Lecture 18: 2D Hyperbolic CG

Introduction Now that we have learned how to discretize elliptic eqs. in 2D, we can now tackle time-dependent problems such as those represented by hyperbolic eqs of the type:

$$\frac{\partial q}{\partial t} + \bar{u} \cdot \bar{\nabla} q = \nu \nabla^2 q \tag{18.1}$$

where $Re = \frac{uL}{\nu}$ is the Reynolds number (ratio of inertial to viscous forces) & u & L are the characteristic velocity (u) & length (L) of the problem, & ν is the kinematic viscosity.

$\mu = \rho \nu$ is the dynamic viscosity

Integral Form

To solve (18.1) we approximate q & $\bar{u}$ as follows:

(18.2)

$$q_N^{(e)}(\bar{x}, t) = \sum_{j=1}^{M_N} \psi_j(\bar{x})\, q_j^{(e)}(t)$$

$$\& \quad \bar{u}_N^{(e)}(\bar{x}) \quad \sum_{j=1}^{M_N} \psi_j(\bar{x})\, \bar{u}_j^{(e)}$$

→ assume $\bar{u}$ is time-independent

& $\bar{u}^{(e)}(\bar{x}) = u^{(e)}(\bar{x})\,\hat{i} + v^{(e)}(\bar{x})\,\hat{j}$ is the velocity vector along the unit directional vectors $\hat{i}$ & $\hat{j}$.

Subbing (18.2) into (18.1), multiplying by ψ_i & integrating yields: find $\zeta_N^{(e)} \in H^1$ s.t.

$$(18.3)\quad \int_{\Omega_e} \psi_i \frac{\partial \zeta_N^{(e)}}{\partial t}\, d\Omega_e + \int_{\Omega_e} \psi_i\, \bar{u}_N^{(e)} \cdot \bar{\nabla} \zeta_N^{(e)}\, d\Omega_e$$

$$= \nu \int_{\Omega_e} \psi_i\, \nabla^2 \zeta_N^{(e)}\, d\Omega_e \qquad \forall\ \psi \in H^1$$

where, for convenience, we have assumed ν = constant.

Element Eqs. on the Physical Element

Starting from (18.3) we then apply IBP as follows:

$$\psi_i \nabla^2 \zeta_N^{(e)} = \bar{\nabla}\cdot\left(\psi_i \bar{\nabla} \zeta_N^{(e)}\right) - \bar{\nabla}\psi_i \cdot \bar{\nabla} \zeta_N^{(e)}$$

which allows us to write (18.3) as follows:

$$(18.4)\quad \int_{\Omega_e} \psi_i \frac{\partial \zeta_N^{(e)}}{\partial t}\, d\Omega_e + \int_{\Omega_e} \psi_i\, \bar{u}_N^{(e)} \cdot \bar{\nabla} \zeta_N^{(e)}\, d\Omega_e$$

$$= \nu \int_{\Gamma_e} \psi_i\, g(\bar{x})\, d\Gamma_e - \nu \int_{\Omega_e} \bar{\nabla}\psi_i \cdot \bar{\nabla} \zeta_N^{(e)}\, d\Omega_e$$

Subbing the expansions for $\zeta_N^{(e)}$ & $\bar{u}_N^{(e)}$ from (18.2) yields:

$$(18.5)\quad \int_{\Omega_e} \psi_i \psi_j \, d\Omega_e \, \frac{d\zeta_j^{(e)}}{dt} + \int_{\Omega_e} \psi_i \left(\sum_{k=1}^{M_N} \psi_k \bar{u}_k^{(e)} \right) \cdot \bar{\nabla}\psi_j \, d\Omega_e \, \zeta_j^{(e)}$$
$$= \nu \int_{\Gamma_e} \psi_i \, g(\bar{x}) \, d\Gamma_e - \nu \int_{\Omega_e} \bar{\nabla}\psi_i \cdot \bar{\nabla}\psi_j \, d\Omega_e \, \zeta_j^{(e)}$$

Matrix-Vector Problem w/ Exact Integration

We can write (18.5) in matrix-vector form as follows:

$$(18.6)\quad M_{ij}^{(e)} \frac{d\zeta_j^{(e)}}{dt} + A_{ij}^{(e)}(\bar{u}) \, \zeta_j^{(e)} = \nu B_i^{(e)} - \nu L_{ij}^{(e)} \zeta_j^{(e)}$$

where

$$M_{ij}^{(e)} \equiv \int_{\Omega_e} \psi_i \psi_j \, d\Omega_e = \sum_{l=1}^{M_Q} \omega_l J_l^{(e)} \psi_{il} \psi_{j,l}$$

$$A_{ij}^{(e)}(\bar{u}) \equiv \int_{\Omega_e} \psi_i \left(\sum_{k=1}^{M_N} \psi_k \bar{u}_k^{(e)} \right) \cdot \bar{\nabla}\psi_j \, d\Omega_e$$
$$= \sum_{l=1}^{M_Q} \omega_l J_l^{(e)} \psi_{il} \left(\sum_{k=1}^{M_N} \psi_{kl} \bar{u}_k^{(e)} \right) \cdot \bar{\nabla}\psi_{j,l}$$

$$B_i^{(e)} \equiv \int_{\Gamma_e} \psi_i \, g(\bar{x}) \, d\Gamma_e = \sum_{l=0}^{Q} \omega_l^{(f)} J_l^{(f)} \psi_{il} \, g(\bar{x}_l)$$

&

$$L_{ij}^{(e)} \equiv \int_{\Omega_e} \bar{\nabla}\psi_i \cdot \bar{\nabla}\psi_j \, d\Omega_e = \sum_{l=1}^{M_Q} \omega_l J_l^{(e)} \bar{\nabla}\psi_{il} \cdot \bar{\nabla}\psi_{j,l}$$

Matrix Problem w/ Inexact Integration

Note that the mass matrix is comprised of $O(2N)$ polynomials & so is the Laplacian matrix. The advection matrix is an $O(3N)$ polynomial & the boundary vector is an $O(2N-1)$ polynomial.

Using inexact integration ($Q = N$ & $M_Q = M_N$) we get:

$$(18.7) \qquad M_{ij}^{(e)} \frac{dq_j^{(e)}}{dt} + \bar{u}_i^{(e)T} D_{ij}^{(e)} q_j^{(e)} = \nu B_i^{(e)} - \nu L_{ij}^{(e)} q_j^{(e)}$$

where

$$M_{ij}^{(e)} = \sum_{l=1}^{M_N} w_l J_l^{(e)} \psi_{i,l} \psi_{j,l} = w_i J_i^{(e)} \delta_{ij}$$

or $M_i^{(e)} = w_i J_i^{(e)}$,

$$B_i^{(e)} = w_i^{(f)} J_i^{(f)} g(\bar{x}_i),$$

$$L_{ij}^{(e)} = \sum_{l=1}^{M_N} w_l J_l^{(e)} \bar{\nabla}\psi_{i,l} \cdot \bar{\nabla}\psi_{j,l}$$ which does not

simplify. The advection matrix:

$$A_{ij}^{(e)}(\bar{u}) = \int_{\Omega_e} \psi_i \left(\sum_{n=1}^{M_N} \psi_n \bar{u}_n^{(e)} \right) \cdot \bar{\nabla}\psi_j \, d\Omega_e$$

becomes:

$$A_{ij}^{(e)}(\bar{u}) = \sum_{l=1}^{M_N} w_l J_l^{(e)} \psi_{i,l} \left(\sum_{k=1}^{M_N} \psi_{k,l} \bar{u}_k^{(e)} \right) \cdot \bar{\nabla} \psi_{j,l}$$

$$= \omega_i J_i^{(e)} \bar{u}_i^{(e)} \cdot \bar{\nabla} \psi_{j,i}$$

which we can write as:

$$A_{ij}^{(e)}(\bar{u}) = \bar{u}_i^{(e)T} \omega_i J_i^{(e)} \bar{\nabla} \psi_{j,i}$$

$$= \bar{u}_i^{(e)T} D_{ij}^{(e)}$$

where $D^{(e)}$ is the strong form differentiation matrix.

Global Matrix Problem

Using DSS we can now write:

$$M_{IJ} \frac{d f_J}{dt} + A_{IJ} f_J = \nu B_I - \nu L_{IJ} f_J$$ → exact integration

& $$M_I \frac{d f_I}{dt} + \bar{u}_I^T D_{IJ} f_J = \nu B_I - \nu L_{IJ} f_J$$ → inexact

Left-multiplying by M^{-1} yields:

$$\frac{d f_I}{dt} + M^{-1}_{IK} A_{KJ} f_J = \nu M^{-1}_{IK} B_K - \nu M^{-1}_{IK} L_{KJ} f_J$$

&

$$\frac{dq_I}{dt} + M^{-1}_I A_{IJ} q_J = \nu M^{-1}_I B_I - \nu M^{-1}_I L_{IJ} q_J$$

Letting $\hat{A} = M^{-1} A$, $\hat{B} = M^{-1} B$, $\hat{L} = M^{-1} L$ yields:

$$\frac{dq_I}{dt} + \hat{A}_{IJ} q_J = \nu \hat{B}_I - \nu \hat{L}_{IJ} q_J \rightarrow \text{exact}$$

&

$$\frac{dq_I}{dt} + \bar{u}^T_I \hat{D}_{IJ} q_J = \nu \hat{B}_I - \nu \hat{L}_{IJ} q_J \rightarrow \text{inexact}$$

Example using N=1

Let us now discretize the advection-diffusion equation:

$$M \frac{dq}{dt} + A(u)\, q = -\nu L q$$

using the CG method w/ N=1

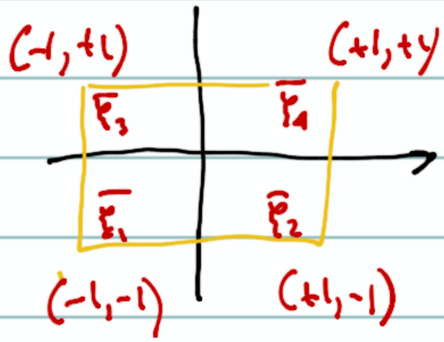
$\boxed{\text{Basis Functions}}$ Recall that for N=1 on tensor-product elements we write:

$$\psi_i(\xi,\eta) = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)$$ which yields:

$$\frac{\partial \psi_i}{\partial \xi}(\xi,\eta) = \frac{1}{4}\xi_i (1+\eta_i\eta) \quad \&$$

$$\frac{\partial \psi_i}{\partial \eta}(\xi,\eta) = \frac{1}{4}\left(1+\xi_i \xi\right)\eta_i$$

where (ξ_i, η_i) are defined as follows:



$\boxed{\text{Metric Terms}}$ From $x_N^{(e)}(\xi,\eta) = \sum_{j=1}^{M_N} \psi_j(\xi,\eta)\, x_j^{(e)}$ we write

$$\frac{\partial x_N^{(e)}}{\partial \xi}(\xi,\eta) = \sum_{j=1}^{M_N} \frac{\partial \psi_j}{\partial \xi}(\xi,\eta)\, x_j^{(e)} \quad \&$$

$$\frac{\partial x_N^{(e)}}{\partial \eta}(\xi,\eta) = \sum_{j=1}^{M_N} \frac{\partial \psi_j}{\partial \eta}(\xi,\eta)\, x_j^{(e)}$$

Where in Ch. 12 we obtained:

$$\frac{\partial x_N^{(e)}}{\partial \xi} = \frac{\Delta x^{(e)}}{2}, \quad \frac{\partial y_N^{(e)}}{\partial \eta} = \frac{\Delta y^{(e)}}{2}, \quad \frac{\partial x_N^{(e)}}{\partial \eta} = \frac{\partial y_N^{(e)}}{\partial \xi} = 0$$

$$\& \quad J^{(e)} = \frac{\partial x_N^{(e)}}{\partial \xi}\frac{\partial y_N^{(e)}}{\partial \eta} - \frac{\partial x_N^{(e)}}{\partial \eta}\frac{\partial y_N^{(e)}}{\partial \xi} = \frac{\Delta x^{(e)}\, \Delta y^{(e)}}{4}$$

for $x = x(\xi)$ & $y = y(\eta)$

From Ch. 12 we also found that

$$\frac{\partial \psi_i}{\partial x} = \frac{1}{4}\xi_i\left(1+\eta_i\eta\right)\frac{2}{\Delta x^{(e)}} \quad \& \quad \frac{\partial \psi_i}{\partial y} = \frac{1}{4}\left(1+\xi_i\xi\right)\eta_i\frac{2}{\Delta y^{(e)}}$$

Mass Matrix From Ch. 12:

$$M_{ij}^{(e)} = \frac{\Delta x^{(e)}\Delta y^{(e)}}{36}\begin{pmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 2 & 1 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{pmatrix}$$

Laplacian Matrix From Ch. 12:

$$L_{ij}^{(e)} = \frac{\Delta y^{(e)}}{6\,\Delta x^{(e)}}\begin{pmatrix} 2 & -2 & 1 & -1 \\ -2 & 2 & -1 & 1 \\ 1 & -1 & 2 & -2 \\ -1 & 1 & -2 & 2 \end{pmatrix} + \frac{\Delta x^{(e)}}{6\,\Delta y^{(e)}}\begin{pmatrix} 2 & 1 & -2 & -1 \\ 1 & 2 & -1 & -2 \\ -2 & -1 & 2 & 1 \\ -1 & -2 & 1 & 2 \end{pmatrix}$$

Advection Matrix This matrix is defined as follows:

$$(18.8) \qquad A_{ij}^{(e)}(\bar{u}) = \int_{\Omega_e} \psi_i \left(\sum_{k=1}^{M_N} \psi_k \bar{u}_k^{(e)}\right) \cdot \bar{\nabla}\psi_j \, d\Omega_e$$

& for simplifying the discussion let's assume that
$\bar{u} = u\,\hat{i} + v\,\hat{j}$ with $u = \text{const}$ & $v = \text{const}$

This allows us to simplify (18.8) as follows:

(18.9) $$A_{ij}^{(e)}(\bar{u}) = u \int_{\Omega_e} \psi_i \frac{\partial \psi_j}{\partial x} \, d\Omega_e + v \int_{\Omega_e} \psi_i \frac{\partial \psi_j}{\partial y} \, d\Omega_e$$

which we write as:

(18.10) $$A_{ij}^{(e)}(\bar{u}) = u D_{ij}^{(e,x)} + v D_{ij}^{(e,y)}$$ where

(18.11) $$D_{ij}^{(e,x)} = \int_{\Omega_e} \psi_i \frac{\partial \psi_j}{\partial x} \, d\Omega_e \quad \& \quad D_{ij}^{(e,y)} = \int_{\Omega_e} \psi_i \frac{\partial \psi_j}{\partial y} \, d\Omega_e$$

are the 2D versions of the strong form differentiation matrices that we already seen in ch. 5.

Since $$\frac{\partial \psi_i}{\partial x} = \frac{1}{4} \xi_i (1 + \eta_i \eta) \frac{2}{\Delta x^{(e)}}$$

& $$\frac{\partial \psi_i}{\partial y} = \frac{1}{4} (1 + \xi_i \xi) \eta_i \frac{2}{\Delta y^{(e)}}$$

we can write (18.11) as follows:

$$D_{ij}^{(e,x)} = \int_{-1}^{+1}\int_{-1}^{+1} \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta) \frac{1}{4}\xi_j (1+\eta_j\eta) \frac{2}{\Delta x^{(e)}} \frac{\Delta x^{(e)} \Delta y^{(e)}}{4} d\xi d\eta$$

$$D_{ij}^{(e,y)} = \int_{-1}^{+1}\int_{-1}^{+1} \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta) \frac{1}{4}\eta_j (1+\xi_j\xi) \frac{2}{\Delta y^{(e)}} \frac{\Delta x^{(e)} \Delta y^{(e)}}{4} d\xi d\eta$$

Simplifying & integrating yields:

$$D_{ij}^{(e,x)} = \frac{\Delta y^{(e)}}{32}\left[\xi_j \xi + \frac{1}{2}\xi_i \xi_j \xi^2\right]\Bigg|_{-1}^{+1}\left[\eta + \frac{1}{2}(\eta_i+\eta_j)\eta^2 + \frac{1}{3}\eta_i\eta_j\eta^3\right]\Bigg|_{-1}^{+1}$$

$$D_{ij}^{(e,y)} = \frac{\Delta x^{(e)}}{32}\left[\eta_j \eta + \frac{1}{2}\eta_i \eta_j \eta^2\right]\Bigg|_{-1}^{+1}\left[\xi + \frac{1}{2}(\xi_i+\xi_j)\xi^2 + \frac{1}{3}\xi_i\xi_j\xi^3\right]\Bigg|_{-1}^{+1}$$

evaluating the matrices yield:

$$D_{ij}^{(e,x)} = \frac{\Delta y^{(e)}}{24}\xi_j(3+\eta_i\eta_j) \qquad D_{ij}^{(e,y)} = \frac{\Delta x^{(e)}}{24}\eta_j(3+\xi_i\xi_j)$$

Subbing in values for $(\xi_i, \eta_i, \xi_j, \eta_j)$ yields:

$$D_{ij}^{(e,x)} = \frac{\Delta y^{(e)}}{12}\begin{pmatrix} -2 & 2 & -1 & 1 \\ -2 & 2 & -1 & 1 \\ -1 & 1 & -2 & 2 \\ -1 & 1 & -2 & 2 \end{pmatrix} \quad \&$$

$$D_{ij}^{(e,y)} = \frac{\Delta x^{(e)}}{12}\begin{pmatrix} -2 & -1 & 2 & 1 \\ -1 & -2 & 1 & 2 \\ -2 & -1 & 2 & 1 \\ -1 & -2 & 1 & 2 \end{pmatrix}$$

This now allows us to write the advection matrix:

$$A_{ij}^{(e)}(\bar{u}) = u D_{ij}^{(e,x)} + v D_{ij}^{(e,y)}$$ as follows:

$$A_{ij}^{(e)}(\bar{u}) = u \frac{\Delta y^{(e)}}{12} \begin{pmatrix} -2 & 2 & -1 & 1 \\ -2 & 2 & -1 & 1 \\ -1 & 1 & -2 & 2 \\ -1 & 1 & -2 & 2 \end{pmatrix}$$

$$v \frac{\Delta x^{(e)}}{12} \begin{pmatrix} -2 & -1 & 2 & 1 \\ -1 & -2 & 1 & 2 \\ -2 & -1 & 2 & 1 \\ -1 & -2 & 1 & 2 \end{pmatrix}$$

Matrix Eqs. on Reference Element

For the advection-diffusion eq. w/ Dirichlet BCs, we have:

$$M^{(e)} \frac{d\zeta^{(e)}}{dt} + A^{(e)}(\bar{u})\,\zeta^{(e)} = -\nu L^{(e)} \zeta^{(e)}$$

w/ the matrices defined as above.

Algorithms for the CG Global Problem

Non-Tensor-Product Approach

Alg. 15.1: Construction of $A^{(e)}$ matrix Exact Integration

$A^{(e)} = \mathbb{0}$

for $e = 1 : N_e$

  for $l = 1 : M_Q$

    $\bar{u}_l = 0$

    for $k = 1 : M_N$

      $\bar{u}_l \mathrel{+}= \psi_{kl}\, \bar{u}_k^{(e)}$

    end

    for $j = 1 : M_N$, $i = 1 : M_N$

      $$\underbrace{A_{ij}^{(e)} \underbrace{+=}_{1\text{ add}} \underbrace{\omega_l J_l^{(e)} \psi_{il}}_{3\text{ mult}} \underbrace{\left(\bar{u}_l \cdot \nabla \psi_{jl}\right)}_{3\text{ ops: }2\text{ mult} + 1\text{ add}}}_{O(7\text{ ops})}$$

    end

  end

end

Complexity of Alg. 15.1 is $7 \cdot M_N^2 \cdot M_Q \cdot N_e$

$= 7\,(N+1)^4 (Q+1)^2 \cdot N_e$

Let $Q = N+1$ $\longrightarrow$ $= 7\, N^4 \cdot N^2 \cdot N_e$

$= 7\, N^6 \cdot N_e$

In general we get $7\, N^{3d} \cdot N_e$

Alg. 15.2 Construction of $A^{(e)}$ w/ Inexact Integration

$A^{(e)} = \square$

for $e = 1 : N_e$

  for $j = 1 : M_N$

    for $i = 1 : M_N$

$$A^{(e)}_{ij} \mathrel{+}= \underbrace{\underbrace{\omega_i}_{1\text{ op}} \underbrace{J_i^{(e)}}_{2\text{ ops}} \underbrace{\left(\bar{u}_i^{(e)} \cdot \bar{\nabla}\psi_{j,i}\right)}_{3\text{ ops}}}_{6\text{ ops}}$$

    end

  end

end

Complexity is : $6 \cdot M_N^2 \cdot N_e = 6(N+1)^4 \cdot N_e$

in general we get $6 N^{d+2} N_e$

Let's consider building the RHS vector R s.t.

$$\frac{dq_i^{(e)}}{dt} = R_i^{(e)}$$ w/o building elmt matrices (except $M^{(e)}$).

Alg. 15.4 : Construction of $R^{(e)}$ w/ Exact Integration

$R^{(e)} = \square$

for $e = 1 : N_e$

  for $l = 1 : M_Q$

$\bar{u}_l = \bar{\nabla} q_l = [\,]$

for $k = 1 : M_N$

$\quad \bar{u}_l \mathrel{+}= \psi_{kl} \, \bar{u}_k^{(e)}$

$\quad \bar{\nabla} q_l \mathrel{+}= \bar{\nabla} \psi_{kl} \, q_k^{(e)}$

end

for $i = 1 : M_N$

$$R_i^{(e)} \mathrel{+}= \underbrace{\omega_l J_l^{(e)} \psi_{il} \left(\bar{u}_l \cdot \bar{\nabla} q_l \right)}_{7 \text{ ops}}$$

end

end

end

Complexity is $7 \cdot M_N \cdot M_Q \cdot N_e = 7 \cdot (N+1)^2 \cdot (N+1)^2 \cdot N_e$

$= 7 N^4 N_e$

$\boxed{\text{Alg. 13.5: } R^{(e)} \text{ w/ Inexact Integration}}$

$R^{(e)} = [\,]$

for $e = 1 : N_e$

for $i = 1 : M_N$

$\quad \bar{\nabla} q_i = 0$

$\quad$ for $k = 1 : M_N$

$$\bar{\nabla} q_i \mathrel{+}= \bar{\nabla} \psi_{ik} \, q_k^{(e)} \rightarrow \overbrace{\left(\psi_\xi \xi_x + \psi_\eta \eta_x \right)}^{4 \text{ ops}} q$$

$\quad$ end

$\quad R_i^{(e)} \mathrel{+}= \omega_i J_i^{(e)} \left(\bar{u}_i^{(e)} \cdot \bar{\nabla} q_i \right)$

end

end

Complexity is $2 \cdot 4 \text{ ops} \cdot M_N \cdot M_Q \cdot N_e$

$$= 8 N^2 \cdot N^2 \cdot N_e$$

$$= 8 N^4 N_e$$

Tensor-Product Approach

Let us consider the tensor-product approach for the particular case of inexact integration since it is easier to explain.

Recall that for the A-matrix we wrote:

$$(18.12) \quad A_{ij}^{(e)} = \bar{u}_i^{(e)T} w_i J_i^{(e)} \bar{\nabla}\psi_{ji}$$

$$= \bar{u}_i^{(e)T} D_{ij}^{(e)}$$

where:

$$(18.13) \quad D_{I,J}^{(e)} = w_I J_I^{(e)} \bar{\nabla}\psi_{J,I} = w_I J_I^{(e)} \left(\frac{\partial\psi}{\partial x}_{J,I} \hat{i} + \frac{\partial\psi}{\partial y}_{J,I} \hat{j} \right)$$

where $I, J = 1, \ldots, M_N$.

Next, let's write the basis functions ψ as the tensor-product of 1D basis functions as follows:

(18.14) $$\psi_{J,I} = h_{k,i}(\xi)\, h_{l,j}(\eta)$$

where $I = i+1 + j(N+1)$ & $J = k+1 + l(N+1)$
w/ $i,j,k,l = 0,\dots,N$

Using these 1D functions we now write (18.13) as follows:

$$D^{(e)}_{J,I} = \omega_i \omega_j J^{(e)}_{i,j}\left[\frac{\partial}{\partial x}\Big(h_k(\xi_i)\, h_l(\eta_j)\Big)\hat{i} + \frac{\partial}{\partial y}\Big(h_k(\xi_i) h_l(\eta_j)\Big)\hat{j}\right]$$

where

$$\frac{\partial}{\partial x}\Big(h_k(\xi_i)\, h_l(\eta_j)\Big) = \frac{\partial}{\partial \xi}\Big(h_k(\xi_i)\, h_l(\eta_j)\Big)\frac{\partial \xi_{i,j}}{\partial x} + \frac{\partial}{\partial \eta}\Big(h_k(\xi_i) h_l(\eta_j)\Big)\frac{\partial \eta_{i,j}}{\partial x}$$

which simplifies to:

$$\frac{\partial}{\partial x}\Big(h_k(\xi_i)\, h_l(\eta_j)\Big) = \frac{d h_k}{d\xi}(\xi_i)\, h_l(\eta_j)\frac{\partial \xi_{i,j}}{\partial x} + h_k(\xi_i)\frac{d h_l}{d\eta}(\eta_j)\frac{\partial \eta_{i,j}}{\partial x}$$

Using cardinality (since $Q = N$) we get

$$\frac{\partial}{\partial x}\Big(h_k(\xi_i)\, h_l(\eta_j)\Big) = \frac{d h_{k,i}}{d\xi}\,\frac{\partial \xi}{\partial x}_{i,j} + \frac{d h_{l,j}}{d\eta}\,\frac{\partial \eta}{\partial x}_{i,j}$$

With this in mind note that an operator of the type: $\int_{\Omega_e} \psi_i \frac{\partial q_N^{(e)}}{\partial x} d\Omega_e$ can be constructed as follows:

$$\int_{\Omega_e} \psi_i \frac{\partial q_N^{(e)}}{\partial x} d\Omega_e = \sum_{k=1}^{M_N} \omega_k J_k^{(e)} \cancelto{\delta_{i,k}}{\psi_{i,k}} \sum_{j=1}^{M_N} \frac{\partial \psi_{j,k}}{\partial x} q_j \, d\Omega_e$$

$$= \omega_i J_i^{(e)} \sum_{j=1}^{M_N} \frac{\partial \psi_{j,i}}{\partial x} q_j$$

Introducing tensor products yields:

$$\int_{\Omega_e} \psi_i \frac{\partial q_N^{(e)}}{\partial x} d\Omega_e = \omega_i \omega_j J_{ij}^{(e)} \left(\sum_{k=0}^{N} \sum_{l=0}^{N} \frac{dh_{k,i}}{d\xi} h_{l,j} \frac{\partial \xi}{\partial x}_{i,j} + h_{k,i} \frac{dh_{l,j}}{d\eta} \frac{\partial \eta}{\partial x}_{i,j} \right) q_{kl}^{(e)}$$

$$= \omega_i \omega_j J_{ij}^{(e)} \left[\sum_{k=0}^{N} \sum_{l=0}^{N} \frac{dh_{k,i}}{d\xi} \cancelto{\delta_{l,j}}{h_{l,j}} \frac{\partial \xi}{\partial x}_{i,j} q_{k,l}^{(e)} + \sum_{k=0}^{N} \sum_{l=0}^{N} \cancelto{\delta_{k,i}}{h_{k,i}} \frac{dh_{l,j}}{d\eta} \frac{\partial \eta}{\partial x}_{i,j} q_{k,l}^{(e)} \right]$$

due to Cardinality, we can eliminate the l-loop in the first sum (& set $l=j$) & the k-loop in the 2nd sum (& set $k=i$) as follows:

$$\int_{\Omega_e} \psi_i \frac{\partial q_N^{(e)}}{\partial x} d\Omega_e = \omega_i \omega_j J_{ij}^{(e)} \left[\sum_{k=0}^{N} \frac{dh_{k,i}}{d\xi} \frac{\partial \xi}{\partial x}_{i,j} q_{k,j}^{(e)} \right.$$

$$+ \sum_{l=0}^{N} \frac{dh_{l,j}}{d\eta} \frac{\partial \eta}{\partial x}_{i,j} f_{i,l}^{(e)} \Bigg]$$

which we can combine into one loop if we call $l = k$ in the second sum as follows:

$$\int_{\Omega_e} \psi_i \frac{\partial f_k^{(e)}}{\partial x} d\Omega_e = w_i w_j J_{ij}^{(e)} \sum_{k=0}^{N} \left(\frac{dh_{k,i}}{d\xi} \frac{\partial \xi}{\partial x}_{i,j} f_{k,j}^{(e)} + \frac{dh_{k,j}}{d\eta} \frac{\partial \eta}{\partial x}_{i,j} f_{i,k}^{(e)} \right)$$

This approach is presented in Alg. 15.7.

Let us now exploit this & introduce the Sum Factorization idea:

$\boxed{\text{Alg. 15.7: Sum Factorization for Constructing } R^{(e)}}$

```
R^(e) = []
for e = 1 : Ne
    for i = 0 : N
        for j = 0 : N
            I = i+1 + j(N+1)
            fξ = fη = 0
            for k = 0 : N
                fξ += (dh_{k,i}/dξ) f^(e)_{k,j}
                fη += (dh_{k,j}/dη) f^(e)_{i,k}
            end
            fx = fξ (∂ξ/∂x)_{i,j} + fη (∂η/∂x)_{i,j}
            fy = fξ (∂ξ/∂y)_{i,j} + fη (∂η/∂y)_{i,j}
            R_I = w_i w_j J^(e)_{ij} ( u^(e)_i fx + v^(e)_i fy )
        end
    end
end
```