Lecture 21: Stabilization Methods

Introduction

Recall that high-order methods have little to no inherent dissipation. This is both a virtue & a vice.
The virtue is that they are able to represent all of the interesting information in the problem.
The vice is that they require special care in order to maintain the simulation stable.

But what causes the instabilities?

Aliasing Errors

Aliasing can be best explained by considering the adjective/convective term:

$$u \frac{\partial u}{\partial x}$$ that appears in, e.g., Navier-Stokes.

Using Fourier modes we can write:

$$u_N(x_j) = \sum_{n=0}^{N} v_n \, e^{ij(n\Delta x)}$$

&

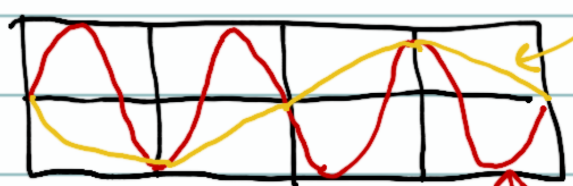
$$\frac{\partial u_N}{\partial x}(x_j) = \sum_{l=0}^{N} v_l \, e^{ij(l\Delta x)}$$

F so

$$u_N \frac{\partial u_N}{\partial x} = \sum_{n=0}^{N} \sum_{l=0}^{N} v_n v_l \, e^{ij(n+l)\Delta x}$$

where, if the method can resolve waves $(n\Delta x)_{max} = \pi$ it will not resolve $(n+l)\Delta x > \pi$.

Here is what this means



The real signal (red) is interpreted as another signal with a different frequency that is wrong & will continue to produce wrong results that may through the model out of balance & produce catastrophic instabilities

Filters

From signal processing, one solution is to throw out signals that cannot be represented. This is achieved through low-pass filters & are a good simple way to resolve the issue.

The idea is to use the Legendre transform as follows:

$$\zeta^{nodal} = V \zeta^{modal}$$ where V is the Vandermonde matrix.

$\therefore$

$$\zeta^{modal} = V^{-1} \zeta^{nodal}.$$

Now ζ_i^{modal} are the amplitudes of the hierarchical frequencies $i = 0, \ldots, N$.

So we can apply a filter as such:

$$\Lambda_i = 1 \quad i < N \quad \& \quad \Lambda_N = 0$$

which cuts off the N^{th} wave.

& so

$$\zeta_F^{modal} = \Lambda \zeta^{modal}$$

& we can map back to nodal space as follows:

(21.1) $$q_F^{Nodal} \equiv V \Lambda \hat{q}^{\,nodal} = V \Lambda V^{-1} q^{Nodal}$$

or $$q_F^{Nodal} = F q^{nodal} \quad \text{where} \quad F = V \Lambda V^{-1}$$

is the filter matrix.

However, in order to retain high-order convergence requires that the Λ function be smoothly varying (Vandeven's theorem).

Diffusion Operators

Another approach used for stabilization is artificial viscosity / diffusion as such:

$$\frac{\partial q}{\partial t} = \ldots \quad \nu \frac{\partial^2 q}{\partial x^2}$$

& more generally as:

$$\frac{\partial q}{\partial t} = \ldots + \nu \nabla^2 q$$

(21.2) $$= \ldots + (-1)^{k+1} \nu \nabla^{2k} q \longrightarrow \text{hyper diffusion}$$

There is also option to fix $k=1$ but modify $\nu = \nu(\bar{x}, t)$ inside each element s.t. the residual:

$$R = \frac{\partial f}{\partial t} - \ldots - \nu \nabla^2 f \longrightarrow \text{minimized.}$$

These are known as Locally Adaptive Viscosity methods & work quite well but are expensive & we cannot know (a priori) how large ν will get.

Exact Integration

Let's consider the Euler eqs. (inviscid Compressible Navier-Stokes):

$$\frac{\partial \rho}{\partial t} + \bar{\nabla} \cdot (\rho \bar{u}) = 0 \qquad (21.3.1)$$

$$\frac{\partial \rho u_i}{\partial t} + \bar{\nabla} \cdot (\rho u_i \bar{u}) + \frac{\partial P}{\partial x_i} = 0 \qquad (21.3.2)$$

$$\frac{\partial \rho e}{\partial t} + \bar{\nabla} \cdot (\rho e \bar{u}) + \frac{\partial}{\partial x_j}(P u_j) = 0 \qquad (21.3.3)$$

where the term $\rho \bar{u} u_i$ in the momentum eq.

is an $O(3N)$ polynomial.

Therefore, the integral

$$\int_{\Omega_e} \psi_i \, \bar{\nabla} \cdot \left(\rho_N \bar{u}_N u_i \right) d\Omega_e \quad \text{is an} \sim O(4N)$$

polynomial. To get it exactly requires:

$$2Q - 1 = 4N \rightarrow Q = \frac{4N+1}{2} = 2N + \tfrac{1}{2}$$

Lobatto points. This becomes prohibitive!

Kinetic Energy Preserving Forms

Let's only consider the convective terms in (21.3) & write them generally as:

$$(21.4) \qquad \bar{\nabla} \cdot \left(\rho \varphi \bar{u} \right) \equiv \frac{\partial}{\partial x_i} \left(\rho \varphi u_i \right)$$

where for (21.3.1), (21.3.2), & (21.3.3) we get:

$$\varphi = 1 \,, \quad u_i \,, \quad \& \quad e$$

Next, we wish to construct an equation

for the conservation of a generalized kinetic energy whereby we shall write it as:

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\mathbf{e}^2_{\rho}\right) + \bar{\nabla}\cdot\left(\tfrac{1}{2}\mathbf{e}^2_{\rho}\bar{u}\right) = 0$$

which can be easily proven to yield:

$$\sum_{e=1}^{N_e}\int_{\Omega_e}\psi_i\,\bar{\nabla}\cdot\left(\tfrac{1}{2}\mathbf{e}^2_{\rho}\bar{u}\right)_N d\Omega_e = 0$$

by virtue of IBP s.t.:

$$\sum_{e=1}^{N_e}\left\{\int_{\Omega_e}\bar{\nabla}\cdot\left[\psi_i\left(\tfrac{1}{2}\mathbf{e}^2_{\rho}\bar{u}\right)\right]d\Omega_e = \int_{\Omega_e}\bar{\nabla}\psi_i\cdot\left(\tfrac{1}{2}\mathbf{e}^2_{\rho}\bar{u}\right)_N d\Omega_e + \int_{\Omega_e}\psi_i\,\bar{\nabla}\cdot\left(\tfrac{1}{2}\mathbf{e}^2_{\rho}\bar{u}\right)_N d\Omega_e\right\}$$

or

$$\sum_{e=1}^{N_e}\left\{\int_{\Omega_e}\psi_i\,\bar{\nabla}\cdot\left(\tfrac{1}{2}\mathbf{e}^2_{\rho}\bar{u}\right)^{(e)}_N d\Omega_e = \int_{\Gamma_e}\psi_i\,\hat{n}\cdot\left(\tfrac{1}{2}\mathbf{e}^2_{\rho}\bar{u}\right)^{(\gamma)}_N d\Gamma_e - \int_{\Omega_e}\bar{\nabla}\psi_i\cdot\left(\tfrac{1}{2}\mathbf{e}^2_{\rho}\bar{u}\right)^{(e)}_N d\Omega_e\right.$$

continuous flux

for $\psi_i = \text{const}$

$$\therefore \int_{\Omega}\frac{\partial}{\partial t}\left(\tfrac{1}{2}\mathbf{e}^2_{\rho}\right)d\Omega = 0$$

Let's derive an equation of this type.
To do so, let's consider the following:

(21.5.1)
$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}u^2\rho\right) = \frac{\partial}{\partial t}\left(\tfrac{1}{2}u\cdot\rho u\right) = \tfrac{1}{2}\rho u\frac{\partial u}{\partial t} + \tfrac{1}{2}u\frac{\partial \rho u}{\partial t}$$

We can also write:

(21.5.2)
$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}u^2\cdot\rho\right) = \rho u\frac{\partial u}{\partial t} + \tfrac{1}{2}u^2\frac{\partial \rho}{\partial t}$$

Subtracting (21.5.2) - (21.5.1) gives:

(21.6)
$$0 = \tfrac{1}{2}\rho u\frac{\partial u}{\partial t} + \tfrac{1}{2}u^2\frac{\partial \rho}{\partial t} - \tfrac{1}{2}u\frac{\partial}{\partial t}(\rho u)$$

$$\therefore\quad \tfrac{1}{2}\rho u\frac{\partial u}{\partial t} = \tfrac{1}{2}u\frac{\partial}{\partial t}(\rho u) - \tfrac{1}{2}u^2\frac{\partial \rho}{\partial t}$$

&

Subbing into (21.5.1) gives:

(21.7)
$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}u^2\rho\right) = u\frac{\partial}{\partial t}(\rho u) - \tfrac{1}{2}u^2\frac{\partial \rho}{\partial t}$$

If both ρ & ρu satisfy conservation laws as follows:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot(\rho\vec{u}) = -M \quad \& \quad \frac{\partial}{\partial t}(\rho u) = -\vec{\nabla}\cdot(\rho u\vec{u}) = -C$$

then (21.7) becomes:

(21.8) $$\frac{\partial}{\partial t}\left(\frac{1}{2}e^2\rho\right) = -eC + \frac{1}{2}e^2 M$$

Split Forms (Coppola et al. 2018)

The question now is: do there exist combinations of M & C s.t. RHS of (21.8) becomes:

$$-eC + \frac{1}{2}e^2 M = \overline{\nabla}\cdot\left(\frac{1}{2}e^2\rho\overline{u}\right)$$

Let $M = \xi M^D + (1-\xi)M^A$ &

$$C = \alpha C^D + \beta C^e + \gamma C^u + \delta C^\rho$$

&

$$M^D = \overline{\nabla}\cdot(\rho\overline{u}), \quad M^A = \overline{u}\cdot\overline{\nabla}\rho + \rho\overline{\nabla}\cdot\overline{u}$$

$$C^D = \overline{\nabla}\cdot(\rho e\overline{u}), \quad C^e = e\overline{\nabla}\cdot(\rho\overline{u}) + \rho\overline{u}\cdot\overline{\nabla}e$$

$$C^u = \overline{u}\cdot\overline{\nabla}(\rho e) + \rho e\overline{\nabla}\cdot\overline{u}$$

$$C^\rho = \rho\overline{\nabla}\cdot(e\overline{u}) + e\overline{u}\cdot\overline{\nabla}\rho$$

Feiereisen form

Assume $\xi = 1$ & $\alpha = \beta = \frac{1}{2}$, $\gamma = \delta = 0$
to give

$$-e C + \frac{1}{2} e^2 M = -e \left[\frac{1}{2} \bar{\nabla} \cdot (\rho e \bar{u}) + \frac{1}{2} e \bar{\nabla} \cdot (\rho \bar{u}) + \frac{1}{2} \rho \bar{u} \cdot \bar{\nabla} e \right]$$

$$+ \frac{1}{2} e^2 \bar{\nabla} \cdot (\rho \bar{u})$$

$$= -\frac{1}{2} e \bar{\nabla} \cdot (\rho e \bar{u}) - \cancel{\frac{1}{2} e^2 \bar{\nabla} \cdot (\rho \bar{u})} - \frac{1}{2} e \rho \bar{u} \cdot \bar{\nabla} e$$

$$+ \cancel{\frac{1}{2} e^2 \bar{\nabla} \cdot (\rho \bar{u})}$$

$$= - \bar{\nabla} \cdot \left(\frac{1}{2} e^2 \rho \bar{u} \right) \longrightarrow \text{using the Product Rule}$$

Kennedy-Gruber Form

Assume $\xi = \frac{1}{2}$, $\alpha = \beta = \gamma = \delta = \frac{1}{4}$ to get

$$M = \frac{1}{2} M^D + \frac{1}{2} M^A = \frac{1}{2} \bar{\nabla} \cdot (\rho \bar{u}) + \frac{1}{2} \left[\bar{u} \cdot \bar{\nabla} \rho + \rho \bar{\nabla} \cdot \bar{u} \right]$$

$$C = \frac{1}{4}\,\bar{\nabla}\cdot(\rho\varphi\bar{u}) + \frac{1}{4}\left[\varphi\,\bar{\nabla}\cdot(\rho\bar{u}) + \rho\,\bar{u}\cdot\bar{\nabla}\varphi\right]$$

$$+ \frac{1}{4}\left[\bar{u}\cdot\bar{\nabla}(\rho\varphi) + \rho\varphi\,\bar{\nabla}\cdot\bar{u}\right] + \frac{1}{4}\left[\rho\,\bar{\nabla}\cdot(\varphi\bar{u}) + \varphi\,\bar{u}\cdot\bar{\nabla}\rho\right]$$

$\therefore$

$$-\varphi C + \tfrac{1}{2}\varphi^2 M = -\frac{1}{4}\varphi\,\bar{\nabla}\cdot(\rho\varphi\bar{u}) - \cancel{\frac{1}{4}\varphi^2\,\bar{\nabla}\cdot(\rho\bar{u})} - \frac{1}{4}\varphi\rho\,\bar{u}\cdot\bar{\nabla}\varphi$$

$$- \frac{1}{4}\varphi\,\bar{u}\cdot\bar{\nabla}(\rho\varphi) - \cancel{\frac{1}{4}\varphi^2\rho\,\bar{\nabla}\cdot\bar{u}}$$

$$- \frac{1}{4}\varphi\rho\,\bar{\nabla}\cdot(\varphi\bar{u}) - \cancel{\frac{1}{4}\varphi^2\,\bar{u}\cdot\bar{\nabla}\rho}$$

$$+ \frac{1}{2}\varphi^2\left[\cancel{\tfrac{1}{2}\bar{\nabla}\cdot(\rho\bar{u})} + \cancel{\tfrac{1}{2}\bar{u}\cdot\bar{\nabla}\rho} + \cancel{\tfrac{1}{2}\rho\bar{\nabla}\cdot\bar{u}}\right]$$

$$= -\frac{1}{4}\left(\underbrace{\varphi\,\bar{\nabla}\cdot(\rho\varphi\bar{u}) + \varphi\rho\,\bar{u}\cdot\bar{\nabla}\varphi}_{} + \varphi\,\bar{u}\cdot\bar{\nabla}(\rho\varphi) + \varphi\rho\,\bar{\nabla}\cdot(\varphi\bar{u})\right)$$

$$= -\frac{1}{2}\left[\bar{\nabla}\cdot\left(\tfrac{1}{2}\varphi^2\rho\bar{u}\right) + \bar{\nabla}\cdot\left(\tfrac{1}{2}\varphi^2\rho\bar{u}\right)\right]$$

$$= -\bar{\nabla}\cdot\left(\tfrac{1}{2}\varphi^2\rho\bar{u}\right)$$ $\rightarrow$ relying on multiple forms averaged to make product

rule be more closely satisfied

$\boxed{\text{Bottom Line}}$ workable but too complicated & difficult to workout optimal choices.

Entropy / Energy Stability for Burger's Eq.

Let's introduce the inviscid Burger's eq:

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0 \tag{21.9}$$

The companion entropy/energy eq. is obtained by multiplying (21.9) by u:

$$u\,u_t + u\left(\tfrac{1}{2}u^2\right)_x = 0 \tag{21.10}$$

which we can write as:

$$\left(\tfrac{1}{2}u^2\right)_t + \underbrace{\left(\tfrac{1}{3}u^3\right)_x}_{=\frac{2}{3}\left(\frac{1}{2}u^2\cdot u\right)_x} = 0$$

which can be shown to yield:

$$\sum_{e=1}^{N_e}\int_{\Omega_e}\psi_i\left(\tfrac{1}{3}u^3\right)_x d\Omega_e = 0$$

& so

$$\int_\Omega \left(\tfrac{1}{2}u^2\right)_t d\Omega = 0$$

so kinetic energy is conserved (will not grow).

Split Form

Let's now consider the split form of (21.9) as follows:

$$(21.11) \qquad u_t + \alpha \left(\tfrac{1}{2}u^2\right)_x + (1-\alpha)\, u u_x = 0$$

Multiplying by u:

$$u u_t + \underbrace{\alpha u \left(\tfrac{1}{2}u^2\right)_x + (1-\alpha)\, u^2 u_x}_{\text{can be combined if:}} = 0$$

we need to find α s.t.:

$$\tfrac{1}{2}\alpha = 1-\alpha$$

$$\alpha = \tfrac{2}{3}$$

$\therefore$

$$\left(\tfrac{1}{2}u^2\right)_t + u\left(\tfrac{1}{3}u^2\right)_x + \tfrac{1}{3}u^2 u_x = 0$$

$$\left(\tfrac{1}{2}u^2\right)_t + \left(\tfrac{1}{3}u^3\right)_x = 0$$

$\therefore$ the split form:

$$(21.12) \qquad u_t + \tfrac{2}{3}\left(\tfrac{1}{2}u^2\right)_x + \tfrac{1}{3} u u_x = 0$$

is energy preserving.

Flux Differencing

For the eq.:

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0 \quad \text{where} \quad f = \tfrac{1}{2}u^2$$

We need to find a new f^{EC} flux s.t. Eq. (21.10) is satisfied. This means that the eq. w/ the new flux must be equivalent to (21.12).

In discrete (strong) form we write:

(21.13) $$M_i^{(e)} \frac{du_i^{(e)}}{dt} + F_i^{(e,l)}\left(f_i^{(*,l)} - f_i^{(e)}\right) + \underbrace{D_{ij}^{(e)} f_j^{(e)}}_{\text{Redefine}} = 0$$

Let's replace

$$D_{ij}^{(e)} f_j^{(e)} \rightarrow 2\, D_{ij}^{(e)} \circ f^{\#}_{ij}$$

where if

(21.14) $$f^{\#}_{ij} = \frac{f_i^{(e)} + f_j^{(e)}}{2} = \{\{f\}\}$$ → two-point flux.

we get:

$$2 D_{ij}^{(e)} \left(\frac{f_i^{(e)} + f_j^{(e)}}{2} \right) = \underbrace{\cancelto{\emptyset}{D_{ij} f_i^{(e)}}}_{\sum_{j=0}^{N} D_{ij} = 0} + D_{ij} f_j^{(e)}$$

$$\therefore \; 2 D_{ij}^{(e)} \{\{f\}\} = D_{ij}^{(e)} f_j^{(e)}$$

$\boxed{\text{Two-Point Flux}}$

Note that (21.14) is consistent:

$$f^{\#}(f_i, f_i) = f_i$$

& is symmetric:

$$f^{\#}(f_i, f_j) = f^{\#}(f_j, f_i)$$

However, we can build other fluxes.

Let's define:

$$(21.15) \quad f_2^{\#}(u_i, u_j) = \frac{u_i^2 + u_i u_j + u_j^2}{6}$$

& so:

$$2 D_{ij}^{(e)} \circ f_2^{\#} = \cancelto{\emptyset}{\frac{1}{3} D_{ij} u_i^2} + \frac{1}{3} D_{ij} u_i u_j + \frac{1}{3} D_{ij} u_j^2$$

$$= \frac{1}{3} u_i D_{ij} u_j + \frac{2}{3} D_{ij} \left(\frac{1}{2} u_j^2\right)$$

Assuming CG w/ periodic BCs (flux term vanishes) yields

$$(21.16) \quad M_i^{(e)} \frac{d u_i^{(e)}}{dt} + \frac{2}{3} D_{ij} \left(\frac{1}{2} u_j^2\right) + \frac{1}{3} u_i D_{ij} u_j = 0$$

which is exactly the split-form we got in (21.12)

To make this applicable to DG requires that we now handle the flux term contribution.

Before tackling this term, let us consider other fluxes that yield, e.g., the Feiereisen & Kennedy-Gruber forms.

Flux Differencing for Feiereisen Form

Recall that for the two eq. system:

$$(21.17) \qquad \frac{\partial \rho}{\partial t} + \bar{\nabla} \cdot (\rho \bar{u}) = 0$$

$$\frac{\partial}{\partial t}(\rho e) + \bar{\nabla} \cdot (\rho e \bar{u}) = 0$$

The Feiereisen split form is:

$$(21.18.1) \quad \frac{\partial \rho}{\partial t} + \bar{\nabla} \cdot (\rho u) = 0$$

$$(21.18.2) \quad \frac{\partial}{\partial t}(\rho e) + \frac{1}{2} \bar{\nabla} \cdot (\rho e \bar{u}) + \frac{1}{2}\left[e \bar{\nabla} \cdot (\rho \bar{u}) + \rho \bar{u} \cdot \bar{\nabla} e \right]$$

Equivalent Flux Difference

Note that replacing $\bar{\nabla} \cdot (\rho \bar{u})$ in (21.18.1) w/ the modified discrete form:

$2 D_{ij}^{(e)} \circ f_{ij}^{\#}$ gives the original conservation form it:

$$f_{ij}^{\#} = \frac{f_i + f_j}{2} \equiv \{\{f\}\} = \{\{\rho \bar{u}\}\}$$

We can show this as follows:

$$2D_{ij}^{(e)} \circ f_{ij}^{\#} = 2D_{ij}^{(e)}\left[\frac{(\rho\bar{u})_i + (\rho\bar{u})_j}{2}\right]$$

$$= (\rho\bar{u}_i)\cancelto{0}{D_{ij}^{(e)}} + D_{ij}^{(e)}(\rho\bar{u})_j$$

$$= D_{ij}^{(e)}(\rho\bar{u})_j \rightarrow$$ strong form discrete divergence operator

This concludes the equivalent 2-point flux for Eq. (21.18.1).

For Eq. (21.18.2) note that it:

$$f^{\#} = \{\{\varphi\}\}\{\{\rho\bar{u}\}\}$$ that we get

$$f_{ij}^{\#} = \left(\frac{\varphi_i + \varphi_j}{2}\right)\left(\frac{(\rho\bar{u})_i + (\rho\bar{u})_j}{2}\right)$$

$$= \frac{1}{4}\left[\varphi_i(\rho\bar{u})_i + \varphi_i(\rho\bar{u})_j + \varphi_j(\rho\bar{u})_i + \varphi_j(\rho\bar{u})_j\right]$$

& we get:

$$2D_{ij}^{\circ} f^{\#}_{ij} = \frac{1}{2} \cancelto{0}{\varphi_i (\rho\bar{u})_i D_{ij}} +$$

$$\frac{1}{2} \varphi_i D_{ij} (\rho\bar{u})_j + \frac{1}{2} (\rho\bar{u})_i D_{ij} \varphi_j$$

$$+ \frac{1}{2} D_{ij} (\rho\varphi u)_j$$

$$= \frac{1}{2} D_{ij} (\rho\varphi u)_j + \frac{1}{2}\left[\varphi_i D_{ij} (\rho\bar{u})_j + (\rho\bar{u})_i D_{ij} \varphi_j \right]$$

which recovers the Feiereisen form.

Flux Differencing for Kennedy-Gruber

Recall that for Kennedy-Gruber we have:

(21.11.1) $$\frac{\partial\rho}{\partial t} + \frac{1}{2}\bar{\nabla}\cdot(\rho\bar{u}) + \frac{1}{2}\left[\bar{u}\cdot\bar{\nabla}\rho + \rho\bar{\nabla}\cdot\bar{u}\right] = 0$$

(21.11.2) $$\frac{\partial}{\partial t}(\rho\varphi) + \frac{1}{4}\bar{\nabla}\cdot(\rho\varphi\bar{u}) + \frac{1}{4}\left[\varphi\bar{\nabla}\cdot(\rho\bar{u}) + \rho\bar{u}\cdot\bar{\nabla}\varphi\right]$$

$$+ \frac{1}{4}\left[\bar{u}\cdot\bar{\nabla}(\rho e) + \rho e\,\bar{\nabla}\cdot\bar{u}\right] + \frac{1}{4}\left[\rho\bar{\nabla}\cdot(e\bar{u}) + e\bar{u}\cdot\bar{\nabla}\rho\right]$$

$$= 0$$

Equivalent Flux Difference Form

We can surmise that (21.19.1) can be recovered by the 2-point flux:

$$f^{\#}_{ij} = \{\{\rho\}\}\{\{\bar{u}\}\} = \left(\frac{\rho_i+\rho_j}{2}\right)\left(\frac{\bar{u}_i+\bar{u}_j}{2}\right)$$

$$= \frac{1}{4}\left[(\rho\bar{u})_i + \rho_i\bar{u}_j + \rho_j\bar{u}_i + (\rho\bar{u})_j\right]$$

& so

$$2D_{ij}\circ f^{\#}_{ij} = \cancelto{0}{\frac{1}{2}(\rho\bar{u}_i)D_{ij}} + \frac{1}{2}\rho_i D_{ij}\bar{u}_j + \frac{1}{2}\bar{u}_i D_{ij}\rho_j$$

$$+ \frac{1}{2}D_{ij}(\rho\bar{u})_j$$

$$= \frac{1}{2}D_{ij}(\rho\bar{u})_j + \frac{1}{2}\left[\bar{u}_i D_{ij}\rho_j + \rho_i D_{ij}\bar{u}_j\right]$$

Let's now see how we can satisfy (21.19.2) for $f = \rho e \bar{u}$

$$f^{\#}_{ij} = \{\{\rho\}\}\{\{e\}\}\{\{\bar{u}\}\}$$

$$= \left(\frac{\rho_i + \rho_j}{2}\right)\left(\frac{e_i + e_j}{2}\right)\left(\frac{\bar{u}_i + \bar{u}_j}{2}\right)$$

$$= \frac{1}{8}\Big[(\rho e)_i + \rho_i e_j + \rho_j e_i + (\rho e)_j \Big](\bar{u}_i + \bar{u}_j)$$

$$= \frac{1}{8}\Big[(\rho e \bar{u})_i + (\rho \bar{u})_i e_j + (e\bar{u})_i \rho_j + \bar{u}_i (\rho e)_j$$

$$+ (\rho e)_i \bar{u}_j + \rho_i (e\bar{u})_j + e_i (\rho \bar{u})_j + (\rho e \bar{u})_j \Big]$$

$\therefore$

$$2 D \circ f^{\#} = \cancelto{\emptyset}{\frac{1}{4} (\rho e \bar{u})_i D_{ij}} + \frac{1}{4} D_{ij} (\rho e \bar{u})_j$$

$$+ \frac{1}{4}\Big[e_i D_{ij} (\rho \bar{u})_j + (\rho \bar{u})_i D_{ij} e_j \Big]$$

$$+ \frac{1}{4}\Big[\bar{u}_i D_{ij} (\rho e)_j + (\rho e)_i D_{ij} \bar{u}_j \Big]$$

$$+ \frac{1}{4}\Big[\rho_i D_{ij} (e\bar{u})_j + (e\bar{u})_i D_{ij} \rho_j \Big]$$

which recovers Kennedy-Gruber.

$\boxed{\text{Summary}}$ Therefore, starting w/ the conservation form of the PDE:

$$\frac{\partial \rho}{\partial t} + \bar{\nabla}\cdot(\rho \bar{u}) = 0$$

$$\frac{\partial}{\partial t}(\rho e) + \bar{\nabla}\cdot(\rho e \bar{u}) = 0$$

the semi-discrete form:

$$M_i^{(e)} \frac{d\rho_i^{(e)}}{dt} + D_{ij}^{(e)} f_j^{\rho} = \square$$

$$M_i^{(e)} \frac{d}{dt}(\rho e)_i^{(e)} + D_{ij}^{(e)} f_j^{\rho e} = \square$$

$\rightarrow$ ignoring the flux

can be rewritten by replacing the flux as follows:

$$M_i^{(e)} \frac{d\rho_i^{(e)}}{dt} + 2D_{ij}^{(e)} f_{ij}^{\#(\rho)} = \square$$

$$M_i^{(e)} \frac{d}{dt}(\rho e)_i^{(e)} + 2D_{ij}^{(e)} f_{ij}^{\#(\rho e)} = \square$$

where different choices of $f^{\#}$ lead to different split forms.

Conservation Laws vs. Balance Laws

What if our PDE does not fit so nicely into the Conservation law form:

$$\frac{\partial q}{\partial t} + \bar{\nabla} \cdot \bar{f} = 0$$

&, instead has:

$$\frac{\partial q}{\partial t} + \bar{\nabla} \cdot \bar{f} + \bar{\nabla} P + \ldots + = 0 \quad ?$$

In this case, one can still consider treating the CL part as before. This will get us to kinetic energy preserving schemes but not to entropy stable ones.