CMSC 651, Spring 2018, University of Maryland
Mid-term exam, due as PDF to the address *cmsc651.umd@gmail.com* by 11:59 **AM** (i.e., just before noon) on March 16, 2018

Notes:

(i) Please work on this **by yourself (not with your group)**. You can consult *all the notes under "Resources" in Piazza*; consulting anyone or other sources (including the Web) is not allowed.

(ii) Write your solutions neatly and *include your name*; if you are able to make partial progress by making some additional assumptions, then **state these assumptions clearly and submit your partial solution**.

(iii) **Please make the subject line of your email** "*651 Mid-term*" followed by your full name, and email a PDF to *cmsc651.umd@gmail.com*: PDF generated from Word or LaTeX strongly encouraged.

1. Let X and Y be *independent* random variables with Y never zero, such that $\mathbf{E}[X] = a$, $\mathbf{E}[Y] = b$, and $\mathbf{E}\left[\frac{1}{Y}\right] = c$. What is $\mathbf{E}\left[\frac{X}{Y}\right]$? Justify your answer. **(5 points)**

2. Let $p_n(k)$ be the number of permutations of the set $\{1, 2, \ldots, n\}$, which have exactly k fixed points. Prove that $\sum_{k=0}^{n} k \cdot p_n(k) = n!$. (An element i is called a fixed point of the permutation f if $f(i) = i$.) **Hint: Use the linearity of expectation. (10 points)**

3. While greedy algorithms are often good when working with a single objective function, they are usually not so in the presence of *multiple* objective functions: randomization often helps here.

Suppose we have a set-cover instance as in Section 1.2 of Williamson-Shmoys, but now with *two* weight functions:

- we have two non-negative weights v_j and w_j for each given S_j;
- we are given reals V and W and are promised that there exists some cover I (i.e., as in Williamson-Shmoys, $I \subseteq \{1, 2, \ldots, m\}$ is such that $\bigcup_{j \in I} S_j = E$), with $\sum_{j \in I} v_i \leq V$ and $\sum_{j \in I} w_i \leq W$.

Our task is to efficiently find a cover that is "good" with respect to both weight functions. Give a randomized polynomial-time algorithm that finds (with high probability) a cover $C \subseteq \{1, 2, \ldots, m\}$ such that $\bigcup_{j \in C} S_j = E$, with $\sum_{j \in C} v_i \leq 3V \ln n$ and $\sum_{j \in C} w_i \leq 3W \ln n$; here, $n = |E|$ as in Williamson-Shmoys. **(15 points)**

4. The proof of Theorem 3.7 in Williamson-Shmoys required a rounding of p_j for all long jobs j: we implicitly constructed the intervals $[T/k^2, 2T/k^2)$, $[2T/k^2, 3T/k^2)$, $[3T/k^2, 4T/k^2), \ldots$, and whichever interval p_j fell in, we rounded p_j to the left end-point of this interval. Note that all interval-lengths here are the same (i.e., equal to T/k^2). Design a different rounding-down scheme for the p_j in which these interval-lengths are approximately geometrically increasing (i.e., the i^{th} interval has length roughly $a \cdot b^i$, where $b > 1$) to design a faster PTAS: one in which the $n^{O(k^2)}$ term in the run-time decreases to $n^{O(k \log k)}$. **(10 points)**