MATH 131AH: Homework #4

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Problem 1

(a) Prove that there is a constant C such that $d(x,y) \leq Cd_1(x,y)$, where d_1 is the distance function $d_1(x,y) = |x_1 - y_1| + \cdots + |x_n - y_n|$.

Solution
Let $x \in \mathbb{R}^n$. Then $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n$. Then

$$\begin{aligned} \| x \| = \| x_1e_1 + \cdots + x_ne_n \| &\leq \| x_1e_1 \| + \cdots + \| x_ne_n \| \\ &= |x_1| \, \| e_1 \| + \cdots |x_n| \, \| e_n \| \\ &\leq \| x \|_2 \sqrt{\| e_1 \|^2 + \cdots \| e_n \|^2} \\ &\leq \| x \|_1 \sqrt{\| e_1 \|^2 + \cdots \| e_n \|^2} \end{aligned}$$

The second to last inequality is a result of the Schwarz Inequality, which holds d is a normed distance function. The last inequality holds because the $\| x \|_2 \leq \| x \|_1$, as shown in exercise 13 of homework 2. Then take $C = \sqrt{\| e_1 \|^2 + \cdots \| e_n \|^2}$. Then we have $\| x \| \leq C \| x \|_1$, so $d(x,y) \leq Cd_1(x,y)$. ■

(b) Show that there is $\widetilde{C}$ such that $d_1(x,y) \leq \widetilde{C}d(x,y)$.

Solution
Suppose for contradiction that there is no such $\widetilde{C}$. If there is no $\widetilde{C}$, then there must be an $x_m \in \mathbb{R}^n, x_m \neq 0$, such that $\| x_m \|_1 \geq m \| x_m \|$ for each $n \in N$. Note that $x_m = x_me_m$. Then

$$\| x_m \| = \| x_me_m \| = |x_m| \, \| e_m \|$$

For some $x \in \mathbb{R}$, let $x = |x_m| \, \| e_m \|$ Then we have $\| x_m \|_1 = |x_m| \geq mx$ for all $m \in \mathbb{N}$, which contradicts the Archimedean Property. This contradiction establishes the desired result. ■

Problem 2

Consider the set S of all infinite sequences $(a_1, a_2, \cdots)$ with $a_j \in \mathbb{N}$ such that $a_j = 0$ for all but finitely many j. Is this countable?

Solution
The sequences $s \in S$ consist of infinitely 0s together with finitely many nonzero terms in $\mathbb{N}$. Thus for the finite sequences of nonzero terms, there is some $n \in N$ such that $a_j \neq 0$ for $j \leq n$ and $a_j = 0$ for $j \geq n$. More over, there are only finitely many of these sequences, as these sequences eventually end in infinitely many trailing 0s. If we then take the countable union of these sets, we have a countable union of finite sets, which is countable. ■

Problem 3

A set $C \subset \mathbb{R}^n$ is defined to be convex, if given any $p_1, p_2, \in S$, the points on the line $(1-t)p_1 + tp_2$ are in C for $0 \leq t \leq 1$. Show that convex sets are always connected.

Solution
Suppose for contradiction that C is not connected. Then C is the union of two nonempty, disjoint open sets, say A and B. Then $C = A \cup B$. For points $a \in A, b \in B$, let L be the line that connects these two points, and let $A^* = A \cap L, B^* = B \cap L$. Then A^* and B^* are open sets relative to C, and since $A^* \subset A \cap L, B^* \subset B \cap L$, then we know that $A^* \cap B^* = \emptyset$. Thus A^*, B^* are nonempty, disjoint open sets in C. Then

$$A^* \cup B^* = (A \cap L) \cup (B \cap L) = (A \cup B) \cap L$$

Since we have $C = A \cup B$, we can further write

$$(A \cup B) \cap L = C \cap L = L$$

The last equality holds because C is convex, and L is the line segment joining two points in C, so $L \subset C$. Hence, we've shown that L can be written as the disjoint union of nonempty open sets, A^*, B^* which implies that L is not connected. However, L is a line segment in $\mathbb{R}^n$, which is homeomorphic to a closed interval in $\mathbb{R}$, which is connected by theorem 2.47 in Rudin. This contradiction establishes our desired result. ■

Problem 4

(a) Assume that there is a p in a metric space (χ, d) such that the function $f(q) = d(p, q), q \in \chi$ omits the value $c > 0$, but takes values greater than c. Show that (χ, d) is not connected.

Solution
We're given that for $p \in (\chi, d)$, $c \notin \{d(p,q) : q \in \chi\}$. In other words, this means that for some point $q \in \chi$, $c \neq d(p,q)$. Then $\chi = \{q \in \chi : d(p,q) < c\} \cup \{q \in \chi : d(p,q) > c\}$, which is a disjoint union of open sets. Hence χ is not connected. ■

(b) Show that any connected metric spaces that does not consist of a single point must have uncountably many points.

Solution
From part (a), we see that if for $c > 0$, $c \in \{d(p,q) : q \in \chi\}$ then it contains all the points from 0 to $d(p,q)$, which forms a closed interval, which, by the corollary to theorem 2.43 in Rudin, contains uncountably many points. ■

Problem 5

For positive numbers b, c compute $\lim_{n\to\infty}(\sqrt{n^2+bn+c}-n)$.

Solution
Consider:

$$\begin{aligned}
\lim_{n\to\infty}(\sqrt{n^2+bn+c}-n) &= \lim_{n\to\infty}\left((\sqrt{n^2+bn+c}-n)\cdot\frac{\sqrt{n^2+bn+c}+n}{\sqrt{n^2+bn+c}+n}\right)\\
&= \lim_{n\to\infty}\frac{(n^2+bn+c)-n^2}{\sqrt{n^2+bn+c}+n}\\
&= \lim_{n\to\infty}\frac{bn+c}{\sqrt{n^2+bn+c}+n}\\
&= \lim_{n\to\infty}\frac{b+\frac{c}{n}}{\sqrt{1+\frac{b}{n}+\frac{c}{n^2}}+1}
\end{aligned}$$

We claim that $\lim_{n\to\infty}\frac{1}{n^p}=0$ for $p>0$. Given $\epsilon>0$, we choose $N=\frac{1}{\epsilon}^{1/p}$. Then for $n>N$, we have $n>(\frac{1}{\epsilon})^{1/p}\Rightarrow n^p>\frac{1}{\epsilon}\to\frac{1}{n^p}<\epsilon\Rightarrow|\frac{1}{n^p}-0|<\epsilon$, thus proving our claim. Then

$$\lim_{n\to\infty}\frac{b+\frac{c}{n}}{\sqrt{1+\frac{b}{n}+\frac{c}{n^2}}+1}=\frac{b+c\cdot 0}{\sqrt{1+b\cdot 0+c\cdot 0}+1}=\frac{b}{2}.$$

Thus, $\lim_{n\to\infty}(\sqrt{n^2+bn+c}-n)=\frac{b}{2}$. ■

Problem 6

Let $\{a_n\}, \{b_n\}$ be bounded sequences of real numbers. Prove that

$$\liminf\,(a_n+b_n)\geq\liminf\,(a_n)+\liminf\,(b_n)$$

and give an example showing that the inequality is not strict.

Solution
Since both $\{a_n\}, \{b_n\}$ are nonempty and bounded in $\mathbb{R}$, $\inf a_n$, $\inf b_n$ exist in $\mathbb{R}$. Let $\alpha=\inf a_n$ and $\beta=\inf b_n$. Then $a_n\geq\alpha, b_n\geq\beta$ for all n. Then the set consisting of a_n+b_n is bounded below, as $\alpha+\beta\leq a_n+b_n$ for all n, so $\alpha+\beta\leq\inf\,(a_n+b_n)$. By a result we proved during lecture, we know that if the limits of sequences s_n, t_n exist and $s_n\leq t_n$ for all n, then $\lim s_n\leq\lim t_n$. Since the limits on both sides of the inequality exist, we can take the limit on both sides

$$\begin{aligned}
\lim\,(\alpha+\beta) &\leq \liminf\,(a_n+b_n)\\
\lim\,\alpha+\lim\,\beta &\leq \liminf\,(a_n+b_n)\\
\liminf a_n+\liminf b_n &\leq \liminf\,(a_n+b_n)
\end{aligned}$$

If we take $a_n=(-1)^n, b_n=(-1)^{n+1}$, then $\lim a_n=-1, \lim b_n=-1$, so $\lim a_n+\lim b_n=-2$. In this case, $\liminf\,(a_n+b_n)=0$, so the inequality is strict. ■

Problem 7

Given a bounded sequences $\{a_n\}$, prove that $L = \text{lim sup}(a_n) \Leftrightarrow$ for every $\epsilon > 0$ there is an $N(\epsilon)$ such that $a_n < L + \epsilon$ for all $n \geq N(\epsilon)$ and there is no $N(\epsilon)$ such that $a_n < L - \epsilon$ for all $n \geq N(\epsilon)$.

Solution

Suppose $L = \text{lim sup}(a_n)$. Then given $\epsilon > 0$, $L < L + \epsilon$, so by Theorem 3.17 (b), there exists an integer $N(\epsilon)$ such that $n \geq N(\epsilon)$ implies $a_n < L + \epsilon$. For the second result, suppose that there exists some $N(\epsilon)$ such that $a_n < L - \epsilon$ for all $n \geq N(\epsilon)$. Then a_n is nonempty, bounded above, so $a_n \leq \text{sup } a_n \leq L - \epsilon < L$, which contradicts $\text{lim sup}(a_n) \leq \text{sup } a_n$. This contradiction establishes the forward direction of the proof.

Conversely, suppose that for every $\epsilon > 0$ there is an $N(\epsilon)$ such that $a_n < L + \epsilon$ for all $n \geq N(\epsilon)$ and there is no $N(\epsilon)$ such that $a_n < L - \epsilon$ for all $n \geq N(\epsilon)$. Then we have $a_n - \epsilon < L$ for $n \geq N(\epsilon)$. Clearly, $\{a_n\}$ is bounded above, so $a_n \leq \text{sup } a_n \leq L + \epsilon$ for all $n \geq N(\epsilon)$. This implies $\text{sup } a_n - L \leq \epsilon$, so $|\text{sup } a_n - L| \leq \epsilon$, so $\text{lim sup } a_n = L$. ■

Problem 8

Suppose that $a_n \geq 0$ for $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n = \infty$. Discuss the convergence of the sums $\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$ and $\sum_{n=1}^{\infty} \frac{a_n}{1+n^2 a_n}$.

Solution

The sum $\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$ can be divergent. For example, if na_n is bounded above by some $B \in \mathbb{R}$, then the terms $\frac{a_n}{1+B}$ tend toward infinity, so the partial sums also approach infinity, so the series diverges.

On the other hand, we can construct a_n so that when $n \in N$ is a perfect square $(1, 4, 9, ...)$, we let $a_n = 1$. Else, let $a_n = \frac{1}{n^2}$. Then $\sum a_n$ diverges because the partial sums tend toward infinity. For squared terms, we have $\sum \frac{1}{1+n^2}$, which converges by the comparison test with the p series for $p = 2$. For nonsquared terms, we have $\sum \frac{1}{n^2+n}$ which also converges by the comparison test with the p series for $p = 2$. Thus, $\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$ converges.

Consider $\sum_{n=1}^{\infty} \frac{a_n}{1+n^2 a_n}$. Since this sum is bounded above by the p series, for $p = 2$, then by the comparison test, we see that $\sum_{n=1}^{\infty} \frac{a_n}{1+n^2 a_n}$ converges. ■