Lecture 20: 2D CG/DG Hyperbolic

Let's consider the PDE

$$\frac{\partial q}{\partial t} + \bar{\nabla} \cdot \bar{f} = \bar{\nabla} \cdot (\nu \bar{\nabla} q) \tag{20.1}$$

& rewrite as $\bar{F} = \bar{f} - \nu \bar{\nabla} q$, which then yields the conservation law:

$$\frac{\partial q}{\partial t} + \bar{\nabla} \cdot \bar{F} = 0 \tag{20.2}$$

Weak Integral Form

Let us use the basis function expansion:

$$q_N^{(e)}(\bar{x},t) = \sum_{j=1}^{M_N} \psi_j(\bar{x})\, q_j^{(e)}(t)$$

$$\bar{u}_N^{(e)}(\bar{x}) = \sum_{j=1}^{M_N} \psi_j(\bar{x})\, \bar{u}_j^{(e)} \quad \text{w/} \quad \bar{u} = u\hat{i} + v\hat{j}$$

& $\bar{f}_N^{(e)} = q_N^{(e)} \bar{u}_N^{(e)} \longrightarrow O(2N)$ polynomial

Substituting these expansions into (20.1), multiplying by ψ_i & integrating yields: find $q \in S$ $\forall \psi \in S$ s.t.:

(20.3) $$\int_{\Omega_e} \psi_i \frac{\partial q_N^{(e)}}{\partial t} d\Omega_e + \int_{\Gamma_e} \psi_i \hat{n}\cdot \bar{f}_N^{(*)} d\Gamma_e - \int_{\Omega_e} \bar{\nabla}\psi_i \cdot \bar{f}_N^{(e)} d\Omega_e$$

$$= \nu \int_{\Gamma_e} \psi_i \hat{n}\cdot \bar{\nabla} q_N^{(*)} d\Gamma_e + \nu \int_{\Gamma_e} \hat{n}\cdot\bar{\nabla}\psi_i \left[q_N^{(e)} - q_N^{(*)} \right] d\Gamma_e$$

$$- \nu \int_{\Omega_e} \bar{\nabla}\psi_i \cdot \bar{\nabla} q_N^{(e)} d\Omega_e$$

where we have used SIPG on the 2nd order terms

$\forall \quad i = 1, \ldots, M_N \quad , \quad e = 1, \ldots, N_e \quad \forall$

$$\bar{f}^{(*)} = \left\{ f^{(e,l)} \right\} - \frac{1}{2}\hat{n}\,|\lambda|\, [\![q^{(e,l)}]\!] , \qquad \lambda = \hat{n}\cdot\bar{u}$$

$$q_N^{(*)} = \left\{ q^{(e,l)} \right\}$$

$$\bar{\nabla} q_N^{(*)} = \left\{ \bar{\nabla} q^{(e,l)} \right\} - \hat{n}\,\mu\, [\![q^{(e,l)}]\!] , \quad \mu \text{ was defined in ch. 14.}$$

Basis Functions

Let us now switch from the monolithic to the tensor-product form as follows:

(20.4) $$\psi_i(\xi,\eta) = h_j(\xi) \otimes h_k(\eta)$$

where $j,k = 0,\ldots,N \quad \forall \quad i = 1,\ldots,M_N$

$\forall \quad i = j+1+k(N+1)$

Reference Element Equations (Weak Form)

Let us only consider the resulting form using inexact integration.

Eq. (20.3) in matrix-vector form is

$$(20.5) \quad M_{ij}^{(e)} \frac{dq_j^{(e)}}{dt} + F_{ij}^{(e,l)} \cdot f_j^{(*,l)} - \tilde{D}_{ij}^{(e)} \cdot f_j^{(e)}$$

$$= \nu \bar{F}_{ij}^{(e,l)} \{q^{(e,l)}\}_j - \nu \bar{\bar{F}}_{ij}^{(e,l)} [\![q^{(e,l)}]\!]_j + \nu \bar{\bar{F}}_{ij} \left(q^{(e)} - \{q^{(e,l)}\} \right)_j$$

$$- \nu L_{ij} q_j^{(e)}$$

where the matrices are:

$$M_{ij}^{(e)} = w_i^{(e)} J_i^{(e)} \delta_{ij} \longrightarrow M_i^{(e)} = w_i^{(e)} J_i^{(e)}$$

$$F_{ij}^{(e,l)} = w_i^{(l)} J_i^{(l)} \hat{n}_i^{(e,l)} \delta_{ij} \rightarrow F_i = w_i^{(l)} J_i^{(l)} \hat{n}_i^{(e,l)}$$

$$\tilde{D}_{ij}^{(e)} = \sum_{k=1}^{M_N} w_k^{(e)} J_k^{(e)} \bar{\nabla}\psi_{ik} \psi_{jk}$$

$$\bar{F}_{ij}^{(e,l)} = w_i^{(l)} J_i^{(l)} \hat{n}_i^{(e,l)} \cdot \bar{\nabla}\psi_{ji}$$

$$\bar{\bar{F}}_{ij}^{(e,l)} \equiv \bar{\bar{F}}_i^{(e,l)} = w_i^{(l)} J_i^{(l)} \mu_i$$

$$\overline{\overline{F}}_{ij}^{(e,\ell)} = \sum_{j=0}^{N} w_j^{(\ell)} J_j^{(\ell)} \hat{n}_j^{(e,\ell)} \cdot \overline{\nabla}\psi_{ij}$$

$$L_{ij}^{(e)} = \sum_{k=1}^{M_N} w_k^{(e)} J_k^{(e)} \overline{\nabla}\psi_{ik} \cdot \overline{\nabla}\psi_{jk}$$

Reference Element Eqs. (Strong Form)

The strong form of the divergence term is written as:

$$\int_{\Omega_e} \psi_i \overline{\nabla}\cdot \overline{f}_N^{(e)} \, d\Omega_e = \int_{\Gamma_e} \psi_i \hat{n} \cdot \left(\overline{f}_N^{(*,\ell)} - \overline{f}_N^{(e)} \right) d\Gamma_e \tag{20.6}$$

$$+ \int_{\Omega_e} \psi_i \overline{\nabla}\cdot \overline{f}_N^{(e)} \, d\Omega_e$$

And substituting into the original PDE, for the 1st term only, yields:

$$\int_{\Omega_e} \psi_i \frac{\partial q_N^{(e)}}{\partial t} d\Omega_e + \int_{\Gamma_e} \psi_i \hat{n} \cdot \left(\overline{f}_N^{(*,\ell)} - \overline{f}_N^{(e)} \right) d\Gamma_e \tag{20.7}$$

$$+ \int_{\Omega_e} \psi_i \overline{\nabla}\cdot \overline{f}_N^{(e)} \, d\Omega_e = \nu \int_{\Gamma_e} \psi_i \hat{n}\cdot\overline{\nabla}\psi_j \, d\Gamma_e \left\{ q^{(e,\ell)} \right\}_j$$

$$- \nu \int_{\Gamma_e} \mu \psi_i \psi_j \, d\Gamma_e [\![q^{(e,\ell)}]\!]_j + \nu \int_{\Gamma_e} \hat{n}\cdot\overline{\nabla}\psi_i \psi_j \, d\Gamma_e \left(q^{(e)} - \left\{ q^{(e,\ell)} \right\} \right)_j$$

$$- \nu \int_{\Omega_e} \overline{\nabla}\psi_i \cdot \overline{\nabla} q_N^{(e)} \, d\Omega_e$$

Matrix-Vector Product (Inexact Integration)

Going from weak to strong form, the only difference is in the differentiation matrix which we write as follows:

$$\bar{D}_{ij}^{(e)} = w_i^{(e)} J_i^{(e)} \bar{\nabla}\psi_{ji} \tag{20.8}$$

Replacing the matrices into (20.7) yields:

$$w_i^{(e)} J_i^{(e)} \frac{dq_i^{(e)}}{dt} + w_i^{(\ell)} J_i^{(\ell)} \hat{n}_i^{(e,\ell)} \cdot \left(\bar{f}^{(*,\ell)} - \bar{f}^{(e)}\right)_i \tag{20.9}$$

$$+ w_i^{(e)} J_i^{(e)} \bar{\nabla}\psi_{ji} \cdot \bar{f}_j^{(e)}$$

$$= \nu \left[w_i^{(\ell)} J_i^{(\ell)} \hat{n}_i^{(e,\ell)} \cdot \bar{\nabla}\psi_{ji} \left\{ q^{(e,\ell)} \right\}_j - w_i J_i \mu_i [\![q^{(e,\ell)}]\!]_i \right.$$

$$\left. + \sum_{j=0}^{N} w_j^{(\ell)} J_j^{(\ell)} \hat{n}_j^{(e,\ell)} \cdot \bar{\nabla}\psi_{ij} \left(q^{(e)} - \left\{ q^{(e,\ell)} \right\} \right)_j \right]$$

$$- \nu \sum_{k=1}^{M_N} w_k^{(e)} J_k^{(e)} \bar{\nabla}\psi_{ik} \cdot \bar{\nabla}\psi_{jk} \, q_j^{(e)}$$

Algorithm for Unified CG/DG

Let's now discuss the construction of the algorithm for Eq. (20.1). We shall accomplish this by first considering the 1st order operators (volume & flux terms), followed by the 2nd order terms (volume & flux).

First Order Operators

Volume Integral

Let's begin with the volume integral written in monolithic form as follows:

$$\int_{\Omega_e} \psi_I \, \bar{\nabla} \cdot \bar{f}_N^{(e)} \, d\Omega_e \simeq \sum_{k=1}^{M_N} w_k^{(e)} J_k^{(e)} \psi_{Ik} \left(\bar{\nabla} \cdot \bar{f}^{(e)}\right)_k$$

$$= w_I^{(e)} J_I^{(e)} \psi_{II} \left(\bar{\nabla} \cdot \bar{f}^{(e)}\right)_I$$

where $I, k = 1, \ldots, M_N$ &

$$\left(\bar{\nabla} \cdot \bar{f}\right)_k = \sum_{I=1}^{M_N} \bar{\nabla} \psi_{Ik} \cdot \bar{f}_I$$

Rewriting in tensor-product form, let:

$$\psi_{IK} = h_{ik}(\xi)\, h_{jl}(\eta) \quad w/ \quad i,j,k,l = 0,\dots,N$$

$$\& \quad \bar{f} = f^{(x)}\hat{i} + f^{(y)}\hat{j}.$$

We can now write the volume integral as:

$$w_I^{(e)} J_I^{(e)} \psi_{II} \left(\bar{\nabla}\cdot\bar{f}^{(e)}\right)_I$$

$$= w_i w_j J_{ij}^{(e)} \left[\frac{\partial\xi}{\partial x}_{ij} \frac{dh}{d\xi}_{ki}(\xi)\, h_{lj}(\eta) + \frac{\partial\eta}{\partial x}_{ij} h_{ki}(\xi) \frac{dh}{d\eta}_{lj}(\eta) \right] f_{kl}^{(x)}$$

$$+ w_i w_j J_{ij}^{(e)} \left[\frac{\partial\xi}{\partial y}_{ij} \frac{dh}{d\xi}_{ki}(\xi)\, h_{lj}(\eta) + \frac{\partial\eta}{\partial y}_{ij} h_{ki}(\xi) \frac{dh_{lj}}{d\eta}(\eta) \right] f_{kl}^{(y)}$$

Using cardinality s.t.

$$h_{ki}(\xi) = \begin{cases} 1 & k = i \\ 0 & k \neq i \end{cases} \quad \& \quad h_{lj}(\eta) = \begin{cases} 1 & l = j \\ 0 & l \neq j \end{cases}$$

We get:

$$w_I^{(e)} J_I^{(e)} \psi_{II} \left(\bar{\nabla}\cdot\bar{f}^{(e)}\right)_I$$

$$= \omega_i \omega_j J_{ij}^{(e)} \left[\frac{\partial \xi}{\partial x}_{ij} \frac{dh}{d\xi}_{ki} f_{kj}^{(x)} + \frac{\partial \eta}{\partial x}_{ij} \frac{dh}{d\eta}_{\ell j} f_{i\ell}^{(x)} \right]$$

$$+ \omega_i \omega_j J_{ij}^{(e)} \left[\frac{\partial \xi}{\partial y}_{ij} \frac{dh}{d\xi}_{ki} f_{kj}^{(y)} + \frac{\partial \eta}{\partial y}_{ij} \frac{dh}{d\eta}_{\ell j} f_{i\ell}^{(y)} \right]$$

Finally, replacing $\ell \to k$ gives the final form:

Vol

$$= \omega_i \omega_j J_{ij}^{(e)} \left[\frac{\partial \xi}{\partial x}_{ij} \frac{dh}{d\xi}_{ki} f_{kj}^{(x)} + \frac{\partial \eta}{\partial x}_{ij} \frac{dh}{d\eta}_{kj} f_{ik}^{(x)} \right]$$

$$+ \omega_i \omega_j J_{ij}^{(e)} \left[\frac{\partial \xi}{\partial y}_{ij} \frac{dh}{d\xi}_{ki} f_{kj}^{(y)} + \frac{\partial \eta}{\partial y}_{ij} \frac{dh}{d\eta}_{kj} f_{ik}^{(y)} \right]$$

(Covariant Form)

Contravariant Form

Noting that $\bar{\nabla}\xi \cdot \bar{f} = f^{(\xi)}$ & $\bar{\nabla}\eta \cdot \bar{f} = f^{(\eta)}$

we can rewrite as follows:

$$\text{Vol} = \omega_i \omega_j J_{ij}^{(e)} \left(\frac{dh}{d\xi}_{ki} \bar{\nabla}\xi_{ij} \cdot \bar{f}_{kj} + \frac{dh}{d\eta}_{kj} \bar{\nabla}\eta_{ij} \cdot \bar{f}_{ik} \right)$$

The algorithm using the covariant form is given in Alg. 17.1

$R^{(e)} = 0$ (Strong Form)

for $e = 1 : N_e$

  for $i = 0 : N$

    for $j = 0 : N$

      $I = i+1 + j(N+1)$

      $\frac{\partial \tilde{f}}{\partial \xi} = 0, \quad \frac{\partial \tilde{f}}{\partial \eta} = 0$

      for $k = 0 : N$

        $\frac{\partial \tilde{f}}{\partial \xi} \mathrel{+}= \frac{d h_{ki}}{d \xi} \tilde{f}_{kj}$

        $\frac{\partial \tilde{f}}{\partial \eta} \mathrel{+}= \frac{d h_{kj}}{d \eta} \tilde{f}_{ik}$

      end

      $\frac{\partial f}{\partial x}^{(x)} = \frac{\partial f}{\partial \xi}^{(x)} \frac{\partial \xi}{\partial x}_{ij} + \frac{\partial f}{\partial \eta}^{(x)} \frac{\partial \eta}{\partial x}_{ij}$

      $\frac{\partial f}{\partial y}^{(y)} = \frac{\partial f}{\partial \xi}^{(y)} \frac{\partial \xi}{\partial y}_{ij} + \frac{\partial f}{\partial \eta}^{(y)} \frac{\partial \eta}{\partial y}_{ij}$

      $R_I^{(e)} \mathrel{-}= w_i w_j J_{ij}^{(e)} \left(\frac{\partial f}{\partial x}^{(x)} + \frac{\partial f}{\partial y}^{(y)} \right)$

    end

  end

end

Volume Integral (Weak Form)

$$\int_{\Omega_e} \bar{\nabla}\psi_I \cdot \bar{f}_N^{(e)} \, d\Omega_e = \sum_{k=1}^{n_v} w_k^{(e)} J_k^{(e)} \bar{\nabla}\psi_{I,k} \cdot \left(\sum_{J=1}^{n_v} \psi_{J,k} \bar{f}_J^{(e)} \right)$$

via cardinality $\rightarrow$ $$= \sum_{J=1}^{n_v} w_J J_J^{(e)} \bar{\nabla}\psi_{IJ} \cdot \bar{f}_J^{(e)}$$

Using the chain rule:

$$\bar{\nabla}\psi_{IJ} = \frac{\partial \psi}{\partial \xi}_{IJ} \bar{\nabla}\xi_{IJ} + \frac{\partial \psi}{\partial \eta}_{IJ} \bar{\nabla}\eta_{IJ}$$

Using tensor-products:

$$\frac{\partial \psi}{\partial \xi}_{IJ} = \frac{\partial}{\partial \xi}\left(h_i(\xi_k) h_j(\eta_l) \right) = \frac{dh_i}{d\xi}(\xi_k) h_j(\eta_l)$$

$j = l$, $i = k$ } via cardinality

$$\frac{\partial \psi}{\partial \eta}_{IJ} = \frac{\partial}{\partial \eta}\left(h_i(\xi_k) h_j(\eta_l) \right) = h_i(\xi_k) \frac{dh_j}{d\eta}_{l}$$

Flux Integral

To complete the first order operators, we next discuss the flux integral term:

$$\text{Flux} = w_i^{(l)} J_i^{(l)} \hat{n}_i^{(e,l)} \cdot \left(\bar{f}^{(*,l)} - \bar{f}^{(e)} \right)_i$$

The algorithm is described in Alg. 17.2:

for $l = 1 : N_{Faces}$

$$\left.\begin{aligned} P_L &= face(1, l) \\ P_R &= face(2, l) \end{aligned}\right\} \text{Position of face on Ref. Element}$$

$$\left.\begin{aligned} L &= face(3, l) \\ R &= face(4, l) \end{aligned}\right\} \text{Element ID number for the Left \& Right elements sharing this face}$$

for $i = 0 : N$

$$\left.\begin{aligned} &i_L = mapL(1, i, P_L) \; ; \; j_L = mapL(2, i, P_L) \\ &I_L = i_L + 1 + j_L (N+1) \end{aligned}\right\} (i,j) \to I$$

$$i_R = mapR(1, i, P_R) \; ; \; j_R = map(2, i, P_R)$$

$$I_R = i_R + 1 + j_R (N+1)$$

$$\bar{f}_i^{(*)} = \frac{1}{2}\left[\bar{f}_{i_L, j_L}^{(L)} + \bar{f}_{i_R, j_R}^{(R)} - |\lambda| \hat{n}_i^{(L)} \left(q_{i_R, j_R}^{(R)} - q_{i_L, j_L}^{(L)} \right) \right]$$

$$R_{I_L}^{(L)} \mathrel{-}= \omega_i^{(\ell)} J_i^{(\ell)} \left[\hat{n}_i^{(\ell)} \cdot \left(\bar{f}_i^{(*)} - \delta \bar{f}_i^{(L)} \right) \right]$$

$$R_{I_R}^{(e)} \mathrel{+}= \omega_i^{(\ell)} J_i^{(\ell)} \left[\hat{n}_i^{(\ell)} \cdot \left(\bar{f}_i^{(*)} - \delta \bar{f}_i^{(R)} \right) \right]$$

end

end

where $|\lambda| = \max\left(\left| \hat{n}_i^{(\ell)} \cdot \bar{u}_{i_L, j_L}^{(L)} \right|, \left| \hat{n}_i^{(\ell)} \cdot \bar{u}_{i_R, j_R}^{(R)} \right| \right)$

&

$$\delta = \begin{cases} 1 & \text{Strong form} \rightarrow \text{IBP twice} \\ 0 & \text{weak form} \rightarrow \text{IBP once} \end{cases}$$

Second Order Operators

Volume Integral Let us describe the discretization of the Laplacian term:

$$\int_{\Omega_e} \bar{\nabla}\psi_I \cdot \bar{\nabla} q_N^{(e)} \, d\Omega_e \approx \sum_{k=1}^{M_N} \omega_k^{(e)} J_k^{(e)} \bar{\nabla}\psi_{Ik} \cdot \bar{\nabla} q_k^{(e)}$$

where

$$\bar{\nabla}\psi_{Iu} = \frac{\partial \psi}{\partial x}_{Iu}\,\hat{i} + \frac{\partial \psi}{\partial y}_{Iu}\,\hat{j}$$

with

$$\frac{\partial \psi}{\partial x}_{Iu} = \frac{\partial \psi_{Iu}}{\partial \xi}\frac{\partial \xi}{\partial x}_u + \frac{\partial \psi}{\partial \eta}_{Iu}\frac{\partial \eta}{\partial x}_u$$

&

$$\frac{\partial \psi}{\partial y}_{Iu} = \frac{\partial \psi}{\partial \xi}_{Iu}\frac{\partial \xi}{\partial y}_u + \frac{\partial \psi}{\partial \eta}_{Iu}\frac{\partial \eta}{\partial y}_u$$

Switching to tensor-products we write:

$$\frac{\partial \psi}{\partial \xi}_{Iu} = \frac{\partial}{\partial \xi}\Big(h_{iu}(\xi)\,h_{je}(\eta)\Big) = \frac{dh_{iu}}{d\xi}(\xi)\,h_{je}(\eta)$$

&

$$\frac{\partial \psi}{\partial \eta}_{Iu} = \frac{\partial}{\partial \eta}\Big(h_{iu}(\xi)\,h_{je}(\eta)\Big) = h_{iu}(\xi)\,\frac{dh}{d\eta}_{je}(\eta)$$

We can now write the derivatives:

$$\frac{\partial \psi}{\partial x}_{Iu} = \frac{dh}{d\xi}_{iu}(\xi)\,h_{je}(\eta)\frac{\partial \xi}{\partial x}_{ue} + h_{iu}(\xi)\,\frac{dh}{d\eta}_{je}(\eta)\,\frac{\partial \eta}{\partial x}_{ue}.$$

Using cardinality & replacing $j \to i$ in the 2^{nd} term:

$$\frac{\partial \psi}{\partial x}_{Iu} = \frac{dh}{d\xi}_{iu}\,\frac{\partial \xi}{\partial x}_{ue} + \frac{dh}{d\eta}_{ie}\,\frac{\partial \eta}{\partial x}_{ue}.$$

Similarly, we write:

$$\frac{\partial \psi}{\partial y}_{Iu} = \frac{dh_{iu}}{d\xi}\,\frac{\partial \xi}{\partial y}_{ue} + \frac{dh_{ie}}{d\eta}\,\frac{\partial \eta}{\partial y}_{ue}$$

to complete $\bar{\nabla}\psi_{In}$.

Next, we write: $\bar{\nabla} q_u = \frac{\partial q}{\partial x}u\,\hat{i} + \frac{\partial q}{\partial y}u\,\hat{j}$ as:

$$\frac{\partial q}{\partial x}u = \sum_{i=0}^{N}\sum_{j=0}^{N}\left[\frac{dh}{d\xi}_{iu}(\xi)\, h_{jl}(\eta)\,\frac{\partial \xi}{\partial x}_{ul} + h_{iu}(\xi)\frac{dh}{d\eta}_{jl}(\eta)\frac{\partial \eta}{\partial x}_{ul}\right] q_{ij}$$

$$= \sum_{i=0}^{N}\frac{dh}{d\xi}_{iu}\,\frac{\partial \xi}{\partial x}_{ul}\; q_{il} + \sum_{j=0}^{N}\frac{dh}{d\eta}_{jl}\,\frac{\partial \eta}{\partial x}_{ul}\; q_{uj}$$

Replacing $j \rightarrow i$ in the 2nd term gives:

$$\frac{\partial q}{\partial x}u = \frac{\partial q}{\partial x}_{ul} = \sum_{i=0}^{N}\left[\frac{dh}{d\xi}_{iu}\,\frac{\partial \xi}{\partial x}_{ul}\, q_{il} + \frac{dh}{d\eta}_{il}\,\frac{\partial \eta}{\partial x}_{ul}\, q_{ui}\right]$$

$$= \frac{\partial q}{\partial \xi}_{ul}\,\frac{\partial \xi}{\partial x}_{ul} + \frac{\partial q}{\partial \eta}_{ul}\,\frac{\partial \eta}{\partial x}_{ul}$$

Similarly, we write:

$$\frac{\partial q}{\partial y}u = \frac{\partial q}{\partial y}_{ul} = \sum_{i=0}^{N}\left[\frac{dh}{d\xi}_{iu}\,\frac{\partial \xi}{\partial y}_{ul}\, q_{il} + \frac{dh}{d\eta}_{il}\,\frac{\partial \eta}{\partial y}_{ul}\, q_{ui}\right]$$

$$= \frac{\partial q}{\partial \xi}_{ul}\,\frac{\partial \xi}{\partial y}_{ul} + \frac{\partial q}{\partial \eta}_{ul}\,\frac{\partial \eta}{\partial y}_{ul}$$

We can now construct

$$\bar{\nabla}\psi_{In}\cdot\bar{\nabla} q_u = \frac{\partial \psi}{\partial x}_{In}\,\frac{\partial q}{\partial x}u + \frac{\partial \psi}{\partial y}_{In}\,\frac{\partial q}{\partial y}u$$

where

$$\frac{\partial \psi}{\partial x}_{Ik}\ \frac{\partial \varphi}{\partial x}_{k} = \left(\frac{dh}{d\xi}_{ik}\ \frac{\partial \xi}{\partial x}_{kl} + \frac{dh}{d\eta}_{il}\ \frac{\partial \eta}{\partial x}_{kl} \right) \frac{\partial \varphi}{\partial x}_{kl}$$

&

$$\frac{\partial \psi}{\partial y}_{Ik}\ \frac{\partial \varphi}{\partial y}_{k} = \left(\frac{dh}{d\xi}_{ik}\ \frac{\partial \xi}{\partial y}_{kl} + \frac{dh}{d\eta}_{il}\ \frac{\partial \eta}{\partial y}_{kl} \right) \frac{\partial \varphi}{\partial y}_{kl}$$

With all this in mind, we can now build the 2nd order volume integral term as follows:

$$\int_{\Omega_e} \bar{\nabla}\psi_I \cdot \bar{\nabla}\varphi_n^{(e)}\, d\Omega_e \simeq \sum_{k=0}^{N}\sum_{l=0}^{N} w_k w_l J_{kl}^{(e)} \left(\frac{dh}{d\xi}_{ik} \frac{\partial \xi}{\partial x}_{kl} + \frac{dh}{d\eta}_{il} \frac{\partial \eta}{\partial x}_{kl} \right) \frac{\partial \varphi_{kl}}{\partial x}$$
$$+ \sum_{k=0}^{N}\sum_{l=0}^{N} w_k w_l J_{kl}^{(e)} \left(\frac{dh}{d\xi}_{ik} \frac{\partial \xi}{\partial y}_{kl} + \frac{dh}{d\eta}_{il} \frac{\partial \eta}{\partial y}_{kl} \right) \frac{\partial \varphi_{kl}}{\partial x}$$

Rearranging:

$$\int_{\Omega_e} \bar{\nabla}\psi_I \cdot \bar{\nabla}\varphi_n^{(e)}\, d\Omega_e \approx \sum_{k=0}^{N}\sum_{l=0}^{N} w_k w_l J_{kl}^{(e)} \frac{dh}{d\xi}_{ik} h_{ll} \left(\bar{\nabla}\xi_{kl} \cdot \bar{\nabla}\varphi_{kl} \right)$$
$$+ \sum_{k=0}^{N}\sum_{l=0}^{N} w_k w_l J_{kl}^{(e)} h_{kk} \frac{dh}{d\eta}_{il} \left(\bar{\nabla}\eta_{kl} \cdot \bar{\nabla}\varphi_{kl} \right)$$

This leads to Alg. 17.3:

for $e = 1 : N_e$

for $n = 0 : N$

for $l = 0 : N$

$\frac{\partial \xi}{\partial \gamma}_{nl} = \frac{\partial \xi}{\partial \eta}_{nl} = 0$

for $i = 0 : N$

$\frac{\partial \xi}{\partial \gamma}_{nl} \mathrel{+}= \frac{dh}{d\gamma}_{in} \; \xi_{il}$

$\frac{\partial \xi}{\partial \eta}_{nl} \mathrel{+}= \frac{dh}{d\eta}_{il} \; \xi_{ni}$

end

$\frac{\partial \xi}{\partial x}_{nl} = \frac{\partial \xi}{\partial \gamma}_{nl} \frac{\partial \gamma}{\partial x}_{nl} + \frac{\partial \xi}{\partial \eta}_{nl} \frac{\partial \eta}{\partial x}_{nl}$

$\frac{\partial \xi}{\partial y}_{nl} = \frac{\partial \xi}{\partial \gamma}_{nl} \cdot \frac{\partial \gamma}{\partial y}_{nl} + \frac{\partial \xi}{\partial \eta}_{nl} \frac{\partial \eta}{\partial y}_{nl}$

$(\bar{\nabla}\gamma \cdot \bar{\nabla}\xi)_{nl} = \frac{\partial \gamma}{\partial x}_{nl} \frac{\partial \xi}{\partial x}_{nl} + \frac{\partial \gamma}{\partial y}_{nl} \frac{\partial \xi}{\partial y}_{nl}$

$(\bar{\nabla}\eta \cdot \bar{\nabla}\xi)_{nl} = \frac{\partial \eta}{\partial x}_{nl} \frac{\partial \xi}{\partial x}_{nl} + \frac{\partial \eta}{\partial y} \frac{\partial \xi}{\partial y}_{nl}$

for $i = 0 : N$

$I_\gamma = i + 1 + l(N+1)$

$I_\eta = n + 1 + i(N+1)$

$R^{(e)}_{I_\gamma} \mathrel{+}= w_n w_l J^{(e)}_{nl} \frac{dh}{d\gamma}_{in} h_{ll} (\bar{\nabla}\gamma \cdot \bar{\nabla}\xi)_{nl}$

$R^{(e)}_{I_\eta} \mathrel{+}= w_n w_l J^{(e)}_{nl} h_{nn} \frac{dh}{d\eta}_{il} (\bar{\nabla}\eta \cdot \bar{\nabla}\xi)_{nl}$

end

end % l

end % n

end % e

$\boxed{\text{Flux Terms}}$ The flux terms are given by Algs 17.4, 17.5, & 17.6.

Final Algorithm for CG/DG

Alg 17.7 identifies the steps required to solve the advection-diffusion eq. using a unified CG/DG method. It is as follows:

Construct M → using Algs. 12.6 & 12.11

for $n = 1 : N_{time-final}$

  Construct $R^{(e)}$ → using Algs. 17.1, 17.2, 17.3, & 17.4

  Construct R_I using DSS → use Alg. 12.12

  $\hat{R}_I = M_I^{-1} R_I$

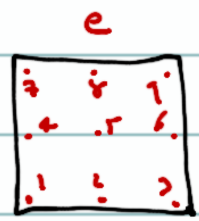
  $\frac{dq_I}{dt} = \hat{R}_I$

end

Note on Data Storage

The algorithms described in this Lecture assumes that the data is stored in what we call the Local

Element-wise (LEW) storage. This means that we store the data as follows:



$$q_i^{(e)} \quad \text{where} \quad i = 1, \dots, M_N$$

$$q_i^{(e)} = q(i,e)$$

However, using INTMA$(i,j,e) \rightarrow I$ we can store the data as a long vector as follows:

$$q_I = q(I) \quad \text{where} \quad I = \text{intma}(i,j,e) \quad \text{where}$$

$$i,j = 0, \dots, N$$

$$e = 1, \dots, N_e$$

We call this storage Augmented Global Gridpoint (AGGP) storage because we use the typical storage used to store C^0 CG methods but also include DG. AGGP is the most general storage scheme & is what I recommend (NUMA is based on AGGP).