Lecture 19: 2D DG Hyperbolic

In Lect 18 we discussed the discretization of the following PDE using CG:

(19.1) $$\frac{\partial q}{\partial t} + \bar{u}\cdot\bar{\nabla} q = \bar{\nabla}\cdot(\nu\bar{\nabla} q)$$ w/ appropriate BCs

We can rewrite the advection term in (19.1) as follows:

(19.2) $$\bar{\nabla}\cdot(q\bar{u}) = \bar{u}\cdot\bar{\nabla} q + q\bar{\nabla}\cdot\bar{u}.$$

Substituting (19.2) into (19.1) yields

(19.3) $$\frac{\partial q}{\partial t} + \bar{\nabla}\cdot(q\bar{u}) - q\bar{\nabla}\cdot\bar{u} = \bar{\nabla}\cdot(\nu\bar{\nabla} q)$$

which can be arranged as follows:

$$\frac{\partial q}{\partial t} + \bar{\nabla}\cdot\left(q\bar{u} - \nu\bar{\nabla} q\right) = q\bar{\nabla}\cdot\bar{u}$$

If the velocity is divergence-free then $\bar{\nabla}\cdot\bar{u} = 0$ & we can write (19.3) in the final form:

(19.4) $$\frac{\partial q}{\partial t} + \bar{\nabla}\cdot(q\bar{u} - \nu\bar{\nabla} q) = 0$$

which describes a conservation law whereas (19.3) represents a balance law.

If we let $\bar{f} = q\bar{u} - \nu \bar{\nabla} q$ then we can write (11.4) as:

$$(11.5) \qquad \frac{\partial q}{\partial t} + \bar{\nabla} \cdot \bar{f} = 0.$$

Integral Form

Let us now introduce the basis function expansion

$$q_N^{(e)}(\bar{x}, t) = \sum_{i=1}^{M_N} \psi_i(\bar{x})\, q_i^{(e)}(t) \quad \&$$

$$\bar{u}_N^{(e)}(\bar{x}) = \sum_{i=1}^{M_N} \psi_j(\bar{x})\, \bar{u}_i^{(e)}$$

where:

$$\bar{f}_N^{(e)} = q_N^{(e)}\, \bar{u}_N^{(e)} - \nu\, \bar{Q}_N^{(e)} \qquad \text{(assuming LDG)}$$

$\&$ $\bar{u} = u\hat{i} + v\hat{j}$ $\&$

$$M_N = (N_\xi + 1)(N_\eta + 1) \equiv (N+1)^2$$

Introducing the expansions into (11.5), multiplying by ψ_i $\&$ integrating yields:

$$(11.6) \quad \int_{\Omega_e} \psi_i \frac{\partial q_N^{(e)}}{\partial t}\, d\Omega_e + \sum_{l=1}^{2d} \int_{\Gamma_e} \psi_i\, \hat{n}^{(e,l)} \cdot \bar{f}_N^{(*,l)}\, d\Gamma_e$$

$$- \int_{\Omega_e} \bar{\nabla}\psi_i \cdot \bar{f}_N^{(e)}\, d\Omega_e = 0 \qquad \forall \quad \psi \in L^2$$

with $i = 1, \dots, M_N$ & $e = 1, \dots, N_e$, $\bar{f}_N^{(*,l)} = \bar{f}_{inv}^{(*,l)} + \bar{f}_{visc}^{(*,l)}$

$$\bar{f}_{inv}^{(*,l)} = \frac{1}{2}\left[\bar{f}_{inv}^{(l)} + \bar{f}_{inv}^{(e)} - |\lambda|\, \hat{n}^{(e,l)} \left(q^{(l)} - q^{(e)} \right) \right].$$

with $|\lambda| = |\hat{n} \cdot \bar{u}|_{max}$

&

$$\bar{f}_{visc}^{(*,l)} = \frac{1}{2}\left[\bar{f}_{visc}^{(l)} + \bar{f}_{visc}^{(e)} \right]$$

Equations on Reference Element: Weak Form

$\bar{Q}$ for LDG

Since the flux is defined as:

$$\bar{f} = q\bar{u} - \nu \bar{\nabla} q$$

then we learned already that using LDG requires us to solve this term as follows:

$$\bar{Q} = \bar{\nabla} q \longrightarrow \bar{\nabla} q_N^{(e)}(\bar{x}, t) = \sum_{j=1}^{M_N} \bar{\nabla}\psi_j(\bar{x})\, q_j^{(e)}(t)$$

which, in matrix form, is:

$$(11.7)\quad M^{(e)}\left(\bar{Q}^{(e)}\cdot \bar{\bar{I}}_d\right) = \sum_{l=1}^{2d} \left(\bar{F}^{(e,l)}\right)^T \left(q^{(*,l)}\,\bar{\bar{I}}_d\right) - \tilde{D}^{(e)}\, q^{(e)}$$

where $q^{(*,l)} = \frac{1}{2}\left(q^{(l)} + q^{(e)}\right)$ & we know q on the RHS & so we can solve for $\bar{Q}^{(e)}$ by inverting the block diagonal mass matrix.

Matrix-Vector Problem for Exact Integration

For simplicity, let us rewrite (11.6) as follows:

$$(11.8)\quad M_{ij}^{(e)} \frac{dq_j^{(e)}}{dt} + \bar{F}_{ij}^{(e,l)\,T}\, \bar{f}_j^{(*,l)} - \tilde{D}_{ij}^{(e)\,T}\, \bar{f}_j^{(e)} = 0$$

where the matrices are defined as:

$$M_{ij}^{(e)} \equiv \int_{\Omega_e} \psi_i \psi_j \, d\Omega_e = \sum_{k=1}^{M_Q} w_k J_k^{(e)} \psi_{ik} \psi_{jk}$$

$$\bar{F}_{ij}^{(e,l)} \equiv \int_{\Gamma_e} \psi_i \psi_j \, \hat{n}^{(e,l)} \, d\Gamma_e = \sum_{k=0}^{Q} w_k^{(l)} J_k^{(l)} \psi_{ik} \psi_{jk}$$

$$\tilde{D}_{ij}^{(e)} \equiv \int_{\Omega_e} \bar{\nabla}\psi_i \, \psi_j \, d\Omega_e = \sum_{k=1}^{M_Q} w_k J_k^{(e)} \bar{\nabla}\psi_{ik} \, \psi_{jk}$$

where $M_Q = (Q+1)^2$ & to integrate $3N$ polynomials exactly requires:

$$2Q - 1 = 3N \rightarrow Q = \frac{3N+1}{2} \text{ Lobatto points}$$

& $2Q+1 = 3N \rightarrow \quad Q = \frac{3N-1}{2}$ Legendre points

Inverting the mass matrix in (11.8) yields:

$$\frac{dq_i^{(e)}}{dt} + \hat{F}_{ij}^{(e,l)\,T} \bar{f}_j^{(*,l)} - \hat{D}_{ij}^{(e)\,T} f_j^{(e)} = 0 \qquad (11.9)$$

where $\hat{F} = M^{-1}\bar{F}$ & $\hat{D} = M^{-1}\tilde{D}$

Matrix-Vector Problem for Inexact Integration

Let's revisit (11.8) but assume that $Q=N$ & $M_Q = M_N$ which results in the simplification of the element matrices as follows:

$$M_{ij}^{(e)} = w_i J_i^{(e)} \delta_{ij} \longrightarrow M_i^{(e)} = w_i J_i^{(e)}$$

$$\bar{F}_{ij}^{(e,l)} = w_i^{(l)} J_i^{(l)} \hat{n}_i^{(e,l)} \delta_{ij} \longrightarrow \bar{F}_i^{(e,l)} = w_i^{(l)} J_i^{(l)} \hat{n}_i^{(e,l)}$$

$$\tilde{D}_{ij}^{(e)} = w_j J_j^{(e)} \nabla\psi_{ji}$$

Equations on Reference Element: Strong Form

Now that we know how to construct the weak form approximation for the divergence operator, let us now consider constructing the strong form.

Let's begin with the product rule

$$(11.11) \quad \bar{\nabla} \cdot \left(\psi_i \bar{f}_N^{(e)} \right) = \bar{\nabla} \psi_i \cdot \bar{f}_N^{(e)} + \psi_i \bar{\nabla} \cdot \bar{f}_N^{(e)}$$

& integrating yields:

$$(11.12) \quad \int_{\Gamma_e} \psi_i \, \hat{n} \cdot \bar{f}_N^{(e)} \, d\Gamma_e = \int_{\Omega_e} \bar{\nabla} \psi_i \cdot \bar{f}_N^{(e)} \, d\Omega_e + \int_{\Omega_e} \psi_i \bar{\nabla} \cdot \bar{f}_N^{(e)} \, d\Omega_e$$

Rearranging (& introducing a numerical flux) yields:

$$(11.13) \quad \int_{\Omega_e} \psi_i \, \bar{\nabla} \cdot \bar{f}_N^{(e)} \, d\Omega_e = \int_{\Gamma_e} \psi_i \, \hat{n} \cdot \bar{f}_N^{(*,e)} \, d\Gamma_e - \int_{\Omega_e} \bar{\nabla} \psi_i \cdot \bar{f}_N^{(e)} \, d\Omega_e$$

where, from (11.12) we note that:

$$(11.14) \quad \int_{\Omega_e} \bar{\nabla} \psi_i \cdot \bar{f}_N^{(e)} \, d\Omega_e = \int_{\Omega_e} \psi_i \, \bar{\nabla} \cdot \bar{f}_N^{(e)} \, d\Omega_e - \int_{\Gamma_e} \psi_i \, \hat{n} \cdot \bar{f}_N^{(e)} \, d\Gamma_e$$

Substituting (11.14) into (11.13) yields:

(19.15) $$\int_{\Omega_e} \psi_i \, \bar{\nabla} \cdot \bar{f}_N^{(e)} \, d\Omega_e = \int_{\Omega_e} \psi_i \, \bar{\nabla} \cdot \bar{f}_N^{(e)} \, d\Omega_e + \int_{\Gamma_e} \psi_i \, \hat{n} \cdot \left(\bar{f}_N^{(*,\ell)} - \bar{f}_N^{(e)} \right) d\Gamma_e$$

Therefore, the discretization of the conservation law:

$$\frac{\partial q}{\partial t} + \bar{\nabla} \cdot \bar{f} = 0 \rightarrow \int_{\Omega_e} \psi_i \frac{\partial q_N^{(e)}}{\partial t} + \int_{\Omega_e} \psi_i \, \bar{\nabla} \cdot \bar{f}_N^{(e)} \, d\Omega_e = 0$$

becomes:

(19.16) $$\int_{\Omega_e} \psi_i \frac{\partial q_N^{(e)}}{\partial t} d\Omega_e + \int_{\Omega_e} \psi_i \, \bar{\nabla} \cdot \bar{f}_N^{(e)} \, d\Omega_e + \sum_{\ell=1}^{2d} \int_{\Gamma_e} \psi_i \, \hat{n} \cdot \left(\bar{f}_N^{(*,\ell)} - \bar{f}_N^{(e)} \right) d\Gamma_e = 0$$

& represents the strong form DG representation.

Matrix-Vector Problem for Exact Integration

In matrix form, (19.16) is written as:

(19.17) $$M_{ij}^{(e)} \frac{dq_j^{(e)}}{dt} + \sum_{\ell=1}^{2d} \bar{F}_{ij}^{(e,\ell)\,T} \left(\bar{f}^{(*,\ell)} - \bar{f}^{(e)} \right)_j + \bar{D}_{ij}^{(e)\,T} \bar{f}_j^{(e)} = 0$$

where:

$$M_{ij}^{(e)} \equiv \int_{\Omega_e} \psi_i \psi_j \, d\Omega_e = \sum_{k=1}^{M_Q} w_k J_k^{(e)} \psi_{ik} \psi_{jk}$$

$$\bar{F}_{ij}^{(e,l)} \equiv \int_{\Gamma_e} \psi_i \psi_j \hat{n}^{(e,l)} \, d\Gamma_e = \sum_{k=0}^{Q} w_k^{(l)} J_k^{(l)} \psi_{ik} \psi_{jk} \hat{n}_k^{(e,l)}$$

$$\bar{D}_{ij}^{(e)} \equiv \int_{\Omega_e} \psi_i \bar{\nabla} \psi_j \, d\Omega_e = \sum_{k=1}^{M_Q} w_k J_k^{(e)} \psi_{ik} \bar{\nabla} \psi_{jk}$$

is the strong form differentiation matrix.

Left-multiplying (11.17) by M^{-1} yields

(11.18) $$\frac{dq_i^{(e)}}{dt} + \sum_{l=1}^{2d} \hat{F}_{ij}^{(e,l)T} \left(\bar{f}^{(*,l)} - \bar{f}^{(e)} \right)_j + \hat{D}_{ij}^{(e)T} \bar{f}_j^{(e)} = 0$$

where
$$\hat{F}_{ij}^{(e,l)} = \left(M_{ik}^{(e)} \right)^{-1} \bar{F}_{kj}^{(e,l)} \quad \& \quad \hat{D}_{ij}^{(e)} = \left(M_{ik}^{(e)} \right)^{-1} \bar{D}_{kj}^{(e)}$$

Matrix-Vector Problem for Inexact Integration

Let us now consider the special case for $Q = N$ & $M_Q = M_N$. With this, we get:

(11.19) $$M_i^{(e)} \frac{dq_i^{(e)}}{dt} + \sum_{l=1}^{2d} \bar{F}_i^{(e,l)T} \left(\bar{f}^{(*,l)} - \bar{f}^{(e)} \right)_i + \bar{D}_{ij}^{(e)T} \bar{f}_j^{(e)} = 0$$

where:

$$M_i^{(e)} = w_i J_i^{(e)}$$

(11.20) $$\bar{F}_i^{(e,\ell)} = w_i^{(\ell)} J_i^{(\ell)} \hat{n}_i^{(e,\ell)}$$

$$\bar{D}_{ij}^{(e)} = w_i J_i^{(e)} \bar{\nabla}\psi_{ji}$$

Substituting the matrices in (11.20) into (11.11) yields:

(11.21) $$w_i^{(e)} J_i^{(e)} \frac{dq_i^{(e)}}{dt} + \sum_{\ell=1}^{2d} w_i^{(\ell)} J_i^{(\ell)} \hat{n}_i^{(e,\ell)} \cdot \left(\bar{f}^{(*,\ell)} - \bar{f}^{(e)}\right)_j$$
$$+ w_i^{(e)} J_i^{(e)} \bar{\nabla}\psi_{ji} \cdot \bar{f}_j^{(e)} = 0$$

Dividing by $w_i^{(e)} J_i^{(e)}$:

(11.22) $$\frac{dq_i^{(e)}}{dt} + \sum_{\ell=1}^{2d} \frac{w_i^{(\ell)} J_i^{(\ell)}}{w_i^{(e)} J_i^{(e)}} \hat{n}_i^{(e,\ell)} \cdot \left(\bar{f}^{(*,\ell)} - \bar{f}^{(e)}\right)_j + \bar{\nabla}\psi_{ji} \cdot \bar{f}_j^{(e)} = 0$$

which looks like a FDM w/ a special penalty term at the element interfaces (fluxes).

Example using $N=1$

Since we already covered CGD & SIPG previously & know how to handle diffusion operators, let us only consider inviscid flow below such that: $\bar{f} = g\bar{u}$.

Furthermore, let us write the element eqs as follows:

$$(19.23) \quad M_{ij}^{(e)} \frac{d q_j}{dt} + \overline{F}_{ij}^{(e,l)} \cdot \bar{f}_j^{(*,l)} - \tilde{D}_{ij}^{(e)} \cdot \bar{f}_j^{(e)} = 0$$

where it should be understood that each element $e = 1, \ldots, N_e$ has $l = 1, \ldots, 2d$ face neighbors (e,l).

Basis Functions & Metric Terms

Recall that for $N=1$ we have:

$$(19.24) \quad \psi_i(\xi,\eta) = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)$$

&

$$\frac{\partial \psi_i}{\partial \xi}(\xi,\eta) = \frac{1}{4}\xi_i(1+\eta_i\eta), \quad \frac{\partial \psi_i}{\partial \eta}(\xi,\eta) = \frac{1}{4}(1+\xi_i\xi)\eta_i.$$

& assuming $x = x(\xi)$, $y = y(\eta)$

$$\frac{\partial x}{\partial \xi} = \frac{\Delta x}{2}, \quad \frac{\partial x}{\partial \eta} = \frac{\partial y}{\partial \xi} = 0, \quad \frac{\partial y}{\partial \eta} = \frac{\Delta y}{2}, \quad J^{(e)} = \frac{\Delta x^{(e)} \Delta y^{(e)}}{4}$$

Finally, we can write:

$$\frac{\partial \psi_i}{\partial x} = \frac{\partial \psi_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \psi_i}{\partial \eta}\cancelto{0}{\frac{\partial \eta}{\partial x}}$$

(19.25.1)
$$= \frac{1}{4}\,\xi_i\,(1+\eta_i\eta)\,\frac{2}{\Delta x}$$

$$\frac{\partial \psi_i}{\partial y} = \frac{\partial \psi_i}{\partial \xi}\cancelto{0}{\frac{\partial \xi}{\partial y}} + \frac{\partial \psi_i}{\partial \eta}\frac{\partial \eta}{\partial y}$$

(19.25.2)
$$= \frac{1}{4}\,(1+\xi_i\xi)\,\eta_i\,\frac{2}{\Delta y}$$

Mass Matrix

Recall that for CG we found the mass matrix to be:

$$M_{ij}^{(e)} = \frac{\Delta x^{(e)}\Delta y^{(e)}}{36}\begin{pmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 2 & 1 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{pmatrix}$$

Differentiation Matrix (Weak Form)

We already found the strong form matrix in Ch. 15 for

$$D_{ij}^{(e)} = \int_{\Omega_e} \psi_i \,\vec{\nabla}\psi_j \, d\Omega_e$$ so now let us find

$$\tilde{D}_{ij}^{(e)} = \int_{\Omega_e} \bar{\nabla}\psi_i \, \psi_j \, d\Omega_e \, .$$

Let's begin by writing the following two scalar matrices

$$\tilde{D}_{ij}^{(e,x)} = \int_{\Omega_e} \frac{\partial \psi_i}{\partial x} \psi_j \, d\Omega_e \, , \qquad \tilde{D}_{ij}^{(e,y)} = \int_{\Omega_e} \frac{\partial \psi_i}{\partial y} \psi_j \, d\Omega_e$$

(11.26)

Substituting (11.25) into (11.26) yields,

$$\tilde{D}_{ij}^{(e,x)} = \int_{-1}^{+1} \int_{-1}^{+1} \frac{1}{4} \xi_i (1+\eta_i \eta) \frac{1}{4} (1+\xi_j \xi)(1+\eta_j \eta) \frac{2}{\Delta x^{(e)}} \frac{\Delta x^{(e)} \Delta y^{(e)}}{4} d\hat{\Omega}$$

Simplifying & integrating,

$$\tilde{D}_{ij}^{(e,x)} = \frac{\Delta y^{(e)}}{32} \left[\xi_i \xi + \frac{1}{2} \xi_i \xi_j \xi^2 \right] \Big|_{-1}^{+1} \left[\eta + \frac{1}{2}(\eta_i + \eta_j)\eta^2 + \frac{1}{3}\eta_i \eta_j \eta^3 \right] \Big|_{-1}^{+1}$$

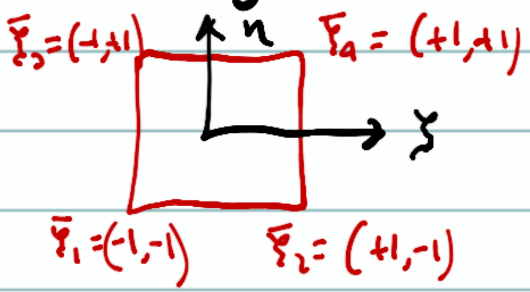
Evaluating:

$$\tilde{D}_{ij}^{(e,x)} = \frac{\Delta y^{(e)}}{24} \xi_i (3 + \eta_i \eta_j)$$

& similarly

$$\tilde{D}_{ij}^{(e,y)} = \frac{\Delta x^{(e)}}{24} \eta_i (3 + \xi_i \xi_j)$$

Substituting the values of $\bar{\xi}_i$ & $\bar{\xi}_j$ as such:



we get:

$$\tilde{D}_{ij}^{(e,x)} = \frac{\Delta y^{(e)}}{12} \begin{pmatrix} -2 & -2 & -1 & -1 \\ 2 & 2 & -1 & 1 \\ -1 & -1 & -2 & -2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

&

$$\tilde{D}_{ij}^{(e,y)} = \frac{\Delta x^{(e)}}{12} \begin{pmatrix} -2 & -1 & -2 & -1 \\ -1 & -2 & -1 & -2 \\ 2 & -1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

Note that $\sum_{j=1}^{4} \tilde{D}_{ij}^{(e)} = 0$ (column sum = 0)

& $\sum_{i,j=1}^{4} \tilde{D}_{ij}^{(e)} = 0$

Flux Matrix

Recall that the flux matrix is defined as:

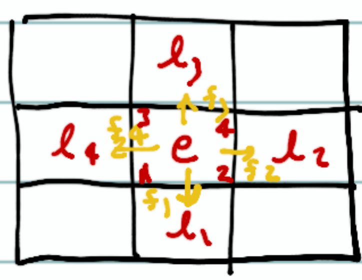
$$F_{ij}^{(e,l)} = \int_{\Gamma_e} \psi_i \psi_j \, \hat{n}^{(e,l)} \, d\Gamma_e$$

Subbing in the basis functions:

$$\psi_i(\xi,\eta) = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta) \quad \text{gives:}$$

$$F_{ij}^{(e,l)} = \int_{-1}^{+1} \frac{1}{16}(1+\xi_i\xi)(1+\eta_i\eta)(1+\xi_j\xi)(1+\eta_j\eta)\,\hat{n}^{(e,l)}\,ds$$

where ds is either $d\xi$ or $d\eta$ depending on which face of the element e we are evaluating. Let's consider the following schematic.



Face f_1

For face 1 we have $ds = d\xi$ when $\eta = -1$ $\hat{n} = -\hat{c}$

$$F_{ij}^{(e,l_1)} = \int_{-1}^{+1} \frac{1}{16}\left[1+(\xi_i+\xi_j)\xi + \xi_i\xi_j\xi^2\right](1-\eta_i)(1-\eta_j)\frac{\Delta x}{2}\,\hat{n}^{(e,l_1)}\,d\xi^{(1)}$$

Integrating:

$$F_{ij}^{(e,l_1)} = \frac{\Delta x^{(1)}}{32}\left[\xi + \frac{1}{2}(\xi_i+\xi_j)\xi^2 + \frac{1}{3}\xi_i\xi_j\xi^3\right]_{-1}^{+1}(1-\eta_i)(1-\eta_j)$$

$$= \frac{\Delta x^{(1)}}{48}(3+\xi_i\xi_j)(1-\eta_i)(1-\eta_j)$$

& subbing in $(\xi,\eta)_i$ & $(\xi,\eta)_j$ & simplifying:

$$F_{ij}^{(e,l_1)} = \frac{\Delta x^{(1)}\hat{n}^{(e,l_1)}}{6}\begin{array}{cl}\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} \rightarrow \text{row 1} \\ \rightarrow \text{row 2} \\ \\ \\ \end{array}\end{array}$$

where $\hat{n}^{(e,l_1)} = 0\hat{i} - \hat{j}$

For f_2

$$F_{ij}^{(e,l_2)} = \frac{\Delta y^{(2)}\hat{n}^{(e,l_2)}}{6}\begin{array}{cl}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} & \begin{array}{l} \\ \rightarrow \text{row 2} \\ \\ \rightarrow \text{row 4} \end{array}\end{array}$$

$\hat{n}^{(e,l_2)} = \hat{i} + 0\hat{j}$

For f_3

$$F_{ij}^{(e,l_3)} = \frac{\Delta x^{(3)}\hat{n}^{(e,l_3)}}{6}\begin{array}{cl}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix} & \begin{array}{l} \\ \\ \rightarrow \text{row 3} \end{array}\end{array}$$

$$\hat{n}^{(e,l_3)} = 0\hat{\imath} + \hat{\jmath} \qquad \begin{array}{cl} \begin{pmatrix} 0 & 0 & 1 & 2 \end{pmatrix} & \rightarrow \text{row 4} \end{array}$$

$\boxed{\text{For } f_4}$

$$F_{ij}^{(e,l_4)} = \frac{\Delta x^{(4)}}{6} \hat{n}^{(e,l_4)} \begin{array}{cl} \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} \rightarrow \text{row 1} \\ \\ \rightarrow \text{row 3} \\ \\ \end{array} \end{array}$$

$$\hat{n}^{(e,l_4)} = -\hat{\imath} + 0\hat{\jmath}$$

Numerical Flux

The numerical flux is defined as follows:

$$f_j^{(*,l)} = \frac{1}{2}\left[f_j^{(l)} + f_j^{(e)} - |\lambda| \, \hat{n}^{(e,l)} \left(q_j^{(l)} - q_j^{(e)} \right) \right]$$

for Rusanov, where $\hat{n}^{(e,l)}$ has already been defined &

$$f_j^{(e)} = \left(\beta_N^{(e)} \bar{u}_N^{(e)} \right)_j$$ where j is an interpolation point.

More details are given below when we discuss the algorithms.

Algorithms for the DG Matrix-Vector Problem

Exact Integration

Let's first consider the non-tensor product (monolithic) form. The algorithm for solving the PDE w/ DG is given in Alg. 16.6 as follows:

Construct $M^{(e)}$

for $n = 1 : N_{time}$

  Construct $R_i^{(e)} = R_i^{(e)}(q^n)$

  $\hat{R}_i^{(e)} = \left(M_{ij}^{(e)}\right)^{-1} R_j^{(e)}$

  $\frac{dq_i^{(e)}}{dt} = \hat{R}_i^{(e)}$

end

Let us now consider how to build the RHS vector $R_i^{(e)}$. Recall that the matrix-vector problem is:

$$M_{ij}^{(e)} \frac{dq_j^{(e)}}{dt} + F_{ij}^{(e,l)} f_j^{(*,l)} - \tilde{D}_{ij}^{(e)} f_j^{(e)} = 0$$

$\therefore$ let:

(11.27) $$R_i^{(e)} = \tilde{D}_{ij}^{(e)} f_j^{(e)} - F_{ij}^{(e,l)} f_j^{(*,l)}$$

which gives:

$$M_{ij}^{(e)} \frac{dq_j^{(e)}}{dt} = R_i^{(e)}$$

Let's consider separately the 2 contributions to Eq. (11.27). The weak form Differentiation matrix contribution can be computed according to

Alg. 16.7 :

```
R_i^(e) = 0
for e = 1 : N_e
    for n = 1 : M_Q
        f_n^(e) = 0
        for j = 1 : M_N
            f_n^(e) += ψ_nj f_j^(e)
        end
        for i = 1 : M_N
            R_i^(e) += ω_n J_n^(e) ∇ψ_in · f_n^(e)        = D̃_ij^(e) · f_j^(e)
        end
    end
end
```

Next, we include the contribution of the flux matrix according to Alg. 16.8 :

```
for l = 1 : N_faces
    for n = 1 : M_Q^(l)
        f_n^(L) = f_n^(R) = 0
        for j = 1 : M_N
            f_n^(L) += ψ_nj f_j^(L)
            f_n^(R) += ψ_nj f_j^(R)
        end
        f_n^(*) = 1/2 [ f_n^(L) + f_n^(R) - |λ| n̂_n^(l) ( q_n^(R) - q_n^(L) ) ]
```

for $i=1:M_N$

$$R_i^{(L)} -= \omega_k^{(\ell)} J_k^{(\ell)} \psi_{ik} \left(\hat{n}_k^{(\ell)} \cdot f_k^{(*)} \right)$$

$$R_i^{(R)} += \omega_k^{(\ell)} J_k^{(\ell)} \psi_{ik} \left(\hat{n}_k^{(\ell)} \cdot f_k^{(*)} \right)$$

end

end % $M_Q^{(\ell)}$

end % N_e

$\boxed{\text{Note}}$: Boundary conditions need to be included here via $f^{(R)}$ to satisfy the flux boundary condition of choice. E.g., if the boundary is a hard wall then $\bar{u}^{(R)} = \underbrace{\left(I - 2\hat{n}\hat{n}^T \right)}_{\text{Householder Reflector will knock-out } \bar{u}|_{\Gamma_{wall}}} \bar{u}^{(L)}$

Let us now consider inexact integration.

Inexact Integration

For inexact integration, the algorithm for computing the differentiation matrix contribution is given by Alg 16.9:

$R^{(e)} = [\,]$

for $e=1:N_e$

for $j=1:M_N$

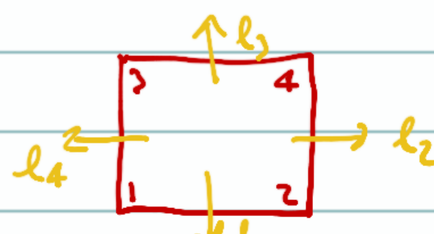
```
            for i = 1 : M_N
                R_i^(e) += w_j J_j^(e) ∇ψ_ji · f_j^(e)
            end
      end
end
```

The contribution from the flux matrix is computed according to Alg. 16.10:

```
for l = 1 : N_faces
      for i = 1 : M_N
            f_i^(*) = 1/2 [ f_i^(L) + f_i^(R) - |λ| n̂_i^(l) ( q_i^(R) - q_i^(L) ) ]

            R_i^(L) -= w_i^(l) J_i^(l) ( n̂_i^(l) · f_i^(*) ) τ_i^(e,l)

            R_i^(R) += w_i^(l) J_i^(l) ( n̂_i^(l) · f_i^(*) ) τ_i^(e,l)
      end
end
```



where $\tau_i^{(e,l)}$ is a trace array w/ values:

$$\tau^{(e,l_1)} = \begin{array}{cl} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} & \begin{array}{l} \to \text{row } 1 \\ \to \text{row } 2 \\ \\ \\ \end{array} \end{array}, \quad \tau^{(e,l_2)} = \begin{array}{cl} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} & \begin{array}{l} \\ \to \text{row } 2 \\ \\ \to \text{row } 4 \end{array} \end{array}$$

$$\tau^{(e,l_3)} = \begin{array}{c} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \rightarrow \text{row } 3 \\ \rightarrow \text{row } 4 \end{array} , \qquad \tau^{(e,l_4)} = \begin{array}{c} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \rightarrow \text{row } 1 \\ \\ \rightarrow \text{row } 3 \\ \\ \end{array}$$