Problem Set 2

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1 Definition of the Competitive Equilibrium

Given k_0, a competitive equilibrium consists of prices $\{\omega_t, r_t\}_{t=0}^{\infty}$, the allocation for the representative agent $\left\{c_t, k_{t+1}^s, \ell_t^s\right\}_{t=0}^{\infty}$ and the allocation for the representative firm $\left\{k_t^d, \ell_t^d\right\}_{t=0}^{\infty}$ such that

1. Given prices $\{\omega_t, r_t\}_{t=0}^{\infty}$, the allocation for the representative agent $\left\{c_t, k_{t+1}^s, \ell_t^s\right\}_{t=0}^{\infty}$ solves

$$\max_{\{c_t, k_{t+1}, \ell_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{\ell_t^{1+\eta}}{1+\eta} \right)$$

subject to

$$c_t + k_{t+1} - (1-\sigma) k_t \leq r_t k_t + \omega_t \ell_t \quad \forall t,$$

$$0 \leq \ell_t \leq 1 \quad \forall t,$$

$$k_0 \quad \text{given.}$$

2. Given prices $\{\omega_t, r_t\}_{t=0}^{\infty}$, the allocation for the representative firm $\left\{k_t^d, \ell_t^d\right\}_{t=0}^{\infty}$ solves

$$\max_{\{k_t, \ell_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} z k_t^{\alpha} \ell_t^{1-\alpha} - r_t k_t - \omega_t \ell_t$$

subject to

$$k_t \geq 0 \quad \forall t,$$

$$\ell_t \geq 0 \quad \forall t.$$

3. Marker Clear

$$c_t + k_{t+1} - (1-\sigma) k_t = z k_t^{\alpha} \ell_t^{1-\alpha} \quad \forall t;$$

$$\ell_t^d = \ell_t^s \quad \forall t;$$

$$k_t^d = k_t^s \quad \forall t.$$

2 Find the Steady State Values

F.O.C.s for the representative agent's problem

$$\beta^t c_t^{-\sigma} - \lambda_t^1 = 0;$$

$$-\lambda_t^1 + \lambda_{t+1}^1 (r_t + 1 - \sigma) = 0;$$

$$-\beta^t \chi \ell_t^\eta + \lambda_t^1 \omega_t - \lambda_t^3 = 0;$$

$$\lambda_t^3 (1 - \ell_t) = 0.$$

F.O.C.s for the representative firm's problem

$$r_t = \alpha z k_t^{\alpha-1} \ell_t^{1-\alpha};$$

$$\omega_t = (1-\alpha) z k_t^\alpha \ell_t^{-\alpha}.$$

After some manipulations, there are two cases and the following equations characterize the equilibria.
Case 1: $\ell_t < 1$ **and** $\lambda_t^3 = 0$

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \left(\alpha z k_t^{\alpha-1} \ell_t^{1-\alpha} + 1 - \sigma\right);$$

$$c_t + k_{t+1} - (1-\sigma) k_t = z k_t^\alpha \ell_t^{1-\alpha};$$

$$\chi \ell_t^\eta = c_t^{-\sigma} (1-\alpha) z k_t^\alpha \ell_t^{-\alpha} \quad \text{and} \quad \ell_t < 1.$$

Case 2: $\ell_t = 1$ **and** $\lambda_t^3 > 0$

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \left(\alpha z k_t^{\alpha-1} \ell_t^{1-\alpha} + 1 - \sigma\right);$$

$$c_t + k_{t+1} - (1-\sigma) k_t = z k_t^\alpha \ell_t^{1-\alpha};$$

$$-\chi \ell_t^\eta + c_t^{-\sigma} (1-\alpha) z k_t^\alpha \ell_t^{-\alpha} > 0 \quad \text{and} \quad \ell_t = 1.$$

In steady state, under case 1, $\{c, \ell, k, y, r, w\}$ solves

$$1 = \beta \left(\alpha z k^{\alpha-1} \ell^{1-\alpha} + 1 - \sigma\right);$$

$$c + \sigma k = z k^\alpha \ell^{1-\alpha};$$

$$\chi \ell^\eta = c^{-\sigma} (1-\alpha) z k^\alpha \ell^{-\alpha} \quad \text{and} \quad \ell < 1;$$

$$r = \alpha z k^{\alpha-1} \ell^{1-\alpha};$$

$$\omega = (1-\alpha) z k^\alpha \ell^{-\alpha};$$

$$y = z k^\alpha \ell^{1-\alpha}.$$

In steady state, under case 2, $\{c, \ell, k, y, r, w\}$ solves

$$1 = \beta \left(\alpha z k^{\alpha-1} + 1 - \sigma \right);$$

$$c + \sigma k = z k^{\alpha};$$

$$-\chi + c^{-\sigma} (1-\alpha) z k^{\alpha} > 0;$$

$$r = \alpha z k^{\alpha-1};$$

$$\omega = (1-\alpha) z k^{\alpha};$$

$$y = z k^{\alpha}.$$

3 Pose the Planner's Problem and Write Down the Bellman equation

Planner's Problem

$$\max_{\{c_t, k_{t+1}, \ell_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{\ell_t^{1+\eta}}{1+\eta} \right)$$

subject to

$$c_t + k_{t+1} - (1-\sigma) k_t = z k_t^{\alpha} \ell_t^{1-\alpha} \quad \forall t,$$

$$c_t \geq 0 \quad \forall t,$$

$$k_{t+1} \geq 0 \quad \forall t,$$

$$0 \leq \ell_t \leq 1 \quad \forall t.$$

Bellman equation

$$V(k) = \max_{c, \ell, k'} \left[\frac{c^{1-\sigma}}{1-\sigma} - \chi \frac{\ell^{1+\eta}}{1+\eta} + \beta V(k') \right]$$

subject to

$$c = z k^{\alpha} \ell^{1-\alpha} + (1-\delta) k - k',$$

$$1 \leq \ell \leq 1,$$

$$c \geq 0,$$

$$k' \geq 0.$$

4 Find χ such that $\ell_{ss} = 0.4$

For the case $\ell_{ss} = 0.4 < 1$, we are under case 1. First, we can solve for $\frac{k_{ss}}{\ell_{ss}}$ as

$$\frac{k_{ss}}{\ell_{ss}} = \left[\frac{1}{\alpha z}\left(\frac{1}{\beta} - 1 + \sigma\right)\right]^{\frac{1}{\alpha-1}}.$$

Then, c_{ss} can be solved by

$$c_{ss} = \left[z\left(\frac{k_{ss}}{\ell_{ss}}\right)^{\alpha} - \sigma\frac{k_{ss}}{\ell_{ss}}\right]\ell_{ss}.$$

Finally, χ that gives a steady state labor as $\ell_{ss} = 0.4$ can be solved by

$$\chi = c_{ss}^{-\sigma}(1-\alpha) z\left(\frac{k_{ss}}{\ell_{ss}}\right)^{\alpha}\frac{1}{(\ell^{ss})^{\eta}}.$$

Use Julia, we can solve for $\chi = 57.1236$.

5 Numerical Solutions

Euler equation (obtained with envelop theorem). First, the F.O.C.s can be written as

$$-\left[zk^{\alpha}\ell^{1-\alpha} + (1-\delta)k - k'\right]^{-\sigma} + \beta V'(k') = 0,$$

$$(1-\alpha)\left[zk^{\alpha}\ell^{1-\alpha} + (1-\delta)k - k'\right]^{-\sigma} zk^{\alpha}\ell^{-\alpha} - \chi\ell^{\eta} = 0.$$

By envelop theorem, we can also obtain

$$V'(k) = \left[zk^{\alpha}\ell^{1-\alpha} + (1-\delta)k - k'\right]^{-\sigma}\left(\alpha zk^{\alpha-1}\ell^{1-\alpha} + 1 - \sigma\right).$$

Thus, we obtain

$$\left[zk^{\alpha}\ell^{1-\alpha} + (1-\delta)k - k'\right]^{-\sigma} = \beta\left[zk'^{\alpha}\ell^{1-\alpha} + (1-\delta)k' - k''\right]^{-\sigma}\left(\alpha zk'^{\alpha-1}\ell^{1-\alpha} + 1 - \sigma\right).$$

Thus, denote the policy function as $k' = g_k(k)$ and $\ell = g_\ell(k)$, we can rewrite the Euler equation as

$$\begin{aligned}&\left[zk^{\alpha}g_\ell(k)^{1-\alpha} + (1-\delta)k - g_k(k)\right]^{-\sigma}\\ &= \beta\left[zg_k(k)^{\alpha}g_\ell(k)^{1-\alpha} + (1-\delta)g_k(k) - g_k(g_k(k))\right]^{-\sigma}\left(\alpha zg_k(k)^{\alpha-1}g_\ell(k)^{1-\alpha} + 1 - \sigma\right)\end{aligned}.$$

We can base on this Euler equation to compute the Euler error.