MATH 128A: Homework #7

Professor John Strain
Assignment: 1-3

Eric Chuu

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Problem 1

Consider a differential equation

$$y'(t) = f(t, y(t)) \tag{1}$$

where f satisfies the condition

$$(u - v)(f(t, u) - f(t, v)) \leq 0 \tag{2}$$

for all u and v.
(a) Suppose $U(t)$ and $V(t)$ are exact solutions. Show that

$$|U(t) - V(t)| \leq |U(0) - V(0)| \tag{3}$$

for all $t \geq 0$.
(b) Suppose W satisfies a perturbed differential equation

$$W'(t) = f(t, W(t)) + r(t) \tag{4}$$

for $t \geq 0$. Show that

$$|U(t) - W(t)| \leq |U(0) - W(0)| + \int_0^t |r(s)| ds \tag{5}$$

for $t \geq 0$.
(c) Show that two numerical solutions u_n and v_n generated by implicit Euler satisfy

$$|u_n - v_n| \leq |u_0 - v_0| \tag{6}$$

for all $n \geq 0$.
(d) Show that the local truncation error τ_{n+1} of the implicit Euler method

$$u_{n+1} = u_n + hf(t_{n+1}, u_{n+1}) \tag{7}$$

is given by

$$\tau_{n+1} = \frac{y_{n+1} - y_n}{h} - f(t_{n+1}, y_{n+1}) = -\frac{h}{2} y''(\zeta) \tag{8}$$

where $y_n = y(t_n)$ is the exact solution and Σ an unknown point.
(e) Show that the numerical solution u_n generated by implicit Euler with $u_0 = y_0$ satisfies

$$|u_n - y_n| \leq nh\tau \tag{9}$$

for $0 \leq nh < \infty$, where $\tau = Mh/2$ and $|y''| \leq M$

Solution
(a) Since $U(t)$ and $V(t)$ are exact solutions,

$$\begin{aligned}U'(t) &= f(t, U(t)) \\ V'(t) &= f(t, V(t)).\end{aligned}$$

Substituting these into the inequality given in (2), we get

$$\begin{aligned}(U-V)(U'-V') &\leq 0 \\ UU' - UV' - VU' + VV' &\leq 0 \\ \int_0^t UU' + VV' ds &\leq \int_0^t UV' + VU' ds \\ \frac{1}{2}(U^2+V^2)\Big|_0^t &\leq \int_0^t (UV)' ds \\ U(t)^2 + V(t)^2 - U(0)^2 - V(0)^2 &\leq 2U(t)V(t) - 2U(0)V(0) \\ U(t)^2 - 2U(t)V(t) + V(t)^2 &\leq U(0)^2 - 2U(0)V(0) + V(0)^2 \\ (U(t)-V(t))^2 &\leq (U(0)-V(0))^2 \\ |U(t)-V(t)| &\leq |U(0)-V(0)|,\end{aligned}$$

for all $t \geq 0$, so the inequality in (3) holds. □

(b) We consider the following product

$$\begin{aligned}(U-W)(U'-W') &= (U-W)(f(t,U) - f(t,W) - r) \\ &= (V-W)(f(t,U) - f(t,W)) - (U-W)r \\ &\leq -(U-W)r\end{aligned}$$

The last inequality follows from condition (2), which implies that $(V-W)(f(t,U) - f(t,W)) \leq 0$. Since $(U-W) \neq 0$, we can divide both sides of the inequality,

$$(U'-W') \leq -r$$

Integrating the left hand side from 0 to t,

$$\begin{aligned}&\int_0^t U' - W' ds = U(t) - W(t) - U(0) + W(0) \leq \int_0^t |U'-W'| \leq \int_0^t |r| ds \\ &\implies |U(t) - W(t)| \leq |U(0) - W(0)| + \int_0^t |r(s)| ds\end{aligned}$$

so the inequality in (5) holds. □

(c) We first express the solutions generated by implicit Euler as a recurrence

$$\begin{aligned}u_{n+1} &= u_n + hf(u_{n+1} \\ v_{n+1} &= v_n + hf(v_{n+1}\end{aligned}$$

Subtracting these and taking the dot product of both sides with $(u_{n+1} - v_{n+1})$, we get

$$\begin{aligned}u_{n+1} - v_{n+1} &= u_n - v_n + h\left[f(u_{n+1} - v_{n+1}\right] \\ (u_{n+1} - v_{n+1})^T(u_{n+1} - v_{n+1}) &= (u_{n+1} - v_{n+1})^T(u_n - v_n) + h(u_{n+1} - v_{n+1})^T\left[f(u_{n+1} - v_{n+1}\right]\end{aligned}$$

Using the hypothesis given in (2) above and using the Cauchy-Schwarz Inequality, we can express this as an inequality

$$\begin{aligned}
\|(u_{n+1} - v_{n+1})\|^2 &\leq u_{n+1} - v_{n+1})^T (u_n - v_n) \\
&\leq \|u_{n+1} - v_{n+1}\| \|u_n - v_n\| \\
\|(u_{n+1} - v_{n+1})\| &\leq \|u_n - v_n\|
\end{aligned}$$

for all $n \geq 0$. We can continue this inequality to see

$$\begin{aligned}
&\|(u_{n+1} - v_{n+1})\| \leq \|u_n - v_n\| \leq \|u_{n-1} - v_{n-1}\| \leq \ldots \leq \|u_0 - v_0\| \\
&\implies |u_n - v_n| \leq |u_0 - v_0|
\end{aligned}$$

which is exactly what we wanted to show. □

(d) We calculate the local truncation error of implicit Euler using the Taylor expansion of $y(t_n) = y(t_{n+1} - h)$ and the formula

$$\begin{aligned}
\tau_{n+1} &= \frac{y_{n+1} - y_n - hf(t_{n+1}, y_{n+1})}{h} \\
&= \frac{y_{n+1} - (y_{n+1} - hy'(t_{n+1}) + \frac{h^2}{2})y''(\zeta))}{h} - y'(t_{n+1}) \\
&= \boxed{-\frac{h^2}{2} y''(\zeta)}
\end{aligned}$$

as desired, and we conclude that □

(e) Since $u_0 = y_0$, the inequality given in (9) holds trivially. Suppose that the inequality holds for $n-1$. That is,

$$|u_{n-1} - y_{n-1}| \leq (n-1)h\tau.$$

To show that it holds for n, consider

$$\begin{aligned}
u_n - y_n &= u_{n-1} - y_{n-1} + h\left[f(t_{n+1}, u_{n+1}) - f(t_{n+1}, y_{n+1})\right] - h\tau_n \\
|u_n - y_n|^2 &\leq (u_n - y_n)(u_{n-1} - y_{n-1}) - h(u_n - y_n)\tau_n \\
|u_n - y_n| &\leq (u_{n-1} - y_{n-1}) - h\tau_n \\
|u_n - y_n| &\leq (u_{n-1} - y_{n-1}) + h\tau_n \\
&\leq (n-1)h\tau_n + h\tau_n \\
&= h\tau n
\end{aligned}$$

The second inequality holds after applying the inequality given to us in (2), and the inequality before the last equal sign is a result of the inductive hypothesis, so by induction, we've shown that the inequality given in (9) holds for $0 \leq nh < \infty$, where $\tau = Mh/2$ and $|y''| \leq M$. □

Problem 2

Define a family of implicit Runge-Kutta methods parametrized by order p, by applying up to $p-1$ passes of deferred correction to p steps of the implicit Euler method, i.e., starting from $u_n^1 = u_n$, define the uncorrected solution by solving

$$u_{n+j+1}^1 = u_{n+j}^1 + hf(t_{n+j+1}, u_{n+j+1}^1) \tag{10}$$

for $0 \leq j \leq p-1$. Let $u(t) = U_1(t)$ be the degree-p polynomial that interpolates the $p+1$ values u_{n+j}^1 at the p points $t = t_{n+j}$ for $0 \leq j \leq p$. Solve the error equation from problem set 6 by the implicit Euler method, yielding approximate errors $e_{n+1}^1, e_{n+2}^1, \ldots, e_{n+p}^1$. Produce a second-order corrected accuracy corrected solution

$$u_{n+j}^2 = u_{n+j}^1 + e_{n+j}^1$$

for $1 \leq j \leq p$. Repeat the procedure to produce $u_{n+j}^3, \ldots u_{n+j}^p$.

(a) Verify that $p = 1$ gives the implicit Euler method. Taylor expand $k_1(h)$. Show that your method has local truncation error $\tau = O(h)$ and find the coefficient of the $O(h)$ term.

(b) For $p = 2$ express your method as a 4-stage Runge-Kutta method in the form

$$k_i = f\left(t_n + 2hc_i, u_n + 2h\sum_{j=1}^{4} a_{ij}k_j)\right) \tag{11}$$

for $1 \leq i \leq 4$,

$$u_{n+2}^2 = u_n + 2h\sum_{i=1}^{4} b_i k_i \tag{12}$$

Find all the constants c_i, a_{ij}, b_j and arrange them in a Butcher array.

Solution

(a) For $p = 1$, there are no steps of deferred correction, so equation (10) just gives us

$$u_{n+j+1} = u_{n+j} + hf(t_{n+j+1}, u_{n+j+1})$$

which is the implicit Euler method. If we consider the Taylor expansion of $k_1(h)$, we get

$$k_1(h) = k_1 + h \cdot \frac{\partial f}{\partial t} + hk_1\frac{\partial f}{\partial u} + O(h)$$

(b) For $p = 2$, we first solve the error equation by the implicit Euler method, and consider the following

$$\begin{aligned}
u_{n+1} &= u_n + hf(t_{n+1}, u_{n+1}) \\
u_{n+2} &= u_{n+1} + hf(t_{n+2}, u_{n+2}) \\
e_{n+1} &= e_n + h\left[f(t_{n+1}, u_{n+1} + e_{n+1}) - u't(t_{n+1})\right] \\
e_{n+2} &= e_{n+1} + h\left[f(t_{n+2}, u_{n+2} + e_{n+2}) - u't(t_{n+2})\right]
\end{aligned}$$

Then we can write $k_1, \ldots, k_4$,

$$\begin{aligned}
k_1 &= f(t_{n+1}, u_{n+1}) \\
&= f(t_{n+1}, u_n + hk_1) \\
k_2 &= f(t_{n+2}, u_{n+2}) \\
&= f(t_{n+2}, u_n + hk_1 + hk_2) \\
k_3 &= f(t_{n+1}, u_{n+1} + e_{n+1}) \\
k_4 &= f(t_{n+2}, u_{n+2} + e_{n+2})
\end{aligned}$$

We use these to expand u^2_{n+2},

$$\begin{aligned}
u^2_{n+2} = u_n + 2h\,&[b_1 f(t_{n+1}, u_n + hk_1) \\
&+ b_2 f(t_{n+2}, u_{n+2}) \\
&+ b_3 f(t_{n+1}, u_{n+1} + e_{n+1}) \\
&+ b_4 f(t_{n+2}, u_{n+2} + e_{n+2})
\end{aligned}$$

Problem 3

Consider the linear initial value problem

$$y' = -L(y(t) - \varphi(t)) + \varphi'(t) \tag{13}$$
$$y(0) = y_0 \tag{14}$$

where $\varphi(t) = \cos(30t)$.

(a) Solve the initial value problem exactly.

(b) Use `euler.m` to solve the initial value problem with $y(0) = 2$ for $0 \leq t \leq 1$ with $L = 10^k$ for $k = 1$ to 5. For each L, use $h = 10^j$ with $j = 1$ to 6. Tabulate the errors.

(c) Write a MATLAB script which uses the method you derived in question 2 with $p = 2$ to solve the initial value problem with $y(0) = 2$ for $0 \leq t \leq 1$ with $L = 10^k$ for $k = 1$ to 5. For each L, use $h = 10^j$ with $j = 1$ to 6. Tabulate the errors. Plot an accurate solution for each L.

Solution

(a) We first solve the associated homogeneous equation

$$\begin{aligned} y' + Ly &= 0 \\ y &= Ce^{-Lt} \end{aligned}$$

We can then use the method of undetermined coefficients

$$y = A\sin(30t) + B\cos(30t) \tag{15}$$
$$y' = 30A\cos(30t) - 30B\sin(30t) \tag{16}$$

Plugging these back into the differential equation given by

$$y' + Ly = L\cos(30t) - 30\sin(30t)$$
$$30A\cos(30t) - 30B\sin(30t) + LA\sin(30t) + LB\cos(30t) = L\cos(30t) - 30\sin(30t)$$

If we match the coefficients of $\cos(30t)$ and $\sin(30t)$, then we see that

$$\begin{aligned} 30LA + LB &= L \\ LA - 30B &= -30 \end{aligned}$$

and after solving for A and B, we get $A = 0, B = 1$. Plugging these back into the equation in (15), we get

$$\begin{aligned} y_p &= \cos(30t) \\ y &= y_h + y_p = Ce^{-Lt} + \cos(30t) \end{aligned}$$

Using the initial condition given in (14),

$$y_0 = y(0) = C + 1 \implies C = y_0 - 1$$
$$\boxed{y = (y_0 - 1)e^{-Lt} + \cos(30t)}$$

□

(b) The MATLAB code for `euler.m` is below and the results are tabulated on the following page.

```

% a, b: interval endpoints with a < b
% n: number of steps with h = (b-a)/n
% ya: vector y(a) of initial conditions
% f : function handle f(t,y) to integrate (y is vector)
% u: output approximation to the final solution y(b)

% approximate the solution of the ivp:
% y' = f(t,y) a <= t <= b, y(a) = alpha
function u = my_euler(a, b, ya, n)
    h = (b - a) / n;
    t = a;
    w_0 = ya;                   % initial condition
    u = zeros(n+1, 1);

    for k = 1:6
        %u(1,1) = t;
        u(1,1) = w_0;
        for i = 2:n+1
            w_i = u(i-1,1) + h * f(u(i-1,1), t, k);
            t = t + h;
            % u(i, 1) = t;           % update time
            u(i, 1) = w_i;        % update approximation
        end
        display(u(95:101,1))
    end

    function y_p = f(y, t, k)
        y_p = -10^(-k) * (y - cos(30*t)) - 30 * sin(30*t);
```