MATH 131AH: Homework #1

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Assignment: 1, 2, 3, 4, 5, 6

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Problem 1

(a) Prove that in an ordered field, if $a^2 + b^2 = 0$, then $a = b = 0$.

Solution
Suppose for contradiction that the conclusion $a = b = 0$ is false. Then we consider cases:
Case 1: $a = 0, b \neq 0$.
Evaluating both sides, we get $b^2 = 0$, but by Proposition 1.18d, we have that if $b \neq 0$, then $b^2 > 0$, so case 1 fails.

Case 2: $a \neq 0, b = 0$.
Evaluating both sides, we get $a^2 = 0$. By the same reasoning used in case 1, we see that case 2 fails as well.

Case 3: $a \neq 0, b \neq 0$.
Since neither a nor b are 0, we have $a^2 + b^2 = 0$. Rearranging, we get $a^2 = -b^2$. However, this contradicts Proposition 1.18d, which states that $x \neq 0$, then $x^2 > 0$. In this case, neither term is equal to 0, so $a^2 > 0$ and $b^2 > 0$. Thus, case 3 fails as well.

The 3 alternative cases failed, and the only case left is: $a = b = 0$. Our assumption was false, thus establishing the result. ■

(b) Prove that it is not possible to make the complex numbers into an ordered field.

Solution
The element i exists in the complex field $\mathbb{C}$, and $i^2 = -1$, which contradicts the necessity for squares to be nonnegative in ordered fields. Thus, we cannot make the complex numbers into an ordered field. ■

Problem 2

(a) If $a \in \mathbb{R}$ and $1 + a > 0$, prove by induction that $(1 + a)^n \geq 1 + na$ for all $n \in \mathbb{N}$.

Base Case: $n = 1$. Then the LHS $= (1+a)^1 = 1+a$, and the RHS $= 1+a$.The LHS = RHS, thus establishing the base case.

Inductive Step: Suppose the inequality holds for $n = k$. To complete the proof, we show that it is also true for $n = k + 1$, so we try to prove that

$$(1 + a)^{k+1} \geq 1 + (k + 1)a$$

Starting with the LHS, we have

$$(1 + a)^{k+1} = (1 + a)(1 + a)^k.$$

By the inductive hypothesis, we can then write

$$(1 + a)(1 + a)^k \geq (1 + a) * (1 + ka) = 1 + a(k + 1) + ka^2.$$

By assumption, we $1 + a > 0$, so $a > -1$, so $ka^2 \geq 1$. Therefore,

$$1 + a(k + 1) + ka^2 = \text{ RHS } + ka^2 \geq \text{ RHS }.$$

Having shown the inequality holds for $n = k + 1$, we have shown that it holds for all $n \in \mathbb{N}$. ■

(b) Prove that $n < 2^n$ for all $n \in \mathbb{N}$.

Base Case: $n = 1$. Evaluating the inequality, we get $1 < 2$, thus satisfying the base case.

Inductive Step: Suppose the inequality holds for $n = k$. To complete the proof, we show that it is also true for $n = k + 1$, so we try to prove that $(k + 1) < 2^{k+1}$. We can then write

$$k + 1 \leq k + k < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

Having shown the inequality holds for $n = k + 1$, we have shown that it holds for all $n \in \mathbb{N}$. ■

Problem 3

Prove that if $x, y \in \mathbb{Q}$ and $x < y$, then there is an irrational number r such that $x < r < y$.

Solution

We first show that if q is rational ($q \neq 0$) and z is irrational, then qz is irrational. Suppose for contradiction that $qz \in \mathbb{Q}$. Then we can write $qz = \frac{a}{b}$ for integers a, b ($b \neq 0$). Since $q \neq 0$ by assumption, we can rewrite this:

$$z = \frac{a}{b} \cdot \frac{1}{q} \tag{1}$$

Since $q \in \mathbb{Q}$, we can write $q = \frac{c}{d}$, where $c, d \in \mathbb{Z}$ and $d \neq 0$. Rewriting (1), we have

$$z = \frac{a}{b} \cdot \frac{1}{(c/d)} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

where $ad, bc \in \mathbb{Z}$. However, this contradicts the irrationality of z, thus proving that the product of an irrational number and a nonzero rational number is irrational.

Now, given $x, y \in \mathbb{Q}$ and $x < y$, we know from the previous result that if we multiply x, y by an irrational number, say $\frac{1}{\sqrt{2}}$, the product would be irrational. So, $\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}$ are both irrational and real numbers. Since the rationals are dense in $\mathbb{R}$, we can find a number $q \neq 0$ such that

$$\frac{x}{\sqrt{2}} < q < \frac{y}{\sqrt{2}}.$$

Multiplying through the inequality by $\sqrt{2}$, we get

$$x < \sqrt{2}q < y$$

and again applying the previous result, we see that $\sqrt{2}q$ is an irrational number. Thus, given $x, y \in \mathbb{Q}$, $x < y$, we can find an irrational number r such that $x < r < y$. ■

Problem 4

Show that the bounded open intervals $I_n = (0, 1/n), n \in \mathbb{N}$, do not have a common point, and show that the closed, unbounded sets $F_n = [n, \infty), n \in \mathbb{N}$ also do not have a common point.

Problem 4 continued on next page...

Solution
We first consider the bounded open intervals $I_n = (0, 1/n), n \in \mathbb{N}$. We want to show that the intersection of the sets I_n for $n \in \mathbb{N}$ is $\emptyset$, that is: $\cap_{n=1}^{\infty}(0, 1/n) = \emptyset$.

Suppose for contradiction that the intersection is not empty. Then, $\exists x$ such that $x \in I_n, \forall n \in \mathbb{N}$. This implies that $x \in (0, 1/n), \forall n \in \mathbb{N}$. Note here that $x \in \mathbb{R}, x > 0$, so $1/x \in \mathbb{R}$. $1 \in \mathbb{R}, 1 > 0$ so by the Archimedean Property, we can find a $n_0 \in \mathbb{N}$ such that

$$n_0 \cdot 1 > 1/x.$$

Rewriting this, we get:

$$1/n_0 < x,$$

but this implies that $x \notin (0, 1/n_0)$, which is a contradiction to our assumption that $x \in I_n$ for all $n \in \mathbb{N}$. This contradiction establishes that $\cap_{n=1}^{\infty}(0, 1/n) = \emptyset$.

Now, we try to show that for the closed, unbounded sets $F_n = [n, \infty), n \in \mathbb{N}$, the intersection of these sets is the empty set. That is, $\cap_{n=1}^{\infty} F_n = \emptyset$.

Suppose for contradiction that the intersection is not empty. Then $\exists x$ such that $x \in F_n, \forall n \in \mathbb{N}$. Then $x \in [n, \infty), \forall n \in \mathbb{N}$. Similar to the approach as above, we want to show that there is a number $n_0 \in \mathbb{N}$ such that $x < n_0$, which would imply that $x \notin [n_0, \infty)$, thus contradicting our assumption that $x \in F_n, \forall n \in \mathbb{N}$. Since $1 \in \mathbb{R}, 1 > 0, x \in \mathbb{R}$, then by the Archimedean Property, there exists an $n_0 \in \mathbb{N}$ such that $n_0 \cdot 1 > x$, thus establishing the desired contradiction. Hence, $\cap_{n=1}^{\infty} F_n = \emptyset$. ■

Problem 5

If S is a bounded set of real numbers and S_0 is a nonempty subset of S, then

$$inf(S) \leq inf(S_0) \leq sup(S_0) \leq sup(S).$$

Solution
First, we consider the existence of the lower and upper bounds. $S_0 \subset S$, and S_0 is nonempty, so S is also nonempty. S is both both bounded and nonempty, so $inf(S), sup(S)$ both exist in $\mathbb{R}$. Since $S_0 \subset S$, it is also bounded. Since S_0 is both bounded and nonempty, $inf(S_0), sup(S_0)$ also exist in $\mathbb{R}$ (the infimum exists by theorem 1.11).

We now prove this one inequality at a time. Let $x \in S$. Since $inf(S)$ the greatest lower bound of S, then $inf(S) \leq x, \forall x \in S$. Since we chose an arbitrary x and $S_0 \subset S$, and noting that $inf(S_0)$ is the greatest lower bound for S_0, then we've shown that $inf(S) \leq inf(S_0)$.

Next, we let $x \in S_0$. Then by definition of $inf(S_0)$, we know that $inf(S_0) \leq x, \forall x \in S_0$. Since $sup(S_0)$ is an upper bound for S_0, then $x \leq sup(S_0), \forall x \in S_0$. The real numbers are an ordered field, so by transitivity, $inf(S_0) \leq sup(S_0)$.

Let $x \in S$. $sup(S)$ is the least upper bound for S, so $x \leq sup(S), \forall x \in S$. Since x was arbitrary, $S_0 \subset S$, and $sup(S_0)$ is the least upper bound of S_0, we've shown that $sup(S_0) \leq sup(S)$. Combining the inequalities, we get $inf(S) \leq inf(S_0) \leq sup(S_0) \leq sup(S)$. ■

Problem 6

(a) For any sequence of real numbers, $b_0, b_1, b_2, ...$ with $b_j > 0$ for $j \geq 1$, define sequences $\{P_n\}_{n=-1}^{\infty}$ and $\{Q_n\}_{n=-1}^{\infty}$ recursively by $P_{-1} = 1, P_0 = b_0, P_k = b_k P_{k-1} + P_{k-2}$ for $k \geq 1$, and $Q_{-1} = 0, Q_0 = 1, Q_k = b_k Q_{k-1} + Q_{k-2}$ for $k \geq 1$. Prove by induction that

$$\{b_0; b_1, b_2, ..., b_n\} = P_n/Q_n.$$

Solution
We prove this inductively.
Base Case: $n = 1$. Then

$$\begin{aligned} \text{LHS } &= [b_0; b_1] \\ &= b_0 + \frac{1}{b_1}. \end{aligned}$$

Evaluating the right-hand side, we get

$$\begin{aligned} \text{RHS } &= P_1/Q_1 \\ &= \frac{b_1 \cdot P_0 + P_{-1}}{b_1 \cdot Q_0 + Q_{-1}} \\ &= \frac{b_1 b_0 + 1}{b1} \\ &= \frac{b_1 b_0}{b_1} + \frac{1}{b_1} \\ &= b_0 + \frac{1}{b_1} \\ &= \text{LHS}. \end{aligned}$$

In the 4th equality, we can cancel the b_1 since $b_1 > 0$. Thus, the equality holds for the base case.

Inductive Step: Suppose the equality is true for $n = k$. We show it is also true for $n = k + 1$. Evaluating the LHS, we get

$$\begin{aligned} \text{LHS } &= [b_0; b_1, ..., b_k, b_{k+1}] \\ &= [b_0; b_1, ..., b_k + \frac{1}{b_{k+1}}] \end{aligned}$$

By the inductive hypothesis, we can further evaluate this:

$$\begin{aligned} [b_0; b_1, ..., b_k + \frac{1}{b_{k+1}}] &= P'_k/Q'_k \\ &= \frac{b'_k P_{k-1} + P_{k-2}}{b'_k Q_{k-1} + Q_{k-1}} \\ &= \frac{(b_k + \frac{1}{b_{k+1}})P_{k-1} + P_{k-2}}{(b_k + \frac{1}{b_{k+1}})Q_{k-1} + Q_{k-2}} \\ &= \frac{b_k b_{k+1} P_{k-1} + P_{k-1} + b_{k+1} P_{k-2}}{b_k b_{k+1} Q_{k-1} + Q_{k-1} + b_{k+1} Q_{k-2}} \\ &= \frac{b_{k+1}(b_k P_{k-1} + P_{k-2}) + P_{k-1}}{b_{k+1}(b_k Q_{k-1} + Q_{k-2}) + Q_{k-1}} \\ &= \frac{b_{k+1} P_k + P_{k-1}}{b_{k+1} Q_k + Q_{k-1}} \\ &= \frac{P_{k+1}}{Q_{k+1}} \\ &= \text{RHS}. \end{aligned}$$

Problem 6 continued on next page. . .

We've shown that the equality holds for $n = k + 1$, so by induction we've shown that equality holds for all $n \geq 1$. ■

(b) Prove that $P_{n-1}Q_n - P_nQ_{n-1} = (-1)^n$ when $n \geq 0$.

Solution
We prove this inductively.
Base Case: n = 0. Then

$$\begin{aligned} \text{LHS} &= P_{-1}Q_0 - P_0Q_{-1} \\ &= 1 - b_0 \cdot 0 \\ &= 1 = (-1)^0 = \text{RHS}, \end{aligned}$$

so the equality is satisfied for the base case.
Inductive Step: Suppose true for $n = k$. We show true for $n = k + 1$. Evaluating the LHS, we get

$$\begin{aligned} \text{LHS} &= P_kQ_{k+1} - P_{k+1}Q_k \\ &= (b_{k+1}Q_k + Q_{k-1}) \cdot P_k - (b_{k+1}P_k + P_{k-1}) \cdot Q_k \\ &= b_{k+1}Q_kP_k + Q_{k-1}P_k - b_{k+1}P_kQ_k - P_{k-1}Q_k \\ &= Q_{k-1}P_k - P_{k-1}Q_k \\ &= -(P_{k-1}Q_k - Q_{k-1}P_k) \end{aligned} \tag{2}$$

By the inductive hypothesis, the last equality in (2) above can be written

$$\begin{aligned} -(P_{k-1}Q_k - Q_{k-1}P_k) &= -1 \cdot (-1)^k \\ &= (-1)^{k+1} \\ &= \text{RHS}. \end{aligned}$$

We've shown that the equality holds for $n = k + 1$, so by induction it holds for all $n \geq 0$. ■

(c) Prove $[a_0; a_1, a_2, ..., a_n] < x$ for n even, and $[a_0; a_1, a_2, ..., a_n] > x$ for n odd.

Solution
We first note that $[a0; a1, a2, ..., a_{n-1}, x_n] = x$. Suppose n is even. We know that $a_n < x_n$, so equivalently, $\frac{1}{a_n} > \frac{1}{x_n}$. Then we can write

$$[a_{n-1}; a_n] = a_{n-1} + \frac{1}{a_n} > a_{n-1} + \frac{1}{x_n} = [a_{n-1}; x_n] \tag{3}$$

$$[a_{n-2}; a_{n-1}, a_n] < [a_{n-2}; a_{n-1}, x_n] \tag{4}$$

Inequality (3) shows for an odd increment, $[a_{n-1}; a_n] > [a_{n-1}; x_n]$, and for an even increment as seen in (4), the inequality flips back to <. Continuing these increments and observing that the inequality flips n times, we eventually get to $[a_0; a_1, ..., a_n] < [a_0; a_1, ..., x_n]$ after an at most countable number of steps. As a result of the inequality flipping on every increment, when n is odd, $[a_0; a_1, ..., a_n] > x$. ■