Math 131B: Real Analysis | **Spring 2016**

Final Exam Notes

Lecturer: Dave Penneys | *Name: Eric Chuu*

2.0.1 Definitions

Limit Point: Suppose $(x_n) \subseteq (X, d)$ and let $L \in X$. L is a limit point of $(x_n)_{n=m}^{\infty} \Leftrightarrow \forall N \geq m$ and $\epsilon > 0, \exists n \geq N$ such that $d(x_n, L) \leq \epsilon$.
Closed: $E \subset (X, d)$ closed $\Leftrightarrow E$ contains all boundary points, $\partial E \subseteq E \Leftrightarrow E = \overline{E}$.
Open: $E \subset (X, d)$ open $\Leftrightarrow E$ contains none of its boundary points, $\partial E \cap E = \emptyset$.
Bounded: $S \subseteq X$ bounded $\Leftrightarrow \forall x \in X, \exists R > 0$ such that $S \subseteq B_R(x) \Leftrightarrow \exists x \in X, R > 0$ such that $B_R(x) \supset S \Leftrightarrow \text{diam}(S) = \sup\{d(x, y) : x, y \in S\} < \infty$.
Dense: $S \subseteq X$ dense $\Leftrightarrow \forall$ nonempty $U \subseteq X, S \cap U \neq \emptyset \Leftrightarrow \forall \epsilon > 0 B_\epsilon(x) \cap S \neq \emptyset \forall x \in X \Leftrightarrow X = \overline{S}$.
Totally Bounded: (X, d) totally bounded $\Leftrightarrow \forall \epsilon > 0, \exists x_1, x_2, \ldots, x_n$, such that $X \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$.

Pointwise Convergence: (f_n) converges to f (both functions from $(X, d_X) \to (Y, d_Y)$) pointwise $\Leftrightarrow \forall x \in X$ and every $\epsilon > 0, \exists N > 0$ such that $\forall n > N$, $d_Y(f_n(x), f(x)) < \epsilon$.

Uniform Convergence:(f_n) converges uniformly to f (both functions from $(X, d_X) \to (Y, d_Y)$) uniformly $\Leftrightarrow \forall \epsilon > 0, \exists N > 0$ such that $\forall n > N, d_Y(f_n(x), f(x)) < \epsilon, \forall x \in X \Leftrightarrow$ converges in the d_∞ metric.

Limit of function: $\lim_{x \to x_0} f(x) = L \Leftrightarrow \forall \epsilon > 0 \exists \delta > 0$ such that $0 < d_X(x, x_0) < \delta \implies d_Y(f(x), L) < \epsilon \Leftrightarrow x \in B_\delta(x_0) \setminus \{x_0\} \implies f(x) \in B_\epsilon(L)$.

Strongly Equivalent: Two metrics d_1, d_2 on X are strongly equivalent if there are $c_1, c_2 > 0$ such that $d_1(x, y) \leq c_2 d_2(x, y)$ and $d_2(x, y) \leq c_1 d_1(x, y)$, for all $x, y \in X$.

2.0.2 Propositions/Lemmas/Theorems

Prop: Let $(x_n)_{n=m}^{\infty} \subseteq (X, d)$. TFAE: (1) L is limit point of $(x_n)_{n=m}^{\infty}$. (2) $\exists$ a subsequence (x_{n_j}) of the original sequence which converges to L.
Lemma: Let $(x_n)_{n=m}^{\infty}$ be a Cauchy subsequence in (X, d). Suppose that there is some $(x_{n_j})_{j=1}^{\infty}$ of this subsequence which converges to some $x_0 \in X$. Then the original sequence converges to x_0.
Theorem: Let (X, d_X) be a metric space, let (Y, d_Y) be complete. Then the space $(C(X, Y), d_\infty|_{C(X,\mathbb{R}) \times C(X,\mathbb{R})})$ is a complete subspace of $L^\infty(X, Y), d_\infty$. In other words, every Cauchy sequence of functions in $C(X, Y)$ converges to a function in $C(X, Y)$.

2.0.3 More Definitions

Space of Bounded Functions: $L^\infty(X, Y) = \{f | f : X \to Y \text{ bounded}\}$
Space of Continuous Functions: $C(X, Y) = \{f | f : X \to Y \text{ continuous}\}$
Space of Continuous, Bounded Functions: $C_B(X, Y) = \{f | f : X \to Y \text{ continuous, bounded}\}$

Sup-Norm($L^\infty(X, Y)$) Metric: $d_\infty(f, g) : L^\infty(X, Y) \times L^\infty(X, Y) \to \mathbb{R}^+$,

$$d_\infty(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$$

Sup Norm: If $f : X \to \mathbb{R}$ is a bounded real-valued function, the sup norm

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}$$

L^2 **metric**: Define the L^2 metric

$$d_{L^2}(f, g) := \|f - g\|_2 = \Big(\int_{[0,1]} |f(x) - g(x)|^2 dx \Big)^{1/2}$$

Lebesgue Number Lemma Suppose that (X, d) is a (sequentially) compact metric space. Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of X. Then there exists a constant $\delta > 0$ which satisfies: for all $x \in X$ there exists $\alpha \in I$ such that $B_\delta(x) \subseteq U_\alpha$.

Weierstrass M-Test Let (X, d) be a metric space and let (f_n) be a sequence of bounded, real-valued continuous functions on X such that the series $\sum_{n=1}^{\infty} \|f_n\|_\infty$ is absolutely convergent. Then the series $\sum_{n=1}^{\infty}$ converges uniformly to some function f on X, and the function is also continuous.

Fourier Transform For all $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{R})$ and any $n \in \mathbb{Z}$ define the n-th Fourier coefficient (2.1) and the Fourier inversion formula (2.2) of f

$$\hat{f}(n) := c_n = \langle f, e_n\rangle = \int_0^1 f(x)e^{-2\pi inx}dx \tag{2.1}$$

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n)e_n = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx} \tag{2.2}$$

Fourier Theorem For any $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, the series $\sum_{n=\infty}^{\infty} \hat{f}(n)e_n$ converges in L^2 metric to f. In other words, we have

$$\lim_{N\to\infty} \left\| f - \sum_{n=-N}^{N} \hat{f}(n)e_n \right\|_2 = 0$$

Plancherel Theorem For any $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, the series $\sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2$ is absolutely convergent, and $\left\|\hat{f}\right\|_2^2 = \sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2$

Power series A power series centered at a is: $\sum_{n=0}^{\infty} c_n(x-a)^n$

Real Analytic Let $f : E \to \mathbb{R}$. If $a \in Int(E)$, f is real analytic at a if there exists a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ centered at a with radius of convergence greater than or equal to r which converges to f on $(a-r, a+r)$.

Theorem (Convolution): Let $f : (a-r, a+r) \to \mathbb{R}$, $g : (a-r, a+r) \to \mathbb{R}$ be functions analytic on $(a-r, a+r)$, with power series expansions $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ and $g(x) = \sum_{n=0}^{\infty} d_n(x-a)^n$. Then $fg : (a-r, a+r) \to \mathbb{R}$ is analytic on $(a-r, a+r)$ power series expansion:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} c_n d_{n-m}(x-a)^n$$

Dini's Theorem Suppose (f_n), f are functions in $C(X, \mathbb{R})$, where X is compact. If $f_n \to 0$ pointwise, and f_n is monotonically increasing, then $f_n \to 0$ uniformly.

Weierstrass Approximation (Polynomials) If $[a, b]$ is an interval, $f : [a, b] \to \mathbb{R}$ is a continuous function, and $\epsilon > 0$, then there exists a polynomial P on $[a, b]$ such that $d_\infty(P, f) \le \epsilon(i.e., |P(x) - f(x)| \le \epsilon$ for all $x \in [a, b])$

Trigonometric Functions: If $z \in \mathbb{C}$, then $\cos(z) := \frac{e^{iz}+e^{-iz}}{2}$, $\sin(z) := \frac{e^{iz}-e^{-iz}}{2i}$

Trig Identities: For $x, y \in \mathbb{R}$, $e^{ix} = \cos(x) + i\sin(x)$ and $e^{-ix} = \cos(x) - i\sin(x)$.

Theorem (Extreme Value): Let (X, d) be a compact metric space and let $f : X \to \mathbb{R}$ be continuous. Then f is bounded. Also, f attains its max at some point $x_0 \in X$ and its min at some point $x_1 \in X$.

Theorem (IVT): Let $f : X \to \mathbb{R}$ be a continuous map from (X, d_X) to the real line. Let $E \subset X$ be connected, $a, b \in E$. Let $y \in [f(a), f(b)]$. Then $\exists c \in E$ such that $f(c) = y$.