MATH 131B: Homework #5

Professor Dave Penneys
Assignment: 46, 49, 50, 55

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Problem 0

Suppose $(f_n), f$ are in $C_b(X, \mathbb{R})$, the space of continuous, bounded functions, with $f_n \to f$ uniformly. Then f_n is uniformly bounded for all $n \in \mathbb{N}$.

Solution
In order to show that (f_n) is uniformly bounded, it suffices to show that there exists $M \in \mathbb{R}$ such that $|f_n(x)| \leq M$ for all $n \in N$ and for all $x \in X$. Since each f_n is bounded, there exists $M_n \in \mathbb{R}$ such that $f_n \leq M_n$ for each n. Since $f_n \to f$ uniformly, there exists $N > 0$ such that for all $n \geq N$, $|f_n(x) - f(x)| < 1$ for all $x \in X$. Then for all $n \geq N$, and all $x \in X$,

$$\begin{aligned} |f_n(x)| &= |f_n(x) - f(x) + f(x) - f_N(x) + f_N(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_N(x)| + |f_N(x)| \\ &< 1 + 1 + M_N = 2 + M_N. \end{aligned}$$

We've shown that we can bound all but finitely many of the f_n's. Let $M := \max\{M_1, M_2, \ldots, M_{N-1}, 2+M_N\}$. Then for all $n \in N$ and all $x \in X$, $|f_n(x)| \leq M$, and we conclude that (f_n) is uniformly bounded. $\square$

Problem 46

For all $n \in \mathbb{N}$, suppose $f_n : (X, d_X) \to (Y, d_Y)$, and suppose $f : (X d_X) \to (Y, d_Y)$. Let $g : (Y, d_Y) \to (Z, d_Z)$ be continuous.

(1) Show that if $f_n \to f$ pointwise, then $g \circ f_n \to g \circ f$ pointwise.
(2) Show that if $f_n \to f$ uniformly, and g is uniformly continuous, then $g \circ f_n \to g \circ f$ uniformly.

Solution
(1) Since g is continuous, then for all $\epsilon > 0$, and all $y_0 \in Y$, there exists $\delta > 0$ such that $d_Y(y, y_0) < \delta \implies d_Z(g(y), g(y_0)) < \epsilon$. Given $\delta > 0$, let $x \in X$. Since $f_n \to f$ pointwise, then there exists $N > 0$ such that for all $n > N$, $d_Y(f_n(x), f_n(x)) < \delta$. Since $f(x) \in Y$, $f_n(x) \in Y$ for all $x \in X$, then by continuity of g, this implies that $d_Z(g(f_n(x)), g(f(x))) < \epsilon$ for all $n > N$. Thus $g \circ f_n \to g \circ f$ pointwise. $\square$

(2) Let $\epsilon > 0$. Since g is uniformly continuous on Y, then there exists $\delta > 0$ such that $d_Y(y_1, y_2) < \delta \implies d_Z(g(y_1), g(y_2)) < \epsilon$ for all $y_1, y_2, \in Y$. Since $f_n \to f$ uniformly, given $\delta > 0$, there exists $N > 0$ such that for all $n > N$, $d_Y(f_n(x), f(x)) < \delta$ for all $x \in X$. Since $f(x) \in Y$ and $f_n(x) \in Y$ for all $x \in X$, then by uniform continuity of g, this implies that $d_Z(g(f_n(x)), g(f(x))) < \epsilon$ for all $n > N$ and all $x \in X$, and we conclude that $g \circ f_n \to g \circ f$ uniformly. $\square$

Problem 49

Suppose $(f_n), f, (g_n), g$ are in $C_b(X, \mathbb{R})$, the continuous bounded functions, $X \to \mathbb{R}$, with $f_n \to f$ uniformly and $g_n \to g$ uniformly.

(1) Show that $f_n + g_n \to f + g$ uniformly, where $(f_n + g_n)(x) = f_n(x) + g_n(x)$ and similarly for $f + g$.

(2) Show that $f_n g_n \to fg$ uniformly, where $(f_n g_n)(x) = f_n(x) f_n(x)$ and similarly for fg.

(3) Show that for every $r \in R$, $rf_n \to rf$ uniformly where $(rf_n)(x) = r \cdot f_n(x)$ and similarly for rf.

Solution

(1) Let $\epsilon > 0$. Since $f_n \to f$ uniformly, then there exists $N_1 > 0$ such that for all $n > N_1$, $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in X$. Since $g_n \to g$ uniformly, then there exists $N_2 > 0$ such that for all $n > N_2$, $|g_n(x) - g(x)| < \epsilon/2$ for all $x \in X$. Then taking $N := \max\{N_1, N_2\}$ we get that for all $n > N$,

$$\begin{aligned}
|(f_n(x) + g_n(x)) - (f(x) + g(x))| &= |(f_n(x) - f(x)) + (g_n(x) - g(x))| \\
&\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon,
\end{aligned}$$

for all $x \in X$, and we conclude that $f_n + g_n \to f + g$ uniformly. □

(2) In Problem 0, we showed that if (f_n) is a sequence of bounded functions that converge uniformly to f, then (f_n) is uniformly bounded. Then, it follows that $(f_n), (g_n)$ are uniformly bounded. That is, there exist $M, L \in \mathbb{R}$ such that $|f_n(x)| \leq M, |g_n(x)| \leq L$ for all $x \in X, n \in \mathbb{N}$. Since f is bounded, then there exists R such that $|f(x)| \leq R$ for all $x \in X$. Set $M' := \max\{M, L, R\}$. Let $\epsilon > 0$. Then there exists $K_1 > 0$ such that for all $n \geq K_1$, $|f_n(x) - f(x)| < \frac{\epsilon}{2M'+1}$, for all $x \in X$. Similarly, there exists $K_2 > 0$ such that for all $n \geq K_2$, $|g_n(x) - g(x)| < \frac{\epsilon}{2M'+1}$ for all $x \in X$. Set $K := \max\{K_1, K_2\}$. Then for all $n \geq K$,

$$\begin{aligned}
|f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - f(x)g_n(x) + f(x)g_n(x) - f(x)g(x)| \\
&\leq |f_n(x)g_n(x) - f(x)g_n(x)| + |f(x)g_n(x) - f(x)g(x)| \\
&\leq |g_n(x)||f_n(x) - f(x)| + |f(x)||g_n(x) - g(x)| \\
&< M' \cdot \frac{\epsilon}{2M'+1} + M' \cdot \frac{\epsilon}{2M'+1} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

for all $x \in X$, and we conclude that $f_n g_n \to fg$ uniformly. □

(3) Let $\epsilon > 0$. We consider cases. If $r = 0$, then $|rf_n(x) - rf(x)| = 0 < \epsilon$ for all $x \in X$, so uniform convergence of rf_n holds trivially for $r = 0$. If $r \neq 0$, then $f_n \to f$ uniformly means that there exists $N > 0$ such that for all $n > N$, $|f_n(x) - f(x)| < \epsilon/|r|$ for all $x \in X$. Then for all $n > N$,

$$|rf_n(x) - rf(x)| = |r||f_n(x) - f(x)| \leq |r| \cdot \frac{\epsilon}{|r|} = \epsilon,$$

for all $x \in X$, and we conclude that $rf_n \to rf$ uniformly. □

Problem 50

Suppose $(f_n), f$ are functions in $C(X, \mathbb{R})$, where X is compact. Prove that if $f_n \to 0$ pointwise and f_n is monotonically decreasing, then $f_n \to 0$ uniformly.

Solution

Let $\epsilon > 0$. Then consider the sets $U_n = \{x \in X | f_n(x) < \epsilon\}$. We first show that:

(a) U_n is open for all n.

If $x \in U_n$ then $f_n(x) < \epsilon$, and we see that U_n is the pre-image of the the interval $(-\infty, \epsilon)$, an open subset of $\mathbb{R}$. Then by continuity of f, we conclude that U_n is open.

(b) $U_n \subset U_{n+1}$ for all n.

We show that if $x \in U_n$, then $x \in U_{n+1}$. If $x \in U_n$, then $f(x) < \epsilon$. Since f_n is monotonically decreasing, then $f_{n+1}(x) \leq f_n(x) < \epsilon$, for all $x \in X$, so $x \in U_{n+1}$, and we conclude that $U_n \subset U_{n+1}$.

(c) $\bigcup_n U_n = X$.

Let $\epsilon > 0, x \in X$. Since $f_n \to 0$ pointwise, there exists $N > 0$ such that $|f_n(x) - 0| < \epsilon \implies |f_n(x)| < \epsilon \implies f_n(x) < \epsilon$. Then $x \in U_n \subset \bigcup_n U_n$, so $X \subseteq \bigcup_n U_n$. Conversely, suppose $x \in \bigcup_n U_n$. Then $x \in U_n$ for some n. Then by construction of $U_n = \{x \in X : f_n(x) < \epsilon\}$, $x \in X$, so $\bigcup_n U_n \subseteq X$. Inclusion in both directions gives us $X = \bigcup_n U_n$.

By (c), we see that the collection of U_n's form an open cover of X. By compactness of X, we can find a finite subcover of X from the U_n's, say $U_1, U_2, \ldots, U_N$. Since the U_n's are nested, then $U_1 \subseteq U_2 \subseteq \ldots \subseteq U_N \subseteq U_{N+1} \subseteq \ldots$, so $X = U_N$. By the nesting property again, for all $n > N$, $X = U_N \subseteq U_n \subseteq X$. This implies that there exists an $N > 0$ such that for all $n > N$, $f_n(x) < \epsilon \implies |f_n(x)| < \epsilon \implies |f_n(x) - 0| < \epsilon$, for all $x \in X$, and we conclude that $f_n \to 0$ uniformly. $\square$

Problem 55

Let X, Y be metric spaces. A function $f : X \to Y$ is called proper if $f^{-1}(K) \subset X$ is compact whenever $K \subset Y$ is compact.

(0) Show that if X is compact, then any continuous function $f : X \to Y$ is proper.

Solution
We use the following results:

(0.1) If $K \subset (X, d)$, and K is compact, then K is closed.

Proof
We will show that K^c is open. Let $x \in K^c$ For each $k \in K$, let $B_{r_k}(k)$ be the open ball of radius $r_i := \frac{1}{2}d(k, x) > 0$ centered about k. $K \subseteq \bigcup_{k \in K} B_{r_k}(k)$, so the collection open balls about points in K is an open cover of K. By compactness of K, we can extract a finite subcover. That is, there exist finitely many points, $k_1, k_2, \ldots, k_n$, such that $K \subseteq \bigcup_{i=1}^n B_{r_{k_i}}(k_i)$. Then set $r := \min\{r_{k_1}, r_{k_2}, \ldots, r_{k_n}\}$. We claim that $B_r(x) \cap K = \emptyset$. Suppose for contradiction that there exists $x_0 \in B_r(x) \cap K$. Then $x_0 \in K \subseteq \bigcup_{i=1}^n B_{r_i}(k_i)$. Without loss of generality, suppose $x_0 \in B_{r_1}(k_1)$. Then

$$d(x, k_1) \leq d(x, x_0), d(x_0, k_1) < r + r_1 \leq r_1 + r_1 = 2r_1 = d(x, k_1),$$

which is a contradiction, so $B_r(x) \cap K = \phi \implies B_r(x) \subset K^c$, so K^c open, and K is closed.

(0.2) If $K \subset X$, where K is closed, and X is compact, then K is compact.

Proof
Let $\{U_\alpha\}_{\alpha \in I}$ be a collection of open sets that form an open cover of K Since K^c is open, then the collection of open sets: $\{U_\alpha\}_{\alpha \in I}$ together with $\{K^c\}$ form an open cover of X and hence of K. By compactness of X, we can find a finite subcover of X from this open cover, $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}, K^c\}$. Omitting K^c from this open cover, we get $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$, a finite subcover of K, so K is compact.

(0.3) $f : X \to Y$ is continuous if and only if for every closed set $K \subset Y$, $f^{-1}(K)$ is closed in X.

Proof
Suppose $f : X \to Y$ is continuous. Let $K \subset Y$ be closed. Then $Y \setminus K$ is open in Y. By continuity of f, $f^{-1}(Y \setminus K)$ is open in X, so $X \setminus f^{-1}(Y \setminus K)$ is closed in X, but $X \setminus f^{-1}(Y \setminus K) = f^{-1}(K)$, so $f^{-1}(K)$ is closed in X. Conversely, suppose for every closed set $K \subset Y$, $f^{-1}(K)$ is closed in X. Let $V \subset Y$ be open. Then $Y \setminus V$ is closed in Y, and $f^{-1}(Y \setminus V)$ is closed in X, so $X \setminus f^{-1}(Y \setminus V)$ is open in X. However, $X \setminus f^{-1}(Y \setminus V) = f^{-1}(V)$, so $f^{-1}(V)$ is open, and we conclude that f is continuous.

To prove the statement in (0), let K be a compact subset of Y. Then $K \subset Y$ is closed. By continuity of f, $f^{-1}(K) \subset X$ is closed. Since X is compact, this means that $f^{-1}(K)$ closed $\implies$ $f^{-1}(K)$ compact, so $f : X \to Y$ is proper. $\square$

Now suppose $f : X \to Y$ is a continuous proper map, with X not necessarily compact.

(1) Use f to construct a function $f^* : C_0(Y) \to C_0(X)$. Show that f^* is well defined.

Solution
Let $g \in C_0(Y)$. Then $g : Y \to \mathbb{R}$, and g vanishes at infinity. That is, for every $\epsilon > 0$, there is a compact subset $H \subset Y$ such that $|g(y)| < \epsilon$ for all $y \in H^c$. Let $f^* := g \circ f : C_0(Y) \to C_0(X)$. We will show that it takes functions that vanish at infinity to functions that vanish at infinity.

Let $\epsilon > 0$. Since $g \in C_0(Y)$, there exists a compact subset $H \subset Y$, such that for all $y \in H^c$, $|g(y)| < \epsilon$. Since f is proper, $f^{-1}(H) =: K$ is a compact subset of X. Let $x \in K^c$. Then $f(x) \in H^c$. $f(x) \in Y$ for all $x \in X$, and since $g \in C_0(Y)$, $|g(f((x))| < \epsilon$, thus vanishing at infinity. Moreover, since g and f are continuous functions, $f^*(g)$ is continuous since the composition of continuous functions is continuous. We conclude that f^* is well-defined. $\square$

(2) Show that f^* is continuous with respect to the L^∞-metrics on $C_0(X)$ and $C_0(Y)$.

Solution To show that f^* is continuous with respect to the L^∞ metrics on $C_0(X)$ and $C_0(Y)$, it suffices to show that it takes uniformly convergent sequences to uniformly convergent sequences. Suppose $g_n \to g$ uniformly. Then for all $\epsilon > 0$ there exists $N > 0$ such that for all $n > N$, $|g_n(y) - g(y)| < \epsilon$ for all $y \in Y$. Since $f(x) \in Y$, for all $x \in X$, then $|g_n(f(x)) - g(f(x))| < \epsilon$ for all $x \in X$, so $g_n \circ f \to g \circ f$ uniformly, and we conclude that f^* is continuous with respect to the L^∞ metrics on $C_0(X)$ and $C_0(Y)$. $\square$