A Remark on Connectedness

The final topological concept in this course is "connectedness". Intuitively a set is connected if you cannot divide it into separated pieces: $[0,2]$ is connected, but $[0,1) \cup (1,2]$ is not. Formalizing this idea is tricky. First of all one always defines "disconnected" and says a set is connected if it is not disconnected. The simplest statement is:

(a) A metric space (χ, d) is disconnected if the there are two nonempty open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ in χ such that $\mathcal{O}_1 \cup \mathcal{O}_2 = \chi$ and $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$,

The problem with this definition is that one can only talk about connectedness of a whole metric space. So for $[0,1) \cup (1,2]$ we would have to take $\chi = [0,1) \cup (1,2]$ with the metric $d(x,y) = |x-y|$. The definition works, $\mathcal{O}_1 = [0,1)$ and $\mathcal{O}_2 = (1,2]$ are open sets in that metric space, but it is more natural to think of $[0,1) \cup (1,2]$ as a subset of $\mathbb{R}$. To use the definition in this situation one needs to have a way to recognize open sets in subspaces of a metric space. Here it is:

Theorem: If the metric space (χ, d) is actually a subset of a larger metric space $(\tilde{\chi}, d)$, i.e. $\chi \subset \tilde{\chi}$ but d is the same, then a set E in (χ, d) is open if and only if there is an open set $\mathcal{O}$ in $(\tilde{\chi}, d)$ such that $\mathcal{O} \cap \chi = E$. [This is Theorem 2.30 on page 36 in Rudin, and I have put an example of how it works at the end of these notes.]

Proof: To see that E is open in (χ, d) if $E = \mathcal{O} \cap \chi$ for some open set $\mathcal{O}$ in $(\tilde{\chi}, d)$, take $p \in E$. We need $B_r(p) = \{q \in \chi : d(q,p) < r\} \subset E$. Since $\mathcal{O}$ is open, there is a $\tilde{B}_{r_0}(p) = \{q \in \tilde{\chi} : d(q,p) < r_0\} \subset \mathcal{O}$. Then $\tilde{B}_{r_0}(p) \cap \chi = B_{r_0}(p)$. Since $\mathcal{O} \cap \chi = E$, this implies $B_{r_0}(p) \subset E$.

For other direction suppose that E is open in (χ, d). Then for each $p \in E$, there is a $B_{r(p)}(p) \subset E$ and $E = \cup_{p \in E} B_{r(p)}(p)$. If we let $\tilde{B}_{r(p)}(p)$ be the ball in $\tilde{\chi}$ with radius $r(p)$ and center p, then $\mathcal{O} = \cup_{p \in E} \tilde{B}_{r(p)}(p)$ is open in $(\tilde{\chi}, d)$, and $\mathcal{O} \cap E = E$.

Rudin defines connectedness differently:

(b) Two nonempty sets A and B in a metric space $(\tilde{\chi}, d)$ are said to be "separated " if $A \cap \overline{B} = \overline{A} \cap B = \emptyset$. A set is connected if and only if it cannot be divided in separated pieces.

We can use the theorem above to show that definition (b) is equivalent to definition (a). To see that disconnected in the sense of definition (b) implies disconnected in the sense of definition (a), take $\mathcal{O}_A = \overline{B}^c$ and $\mathcal{O}_B = \overline{A}^c$. Both those are complements of closed sets, and have to be open. Since $\mathcal{O}_A \cap (A \cup B) = A$ and $\mathcal{O}_B \cap (A \cup B) = B$, the nonempty, disjoint sets A and B are open in the metric space (χ, d) when we take $\chi = A \cup B$. So χ is disconnected is the sense of definition (a).

Conversely, assume that $\chi = A \cup B$ where A and B are nonempty, open and $A \cap B = \emptyset$. So χ is disconnected in the sense of definition (a). If A and B are sets in the larger metric space $(\tilde{\chi}, d)$, we need to show that they satisfy definition (b). Suppose that $p \in \overline{A} \cap B$, where the closure of A is taken in $\tilde{\chi}$. Then $\{q \in \tilde{\chi} : d(p,q) < r\} \cap A$ is nonempty for every $r > 0$. But, since $A \subset A \cup B = \chi$, that says $\{q \in \chi : d(p,q) < r\} \cap A$ is nonempty for every $r > 0$, and $p \in \overline{A}$ in χ. A is closed in (χ, d): its complement B is open. So $p \in A$, contradicting $A \cap B = \emptyset$.

Example: The set $I = \{(x, y) \in \mathbb{R}^2 : -1 < x < 1, y = 0\}$ in $\mathbb{R}^2$ is definitely not open: for any of our metrics, d_1, d_2 or d_∞, $Int(I) = \emptyset$. However, for any of those metrics on $\mathbb{R}^2$, I is the intersection of the open ball $B_1((0, 0))$ with $\chi = \{(x, 0)) : x \in \mathbb{R}\}$. So I is not open as a subset of $(\mathbb{R}^2, d)$ but it is open as a subset of (χ, d).