MA4245 Mathematical Principles of Galerkin Methods
Project 3: 1D Linearized Shallow Water Equations

Prof. Frank Giraldo
Department of Applied Mathematics
Naval Postgraduate School
Monterey, CA 93943-5216

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1 Continuous Problem

The governing partial differential equation (PDE) is

$$\frac{\partial}{\partial t}\begin{pmatrix} h_S \\ U \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} U \\ g h_B h_S \end{pmatrix} = \begin{pmatrix} 0 \\ h_S \frac{dh_B}{dx} \end{pmatrix} \tag{1}$$

1.1 Initial Condition

Since the governing PDE is a hyperbolic system, then this problem represents an initial value problem (IVP or Cauchy Problem). We, therefore, need an initial condition. Let it be the following sinusoidal

$$h_S(x,0) = \frac{1}{2}\cos 2\pi x \quad U(x,0) = 0 \tag{2}$$

where $(x,t) \in [0,1]^2$. The exact solution to this system of equations is

$$h_S(x,t) = \frac{1}{2}\cos 2\pi x \cos 2\pi t \quad U(x,t) = \frac{1}{2}\sin 2\pi x \sin 2\pi t \tag{3}$$

when we let $g = h_B = 1$.

1.2 Boundary Condition

This problem also requires a boundary condition: let us impose no-flux boundary conditions at the endpoints. That is at $x = 0$ and $x = 1$ we let $U = 0$.

2 Simulations

Write one code (will accept two but prefer one) that uses both CG and DG. It is better to use the same code to do both CG and DG with a switch (if statement) to handle the communicator in both CG and DG. You need to show results for exact (let Q=N+1 be exact) AND inexact integration (Q=N) so write your codes in a general way.

2.1 Results You Need to Show

You must show results for linear elements $N = 1$ with increasing number of elements N_e and then show results for $N = 4$, $N = 8$, and $N = 16$ with increasing numbers of elements. Plot all the convergence rates on one plot (That is, N=1, 2, 4, 8, and 16 each produce one curve). I used a time-step of $\Delta t = 1 \times 10^{-4}$ with an RK3 method to get the rates in my manuscript.

N=1 Simulations For linear elements, use $N_e = 32, 64, 96, 128$ elements.

N=2 Simulations For $N = 2$, use $N_e = 16, 32, 48, 64$ elements.

N=4 Simulations For $N = 4$ use $N_e = 8, 16, 24, 32$ elements.

N=8 Simulations For $N = 8$ use $N_e = 4, 8, 12, 16$, elements.

N=16 Simulations For $N = 16$ use $N_e = 2, 4, 6, 8$ elements.

3 Helpful Relations

Error Norm The following expressions may be helpful

$$L^2 = \sqrt{\sum_{e=1}^{N_e} \sum_{i=0}^{N} \left(q_{N,i}^{(e)} - q_{E,i}^{(e)} \right)^2} \tag{4}$$

where $e = 1, ..., N_e$ are the number of elements and $i = 0, ..., N$ are the interpolation points and q_N and q_E denote the numerical and exact solutions.

In addition, let us define the mass conservation measure as follows

$$\Delta M = \mid \text{Mass(t)} - \text{Mass(0)} \mid .$$

where Mass(t) is the total mass at time t and M(0) the mass at the initial time. The mass is defined as follows:

$$\text{Mass(t)} = \sum_{e=1}^{N_e} \sum_{i=0}^{N} \sum_{k=0}^{Q} w_k \left(h_{S,i}^{(e)}(t) + h_{B,i}^{(e)} \right) \psi_j(x_k).$$

Time-Integrator To solve the time-dependent portion of the problem use the 2nd order RK method: for $\frac{\partial q}{\partial t} = R(q)$ let

$$q^{n+1/2} = q^n + \frac{\Delta t}{2} R(q^n)$$

$$q^{n+1} = q^n + \Delta t R(q^{n+1/2})$$

or a better time-integrator of your choice (DO NOT USE FORWARD EULER); feel free to use ODE45 in Matlab if you know how to use it.

Make sure that your time-step Δt is small enough to ensure stability. Recall that the Courant number

$$C = u \frac{\Delta t}{\Delta x}$$

must be within a certain value for stability. For the 2nd order RK method I give you, it should be below $\frac{1}{4}$. For Δx take the difference of the first point in your domain $x_1 = 0$ and the next point x_2 since this will be the tightest clustering of points in your model.

4 Extra Credit

Vary the initial conditions to make the simulation more interesting. Show me plots of the results (no convergence rates since you don't know the exact solution!). For example, make $h_B \neq 0$. Perhaps make it parabolic or triangular. Make h_S initially a Gaussian bump (with $U = 0$) and see how this bump is affected by the submarine ridge defined by $h_B \neq 0$. Again, show me plots of the solution and give a brief discussion of the pros of CG versus DG.