Math 131B: Real Analysis | Midterm 2: May 13, 2016 | **Spring 2016**

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2.0.1 Definitions

Limit Point: Suppose $(x_n) \subseteq (X, d)$ and let $L \in X$. L is a limit point of $(x_n)_{n=m}^{\infty} \Leftrightarrow \forall N \geq m$ and $\epsilon > 0, \exists n \geq N$ such that $d(x_n, L) \leq \epsilon$.
Adherent (Accumulation) Point: x_0 is an adherent point to $E \subset (X, d)$ if $\forall r > 0, B_r(x_0) \cap E \neq \emptyset$. The set of all adherent points is the **closure** of E, $\overline{E}$.
Closed: $E \subset (X, d)$ closed $\Leftrightarrow E$ contains all boundary points, $\partial E \subseteq E \Leftrightarrow E = \overline{E}$.
Open: $E \subset (X, d)$ open $\Leftrightarrow E$ contains none of its boundary points, $\partial E \cap E = \emptyset$.
Compact: $K \subset (X, d)$ compact if for every open cover of K, we can find a finite sub-cover: if $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets in X such that $K \subseteq \bigcup_{\alpha \in I} U_\alpha$, then we can find $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that $K \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_n}$.
Sequentially Compact: Every sequence in (X, d) has a convergent subsequence.
Bounded: $S \subseteq X$ bounded $\Leftrightarrow \forall x \in X, \exists R > 0$ such that $S \subseteq B_R(x) \Leftrightarrow \exists x \in X, R > 0$ such that $B_R(x) \supset S \Leftrightarrow \text{diam}(S) = \sup\{d(x, y) : x, y \in S\} < \infty$.
Dense: $S \subseteq X$ dense $\Leftrightarrow \forall$ nonempty $U \subseteq X, S \cap U \neq \emptyset \Leftrightarrow \forall \epsilon > 0 B_\epsilon(x) \cap S \neq \emptyset \forall x \in X \Leftrightarrow X = \overline{S}$.
Totally Bounded: (X, d) totally bounded $\Leftrightarrow \forall \epsilon > 0, \exists x_1, x_2, \ldots, x_n$, such that $X \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$.
Continuous: $f : (X, d_X) \to (X, d_Y)$ is continuous $\Leftrightarrow \forall \epsilon > 0, \forall x_0 \in X, \exists \delta > 0$ such that $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon \forall x \in X \Leftrightarrow \forall$ open $\subseteq Y, f^{-1}(V) = \{x \in X | f(x) \in V\}$ open in $X \Leftrightarrow$ if $(x_n) \subset X$ converges to $x_0 \in X$ with respect to d_X, the sequence $(f(x_n))$ converges to $f(x_0) \in Y$ with respect to d_Y.
Uniformly Continuous: $f : (X, d_X) \to (Y, d_Y)$ is uniformly continuous $\Leftrightarrow$ if for all $\epsilon > 0, \exists \delta > 0$ such that $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon, \forall x, y \in X$.
Connected X is disconnected $\Leftrightarrow \exists$ disjoint, nonempty open sets $V, W \subset X$ such that $V \cup W = X$. (X, d_X) connected $\Leftrightarrow X \neq \emptyset$ and not disconnected $\Leftrightarrow$ every continuous two-valued function is constant.
Pointwise Convergence: (f_n) converges to f (both functions from $(X, d_X) \to (Y, d_Y)$) pointwise $\Leftrightarrow \forall x \in X$ and every $\epsilon > 0, \exists N > 0$ such that $\forall n > N$, $d_Y(f_n(x), f(x)) < \epsilon$.
Uniform Convergence:(f_n) converges uniformly to f (both functions from $(X, d_X) \to (Y, d_Y)$) uniformly $\Leftrightarrow \forall \epsilon > 0, \exists N > 0$ such that $\forall n > N, d_Y(f_n(x), f(x)) < \epsilon, \forall x \in X \Leftrightarrow$ converges in the d_∞ metric.
Convergence (functions): $\lim_{x \to x_0} f(x) = L \Leftrightarrow \forall \epsilon > 0 \exists \delta > 0$ such that $0 < d_X(x, x_0) < \delta \implies d_Y(f(x), L) < \epsilon \Leftrightarrow x \in B_\delta(x_0) \setminus \{x_0\} \implies f(x) \in B_\epsilon(L)$.
Strongly Equivalent: Two metrics d_1, d_2 on X are strongly equivalent if there are $c_1, c_2 > 0$ such that $d_1(x, y) \leq c_2 d_2(x, y)$ and $d_2(x, y) \leq c_1 d_1(x, y)$, for all $x, y \in X$.

2.0.2 Propositions/Lemmas/Theorems

Prop: Let $(x_n)_{n=m}^{\infty} \subseteq (X, d)$. TFAE: (1) L is limit point of $(x_n)_{n=m}^{\infty}$. (2) $\exists$ a subsequence (x_{n_j}) of the original sequence which converges to L.
Lemma: Let $(x_n)_{n=m}^{\infty}$ be a Cauchy subsequence in (X, d). Suppose that there is some $(x_{n_j})_{j=1}^{\infty}$ of this subsequence which converges to some $x_0 \in X$. Then the original sequence converges to x_0.
Theorem: (X, d) compact $\Leftrightarrow (X, d)$ sequentially compact $\Leftrightarrow (X, d)$ totally bounded and complete.
Theorem: X compact $\implies X$ closed, bounded.
Theorem: X compact, $K \subset X$, K closed, then K compact.
Theorem (Lebesgue): Suppose (X, d) is a compact metric space. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X. Then $\exists \delta > 0$ such that $\forall x \in X, \exists \alpha \in I$ such that $B_\delta(x) \subseteq U_\alpha$.
Theorem (Extreme Value): Let (X, d) be a compact metric space and let $f : X \to \mathbb{R}$ be continuous. Then f is bounded. Also, f attains its max at some point $x_0 \in X$ and its min at some point $x_1 \in X$.
Theorem (IVT): Let $f : X \to$ be a continuous map from (X, d_X) to the real line. Let $E \subset X$ be connected, $a, b \in E$. Let $y \in [f(a), f(b)]$. Then $\exists c \in E$ such that $f(c) = y$.

Theorem: Let (X, d_X) be a metric space, let (Y, d_Y) be complete. Then the space $(C(X,Y), d_\infty|_{C(X,\mathbb{R})\times C(X,\mathbb{R})})$ is a complete subspace of $L^\infty(X,Y), d_\infty$. In other words, every Cauchy sequence of functions in $C(X,Y)$ converges to a function in $C(X,Y)$.
Theorem: f continuous at $x_0 \Leftrightarrow \lim_{x\to x_0} f(x) = f(x_0)$.
Theorem: If (f_n) sequence of continuous functions with $f_n \to f$ uniformly, then f is continuous.
Theorem: If (f_n) sequence of bounded functions with $f_n \to f$ uniformly, then f is bounded.

2.0.3 More Definitions

Space of Bounded Functions: $L^\infty(X,Y) = \{f | f : X \to Y \text{ bounded}\}$
Space of Continuous Functions: $C(X,Y) = \{f | f : X \to Y \text{ continuous}\}$
Space of Continuous, Bounded Functions: $C_B(X,Y) = \{f | f : X \to Y \text{ continuous, bounded}\}$
Sup-Norm($L^\infty(X,Y)$) Metric: $d_\infty(f,g) : L^\infty(X,Y)\times L^\infty(X,Y) \to \mathbb{R}^+$, $d_\infty(f,g) = \sup_{x\in X} d_Y(f(x), g(x))$