MATH 131AH: Homework #7

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Problem 44

If f is a real-valued continuous function on a metric space, prove that $Z = \{p \in \chi : f(p) = 0\}$ is closed. More generally, for a continuous function on (χ, d_χ) with values in (Y, d_Y), the set $Q_q = \{p \in \chi : f(p) = q \in Y\}$ is closed, but you only need to prove the first statement since the proofs are the same.

Solution

The set consisting of all $f(Z)$ is exactly $\{0\}$, which is a closed set. Since f is continuous on χ, then for all closed sets in the range of f, the inverse image of those sets are closed. Thus, for $f(Z) = \{0\}$ closed in the range, we also have Z closed in χ. ■

Problem 45

If $f : \mathbb{R} \to \mathbb{R}$ continuous and $f(I)$ is open for every open interval I, prove that f is monotonic: either $x < y \Leftarrow f(x) < f(y)$ for all $x, y \in \mathbb{R}$ or $x < y \Leftarrow f(x) > f(y)$ for all $x, y \in \mathbb{R}$.

Solution

Suppose that f is not monotonic. Without loss of generality, we can find some open interval I $= (x, z)$ with $x < y < z$ with $f(x) < f(y) > f(z)$. If we consider the closed and bounded set $[x, z]$, then by continuity over a closed and bounded set in $\mathbb{R}$, we can find a maximum in $[x, z]$. Since $f(x) < f(y) > f(z)$, we can further say that f achieves its maximum, call it M in the open interval (x, z). However, this implies that for any $\delta > 0$, $M + \delta \notin f(I) = f((x, z))$, contradicting openness of $f(I)$. Hence, f must be monotonic. ■

Problem 46

There are lots strange examples of functions of two variables which are almost continuous. Here's one:

$$f(0,0) = 0 \text{ and } f(x,y) = \frac{xy^2}{x^2+y^6} \text{ when } (x,y) \neq (0,0).$$

Show that f is not bounded on any disk $x^2+y^2 < \delta^2$, but it is continuous on every straight the line through the $(0,0)$.

Solution

To show that f is not bounded for any disk $x^2+y^2 < \delta^2$, take $x = y^3$, and evaluating the limit of f as $y \to 0$, we see that

$$\lim_{y\to 0} f(y^3, y) = \lim_{y\to 0} \frac{y^5}{2y^6} = \lim_{y\to 0} \frac{1}{2y} = \infty$$

so for any neighborhood disk, we have that f is unbounded.

For continuity on every straight line through the origin, first consider the y-axis: $x = 0$. Evaluating the limit of f at $x = 0$, we see that

$$\lim_{y\to 0} f(0,y) = \frac{0}{y^6} = 0$$

where $y \neq 0$, which shows continuity at $x = 0$.

Consider the line $y = cx$ where $c \in \mathbb{R}$, that passes through the origin. Then we can write the function as

$$f(x,cx) = \frac{c^2x^3}{x^2+c^6x^6} = \frac{c^2x}{1+c^6x^4}$$

Then, taking the limit as $x \to 0$, we see that

$$\lim_{x\to 0} \frac{c^2x}{1+c^6x^4} = \frac{0}{1} = 0$$

so it is clear that for straight lines through the origin, f is continuous.

Problem 47

Suppose that f is a continuous function that maps the interval $[0, 1]$ into, but possibly not onto, itself. Prove that there is an $x \in [0, 1]$ such that $f(x) = x$.

Solution
We first show that the identity function $I_X : (X, d) \to (X, d)$, where $I_X(x) = x$ for all $x \in X$, is continuous on X. Given $\epsilon > 0$, take $\delta = \epsilon$. Then for $x, y \in X$, $d(x, y) < \delta \Rightarrow d(I_X(x), I_X(y)) = d(x, y) < \delta = \epsilon$, so the identity mapping is continuous.

Now consider the function g, where $g(x) = f(x) - x$, where x is essentially the identity function. Note that by theorem 4.9, g is continuous on $[0, 1]$ since it is the sum of two functions continuous on $[0, 1]$. Then

$$\begin{aligned} g(0) &= f(0) - 0 = f(0) \geq 0 \\ g(1) &= f(1) - 1 \leq 0 \end{aligned}$$

Since g is continuous on the connected set $[0, 1]$, and $g(1) \leq 0 \leq g(0)$, then by theorem 4.23 there exists some point $x \in [0, 1]$ such that $g(x) = 0$. Then we get $g(x) = 0 = f(x) - x$, which implies that $f(x) = x$ for some $x \in [0, 1]$, and we are done. ■

Problem 48

Suppose that the derivatives of f and g exist at $t = x \in \mathbb{R}$, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}. \tag{1}$$

Solution
Consider

$$\frac{f(t)}{g(t)} = \frac{f(t) - 0}{g(t) - 0} = \frac{f(t) - f(x)}{g(t) - g(x)} = \frac{f(t) - f(x)}{g(t) - g(x)} \cdot \frac{\frac{1}{t-x}}{\frac{1}{t-x}} = \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}}$$

Then we can take the limit of this quantity,

$$\lim_{t \to x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} = \frac{f'(x)}{g'(x)}$$

which gives us the result in (1). ■

Problem 49

Suppose that f is a real-valued function on the line with derivative $f'(x)$ that satisfies $\lim_{x\to\infty} f'(x) = 0$ that means every $\epsilon > 0$ there is an N such that $|f'(x)| < \epsilon$ when $x > N$. Prove that $\lim_{x\to\infty}[f(x+1) - f(x)] = 0$. Does $\lim_{x\to\infty}[f(x+\sqrt{x}) - f(x)]$ have to be zero?

Solution

Since $\lim_{x\to\infty} f'(x) = 0$, then given $\epsilon > 0$, we have $|f'(x)| < \epsilon$ when $x > x_N$. Since f differentiable for every $x > 0$, by the mean value theorem, we can find an $x_0 \in (x, x+1)$ such that

$$f(x+1) - f(x) = f'(x_0)(x+1-x) = f'(x_0)$$

This gives us $|f(x+1) - f(x)| < \epsilon$, so $\lim_{x\to\infty}[f(x+1) - f(x)] = 0$.

Consider $\lim_{x\to\infty}[f(x+\sqrt{x}) - f(x)]$, where $f(x) = \sqrt{|x|}$. This satisfies the hypothesis of the question, as $\lim_{x\to\infty} f'(x) = 0$. Then

$$\begin{aligned}
\lim_{x\to\infty}[f(x+\sqrt{x}) - f(x)] &= \lim_{x\to\infty}\left(\sqrt{|x+\sqrt{x}|} - \sqrt{|x|}\right) \\
&= \lim_{x\to\infty}\left(\sqrt{|x+\sqrt{x}|} - \sqrt{|x|}\right) \cdot \frac{\sqrt{|x+\sqrt{x}|} + \sqrt{|x|}}{\sqrt{|x+\sqrt{x}|} + \sqrt{|x|}} \\
&= \lim_{x\to\infty} \frac{|x| + \sqrt{|x|} - |x|}{\sqrt{|x+\sqrt{x}|} + \sqrt{|x|}} \\
&= \lim_{x\to\infty} \frac{\sqrt{|x|}}{\sqrt{|x+\sqrt{x}|} + \sqrt{|x|}} \cdot \frac{1/\sqrt{|x|}}{1/\sqrt{|x|}} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{1 + 1/\sqrt{|x|}} + 1} \\
&= \frac{1}{2} \neq 0
\end{aligned}$$

Thus, the $\lim_{x\to\infty}[f(x+\sqrt{x}) - f(x)]$ does not necessarily equal 0. ■

Problem 50

Suppose that f is real-valued, continuous function on $\mathbb{R}$, and $f'(x)$ exists for $x \neq 0$. If $\lim_{x\to 0} f'(x) = 4$, does $f'(0)$ necessarily exist?

Solution
Consider the derivative of f at x

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Then, evaluated at $x = 0$, we get

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

Since both the numerator and denominator tend to 0, we can use L'Hopital's rule,

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f'(h)}{1} = 4.$$

Since the LHS is exactly $f'(0)$, then it must exist and it equals 4. ∎