Problem Set 1

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1 Definition of the Competitive Equilibrium[1]

Given k_0, a competitive equilibrium consists of prices $\{p_t, \omega_t, r_t\}_{t=0}^{\infty}$, the allocation for the representative agent $\{c_t, i_t, k_t^s, l_t^s\}_{t=0}^{\infty}$ and the allocation for the representative firm $\left\{k_t^d, l_t^d\right\}_{t=0}^{\infty}$ such that

1. Given prices $\{p_t, \omega_t, r_t\}_{t=0}^{\infty}$, the allocation for the representative agent $\{c_t, i_t, k_t^s, l_t^s\}_{t=0}^{\infty}$ solves

$$\max_{\{c_t, i_t, k_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

subject to

$$\sum_{t=0}^{\infty} p_t\left(c_t + i_t\right) \leq \sum_{t=0}^{\infty} p_t\left(r_t k_t + \omega_t l_t\right) + \pi$$

$$k_{t+1} = i_t \geq 0 \quad \forall t$$

$$0 \leq l_t \leq 1 \quad \forall t$$

$$c_t \geq 0 \quad \forall t$$

$$k_0 \quad \text{given}$$

2. Given prices $\{p_t, \omega_t, r_t\}_{t=0}^{\infty}$, the allocation for the representative firm $\left\{k_t^d, l_t^d\right\}_{t=0}^{\infty}$ solves

$$\pi = \max_{\{k_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t\left[F\left(k_t, l_t\right) - r_t k_t - \omega_t l_t\right]$$

subject to

$$k_t \geq 0 \quad \forall t$$

$$l_t \geq 0 \quad \forall t$$

3. Marker Clear

$$y_t = c_t + i_t \quad \forall t$$

$$l_t^d = l_t^s \quad \forall t$$

[1] For simplicity, I assume all the standard neoclassical assumptions holds for this problem set.

$$k_t^d = k_t^s \quad \forall t$$

2 Definition of the Social Planner's Problem

Given k_0, a social planner's problem consists of a feasible allocation $\{c_t, k_t, l_t\}_{t=0}^{\infty}$ which solves

$$\max_{\{c_t,k_t,l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$F(k_t, l_t) = c_t + k_{t+1} \quad \forall t$$

$$c_t \geq 0 \quad \forall t$$

$$k_t \geq 0 \quad \forall t$$

$$0 \leq l_t \leq 1 \quad \forall t$$

$$k_0 \quad \text{given}$$

3 Show the Equilibrium Allocation of Consumption, Capital, and Labor Coincides with Those of the Planner's

The Welfare Theorems hold, thus there exists the equivalence between the allocation of a competitive equilibrium and the Pareto Optimal allocation.

4 Pose the Planner's Dynamic Programming Problem and Write Down the Appropriate Bellman Equation

Recall that I have assumed that $F(\cdot,\cdot)$ is strictly increasing in both augments. Thus, $l_t = 1$ for all t. Define $f(k_t) = F(k_t, 1)$, then we can rewrite the social planner's problem as

$$W(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty} \text{ s.t. } 0 \leq k_{t+1} \leq f(k_t) \ \forall t} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$$

$$\Rightarrow \quad W(k_0) = \max_{k_1 \text{ s.t. } 0 \leq k_1 \leq f(k_0)} \left\{ u(f(k_0) - k_1) + \beta \left[\max_{\{k_{t+1}\}_{t=1}^{\infty} \text{ s.t. } 0 \leq k_{t+1} \leq f(k_t) \ \forall t} \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1}) \right] \right\}$$

This suggests

$$W(k_0) = \max_{k_1 \text{ s.t. } 0 \leq k_1 \leq f(k_0)} \left[u(f(k_0) - k_1) + \beta W(k_1) \right]$$

This is the sequential formulation of the social planner's problem, and the recursive formulation of the planner's problem, i.e. the Bellman equation, can be written as

$$V(k) = \max_{0 \le k' \le f(k)} \left[u\left(f(k) - k'\right) + \beta V(k') \right]$$

5 Solve the Planner's Dynamic Programming Problem given $u(c) = \log c$ and $F(k, l) = zk^{\alpha} l^{1-\alpha}$

Guess and verify (of the value function) method.

Step 1: Guess $V(k) = A + B \log k$, and solve the maximization problem on the RHS of the Bellman equation given the guess yields

$$\frac{1}{zk^{\alpha} - k'} = \frac{\beta B}{k'} \quad \Rightarrow \quad k' = \frac{\beta B z k^{\alpha}}{1 + \beta B}$$

Step 2: Evaluate the RHS at the optimum $k' = \frac{\beta B k^{\alpha}}{1+\beta B}$, then

$$\text{RHS} = \log\left(\frac{zk^{\alpha}}{1+\beta B}\right) + \beta A + \beta B \left(\frac{\beta B z k^{\alpha}}{1+\beta B}\right)$$

Rewrite it as

$$\text{RHS} = \log\left(\frac{z}{1+\beta B}\right) + \alpha \log k + \beta A + \beta B \log\left(\frac{\beta B z}{1+\beta B}\right) + \alpha\beta B \log(k)$$

Step 3: Verify that LHS=A+Blog k=RHS, solve for the undetermined coefficients A and B

$$B = \frac{\alpha}{1 - \alpha\beta} \tag{1}$$

$$A = \frac{1}{1-\beta}\left[\log\left(z(1-\alpha\beta)\right) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta z)\right] \tag{2}$$

I can conclude that the value function is $V(k) = A + B\log k$ with A and B given in equations (2) and (1), respectively. While the policy function $k' = g(k) = \frac{\beta B z k^{\alpha}}{1+\beta B} = \alpha\beta z k^{\alpha}$.

6 Steady State Value

In steady state, $\bar{k} = g\left(\bar{k}\right)$,

$$\bar{k} = \alpha\beta z \bar{k}^{\alpha} \quad \Rightarrow \quad \bar{k} = (\alpha\beta z)^{\frac{1}{1-\alpha}}$$

$$\bar{c} = f\left(\bar{k}\right) - \bar{k} = z(\alpha\beta z)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta z)^{\frac{1}{1-\alpha}}$$

$$r = z\alpha\bar{k}^{\alpha-1} = \frac{1}{\beta}$$

$$\omega = z(1-\alpha)\bar{k}^{\alpha} = z(1-\alpha)(\alpha\beta z)^{\frac{\alpha}{1-\alpha}}$$

$$y = z\bar{k}^{\alpha} = z(\alpha\beta z)^{\frac{\alpha}{1-\alpha}}$$

7 Experiments: Transition Path with $\alpha = \frac{1}{3}$ and $z = 1$