CMSC651 Assignment 4

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Problem 1

Consider the following algorithm: initially, we have n elements to add up. Task processor i for $i \in \{1, 2, \ldots, \lceil n/2 \rceil\}$ with adding up elements $2i-1$ and $2i$ (and possibly only $2i-1$ for the last processor if n is odd). Now we add these $\lceil n/2 \rceil$ partial sums outputted by each processor in a similar fashion: task processor $i \in \{1, 2, \ldots, \lceil \lceil n/2 \rceil / 2 \rceil\}$ with adding up the partial sums $2i$ and $2i-1$. We continue this process until we have 2 partial sums remaining, at which point a single processor adds these up and then outputs the final sum.

Note that at each step we reduce the problem size by 2. Moreover, at each step, each processor executes $\mathcal{O}(1)$ operations, all of which without any memory contention. Thus each step is done in $\mathcal{O}(1)$ time on a EREW PRAM. Thus the total runtime of the algorithm is dictated by $T(n) = T(n/2) + \mathcal{O}(1),\ T(1) = \mathcal{O}(1)$. We can see that this is $\mathcal{O}(\log(n))$ as follows: Suppose that $k = \log_2(n)$. Then we have:

$$\begin{aligned} T(n) &= T(2^k) = T(2^{k-1}) + \mathcal{O}(1) = T(2^{k-2}) + 2 \cdot \mathcal{O}(1) = \cdots = T(2^0) + k \cdot \mathcal{O}(1) = (k+1) \cdot \mathcal{O}(1) \\ &= \mathcal{O}(k) = \mathcal{O}(\log(n)). \end{aligned}$$

Problem 2

Let N be the number of vertices in the longest path in G. We show $\ell = N$ by showing $\ell \leq N$ and $\ell \geq N$.

First, observe that at iteration i of the while loop, each of the vertices $v^{(i)}$ we remove is part of a path consisting of at least i vertices in our original graph G. To see this, note that $v^{(i)}$ must have not been a source in any previous iteration (otherwise we would have removed it then). Moreover, this means that at the previous iteration, there was an edge to $v^{(i)}$ from one of the vertices $v^{(i-1)}$ we removed in iteration $i-1$. By similar reasoning, we see that we must also have an edge to $v^{(i-1)}$ from one of the vertices $v^{(i-2)}$ removed in iteration $i-2$. Tracing this back all the way to iteration 1, we see that G must have a path with vertices $v^{(1)}, v^{(2)}, \ldots, v^{(i)}$, hence $v^{(i)}$ is part of a path consisting of at least i vertices. Since A partitions G into $L(1), L(2), \ldots, L(\ell)$, then it must have reached iteration ℓ. Therefore there must be a path consisting of at least ℓ vertices in G. Hence, by definition, the longest path in G must also have at least ℓ vertices, so $\ell \leq N$.

Next, suppose $\ell < N$. Thus there exists a path P consisting of vertices $v_1, v_2, \ldots, v_N$ in G, where v_1 is a source and v_N a sink. If A terminates after iteration ℓ, then all of these vertices were removed in these iterations. But note that in the first iteration, v_1 is the only vertex in P that can be removed. Moreover, this removal can only make v_2 a source, as there are still edges to $v_i,\ i \in \{3, 4, \ldots, N\}$ from v_{i-1}. Hence in the second iteration, only v_2 can be removed, which by the same logic can only make v_3 a source. Thus we see that after iteration ℓ, at best we have removed $v_1, v_2, \ldots, v_\ell$. Since $\ell < N$, v_N must not have been removed, which contradicts the termination condition of A. Hence $\ell \geq N$.

We have shown $\ell \leq N$ and $\ell \geq N$, thus it must be that $\ell = N$.

Problem 3

(a) Given that $M = (S, \mathcal{I})$ is a matroid, we show that the three properties of a matroid hold for $M^* = (S, \mathcal{I}^*)$.

(i) $\emptyset$ is clearly a subset of S, and $S \setminus \emptyset = S$, which clearly contains a basis for M. Hence $\emptyset \in \mathcal{I}$.

(ii) Suppose $A \in \mathcal{I}^*$ and $B \subseteq A$. By the definition of $\mathcal{I}^*$, we know that $S \setminus A$ contains a basis of M. Since $B \subseteq A$, $S \setminus B$ contains that same basis of M, and therefore $B \in \mathcal{I}^*$ by the definition of $\mathcal{I}^*$.

(iii) Suppose $A, B \in \mathcal{I}^*$ and $|B| > |A|$. Then $S \setminus A$ has a basis and $S \setminus B$ has a basis. There are two possibilities we wish to consider:

(1) Suppose there exists a basis in $S \setminus A$, for which it fails to contain some element $x^* \in B \setminus A$. Then if we take $A \cup \{x^*\}$, clearly $S \setminus A \cup \{x^*\}$ has a basis, namely the one we started with originally, and so $A \cup \{x^*\} \in \mathcal{I}$ and we are done.

(2) Suppose that every basis in $S \setminus A$ contains $B \setminus A$. We will prove there is a contradiction, and hence that this case is impossible.

Let $\mathcal{B}_1$ be a basis in $S \setminus A$. Let $\mathcal{B}_2$ be a basis in $S \setminus B$. By our supposition, $\mathcal{B}_1 \supseteq (B \setminus A)$. Let $D := (A \setminus B) \cap \mathcal{B}_2$, in other words, simply all elements in the basis $\mathcal{B}_2$ which lie in A (since by assumption, $\mathcal{B}_2$ has already excluded elements of B). By the fact that B is bigger than A, $|B \setminus A| > |A \setminus B|$, which then clearly implies that $|B \setminus A| > |D|$.

We perform an iterative process: we remove each element in D from our basis $\mathcal{B}_2$. Using the third property of matroids, we replace each with an element from $\mathcal{B}_1$ (that doesn't already exist in $\mathcal{B}_2$). Each element from $\mathcal{B}_1$ will necessarily be in $S \setminus A$. So at the end of this iterative process, the basis $\mathcal{B}_2$ will no longer contain any elements from A anymore, and so $\mathcal{B}_2$ will also be a basis in $S \setminus A$.

Since the iterative process proceeded only D times and $\mathcal{B}_1$ contains all of the comparatively greater number of elements in $B \setminus A$, there must exist an element $x' \in (B \setminus A)$ which lies in $\mathcal{B}_1$ but which was not introduced into $\mathcal{B}_2$. We remove x' from $\mathcal{B}_1$ and, using the third property of matroids, replace it with x'' from $\mathcal{B}_2$ (that doesn't already exist in $\mathcal{B}_1$). First note that $x'' \in S \setminus A$ since we showed above that $\mathcal{B}_2$ is now a basis in $S \setminus A$ after the iterative process concluded. Furthermore, $x'' \neq x'$ since $x' \notin \mathcal{B}_2$. Hence, our newly produced basis $\mathcal{B}_1^*$ is both a basis in $S \setminus A$, but is also missing an element $x' \in (B \setminus A)$, which contradicts our original assumption that every basis in $S \setminus A$ contains $B \setminus A$ in its entirety.

(b) We consider two cases:

(1) Suppose T is independent, or $T \in \mathcal{I}^*$. Then, by definition of $\mathcal{I}^*$, $S \setminus T$ contains a basis of M, so $r(S \setminus T) = r(S)$. According to the definition of the rank function, $r^*(T)$ is the size of the largest subset of T which is independent, which in this case we know is $|T|$. On the right side of the equation, we verify indeed that $|T| - r(S) + r(S \setminus T) = |T| - 0 = |T|$ and hence the equation holds.

(2) Suppose T is dependent, or $T \notin \mathcal{I}^*$. Then by definition of $\mathcal{I}^*$, $S \setminus T$ does not contain a basis of M. Now consider that $r(S)$ refers to the size of a basis of M, and $r(S \setminus T)$ refers to the largest independent subset of $S \setminus T$. Since by the discussion in the textbook we are always able to iteratively build up a basis from any independent set, $r(S) - r(S \setminus T)$ is therefore the number of elements we would need to (and are able to) move over from T to $S \setminus T$ in order to give $S \setminus T$ a basis. So, the size of the largest subset of T which would qualify to be in $\mathcal{I}^*$ – which is by definition referred to as $r^*(T)$ – would then be equal to the total number of elements of T minus the number of elements we would need to move over, or expressed as $T - (r(S) - r(S \setminus T)) = T - r(S) + r(S \setminus T)$. And so the equation holds here.