Metric Spaces Light

These notes are intended to relate the discussion in the lectures to pages 30-40 in Rudin.

A general metric space is a set χ with elements called points, and a distance function $d(x, y)$ defined on $\chi \times \chi$. To be a distance function d has to be real-valued and satisfy three properties:

$$(i)\ d(x, y) \geq 0 \text{ and } d(x, y) = 0 \Leftrightarrow x = y,\ (ii)\ d(x, y) = d(y, x) \text{ and}$$

$$(iii)\ d(x, y) \leq d(x, z) + d(z, y) \text{ for all } x, y, z \in \chi.$$

I am starting with $\chi = \mathbb{R}^n$ and d one of three choices $d_1(x, y) = ||x-y||_1$, $d_2(x, y) = ||x-y||_2$ or $d_\infty(x, y) = ||x-y||_\infty$, where $||\cdot||_1$, $||\cdot||_2$ and $||\cdot||_\infty$ are the three norms on $\mathbb{R}^n$ from Assignment Two. So d will always stand for one of those distance functions on $\mathbb{R}^n$.

The essential definitions for any subset S of $\mathbb{R}^n$ with complement S^c are:

$$B_r(x) = \{y \in \mathbb{R}^n : d(x, y) < r\} \text{ with } r > 0, \text{ the open ball of radius r centered at x}$$

$$Int(S) = \{y \in \mathbb{R}^n : B_r(y) \subset S, \text{ for } some\ r > 0\}, \text{ the interior of S,}$$

$$Ext(S) = \{y \in \mathbb{R}^n : B_r(y) \subset S^c, \text{ for } some\ r > 0\}, \text{ the exterior of S, and}$$

$$Bdy(S) = \{y \in \mathbb{R}^n : B_r(y) \cap S \neq \emptyset \text{ and } B_r(y) \cap S^c \neq \emptyset \text{ for } all \text{ L } r > 0\},$$

the boundary of S. There are several immediate logical conclusions from those definitions:

$$Int(S^c) = Ext(S), \quad Bdy(S) = Bdy(S^c) \text{ and } Int(S) \cup Bdy(S) \cup Ext(S) = \mathbb{R}^n.$$

It is worth checking (use problem 13) that for any set $S \subset \mathbb{R}^n$, $Int(S)$, $Ext(S)$ and $Bdy(S)$ do not depend on which choice for d we use to define them.

More definitions:

$$S \text{ is } open \quad \Leftrightarrow S \cap Bdy(S) = \emptyset, \text{ and } S \text{ is } closed \quad \Leftrightarrow S \cap Bdy(S) = Bdy(S).$$

In other words S is open $\Leftrightarrow S = Int(S)$, and S is closed $\Leftrightarrow S = Int(S) \cup Bdy(S)$. That has the immediate logical conclusion: S is open $\Leftrightarrow S^c$ is closed.

Some basic results:

Prop. 1: $B_r(x)$ is open for all $x \in \mathbb{R}^n$ and $r > 0$.

Proof: We need to show that all points in $B_r(x)$ are interior points. So, given $y \in B_r(x)$, we need to find $s > 0$ such that $B_s(y) \subset B_r(x)$. Since $y \in B_r(x)$, $d(x, y) < r$. If we choose $s = r - d(x, y)$, we have $s > 0$, and, for any $z \in B_s(y)$, the triangle inequality (property (iii) of d) implies

$$d(z,x) \leq d(z,y) + d(y,x) < s + d(y,x) = r.$$

So $B_s(y) \subset B_r(x)$, and we can conclude that $B_r(x)$ is open.

Prop. 2: For any set $S \subset \mathbb{R}^n$, $Int(S)$ is open.

Proof: If $x \in Int(S)$, there is an $r > 0$ such that $B_r(x) \subset S$. I claim $B_r(x) \subset Int(S)$. To see that you can use the argument that showed $B_r(x)$ was open: given $y \in B_r(x)$, we need to find $s > 0$ such that $B_s(y) \subset B_r(x)$. Since $y \in B_r(x)$, $d(x,y) < r$. If we choose $s = r - d(x,y)$, we have $s > 0$, and, for any $z \in B_s(y)$, $d(z,x) \leq d(z,y) + d(y,x) < s + d(y,x) = r$. So $B_r(x) \subset Int(S)$. This shows all points in $Int(S)$ are interior points. So $Int(S)$ is open.

Prop. 3: Finite intersections and arbitrary unions of open sets are open.

Proof: Given open sets $\mathcal{O}_j,\ j = 1, \dots N$, let $x \in \cap_{j=1}^N \mathcal{O}_j$ We need to show that x is an interior point. Since each $\mathcal{O}_j$ is open, we have positive $r_j,\ j = 1, \dots, N$, such that $B_{r_j}(x) \subset \mathcal{O}_j$. Let $r = \min\{r_j,\ j = 1, \dots, N\}$. Since r is the minimum of a finite set of positive numbers, $r > 0$, and $B_r(x) \subset \mathcal{O}_j$ for all j. Thus $B_r(x) \subset \cap_{j=1}^N \mathcal{O}_j$, and we conclude that $\cap_{j=1}^N \mathcal{O}_j$ is open.

To write an arbitrary union of open sets let A be any set and label the sets with the elements of A. So $\cup_{\alpha \in A} \mathcal{O}_\alpha$ is an arbitrary union of open sets. If $x \in \cup_{\alpha \in A} \mathcal{O}_\alpha$, then x is in one of the $\mathcal{O}_\alpha$'s, $\mathcal{O}_{\alpha_0}$. Since $\mathcal{O}_{\alpha_0}$ is open, $B_{r_0}(x) \subset \mathcal{O}_{\alpha_0}$ for some $r_0 > 0$. So $B_{r_0}(x) \subset \cup_{\alpha \in A} \mathcal{O}_\alpha$ and $\cup_{\alpha \in A} \mathcal{O}_\alpha$ is open.

A basic theorem in set theory is $(\cup_{\alpha \in A} S_\alpha)^c = \cap_{\alpha \in A} S_\alpha^c$ (Theorem 2.22 in Rudin). If you use that and remember S is open $\Leftrightarrow$ S^c is closed, Prop. 2 becomes

Prop. 4: Finite unions and arbitrary intersections of closed sets are closed.

If you combine Prop. 2 with $Ext(S) = Int(S^c)$, you see that $Ext(S)$ is also open. Finally, writing $Bdy(S) = (Int(S) \cup Ext(S))^c$, you see that $Bdy(S)$ is closed.

Closure: For any set $S \subset \mathbb{R}^n$, the set $\overline{S} = S \cup Bdy(S)$ is called the "closure" of S. Since any smaller set that contains S, will have to omit part of $Bdy(S)$, $\overline{S}$ is the smallest closed set containing S.

This is the end of the basic picture. There is more in pages 30-36 in Rudin, but this is the part that we will use.

Compact Sets

Compact sets are important, and there are two equivalent ways to define them:

Definition 1(Bolzano-Weierstrass): A set S is compact $\Leftrightarrow$ Given any sequence of points, $\{p_n\}$, $p_n \in S$, there is at least one $p_\infty \in S$ such that for every $r > 0$, $p_n \in B_r(p_\infty)$ for *infinitely many n*.

Definition 2 (Heine-Borel): A set S is compact $\Leftrightarrow$ Given *any* set of open sets $\mathcal{O}_\alpha,\ \alpha \in A$ such that $S \subset \cup_{\alpha\in A}\wr_\alpha$, there is a *finite* subset of the $\mathcal{O}_\alpha$'s, $\mathcal{O}_{\alpha_j},\ j = 1,\ldots,N$, such that one has $S \subset \cup_{j=1}^{N}\mathcal{O}_{\alpha_j}$.

Definition 2 is usually stated "Every open cover of S has a finite subcover". The most important theorem for us is the following:

Theorem: Every closed, bounded subset of $\mathbb{R}^n$ is compact. This is $(a) \Rightarrow (b)$ in Theorem 2.41 in Rudin, and we will prove it soon.

To me this theorem is more plausible with Definitions 1: If you pack an infinite number of points into a bounded set, they will have to bunch up somewhere (that's p_∞). They might bunch up at some point on the boundary, but that's OK because $Bdy(S)$ is contained in S. Rudin likes Definition 2. In many other places it is the easiest definition to use. I will call Definition 1 (BW) and Definition 2 (HB). As I said above it is a theorem that in any metric space (BW)-compact sets are the same as (HB)-compact sets. One half of that equivalence is pretty easy: I will postpone the other half, (BW)-compact sets are (HB)-compact, because it's quite a bit harder.

Theorem: (HB)-compact sets are (BW)-compact.

Proof: One does this by proving the contra-positive: if a set is not (BW)-compact, then it is not (HB)-compact. So suppose that we have $\{p_n\} \subset S$, but there is no point p_∞ in S. Note first that $\{p_n\}$ must contain an infinite number of *different* points, or p_n would have to be the *same* point for infinitely many n, and we could take that point to be p_∞. Since there is no p_∞, we can assume that for every $p \in S$ there is an $r > 0$ such that $B_r(p)$ contains p_n for only *finitely* many n. That means that we can take r smaller so that no p_n's are in $B_r(p)$, except for p itself if it happens to be a p_n. The conclusion from that is that for every $p \in S$ there is an $r(p) > 0$ such that $B_{r(p)}(p)$ contains no p_n's or one p_n. Now by Prop. 1 $\cup_{p\in S}B_{r(p)}(p)$ is an open cover of S, but any finite subset of it could not cover all the p_n's because there are an infinite number of different p_n's and at most one of them is in each $B_{r(p)}(p)$.

I will postpone the other half of the equivalence, (BW)-compact sets are (HB)-compact, because it's quite a bit harder, and end this installment of notes here. For the rest of pages 36-40 I intend to follow Rudin.