MATH 131B FINAL PREPARATORY PROBLEMS

Problem 1. Prove that $\mathbb{R}^n$ is a metric space under one the d_1, d_2, d_∞ metrics.

Problem 2. State and prove the extreme value theorem.

Problem 3. What are the compact subsets of (X, d_X) if d_X is the discrete metric? Which sequences converge?

Problem 4. Show that the connected subsets of $\mathbb{R}$ are exactly the intervals.

Problem 5. State and prove the Heine-Borel property for $\mathbb{R}$.

Problem 6. Find a closed and bounded subset of either $L^\infty(\mathbb{Z}, \mathbb{R})$ or $C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ which is not compact. (Try $\{\delta_n | n \in \mathbb{Z}\}$ where $\delta_n(m) = \delta_{m,n}$ and $\{e_n | n \in \mathbb{Z}\}$ respectively.)

Problem 7. Let $B_{\leq\varepsilon}(x) = \{y \in X | d_X(x, y) \leq \varepsilon\}$. Show that for all $x \in X$, $\overline{B_\varepsilon(x)} \subseteq B_{\leq\varepsilon}(x)$. Do we have the other containment?

Problem 8. Let X, Y be metric spaces with Y compact.

1. Show that $\{x_0\} \times Y$ is compact for every $x_0 \in X$.
2. Show that if $\{x_0\} \times Y \subset U$ where $U \subset X \times Y$ is open, then there is a $\varepsilon > 0$ such that $B_\varepsilon(x_0) \times Y \subset U$.

Problem 9. Prove that the uniform limit of bounded functions is bounded.

Problem 10. Prove that a power series converges uniformly on every compact subset within its radius of covergence.

Problem 11. Where is the function $f(x) = \dfrac{1}{1-x}$ analytic?

Problem 12. Prove that the continuous image of a compact set is compact. Then prove that the continuous image of a connected set is connected.

Problem 13. Consider the $\mathbb{Z}$-periodic functions $e_n(z) = e^{2\pi niz} \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. Show that $\langle e_m, e_n \rangle = \delta_{m,n}$.

Problem 14. Prove that under the Fourier Transform, the trigonometric polynomials are in bijective correspondence with the functions $\mathbb{Z} \to \mathbb{C}$ of finite support.

Problem 15.

Problem 16.

Problem 17.

Problem 18.

Problem 19.

Problem 20.