MATH 131AH: Homework #3

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Problem 1

Assume that $d(x, y)$ is a distance function on some set X. Prove that $\widetilde{d}(x, y) = d(x, y)(d(x, y) + 1)^{-1}$ is a distance function, too.

Solution

In order to show that $\widetilde{d}(x, y)$ is a distance function, we show that it satisfies the three properties of a distance function. Consider $x \neq y$. Then $d(x, y) > 0, d(x, y) + 1 > 0$, so $\widetilde{d}(x, y) = \frac{d(x,y)}{d(x,y)+1} > 0$. Let $\widetilde{d}(x, y) = \frac{d(x,y)}{d(x,y)+1} = 0$. Then $d(x, y) = 0 \Leftrightarrow x = y$. Conversely, suppose $x = y$. Then $d(x, y) = 0$, which implies that $\widetilde{d}(x, y) = \frac{d(x,y)}{d(x,y)+1} = 0$. Hence we've shown that $\widetilde{d}(x, y) \geq 0$; $\widetilde{d}(x, y) = 0 \Leftrightarrow x = y$, satisfying the first property of a metric. Since we know that $d(x, y)$ is a metric, then $d(x, y) = d(y, x)$, so

$$\widetilde{d}(x, y) = \frac{d(x, y)}{d(x, y) + 1} = \frac{d(y, x)}{d(y, x) + 1} = \widetilde{d}(y, x)$$

so $\widetilde{d}(x, y)$ satisfies the reflexive property of a metric as well. In order to prove the triangle inequality, we first consider

$$\begin{gathered}
d(a, b) \leq d(c, e) \\
d(a, b) + d(a, b) \cdot d(c, e) \leq d(c, e) + d(a, b) \cdot d(c, e) \\
d(a, b)(1 + d(c, e)) \leq d(c, e)(1 + d(a, b)) \\
\frac{d(a, b)}{1 + d(a, b)} \leq \frac{d(c, e)}{1 + d(c, e)} \\
\widetilde{d}(a, b) \leq \widetilde{d}(c, e)
\end{gathered}$$

Now, we consider $d(x, z) \leq d(x, y) + d(y, z)$. From the above result, we can write

$$\begin{aligned}
\widetilde{d}(x, z) &= \frac{d(x, z)}{1 + d(x, z)} \\
&\leq \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)} \\
&= \frac{d(x, y)}{1 + d(x, y) + d(y, z)} + \frac{d(y, z)}{1 + d(x, y) + d(y, z)} \\
&\leq= \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} \\
&= \widetilde{d}(x, y) + \widetilde{d}(y, z)
\end{aligned}$$

Therefore, we've shown that $\widetilde{d}(x, z) \leq \widetilde{d}(x, y) + \widetilde{d}(y, z)$, thus satisfying the third property of the metrix, so $\widetilde{d}(x, y)$ is a metric. ■

Problem 2

Find sets of points in the plane $\mathbb{R}^2$ with the following properties.

(a) A set with nonempty interior and nonempty boundary, but empty interior.

(b) A set with nonempty boundary, nonempty exterior, but empty interior.

(c) Prove that there is no set in the plane with nonempty interior and nonempty exterior but empty boundary.

Problem 3

(a) Give an example of an infinite family of closed sets F_n such that $\cup_{n=1}^{\infty} F_n$ is not closed.

Solution
Consider $F_n = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. F_n is closed for each n, but the infinite union of these closed sets is not closed, because 0 is a limit point of $\cup_{n=1}^{\infty} F_n$, but $0 \notin \cup_{n=1}^{\infty}$, so it does not contain all of its limit points. ■

(b) For any family of sets $\{S_n\}$ show that the closure of $\cup_{n=1}^{\infty} S_n$ contains the union of the closures, $\cup_{n=1}^{\infty} \overline{S_n}$.

Solution
We define $B = \cup_{i=1}^{\infty} A_i$. To show containment, we let $x \in \cup_{i=1}^{\infty} \overline{A}$, and we want to show that $x \in \overline{B}$ for all x. Since x is in the infinite union, $x \in \overline{A_i}$ for some i, where $\overline{A_i} = A_i \cup A_i'$, where A_i' denotes the set of limit points of A_i. We consider cases. Case 1: if $x \in A_i$, then $x \in B \subset \overline{B}$, so $x \in \overline{B}$. Case 2: If $x \in A_i'$, then x is a limit point of the set A_i, which means that for every open ball $B_r(x)$ of radius $r > 0$, we can find a $y \neq x$ such that $y \in A_i$. This implies that x is also a limit point of B, so $x \in B' \subset \overline{B}$. Thus, in both cases 1 and 2, we end up with $x \in \overline{B}$, so containment is shown. ■

(c) Explain why part (a) implies that $\cup_{n=1}^{\infty} \overline{S_n}$ may be strictly smaller than $\overline{\cup_{n=1}^{\infty} S_n}$.

Solution
Part (a) shows that the union of the closures may be strictly smaller than the closure of the union because we see that $0 \in \overline{\cup_{n=1}^{\infty} F_n}$, where F_n is as defined above, but 0 is not in the individual closures, so $0 \notin \cup_{n=1}^{\infty} \overline{F_n}$. Therefore, the containment may be strict. ■

Problem 4

Show that, if S is a nonempty set, then $|d(S, x) - d(S, y)| \leq d(x, y)$.

Solution
Let $x, y \in X$ and $z_0 \in S$. Then by the triangle inequality, we have

$$d(x, z_0) \leq d(x, y) + d(y, z_0). \tag{1}$$

We can take the infimum over all $z \in S$ on the LHS to get

$$d(x, S) = \inf_{z \in S} \{d(x, z)\} \leq d(x, z_0) \leq d(x, y) + d(y, z_0).$$

We can then take the infimum over all of $z \in S$ on the RHS, we get

$$d(x, S) \leq d(x, z_0) \leq d(x, y) + \inf_{z \in S} \{d(y, z)\} = d(x, y) + d(y, S).$$

Therefore, we get $d(x, S) - d(y, S) \leq d(x, y)$. To prove this is true when taking the absolute value, we show need to show that $d(y, S) - d(x, S) \leq d(x, y)$. This is a consequence of applying the triangle inequality on y and S in inequality (1). Consider $d(y, z_0) \leq d(y, x) + d(x, z_0)$. Then we take infimum over all of $z \in S$ on the LHS to get

$$d(y, S) = \inf_{z \in S} \{d(y, z)\} \leq d(y, z_0) \leq d(y, x) + d(x, z_0).$$

Taking the infimum over all $z \in S$ on the RHS we get

$$d(y, S) \leq d(y, z_0) \leq d(x, y) + \inf_{z \in S} \{d(x, z)\} = d(x, y) + d(x, S),$$

so $d(y, S) - d(x, S) \leq d(x, y)$. We've shown that the both differences are less than or equal to $d(x, y)$, so we conclude that $|d(x, S) - d(y, S)| \leq d(x, y)$. ■

Problem 5

Show that, when K is a compact set in $\mathbb{R}^n$ and x is a point, there is at least one $y \in K$ such that $d(x, y) = d(x, K)$.

Solution

Problem 6

Show that the conclusion of problem 19 is false if we only assume that K is closed.

Solution