MATH 131AH: Homework #2

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Problem 1

For each of the subsets of $M_n(\mathbb{R})$, state exactly which field axioms it fails to satisfy.

(a) The diagonal matrices, $A \sim (a_{ij})$ with $a_{ij} = 0$ when $i \neq j$.

Solution
If A is a diagonal matrix with any 0s on the diagonal, then it fails M5, as it does not have a multiplicative inverse. ■

(b) The diagonal matrices A with positive entries on the diagonal, $a_{ij} = 0$ when $i \neq j$ and $a_{ii} > 0$.

Solution
The set of matrices described in (b) fails A4 since there is no 0 element such that $0 + A = A$, since A must have positive real numbers on the diagonal. It also fails A5, which states that to every $x \in F$ corresponds a $-x \in F$ such that $x + (-x) = 0$. Since all diagonal elements in a matrix A have to be positive, the additive inverse of A does not exist in the set of matrices described. ■

(c) The diagonal matrices with constant entries, $a_{ij} = 0$ when $i \neq j$ and $a_{ii} = a_{jj}$ for all i, j.

Solution
All field axioms are satisfied. ■

(d) The invertible matrices plus the zero matrix.

Solution
Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then A1, closure under addition is violated when we consider $A + B$. ■

Problem 2

(a) Prove that $\mathbb{Q}(x)$ is an ordered field.

Solution
First we show that $\mathbb{Q}(x)$ satisfies the properties of ordered sets. Let $x = \frac{p_1}{q_1}, y = \frac{p_2}{q_2} \in \mathbb{Q}(x)$. Note that in the solution below, when we write $p_1 \cdot q_1 > p_2 \cdot q_2$, we are comparing the highest order coefficients of the respective products. We then consider cases.
Case 1: $y - x$ is positive.

$$\frac{p_2}{q_2} - \frac{p_1}{q_1} = \frac{p_2q_1 - p_1q_2}{q_2q_1}$$

which is positive $\Leftrightarrow$ the highest order coefficient of $(p_2q_1 - p_1q_2) \cdot (q_2q_1)$ is greater than 0 $\Leftrightarrow p_2 \cdot q_2 \cdot q_1 \cdot q_1 - p_1 \cdot q_2 \cdot q_2 \cdot q_1 > 0$. Since the highest order coefficients of $q_1 \cdot q_1$ and $q_2 \cdot q_2$ are positive, we only need to consider the highest order coefficient of $p_2 \cdot q_2 - p_1 \cdot q_1$, and since it is greater than 0, we see that $y > x$.

Case 2: $x - y$ is positive.

$$\frac{p_1}{q_1} - \frac{p_2}{q_2} = \frac{p_1q_2 - p_2q_1}{q_2q_1}$$

which is positive $\Leftrightarrow$ the highest order coefficient of $(p_1q_2 - p_2q_1) \cdot q_2q_1 > 0 \Leftrightarrow p_1 \cdot q_2 \cdot q_1 \cdot q_2 - p_2 \cdot q_1 \cdot q_1 \cdot q_2 > 0$. Since the leading coefficients of $q_2 \cdot q_2$ and $q_1 \cdot q_1$ are positive, we only consider the highest order coefficient of $p_1 \cdot q_1 - p_2 \cdot q_2$, and since it is greater than 0, we see that $x > y$.

Case 3: $y - x$ is not positive and $x - y$ is not positive. So $y - x = 0$.

$$\frac{p_1}{q_1} - \frac{p_2}{q_2} = 0$$

Performing similar calculations as above, we consider $p_1 \cdot q_1 - p_2 \cdot q_2 = 0 \Leftrightarrow p_1 \cdot q_1 = p_2 \cdot q_2$, so the highest order coefficients (and all coefficients following) of both $p_1 \cdot q_1$ and $p_2 \cdot q_2$ are the same, so $x = y$. With these three cases, we have shown that one of the three cases occurs for any $x, y \in \mathbb{Q}(x)$, so$\mathbb{Q}(x)$ satisfies the first property of ordered sets.

To show that the second property (transitivity) holds, we let $z = \frac{p_3}{q_3} \in \mathbb{Q}(x)$. If $x < y$, then $y - x$ is positive, and by the results from the cases above, we know that this implies $p_2 \cdot q_2 > p_1 \cdot q_1$. Similarly, if $y < z$, then $z - y$ is positive, and $p_3 \cdot q_3 > p_2 \cdot q_2$. Since $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{Q}$, and since $\mathbb{Q}$ is ordered, if $p_1 \cdot q_1 < p_2 \cdot q_2$ and $p_2 \cdot q_2 < p_3 \cdot q_3$, then by transitivity, we have $p_1 \cdot q_1 < p_3 \cdot q_3$, which implies that $p_3 \cdot q_3 - p_1 \cdot q_1 > 0$, so $x < z$. Hence the transitivity property is satisfied for $\mathbb{Q}(x)$ as well, so $\mathbb{Q}(x)$ satisfies the second property of an ordered set.

In order for $\mathbb{Q}(x)$ to be an ordered field, we must show that it satisfies (i) $x + y < x + z$ if $x, y, z \in \mathbb{Q}(x)$ and $y < z$ and (ii) $xy > 0$ if $x, y \in \mathbb{Q}(x), x > 0, y > 0$. Like before, we let $y = \frac{p_2}{q_2} < z = \frac{p_3}{q_3}$. Then the highest order coefficient of $z - x$ is positive. Drawing from our results from proving that $\mathbb{Q}(x)$ satisfies the properties of an ordered set, we know that $z - y$ being positive implies that the $p_3 \cdot q_3 > p_2 \cdot q_2$. Then we take

$$(x + z) - (x + y) = \frac{p_3}{q_3} - \frac{p_2}{q_2} = \frac{p_3 q_2 - p_2 q_3}{q_3 q_2}$$

which is positive $\Leftrightarrow$ the highest order coefficient of $(p_3 \cdot q_2 - p_2 \cdot q_3) q_3 q_2$ is positive $\Leftrightarrow p_3 \cdot q_2 \cdot q_3 \cdot q_2 - p_2 \cdot q_3 \cdot q_3 \cdot q_2 > 0$. Since the highest order coefficients of $q_2 \cdot q_2$ and $q_3 \cdot q_3$ are positive, we need only consider $p_3 \cdot q_3 - p_2 \cdot q_2$, which we know is positive because $z > y$, so property (i) is proven. Now, consider $x > 0, y > 0 \Leftrightarrow$ the highest order coefficients of $p_1 \cdot q_1$ and $p_2 \cdot q_2$ are positive. Then $xy = \frac{p_1 \cdot q_1}{p_2 \cdot q_2}$ is positive $\Leftrightarrow$ the highest order coefficient of $\mathrm{p}_1 p_2 q_1 q_2$ is positive. Since $x > 0, y > 0$, we know that $p_1 q_1 > 0$ and $p_2 q_2 > 0$, so the highest order coefficient of $p_1 p_2 q_1 q_2$ is also positive, so $xy > 0$. Thus, $\mathbb{Q}(x)$ is an ordered field. ∎

(b) Prove that if $r(x)$ and $w(x)$ are elements in $\mathbb{Q}(x)$ and $r(x) > 0$, there may be no $n \in N$ such that $nr(x) > w(x)$. In other words $\mathbb{Q}(x)$ is not archimedean.

Solution
Suppose for contradiction that $\mathbb{Q}(x)$ is archimedean. Then for some $f(x) \in \mathbb{Q}(x), g(x) \in \mathbb{Q}(x), f(x) > 0$, there exists some $n \in \mathbb{N}$ such that $nf > g$. Let $f(x) := \frac{1}{x}, g(x) := 1$. Note that by the way order is defined in $\mathbb{Q}(x)$, we have $\frac{1}{x} < 1$ since $1 - \frac{1}{x} = \frac{x-1}{x} > 0$. Then we can find some $n \in N$ such that $n \cdot \frac{1}{x} > 1$, but this implies that $\frac{n}{x} - 1 > 0 \Rightarrow \frac{-x+n}{x} > 0$, which contradicts the way order is defined in $\mathbb{Q}(x)$. This contradiction establishes the result. Hence $\mathbb{Q}(x)$ is not archimedean. ∎

Problem 3

If $\alpha \in S$ and α is an upper bound for S, prove $\alpha = sup(S)$. We say $\alpha = max(S)$.

Solution
It is sufficient to show that α satisfies the properties of a supremum of a set. S is bounded above, and since $\alpha \in S$, S is nonempty, so $sup(S)$ exists in $\mathbb{R}$. If S contains only the single element α, then α is trivially the

Problem 3 continued on next page...

supremum of the set. If S contains more than just the element α, then for some $\epsilon > 0$, we can find $\alpha - \epsilon \in S$ as well. Since α is an upper bound of S, then $\alpha - \epsilon < \alpha$. Then $\alpha - \epsilon$ fails to be an upper bound for S. Hence, α satisfies the definition of $sup(S)$, so $\alpha = sup(S) = max(S)$. ∎

Problem 4

Give an example of a set of irrational numbers with a rational supremum.

Solution

Consider the set $S = \{x \in (0,1), : x \notin \mathbb{Q}\}$. We claim that $sup(S) = 1$. Thus, we need to show that 1 satisfies the properties of the supremum of S. It is clear that 1 is an upper bound since $x < 1$ for all $x \in S$. Then for some $\epsilon > 0$, where $\epsilon \notin \mathbb{Q}$, we can find a $\gamma > 0$ such that $\gamma = 1 - \epsilon$. Then $\gamma \notin \mathbb{Q}$, $\gamma < 1$, and $\gamma \in S$. Since $\gamma \in \mathbb{R}, 1 \in \mathbb{R}$, then by the density of rationals, there exists some $r \in \mathbb{Q}$ such that $\gamma < r < 1$. As a result of problem 3 from the previous homework which states that we can find an irrational number between two rational numbers x, y, we can then find an irrational number q between the rationals r and 1, so $r < q < 1$. Hence we've shown that when $\gamma < 1$, γ fails to be an upper bound for S, as we're able to find a $q \in S$ such that $\gamma < q < 1$. Hence 1 satisfies both properties of the supremum of the set S. ∎

Problem 5

For non-empty sets X and Y define $X \times Y$ to be the set of ordered pairs $\{(x, y) : x \in X : y \in Y\}$. Let f be the real valued function on $X \times Y$. If the range of f, $\{f(x, y) : x \in X, y \in Y\}$ is bounded, you have two more functions,

$$f_1(x) = \inf \{f(x, y), y \in Y\}$$
$$f_2(y) = \sup \{f(x, y) : x \in X\}$$

Prove:

$$\sup \{f_1(x) : x \in X\} \leq \inf \{f_2(y) : y \in Y\}, \tag{1}$$

and this inequality can be strict.

Solution

Suppose for contradiction that the inequality given in (1) is false. Then we have

$$\inf \{f_2(y) : y \in Y\} < \sup \{f_1(x) : x \in X\}.$$

Since $\sup \{f_1(x) : x \in X\}$ is the least upper bound for $f_1(x)$, then $\inf \{f_2(y) : y \in Y\}$ fails to be an upper bound for $f_1(x)$, which means that there exists some $x_0 \in X$ such that $\inf \{f_2(y) : y \in Y\} < f_1(x_0) < \sup \{f_1(x) : x \in X\}$. Since $\inf \{f_2(y) : y \in Y\}$ is the greatest lower bound for $f_2(y)$, then $f_1(x_0)$ fails to be a lower bound for $f_2(y)$. This means that there exists some $y_0 \in Y$ such that $\inf \{f_2(y) : y \in Y\} < f_2(y_0) < f_1(x_0)$. However, this implies that $\sup \{f(x, y_0) : x \in X\} < \inf \{f(x_0, y) : y \in Y\}$, but this contradicts the definition of the supremum of a set. Hence, the result in (1) is proven. Consider the sequence $f(x, y) = \{1, -1, 1, -1, -1, -1, 1, \cdots\}$, consisting of only -1 and 1s. Clearly, f is bounded, but we see that $\sup \{f_1(x) : x \in X\} = -1$ and $\inf \{f_2(y) : y \in Y\} = 1$, so in this case, the inequality is strict. ∎

Problem 6

For $v = (v_1, v_2, ..., v_n)$ these norms are given by

$$\|v\|_1 = |v_1| + |v_2| + \cdots + |v_n| \tag{2}$$

$$\|v\|_2 = (v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2} \tag{3}$$

$$\|v\|_\infty = \max\{|v_1|, |v_2|, \cdots, |v_n|\} \tag{4}$$

Prove that each of these is a norm. Prove that if $\|v\|$ is a norm, then $d(v, w) = \|v - w\|$ is a metric.

Solution

We prove (2) first. Suppose $\|v\|_1 = 0$. Then $\|v\|_1 = |v_1| + |v_2| + \cdots + |v_n| = 0$. Since $|v_i|_1 \geq 0$ for all $i = 1, ..., n$, it's clear that $v_1 = v_2 = v_n = 0$, so $v = (v_1, v_2, ...v_n) = 0$. Conversely, suppose $v = 0$. Then $(v_1, ..., v_n) = 0$, so $|v_1| + |v_2| + \cdots + |v_n| = 0$, so $\|v\|_1 = 0$. In all other cases when $v \neq 0$, since $|v_i|_1 \geq 0$ for $i = 1, ..., n$, we see that $\|v\|_1 > 0$. Let $c \in \mathbb{R}$. Then

$$\begin{aligned}\|cv\|_1 &= |cv_1| + |cv_2| + \cdots |cv_n| \\ &= |c| |v_1| + |c| |v_2| + \cdots + |c| |v_n| \\ &= |c| (|v_1| + |v_2| + \cdots + |v_n|) \\ &= |c| \|v\|_1\end{aligned}$$

Finally, let $w = (w_1, \cdots, w_n)$. Then,

$$\|v + w\|_1 = |v_1 + w_1| + \cdots |v_n + w_n| \leq (|v_1| + |w_1| + \cdots + |v_n| + |w_n|) = \|v\|_1 + \|w\|_1$$

Hence we see that (2) satisifies the 3 properties of a norm, so (2) is a norm.

We now consider (3). Suppose $\|v\|_2 = 0$. Then $(v_1^2 + \cdots v_n^2)^{1/2} = 0$, so $(v_1^2 + \cdots v_n^2) = 0$. Since squares are non-negative, it's clear that $v_1 = v_2 = \cdots = v_n = 0$, so $v = 0$. Conversely, suppose $v = 0$. Then $v_i = v_i^2 = 0$ for $i = 1, ..., n$. Thus, $(v_1^2 + \cdots v_n^2)^{1/2} = 0$, so $\|v_n\|_2 = 0$. For other nonzero values of v, we see that $v_i^2 > 0$, so $(v_1^2 + \cdots v_n^2)^{1/2} = \|v_n\|_2 = 0$ as well. Let $c \in \mathbb{R}$. Then

$$\begin{aligned}\|cv\|_2 &= ((cv_1)^2 + (cv_2)^2 + \cdots + (cv_n)^2)^{1/2} \\ &= (c^2(v_1^2 + v_2^2 + \cdots + v_n^2))^{1/2} \\ &= |c| \cdot (v_1^2 + \cdots + v_n^2)^{1/2} \\ &= |c| \|v\|_2\end{aligned}$$

Let $w = (w_1, w_2, \cdots, w_n)$. Then

$$\begin{aligned}\|v + w\|_2 &= ((v_1 + w_1)^2 + (v_2 + w_2)^2 + \cdots + (v_n w_n)^2)^{1/2} \\ \|v + w\|_2^2 &= (v_1 + w_1)^2 + (v_2 + w_2)^2 + \cdots + (v_n w_n)^2 \\ &= (v_1^2 + 2v_1 w_1 + w_1^2 + \cdots + v_n^2 + 2v_n w_n + w_n^2) \\ &= \|v\|_2^2 + 2\langle v, w\rangle + \|w\|_2^2\end{aligned}$$

By the Cauchy-Schwarz Inequality, we can then write

$$\|v\|_2^2 + 2\langle v, w\rangle + \|w\|_2^2 \leq \|v\|_2^2 + 2\|v\|_2 \cdot \|w\|_2 + \|w\|_2^2 = (\|v\|_2 + \|w\|_2)^2$$

Problem 6 continued on next page. . .

$\therefore \|v+w\|_2^2 \le (\|v\|_2 + \|w\|_2)^2 \implies \|v+w\|_2 \le \|v\|_2 + \|w\|_2$. The three properties of a norm are satisfied, so (3) is a norm.

We now consider (4). Suppose $\|v\|_\infty = 0$. Then $\max\{|v_1|, |v_2|, \cdots, |v_n|\} = 0$, and since $|v_i| \ge 0$ for all $i = 1, ..., n$, we see that $v_1 = v_2 = \cdots = v_n = 0$, so $v = 0$. Conversely, suppose $v = 0$. Then $v_1 = v_2 = \cdots = 0$, which implies that $\max\{|v_1|, |v_2|, \cdots, |v_n|\} = 0$, so $\|v\|_\infty = 0$. Thus, for the other cases $v \ne 0$, we see that $\max\{|v_1|, |v_2|, \cdots, |v_n|\} > 0$, since $|v_i| \ge 0$ for nonzero v_i. Hence, $\|v\|_\infty > 0$. Let $c \in \mathbb{R}$. Then

$$\begin{aligned}\|cv\|_\infty &= \max\{|cv_1|, |cv_2|, \cdots, |cv_n|\} \\ &= \max\{|c|\,|v_1|, |c|\,|v_2|, \cdots, |c|\,|v_n|\} \\ &= |c| \cdot \max\{|v_1|, |v_2|, \cdots, |v_n|\} \\ &= |c| \cdot \|v\|_\infty\end{aligned}$$

Let $w = (w_1, w_2, \cdots, w_n)$. Then

$$\|v+w\|_\infty = \max\{|v_1+w_1|, |v_2+w_2|, \cdots, |v_n+w_n|\} \tag{5}$$

We know that $|v+w| \le |v| + |w|$, so $|v_i + w_i| \le |v_i| + |w_i|$ for all $i = 1, ..., n$. Thus, we can evaluate (5) as

$$\begin{aligned}\max\{|v_1+w_1|, |v_2+w_2|, \cdots, |v_n+w_n|\} &\le \max\{(|v_1|+|w_1|), (|v_2|+|w_2|), \cdots, (|v_n|+|w_n|)\} \\ &\le \max\{|v_1|, |v_2|, \cdots, |v_n|\} + \max\{|w_1|, |w_2|, \cdots, |w_n|\} \\ &= \|v\|_\infty + \|w\|_\infty\end{aligned}$$

$\therefore \|v+w\|_\infty \le \|v\|_\infty + \|w\|_\infty$. The three properties of a norm are satisfied, so (4) is a norm.
Now, we prove that if $\|v\|$ is a norm, then

$$d(v,w) = \|v-w\| \tag{6}$$

is a metric for $v, w \in V$. Since $\|v\|$ is a norm, then $\|v-w\| \ge 0$, and $\|v-w\| = 0$ if and only if $v - w = 0 \Leftrightarrow v = w$. Since $d(v,w) = \|v-w\|$, then $d(v,w) \ge 0, d(v,w) = 0$ if and only if $v = w$, satisfying property 1 of a metric. Consider:

$$\begin{aligned}d(w,v) &= \|w-v\| \\ &= \|(-1)\cdot(v-w)\| \\ &= |-1| \cdot \|v-w\| \\ &= \|v-w\| \\ &= d(v,w)\end{aligned}$$

Hence, we've shown that $d(v,w) = d(w,v)$ for any $v, w \in V$, satisfying property 2 of a metric. Let $x, y, z \in V$. Then evaluating $\|x-z\|$ and using the triangle inequality property of norms, we see that

$$\begin{aligned}d(x,z) &= \|x-z\| \\ &= \|x-y+y-z\| \\ &= \|(x-y)+(y-z)\| \\ &\le \|x-y\| + \|y-z\| \\ &= d(x,y) + d(y,z)\end{aligned}$$

$\therefore d(x,z) \le d(x,y) + d(y,z)$ for $x, y, z \in V$, so $\|v-w\|$ satisfies property 3 of a metric. Thus, we've shown that the distance function defined in (6) is a metric. ■

Problem 7

Prove the extended inequality for $v \in \mathbb{R}^n$.

$$\|v\|_\infty \leq \|v\|_2 \leq \|v\|_1 \leq \sqrt{n}\,\|v\|_2 \leq n\,\|v\|_\infty\,, \tag{7}$$

and show that each of the individual inequalities in (7) can be an equality for some (nonzero) choice of v.

Solution

Consider $\|v\|_\infty$. Without loss of generality, suppose $\max\{|v_1|, |v_2|, \cdots, |v_n|\} = |v_1|$, so $\|v\|_\infty = |v_1|$. In order to compare $\|v\|_\infty$ to $\|v\|_2 = (v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2}$, we first square both of these norms. We see that

$$\|v\|_\infty^2 = |v_1|^2 \leq v_1^2 + v_2^2 + \cdots + v_n^2 = \|v\|_2^2$$

$\therefore \|v\|_\infty^2 \leq \|v\|_2^2$. Taking the square root, we then get $\|v\|_\infty \leq \|v\|_2$.

Continuing with $\|v\|_2$, we see that

$$\begin{aligned}\|v\|_2^2 &= v_1^2 + v_2^2 + \cdots + v_n^2 \\ &= |v_1|^2 + |v_2|^2 + \cdots + |v_n|^2 \\ &\leq (|v_1| + |v_2| + \cdots + |v_n|)^2 \\ &= \|v\|_1^2\end{aligned}$$

$\therefore \|v\|_2^2 \leq \|v\|_1^2 \implies \|v\|_2 \leq \|v\|_1$.

Consider $\|v\|_1 = \sum_{j=1}^n |v|$. We can square the norm and evaluate it using the Schwarz Inequality to get

$$\|v\|_1^2 = \left(\sum_{j=1}^n |v|\right)^2 \leq \sum_{j=1}^n 1 \cdot \sum_{j=1}^n |v_n|^2 = n \cdot \sum_{j=1}^n v_j^2 = n \cdot \|v\|_2^2\,.$$

$\therefore \|v\|_1^2 \leq n \cdot \|v\|_2^2 . \Rightarrow \|v\|_1 \leq \sqrt{n}\,\|v\|_2$.

In order to prove the last inequality, we consider $\|v\|_\infty = \max\{|v_1|, \cdots |v_n|\}$. Without loss of generality, suppose $\max\{|v_1|, \cdots |v_n|\} = |v_1|$. Then squaring $\sqrt{n}\,\|v\|_2$, we get

$$n\,\|v\|_2^2 = \sum_{j=1}^n 1 \cdot \sum_{j=1}^n v_j^2 \leq \sum_{j=1}^n 1 \cdot \sum_{j=1}^n v_1^2 = n \cdot \|v\|_\infty^2 \leq n^2 \cdot \|v\|_\infty^2$$

$\therefore n\,\|v\|_2^2 \leq n^2\,\|v\|_\infty^2 \Rightarrow \sqrt{n}\,\|v\|_2 \leq n\,\|v\|_\infty$. By transitivity, we get

$$\|v\|_\infty \leq \|v\|_2 \leq \|v\|_1 \leq \sqrt{n}\,\|v\|_2 \leq n\,\|v\|_\infty\,.$$

Hence, the inequality is proven. Now we show that each of the individual inequalities in (7) can be an equality for some nonzero choice of v. For $v = (1, 0, 0, \cdots, 0) \in \mathbb{R}^n$ consisting of all 0s except for a 1 in the first element, we see that $\|v\|_\infty = 1 = \|v\|_2$. For the same v, we see that $\|v\|_2 = 1 = |1| + |0| + \cdots + |0| = \|v\|_1$. For $v = (1, 1, 1, \cdots, 1) \in \mathbb{R}^n$, we see that $\|v\|_1 = n$, and $\sqrt{n}\,\|v\|_2 = \sqrt{n} \cdot (1 + \cdots + 1)^{1/2} = \sqrt{n} \cdot \sqrt{n} = n = \|v\|_1$. For the same value of v we see that $n\,\|v\|_\infty = n \cdot 1 = n = \sqrt{n}\,\|v\|_2$. ■