MATH 128A: Homework #2

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Assignment: 1-13

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Problem 1 (BF 2.2.1)

Use algebraic manipulation to show that each of the following functions has a fixed point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

(a) $g_1(x) = (3 + x - 2x^2)^{1/4}$
(b) $g_2(x) = \left(\frac{x+3-x^4}{2}\right)^{1/2}$
(c) $g_3(x) = \left(\frac{x+3}{x^2+2}\right)^{1/2}$
(d) $g_4(x) = \left(\frac{3x^4+2x^2+3}{4x^3+4x-1}\right)$

Solution When $f(p) = 0$, we have that

$$p^4 + 2p^2 - p - 3 = 0, \tag{1}$$

and we can use this to find functions that have a fixed point at p. From (1), we have

$$0 = p^4 + px^2 - p - 3 \implies p = (3 + p - 2p^2)^{1/4},$$

which implies that $p = g_1(p)$, so $g_1(x)$ has a fixed point at p.
Using (1) again, we see that

$$2p^2 = p + 3 - p^4 \implies p^2 = \frac{p + 3 - p^4}{2} \implies p = \left(\frac{p + 3 - p^4}{2}\right)^{1/2}$$

Taking the positive answer, we get $p = g_2(p)$, so $g_2(x)$ has a fixed point at p.
Using (1) again, we see that

$$p^4 + 2p^2 = p + 3 \implies p^2(p^2 + 2) = p + 3 \implies p = \left(\frac{p + 3}{p^2 + 2}\right)^{1/2},$$

which implies that $p = g_3(p)$, so $g_3(x)$ has a fixed point at p.
Using (1) again, we see that

$$\begin{aligned}
0 &= p^4 + 2p^2 - p - 3 = (4p^4 - 3p^4) + (4p^2 - 2p^2) - p - 3 \\
&\implies 4p^4 + 4p^2 - p = 3p^4 + 2p^2 + 3 \\
&\implies p(4p^3 + 4p - 1) = 3p^2 + 2p^3 + 3 \\
&\implies p = \frac{3p^2 + 2p^2 + 3}{4p^2 + 4p - 1},
\end{aligned}$$

so $p = g_4(p)$, and $g_4(x)$ has a fixed point at p. $\square$

Problem 2 (BF 2.3.15)

Suppose that $f'(x)$ exists on $[a,b]$ and that $f'(x) \neq 0$ on $[a,b]$. Further, suppose there exists one $p \in [a,b]$ such that $f(p) = 0$, and let $p_0 \in [a,b]$ be arbitrary. Let p_1 be the point at which the tangent line to f at $(p_0, f(p_0))$ crosses the x-axis. For each $n \geq 1$, let p_n be the x-intercept of the line tangent to f at $(p_{n-1}, f(p_{n-1}))$. Derive the formula describing this method.

Solution We can approximate the original function f with the line tangent to f at the point $(p_0, f(p_0))$ that intersects the x-axis at p_1, given by

$$\frac{f(p_0) - f(x)}{p_0 - x} = f'(p_o) \tag{2}$$

$$\Rightarrow f(x) \approx f(p_0) - f'(p_0)(p_0 - x) \tag{3}$$

Evaluating (3) at $x = p_1$, we get

$$\begin{aligned} &f(p_1) = 0 = f(p_0) - f'(p_0)(p_0 - p_1) \\ &\Rightarrow f(p_0) = f'(p_0)(p_0 - p_1) \\ &\Rightarrow p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} \end{aligned}$$

In solving for p_1 we have a new approximation for the root of f. We then compare p_1 to p_1, and should this difference in absolute value be too large, we use p_1 as our new starting point, evaluate f at this point, and find another approximation for f at the line tangent to f at $(p_1, f(p_1))$ that intersects the x-axis at a point p_2. Using our work from above, we can again solve for p_2:

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)}$$

Again, we compare p_2 to the previous point p_1, and if this value is too large, then we continue iterating. If we let p_n be the x-intercept of the line tangent to f at $(p_{n-1}, f(p_{n-1}))$ for each $n \geq 1$, then from our first two iterations above, we see that after n iterations, we have that

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})},$$

which is exactly the formula for Newton's method. □

Problem 3 (BF 2.4.13)

The iterative method to solve $f(x) = 0$, given by the fixed-point method $g(x) = x$, where

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1}}{f'(p_{n-1})} - \frac{f''(p_{n-1})}{2f'(p_{n-1}} \left[\frac{f(p_{n-1})}{f'(p_{n-1})}\right]^2 \qquad \text{for } n = 1, 2, 3, \dots \tag{4}$$

has $g'(p) = g''(p) = 0$. This will generally yield cubic ($\alpha = 3$) convergence. Expand the analysis of Example 1 to compare quadratic and cubic convergence.

Solution We consider sequences of the form (p_n) that converge to p, with $p_n \neq p$ for all n, with positive constants λ, α such that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda$$

In addition, $p_n \to p$ of order α, with asymptotic constant λ. If we consider one such sequence (p_n) converging to 0, with $\alpha = 2$ and $\lambda = 0.25$, and another sequence $(\tilde{p}_n)$ converging to 0 with $\alpha = 3$ and $\lambda = 0.25$, and if both (p_n) and $(\tilde{p}_n)$ satisfy the properties defined above, then p_n is quadratically convergent, and $\tilde{p}_n$ is cubically convergent.

If we first analyze (p_n), we see with (4), we can write multiply both sides by the denominator and see that

$$|p_{n+1}| \approx 0.25|p_n|^2$$

Again, we can use the relationship defined in (4) to recursively rewrite $|p_n|$, and we get

$$\begin{aligned} |p_{n+1}| &\approx 0.25^3|p_n|^4 \\ &\approx 0.25^7|p_{n-2}|^8 \\ &\vdots \\ &\approx 0.25^{2^n-1}|p_0|^{2^n} \end{aligned}$$

We can do a similar approximation for $|\tilde{p}_{n+1}|$

$$\begin{aligned} |\tilde{p}_{n+1}| &\approx 0.25^3|\tilde{p}_n|^4 \\ &\approx 0.25^7|\tilde{p}_{n-2}|^8 \\ &\vdots \\ &\approx 0.25^{\frac{3^n-1}{2}}|\tilde{p}_0|^{3^n} \end{aligned}$$

If $|p_0| = |\tilde{p}_0| = 1$, then we can better see the differences in the rates of convergence. In the MATLAB program shown on the following page, we see that in the first 10 iterations, cubic convergence converges to much more rapidly than does quadratic convergence, as shown by the higher number of leading 0's in each iteration. $\square$

Quadratic and Cubic Convergence MATLAB Code

```
% converge.m

% tabulates quadratic and cubic convergence to 0
% for 10 iterations
% Note we take |p_0| = |\tilde{p}_0|

function converge(lambda)

    quad_matrix = zeros(10,1); % store values per iteration for quadratic
    cube_matrix = zeros(10,1); % store values per iteration for cubic

    for n = 1:10
        quad_matrix(n,1) = lambda^(2^n - 1);
        cube_matrix(n,1) = lambda^((3^n - 1) / 2);
    end

    fprintf('Quadratic Convergence: %s', 10);
    disp(quad_matrix);
    fprintf('Cubic Convergence: %s', 10);
    disp(cube_matrix);

%% Output for lambda = 0.25

>> converge(0.25)

Quadratic Convergence:              Cubic Convergence:
0.250000000000000                   0.250000000000000
0.015625000000000                   0.003906250000000
0.000061035156250                   0.000000014901161
0.000000000931323                   0.000000000000000
0.000000000000000                   0.000000000000000
0.000000000000000                   0.000000000000000
0.000000000000000                   0
0.000000000000000                   0
0.000000000000000                   0
0                                   0

%%
```

Problem 4

Consider the fixed point iteration

$$x_{n+1} = \frac{-x_n^2 - c}{2b}, \tag{5}$$

where b and c are fixed real parameters.
(a) If $x_n \to x$, what does x solve?
(b) Analyze and sketch the region of (b, c) values where (5) converges at a rate of $O(2^{-n})$ or better from an interval of starting values x_0 near x.

Solution
(a) If $x_n \to x$, then $x_{n+1} \to x$, then x solves the fixed point problem, which we define to be $g(x)$

$$x = \frac{-x^2 - c}{2b} =: g(x). \tag{6}$$

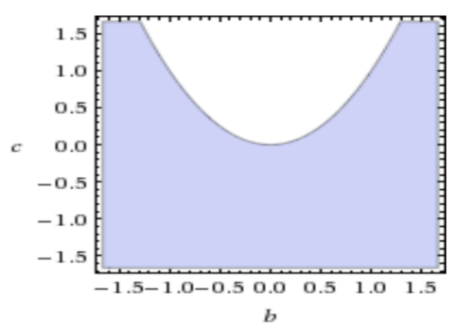
(b) To see where (5) converges at a rate of $O(2^{-n})$ or better from an interval of starting values x_0 near x, it suffices to show that $|g'(x)| \le \frac{1}{2}$. Since $g'(x) = -x/b$, we want to show that $\left|\frac{-x}{b}\right| \le \frac{1}{2}$. Using invariance of $g(x)$, we see that

$$\begin{aligned} \left|\frac{-x}{b}\right| \le \frac{1}{2} \Leftrightarrow \ |x| \le \frac{b}{2} \Leftrightarrow |g(x)| \le \frac{b}{2} \Leftrightarrow \left|\frac{-x^2 - c}{2b}\right| \le \frac{b}{2} &\Leftrightarrow \left|x^2 + c\right| \le b^2 \\ &\Leftrightarrow -b^2 - c \le x^2 \le b^2 - c \\ &\Leftrightarrow b^2 - c > 0 \wedge -b^2 - c > 0 \\ &\Leftrightarrow b^2 > c \wedge -b^2 > c \end{aligned}$$

Note that in the final equivalence, we can make the inequality strict because $x = 0$ is not a solution to $g(x)$. If we rearrange the equation in (6) and solve for x in the quadratic equation: $x^2 + 2bx + c = 0$, then we see that

$$x = -b \pm \sqrt{b^2 - c}$$

and since we are interested in real values of x, we require $b^2 - c > 0$, which is the condition in the last equivalence above, and we conclude that $|gx)| \le \frac{1}{2}$. Thus, (5) converges in the region $c < b^2$ at a rate of at least $O(2^{-n})$. The region is shaded below in (b, c)-space.



□

Problem 5

Consider the fixed point iteration

$$x_{n+1} = -b - \frac{c}{x_n} = g(x_n) \tag{7}$$

(a) Show that $|g'(x)| \leq \frac{1}{2}$ whenever $x^2 \geq 2|c|$.
(b) Show that $g(x)^2 \geq 2|c|$ whenever $x^2 \geq 2|c|$ and $b^2 \geq \frac{9}{2}|c|$.
(c) Draw the region of (b, c)-space where (7) converges at a rate at least $O(2^{-n})$ from any starting point x_0 with $x_0^2 \geq 2|c|$.

Solution
(a) Observe that $g'(x) = \frac{c}{x^2}$. If $x^2 \geq 2|c|$, then

$$\left|\frac{c}{x^2}\right| \leq \frac{1}{2} \Leftrightarrow |g'(x)| \leq \frac{1}{2}.$$

(b) We first calculate $(g(x))^2$, and if $b^2 \geq \frac{9}{2}|c|$, then

$$(g(x))^2 = b^2 + \frac{2bc}{x} + \frac{c^2}{x^2} \geq \frac{9}{2}|c| + \frac{2bc}{x} + \frac{c^2}{x}$$

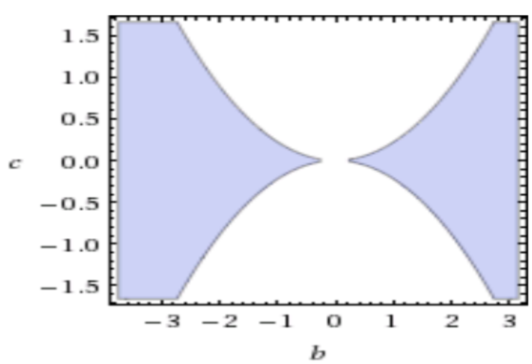
If we consider the derivative of $b^2 + \frac{2bc}{x} + \frac{c^2}{x^2}$ and set it equal to 0 and solve for x, then we see that $x = \frac{-|c|}{2b}$, and since g is a decreasing function, g attains its minimum at $x = \frac{-|c|}{2b}$, so we can again bound the inequality above below

$$\begin{aligned}\frac{9}{2}|c| + \frac{2bc}{x} + \frac{c^2}{x} &\geq 4|c| + \frac{2bc}{\frac{-|c|}{2b}} + \frac{c^2}{\frac{c^2}{4b^2}} \\ &\geq 4|c| - \frac{4b^2c}{|c|} + 4b^2\end{aligned}$$

Using the hypothesis again that $b^2 \geq \frac{9}{2}|c|$, we get

$$4|c| - \frac{4b^2c}{|c|} + 4b^2 \geq 4|c| - 16c + 16|c| = 20|c| - 16c \geq 2|c|,$$

so $(g(x))^2 \geq 2|c|$ holds.
(c) The region of (b, c)-space where (7) converges at a rate at least $O(2^{-n})$ is given by



Problem 6

(a) Write down Newton's method in the form

$$x_{k+1} = g(x_k)$$

for solving

$$f(x) = x^2 - 2bx + b^2 - d^2 \tag{8}$$

where $b > 0$, $d > 0$ are parameters.
(b) Show that $|g'(x)| \leq 1/2$ whenever $|x - b| \geq d/\sqrt{2}$.
(c) Show that $|g(x) - b| \geq d/\sqrt{2}$ whenever $|x - b| \geq d/\sqrt{2}$.
(d) Sketch the graph of $f(x)$ with the roots of $f(x) = 0$ and the intervals of x where Newton's method is guaranteed to converge.

Solution
(a) We rewrite $f(x)$ and calculate the first derivative of f:

$$\begin{aligned} f(x) &= \frac{(x-b)^2 - d^2}{2(x-b)} \\ f'(x) &= 2(x-b) \end{aligned}$$

For solving equation (8), we can use Newton's method, where

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} \\ &= x - \frac{(x-b)^2 - d^2}{2(x-b)} \\ &= x \cdot \frac{2(x-b)}{2(x-b)} - \frac{(x-b)^2 - d^2}{2(x-b)} \\ &= \frac{x^2 - b^2 + d^2}{2(x-b)} \\ &= \frac{x+b}{2} + \frac{d^2}{2(x-b)} \end{aligned}$$

(b) Observe that $g'(x) = \frac{1}{2} - \frac{d^2}{2(x-b)^2}$. To show that $|g'(x)| \leq 1/2$ whenever $|x-b| \geq d/\sqrt{2}$, we see that

$$\left|\frac{1}{2} - \frac{d^2}{2(x-b)^2}\right| \leq \frac{1}{2} \Leftrightarrow -\frac{1}{2} \leq \frac{1}{2 - \frac{d^2}{2(x-b)^2}} \leq \frac{1}{2} \Leftrightarrow 0 \leq \frac{d^2}{2(x-b)^2} \leq \frac{1}{4} \Leftrightarrow 2d^2 \leq (x-b)^2 \Leftrightarrow \sqrt{2}d \leq |x-d|$$

Since $|x - b| \geq d\sqrt{2} \geq d \cdot \frac{\sqrt{2}}{2} = d/\sqrt{2}$, then we can follow the equivalences back to what we wanted to prove.
(c) In part (*b*) we calculated $g'(x)$. If we set $g'(x) = 0$ and solve for x, then we get $x = b \pm d$, where $g(x) \geq g(b+d)$ when $x \geq b + d/\sqrt{2}$. Then for $x = b + d$, we see that

$$g(x) - b \geq g(b+d) - b = \frac{d}{2} + \frac{d^2}{2d} = d \geq \frac{d}{\sqrt{2}}$$

when $x \geq b + d\sqrt{2}$. For $x = b - d$, $g(x) \leq g(x-b)$ when $x - b \leq -d/\sqrt{2}$, so

$$g(x) - b \leq g(b-d) - b = -\frac{d}{2} + \frac{d}{2(-d)} = -d \leq \frac{-d}{\sqrt{2}}.$$

We can then conclude that $|g(x) - b| \geq d/\sqrt{2}$ whenever $|x - b| \geq d/\sqrt{2}$.

(d) We saw from part (b) that when $|x - b| \geq d/\sqrt{2}$, $|g'(x)| \leq 1/2$, so Newton's method will converge if we consider intervals of the form $\left[b + d/\sqrt{2}, b + k \cdot d\right]$ for $k \in \mathbb{R}, k > 2$. The plot is shown above. $\square$

Problem 7

Fix $a > 0$ and consider the fixed point iteration

$$x_{n+1} = x_n(2 - ax_n) \tag{9}$$

(a) Show that if $x_n \to x$, then $x = \frac{1}{a}$ or $x = 0$.
(b) Find an interval (α, β) containing $\frac{1}{a}$ such that (9) converges to $1/a$ whenever $x_0 \in (\alpha, \beta)$.
(c) Find the rate of convergence in (b).
(d) Generalize (a) to 2×2 matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

Solution
(a) If $x_n \to x$, then (9) becomes $x = x(2 - ax) \implies x - x(2 - ax) = 0 \implies x(ax - 1) = 0$, so $x = 0$ or $x = 1/a$, where $a > 0$ is given.

(b) Then we show that

$$|g'(x)| = |2(1 - ax)| \leq 1/2$$

when $x = \frac{1}{a}$. Note that the above equality happens if and only if

$$|1 - ax| \leq \frac{1}{4} \Leftrightarrow \frac{3}{4} \leq ax \leq \frac{5}{4} \Leftrightarrow \frac{3}{4a} \leq x \leq \frac{5}{4a}$$

Since $g(\frac{1}{a}) = \frac{1}{a}$, we see that $g(x)$ satisfies the last inequality as well, so

$$\frac{3}{4a} \leq g(a) \leq \frac{5}{4a}$$

so an interval containing $1/a$ such that the equation in (9) converges to $1/a$ whenever x_0 is in this interval is $(0, \frac{2}{a})$.

(c) Since $g'(a) = 0$ and we bounded the derivative above by $1/2$, then (9) is Newton and converges at a rate of $O(2^{-n})$.

(d) Generalizing (a) to 2×2 matrices of the above form, then we are solving for $X = IA^{-1} = A^{-1}$. $\square$

Problem 8

Use Newton's method in MATLAB to solve the equation

$$f(x) = \frac{1}{x} + \ln x - 2 = 0$$

for $x > 0$. Characterize the convergence as linear or quadratic by tabulating the number of correct bits at each step of the iteration.

Solution
Solving the function above using Newton's method in MATLAB, we get the results tabulated below

Iteration	Approximation 1	Approximation 2
0	5	0.5
1	**6**.191013047286873	0.153426409720027
2	**6.30**4550119075867	0.226924070540923
3	**6.305395**233304984	0.288446371473163
4	**6.305395279271691**	**0.31**4592046632893
5	**6.305395279271691**	**0.3178**03540977670
6		**0.3178444**26413115
7		**0.317844432899373**
8		**0.317844432899373**

Iteration 0 is the initial guess, and we see that there are two solutions to $f(x) = \frac{1}{x} + \ln x - 2 = 0$. In the first approximation of the root, we see that after the first iteration, the number of correct bits roughly doubles with each iteration, as indicated by the number of digits in bold. In the second approximation of the root, it takes three iterations until we see correct digits, but starting with the 4th iteration onward, we see the number of correct digits doubling at each iteration as well, so we conclude that in both cases, convergence is quadratic, though it takes a bit longer in the case of the second approximation. We conclude that f has roots at $x = 6.305395279271691$ and $x = 0.317844432899373$The MATLAB code used for this problem is can be found on the following page. □

Newton's Method MATLAB Code

```
% Newton's Method'


% use Newton's method to solve the function
% f(x) = 1/x + ln(x) - x = 0

function [r,guess] = solve_f(x_0)
    max_iter = 10;
    TOL = 10^(-12);
    guess_matrix = x_0; % matrix to store each guess
    for i = 2:(max_iter+1)
        x = x_0 - f(x_0) / f_prime(x_0);
        guess_matrix = [guess_matrix; x];
        if abs(x - x_0) < TOL
            r = x_0;
            guess = guess_matrix;
            return;
        end
        x_0 = x;
    end
    guess = guess_matrix;


% evaluate the function f(x) = 1/x + ln(x) - 2
function y = f(x)
    y = 1 / x + log(x) - 2;


% evaluate the derivatve of f
function y_prime = f_prime(x)
    y_prime = -1 / x^2 + 1 / x;
```

Problem 9

Use Newton's method in MATLAB to solve the equation

$$f(x) = x^3 = 0$$

Characterize the convergence as linear or quadratic by tabulating the number of correct bits at each step of the iteration. Explain your results.

Solution

The tabulated results of Newton's Method are as shown in the table below.

Iteration	Approximation	Correct Bits
0	1	0
1	0.500000000000000	1
2	0.250000000000000	1
3	0.125000000000000	1
4	0.062500000000000	2
5	0.031250000000000	2
6	0.015625000000000	2
7	0.007812500000000	3
8	0.003906250000000	3
9	0.001953125000000	3
10	0.000976562500000	4
11	0.000488281250000	4
12	0.000244140625000	4
13	0.000122070312500	4
14	0.000061035156250	5
15	0.000030517578125	5
16	0.000015258789062	5
17	0.000007629394531	6
18	0.000003814697266	6
19	0.000001907348633	6
20	0.000000953674316	7
21	0.000000476837158	7
22	0.000000238418579	7
23	0.000000119209290	7
24	0.000000059604645	8
25	0.000000029802322	8
26	0.000000014901161	8
27	0.000000007450581	9
28	0.000000003725290	9
29	0.000000001862645	9
30	0.000000000931323	10

We see that in the case of solving $f(x) = x^3 = 0$, Newton's method gives only linear convergence, as we see the correct number of bits increases by one every few iterations. This is because $f(0) = f'(0) = f''(0) = 0$, so quadratic convergence is not guaranteed with Newton's method. The MATLAB Code used to solve this equation is on the following page. □

MATLAB Code for solving $f(x) = x^3 = 0$

```
% Solve f(x) = x^3 via Newton's Method
function [r,guess,iters] = solve_f(x_0)
    max_iter = 30; TOL = 10^(-9);
    guess_matrix = x_0;
    for i = 1:(max_iter)
        x = x_0 - f(x_0) / f_prime(x_0);
        guess_matrix = [guess_matrix; x];
        fprintf('Iteration: %d %s', i, 10);
        disp(guess_matrix); % display the guesses at each iteration
        if abs(x - x_0) < TOL
            r = x_0;
            guess = guess_matrix;
            iters = i;
            return;
        end
        x_0 = x;
        iters = i;
    end
    guess = guess_matrix;

% evaluate the function:
% f(x) = x^3
function y = f(x)
    y = x^3;

% evaluate derivative of the function f as defined above
function y_prime = f_prime(x)
    y_prime = 2 * x^2;
```

Problem 10

Use Newton's method in MATLAB to solve the equation

$$f(x) = \arctan x = 0$$

for a diverse selection of starting values. Find starting values which lead to convergence, divergence, and oscillation.

Solution
The MATLAB code on the following page give uses Newton's method to solve the equation $f(x) = \arctan x = 0$. Some values of x that lead to convergence are $x = 0.25, 0.5, 1, 1.25$. Some values of x that lead to oscillation/divergence are $x = 1.5, 1.75, 3$. The output of the MATLAB code when our initial guess is $x = 0.25$ is shown below the code on the following page, and more comprehensive results are tabulated below.

Convergence for initial guess $x = 0.25, 0.5, 1.25$

Iteration	Approx 1	Approx 2	Approx 3
1	0.25	0.5	1.25
1	-0.010289829572293	-0.079559511251008	-1.046141922964069
2	0.000000726313455	0.000335302204005	0.646028604062692
3	-0.000000000000000	-0.000000000025131	-0.166934309609309
4	0	0	0.003084231093750
5		0	-0.000000019559090
6			0.000000000000000
7			0

Divergence for initial guess $x = 1.5, 3$

Iteration	Approx 1	Approx 2
1	1.5	3
1	-0.010289829572293	-9.490457723982544
2	-1.694079600553819	1.0e+02 * 1.239995111788842
3	2.321126961438388	1.0e+04 * -2.390594029492087
4	-5.114087836777514	1.0e+08 * 8.976528364340615
5	32.295683914210016	1.0e+18 * -1.265717228069277
6	1.0e+3 * -1.575316950821204	1.0e+36 * 2.516478706706525
10	1.0e+108 * 2.453994637498496	Inf
12	Inf	NaN

We see that in the case of convergence, it takes more steps for Newton's method to converge as we increase the value of our initial guess. However, starting from about $x = 1.5$, Newton's method no longer converges, and in fact eventually diverges on the 12th iteration. At $x = 3$, it diverges even more quickly, and goes to $+\infty$ on the 10th iteration.

To find the point x_0 at which $f(x_0)$ would oscillate, we would search for a point that satisfied the conditions

$$x_{n+1} \neq x_n$$
$$x_{n+1} = x_{n-1}$$

Then we want to find the solution to the problem $x = g(g(x))$, where

$$g(x) = x - \frac{f(x)}{f'(x)},$$

so we want to find a solution to the fixed point problem

$$x = g(g(x)) = g(x) - (1 + g(x)^2)\arctan(g(x))$$

The code used to find the solution to this problem is on the page following the MATLAB code for finding the solution to $f(x) = \arctan x = 0$. We use the bisection method to find the root of the equation

$$\varphi(x) = x - (g(x) - (1 + g(x)^2)\arctan(g(x))) \tag{10}$$

The solution we get for the equation in (10) is

$$\mathbf{x = 1.391745200270734894587576491176150739192962646484}$$

, and we see that when we plug this in as an initial guess for Newton's method for solving $f(x) = \arctan(x) = 0$, we get an oscillating value. The first eight iterations are shown in the table below.

Iteration	Approximation 1
0	1.391745200270735
1	-1.391745200270735
2	1.391745200270735
3	-1.391745200270735
4	1.391745200270735
5	-1.391745200270735
6	1.391745200270735
7	-1.391745200270735
8	1.391745200270735

□

MATLAB Code for solving $f(x) = \arctan x = 0$

```
function [r,guess,iters] = solve_f(x_0)
    max_iter = 20; TOL = 10^(-16);
    guess_matrix = x_0;
    for i = 1:(max_iter)
        x = x_0 - f(x_0) / f_prime(x_0);
        guess_matrix = [guess_matrix; x];
        fprintf('Iteration: %d %s', i, 10);
        disp(guess_matrix); % display the guesses at each iteration
        if abs(x - x_0) < TOL
            r = x_0;
            guess = guess_matrix;
            iters = i;
            return;
        end
        x_0 = x;
        iters = i;
    end
    guess = guess_matrix;
% evaluate the function: f(x) = arctan(x)
function y = f(x)
    y = atan(x);
% evaluate derivative of the function f as defined above
function y_prime = f_prime(x)
    y_prime = 1 / (1 + x^2);
%% output for x = 0.25
>> [r,guess,iters] = solve_f(0.25);
Iteration: 1
 0.250000000000000
-0.010289829572293

Iteration: 2
 0.250000000000000
-0.010289829572293
 0.000000726313455

Iteration: 3
 0.250000000000000
-0.010289829572293
 0.000000726313455
-0.000000000000000

Iteration: 4
 0.250000000000000
-0.010289829572293
 0.000000726313455
-0.000000000000000
 0
```

MATLAB Code for finding oscillating value

```
function bisection()
    low = 1.35;
    high = 1.4;
    mid = (low + high) / 2;
    tol = 10^(-52);
    f = oscillate(mid);
    iter = 0;

    while abs(f) > tol
        if f > 0
            low = mid;
        else
            high = mid;
        end
        mid = (low + high) / 2;
        f = oscillate(mid);

        iter = iter + 1;
    end

    root = mid;
    n = iter;

    fprintf('root: %.50f, f = %.3f, iter = %d', root, f, iter);

function y = oscillate(x)
    y = x - (g(x) - (1 + (g(x))^2) * atan(g(x)));

function g = g(x)
    g = x - f(x) / f_prime(x);

function y = f(x)
    y = atan(x);

function y_prime = f_prime(x)
    y_prime = 1 / (1 + x^2);
```

Problem 11

Derive a new iteration method for solving $f(x) = 0$ by solving the quadratic equation

$$f(x_k) + f'(x_k)(x - x_k) + \frac{1}{2}f''(x_k)(x - x_k)^2 = 0 \tag{11}$$

Complete your algorithm by specifying which root to choose, and prove cubic convergence under appropriate assumptions on f, x, and the starting value x_0.

Solution
Suppose $x = g(x)$. Then $f(x) = 0$, and we can rewrite (11)

$$f(x) + f'(x)(g(x) - x) + \frac{1}{2}f''(g(x) - x)^2 = 0$$

To show cubic convergence, it suffices to show that at the root $x = p$, we have $g'(p) = g''(p) = 0$. Suppose that $f'(p) \neq 0$. Then taking the derivative of the above equation, we have

$$f'(x) + f''(x)(g(x) - x) + f'(x)(g'(x) - 1) + \frac{1}{2}f'''(x)(g(x) - x)^2 + f''(x)(g(x) - x)(g'(x) - 1) = 0$$

At $x = p$, $g(p) = p$, so this simplifies to

$$f'(p)g'(p) = 0,$$

but we assumed that $f'(p) \neq 0$, so $g'(p) = 0$. Taking the derivative again, we get a similarly complicated equation that is equal to 0. As in the case of the first derivative, most of the terms cancel at $x = p$, and we are left with

$$f'(p)g''(p) = 0,$$

which implies $g''(p) = 0$, so $g'(p) = g''(p) = 0$, and we have cubic convergence. □

Problem 12

Implement a MATLAB function `schroderbisection.m` which combines the fast convergence of the Schroder iteration for multiple roots

$$g(x) = x - \frac{f(x)}{f'(x)} \frac{1}{1 - \frac{f(x)f''(x)}{f'(x)^2}} = x - \frac{f(x)f'(x)}{f'(x)^2 - f(x)f;''(x)} \tag{12}$$

with the bracketing guarantee of bisection. At each step $j = 1$ to n, carefully choose m as in geometric mean bisection. Define

$$\epsilon = \min(|f(b) - f(a)|/8, |f''(m)||b - u|^2)$$

```
% Arguments:
    % a: beginning of interval [a,b]
    % b: end of interval [a,b]
    % f: function handle [f, fp, fpp]
       % changed functionality: f{1}(x) returns f(x),
       % f{2}(x) returns f'(x), etc.
    % tol: user specified tolerance for interval width
function [r,h] = schroderbisection(a, b, f, tol)
    j = 1;
    while(abs(b - a) > tol * max(abs(a), abs(b)))
        mid = (a + b) / 2;

        d = abs((f{1}(b) - f{1}(a)) / 8);
        m = abs(f{3}(mid) * (b-a)^2);

        % set epsilon to minimum of d, m
        eps = min(d, m);

        % 6 candidates for new intervals
        q_plus  = g(f, mid,  eps);
        q_minus = g(f, mid, -eps);
        a_plus  = g(f, a,    eps);
        a_minus = g(f, a,   -eps);
        b_plus  = g(f, b,    eps);
        b_minus = g(f, b,   -eps);

        %fprintf('value: %.5f %s', q_plus, 10);



        a = max([a_minus, a, a_plus, q_minus]);
        b = min([b_plus, b, b_minus, q_plus]);


        if f{1}(mid) * f{1}(a) > 0
            a = mid;
        else
            b = mid;
```

```
    end

    % store numbers into j-th column of 3 x n matrix
    h(1,j) = a;
    h(2,j) = b;
    h(3,j) = f{1}(mid);

    j = j + 1;

    end

    % final approximation of root
    r = (b + a) / 2;

    % f is function handle from above
    function  y = g(f, x, eps)
        y = x - (f{1}(x) + eps) * f{2}(x) / ((f{2}(x))^2 - (f{1}(x)
            + eps) * f{3}(x));
```

Problem 13

Let λ_k be $n+1$ distinct real numbers. Let t_j be $n+1$ distinct real numbers.
(a) Show that

$$a(t) = \sum_{k=0}^{n} a_k e^{\lambda_k t} \tag{13}$$

can vanish for all real t only if $a_0 = a_1 = \cdots = a_n = 0$.
(b) Show that for the exponential interpolation problem

$$a(t_j) = \sum_{k=0}^{n} a_k e^{\lambda_k t_j} = f_j, \quad 0 \leq j \leq n$$

there exists a unique solution $a(t)$ for any data values f_j.
(c) For equality spaced $\lambda_k = -k/n$, find an explicit formula and an error estimate for $a(t)$.
(d) Interpolate Runge's function

$$f(t) = \frac{1}{1+t^2}$$

at $n+1$ equidistant points on $[0,5]$ by your formula from (c) and tabulate the error for $n = 3, 5, 9, 17, 33$.

Solution
(a) Suppose the coefficients $a_0, a_1, \ldots, a_n$ are not all equal to 0. Without loss of generality, suppose $a_0 \neq 0$. Then even if $a_1 = a_2 = \ldots = a_n = 0$, we see that the sum in (13)

$$a(t) = \sum_{k=0}^{n} a_k e^{\lambda_k t} = a_0 e^{\lambda_0 t} + a_1 e^{\lambda_1 t} + \ldots + a_n e^{\lambda_n t} = a_0 e^{\lambda_0 t} + 0 + \ldots 0$$

is nonzero since $a_0 \neq 0$ and $e^{o t}$ since the exponential function e^x is strictly positive for all values of $x \in \mathbb{R}$. Then $a(t)$ as defined above cannot vanish, and we have shown the contrapositive of (a).

(b)