for d_1 and d_∞. In any case we can conclude that $B_r(p_\infty)$ contains p_j for infinitely many j, and B_R is compact in the (BW)-sense.