MATH 131B: Homework #1

Professor Dave Penneys
Assignment: 1, 2, 4, 5

Eric Chuu
UID: 604406828

June 30, 2016

Problem 1

Let X be a set, and let $d : X \times X \to [0, \infty)$ be a function.

(a) Give an example of a pair (X, d) which obeys axioms (bcd) of Definition 1.1.2, but not (a).

Solution
Following the hint, we consider the pair $(\mathbb{R}, d)$, where $d(x, y) = 1$ for all $x, y \in \mathbb{R}$. Then for any $x \in R$, $d(x, x) = 1 \neq 0$, so axiom (a) fails. We check the other three axioms. For any distinct $x, y \in \mathbb{R}$, $d(x, y) = 1 > 0$, so (b) holds. For any $x, y \in \mathbb{R}$, $d(x, y) = 1 = d(y, x)$, so (c) holds. Lastly, let $x, y, z \in \mathbb{R}$. In the case that $x \neq y \neq z$, then $d(x, z) = 1 \leq 1 + 1 = d(x, y) + d(y, z)$ for all $x, y, z \in \mathbb{R}$, so (d) holds. In the other case, without loss of generality, suppose that $x = y, x \neq z$, then $d(x, z) = 1 \leq d(x, y) + d(y, z) = 0 + 1$. The cases when the LHS is 0 are trivial, so we see that axiom (d) holds for all $x, y, z \in \mathbb{R}$. $\square$

(b) Give an example of a pair (X, d) which obeys axioms (abd) of Definition 1.1.2, but not (b).

Solution
Consider the pair $(\mathbb{R}, d)$, where $d(x, y) = 0$ for all $x, y \in \mathbb{R}$. Then for distinct $x, y, \in \mathbb{R}$, $d(x, y) = 0$, so axiom (b) fails. We check the other three axioms. For any $x \in \mathbb{R}$, $d(x, x) = 0$, so (a) holds. For all $x, y \in \mathbb{R}, d(x, y) = 0$ and $d(y, x) = 0$, so $d(x, y) = d(y, x)$, so (c) holds. Finally, for all $x, y, z \in \mathbb{R}$, $d(x, z) = 0 \leq 0 + 0 = d(x, y) + d(y, z)$, so (d) holds. $\square$

(c) Give an example of a pair (X, d) which obeys axioms (abd) of Definition 1.1.2, but not (c).

Solution
Consider the pair $(\mathbb{R}, d)$, where

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{for } x < y \\ 2 & \text{for } x > y \end{cases}$$

Then let $a, b \in \mathbb{R}$, and suppose $a < b$. Then $d(a, b) = 1$, but $d(b, a) = 2$, so (c) fails. We check the other three axioms. For any $x \in X$, $d(x, x) = 0$, so (a) holds. For distinct x, y, $d(x, y) = 1$ or $d(x, y) = 2$, both of which are strictly positive, so (b) holds. Let $x, y, z \in \mathbb{R}$. Consider cases:
Case 1: Without loss of generality, suppose $x < y < z$. Since d is not symmetric, we must consider the six possible distances that arise from these three points. We do not need to check the distance for any point

from itself because 0 is trivially less than or equal to anything on the right hand side.

Note that $d(x,y) = 1, d(x,z) = 1, d(y,x) = 2, d(y,z) = 1, d(z,y) = 2, d(z,x) = 2$

$$\begin{aligned}
d(x,y) &= 1 \leq d(x,z) + d(z,y) = 1 + 2 = 3 \\
d(x,z) &= 1 \leq d(x,y) + d(y,z) = 1 + 1 = 2 \\
d(y,x) &= 1 \leq d(y,z) + d(z,x) = 1 + 2 = 3 \\
d(y,z) &= 1 \leq d(y,x) + d(x,z) = 2 + 1 = 3 \\
d(z,x) &= 2 \leq d(z,y) + d(y,x) = 2 + 2 = 4 \\
d(z,y) &= 2 \leq d(z,x) + d(x,y) = 2 + 1 = 3
\end{aligned}$$

Case 2: Without loss of generality, suppose $x = y, y < z$.
Then: $d(x,y) = d(y,x) = 0, d(x,z) = d(y,z) = 1, d(z,x) = d(z,y) = 2$. We do not need to check any distances between x and y since they would be trivially less than or equal to than anything on the right hand side.

$$\begin{aligned}
d(x,z) &= 1 \leq d(x,y) + d(y,z) = 0 + 1 = 1 \\
d(y,z) &= 1 \leq d(y,x) + d(x,z) = 0 + 1 = 1 \\
d(z,x) &= 2 \leq d(z,y) + d(y,x) = 2 + 0 = 2 \\
d(z,y) &= 2 \leq d(z,x) + d(x,y) = 2 + 0 = 2
\end{aligned}$$

Case 3: $x = y = z$. This case is trivial, since every distance would be 0. Hence, we've shown exhaustively that axiom (d) holds. □

(d) Give an example of a pair (X, d) which obeys axioms (abc) of Definition 1.1.2, but not (d).

Solution
Consider the pair $(\mathbb{R}, d)$, where $d = (x-y)^2$. Then, if we take $x = -5, y = 0, z = 6$, we see that $d(x,z) = (-5-6)^2 = 121$, while $d(x,y) + d(y,z) = 25 + 36 = 61$, and since $121 > 61$, the axiom (d) fails. We check the other three axioms. For any $x \in \mathbb{R}$, $d(x,x) = (x-x)^2 = 0$, so (a) holds. For distinct $x, y \in \mathbb{R}, d(x,y) > 0$ since the square of a difference of distinct numbers is positive, so (b) holds. For any $x, y \in \mathbb{R}$, $d(x,y) = (x-y)^2 = (y-x)^2 = d(y,x)$, so (3) holds. □

Source used: I read the Wikipedia page on Quasimetrics.

Problem 2

Let $n \geq 1$, and let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, , b_n$ be real numbers. Verify the identity:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right)$$

and conclude the Cauchy-Schwarz inequality.

Then use the Cauchy-Schwarz inequality to prove the triangle inequality

$$\left(\sum_{i=1}^{n}(a_i + b_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}$$

Solution

We first verify the identity. If we expand the second term on the LHS, we get

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_i^2 b_j^2 - 2a_i b_j a_j b_i + a_j^2 b_i^2\right) &= \frac{1}{2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2 b_j^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} a_j^2 b_i^2 - 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j a_j b_i\right] \\
&= \frac{1}{2}\left[2 \cdot \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2 b_j^2 - 2\sum_{i=1}^{n} a_i b_i \sum_{j=1}^{n} a_j b_j\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2 b_j^2 - \sum_{i=1}^{n} a_i b_i \sum_{j=1}^{n} a_j b_j \\
&= \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2
\end{aligned}$$

Rearranging the terms, we get

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) \implies \left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{j=1}^{n} b_j^2\right)^{1/2}$$

which is exactly the identity above, and since $\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 \geq 0$, the LHS without the second sum is less than or equal to the RHS, and taking square roots on both sides gives us the Cauchy-Schwarz Inequality. Now, we prove the triangle inequality. Squaring both sides, we get

$$\text{LHS} = \sum_{i=1}^{n}(a_i + b_i)^2 = \sum_{i=1}^{n}\left(a_i^2 - 2a_i b_i + b_i^2\right) = \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sum_{i=1}^{n} a_i b_i$$

$$\text{RHS} = \sum_{i=1}^{n} a_i^2 + \sum_{j=1}^{n} b_j^2 + 2\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{j=1}^{n} b_j^2\right)^{1/2} = \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{j=1}^{n} b_j^2\right)^{1/2}$$

Two of the sums in LHS and RHS are the same, so we need only show that

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{j=1}^{n} b_j^2\right)^{1/2}$$

Using the Cauchy-Schwarz inequality, we see that

$$\sum_{i=1}^{n} a_i b_i \leq \left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{j=1}^{n} b_j^2\right)^{1/2},$$

so the triangle inequality follows. □

Problem 4

Two metrics d_1 and d_2 are called strongly equivalent if there are $c_1, c_2 > 0$ such that

$$d_1(x, y) \leq c_2 d_2(x, y) \qquad \text{and} \qquad d_2(x, y) \leq c_1 d_1(x, y)$$

for all $x, y \in \mathbb{R}^n$.

(a) Prove that d_1, d_2, and d_∞ metrics on $\mathbb{R}^n$ are all strongly equivalent by showing

$$d_\infty(x, y) \leq d_2(x, y) \leq d_1(x, y) \leq n \cdot d_\infty(x, y) \tag{1}$$

for all $x, y \in \mathbb{R}^n$.
(b) Prove:

$$d_1(x, y) \leq \sqrt{n} \cdot d_2(x, y) \tag{2}$$
$$d_2(x, y) \leq \sqrt{n} \cdot d_\infty(x, y) \tag{3}$$

for all $x, y \in \mathbb{R}^n$.

Solution
(a) We prove inquality (1) one at a time. We first show that $d_\infty(x, y) \leq d_2(x, y)$ for all $x, y \in \mathbb{R}^n$. Without loss of generality, suppose $\max\{|x_1 - y_1| + |x_2 - y_2| + \ldots |x_n - y_n|\} = |x_1 - y_1|$. Since $|x_i - y_i| \geq 0$ for $1 \leq i \leq n$, so we have

$$(d_\infty(x, y))^2 = |x_1 - y_1|^2 = (x_1 - y_1)^2 \leq \left(\sum_{i=1}^{n}(x_i - y_i)\right)^2 = (d_2(x, y))^2$$

Since both distances are non-negative, we take the square root, and we get

$$d_\infty(x, y) \leq d_2(x, y),$$

for all $x, y \in \mathbb{R}^n$. Next, we show that $d_2(x, y) \leq d_1(x, y)$ for all $x, y \in \mathbb{R}^n$. If we square $d_2(x, y)$, then we get

$$(d_2(x, y))^2 = \sum_{i=1}^{n}(x_i - y_i)^2 = \sum_{i=1}^{n}|x_i - y_i|^2 \leq (|x_1 - y_1| + \cdots + |x_n - y_n|)^2 = \left(\sum_{i=1}^{n}|x_i - y_i|\right)^2 = (d_1(x, y))^2$$

Taking the square root, we get

$$d_2(x, y) \leq d_1(x, y)$$

for all $x, y \in \mathbb{R}^n$. Finally, we show that $d_1(x, y) \leq n \cdot d_\infty(x, y)$ for all $x, y \in \mathbb{R}^n$. Without loss of generality suppose that $\max\{|x_1 - y_1|, \cdots, |x_n - y_n|\} = |x_1 - y_1|$. Then

$$\begin{aligned} d_1(x, y) &= |x_1 - y_1| + \cdots + |x_n - y_n| \leq n \cdot |x_1 - y_1| = n \cdot d_\infty(x, y) \\ \implies d_1(x, y) &\leq n \cdot (d_\infty(x, y) \end{aligned}$$

for all $x, y \in \mathbb{R}^n$. Piecing together the inequalities, we get

$$d_\infty(x, y) \leq d_2(x, y) \leq d_1(x, y) \leq n \cdot d_\infty(x, y),$$

which is exactly the inequality in (1) and we are done. □

(b) We first prove inequality (2), that is, $d_1(x,y) \le \sqrt{n} \cdot d_2(x,y)$. Note that we if we square $d_1(x,y)$, we can write it as

$$(d_1(x,y))^2 = \left(\sum_{i=1}^{n} |x_i - y_i|\right)^2 = \left(\sum_{i=1}^{n} 1 \cdot |x_i - y_i|\right)^2$$

Then by the Cauchy-Schwarz Inequality, we know that

$$\left(\sum_{i=1}^{n} 1 \cdot |x_i - y_i|\right)^2 \le \sum_{i=1}^{n} 1^2 \cdot \sum_{i=1}^{n} |x_i - y_i|^2 = n \cdot \sum_{i=1}^{n} |x_i - y_i|^2 = n \cdot (d_2(x,y))^2$$

for all $x, y \in \mathbb{R}^n$. Taking the square root, we get $d_1(x,y) \le \sqrt{n} \cdot d_2(x,y)$ for all $x, y \in \mathbb{R}^n$, so inequality (2) holds.

Now, we show that $d_2(x,y) \le \sqrt{n} \cdot d_\infty(x,y)$ holds for all $x, y \in \mathbb{R}^n$. Without loss of generality, suppose $\max\{|x_1 - y_1| + |x_2 - y_2| + \ldots |x_n - y_n|\} = |x_1 - y_1|$. Then squaring $d_2(x,y)$, we get

$$(d_2(x,y))^2 = \sum_{i=1}^{n} (x_i - y_i)^2 = |x_1 - y_1|^2 + \cdots + |x_n - y_n|^2 \le n \cdot |x_1 - y_1|^2 = n \cdot (d_\infty(x,y))^2$$

Taking the square root, we get $d_2(x,y) \le \sqrt{n} \cdot d_\infty(x,y)$ for all $x, y \in \mathbb{R}^n$, so inequality (3) holds, and we are done. $\square$

Problem 5

Let $\mathbb{R}^n$ be a Euclidean space, and let $(x^{(k)})_{k=m}^{\infty}$ be a sequence of points in $\mathbb{R}^n$. We write $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)})$, i.e., for $j = 1, 2, \dots, n$, $x_j^{(k)} \in \mathbb{R}$ is the j^{th} coordinate of $x^{(k)} \in \mathbb{R}^n$. Let $x = (x_1, x_2, \dots, x_n)$ be a point in $\mathbb{R}^n$. Then the following four statements are equivalent:

(a) $(x^{(k)})_{k=m}^{\infty}$ converges to x with respect to the Euclidean metric d_2

(b) $(x^{(k)})_{k=m}^{\infty}$ converges to x with respect to the taxi-cab metric d_1

(c) $(x^{(k)})_{k=m}^{\infty}$ converges to x with respect to the sup norm metric d_∞.

(d) For every $1 \leq j \leq n$, the sequence $(x_j^{(k)})_{k=m}^{\infty}$ converges to x_j.

Solution
We show that (a) $\Leftrightarrow$ (b):
Note that we showed in Problem 4 that d_1 and d_2 are strongly equivalent, so there exists $c_1, c_2 > 0$ such that $d_1(x,y) \leq c_2 d_2(x,y)$ and $d_2(x,y) \leq c_1 d_1(x,y)$ for all $x, y \in \mathbb{R}^n$. Now, suppose $(x^{(k)})_{k=m}^{\infty}$ converges to x with respect to d_2. Then given $\epsilon > 0$, there exists $N > m$ such that for all $k \geq N$, $d_2((x^{(k)}), x) < \epsilon/c_2$. Then, $d_1((x^{(k)}), x) \leq c_2 d_2((x^{(k)}), x) < c_2 \cdot \epsilon/c_2 = \epsilon$, so $(x^{(k)})_{k=m}^{\infty}$ converges with respect to d_1.
Now suppose that $(x^{(k)})_{k=m}^{\infty}$ converges with respect to d_1. We will show that this implies convergence with respect to d_2. Given $\epsilon > 0$, there exists $N > m$ such that for all $k \geq N$, we have $d_1((x^{(k)}, x) < \epsilon/c_1$, where c_1 is the constant defined above. Then, $d_2(x^{(k)}, x) \leq c_1 d_1(x^{(k)}, x) < c_1 \cdot \epsilon/c_1 = \epsilon$, so $(x^{(n)})_{k=m}^{\infty}$ converges with respect to d_2.

We show that (b) $\implies$ (d)
Suppose $(x^{(k)})_{k=m}^{\infty}$ converges with respect to d_1. We want to show that for every $1 \leq j \leq n$, the sequence $(x_j^{(k)})_{k=m}^{\infty}$ in $\mathbb{R}$ converges to x_j. Given $\epsilon > 0$, there exists $N > m$ such that for all $k \geq N$, $d_1((x^{(k)}), x) < \epsilon$, which means that

$$\sum_{j=1}^{n} |x_j^{(k)} - x_j| < \epsilon,$$

which means that $d_1(x_j^{(k)}, x_j) = |x_j^{(k)} - x_j| < \epsilon$ for all $1 \leq j \leq n$, so the sequence $(x_j^{(k)})_{k=m}^{\infty}$ converges to x_j, satisfying (d).

We show that (d) $\implies$ (c)
Suppose that for every $1 \leq j \leq n$, the sequence $(x_j^{(k)})_{k=m}^{\infty}$ converges to x_j. Then we show that $(x^{(k)})_{k=m}^{\infty}$ converges with respect to d_∞. By the hypothesis, for each $j \in [1, n]$, given $\epsilon > 0$, there exists $N_j > m$ such that for all $n \geq N_j$, we have $d_1(x_j^{(k)}, x_j) = |x_j^{(k)} - x_j| < \epsilon$. Take N = $\max\{N_1, N_2, \dots, N_n\}$; note we can do this because the set of N_j's is finite. Then for all $k \geq N$, $|x_j^{(k)} - x_j| < \epsilon$ for $1 \leq j \leq n$. Since these distances form a finite set, there exists a maximum distance in the set. Then $\max\{|x_j^{(k)} - x_j| : 1 \leq j \leq n\} < \epsilon$, so $(x^{(k)})_{k=m}^{\infty}$ converges with respect to d_∞, satisfying (c).

We show that (c) $\implies$ (b)
Again from the previous problem, we know that $d_1(x,y) \leq n \cdot d_\infty(x,y)$ for all $x, y \in \mathbb{R}^n$. Suppose that $(x^{(k)})_{k=m}^{\infty}$ converges with respect to d_∞. Then given $\epsilon > 0$, there exists $N > m$ such that for all $k \geq N$, $d((x^{(k)}, x) < \epsilon/n$. Then $d_1(x,y) \leq n \cdot d_\infty(x,y) \leq n \cdot \epsilon/n = \epsilon$, so $(x^{(k)})_{k=m}^{\infty}$ converges with respect to d_1.
Hence, we have shown that all four statements are equivalent, and we are done. $\square$