CALCULUS

PRACTICAL NO. 1 :-

TOPIC :- LIMIT AND CONTINUITY.

1) $\lim_{x \to a} \left[\dfrac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$

2) $\lim_{y \to 0} \left[\dfrac{\sqrt{a-y} - \sqrt{a}}{y\sqrt{a+y}} \right]$

3) $\lim_{x \to \pi/6} \left[\dfrac{\cos x - \sqrt{3}\sin x}{\pi - 6x} \right]$

4) $\lim_{x \to \infty} \left[\dfrac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}} \right]$

5) Examine the continuity of the following f^n :-

i) $F(x) = \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$ at $0 < x \leq \dfrac{\pi}{2}$

$= \dfrac{\cos x}{\pi - 2x}$ at $\dfrac{\pi}{2} < x < \pi$

at $x = \pi/2$.

ii) $f(x) = \dfrac{x^2-9}{x-3}$, $0 < x < 3$ ~~$= \dfrac{x^2-9}{x+3}$~~, ~~$6 \le x < 9$~~

$= x+3$, $3 \leq x < 6$

$= \dfrac{x^2-9}{x+3}$, $6 \leq x < 9$.

at $x = 3$ and $x = 6$.

6) Find value of k, so that the f^n $f(x)$ is continuous at the indicated point:

i) $f(x) = \frac{1-\cos 4x}{x^2}$, $x<0$ } $x=0$
$= k$, $x=0$

ii) $f(x) = (\sec^2 x)^{\cot^2 x}$, $x \neq 0$ } at $x=0$.
$= k$, $x=0$

iii) $f(x) = \frac{\sqrt{3}-\tan x}{\pi - 3x}$, $x \neq \frac{\pi}{3}$ } at $x = \frac{\pi}{3}$
$= k$, $x = \frac{\pi}{3}$

7) Discuss the continuity of the following functions which of these function have removable discontinuity? Redefine function so as to remove the discontinuity.

i) $f(x) = \frac{1-\cos 3x}{x\tan x}$, $x \neq 0$ } at $x=0$
$= 9$, $x=0$

ii) $f(x) = \frac{(e^{3x}-1)\sin x^\circ}{x^2}$, $x \neq 0$ } at $x=0$
$= \frac{\pi}{60}$, $x=0$

8) If $f(x) = \dfrac{e^{x^2} - \cos x}{x^2}$ ~~x ≠ 0~~

for $x \neq 0$ is continuous at $x = 0$ find $f(0)$

9) If $f(x) = \dfrac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x}$ for $x \neq \dfrac{\pi}{2}$

is continuous at $x = \dfrac{\pi}{2}$. find $f\left(\dfrac{\pi}{2}\right)$

Solutions :-

1) $$\lim_{x\to a}\left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}\right]$$

$$= \lim_{x\to a}\left[\frac{(\sqrt{a+2x} - \sqrt{3x})(\sqrt{3a+x} + 2\sqrt{x})}{(\sqrt{3a+x} - 2\sqrt{x})(\sqrt{3a+x} + 2\sqrt{x})}\right]$$

$$= \lim_{x\to a}\left[\frac{(\sqrt{a+2x} - \sqrt{3x})(\sqrt{3a+x} + 2\sqrt{x})(\sqrt{a+2x} + \sqrt{3x})}{(3a + x - 4x)\{(\sqrt{a+2x} + \sqrt{3x})}\right]$$

$$= \lim_{x\to a}\left[\frac{(a+2x-3x)(\sqrt{3a+x} + 2\sqrt{x})}{(3a-3x)(\sqrt{a+2x} + \sqrt{3x})}\right]$$

$$= \lim_{x\to a}\left[\frac{\cancel{(a-x)}(\sqrt{3a+x} + 2\sqrt{x})}{3\cancel{(a-x)}(\sqrt{a+2x} + \sqrt{3x})}\right] \cdots \left[\text{As } x\to a,\ x\neq a,\ a-x\neq 0\right]$$

$$= \frac{1}{3}\lim_{x\to a}\left[\frac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{a+2x} + \sqrt{3x}}\right]$$

$$= \frac{1}{3}\left[\frac{\sqrt{3a+a} + 2\sqrt{a}}{\sqrt{a+2a} + \sqrt{3a}}\right]$$

$$= \frac{1}{3}\left[\frac{\sqrt{4a} + 2\sqrt{a}}{\sqrt{3a} + \sqrt{3a}}\right]$$

$$= \frac{1}{3}\left[\frac{2\sqrt{a}+2\sqrt{a}}{2\sqrt{3a}}\right] = \frac{1}{3}\left[\frac{\cancel{4}^{2}\sqrt{a}}{\cancel{2}\sqrt{3a}}\right]$$

$$= \frac{2\sqrt{a}}{3\sqrt{3a}} = \frac{2\sqrt{\cancel{a}}}{3\sqrt{\cancel{a}}\sqrt{3}} = \frac{2}{3\sqrt{3}}$$

2) $$\lim_{y\to 0}\left[\frac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\right]$$

$$= \lim_{y\to 0}\left[\frac{(\sqrt{a+y}-\sqrt{a})(\sqrt{a+y}+\sqrt{a})}{(y\sqrt{a+y})(\sqrt{a+y}+\sqrt{a})}\right]$$

$$= \lim_{y\to 0}\left[\frac{\cancel{a}+\cancel{y}-\cancel{a}}{\cancel{y}(\sqrt{a+y})(\sqrt{a+y}+\sqrt{a})}\right]$$

$$= \lim_{y\to 0}\left[\frac{1}{\sqrt{a+y}\,(\sqrt{a+y}+\sqrt{a})}\right]$$

$$= \frac{1}{\sqrt{a+0}}\left(\frac{1}{\sqrt{a+0}+\sqrt{a}}\right)$$

$$= \frac{1}{\sqrt{a}}\left(\frac{1}{\sqrt{a}+\sqrt{a}}\right) = \frac{1}{\sqrt{a}}\left(\frac{1}{2\sqrt{a}}\right) = \frac{1}{2a}.$$

3) $\lim_{x \to \pi/6} \left[\dfrac{\cos x - \sqrt{3} \sin x}{\pi - 6x} \right]$

Put $x = \dfrac{\pi}{6} + h$

$\therefore x - \dfrac{\pi}{6} = h$

As, $x \to \dfrac{\pi}{6}$, $x \neq \dfrac{\pi}{6}$, $x - \dfrac{\pi}{6} \neq 0$, $h \neq 0, h \to 0$.

$\cancel{= \lim_{x \to \pi/6}}$

$$= \lim_{h \to 0} \left[\frac{\cos\left(\frac{\pi}{6} + h\right) - \sqrt{3} \sin\left(\frac{\pi}{6} + h\right)}{\pi - 6\left(\frac{\pi}{6} + h\right)} \right]$$

$$= \lim_{h \to 0} \left[\frac{(\cos \pi/6 \cosh - \sin \pi/6 \sinh) - \sqrt{3}(\sin \pi/6 \cosh + \cos \pi/6 \sinh)}{\cancel{\pi} - \cancel{\pi} - 6h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\left(\frac{\sqrt{3}}{2} \cosh - \frac{1}{2} \sinh\right) - \sqrt{3}\left(\frac{1}{2} \cosh + \frac{\sqrt{3}}{2} \sinh\right)}{-6h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\left(\frac{\sqrt{3}}{3} \cosh - \frac{1}{2} \sinh\right) - \left(\frac{\sqrt{3}}{2} \cosh + \frac{3}{2} \sinh\right)}{-6h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\cancel{\frac{\sqrt{3}}{2} \cosh} - \frac{1}{2} \sinh - \cancel{\frac{\sqrt{3}}{2} \cosh} - \frac{3}{2} \sinh}{-6h} \right]$$

$$= \lim_{h\to 0}\left[\frac{+4/2 \sin h}{+6h}\right]$$

$$= \frac{1}{6}\times\frac{4}{2}\left[\lim_{h\to 0}\frac{\sin h}{h}\right]$$

$$= \frac{1}{\cancel{6}\,3}\times\frac{\cancel{4}\,2}{\cancel{2}} = \frac{1}{3}$$

4) $$\lim_{x\to\infty}\left[\frac{\sqrt{x^2+5}-\sqrt{x^2-3}}{\sqrt{x^2+3}-\sqrt{x^2+1}}\right]$$

$$= \lim_{x\to\infty}\left[\frac{\left(\sqrt{x^2+5}-\sqrt{x^2-3}\right)\left(\sqrt{x^2+3}+\sqrt{x^2+1}\right)}{\left(\sqrt{x^2+3}-\sqrt{x^2+1}\right)\left(\sqrt{x^2+3}+\sqrt{x^2+1}\right)}\right]$$

$$= \lim_{x\to\infty}\left[\frac{\left(\sqrt{x^2+5}-\sqrt{x^2-3}\right)\left(\sqrt{x^2+5}+\sqrt{x^2-3}\right)\left(\sqrt{x^2+3}+\sqrt{x^2+1}\right)}{\left(\cancel{x^2}+3-\cancel{x^2}-1\right)\left(\sqrt{x^2+5}+\sqrt{x^2-3}\right)}\right]$$

$$= \lim_{x\to\infty}\left[\frac{\left(\cancel{x^2}+5-\cancel{x^2}+3\right)\left(\sqrt{x^2+3}+\sqrt{x^2+1}\right)}{2\left(\sqrt{x^2+5}+\sqrt{x^2-3}\right)}\right]$$

$$= \lim_{x\to\infty}\left[\frac{\cancel{8}\,4}{\cancel{2}}\left(\frac{\sqrt{x^2+3}+\sqrt{x^2+1}}{\sqrt{x^2+5}+\sqrt{x^2-3}}\right)\right]$$

$$= 4\lim_{x\to\infty}\left[\frac{\sqrt{1+3/x^2}+\sqrt{1+1/x^2}}{\sqrt{1+5/x^2}+\sqrt{1-3/x^2}}\right]$$

$$\cancel{= 4\left[\sqrt{1+}\right.}\qquad = 4\left(\frac{\sqrt{1}+\sqrt{1}}{\sqrt{1}+\sqrt{1}}\right)$$

$$= 4\left(\frac{\cancel{1+1}}{\cancel{1+1}}\right) = 4.$$

5) i) $f(x) = \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$, $0 < x \leq \dfrac{\pi}{2}$ } at $x = \dfrac{\pi}{2}$ 030

$= \dfrac{\cos x}{\pi - 2x}$, $\dfrac{\pi}{2} < x < \pi$

~~For $x \to \frac{\pi}{2}^+$, $f(x) = \lim_{x\to\frac{\pi}{2}} \frac{\sin 2x}{\sqrt{1-\cos 2x}} = \lim_{x\to\frac{\pi}{2}^-}$~~

for $x = \dfrac{\pi}{2}$,

$f\left(\dfrac{\pi}{2}\right) = \dfrac{\sin 2\frac{\pi}{2}}{\sqrt{1-\cos 2\frac{\pi}{2}}} = \dfrac{\sin\pi}{\sqrt{1-\cos\pi}} = 0$

For $x \to \dfrac{\pi}{2}^-$,

$f(x) = \lim_{x\to\frac{\pi}{2}^-} \dfrac{\sin 2x}{\sqrt{1-\cos 2x}} = \lim_{x\to\frac{\pi}{2}^-} \dfrac{\cancel{2\sin x\cos x}\ \sin 2x}{\sqrt{2\sin^2 x}}$

$= \lim_{x\to\frac{\pi}{2}^-} \dfrac{\cancel{2\sin x\cos x}\ \sin 2x}{\sqrt{2}\sin x} = \lim_{x\to\frac{\pi}{2}^-} \dfrac{\frac{\sin 2x}{x}}{\frac{\sqrt{2}\sin x}{x}}$

$= \dfrac{2}{\sqrt{2}} = \sqrt{2}$

for $x \to \dfrac{\pi}{2}^+$,

$f(x) = \lim_{x\to\frac{\pi}{2}^+} \dfrac{\cos x}{\pi - 2x}$ => Put $x = \dfrac{\pi}{2} + h$, $x - \dfrac{\pi}{2} = h$

As $x \to \dfrac{\pi}{2}$, $x - \dfrac{\pi}{2} \to 0$, $h \to 0$.

$= \lim_{h\to 0} \dfrac{\cos\left(\frac{\pi}{2}+h\right)}{\pi - 2\left(\frac{\pi}{2}+h\right)} = \lim_{h\to 0} \dfrac{-\sin h}{\pi - \pi - 2h} = \lim_{h\to 0} \dfrac{+\sin h}{+2h}$

$= \dfrac{1}{2}\lim_{h\to 0} \dfrac{\sin h}{h} = \dfrac{1}{2}$

since, $f\left(\dfrac{\pi}{2}\right) \neq \lim_{x\to\frac{\pi}{2}^-} f(x) \neq \lim_{x\to\frac{\pi}{2}^+} f(x)$

$\therefore$ Function is not continuous.

ii) For $x=3$,

$f(3) = 3+3 = 6$

$\therefore$ function is defined at $x=3$.

For $x=3^+$,

$f(x) = \lim_{x\to 3^+} x+3$

$= 3+3 = 6$.

For $x=3^-$,

$f(x) = \lim_{x\to 3^-} \frac{x^2-9}{x-3}$

$= \lim_{x\to 3^-} \frac{x^2-3^2}{x-3} = \lim_{x\to 3^-} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$

$= \lim_{x\to 3^-} x+3 = 3+3 = 6$.

Since, $f(3) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x)$

$\therefore$ Function is continous at $x=3$.

For $x=6$,

$f(6) = \frac{x^2-9}{x+3} = \frac{6^2-9}{6+3} = \frac{36-9}{9} = \frac{27}{9} = 3$.

For $x=6^+$,

$f(x) = \lim_{x\to 6^+} f(x) = \lim_{x\to 6^+} \frac{x^2-9}{x+3} = \lim_{x\to 6^+} \frac{x^2-3^2}{x+3} = \lim_{x\to 6^+} \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}}$

$= \lim_{x\to 6^+} (x-3) = 6-3 = 3$

For $x=6^-$,

$f(x) = \lim_{x\to 6^-} f(x) = \lim_{x\to 6^+} x+3 = 6+3 = 9$.

Since, $f(6) = \lim_{x\to 6^+} f(x) \neq \lim_{x\to 6^-} f(x)$

$\therefore$ Function is not continous at $x=6$.

6) i) Since, the function is continous at $x=0$

$\therefore f(0) = \lim_{x\to 0} \frac{1-\cos 4x}{x^2}$

$\therefore k = \lim_{x\to 0} \frac{2\sin^2 2x}{x^2}$

$= 2\lim_{x\to 0} \left(\frac{\sin 2x}{x}\right)^2$

$= 2(2)^2 = 2(4) = 8$

$\boxed{\therefore k=8}$

Q) ii) Since, the function is continous at $x=0$

$\therefore f(0) = \lim_{x \to 0} f(x)$

$\therefore k = \lim_{x \to 0} (\sec^2 x)^{\cot^2 x}$

$= \lim_{x \to 0} (1 + \tan^2 x)^{1/\tan^2 x}$

$= e$

$\boxed{\therefore k = e}$

iii) Since, the function is continous at $x = \frac{\pi}{3}$

$\therefore f\left(\frac{\pi}{3}\right) = \lim_{x \to \frac{\pi}{3}} f(x)$

$\therefore k = \lim_{x \to \pi/3} \frac{\sqrt{3} - \tan x}{\pi - 3x}$

Put $x = \frac{\pi}{3} + h$, $x - \frac{\pi}{3} = h$

As $x \to \frac{\pi}{3}$, $x - \frac{\pi}{3} \to 0$, $h \to 0$

$\therefore k = \lim_{\substack{\cancel{x \to \pi/3} \\ h \to 0}} \frac{\sqrt{3} - \tan\left(\frac{\pi}{3} + h\right)}{\pi - 3\left(\frac{\pi}{3} + h\right)}$

$= \lim_{h \to 0} \frac{\sqrt{3} - \left(\frac{\tan\frac{\pi}{3} + \tan h}{1 - \tan\frac{\pi}{3} \tan h}\right)}{\cancel{\pi} - \cancel{\pi} - 3h}$

$= \lim_{h \to 0} \frac{\sqrt{3} - \left(\frac{\sqrt{3} + \tan h}{1 - \sqrt{3} \tan h}\right)}{-3h}$

$$\therefore k = \lim_{h\to 0} \frac{\sqrt{3}(1-\sqrt{3}\tan h) - (\sqrt{3}+\tan h)}{-3h(1-\sqrt{3}\tan h)}$$

$$= \lim_{h\to 0} \frac{\cancel{\sqrt{3}} - 3\tan h - \cancel{\sqrt{3}} - \tan h}{-3h(1-\sqrt{3}\tan h)}$$

$$= \lim_{h\to 0} \frac{\cancel{-}4\tan h}{\cancel{-}3h(1-\sqrt{3}\tan h)}$$

$$= \frac{4}{3}\lim_{h\to 0} \frac{\tan h}{h} \times \lim_{h\to 0} \frac{1}{1-\sqrt{3}\tan h}$$

$$= \frac{4}{3} \times 1 \times \frac{1}{1-0} = \frac{4}{3} \times 1 \times 1 = \frac{4}{3}$$

$$\boxed{\therefore k = \frac{4}{3}}$$

7) i) $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-\cos 3x}{x\tan x}$

$$= \lim_{x\to 0} \frac{2\sin^2 3x/2}{x\tan x}$$

Dividing Numerator and Denominator by x^2

$$= 2\lim_{x\to 0} \frac{\cancel{2}(\sin^2 3x/2)/x^2}{\frac{x\tan x}{x^2}}$$

$$= 2\lim_{x\to 0}\left(\frac{\sin\frac{3x}{2}}{x}\right)^2 \times \lim_{x\to 0} \frac{1}{\frac{\tan x}{x}}$$

$$= 2\left(\frac{3}{2}\right)^2 = 2\left(\frac{9}{4}\right) = \frac{9}{2}$$

But, $f(0) = 9$

$\because f(0) \neq \lim_{x\to 0} f(x)$

$\therefore$ Function is not dcontinous at $x=0$

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Redefining the function we get,

$$f(x) = \frac{1-\cos 3x}{x \tan x}, \quad x \neq 0$$
$$= \frac{9}{2}, \quad x = 0 \quad \Big\} \text{ at } x=0.$$

ii) Now, $f(0) = \lim_{x\to 0} f(x)$

$\therefore$ Function has removable discontinuity at $x=0$.

2) ii) $f(0) = \frac{\pi}{60}$ ---- (given)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(e^{3x}-1)\sin x^\circ}{x^2}$$

$$= \lim_{x\to 0} \frac{3(e^{3x}-1)}{3x} \times \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$$

$$= 3\log e \times \frac{\pi}{18} = 3 \times \frac{\pi}{180} = \frac{\pi}{60}.$$

$\therefore$ Function is continous at $x=0$.

8) Since, the function is continous at $x=0$.

$\therefore f(0) = \lim_{x\to 0} f(x)$

$$= \lim_{x\to 0} \frac{e^{x^2} - \cos x - 1 + 1}{x^2} = \lim_{x\to 0} \frac{e^{x^2}-1}{x^2} + \lim_{x\to 0} \frac{1-\cos x}{x^2}$$

$$= \cancel{\lim} \log e + \lim_{x\to 0} \frac{2\sin^2 x/2}{x^2} = 1 + 2\lim_{x\to 0}\left(\frac{\sin x/2}{x}\right)^2$$

$$= 1 + 2\left(\frac{1}{2}\right)^2 = 1 + \cancel{2}\left(\frac{1}{\cancel{2}4}\right) = 1 + \frac{1}{2} = \frac{3}{2}$$

$$\boxed{\therefore f(0) = \frac{3}{2}}$$

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Sol

9) Since, the function is continous at $x = \frac{\pi}{2}$.

$$\therefore f\left(\frac{\pi}{2}\right) = \lim_{x \to \frac{\pi}{2}} f(x)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{(\sqrt{2} - \sqrt{1+\sin x})(\sqrt{2} + \sqrt{1+\sin x})}{\cos^2 x(\sqrt{2} + \sqrt{1+\sin x})}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{[2 - (1+\sin x)]}{(1-\sin^2 x)(\sqrt{2} + \sqrt{1+\sin x})}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{(2 - 1 - \sin x)}{(1-\sin x)(1+\sin x)(\sqrt{2} + \sqrt{1+\sin x})}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cancel{(1-\sin x)}}{\cancel{(1-\sin x)}(1+\sin x)(\sqrt{2} + \sqrt{1+\sin x})}$$

[As $x \to \frac{\pi}{2}$, $\frac{\pi}{2} - x \to 0$, $\sin\frac{\pi}{2} - \sin\pi \to \sin 0$, $1 - \sin x \to 0$,
$1 - \sin x \neq 0$:]

$$= \frac{1}{\left(1+\sin\frac{\pi}{2}\right)\left(\sqrt{2} + \sqrt{1+\sin\frac{\pi}{2}}\right)} = \frac{1}{(1+1)(\sqrt{2} + \sqrt{1+1})}$$

$$= \frac{1}{2(\sqrt{2}+\sqrt{2})} = \frac{1}{2(2\sqrt{2})} = \frac{1}{4\sqrt{2}}$$

$$\boxed{\therefore f\left(\frac{\pi}{2}\right) = \frac{1}{4\sqrt{2}}}$$

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PRACTICAL-2

Topic :- Derivation

Q1) Show that the following function defined from R to R are differentiable :-
i) $\cot x$, ii) $\operatorname{cosec} x$, iii) $\sec x$.

Q2) If $f(x) = 4x+1$, $x \le 2$
$= x^2+5$, $x > 0$ } at $x=2$.
Then find f is differentiable or not?

Q3) If $f(x) = 4x+7$, $x < 3$
$= x^2+3x+1$, $x \ge 3$
then find f is differentiable or not?

Q4) If $f(x) = 8x-5$, $x < 2$
$= 3x^2-4x+7$, $x > 2$ } at $x=2$.
Then find f is differentiable or not?

Solutions :-

Q1) i) $f(x) = \cot x$.

$$Df(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x \to a} \frac{\cot x - \cot a}{x-a}$$

$$= \lim_{x \to a} \frac{\tan a - \tan x}{(x-a)\tan x \cdot \tan a}$$

Put $x - a = h$

$x = a+h$

As $x \to a$, $x - a \to 0$, $h \to 0$.

$$Df(h) = \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{(\cancel{a}+h-\cancel{a})\tan(a+h)\tan(a)}$$

$$= \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{h \times \tan(a+h)\tan a}$$

$$= \lim_{h \to 0} \frac{\{\tan(a-a-h)(1+\tan a \cdot \tan(a+h))\}}{h \cdot \tan(a+h) \cdot \tan a}$$

$$= \lim_{h \to 0} -\frac{\tan h}{h} \times \lim_{h \to 0} \frac{1+\tan a \cdot \tan(a+h)}{\tan(a+h)\tan a}$$

$$= -1 \times \frac{1+\tan^2 a}{\tan^2 a}$$

$$= \cancel{-1 \times} -\frac{\sec^2 a}{\tan^2 a}$$

$$= \frac{-1}{\cancel{\cos^2 a}} \times \frac{\cancel{\cos^2 a}}{\sin^2 a}$$

$$= -\operatorname{cosec}^2 a$$

$$\therefore Df(a) = -\operatorname{cosec}^2 a$$

ii) $\operatorname{cosec} x$

$\Rightarrow$ $f(x) = \operatorname{cosec} x$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x \to a} \frac{\operatorname{cosec} x - \operatorname{cosec} a}{x-a}$$

$$= \lim_{x \to a} \frac{\sin a - \sin x}{(x-a)(\sin a \cdot \sin x)}$$

Put $x = a+h \Rightarrow x - a = h$

As $x \to a$; $x - a \to 0$, $h \to 0$.

$$Df(h) = \lim_{h \to 0} \frac{\sin a - \sin(a+h)}{(a+h-a)\sin a \cdot \sin(a+h)}$$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{a+a+h}{2}\right)\cdot \sin\left(\frac{a-a-h}{2}\right)}{h \sin a \sin(a+h)}$$

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$$= \lim_{h\to 0} -\frac{\sin h/2}{h/2}\cdot\frac{1}{2} \times \frac{2\cos\left(\frac{2a+h}{2}\right)}{\sin a\cdot\sin(a+h)}$$

$$= -\frac{1}{2} \times \frac{2\cos\left(\frac{2a+0}{2}\right)}{\sin(a+0)\cdot\sin a} = -\frac{\cos a}{\sin^2 a} = -\cot a\cdot\operatorname{cosec} a$$

iii) $\sec a$

$\Rightarrow f(x) = \sec x$

$$Df(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\sec x - \sec a}{x-a}$$

$$= \lim_{x\to a} \frac{\cos a - \cos x}{(x-a)(\cos a\cdot\cos x)}$$

Put $x = a+h$, $x-a = h$.

As $x\to a$, $x-a\to 0$, $h\to 0$

$$= \lim_{h\to 0} \frac{\cos a - \cos(a+h)}{(x-a)\cos a\cos(a+h)}$$

$$= \lim_{h\to 0} \frac{-2\sin\left(\frac{a+a+h}{2}\right)\sin\left(\frac{a-a-h}{2}\right)}{h\cos a\cos(a+h)}$$

$$= \lim_{h\to 0} \frac{-2\sin\left(\frac{2a+h}{2}\right)}{\cos a\cos(a+h)} \times \frac{-\sin\frac{h}{2}}{h}$$

$$= -\frac{1}{2} \times \frac{-2\sin\left(\frac{2a+0}{2}\right)}{\cos a\cdot\cos(a+0)} = -\frac{1}{2}\times -2\,\frac{\sin a}{\cos a\cos a}$$

$$= \tan a\cdot\sec a.$$

Q2) If $f(x) = 4x+1 \quad , x \leq 2$
$= x^2+5 \quad , x > 0$ } at $x=2$.

Sol

$\Rightarrow$ LHD $= \lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2}$

$= \lim_{x \to 2^-} \frac{4x+1-(4\times2+1)}{x-2}$

$= \lim_{x \to 2^-} \frac{4x+1-9}{x-2}$

$= \lim_{x \to 2^-} \frac{4x-8}{x-2}$

$= \lim_{x \to 2^-} \frac{4\cancel{(x-2)}}{\cancel{x-2}} = 4.$

$Df(2^-) = 4.$

RHD :- $\lim_{x \to 2^+} \frac{x^2+5-9}{x-2}$

$= \lim_{x \to 2^+} \frac{x^2-4}{x-2}$

$= \lim_{x \to 2^+} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = \lim_{x \to 2^+} x+2 = 4.$

$\therefore$ LHD = RHD.

Q3) RHD $= \lim_{x \to 3^+} \frac{f(x) - f(a)}{x-a}$

$= \lim_{x \to 3^+} \frac{x^2+3x+1-(3^2+9+1)}{x-3}$

$= \lim_{x \to 3^+} \frac{x^2+3x+1-19}{x-3}$

$= \lim_{x \to 3^+} \frac{x^2+3x-18}{x-3}$

$= \lim_{x \to 3^+} \frac{(x+6)\cancel{(x-3)}}{\cancel{(x-3)}} = \lim_{x \to 3^+} x+6$

$= 3+6 = 9.$

LHD :- $\lim_{x \to 3^-} \frac{f(x) - f(a)}{x - a}$

$= \lim_{x \to 3^-} \frac{4x + 7 - 19}{x - a}$

$= \lim_{x \to 3^-} \frac{4x - 12}{x - 3}$

$= \lim_{x \to 3^-} \frac{4\cancel{(x-3)}}{\cancel{(x-3)}} = 4.$

RHD ≠ LHD

∴ f is not differentiable at $x = 3$.

Q4) RHD :- $\lim_{x \to 2^+} \frac{f(x) - f(a)}{x - a}$

$= \lim_{x \to 2^+} \frac{3x^2 - 4x + 7 - 11}{x - 2} = \lim_{x \to 2^+} \frac{3x^2 - 4x - 4}{x - 2}$

$= \lim_{x \to 2^+} \frac{(3x+2)\cancel{(x-2)}}{\cancel{(x-2)}} = \lim_{x \to 2^+} 3x + 2$

$= 3(2) + 2 = 6 + 2 = 8.$

LHD :- $\lim_{x \to 2^-} \frac{f(x) - f(a)}{x - a}$

$= \lim_{x \to 2^-} \frac{8x - 5 - 11}{x - 2} = \lim_{x \to 2^-} \frac{8x - 16}{x - 2} = \lim_{x \to 2} \frac{8\cancel{(x-2)}}{\cancel{(x-2)}}$

$= 8$

∴ RHD = LHD.

∴ f is differentiable at $x = 3$.

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PRACTICAL - 3

TOPIC :- APPLICATION OF DERIVATIVES.

Q1) Find the intervals in which function is increasing or decreasing :-

i) $f(x) = x^3 - 5x - 11$

ii) $f(x) = x^2 - 4x$

iii) $f(x) = 2x^3 + x^2 - 20x + 4$

iv) $f(x) = x^3 - 27x + 5$

v) $f(x) = 69 - 24x - 9x^2 + 2x^3$.

Q2) Find the intervals in which function is concave upwards :-

i) $y = 3x^2 - 2x^3$

ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

iii) $y = x^3 - 27x + 5$

iv) $y = 69 - 24x - 9x^2 + 2x^3$

v) $y = 2x^3 + x^2 - 20x + 4$

Solutions :-

Q1) i) $f(x) = x^3 - 5x - 11$

$\therefore f'(x) = 3x^2 - 5$

$\because$ f is increasing iff $f'(x) > 0$

$3x^2 - 5 > 0$

$3(x^2 - 5/3) > 0$

$(x - \sqrt{5/3})(x + \sqrt{5/3}) > 0$

$+ \qquad - \qquad +$

$-5/\sqrt{3} \qquad 5\sqrt{3}$

$x \in (-\infty, -\sqrt{5/3}) \cup (\sqrt{5/3}, \infty)$

and f is decreasing if $f'(x) < 0$

$\therefore (3x^2 - 5) < 0$

$\therefore 3(x^2 - 5/3) < 0$

$\therefore (x - \sqrt{5/3})(x + \sqrt{5/3}) < 0$

$+ \quad - \quad +$

$-\sqrt{5/3} \qquad \sqrt{5/3}$

$x \in (-\sqrt{5/3}, \sqrt{5/3})$

ii) $f(x) = x^2 - 4x$

$f'(x) = 2x - 4$

$\because f(x)$ is increasing if $f'(x) > 0$

$\therefore 2x - 4 > 0$

$\therefore 2(x-2) > 0$

$\therefore x - 2 > 0$

$\therefore x > 2$

$\therefore x \in (2, \infty)$

and f is decreasing if $f'(x) < 0$.

$\therefore (2x - 4) < 0$

$\therefore 2(x-2) < 0$

$\therefore x - 2 < 0$

$\therefore x < 2$

$\therefore x \in (-\infty, 2)$

iii) $f(x) = 2x^3 + x^2 - 20x + 4$.

$\therefore f'(x) = 6x^2 + 2x - 20$

$\because f$ is increasing iff $f'(x) > 0$.

$\therefore 6x^2 + 2x - 20 > 0$

$\therefore 2(3x^2 + x - 10) > 0$.

$\therefore 3x^2 + x - 10 > 0$

$\therefore 3x^2 + 6x - 5x - 10 > 0$

$\therefore 3x(x+2) - 5(x+2) > 0$.

$\therefore (x+2)(3x-5) > 0$.

$\therefore x \in (-\infty, 2) \cup (5/3, \infty)$.

and f is decreasing if $f'(x) < 0$.

$\therefore 6x^2 + 2x - 20 < 0$.

$\therefore 2(3x^2 + x - 10) < 0$.

$\therefore (3x^2 + x - 10) < 0$.

$\therefore (x+2)(3x-5)<0$

$\therefore x \in (-2, 5/3)$

iv) $f(x) = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$\because f$ is increasing iff $f'(x) > 0$.

$\therefore 3(x^2 - 9) > 0$.

$\therefore (x+3)(x-3) > 0$.

$\therefore x \in (-\infty, -3) \cup (3, \infty)$.

and f is decreasing if $f'(x) < 0$.

$\therefore 3x^2 - 27 < 0$.

$\therefore 3(x^2 - 9) < 0$

$\therefore (x-3)(x+3) < 0$.

$+ \quad - \quad +$

$-3 \quad 3$

$\therefore x \in (-3, 3)$.

v) $f(x) = 2x^3 - 9x^2 - 24x + 69$.

$\therefore f'(x) = 6x^2 - 18x - 24$.

$\because f$ is increasing if $f'(x) > 0$.

$\therefore 6(x^2 - 3x - 4) > 0$.

$\therefore x^2 - 3x - 4 > 0$.

$\therefore (x-4)(x+1) > 0$.

$+ \quad - \quad +$

$-1 \quad 4$

$\therefore x \in (-\infty, -1) \cup (4, \infty)$.

and f is decreasing if $f'(x) < 0$.

$\therefore 6(x^2 - 3x - 4) < 0$.

$\therefore x^2 - 3x - 4 < 0$.

$\therefore (x-4)(x+1) < 0$.

$+ \quad - \quad +$

$-1 \quad 4$

$\therefore x \in (-1, 4)$.

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Q2) i) $y = 3x^2 - 2x^3$

$\therefore f(x) = 3x^2 - 2x^3$

$\therefore f'(x) = 6x - 6x^2$

$\therefore f''(x) = 6 - 12x.$

$\therefore f$ is concave upward if $f''(x) > 0.$

$\therefore (6 - 12x) > 0.$

$\therefore -12(-1/2 + x) > 0.$

$\therefore x - \frac{1}{2} > 0.$

$\therefore x > 1/2.$

$\therefore x \in (1/2, \infty).$

ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7.$

$f'(x) = \cancel{3}\,4x^3 - 18x^2 + 24x + 5.$

$f''(x) = 12x^2 - 36x + 24.$

f is concave upward iff $f''(x) > 0.$

$\therefore 12(x^2 - 3x + 2) > 0.$

$\therefore x^2 - 3x + 2 > 0$

$\therefore (x-2)(x-1) > 0.$

$+ \quad 1 \quad - \quad 2 \quad +$

$x \in (-\infty, 1) \cup (2, \infty).$

iii) $y = x^3 - 27x + 5.$

$f'(x) = 3x^2 - 27.$

$f''(x) = 6x.$

f is concave upward iff $f''(x) > 0.$

$\therefore 6x > 0$

$\therefore x > 0$

$\therefore x \in (0, \infty).$

iv) $y = 69 - 24x - 9x^2 + 2x^3$

$f'(x) = 6x^2 - 18x - 24.$

$f''(x) = 12x - 18.$

f is concave upward iff $f''(x) > 0.$

$\therefore 12(x - 3/2) > 0.$

$\therefore x - 3/2 > 0$

$\therefore x > 3/2.$

$\therefore x \in (3/2, \infty).$

v) $y = 2x^3 + x^2 - 20x + 4.$

$f(x) = 2x^3 + x^2 - 20x + 4.$

$f'(x) = 6x^2 + 2x - 20.$

$f''(x) = 12x + 2.$

f is concave upward iff $f''(x) > 0.$

$\therefore 12x + 2 > 0.$

$\therefore 12(x + 1/6) > 0$

$\therefore x + \frac{1}{6} > 0.$

$\therefore x < -1/6.$

$\therefore f''(x) \not> 0.$

$\therefore$ there exist no interval.

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PRACTICAL-04

Topic: Application of Derivative and Newton's Method.

Q1) Find maximum and minimum value of following functions:

i) $F(x) = x^2 + \frac{16}{x^2}$

ii) $F(x) = 3 - 5x^3 + 3x^5$.

iii) $F(x) = x^3 - 3x^2 + 1$ in $[-1/2, 4]$

iv) $F(x) = 2x^3 - 3x^2 - 12x + 1$. in $[-2, 3]$.

Q2) Find the root of the following eqn :-

i) $F(x) = x^3 - 3x^2 - 55x + 9.5$ (take $x_0 = 0$)

ii) $F(x) = x^3 - 4x - 9$ in $[2, 3]$

iii) $F(x) = x^3 - 1.8x^2 - 10x + 17$ in $[1, 2]$

Q1) i) $f(x) = x^2 + \frac{16}{x^2}$

$\therefore f'(x) = 2x - \frac{32}{x^3}$

Now, consider,

$f'(x) = 0$.

$\therefore 2x - \frac{32}{x^3} = 0$.

$\therefore 2x = \frac{32}{x^3}$.

$\therefore x^4 = \frac{32}{2}$

$\therefore x^4 = 16$.

$\therefore x = \pm 2$.

$f''(x) = 2 + \frac{96}{x^4}$.

$f''(2) = 2 + \frac{96}{\cancel{8}16}$

$= 2 + 6$

$= 8 > 0$.

$\therefore f$ has minimum value at $x = 2$.

$\therefore f(2) = 2^2 + \frac{16}{2^2}$

$= 4 + \frac{16}{4}$

$= 4 + 4$

$= 8$

$f(-2) = 2 + \frac{96}{16}$

$= 2 + 6$

$= 8 > 0$.

$\therefore f$ has minimum value at $x = -2$.

ii) $f(x) = 3 - 5x^3 + 3x^5$

$f'(x) = 15x^4 - 15x^2$.

Consider,

$f'(x) = 0$.

$\therefore 15x^4 = 15x^2$

$\therefore x^2 = 1$

$\therefore x = \pm 1$

$\therefore f''(x) = -30x + 60x^3$.

$f''(1) = -30 + 60$.

$= 30 > 0$.

$\therefore f$ has minimum value at $x = 1$.

$\therefore f(1) = 3 - 5(1) + 3(1)$

$= 3 - 5 + 3$.

$= 1$.

$f''(-1) = -30(-1) + 60(-1)^3$

$= 30 - 60$

$= -30 < 0$.

$\therefore f$ has maximum value at $x = -1$.

$\therefore f(-1) = 3 - 5(-1)^3 + 3(-1)^5$

$= 3 + 5 - 3$.

$= 5$.

$\therefore f$ has the maximum value ~~as~~ 5 at $x = -1$ and has minimum value 1 at $x = 1$.

iii) $f(x) = x^3 - 3x^2 + 1$

$\therefore f'(x) = 3x^2 - 6x$.

consider $f'(x) = 0$

$\therefore 3x^2 - 6x = 0$

$\therefore 3x(x-2) = 0$

$\therefore 3x = 0$ or $x - 2 = 0$

$\therefore x = 0$ or $x = 2$.

$f''(x) = 6x - 6$

$\therefore f''(0) = -6 < 0$.

$\therefore f$ has maximum value at $x = 0$.

$\therefore f(0) = 0 - 0 + 1$
$= 1$.

$f''(2) = 6(2) - 6$
$= 6 > 0$

$\therefore f$ has minimum value at $x = 2$.

$f(2) = 2^3 - 3(2)^2 + 1$
$= 8 - 12 + 1$
$= -3$.

$\therefore f$ has maximum value 1 at $x = 0$, and has minimum value -3 at $x = 2$.

iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$

$\therefore f'(x) = 6x^2 - 6x - 12$.

Consider, $f'(x) = 0$.

$\therefore 6x^2 - 6x - 12 = 0$.

$\therefore \cancel{6x^2 - 6x = 12}$

$\therefore \cancel{6x(x-1) = 12}$

$\therefore \cancel{x(x-1) = 2}$

$\therefore \cancel{x = 2 \text{ or } x - 1 = 2}$

$\therefore \cancel{x = 2 \text{ or } x = 3}$

$\therefore 6(x^2 - x - 2) = 0$

$\therefore x^2 - 2x + x - 2 = 0$.

$\therefore x(x-2) + 1(x-2) = 0$.

$\therefore (x-2)(x+1) = 0$.

$\therefore x = 2$ or $x = -1$.

$\therefore f''(x) = 12x - 6$.

$\therefore f''(2) = 24 - 6$
$= 18 > 0$.

$\therefore f$ has minimum value at $x = 2$

$\therefore f(2) = 2(8) - 3(4) - 12(2) + 1$
$= 16 - 12 - 24 + 1$
$= -19$.

$f''(-1) = 12(-1) - 6$.
$= -18 < 0$.

$\therefore f$ has maximum value at $x = -1$.

$\therefore f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$
$= -2 - 3 + 12 + 1$
$= 8$.

$\therefore f$ has maximum value 8 at $x = -1$. and has minimum value -19 at $x = 2$.

Q2) i) $F(x) = x^3 - 3x^2 - 55x + 9.5$

$x_0 = 0$ ....(given)

$\therefore f'(x) = 3x^2 - 6x - 55$

By Newton's Method,

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$$

$$\therefore x_1 = 0 + \frac{9.5}{55}$$

$$= 0.1727$$

$\therefore F(x_1) = F(0.1727) = (0.1727)^3 - 3(0.1727)^2 - 55(0.1727) + 9.5$

$= 0.0051 - 0.0895 - 9.4985 + 9.5$

$= -0.0829$

$F'(x_1) = 3(0.1727)^2 - 6(0.1727) - 55$

$= 0.0895 - 1.0362 - 55$

$= -55.9467$

$$\therefore x_2 = x_1 - \frac{F(x_1)}{F'(x_1)}$$

$$= 0.1727 - \frac{0.0829}{55.9467}$$

$$= 0.1712$$

$F(x_2) = (0.1712)^3 - 3(0.1712)^2 - 55(0.1712) + 9.5$

$= 0.005 - 0.0879 - 9.416 + 9.5 = 0.0011$

$F'(x_2) = 3(0.1712)^2 - 6(0.1712) - 55$

$= 0.0879 - 1.0272 - 55 = -55.9393$

$$\therefore x_3 = x_2 - \frac{F(x_2)}{F'(x_2)} = 0.1712 + \frac{0.0011}{55.9393} = 0.1712$$

$\therefore$ The root of the equation is 0.1712

ii) $F(x) = x^3 - 4x - 9 \quad \ldots [2, 3]$.

$F'(x) = 3x^2 - 4$.

$F(2) = 2^3 - 4(2) - 9$.

$= 8 - 8 - 9$.

$= -9$.

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$f(3) = 3^3 - 4(3) - 9$

$= 27 - 12 - 9 = 6$.

Let $x_0 = 3$ be the initial approximation.

$\therefore$ By Newton's Method,

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$$

$$= 3 - \frac{6}{23}$$

$$= 2.7392.$$

$\therefore F(x_1) = F(2.7392) = (2.7392)^3 - 4(2.7392) - 9$.

$= 20.5528 - 10.9568 - 9 = 0.596$.

$F'(x_1) = 3(2.7392)^2 - 4 = 22.5096 - 4 = 18.5096$.

$$x_2 = x_1 - \frac{F(x_1)}{F'(x_1)} = 2.7392 - \frac{0.596}{18.5096} = 2.7071.$$

$\therefore F(x_2) = F(2.7071) = (2.7071)^3 - 4(2.7071) - 9$

$= 19.8386 - 10.8284 - 9 = 0.0102$.

$F'(x_2) = 3(2.7071)^2 - 4 = 21.9851 - 4 = 17.9851$.

$$\therefore x_3 = x_2 - \frac{F(x_2)}{F'(x_2)} = 2.7071 - \frac{0.0102}{17.9851} = 2.7015.$$

$\therefore F(x_3) = (2.7015)^3 - 4(2.7015) - 9 = 19.7158 - 10.806 - 9 = -0.0901$

$F'(x_3) = 3(2.7015)^2 - 4 = 21.8943 - 4 = 17.8943$

$$\therefore x_4 = 2.7015 + \frac{0.0901}{17.8943} = 2.7015 + 0.0050 = 2.7065.$$

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iii) $F(x) = x^3 - 1.8x^2 - 10x + 17$ --- [1, 2]

$\therefore F'(x) = 3x^2 - 3.6x - 10$.

$\therefore F(1) = 1^3 - 1.8(1)^2 - 10 + 17$

$= 1 - 1.8 - 10 + 17 = 6.2$.

$F(2) = 2^3 - 1.8(4) - 20 + 17$.

$= 8 - 7.2 - 3 = -2.2$.

Let $x_0 = 2$.

$\therefore$ By Newton's Method,

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)} = 2 - \frac{2.2}{5.2} = 2 - 0.4230 = 1.577$$

$\therefore F(x_1) = F(1.577) = (1.577)^3 - 1.8(1.577)^2 - 10(1.577) + 17$.

$= 3.9219 - 4.4764 - 15.77 + 17 = 0.6755$.

$F'(x_1) = 3(1.577)^2 - 3.6(1.577) - 10 = 7.4608 - 5.6772 - 10$

$= -8.2164$.

$$\therefore x_2 = x_1 - \frac{F(x_1)}{F'(x_1)} = 1.577 + \frac{0.6755}{8.2164} = 1.577 + 0.0822$$

$= 1.6592$.

$\therefore F(x_2) = F(1.6592) = (1.6592)^3 - 1.8(1.6592)^2 - 10(1.6592) + 17$

$= 4.5677 - 4.9553 - 16.592 + 17 = 0.0204$.

$F'(x_2) = 3(1.6592)^2 - 3.6(1.6592) - 10 = 8.2588 - 5.9731 - 10 = -7.7143$.

$$x_3 = x_2 - \frac{F(x_2)}{F'(x_2)} = 1.6592 + \frac{0.0204}{7.7143} = 1.6592 + 0.0026 = 1.6618$$

$\therefore F(x_3) = F(1.6618) = (1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17$

$= 4.5892 - 4.9708 - 16.618 + 17 = 0.0004$.

$F'(x_3) = 3(1.6618)^2 - 3.16(1.6618) - 10 = 8.2847 - 5.9824 - 10 = -7.697$

$$x_4 = x_3 - \frac{F(x_3)}{F'(x_3)} = 1.6618 + \frac{0.0004}{7.6972} = 1.6618$$

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$\therefore$ The root of the equation is 1.6618.

PRACTICAL-05

Topic :- Integration

Q) solve the following integration :-

i) $\int \frac{dx}{\sqrt{x^2+2x-3}}$ ii) $\int (4e^{3x}+1)dx$ iii) $\int (2x^2-3\sin x+5\sqrt{x})dx$

iv) $\int \frac{x^3+3x+4}{\sqrt{x}}dx$ v) $\int t^7 \sin(2t^4)dt$ vi) $\int \sqrt{x}(x^2-1)dx$

vii) $\int \frac{1}{x^3}\sin\left(\frac{1}{x^2}\right)dx$ viii) $\int \frac{\cos x\,dx}{\sqrt[3]{\sin^2 x}}$ ix) $\int e^{\cos^2 x}\sin 2x\,dx$

x) $\int \left(\frac{x^2-2x}{x^3-3x^2+1}\right)dx$

solutions :-

i) $\int \frac{1}{\sqrt{x^2+2x-3}}dx$

$= \int \frac{1}{\sqrt{x^2+2x-4+1}}dx$

$= \int \frac{1}{\sqrt{(x+1)^2-4}}dx$

Put $x+1 = t \Rightarrow dx = dt$

$= \int \frac{dt}{\sqrt{t^2-4}} = \log(|t+\sqrt{t^2-4}|) = \log(|x+1+\sqrt{(x+1)^2-4}|)$

$= \log(|x+1+\sqrt{x^2+2x-3}|)$

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ii) $\int (4e^{3x} + 1)\,dx$

$= 4\int e^{3x}\,dx + \int 1\,dx$

$= \frac{4e^{3x}}{3} + x + c$

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iii) $\int (2x^2 - 3\sin x + 5\sqrt{x})\,dx$

$= 2\int x^2\,dx - 3\int \sin x\,dx + 5\int \sqrt{x}\,dx$

$= \frac{2x^3}{3} + 3\cos x + 5\int x^{1/2}\,dx$

$= 2/3\, x^3 + 3\cos x + 5 \times \frac{2}{3} x^{3/2} + c$

$= 2/3\, x^3 + 3\cos x + 10/3\, x^{3/2} + c$

iv) $\int \frac{x^3 + 3x + 4}{\sqrt{x}}\,dx$

$= \int \left(\frac{x^3}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{4}{x^{1/2}}\right) dx$

$= \int (x^{5/2} + 3x^{1/2} + 4x^{-1/2})\,dx$

$= \int x^{5/2}\,dx + 3\int x^{1/2}\,dx + 4\int x^{-1/2}\,dx$

20 $= \frac{2}{7} x^{7/2} + \cancel{3} \times \frac{2}{\cancel{3}} x^{3/2} + 4 \times 2 x^{1/2} + c$

$= 2/7\, x^{7/2} + 2x^{3/2} + 8x^{1/2} + c$

v) $\int t^7 \sin(2t^4)\, dt$

Put $u = 2t^4$

$du = 8t^3 dt$

$dt = \frac{du}{8t^3}$

$= \int t^7 \times \sin(2t^4) \times \frac{1}{8t^3} du$

$= \int t^4 \times \sin u \times \frac{1}{8} du$

substitute $t^4 = \frac{u}{2}$

$= \int \frac{u}{2} \times \sin u \times \frac{1}{8} du$

$= \frac{1}{16}\left[\int u \sin u \, du\right] = \frac{1}{16}\left[(u \times (-\cos u)) - \int (-\cos u) du\right]$

$= \frac{1}{16}\left[-u\cos u + \int \cos u \, du\right] = \frac{1}{16}(-u\cos u + \sin u) + c$

$= \frac{1}{16}\left[-2t^4 \cos 2t^4 + \sin 2t^4\right] + c = -\frac{t^4 \cos 2t^4}{8} + \frac{\sin 2t^4}{16} + c$

vi) ~~$\int \cos x$~~ $\int \sqrt{x}\,(x^2 - 1)\, dx$

$= \int (x^{1/2} x^2 - x^{1/2})\, dx = \int (x^{5/2} - x^{1/2})\, dx$

$= \frac{2}{7} x^{7/2} - \frac{2}{3} x^{3/2} + c$

vii) $\int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

Put $\frac{1}{x^2} = t$

$\Rightarrow x^{-2} = t$

$\frac{d}{dx}(x^{-2}) = \frac{dt}{dx}$

$-2x^{-3} = \frac{dt}{dx}$

$\frac{-2}{x^3} = \frac{dt}{dx}$

$\frac{1}{x^3} \times dx = \frac{dt}{-2}$

$= \int \frac{\sin t \, dt}{-2}$

$= \frac{1}{-2} \int \sin t \, dt$

$= \frac{1}{-2}(-\cos t) + c$

$= \frac{\cos t}{2} + c$

$= \frac{1}{2} \times \cos\left(\frac{1}{x^2}\right) + c$

viii) $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}\,dx = \int \frac{\cos x}{(\sin x)^{2/3}}\,dx$

Sol

Put $\sin x = t$

$\cos x = \frac{dt}{dx}$

$\cos x\,dx = dt$

$= \int \frac{dt}{t^{2/3}} = \int t^{-2/3}\,dt = \frac{t^{-2/3+1}}{-2/3+1} + c = \frac{t^{1/3}}{1/3} + c = 3t^{1/3} + c$

$= 3\sqrt[3]{t} + c = 3\sqrt[3]{\sin x} + c$

ix) $\int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1}\right) dx$

Put $x^3 - 3x^2 + 1 = t$

$\frac{d}{dx}(x^3 - 3x^2 + 1) = \frac{dt}{dx}$

$3x^2 - 6x = \frac{dt}{dx}$

$3(x^2 - 2x) = \frac{dt}{dx}$

$(x^2 - 2x)\,dx = \frac{dt}{3}$

$\int \frac{dt}{3t}$

$= \frac{1}{3}\int \frac{dt}{t}$

$= \frac{1}{3}\log t + c$

$= \frac{\log(x^3 - 3x^2 + 1)}{3} + c$

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(ix) $\int e^{\cos^2 x} \cdot \sin 2x\,dx$

Put $\cos^2 x = t$

$-2\cos x \sin x = \frac{dt}{dx}$

$\therefore -\sin 2x\,dx = dt$

$\therefore \sin 2x\,dx = -dt$

$= -\int e^t\,dt$

$= -e^t + c$

$= -e^{\cos^2 x} + c$

PRACTICAL-06

Topic :- Application of Integration and Normal Integration.

Q1) Find the length of the following curve :-

1) $x = t - \sin t$, $y = 1 - \cos t$ in $[0, 2\pi]$

→ $\frac{dx}{dt} = 1 - \cos t$

$\frac{dy}{dt} = \sin t$

Length of the curve (L) = $\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$\therefore L = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt$

$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt$

$= \int_0^{2\pi} \sqrt{1 - 2\cos t + 1}\, dt$

$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$

$= \int_0^{2\pi} \sqrt{2(1-\cos t)}\, dt$

$= \int_0^{2\pi} \sqrt{2 \times 2\sin^2 t/2}\, dt = \int_0^{2\pi} \sqrt{4\sin^2 t/2} = \int_0^{2\pi} 2\sin t/2 = 2\left[\frac{\cos t/2}{1/2}\right]_0^{2\pi}$

$= -2 \times 2\,[\cos 2\pi/2 - \cos 0] = -2 \times 2\,[\cos \pi - 1] = -4[-1-1]$

$= -4 \times -2 = 8.$

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2) $y = \sqrt{4-x^2}\, dx$, $x \in [-2, 2]$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cancel{2}\sqrt{4-x^2}} \times -\cancel{2}x = \frac{-x}{\sqrt{4-x^2}}$ d

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{-2}^{2} \sqrt{1 + \frac{x^2}{4-x^2}}\, dx$$

$$= 2\int_0^2 \sqrt{\frac{4-x^2+x}{4-x^2}}\, dx$$

$$= 2\int_0^2 \sqrt{\frac{4}{4-x^2}}\, dx$$

$$= 2 \times 2 \int_0^2 \frac{1}{\sqrt{4-x^2}}\, dx$$

$$= 4 \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= 4 \left[\sin^{-1}\left(\frac{2}{2}\right) - \sin^{-1}\left(\frac{0}{2}\right)\right]$$

$$= 4 \{\sin^{-1}(1) - \sin^{-1}(0)\}$$

$$= 4 \left[\frac{\pi}{2} - 0\right]$$

$$= 4 \times \frac{\pi}{2} = 2\pi.$$

3) $y = x^{3/2}$ in $[0,4]$.

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^{3/2})$

$= \frac{3}{2} x^{1/2}$.

$\therefore L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$.

$= \int \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\, dx$.

$= \int \sqrt{1+\frac{9}{4}x}\quad dx$.

Put $1+\frac{9}{4}x = u$.

$\therefore \frac{9}{4} = \frac{du}{dx}$

$\therefore dx = \frac{4}{9} du$.

$\cancel{\text{Zt}} = \int$

$\therefore L = \frac{4}{9}\int \sqrt{u}\, du$

$= \frac{4}{9} \times \frac{2}{3} [u^{3/2}]$

4) $x = 3\sin t,\ y = 3\cos t$
$t \in [0, 2\pi]$

$\Rightarrow \frac{dx}{dt} = 3\cos t$

$\frac{dy}{dt} = -3\sin t$

$\therefore L = \int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2}\, dt$.

$= \int_0^{2\pi} \sqrt{9\cos^2 t + 9\sin^2 t}\, dt$

$= \int_0^{2\pi} \sqrt{9(\cos^2 t + \sin^2 t)}\, dt$

$= \int_0^{2\pi} \sqrt{9}\quad dt$.

$= 3\int_0^{2\pi} dt$.

$= 3[t]_0^{2\pi}$

$= 3[2\pi - 0]$

$= 3 \times 2\pi = 6\pi$.

45) $x = \frac{1}{6}y^3 + \frac{1}{2y}$

$y \in \{1, 2\}$

$\therefore \frac{dx}{dy} = \frac{1}{\cancel{6}_2} \times \cancel{3}y^2 - \frac{1}{2y^2}$

$= \frac{y^2}{2} - \frac{1}{2y^2}$

$\therefore \frac{dx}{dy} = \frac{y^4 - 1}{2y^2}$

$\therefore L = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

$= \int_1^2 \sqrt{1 + \left(\frac{y^4 - 1}{2y^2}\right)^2}\, dy$

$= \int_1^2 \sqrt{1 + \frac{(y^4 - 1)^2}{4y^4}}\, dy$

$= \int_1^2 \sqrt{\frac{4y^4 + y^8 - 2y^4 + 1}{4y^4}}\, dy$

$= \int_1^2 \sqrt{\frac{y^8 + 2y^4 + 1}{4y^4}}\, dy$

$= \int_1^2 \sqrt{\frac{(y^4 + 1)^2}{4y^4}}\, dy$

$= \int_1^2 \frac{y^4 + 1}{2y^2}\, dy$

$= \frac{1}{2} \int_1^2 \left(\frac{y^4}{y^2} + \frac{1}{y^2}\right) dy$

$= \frac{1}{2} \int_1^2 \left(y^2 + \frac{1}{y^2}\right) dy$

$= \frac{1}{2} \left[\int_1^2 y^2\, dy + \int_1^2 y^{-2}\, dy \right]$

$= \frac{1}{2} \left[\frac{[y^3]_1^2}{3} - \left[\frac{1}{y}\right]_1^2 \right]$

$= \frac{1}{2} \left[\frac{8-1}{3} - \left(\frac{1}{2} - 1\right) \right]$

$= \frac{1}{2} \left[\frac{7}{3} - \left(-\frac{1}{2}\right) \right]$

$= \frac{1}{2} \left[\frac{7}{3} + \frac{1}{2} \right]$

$= \frac{1}{2} \left[\frac{14 + 3}{6} \right]$

$= \frac{1}{2} \times \frac{17}{6}$

$= \frac{17}{12}$

Q2) Using Simpson's Rule find the following:-

i) $\int_0^2 e^{x^2} dx$ with $n=4$.

→ Here, $n=4$, $a=0$, $b=2$.

$\therefore h = \frac{b-a}{n} = \frac{2}{4} = 0.5$.

$f(x) = e^{x^2}$.

x	0	0.5	1	1.5	2
y	1	1.28	2.71	9.48	54.59

$\therefore \int_0^2 e^{x^2} = \frac{0.5}{3}\left[(1+54.59)+4(1.28+9.48)+2(2.71)\right]$

$= \frac{0.5}{3}[55.59+4(10.76)+5.42]$

$= \frac{0.5}{3}[55.59+43.04+5.42]$

$= \frac{0.5}{3}[104.05]$

$= 0.5 \times 34.68$

$= 17.34$.

ii) $\int_0^4 x^2 dx$ with $n=4$.

→ Here, $n=4$, $a=0$, $b=4$.

$\therefore h = \frac{4-0}{4} = \frac{4}{4} = 1$.

$f(x) = x^2$.

x	0	1	2	3	4
y	0	1	4	9	16

$$\therefore \int_0^4 x^2\,dx = \frac{1}{3}\left[(0+16)+4(1+9)+2(4)\right]$$

$$= \frac{1}{3}\left[16+4(10)+8\right]$$

$$= \frac{1}{3}\left[16+40+8\right]$$

$$= \frac{1}{3}\times 64$$

$$= 21.33.$$

iii) $\int_0^{\pi/3} \sqrt{\sin x}\,dx$ with $n=6$.

→ here, $n=6$, $a=0$, $b=\frac{\pi}{3}$

$h = \frac{\pi}{3\times 6} = \frac{\pi}{18}$, $f(x) = \sqrt{\sin x}\,dx$.

~~x 0 1 2 3 4 5 6.~~

~~y 0 0.91 0.95 0.37~~

x	0	$\frac{\pi}{18}$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$
y	0	~~0.17~~ 0.41	0.58	0.70	0.80	0.87	0.93.

$$\int_0^{\pi/3} \sqrt{\sin x}\,dx = \frac{\pi}{18\times 3}\left[(0+0.93)+4(0.41+0.7+0.87)+2(0.58+0.8)\right]$$

$$= \frac{\pi}{54}\left[0.93+4(1.98)+2(1.38)\right]$$

$$= \frac{\pi}{54}\left[0.93 + 7.92 + 2.76\right]$$

$$= \frac{\pi}{54}\left[11.61\right]$$

$$= \frac{11.61\pi}{54}.$$

$$\approx 11.61 \times 0.05$$

$$\approx 0.5805.$$

PRACTICAL-07

Topic :- Differential Equation.

Q1) Solve the following :-

1) $x \frac{dy}{dx} + \frac{1}{x} y = \frac{e^x}{x}$

$P_0(x) = \frac{1}{x}$, $Q(x) = \frac{e^x}{x}$

$y(IF) = \int Q(x)(IF)\,dx + c$

$\therefore xy = \int \frac{e^x}{x} \times x\,dx + c$

$= \int e^x\,dx + c.$

$\therefore xy = e^x + c.$

2) $e^x \frac{dy}{dx} + 2e^x y = 1$

$\Rightarrow \frac{dy}{dx} + \frac{2e^x y}{e^x} = \frac{1}{e^x}$

$\frac{dy}{dx} + 2y = e^{-x}.$

$P(x) = 2$, $Q(x) = e^{-x}.$

$If = e^{\int 2dx}$

$= e^{2x}$

$y(IF) = \int Q(x)(If)\,dx + c$

$e^{2x} y = \int e^x\,dx + c.$

$e^{2x} y = e^x + c.$

3) $x\frac{dy}{dx} = \frac{\cos x}{x} - 2y.$

$\Rightarrow \frac{dy}{dx} + \frac{2y}{x} = \frac{\cos x}{x^2}.$

$P(x) = 2(\frac{1}{x})$, $Q(x) = \frac{\cos x}{x^2}$

$If = e^{\int 2(\frac{1}{x})dx}.$

$= x^2$

$y(IF) = \int Q(x)(IF)\,dx + c.$

$= \int \frac{\cos x}{x^2} \cdot x^2\,dx + c$

$x^2 y = \int \cos x + c.$

$x^2 y = \sin x + c.$

4) $x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

$\Rightarrow \frac{dy}{dx} + \frac{3y}{x} = \frac{\sin x}{x^3}$

$P(x) = 3/x$, $Q(x) = \frac{\sin x}{x^3}$

$\therefore IF = x^3.$

$y(IF) = \int Q(x)(IF)\,dx + c$

$x^3 y = \int \frac{\sin x}{x^3} \times x^3\,dx + c.$

$x^3 y = -\cos x + c.$

5) $e^{2x}\dfrac{dy}{dx} + 2e^{2x}y = 2x$

$\Rightarrow \dfrac{dy}{dx} + \dfrac{2e^{2x}}{e^{2x}}y = \dfrac{2x}{e^{2x}}$

$P(x) = 2$, $Q(x) = \dfrac{2x}{e^{2x}}$

$IF = e^{\int 2\,dx}$
$= e^{2x}$

$y(IF) = \int Q(x)\,IF\,dx + c$

$e^{2x}y = \int \dfrac{2x}{\cancel{e^{2x}}} \times \cancel{e^{2x}}\,dx + c$

$e^{2x}y = \dfrac{2x^2}{2} + c$

$e^{2x}y = x^2 + c$

6) $\sec^2x \tan y\,dx + \sec^2y \tan x\,dy = 0$

$\Rightarrow \sec^2x \tan y\,dx = -\sec^2y \tan x\,dy$

$\int \dfrac{\sec^2x}{\tan x}\,dx = -\int \dfrac{\sec^2y}{\tan y}\,dy$

$\log|\tan x| = -\log|\tan y| + c$

$\tan x + \tan y = e^c$

7) $\dfrac{dy}{dx} = \sin^2(x - y + 1)$

Put $x - y + 1 = v$

$1 - \dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore 1 - \dfrac{dv}{dx} = \sin^2 v$

$\dfrac{dv}{dx} = 1 - \sin^2 v$

$\dfrac{dv}{dx} = \cos^2 v$

$\int \dfrac{dv}{\cos^2 v} = \int dx$

$\int \sec^2 v\,dv = \int dx$

$\tan v = x + c$

$\therefore \tan(x - y + 1) = x + c$

8) $\dfrac{dy}{dx} = \dfrac{2x + 3y - 1}{6x + 9y - 6}$

Put $2x + 3y = v$

$2 + 3\dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\dfrac{dy}{dx} = \dfrac{1}{3}\left(\dfrac{dv}{dx} - 2\right)$

$\therefore \frac{1}{3}\left(\dfrac{dv}{dx} - 2\right) = \frac{1}{3}\left(\dfrac{v-1}{v+2}\right)$

$\therefore \dfrac{dv}{dx} = \dfrac{v-1}{v+2} + 2$

$\therefore \dfrac{dv}{dx} = \dfrac{v - 1 + 2v + 4}{v+2}$

$\therefore dv/dx = \dfrac{3(v+1)}{v+2}$

$\therefore \int \dfrac{v+2}{v+1}\,dv = 3\int dx$

$\therefore \int \dfrac{v+1}{v+1}\,dv + \int \dfrac{1}{v+1}\,dv = 3x$

$\therefore v + \log(v+1) = 3x + c$

$\therefore v + \log(v+1) = 3x + c$

$\therefore 2x + 3y + \log(2x + 3y + 1) = 3x + c$

$\therefore 3y = x - \log|2x + 3y + 1| + c$

PRACTICAL - 08

Topic :- Euler's Method :

Q1) $\frac{dy}{dx} = y + e^{x} - 2$, $y(0) = 2$, $h = 0.5$, find $y(2)$

→ $f(x) = y + e^{x} - 2$, $x_0 = 0$, $y_0 = 2$, $h = 0.5$.

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.1487	3.5743
2	1	3.5743	4.2925	5.7205
3	1.5	5.7205	8.2021	9.8215
4	2	9.8215.		

$Y(2) = 9.8215$.

Q2) $\frac{dy}{dx} = 1 + y^{2}$, $y(0) = 1$, $h = 0.2$, find $y(1)$.

→ $x_0 = 0$, $y_0 = 1$, $h = 0.2$.

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	1		
1	0.2	0.2	2	0.2
2	0.4	0.408	1.04	0.408
3	0.6	0.6412	1.1664	0.6412
4	0.8	0.9234	1.4111	0.9234
5	1	1.2939	1.8526	1.2939.

$Y(1) = 1.2939$.

Q3) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$, $y(0)=1$, $h=0.2$, find ~~&~~ $y(1)=?$

→ $x_0=0$, $y_0=1$, $h=0.2$.

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n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	1	0	1
1	0.2	1	0.4472	1.0894
2	0.4	1.0894	0.6059	1.2105
3	0.6	1.2105	0.7040	1.3513
4	0.8	1.3513	0.7696	1.5051
5	1	1.5051		

$y(1) = 1.5051$.

Q4) $\frac{dy}{dx} = 3x^2+1$, $y(1)=2$, find $y(2)$, $h=0.5$.

$y_0=2$, $x_0=1$, $h=0.5$

n	x_n	y_n	~~y_{n+}~~ $f(x_n, y_n)$	y_{n+1}
0	1	2	4	4
1	1.5	4	7.75	7.875
2	2	7.875		

$y(2) = 7.875$.

Q5) ~~$Y_0 \in R$~~ , $x_0 = 1$, ~~$h=0.25$~~ $\frac{dy}{dx} = \sqrt{xy} + 2$, $y(1)=1$, $h=0.2$.
$y(1.2)=?$

→ $x_0=1$, $y_0=1$, $h=0.2$.

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	1	3	3.6
1	1.2	3.6		

$y(1.2) = 3.6$.

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PRACTICAL - 9

Aim: Limits and Partial order derivative

Q1) Evaluate the following :-

i) $\lim_{(x,y)\to(-4,-1)} \dfrac{x^3-3x+y^2-1}{xy+5}$

$= \cancel{\lim_{(x,y)\to(-4,-1)}}$ Apply Limit

$$= \frac{(-4)^3-3(-4)+1^2-1}{(-4)(-1)+5}$$

$$= \frac{-64+12+1-1}{4+\cancel{9}5} = \frac{-52}{9}$$

ii) $\lim_{(x,y)\to(2,0)} \dfrac{(y+1)(x^2+y^2-4x)}{x+3y}$

Apply limit

$$= \frac{(0+1)(2^2+0-4(2))}{2+0} = \frac{1(4-8)}{2} = -\frac{4}{2} = -2$$

iii) $\lim_{(x,y,z)\to(1,1,1)} \dfrac{x^2-y^2z^2}{x^2-x^2yz}$

Apply limit

$$= \frac{1^2-1^2(1^2)}{1^2-1^2(1)(1)} = \frac{1-1}{1-1} = \frac{0}{0} = 0$$

$\therefore$ Limit does not exist.

Q2) Find f_x, f_y for each of the following:-

i) $f(x,y) = xy e^{x^2+y^2}$

$\Rightarrow f_x = y(1 \cdot e^{x^2+y^2}) + xy(e^{x^2+y^2} \cdot 2x)$

$= y e^{x^2+y^2} + 2x^2 y e^{x^2+y^2}$.

~~ii) f(x,y) =~~ $f_y = x e^{x^2+y^2} + xy(e^{x^2+y^2} \cdot 2y)$

$\therefore f_y = x e^{x^2+y^2} + 2xy^2 \cdot e^{x^2+y^2}$.

ii) $f(x,y) = e^x \cos y$.

$\Rightarrow f_x = e^x \cdot \cos y$.

$f_y = e^x \cdot (-\sin y)$

$= -e^x \sin y$.

$\therefore f_y = -e^x \sin y$

iii) $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$

$\Rightarrow f_x = 3x^2y^2 - 6xy$

$f_y = 2yx^3 - 3x^2 + 3y^2$.

Q3) Find value of f_x, f_y at $(0,0)$ for $f(x,y) = \frac{2x}{1+y^2}$.

$\Rightarrow F_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$

$= \lim_{h \to 0} \frac{2h - 0}{h} = \lim_{h \to 0} \frac{2h}{h} = 2$.

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Q4) Find all second order partial derivative of f. Also verify whether $f_{xy} = f_{yx}$.

i) $f(x,y) = \dfrac{y^2 - xy}{x^2}$.

$$f_x = \frac{x^2(0-y) - (y^2 - xy)2x}{x^4}$$

$$= \frac{-x^2y - 2xy^2 + 2x^2y}{x^4} = \frac{x^2y - 2xy^2}{x^4}$$

$$\cancel{= \frac{2x^2y - 2xy^2 - x^2y}{x^4} = \frac{2xy - 2y^2 - xy}{x^3}}$$

$$\cancel{f_{xx} = \frac{x^4(4xy - 2y^2 - 2x) - (2x^2y - 2xy^2 - x^2)(4x^3)}{x^4}}$$

$$=$$

$$\cancel{f_{xx} = \frac{x^3(2y - y) - (2xy - 2y^2 - xy)(3x^2)}{x^6}}$$

$$\cancel{= \frac{2x^3y - x^3y - 6x^3y + 6x^2y^2 + 3x^3y}{x^6}}$$

$$f_{xx} = \frac{x^4(2xy - 2y^2) - (x^2y - 2xy^2)(4x^3)}{x^8}$$

$$= \frac{2x^5y - 2x^4y^2 - 4x^5y + 8x^4y^2}{x^8}$$

$$= \frac{6x^4y^2 - 2x^5y}{x^8}$$

$$f_{yy} = \frac{2y - x}{x^2}.$$

$$f_{yy} = \frac{x^2(2) - (2y-x)(0)}{x^4}$$

$$= \frac{2x^2}{x^4} = \frac{2}{x^2}.$$

$$f_{xy} = \frac{x^2 - 4xy}{x^4}. \quad \text{— I}$$

$$f_{yx} = \frac{x^2(0-1) - (2y-x)(2x)}{x^4}$$

$$= \frac{-x^2 - 4xy + 2x^2}{x^4} = \frac{x^2 - 4xy}{x^4}. \quad \text{— II}$$

from I, II ---,

$f_{xy} = f_{yx}$.

ii) $f(x,y) = x^3 + 3x^2y^2 - \log(x^2+1)$

$\Rightarrow f_x = 3x^2 + 6xy^2 - \frac{2x}{(x^2+1)}$

$f_y = 6x^2y$.

$$f_{xx} = 6x + \cancel{6y^2} - \frac{(x^2+1)(2) - 2x(2x)}{(x^2+1)^2}$$

$$= 6x + 6y^2 - \frac{2x^2 + 2 - 4x^2}{(x^2+1)^2} = 6x + 6y^2 + \frac{2-2x^2}{(x^2+1)^2}.$$

$f_{yy} = 6x^2$.

$f_{xy} = 12xy$ — I

$f_{yx} = 12xy$ — II

from eq^n I & eq^n II,

$f_{xy} = f_{yx}$.

iii) $F(x,y) = \sin(xy) + e^{x+y}$.

$\Rightarrow$ $F_x = \cos(xy)(y) + e^{x+y}$, $\quad F_y = x\cos(xy) + e^{x+y}$.

$= y\cos(xy) + e^{x+y}$.

$F_{xx} = -y^2\sin(xy) + e^{x+y}$. $\quad f_{yy} = -x^2\cos(xy) + e^{x+y}$.

~~$f_{xy} = -y^2\cos(xy)$~~

$f_{xy} = -y^2\sin(xy) + \cos(xy) + e^{x+y}$ — I

$f_{yx} = -x^2\sin(xy) + \cos(xy) + e^{x+y}$. — II

from eq^n I & eq^n II, $f_{xy} \neq f_{yx}$.

Q5) i) $F(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$

$\Rightarrow$ $F(1,1) = \sqrt{1+1} = \sqrt{2}$.

$F_x = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}x) = \frac{x}{\sqrt{x^2+y^2}}$ $\quad f_y = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}y) = \frac{y}{\sqrt{x^2+y^2}}$

f_x at $(1,1) = \frac{1}{\sqrt{2}}$ $\quad f_y$ at $(1,1) = \frac{1}{\sqrt{2}}$.

$L(x,y) = F(a,b) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1+y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x+y-2)$

$= \cancel{\sqrt{2}} + \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} - \cancel{\sqrt{2}}$

$= \frac{x+y}{\sqrt{2}}$

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ii) $f(x,y) = 1 - x + y\sin x$ at $\left(\frac{\pi}{2}, 0\right)$

→ $f\left(\frac{\pi}{2}, 0\right) = 1 - \frac{\pi}{2}$.

$f_x = -1 + y\cos x$.
$= y\cos x - 1$

$f_y = \sin x$.

$f_x\left(\frac{\pi}{2}, 0\right) = 0 - 1$
$= -1$

$f_y\left(\frac{\pi}{2}, 0\right) = 1$.

$L(a,b) = f\left(\frac{\pi}{2}, 0\right) + f_x\left(\frac{\pi}{2}, 0\right)\left(x - \frac{\pi}{2}\right) + f_y\left(\frac{\pi}{2}, 0\right)(y - 0)$

$= 1 - \frac{\pi}{2} + (-1)\left(x - \frac{\pi}{2}\right) + 1(y)$

$= 1 - \cancel{\frac{\pi}{2}} - x + \cancel{\frac{\pi}{2}} + y = 1 - x + y$.

iii) $f(x,y) = \log x + \log y$ at $(1,1)$

→ $f(1,1) = 0 + 0 = 0$.

$f_x = \frac{1}{x}$

$f_y = \frac{1}{y}$.

$f_x(1,1) = \frac{1}{1} = 1$

$f_y(1,1) = \frac{1}{1} = 1$.

$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$
$= 0 + 1(x-1) + 1(y-1)$
$= x - 1 + y - 1 = x + y - 2$.

PRACTICAL-10

Aim :- Directional derivative, minima & maxima.

Q1) Find the directional derivative :-

i) $F(x,y) = x+2y-3$, $a=(1,-1)$, $u = 3i-j$.

$u = 3i - j$

$|\bar{u}| = \sqrt{3^2+1^2} = \sqrt{9+1} = \sqrt{10}$.

Unit vector along $u = \frac{u}{|u|} = \frac{1}{\sqrt{10}}(3,-1) = \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$F(a+hu) = F(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$F(a) = F(1,-1) = 1+2(-1)-3 = 1-2-3 = -4$.

$f(a+hu) = F(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$= F\left[\left(1+\frac{3h}{\sqrt{10}}\right), \left(-1-\frac{h}{\sqrt{10}}\right)\right]$

$f(a+hu) = \left(1+\frac{3h}{\sqrt{10}}\right) + 2\left(-1-\frac{h}{\sqrt{10}}\right) - 3$.

$= 1 + \frac{3h}{\sqrt{10}} - 2 - \frac{2h}{\sqrt{10}} - 3$.

$= -4 + \frac{h}{\sqrt{10}}$

$D_u f(a) = \lim_{h\to 0} \frac{f(a+hu) - f(a)}{h}$

$= \lim_{h\to 0} \frac{\cancel{-4} + h/\sqrt{10} + \cancel{4}}{h} = \frac{1}{\sqrt{10}}$.

ii) $F(x,y) = y^2 - 4x + 1$ at $(3,4)$, $u = i + 5j$

$u = i + 5j$

$\rightarrow |\bar{u}| = \sqrt{1+25} = \sqrt{26}$

unit vector $= \frac{1}{\sqrt{26}}(1,5) = \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

$f(3,4) = 16 - 12 + 1 = 5.$

$F(a+hu) = f(3,4) + h\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

$= f\left(3 + \frac{h}{\sqrt{26}}, 4 + \frac{5h}{\sqrt{26}}\right).$

$f\left(3+\frac{h}{\sqrt{26}}, 4+\frac{5h}{\sqrt{26}}\right) = \left(4+\frac{5h}{\sqrt{26}}\right)^2 - 4\left(3+\frac{h}{\sqrt{26}}\right) + 1$

$= 16 + \frac{40h}{\sqrt{26}} + \frac{25h^2}{26} - 12 - \frac{4h}{\sqrt{26}} + 1.$

$= 5 + \frac{36h}{\sqrt{26}} + \frac{25h^2}{26}.$

$D_u F(a) = \lim_{h\to 0} \frac{f(a+hu) - f(a)}{h}.$

$= \lim_{h\to 0} \frac{\frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + \cancel{5} - \cancel{5}}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}\left(\frac{25h}{26} + \frac{36}{\sqrt{26}}\right)}{\cancel{h}}$

$= \frac{25h}{26} + \frac{36}{\sqrt{26}}.$

iii) $2x+3y$ $a=(1,2)$, $u=3i+4j$

$\Rightarrow$ $u=3i+4j$

$|\bar{u}| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$

Unit vector along $u = \frac{1}{5}(3,4) = \left(\frac{3}{5}, \frac{4}{5}\right)$.

$F(a) = F(1,2) = 2+6 = 8$.

$F(a+hu) = F(1,2) + h\left(\frac{3}{5}, \frac{4}{5}\right)$

$= F\left(1 + \frac{3h}{5}, 2 + \frac{4h}{5}\right)$.

$F(a+hu) = 2\left(1+\frac{3h}{5}\right) + 3\left(2+\frac{4h}{5}\right)$

$= 2 + \frac{6h}{5} + 6 + \frac{12h}{5} = 8 + \frac{18h}{5}$.

$$D_u F(a) = \lim_{h\to 0} \frac{\frac{18h}{5} + 8 - 8}{h}$$

$= \frac{18}{5}$.

Q2) i) find gradient vector:-

i) $F(x,y) = x^y + y^x$, $a=(1,1)$.

$\Rightarrow$ $F_x = yx^{y-1} + y^x \log y$.

$F_y = x^y \log x + xy^{x-1}$.

$\nabla F(x,y) = (F_x, F_y)$

$= (yx^{y-1} + y^x \log y, x^y \log x + xy^{x-1})$.

$F(1,1) = (1+0, 1+0) = (1,1)$

ii) $F(x,y) = [\tan^{-1}(x)] \cdot y^2$, $a=(1,-1)$

$\Rightarrow$ $F_x = \frac{1}{1+x^2} \cdot y^2$

$F_y = 2y \tan^{-1}(x)$ -

$\nabla f(x,y) = (f_x, f_y)$

$= \left[\frac{y^2}{1+x^2}, 2y\tan^{-1}(x)\right]$

$f(1,-1) = \left[\frac{1}{2}, \tan^{-1}(1)(-2)\right]$

$= \left(\frac{1}{2}, \frac{\pi}{4}(-2)\right) = \left(\frac{1}{2}, -\frac{\pi}{2}\right)$.

iii) $f(x,y,z) = xyz - e^{x+y+z}$, $a = (1,-1,0)$.

$f_x = yz - e^{x+y+z}$.

$f_y = xz - e^{x+y+z}$.

$f_z = xy - e^{x+y+z}$.

$\nabla f(x,y,z) = (f_x, f_y, f_z)$

$= (yz - e^{x+y+z}, xz - e^{x+y+z}, xy - e^{x+y+z})$.

$f(1,-1,0) = [0 - e^0, 0 - e^0, -1 - e^0] = (-1,-1,-2)$.

Q3) Find the equation of tangent and normal :-

i) $x^2\cos y + e^{xy} = 2$ at $(1,0)$.

→ $f_x = \cos y(2x) + e^{xy}(y)$.

$f_y = x^2(-\sin y) + e^{xy}(x)$.

$(x_0, y_0) = (1,0)$ ∴ $x_0 = 1$, $y_0 = 0$.

Eqn of tangent :- $f_x(x - x_0) + f_y(y - y_0) = 0$.

i.e.

$f_x(x_0, y_0) = \cos 0 \times 2(1) + e^0 \times 0$.

$= 1(2) = 2$.

$f_y(x_0, y_0) = 1^2(-\sin 0) + e^0 \times 1 = 0 + 1 \times 1 = 1$.

$\therefore$ Eq^n of tangent :- $2(x-1) + 1(y-0) = 0$.
$2x - 2 + y = 0$.

Eq^n of normal :- $ax + by + c = 0$.
$\therefore bx + ay + d = 0$.
$\therefore 1(1) + 2(0) + d = 0$.
$\therefore 1 + d = 0$.
$\therefore d = -1$.

ii) $x^2 + y^2 - 2x + 3y + 2 = 0$. at $(2, -2)$
$\rightarrow$ $f_x = 2x + 0 - 2 + 0 + 0$.
$= 2x - 2$.
$f_y = 0 + 2y + 3 = 2y + 3$
$(x_0, y_0) = 2, -2 \Rightarrow x_0 = 2, y_0 = -2$.
$f_x(x_0, y_0) = 2(2) - 2 = 4 - 2 = 2$.
$f_y(x_0, y_0) = 2(-2) + 3 = -4 + 3 = -1$.

Eq^n of tangent :- $f_x(x - x_0) + f_y(y - y_0) = 0$.
$\therefore 2(x-2) + (-1)(y+2) = 0$.
$\therefore 2x - 4 - y - 2 = 0$.
$\therefore 2x - y - 6 = 0$.

Eq^n of Normal :- $ax + by + c = 0$.
$bx + ay + d = 0$.
$\therefore -1(x) + 2(y) + d = 0$
$\therefore -x + 2y + d = 0$.
$\therefore -2 + 2(-2) + d = 0$ at $(2, -2)$
$\therefore -2 - 4 + d = 0$
$\therefore -6 + d = 0$
$\therefore d = 6$.

55) Find maxima and ~~max~~ minima:-

i) $f(x,y) = 3x^2 + y^2 - 3xy + 6x - 4y$.

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$f(x) = 6x + 0 - 3y + 6 - 0$

$= 6x - 3y + 6$

$f_y = 0 + 2y - 3x + 0 - 4$

$= 2xy - 3x - 4$.

$f_x = 0$

$6x - 3y + 6 = 0$

$3(2x - y + 2) = 0$.

$2x - y + 2 = 0$

$2x - y = -2$ —I.

$f_y = 0$

$2y - 3x - 4 = 0$

$2y - 3x = 4$ —II.

Multiply eq^n I with 2

$4x - 2y = -4$

$2y - 3x = 4$

$x = 0$.

Put $x = 0$ in eq^n I

$2(0) - y = -2$

$-y = -2$

$y = 2$.

∴ Critical points are $(0,2)$.

$r = f_{xx} = 6$

$t = f_{yy} = 2$

$s = f_{xy} = -3$

Here $r > 0$

$= rt - s^2$

$= 6(2) - (-3)^2$

$= 12 - 9 = 3 > 0$

$\therefore f$ has maximum at $(0,2)$.

$\therefore f$ has maximum $3x^2+y^2-3xy+6x-4y$ at $(0,2)$.

$3(0)+2^2-3(0)(2)+6(0)-4(2)$

$0+4-0+0-8=-4$.

ii) $f(x,y)=2x^4+3x^2y-y^2$.

$f_x=8x^3+6xy$.

$f_y=3x^2-2y$.

$f_x=0$.

$\therefore 8x^3+6xy=0$.

$\therefore 4x^2+3y=0$. — I.

$f_y=0$.

$3x^2-2y=0$ — II.

Multiply eq^n I with 2.

eq^n II with 3

$\cancel{12x^2+6y=8}$

$\cancel{-12x^2-8y=0}$

$8x^2+6y=0$

$9x^2-6y=0$

$17x^2=0$

$x=0$.

Put $x=0$ in eq^n I.

$3y=0$.

$y=0$.

Critical points are $(0,0)$

$r=f_{xx}=24x^2+6x$.

$t=f_{yy}=0-2=-2$.

$s=f_{xy}=6x-0=6x=0$.

r at $(0,0)$

$r=24(0)+6(0)$

$=0$.

$rt - s^2 = 0(-2) - 0^2$
$= 0 - 0 = 0.$

$f(x,y)$ at $(0,0)$
$= 2(0) + 3(0)(0) - 0$
$= 0.$

iii) $f(x,y) = x^2 - y^2 + 2x + 8y - 70.$

$f_x = 2x + 2.$

$f_y = -2y + 8$

$f_x = 0$

$\therefore 2x + 2 = 0.$

$x = -1$

$f_y = 0.$

$-2y = -8$

$y = 4.$

$\therefore$ Critical points are $(-1,4)$.

$r = f_{xx} = 2.$

$t = f_{yy} = -2.$

$s = f_{xy} = 0.$

$r > 0.$

$rt - s^2 = 2(-2) - 0^2$
$= -4 - 0 = -4 < 0.$

$f(x,y)$ at $(-1,4)$
$= (-1)^2 - 4^2 + 2(-1) + 8(4) - 70.$
$= 1 - 16 - 2 + 32 - 70$
$= -15 + 30 - 70.$
$= 15 - 70.$
$= -55.$

$\therefore f$ has minimum value at $(-1,4)$.

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