MATH 131AH: Homework #5

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Problem 1

If $\lim_{n\to\infty} x_n = L$ and $|x_n - c| < 1$ for all $n \in N$, does it follow that $|L - c| < 1$?

Solution
Since $\lim_{n\to\infty} x_n = L$, then given $\epsilon > 0$, there exists N such that $n \geq N$ implies $|x_n - L| < \epsilon$. Consider

$$|L - c| = |L - x_n + x_n - c| \leq |x_n - L| + |x_n - c| < \epsilon + 1$$

Therefore, we have $|L - c| \leq 1$, and we've shown that the inequality need not be strict. ■

Problem 2

If $x_n \geq 0$ and $\lim_{n\to\infty} x_n^m = c^m$ for some $m \in \mathbb{N}$ and $c \geq 0$, prove $\lim_{n\to\infty} x_n = c$. This completes our set of limit theorems for algebraic functions.

Solution
We can use the power rule for limits, i.e., the limit of a positive integral power is the power of the limit. This property of limits can be shown inductively using the property of limits that states the limit of products is the product of the limits. Using this, we can then evaluate $\lim_{n\to\infty} x_n^m$ and write this as

$$\lim_{n\to\infty} x_n^m = (\lim_{n\to\infty} x_n)^m = c^m$$

Taking the mth root on both sides, we see that

$$(\lim_{n\to\infty} x_n)^{m \cdot 1/m} = (c^m)^{1/m} = c.$$

Thus, we've shown that $\lim_{n\to\infty} x_n = c$. ■.

Problem 3

Given the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ make a new sequence $\{c_n\}_{n=1}^{\infty}$ by taking the first term from $\{b_n\}_{n=1}^{\infty}$, then the second term $\{a_n\}_{n=1}^{\infty}$, and so on. In other words $c_n = a_{(n+1)/2}$ if n is odd, and $c_n = b_{n/2}$ if n is even. Prove that $\{c_n\}_{n=1}^{\infty}$ is convergent if and only if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are both convergent and have the same limit.

Solution

Suppose that $\{c_n\}_{n=1}^{\infty}$ converges to L. Since $\{c_n\}$ converges, it is also bounded, and by theorem 3.6b, it contains a convergent subsequence. Since $\{a_n\}$ and $\{b_n\}$ are both subsequences of $\{c_n\}$, it suffices to show that all such subsequences converge to L.

Claim: $\{c_n\}$ converges to L if and only if every subsequence of $\{c_n\}$ converges to L. To prove this, first suppose that $\{c_n\}$ converges to L. Then given $\epsilon > 0$, there exists N such that $n \geq N$ implies $|c_n - L| < \epsilon$. Let $\{c_{nk}\}$ be a subsequence of $\{c_n\}$. Note that $n_k \geq k$ for all k. Then for $k \geq N$, we also have $n_k \geq N$, which implies that $|c_{n_k} - L| < \epsilon$, so $\{c_{n_k}\}$ Thus we've shown that an arbitrary subsequence of $\{c_n\}$ converges to L, so all subsequences of $\{c_n\}$ converge to L. Conversely, suppose that all subsequences of $\{c_n\}$ converge to L. We choose $\{c_n\}$ itself to be a subsequence. Then by assumption, given $\epsilon > 0$, there exists N such that $n \geq N$ implies $|c_n - L| < \epsilon$, which is exactly the definition for $\{c_n\}$ converging to L. Thus, our claim is proven.

We've shown that $\{a_n\}$ and $\{b_n\}$ are both convergent and both converge to L.

Conversely, suppose that $\{a_n\}$ and $\{b_n\}$ are both convergent and have the same limit, that is, they both converge to L. Since the even-indexed elements of $\{c_n\}$ consist of elements in the sequence $\{a_n\}$ and the odd-indexed elements of $\{c_n\}$ consist of elements in the sequence $\{b_n\}$, we denote these subsequences c_{2n} and c_{2n+1}, respectively. By assumption, c_{2n} converges, so given $\epsilon > 0$, there exists N_1 such that $n \geq N_1$ implies $|c_{2n} - L| < \epsilon$. Similarly c_{2n+1} converges, so for $\epsilon > 0$, there exists N_2 such that $n \geq N_2$ implies $|c_{2n+1} - L| < \epsilon$. Take $N = \max\{N_1, N_2\}$. Then $n \geq N$ implies that $|c_n - L| < \epsilon$ so $\{c_n\}$ converges to L, and we are done. ■

Problem 4

Define

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

Show that $\lim_{n\to\infty} x_n$ exists.

Solution
Consider:

$$\frac{1}{2} = \frac{n}{2n} \leq \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \leq \frac{n}{n+1}$$

It is easily shown that $\frac{n}{n+1}$ converges to 1 as n gets large, so we've shown that we can bound $\{x_n\}$. Now consider

$$x_{n+1} - x_n = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1} = \frac{n}{(2n+2)(2n+1)(n+1)} > 0.$$

Thus, $\{x_n\}$ is monotone increasing. Since it is also bounded, it converges. ■

Problem 5

Let $0 < a_1 < b_1$ and define for $n \in \mathbb{N}$

$$a_{n+1} = \sqrt{a_n b_n} \text{ and } b_{n+1} = \frac{a_n + b_n}{2}.$$

Show that $a_n < b_n$ for all n by induction. What are the relations between a_n and a_{n+1} and between b_n and b_{n+1}? Prove that $\lim_n \to \infty a_n = \lim_{n\to\infty} b_n$.

Solution
We make a stronger claim:

$$b_n > b_{n+1} > a_{n+1} > a_n \tag{1}$$

for all $n \in \mathbb{N}$. If we can show this inequality is true, then we will have shown that $a_n < b_n$ for all n by transitivity. We do this by induction.

Base Case: $n = 1$. Then we want to show that

$$b_1 > b_2 > a_2 > a_1. \tag{2}$$

Since $b_1 > a_1$, then $b_1 + b_1 > a_1 + b_1 \Rightarrow 2b_1 > a1 + b_1 \Rightarrow b_1 > \frac{a_1+b_1}{2} = b2$, so $b_1 > b_2$. Since squares are non-negative and we're given that $b_1 > a_1$, we can write

$$\begin{aligned}
&(\sqrt{b_1} - \sqrt{a_1})^2 > 0 \\
&\Rightarrow a_1 - 2\sqrt{a_1}\sqrt{b_1} + b_1 > 0 \\
&\Rightarrow \frac{a_1 + b_1}{2} > \sqrt{a_1 b_1}
\end{aligned}$$

Since $b_2 = \frac{a_1+b_1}{2}$ and $a_2 = \sqrt{a_1 b_1}$, then the last inequality implies that $b_2 > a_2$. Again, since $b_1 > a_1$, then $a_1 b_1 > a_1^2$. Both sides of the inequality are positive, so we take the square root, $a_1 < \sqrt{a_1 b_1} = a_2$, so $a_2 > a_1$. Putting all of the inequalities together, we see that $b_1 > b_2 > a_2 > a_1$, thus satisfying the inequality in (2) and proving the base case.

Inductive Step: Suppose that the inequality shown in (1) holds for n. Then we must show that it holds for $n + 1$, that is:

$$b_{n+1} > b_{n+2} > a_{n+2} > a_{n+1}. \tag{3}$$

From the inductive hypothesis, we see that

$$b_{n+1} > a_{n+1} \Rightarrow b_{n+1} + b_{n+1} > a_{n+1} + b_{n+1} \Rightarrow 2b_{n+1} > a_{n+1} + b_{n+1} \Rightarrow b_{n+1} > \frac{a_{n+1} + b_{n+1}}{2} = b_{n+2} \tag{4}$$

Since $b_{n+1} > a_{n+1}$, then

$$b_{n+1} a_{n+1} > a_{n+1}^2$$

Taking the square root on both sides,

$$a_{n+2} = \sqrt{a_{n+1} b_{n+1}} > a_{n+1} \tag{5}$$

Consider $(\sqrt{b_{n+1}} - \sqrt{a_{b+1}})^2 > 0$. Then

$$\begin{aligned}
&b_{n+1} - 2\sqrt{a_{n+1} b_{n+1}} + a_{n+1} > 0 \\
&\Rightarrow b_{n+1} + a_{n+1} > 2\sqrt{a_{n+1} b_{n+1}} \\
&\Rightarrow \frac{a_{n+1} + b_{n+1}}{2} > \sqrt{a_{n+1} b_{n+1}} \\
&\Rightarrow \frac{a_{n+1} + b_{n+1}}{2} = b_{n+2} > a_{n+2} = \sqrt{a_{n+1} b_{n+1}}
\end{aligned} \tag{6}$$

Problem 5 continued on next page. . .

Combining the inequalities in (4), (5), and (6), we see that $b_{n+1} > b_{n+2} > a_{n+2} > a_{n+1}$, which satisfies the inequality (3). By induction, we've shown that

$$b_n > b_{n+1} > a_{n+1} > a_n$$

for all n, so it follows directly that $b_n > a_n$ for all n.

It remains to show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. We first show the existence of such limits. Both sequences $\{a_n\}, \{b_n\}$ are bounded above by b_1 and bounded below by a_1. $\{a_n\}$ is monotone increasing, and $\{b_n\}$ is monotone decreasing. Both sequences being bounded and monotone imply convergence. Thus, let $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then, we take the limit of b_{n+1}, (a_{n+1} works too),

$$\begin{aligned} &\lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty} \frac{a_{n+1} + b_{n+1}}{2} = \frac{a+b}{2} = b \\ &\Rightarrow 2b = a + b \Rightarrow b = a. \end{aligned}$$

Thus, the limits of the two sequences are equal, and we are done. ■

Problem 6

Suppose $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n < \infty$. Show

$$\sum_{n=1}^{\infty} a_n^p < \infty, \text{ when } p > 1. \tag{7}$$

Solution

We're given that $\sum_{n=1}^{\infty} a_n < \infty$, so $\lim_{n\to\infty} = 0$. Then there exists N such that $n \geq N$ implies $|a_n - 0| = |a_n| < 1$. Note that $0 \leq a \leq 1$, then $a^p \leq a$ for $p > 1$. Then we can rewrite (7) as

$$\sum_{n=1}^{\infty} a_n^p = \sum_{n=1}^{N} a_n^p + \sum_{n=N+1}^{\infty} a_n^p < \infty$$

The inequality holds because the first term is a finite sum, so it is bounded, and we have shown that the terms a_n^p converge when $n \geq N$. Since both terms are bounded, the sum is bounded as well. The sum being bounded, together with the sum in (7) being monotone increasing, implies that $\sum_{n=1}^{\infty} a_n^p$ converges when $p > 1$. ■

Problem 7

Suppose $a_n \geq 0$ for all $n \in \mathbb{N}$. Show $\sum_{n=1}^{\infty} a_n^2 < \infty$ implies $\sum_{n=1}^{\infty} a_n/n < \infty$.

Solution

By the Schwarz inequality, we can write

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \leq \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2} \cdot \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2} < \infty.$$

The inequality is bounded because $\sum_{n=1}^{\infty} a_n^2$ is bounded by the hypothesis, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because it is a $p-$series with $p = 2$, hence bounded. The product of two bounded series is bounded, so we have shown that

$$\sum_{n=1}^{\infty} a_n/n < \infty.$$

■