MATH 131B: Homework #7

Professor Dave Penneys
Assignment: 67, 69, 72, 73

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Problem 67

Theorem 4.5.2 (Basic Properties of exponential).

(a) For every real number x, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent. In particular, $\exp(x)$ exists and is real for every $x \in \mathbb{R}$, the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has as infinite radius of convergence, and exp is a real analytic function on $(-\infty, \infty)$.

(b) exp is differentiable on $\mathbb{R}$, and for every $x \in \mathbb{R}$, $\exp'(x) = \exp(x)$.

(c) exp is continuous on $\mathbb{R}$, and for every interval $[a, b]$, we have $\int_a^b \exp(x)dx = \exp(b) - \exp(a)$.

(d) For every $x, y \in \mathbb{R}$, we have $\exp(x + y) = \exp(x)\exp(y)$.

(e) We have $\exp(0) = 1$. Also, for every $x \in \mathbb{R}$, $\exp(x)$ is positive, and $\exp(-x) = 1/\exp(x)$.

(f) exp is strictly monotone increasing: in other words, if $x, y \in R$, then we have $\exp(y) > \exp(x)$ if and only if $y > x$.

Solution
Let $x \in \mathbb{R}$. To show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent, it suffices to show that $\sum_{n=0}^{\infty} \left|\frac{x^n}{n!}\right|$ converges. By the ratio test, we see that

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty} \left|\frac{x}{n+1}\right| < 1,$$

so we conclude that $\sum_{n=0}^{\infty} \left|\frac{x^n}{n!}\right|$ converges, so $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely in $\mathbb{R}$, so $\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$ exists for every $x \in \mathbb{R}$. Since we've shown that $\exp(x)$ converges absolutely for every $x \in \mathbb{R}$, then we conclude that exp has an infinite radius of convergence, and exp is a real analytic function on $(-\infty, \infty)$, so (a) holds.

If we then consider $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\exp(y) = \sum_{n=0}^{\infty} \frac{y^n}{n!}$, then we know from above that both $\exp(x), \exp(y)$ are real analytic functions on $(-\infty, \infty)$, and we can use Theorem 4.41 to see that the product of $\exp(x), \exp(y)$ is:

$$\exp(x)\exp(y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot \sum_{n=0}^{\infty} \frac{y^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \cdot \frac{y^{n-k}}{(n-k)!} \tag{1}$$

Note that $\frac{1}{k!(n-k)!} = \frac{1}{n!} \cdot \frac{n!}{k!(n-k)!} = \frac{1}{n!}\binom{n}{k}$, so rewriting (1) and using the binomial theorem, we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{x^k}{k!} \cdot \frac{y^{n-k}}{(n-k)!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{n!}\binom{n}{k} x^k y^{n-k} = \sum_{n=0}^{\infty} \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} x^k y^{n-k} = \sum_{n=0}^{\infty}\frac{1}{n!}(x+y)^n = \sum_{n=0}^{\infty}\frac{(x+y)^n}{n!},$$

where the last equality is $\exp(x+y)$, so we've shown that $\exp(x)\exp(y) = \exp(x+y)$, so (d) holds.

Applying (d), we then see that $\exp(x)\exp(-x) = \exp(x-x) = \exp(0) = \sum_{n=0}^{\infty} \frac{0^n}{n!} = 1$, since $0^0 = 1$. Then $\exp(-x)\exp(x) = 1$, both $\exp(-x), \exp(x)$ are nonzero, so $\exp(-x) = 1/\exp(x)$. Moreover, if $x > 0$, then $\exp(x)$ is a sum of positive numbers, so $\exp(-x) = 1/\exp(x)$ together with $\exp(0) = 1 \implies \exp(x) > 0$ for all $x \in \mathbb{R}$, and (e) holds.

Again applying (d), we can evaluate the following:

$$\lim_{h\to 0} \frac{\exp(x+h)-\exp(x)}{h} = \lim_{h\to 0}\frac{\exp(x)\exp(h)-\exp(x)}{h} = \exp(x)\lim_{h\to 0}\frac{\exp(h)-1}{h} \tag{2}$$

If we only consider the limit in the last equality, we see that

$$\begin{aligned}
&\lim_{h\to 0}\frac{\exp(h)-1}{h} = \lim_{h\to 0}\frac{1}{h}(\exp(h)-1) = \lim_{h\to 0}\frac{1}{h}\left(\sum_{n=0}^{\infty}\frac{h^n}{n!} - 1\right) \\
&\implies \lim_{h\to 0}\frac{1}{h}\left(\frac{h^0}{0!} + \frac{h}{1!} + \frac{h^2}{2!} + \ldots - 1\right) = \lim_{h\to 0}\frac{1}{h}\left(1 + \frac{h^1}{1!} + \frac{h^2}{2!} + \ldots - 1\right) \\
&\implies \lim_{h\to 0}\frac{1}{h}\left(\frac{h}{1!} + \frac{h^2}{2!} + \ldots\right) = \lim_{h\to 0}\left(\frac{1}{1!} + \frac{h}{2!} + \ldots\right) = 1
\end{aligned}$$

Substituting this back into (2), we see that

$$\lim_{h\to 0}\frac{\exp(x+h)-\exp(x)}{h} = \exp(x)\lim_{h\to 0}\frac{\exp(h)-1}{h} = \exp(x)\cdot 1 = \exp(x),$$

for any $x \in R$ and we conclude that exp is differentiable on $\mathbb{R}$ and $\exp'(x) = \exp(x)$, so (b) holds. Since differentiability at a point implies continuity at the same point, exp being differentiable on $\mathbb{R}$ implies that exp is continuous on $\mathbb{R}$. Since exp is continuous on $\mathbb{R}$, then it is continuous on every interval $[a,b]$, so exp is integrable on $[a,b]$, and $\int_a^b \exp(x)dx = \exp(b) - \exp(a)$, since $\exp'(x) = \exp(x)$ for all $x \in \mathbb{R}$. We conclude that (c) holds.

For (f), let $x, y \in \mathbb{R}$ such that $x < y$. Note that from part (e), $\exp(x) > 0$ for all $x \in \mathbb{R}$. Then using part (d), we see that $x < y \Leftrightarrow y - x > 0 \Leftrightarrow \exp(y-x) > \exp(0) \Leftrightarrow \exp(y)/\exp(x) > 1 \Leftrightarrow \exp(y) > \exp(x)$, so we conclude that exp is strictly monotone increasing. We have verified that all six of the above properties hold for the exponential, and we are done. $\square$

Problem 69

Theorem 4.5.6 (Logarithm properties).

(a) For every $x \in (0, \infty)$, we have $\ln'(x) = \frac{1}{x}$. In particular, by the fundamental theorem of calculus, we have $\int_a^b \frac{1}{x} dx = \ln(b) - \ln(a)$ for any interval $[a, b]$ in $(0, \infty)$.
(b) We have $\ln(xy) = \ln(x) + \ln(y)$ for all $x, y \in (0, \infty)$.
(c) We have $\ln(1) = 0$ and $\ln(1/x) = -\ln(x)$ for all $x \in (0, \infty)$.
(d) For any $x \in (0, \infty)$ and $y \in \mathbb{R}$, we have $\ln(x^y) = y \ln(x)$.
(e) For any $x \in (-1, 1)$, we have

$$\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}. \tag{3}$$

In particular, ln is analytic at 1, with the power series expansion

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x - 1)^n \tag{4}$$

for $x \in (0, 2)$, with radius of convergence 1.

Solution
For part (a), we note that log is differentiable as mentioned in the book under Definition 4.5.5, so let $x \in (0, \infty)$. Then note that $x = \exp(\ln(x))$. Since exp is differentiable for all $x \in \mathbb{R}$, we can take the derivative, and using the chain rule, we get

$$x' = (\exp(\ln(x)))' = \exp'(\ln(x)) \cdot \ln'(x) = \exp(\ln(x)) \cdot \ln'(x),$$

where the last equality follows from $\exp'(x) = \exp(x)$ for all $x \in \mathbb{R}$. Continuing to evaluate this, we get $\exp(\ln(x)) \cdot \ln'(x) = x \cdot \ln'(x)$, and since the $x' = 1$, we have $\ln'(x) = 1/x$. Moreover, by the fundamental theorem of calculus, we can integrate this, and we have $\int_a^b \frac{1}{x} dx = \ln(b) - \ln(a)$ for any interval $[a, b]$ in $(0, \infty)$, so property (a) holds.

As shown in problem 67, exp exists and is real for every $x \in \mathbb{R}$, let $x = \exp(u), y = \exp(v)$, where $x, y, u, v \in \mathbb{R}$. Then consider:

$$\ln(xy) = \ln(\exp(u) \exp(v)) = \ln(\exp(u + v)) = u + v = \ln(x) + \ln(y),$$

where the third equality holds because of property (d) of the exponential. Thus, property (b) of the logarithm holds.

To show property (d), we use proposition 4.5.4 that for every $x \in \mathbb{R}$, we have $\exp(x) = e^x$. In particular, we can use exponent laws. Let $x \in (0, \infty), y \in \mathbb{R}$. Consider $y \ln(x)$. Then

$$\begin{aligned} &\exp(y \ln(x)) = e^{y \ln(x)} = \left(e^{\ln(x)}\right)^y = x^y \\ &\implies \ln\left(e^{y \ln(x)}\right) = \ln(x^y) \\ &\implies y \ln(x) = \ln(x^y), \end{aligned}$$

so property (d) holds.

To show property (c), we first note that from property (e) from the exponential that $\exp(0) = 1$, so

$$\ln(1) = \ln(\exp(0)) = 0.$$

We can use property (d) of the logarithm to note that since $1/x = x^{-1}$, then

$$\ln(1/x) = \ln(x^{-1}) = -\ln(x),$$

so property (c) holds.

For part (e), we first recall that for any $x \in (-1, 1)$ the power series expansion for $f(x) = \frac{1}{1-x}$ is given by $f(x) = \sum_{n=0}^{\infty} x^n$ around 0. Then by property (a) of logarithms, we can write

$$\ln(1-x) = -\int \frac{1}{1-x}dx = -\int \sum_{n=0}^{\infty} x^n dx = -\sum_{n=0}^{\infty} \int x^n dx = C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

To find C, we consider $\ln(1-x)$ when $x = 0$, and since $\ln(1) = 0$, then $C = 0$, so we have

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$

Note that we can rewrite the last sum by starting the sum from $n = 1$ and we get

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^n}{n},$$

so the equality in (3) holds. To show (4), we first note that $f(x) = \frac{1}{1+x}$, defined on $\mathbb{R} \setminus \{-1\}$ has the power series $f(x) = \sum_{n=0}^{\infty} (-1)^n x^n$ around 0 on the interval $(-1, 1)$. Then

$$\ln(1+x) = \int \frac{1}{1+x}dx = \int \sum_{n=0}^{\infty} (-1)^n x^n dx = \sum_{n=0}^{\infty} (-1)^n \int x^n dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C.$$

Evaluating $\ln(1+x)$ when $x = 0$, we see that $\ln(1) = 0$, so $C = 0$. Then

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

Then, we can write $\ln(x)$ as the power series representation of $\ln(1+x)$ centered around 1,

$$\ln(x) = \ln(1 + x - 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n.$$

To find the radius of convergence, we consider

$$r = \frac{1}{\limsup_{n\to\infty} \left|\frac{(-1)}{n}\right|^{1/n}} = \frac{1}{\limsup_{n\to\infty} (\frac{1}{n})^{1/n}} = 1,$$

so we conclude that $\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$, centered around 1, for $x \in (0, 2)$ with radius of convergence of 1, so ln is analytic at 1 and (4) holds and consequently property (e) holds. $\square$

Problem 72

Prove the algebra $C(\mathbb{R}/\mathbb{Z})$ of continuous $\mathbb{Z}$-periodic functions $\mathbb{R} \to \mathbb{C}$ is isometrically isomorphic to the algebra $C(S^1)$ of continuous functions $S^1 = \{z \in \mathbb{C} : |z| = 1\} \to \mathbb{C}$. That is, find a bijection $\varphi : C(S^1) \to C(\mathbb{R}/\mathbb{Z})$ such that such that for all $f, g \in C(S^1)$ and $\lambda \in \mathbb{C}$,

(a) $\varphi(f+g) = \varphi(f) + \varphi(g)$
(b) $\varphi(\lambda f) = \lambda \varphi(f)$
(c) $\varphi(f(g)) = \varphi(f)\varphi(g)$
(d) $\varphi(\overline{f}) = \overline{\varphi(f)}$
(e) $\|\varphi(f)\|_\infty = \|f\|_\infty$
(f) $d_\infty(\varphi(f), \varphi(g)) = d_\infty(f, g)$

Solution
Define $\phi : [0,1) \to S^1$ to be the bijective function, where $\phi(x) = e^{2\pi i x}$ for $x \in [0,1)$. Then let $\varphi(f) = f \circ \phi$, whenever $f \in C(S^1)$.

We first show that $\varphi(f)$ is well-defined. We check that for $f \in C(\mathbb{R}/\mathbb{Z})$, $\varphi(f)$ uniquely determines a $\mathbb{Z}$-periodic function. Let $x \in [0,1)$. Then $\phi(x) \in S^1$. Then we evaluate the limit:

$$\lim_{x \to 1^-} (f \circ \phi)(x) = \lim_{\phi(x) \to 1^-} (f \circ \phi)(x) = \lim_{\phi(x) \to 1^-} f(\phi(x))$$

Note that the first equality holds because ϕ is a bijective function. By continuity of f, we can then move the limit inside, and we see that

$$f\left(\lim_{\phi(x) \to 1^-} \phi(x)\right) = f(1) = (f \circ \phi)(0),$$

and we see that $\lim_{x \to 1^-} (f \circ \phi)(x) = (f \circ \phi)(0)$ so that $\varphi(f)$ uniquely determines a $\mathbb{Z}$-periodic function and is well-defined.

We show that φ is a bijection. That is, we show that φ is both one-to-one and onto.

(1) Claim: φ is one-to one.
It suffices to show that for $f, g \in C(S^1)$, if $\varphi(f) = \varphi(g)$, then $f = g$. Suppose $\varphi(f) = \varphi(g)$. Then $\varphi(f) = f \circ \phi = g \circ \phi = \varphi(g)$. Let $z \in S^1$. Then

$$\begin{aligned} &(f \circ \phi)(z) = (g \circ \phi)(z) \\ \implies\ &f(\phi(z)) = g(\phi(z)), \end{aligned}$$

and so we conclude that $f = g$ and that φ is one-to-one.

(2) Claim: φ is onto.
It suffices to show that for any $h \in C(\mathbb{R}/\mathbb{Z})$, there exists $f \in C(S^1)$ such that $\varphi(f) = h$. Note that since ϕ is bijective, then there exists an inverse function $\phi^{-1} : S^1 \to [0,1)$. Since $h \in C(\mathbb{R}/\mathbb{Z})$, then $h : [0,1) \to \mathbb{C}$. Then $h \circ \phi^{-1} : S^1 \to \mathbb{C}$, so $h \circ \phi^{-1} \in C(S^1)$. Setting $f := h \circ \phi^{-1}$ and applying φ, we see that $\varphi(f) = \varphi(h \circ \phi^{-1}) = h \circ \phi^{-1} \circ \phi = h$, so φ is onto.

We show that φ satisfies properties (a) to (f). Let $f, g \in C(S^1)$ and $\lambda \in \mathbb{C}$, and $x \in S^1$.

(a) $\varphi(f+g)(x) = ((f+g)\circ\phi)(x) = (f\circ\phi)(x) + (g\circ\phi)(x) = (\varphi(f))(x) + (\varphi(g))(x) = (\varphi(f)+\varphi(g))(x)$
$\implies \varphi(f+g) = \varphi(f) + \varphi(g)$.

(b) $\varphi(\lambda f)(x) = ((\lambda f)\circ\phi)(x) = \lambda(f\circ\phi)(x) = \lambda\varphi(f)(x)$
$\implies \varphi(\lambda f) = \lambda\varphi(f)$.

(c) $\varphi(fg)(x) = ((fg)\circ\phi)(x) = (f\circ\phi)(x)\cdot(g\circ\phi)(x) = \varphi(f(x))\cdot\varphi(g(x)) = (\varphi(f)\cdot\varphi(g))(x)$
$\implies \varphi(fg) = \varphi(f)\varphi(g)$.

(d) $\varphi(\overline{f})(x) = (\overline{f}\circ\phi)(x) = \overline{f}(\phi(x)) = \overline{f}(e^{2\pi i x}) = \overline{f(e^{2\pi i x})} = \overline{f(\phi(x))} = \overline{(f\circ\phi)(x)} = \overline{\varphi(f)}(x)$
$\implies \varphi(\overline{f}) = \overline{\varphi(f)}$.

(e) $\|\varphi(f)\|_\infty = \|f\circ\phi\|_\infty = \sup_{x\in[0,1)}\{(f\circ g)(x)\}$. Since ϕ is a bijection, taking the supremum over $x \in [0,1)$ is the same as taking the supremum over $\{\phi(z) : z \in [0,1)\}$, so we can write

$$\sup_{x\in[0,1)}\{(f\circ g)(x)\} = \sup_{\phi(x)\in S^1}\{f(\phi(x))\} = \sup_{z\in S^1}\{f(z)\} = \|f\|_\infty$$

(f) We apply the results of part (a) and part (e) to the second and third equalities, respectively:

$$\begin{aligned} d_\infty(\varphi(f),\varphi(g)) &= \|\phi(f) - \varphi(g)\|_\infty \\ &= \|\varphi(f-g)\|_\infty \\ &= \|f-g\|_\infty \\ &= d_\infty(f,g) \end{aligned}$$

We've shown that $\varphi : C(S^1) \to C(\mathbb{R}/\mathbb{Z})$, where $\varphi(f)(x) = f\circ\phi(x)$ for all $x \in S^1$ is a bijection that satisfies properties (a) to (f) above. We conclude that the algebra $C(\mathbb{R}/\mathbb{Z})$ of continuous Z-periodic functions $\mathbb{R}\to\mathbb{C}$ is isometrically isomorphic to the algebra $C(S^1)$. $\square$

Problem 73

Show that $(\ell^1, |\cdot|_1)$ is a Banach Algebra, where $(a_n) + (b_n) = (a_n + b_n)$, $\lambda(a_n) = (\lambda a_n), (a_n) * (b_n) = \sum_{k=0}^{n}(a_k b_{n-k})$, and $\|(a_n)\| = \sum_{n=0}^{\infty} |a_n|$. That is, show:

(1) The set ℓ^1 with its addition, scalar multiplication, and multiplication is an associative algebra over $\mathbb{R}$.

(2) The multiplication and norm are related by $\|(a_n) * (b_n)\|_1 \leq \|(a_n)\|_1 \cdot \|(b_n)\|_1$.

What nice property does the compatibility of axiom (2) above ensure?

Solution
To show (1) holds, we show the following:

(a) *Addition is associative and commutative.* Let $(a_n), (b_n), (c_n) \in \ell^1$. Then by definition of addition given in the problem, $(a_n) + ((b_n) + (c_n)) = (a_n + (b_n + c_n)) = (a_n + b_n + c_n)$. Similarly, $((a_n) + (b_n)) + (c_n) = ((a_n + b_n) + c_n) = (a_n + b_n + c_n)$, so addition is associative. By definition of addition, $(a_n) + (b_n) = (a_n + b_n)$, and since we are just adding the sequences component-wise, we can interchange the order of the summing so that $(a_n + b_n) = (b_n + a_n) = (b_n) + (a_n)$, so addition is also commutative.

(b) *There exists a zero element.* The 0 element is the sequence of all 0's: $\mathbf{0} := (0, 0, \ldots)$. Then for any $(a_n) \in \ell^1, (a_n) + \mathbf{0} = (a_n + \mathbf{0}) = (a_n)$, since we do component-wise addition of 0 to each component of (a_n).

(c) *There exists an additive inverse.* Let $(a_n) \in \ell^1$. Then its additive inverse is $-(a_n)$, which is the sequence (a_n) with each component multiplied by -1. Then $(a_n) + -(a_n)) = (a_n + (-1 \cdot a_n)) = 0$, since addition is component-wise.

(d) *Associativity of scalar multiplication and existence of multiplicative identity.* Let $\lambda \in \mathbb{R}$. Then by scalar multiplication is given by multiplying each component of a sequence $(a_n) \in \ell^1$ by λ, so $\lambda \cdot (a_n) = (\lambda a_n)$. The multiplicative identity is given by 1, where $1 \cdot (a_n)$ is given by multiplying each component of the sequence by 1, so $1 \cdot (a_n) = (1 \cdot a_n) = (a_n)$.

(e) *Distributivity of scalar multiplication.* Let $\lambda, \mu \in \mathbb{R}$ and $(a_n), (b_n) \in \ell^1$. Consider $(\lambda + \mu)(a_n)$. Since (a_n) is a sequence in $\mathbb{R}$, then we can use distributivity of each of its components to get $(\lambda + \mu)(a_n) = \lambda(a_n) + \mu(a_n)$, where we scalar multiply the components of the sequence first by λ and then sum that with the product of the components of (a_n) and μ. Similarly, for $\lambda((a_n) + (b_n)) = \lambda(a_n + b_n)$, we first scalar multiply the components of (a_n) and then scalar multiply the components of (b_n). We can distribute the scalar because the terms of the sequences $(a_n), (b_n)$ are sequences in $\mathbb{R}$. Thus, we get $\lambda(a_n + b_n) = \lambda(a_n) + \lambda(b_n)$, so we have distributivity of scalar multiplication.

(f) *Commutativity of* $*$. By definition, $(a_n) * (b_n) = \sum_{k=0}^{n} a_k b_{n-k}$. Let $q = n - k$. Then $\sum_{q=0}^{n} a_{n-q} b_q = \sum_{q=0}^{n} b_q a_{n-q} = (b_n) * (a_n)$, so $*$ is commutative.

(g) *Distributivity of* $*$. $(a_n + b_n) * (c_n) = \sum_{k=0}^{n}(a_k + b_k)c_{n-k} = \sum_{k=0}^{n}(a_k c_{n-k} + b_k c_{n-k})$. Note that we can distribute, since each component of these sequences are elements in $\mathbb{R}$. Moreover, since this is a finite sum, we can split the sum so that $\sum_{k=0}^{n}(a_k c_{n-k} + b_k c_{n-k}) = \sum_{k=0}^{n} a_k c_{n-k} + \sum_{k=0}^{n} b_k c_{n-k} = (a_n) * (c_n) + (b_n) * (c_n)$. Note that by commutativity, we also have $(c_n) * (a_n + b_n) = (c_n) * (a_n) + (c_n) * (b_n)$, so distributivity holds.

(h) *Compatibility of* $*$ *and* $\cdot$. Consider $(\lambda a_n)*(b_n) = \sum_{k=0}^n (\lambda a_k)b_{n-k} = \sum_{k=0}^n \lambda a_k b_{n-k} = \lambda \cdot \sum_{k=0}^n a_k n - k = \lambda((a_n)*(b_n))$. Note we can pull out λ from the sum because of the distributive property of scalars in $\mathbb{R}$. By commutativity of $*$, we also have $(a_n)*(\lambda b_n)) = \lambda((a_n)*(b_n))$, so there is compatibility of $*$ and $\cdot$.

(i) *Existence of Identity.* Let $\mathbf{1} := (1, 0, 0, \ldots)$, the sequence of 1 followed by all 0's, so $\mathbf{1_k}$ is the k-th term of this sequence. Then $(\mathbf{1})*(a_n) = \sum_{k=0}^n (1_k)a_{n-k} = (a_n)$. By commutativity, we also have $(a_n)*(1) = (a_n)$.

We've shown that all properties of an associative algebra over $\mathbb{R}$ hold, so we conclude that $(\ell^1, |\cdot|)$ is an associative algebra over $\mathbb{R}$. $\square$

(2) We consider the left hand side of the inequality shown in (2),

$$\|(a_n)*(b_n)\|_1 = \left\| \sum_{k=0}^n a_k b_{n-k} \right\| = \sum_{n=0}^\infty \left| \sum_{k=0}^n a_k b_{n-k} \right| \leq \sum_{n=0}^\infty \sum_{k=0}^n |a_k b_{n-k}| \leq \sum_{n=0}^\infty \sum_{k=0}^n |a_k||b_{n-k}|$$

Using Fubini's theorem, we can then rewrite the summation as

$$\sum_{n=0}^\infty \sum_{k=0}^n |a_k||b_{n-k}| = \sum_{k=0}^n \sum_{n=0}^\infty |a_k||b_{n-k}| = \sum_{k=0}^n \sum_{n=k}^\infty |a_k||b_{n-k}| = \sum_{k=0}^n |a_k| \sum_{n=k}^\infty |b_{n-k}| \leq \sum_{k=0}^\infty |a_k| \sum_{n=k}^\infty |b_{n-k}|$$

If we adopt the convention that $b_j = 0$ for all $j < 0$, then we can rewrite the final inequality as

$$\sum_{k=0}^\infty |a_k| \sum_{n=k}^\infty |b_{n-k}| = \sum_{k=0}^\infty |a_k| \sum_{n=0}^\infty |b_n| = \|a_n\|_1 \cdot \|b_n\|_1,$$

so the inequality in (2) holds. The compatibility of axiom (2) ensures that the multiplication, $*$, from $\ell^1 \times \ell^1 \to \ell^1$ is continuous and that ℓ^1 is closed under convolution. $\square$