MATH 131B: Homework #6

Professor Dave Penneys
Assignment: 59, 60, 61, 62

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Problem 59

Consider the metric space $C([0,1],\mathbb{R})$, where $[0,1]$ and $\mathbb{R}$ have the absolute value metric. Show that $\{f \in C([0,1],\mathbb{R}) : \|f\|_\infty \leq 1\}$ is not compact.

Solution
Let $B = \{f \in C([0,1],\mathbb{R}) : \|f\|_\infty \leq 1\}$. We will show that B is not sequentially compact, hence not compact.

Let $f_n : [0,1] \to \mathbb{R}$ be a continuous function such that $f_n(x) = x^n$. It is clear that $f_n \in C([0,1],\mathbb{R})$ for all $n \in N$. Since $0 \leq x \leq 1$, then $x^n \leq 1$ for all $n \in N$, so $\|f_n\|_\infty \leq 1$, and we see that $f_n \in B = \{f \in C([0,1],\mathbb{R}) : \|f\|_\infty \leq 1\}$. We've shown in class that $f_n(x)$ converges pointwise to to the function

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Suppose for contradiction that B is compact. Then every sequence in B has a subsequence that converges uniformly in B. In particular, for a subsequence there exists a subsequence (f_{n_k}) of (f_n) that converges uniformly, and hence pointwise, in B. Since $f_n \to f$ pointwise, it follows that every subsequence of (f_n) converges to f pointwise. However, f as defined above, is not a continuous function, i.e., $f \notin B$. This means that there does not exist a subsequence of (f_n) that converges pointwise in B. Consequently, there does not exist a subsequence of (f_n) that converges uniformly in B. We conclude that $B = \{f \in C([0,1],\mathbb{R}) : \|f\|_\infty \leq 1\}$ is not sequentially compact, and hence not compact. $\square$

Problem 60

Let $C^1([0,1],\mathbb{R})$ be the subset of $C([0,1],\mathbb{R})$ consisting of all differentiable functions with continuous derivatives on $[0,1]$.

(a) For $f \in C([0,1],\mathbb{R})$, define $Tf : [0,1] \to \mathbb{R}$ by $Tf(x) = \int_0^x f(t)dt$. Show that $Tf \in C([0,1],\mathbb{R})$. Then show that $Tf \in C^1([0,1],\mathbb{R})$.

Solution
Note that since f is continuous on $[0,1]$, a compact subset of $\mathbb{R}$, then by the extreme value theorem, f attains its minimum and maximum on $[0,1]$. Therefore, there exists $M \geq 0$ such that $|f(x| \leq M$ for all $x \in [0,1]$. Let $\epsilon > 0$. Let $x, y \in [0,1]$ such that $y < x$. Pick $\delta = \frac{\epsilon}{M+1}$Then $|x-y| < \delta$ implies

$$|Tf(x) - Tf(y)| = \left| \int_0^x f(t)dt - \int_0^y f(t)dt \right| = \left| \int_y^x f(t)dt \right| \leq \int_y^x |f(t)|dt$$

Since $|f(x| \leq M$ for all $x \in [0,1]$, then we can write

$$\left| \int_y^x f(t)dt \right| \leq \int_y^x |f(t)|dt \leq \int_y^x Mdt = M|x-y| < M\delta = M \cdot \frac{\epsilon}{M+1} < \epsilon.$$

Thus, Tf is continuous, so $Tf \in C([0,1],\mathbb{R})$.

Next, we show that $Tf \in C^1([0,1],\mathbb{R})$. It suffices to show that Tf is differentiable, so $Tf \in C^1([0,1],\mathbb{R})$, with continuous derivatives on $[0,1]$. Since f is continuous on $[0,1]$ and $Tf(x) = \int_0^x f(t)dt$, then by the fundamental theorem of calculus, it follows that $(Tf)'(x) = f(x)$. Therefore, Tf is differentiable, with continuous derivatives on $[0,1]$, as $f \in C([0,1],\mathbb{R})$. $\square$

(b) Show that $Y = \{Tf : f \in C^1([0,1],\mathbb{R}) \text{ and } \|f\|_\infty \leq M\}$ is equicontinuous. Deduce that every sequence in Y has a convergent subsequence whose limit is in $\overline{Y}$.

Solution
Let $\epsilon > 0$. Pick $\delta = \frac{\epsilon}{M+1}$. Let $x, y \in [0,1]$ such that $y < x$ and $Tf \in Y$. Then $|x-y| < \delta$ implies

$$|Tf(x) - Tf(y)| = \left| \int_0^x f(t)dt - \int_0^y f(t)dt \right| = \left| \int_y^x f(t)dt \right| \leq \int_y^x |f(t)|dt$$

Since $\|f\|_\infty \leq M$, then

$$\int_y^x |f(t)|dt \leq \int_y^x Mdt = M|x-y| < M\delta = M \cdot \frac{\epsilon}{M+1} < \epsilon.$$

Thus, we conclude that Y is equicontinuous.

In order to deduce the claim that every sequence in Y has a convergent subsequence in $\overline{Y}$, we first show that $\overline{Y}$ is equicontinuous. Let $Tf \in \overline{Y}$. Since $\overline{Y}$ is closed, then there exists (Tf_n) such that $Tf_n \to Tf$. Given $\epsilon > 0$, let $x, y \in [0,1]$. Then there exists N such that for all $n > N$, $|Tf_n(x) - Tf(x)| < \epsilon/3$. Note that since f is continuous on a compact set, f is uniformly continuous, so for all $n > N$, we also have that $|Tf_n(y) - Tf(y)| < \epsilon/3$. By equicontinuity of Y, there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|f_n(x), f_n(y)| < \epsilon/3$. Then, using the triangle inequality, we have for all $n > N$ and $|x-y| < \delta$,

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,$$

so we've shown that $\overline{Y}$ is equicontinuous. Since we already know that $\overline{Y}$ is closed and bounded, we apply Arzela-Ascoli Theorem, and conclude that $\overline{Y}$ is compact, so every sequence in Y has a convergent subsequence in $\overline{Y}$. $\square$

(c) Show that $F = \{f \in C^1([0,1], \mathbb{R}) : \|f'\|_\infty \leq M\}$ is equicontinuous. Deduce that every sequence in

$$Z = \{f \in C^1([0,1], \mathbb{R}) : \|f\|_\infty + \|f'\|_\infty \leq M\}$$

has a convergent subsequence whose limit is in $\overline{Z}$.

Solution

Let $\epsilon > 0$. Then pick $\delta := \frac{\epsilon}{M+1}$. Then let $x, y \in [0,1]$, such that $x \neq y$ and $y < x$, and $f \in F$. Since f is continuous on $[0,1]$ and differentiable on $(0,1)$, then by the mean value theorem, there exists $c \in (y, x)$, such that $f'(c) = \frac{f(x)-f(y)}{x-y} \implies f'(c)(x-y) = f(x) - f(y)$. Since $\|f'\|_\infty \leq M$, then $|f(x) - f(y)| \leq M|x-y|$. Then $|x - y| < \delta$ implies

$$|f(x) - f(y)| \leq M|x-y| < M\delta = M \cdot \frac{\epsilon}{M+1} < \epsilon.$$

Thus, F is equicontinuous.

Z bounded by construction, Z is equicontinuous. Then by the Arzela-Ascoli Theorem, since Z is equicontinuous and bounded, then its closure, $\overline{Z}$, is compact, so every sequence in Z has a convergent subsequence in $\overline{Z}$. $\square$

Problem 61

Define $f_n : [1,2] \to \mathbb{R}$ by $f_n(x) = \frac{x}{(1+x)^n}$.

(a) Directly prove that $\sum_{n=0}^{\infty} f_n$ converges pointwise.

Solution
Let $x \in [1,2]$. Then

$$\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x}{(1+x)^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{(1+x)^n} \cdot \frac{x}{1+x} = \frac{x}{1+x} \cdot \sum_{n=0}^{\infty} \frac{1}{(1+x)^n}.$$

Since $x \in [1,2]$, $\left|\frac{1}{1+x}\right| < 1$, so $\sum_n \frac{1}{(1+x)^n}$ is a convergent geometric series, so we can further evaluate the equality above:

$$\frac{x}{1+x} \cdot \sum_{n=0}^{\infty} \frac{1}{(1+x)^n} = \frac{x}{1+x} \cdot \frac{1}{1-\frac{1}{1+x}} = \frac{x}{1+x} \cdot \frac{1+x}{x} = 1,$$

and we conclude that $\sum_{n=0}^{\infty} f_n$ converges pointwise. $\square$

(b) Show that $\|f_n\|_\infty \leq \frac{2}{3}\left(\frac{1}{2}\right)^n$ for all $n \geq 0$.

Solution
It suffices to show that $|f_n(x)| \leq \frac{2}{3}\left(\frac{1}{2}\right)^n$ for all $n \geq 0$, for $x \in [1,2]$. Since $f_n(x) = \frac{x}{(1+x)^{n+1}} = \frac{1}{(1+x)^n} \cdot \frac{x}{1+x}$,

$$\frac{x}{1+x} \leq \frac{2}{3} \tag{1}$$

for $x \in [1,2]$, so we need only show that $\frac{1}{(1+x)^n} \leq \left(\frac{1}{2}\right)^n$ for all $n \geq 0$.
Base Case: $n = 0$. Then LHS $= \frac{1}{(1+x)^0} = 1 \leq 1 = \left(\frac{1}{2}\right)^0 =$ RHS, so the base case holds.
Inductive Step: Suppose the inequality holds for $n = k$. We then show it is true for $n = k+1$.

$$\text{LHS} = \frac{1}{(1+x)^{k+1}} = \frac{1}{(1+x)^k} \cdot \frac{1}{1+x}$$

By the inductive hypothesis, we can further evaluate:

$$\frac{1}{(1+x)^k} \cdot \frac{1}{1+x} \leq \left(\frac{1}{2}\right)^k \cdot \frac{1}{1+x} \leq \left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{k+1},$$

so the inequality holds for $n = k+1$ as well, and we have shown by induction that for $x \in [1,2]$, $\frac{1}{(1+x)^n} \leq \left(\frac{1}{2}\right)^n$ holds for all $n \geq 0$. Combining this with the inequality (1) above, we see that

$$\frac{1}{(1+x)^n} \cdot \frac{x}{1+x} \leq \left(\frac{1}{2}\right)^n \frac{2}{3}$$

for all $n \geq 0$. Thus, $\|f_n\|_\infty \leq \frac{2}{3}\left(\frac{1}{2}\right)^n$ for all $n \geq 0$ $\square$

(c) Show that $\sum_n f_n$ is uniformly convergent on $[1,2]$.

Solution
By part (b), we see that $\|f_n\|_\infty \leq \frac{2}{3}\left(\frac{1}{2}\right)^n$ for all $n \geq 0$. Thus, $f_n \in L^\infty([1,2], \mathbb{R})$. Moreover,

$$\sum_{n=0}^{\infty} \|f_n\|_\infty \leq \sum_{n=0}^{\infty} \frac{2}{3}\left(\frac{1}{2}\right)^n = \frac{2}{3} \cdot \frac{1}{1-1/2} = \frac{2}{3} \cdot 2 = \frac{4}{3} < \infty.$$

Thus, $\sum_{n=0}^{\infty} \|f_n\|_\infty < \infty$, and by the Weierstrass M-test, $\sum_n f_n$ is converges uniformly on $[1,2]$. $\square$

Problem 62

Suppose $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ has a radius of convergence $R > 0$, and suppose $0 < r < R$.

(1) Prove that the partial sums converge uniformly to f on $[x_0 - r, x_0 + r]$.

Solution
Let $0 < r < R$. We want to show that $\sum_n a_n(x-x_0)^n$ converges uniformly on $[x_0 - r, x_0 + r]$. Let $x \in [x_0 - r, x_0 + r]$. Then $|x - x_0| \leq r$. This then implies that $|a_n(x-x_0)^n| \leq |a_n|r^n \implies \sum_n |a_n(x-x_0)^n| \leq \sum_n |a_n|r^n$. note that $\sum_n |a_n|r^n$ converges because $r \in (x_0 - R, x_0 + R)$. Since this convergence is absolute, we can use the Weierstrass M-test and conclude that the partial sums $\sum_n a_n(x-x_0)^n$ converge uniformly to f on $[x_0 - r, x_0 + r]$. □

(2) Prove that f is differentiable on $(x_0 - R, x_0 + R)$ with derivative $f'(x) = \sum_{n\geq 1} na_n(x-x_0)^{n-1}$.

Solution
By defining $g(x) = f(x + x_0) = \sum_{n=0}^{\infty} a_n x^n$, we assume without loss of generality that $x_0 = 0$ so that the power series is centered about $x_0 = 0$. Then it suffices to show that g is differentiable on $(x_0 - R, x_0 + R)$ and that $g'(x) = \sum_{n\geq 1}^{\infty} na_n x^{n-1}$. Let $x \in (-R, R)$. Then $|x| < R$. Then we can find $x_0 \in \mathbb{R}$ such that $|x| < x < R$, and $r := \frac{|x|}{x_0} < 1$. Consider the term:

$$
\begin{aligned}
\left|na_n x^{n-1}\right| &= \left|na_n \cdot x^{n-1} \cdot \frac{x_0^n}{x_0^n}\right| \\
&= \left|na_n x_0^n \cdot \frac{x^{n-1}}{x_0^{n-1} x_0}\right| \\
&= \left|\frac{na_n}{x_0} \cdot x_0^n \cdot \frac{x^{n-1}}{x_0^{n-1}}\right| \\
&= \left|\frac{a_n x_0^n}{x_0} nr^{n-1}\right|
\end{aligned}
$$

Note that since $\sum_n a_n x_0^n$ converges because $x_0 \in (-R, R)$, then $\lim_{\to\infty} a_n x_0^n \to 0$, so there exists $M > 0$ such that $|a_n \cdot x_0^n| \leq M$. Then we can continue evaluating the above equality,

$$
\left|\frac{a_n x_0^n}{x_0} nr^{n-1}\right| \leq \left|\frac{M}{x_0} nr^{n-1}\right|
$$

and by the ratio test, we see that

$$
\lim_{n\to\infty} \left|\frac{\frac{M}{x_0}(n+1)r^n}{\frac{M}{x_0} nr^{n-1}}\right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right) r = r < 1,
$$

so $\sum_n \left|\frac{M}{x_0} nr^{n-1}\right|$ converges. This means that $\sum_n \frac{M}{|x_0|} nr^{n-1}$ converges absolutely, so by the Weierstrass M-test, we conclude that $\sum_n na_n x^{n-1}$ converges uniformly on $(-R, R)$, the same radius of convergence as $\sum_{n=0}^{\infty} a_n x^n$. Using what we've shown in this problem in conjunction with the corollary shown in class, we can conclude that $(\sum_n f_n)' = \sum_n f_n'$, so $g'(x) = \sum_{n\geq 1}^{\infty} nax^{n-1}$. It follows that the original function f is differentiable on $(x_0 - R, x_0 + R)$ with derivative $f'(x) = \sum_{n\geq 1} na_n(x-x_0)^{n-1}$ □

(3) Calculate a_n in terms of n, f, x_0.

Solution

We use the result from (b) to find a formula for $f^{(n)}(x)$. We know that

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$
$$f'(x) = \sum_{n \geq 1} n a_n (x - x_0)^{n-1}$$

We can continue to take the derivative of f to find that the k-th derivative of f is

$$f^{(k)}(x) = \sum_{n \geq k} \frac{n!}{(n-k)!} a_n (x - x_0)^{n-k}$$

If we then evaluate this at x_0, we see that

$$f^{(k)}(x_0) = \sum_{n \geq k} \frac{n!}{(n-k)!} a_n (0)^{n-k} = \frac{k!}{0!} a_n \cdot 0^0 + 0 + 0 + \cdots = k! \cdot a_n$$

Then, we have $a_n = \frac{f^{(n)}(x_0)}{n!}$. $\square$