COMPUTATIONAL FINANCE & RISK MANAGEMENT

UNIVERSITY *of* WASHINGTON

Department of Applied Mathematics

AMATH 460: Mathematical Methods for Quantitative Finance

5. Linear Algebra I

Kjell Konis

Acting Assistant Professor, Applied Mathematics

University of Washington

Outline

Outline

Vectors

- Portfolio: w_1 shares of asset 1, w_2 shares of asset 2
- Think of this pair as a two-dimensional vector

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = (w_1, w_2)$$

- The numbers w_1 and w_2 are the components of the column vector w
- A second portfolio has u_1 shares of asset 1 and u_2 shares of asset 2; the combined portfolio has

$$u + w = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} u_1 + w_1 \\ u_2 + w_2 \end{bmatrix}$$

- Addition for vectors is defined component-wise

Vectors

- Doubling a vector

$$2w = w + w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1 + w_1 \\ w_2 + w_2 \end{bmatrix} = \begin{bmatrix} 2w_1 \\ 2w_2 \end{bmatrix}$$

- In general, multiplying a vector by a <u>scalar</u> value c

$$cw = \begin{bmatrix} cw_1 \\ cw_2 \end{bmatrix}$$

- A <u>linear combination</u> of vectors u and w

$$c_1 u + c_2 w = \begin{bmatrix} c_1 u_1 \\ c_1 u_2 \end{bmatrix} + \begin{bmatrix} c_2 w_1 \\ c_2 w_2 \end{bmatrix} = \begin{bmatrix} c_1 u_1 + c_2 w_1 \\ c_1 u_2 + c_2 w_2 \end{bmatrix}$$

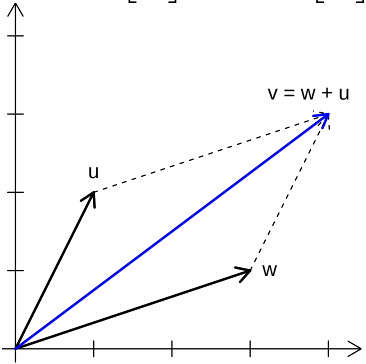
- Setting $c_1 = 1$ and $c_2 = -1$ gives vector subtraction

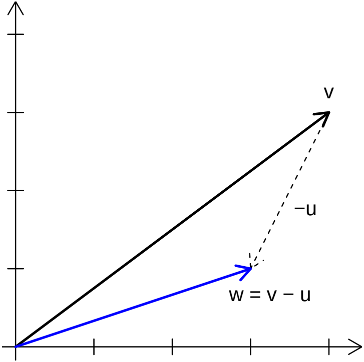
$$u - w = 1u + -1w = \begin{bmatrix} 1u_1 + -1w_1 \\ 1u_2 + -1w_2 \end{bmatrix} = \begin{bmatrix} u_1 - w_1 \\ u_2 - w_2 \end{bmatrix}$$

Visualizing Vector Addition

- The vector $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ is drawn as an arrow with the tail at the origin and the pointy end at the point (w_1, w_2)

- Let $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $w = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$





Vector Addition

- Each component in the sum is $u_i + w_i = w_i + u_i \implies u + w = w + u$
- e.g., $\begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$
- The zero vector has $u_i = 0$ for all i, thus $w + 0 = w$
- For $2u$: preserve direction and double length
- $-u$ is the same length as u but points in opposite direction

Dot Products

- The dot product or inner product of $u = (u_1, u_2)$ and $w = (w_1, w_2)$ is the scalar quantity

$$u \cdot w = u_1 w_1 + u_2 w_2$$

- Example: letting $u = (1, 2)$ and $w = (3, 1)$ gives

$$u \cdot v = (1, 2) \cdot (3, 1) = 1 \times 3 + 2 \times 1 = 5$$

- Interpretation:
  - w is a position in assets 1 and 2
  - Let $p = (p_1, p_2)$ be the prices of assets 1 and 2

$$V = w \cdot p = w_1 p_1 + w_2 p_2$$

value of portfolio

Outline

- The length of a vector w is

$$\text{length}(w) = \|w\| = \sqrt{w \cdot w}$$

- Example: suppose $w = (w_1, w_2, w_3, w_4)$ has 4 components

$$\|w\| = \sqrt{w \cdot w} = \sqrt{w_1^2 + w_2^2 + w_3^2 + w_4^2}$$

- The length of a vector is positive except for the zero vector, where

$$\|0\| = 0$$

- A unit vector is a vector whose length is equal to one: $u \cdot u = 1$
- A unit vector having the same direction as $w \neq 0$

$$\tilde{w} = \frac{w}{\|w\|}$$

Vector Norms

- A norm is a function $\|\cdot\|$ that assigns a *"length"* to a vector
- A norm must satisfy the following three conditions

  i) $\|x\| \geq 0$ and $\|x\| = 0$ only if $x = 0$

  ii) $\|x + y\| \leq \|x\| + \|y\|$

  iii) $\|ax\| = |a|\,\|x\|$ where a is a real number

- Important class of vector norms: the *p*-norms
  - $$\|x\|_p = \left(\sum_{i=1}^{m} |x_i|^p\right)^{\frac{1}{p}} \qquad (1 \leq p < \infty)$$
  - $$\|x\|_\infty = \max_{1 \leq i \leq m} |x_i|$$
  - $$\|x\|_1 = \sum_{i=1}^{m} |x_i|$$

The Angle Between Two Vectors

- The dot product $u \cdot w$ is zero when v is perpendicular to w
- The vector $v = \big(\cos(\phi), \sin(\phi)\big)$ is a unit vector

$$\|v\| = \sqrt{v \cdot v} = \sqrt{\cos^2(\phi) + \sin^2(\phi)} = 1$$

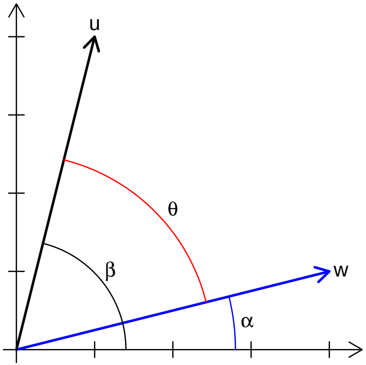
- Let $u = (1, 4)$, then

$$\tilde{u} = \frac{u}{\|u\|} = \big(\cos(\beta), \sin(\beta)\big)$$

- Let $w = (4, 1)$, then

$$\tilde{w} = \frac{w}{\|w\|} = \big(\cos(\alpha), \sin(\alpha)\big)$$

- <u>Cosine Formula</u>: $\tilde{u} \cdot \tilde{w} = \cos(\theta)$
- <u>Schwarz Inequality</u>: $|u \cdot w| \leq \|u\|\|w\|$



Planes

- So far, 2-dimensional
- Everything (dot products, lengths, angles, etc.) works in higher dimensions too
- A plane is a 2-dimensional sheet that lives in 3 dimensions
- Conceptually, pick a normal vector n and define the plane P to be all vectors perpendicular to n
- If a vector $v = (x, y, z) \in P$ then

$$n \cdot v = 0$$

- However, since $n \cdot 0 = 0, \quad 0 \in P$
- The equation of the plane passing through $v_0 = (x_0, y_0, z_0)$ and normal to n is

$$n \cdot (v - v_0) = n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

Planes (continued)

- Every plane normal to n has a linear equation with coefficients n_1, n_2, n_3:

$$n_1 x + n_2 y + n_3 z = n_1 x_0 + n_2 y_0 + n_3 z_0 = d$$

- Different values of d give parallel planes
- The value $d = 0$ gives a plane through the origin

Outline

Systems of Linear Equations

- Want to solve 3 equations in 3 unknowns

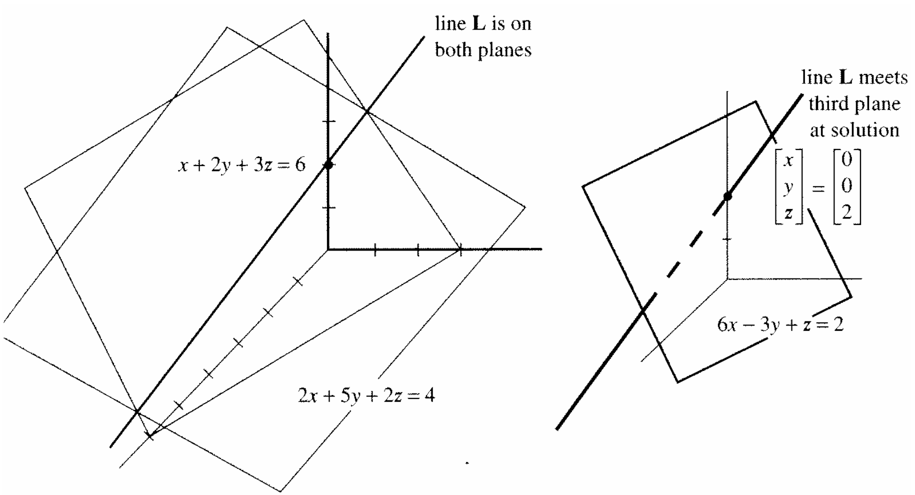
$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 5y + 2z &= 4 \\ 6x - 3y + z &= 2 \end{aligned}$$

- Row picture:

$$\begin{aligned} (1,2,3)\cdot(x,y,z) &= 6 \\ (2,5,2)\cdot(x,y,z) &= 4 \\ (6,-3,1)\cdot(x,y,z) &= 2 \end{aligned}$$

- Column picture:

$$x \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + z \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$


line **L** is on
both planes
$x + 2y + 3z = 6$
$2x + 5y + 2z = 4$
line **L** meets
third plane
at solution
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$
$6x - 3y + z = 2$

Matrix Form

- Stacking rows or binding columns gives the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}$$

- Matrix notation for the system of 3 equations in 3 unknowns

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} \quad \text{is} \quad Av = b$$

where $v = (x, y, z)$ and $b = (6, 4, 2)$

- The left-hand side multiplies A times the unknowns v to get b
- Multiplication rule must give a correct representation of the original system

Matrix-Vector Multiplication

- Row picture multiplication

$$Av = \begin{bmatrix} (\text{row } 1) \cdot v \\ (\text{row } 2) \cdot v \\ (\text{row } 3) \cdot v \end{bmatrix} = \begin{bmatrix} (\text{row } 1) \cdot (x, y, z) \\ (\text{row } 2) \cdot (x, y, z) \\ (\text{row } 3) \cdot (x, y, z) \end{bmatrix}$$

- Column picture multiplication

$$Av = x\,(\text{column } 1) + y\,(\text{column } 2) + z\,(\text{column } 3)$$

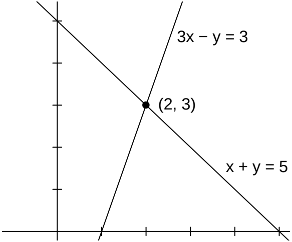
- Examples:

$$Av = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \qquad Av = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

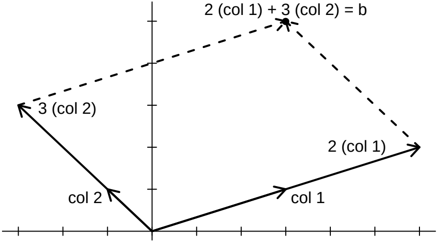
Example

$$\begin{aligned} 3x - y &= 3 \\ x + y &= 5 \end{aligned} \quad \Longleftrightarrow \quad \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Row Picture



Column Picture



Rows $\quad \begin{aligned} 3\,(2) - (3) &= 3 \\ (2) + (3) &= 5 \end{aligned}$

Columns $2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

Matrix $\quad \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

Systems of Equations

- For an equal number of equations and unknowns, there is *usually* one solution
- Not guaranteed, in particular there may be
  - no solution (e.g., when the lines are parallel)
  - infinitely many solutions (e.g., two equations for the same line)

Outline

Elimination

- Want to solve the system of equations

$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned}$$

- High school algebra approach
  - solve for x: $\quad x = 2y + 1$
  - eliminate x: $\quad 3(2y + 1) + 2y = 11$
  - solve for y: $\quad 8y + 3 = 11 \implies y = 1$
  - solve for x: $\quad x = 3$

$$\begin{aligned} 1x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned}$$

Terminology

Pivot	The first nonzero in the equation (row) that does the elimination
Multiplier	(number to eliminate) / (pivot)

- How was x eliminated?

$$\begin{array}{r} 3x + 2y = 11 \\ \underline{-3[1x - 2y = 1\ ]} \\ 0x + 8y = 8 \end{array}$$

- Elimination: subtract a multiple of one equation from another
- Idea: use elimination to make an <u>upper triangular system</u>

Elimination

- An upper triangular system of equations

$$\begin{aligned} 1x - 2y &= 1 \\ 0x + 8y &= 8 \end{aligned}$$

- Solve for x and y using <u>back substitution</u>:
  - solve for y
  - use y to solve for x

Elimination Using Matrices

- The system of 3 equations in 3 unknowns can be written in the matrix form $Ax = b$

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 2 \\ 4x_1 + 9x_2 - 3x_3 &= 8 \\ -2x_1 - 3x_2 + 7x_3 &= 10 \end{aligned} \qquad \sim \qquad \underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}}_{b}$$

- The unknown is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and the solution is $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

- $Ax = b$ represents the row form and the column form of the system
- Can multiply Ax a column at a time

$$Ax = (-1)\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + 2\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

Elimination Using Matrices

- Can represent the original equation as $Ax = b$
- What about the elimination steps?
- Start by subtracting 2 times first equation from the second
- Use elimination matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- The right-hand side Eb becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$$

Two Important Matrices

- The <u>identity matrix</u> has 1's on the diagonal and 0's everywhere else

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- The elimination matrix that subtracts a multiple l of row j from row i has an additional nonzero entry $-l$ in the i, j position

$$E_{3,1}(l) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{bmatrix}$$

- Examples

$$E_{2,1}(2)b = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix} \qquad Ib = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Outline

Matrix Multiplication

- Have linear system $Ax = b$ and elimination matrix E
- One elimination step (introduce one 0 below the diagonal)

$$EAx = Eb$$

- Know how to do right-hand-side
- Since E is an elimination matrix, also know answer to EA
- Column view:
  - The matrix A is composed of n columns $a_1, a_2, \ldots, a_n$
  - The columns of the product EA are

$$EA = \left[Ea_1, Ea_2, \ldots, Ea_n\right]$$

Rules for Matrix Operations

- A matrix is a rectangular array of numbers
- An $m \times n$ matrix A has m rows and n columns
- The entries are denoted by a_{ij}

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

- Matrices can be added when their dimensions are the same
- A matrix can be multiplied by a scalar value c

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 4 & 4 \\ 9 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 7 & 8 \\ 9 & 9 \end{bmatrix} \qquad 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 0 & 0 \end{bmatrix}$$

Rules for Matrix Multiplication

- Matrix multiplication a bit more difficult
- To multiply a matrix A times a matrix B

$$\#\text{ columns of } A = \#\text{ rows of } B$$

- Let A be an $m \times n$ matrix and B an $n \times p$ matrix

$$\underbrace{\begin{bmatrix} \textbf{m rows} \\ n \text{ columns} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} n \text{ rows} \\ \textbf{p columns} \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} \textbf{m rows} \\ \textbf{p columns} \end{bmatrix}}_{AB}$$

- The dot product is extreme case, let $u = (u_1, u_2)$ and $w = (w_1, w_2)$

$$u \cdot w = u^{\mathsf{T}} w = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = u_1 w_1 + u_2 w_2$$

Matrix Multiplication

- The matrix product AB contains the dot products of the rows of A and the columns of B

$$(AB)_{ij} = (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B)$$

- Matrix multiplication formula, let $C = AB$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

- Example

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$$

- Computational complexity: $\{n$ multiplications, $n-1$ additions$\}$ / cell

Matrix Multiplication

- An inner product is a row times a column
- A column times a row is an outer product

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 9 & 6 & 3 \end{bmatrix}$$

- Each column of AB is a linear combination of the columns of A

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{A} \begin{bmatrix} \text{column } j \text{ of } B \end{bmatrix} = \begin{bmatrix} \text{column } j \text{ of } AB \end{bmatrix}$$

- Rows of AB are linear combinations of the rows of B

$$\begin{bmatrix} \text{row } i \text{ of } A \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{B} = \begin{bmatrix} \text{row } i \text{ of } AB \end{bmatrix}$$

Laws for Matrix Operations

Laws for Addition

1. $A + B = B + A$ (commutative law)
2. $c(A + B) = cA + cB$ (distributive law)
3. $A + (B + C) = (A + B) + C$ (associative law)

Laws for Multiplication

1. $C(A + B) = CA + CB$ (distributive law from left)
2. $(A + B)C = AC + BC$ (distributive law from right)
3. $A(BC) = (AB)C$ (associative law; parentheses not needed)

Laws for Matrix Operations

- Caveat: there is one law we don't get

$$AB \neq BA \qquad \text{(in general)}$$

- BA exists only when $p = m$
- If A is an $m \times n$ matrix and B is $n \times m$
  - AB is an $m \times m$ matrix
  - BA is an $n \times n$ matrix
- Even when A and B are square matrices ...

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{but} \quad BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- Square matrices always commute multiplicatively with cI
- Matrix powers commute and follow the same rules as numbers

$$(A^p)(A^q) = A^{p+q} \qquad (A^p)^q = A^{pq} \qquad A^0 = I$$

Block Matrices/Block Multiplication

- A matrix may be broken into blocks (which are smaller matrices)

$$A = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} I & I & I \\ I & I & I \end{bmatrix}$$

- Addition/multiplication allowed when block dimensions appropriate

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \left[\begin{array}{c|c} B_{11} & \ldots \\ \hline B_{21} & \ldots \end{array}\right] = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & \ldots \\ \hline A_{21}B_{11} + A_{22}B_{21} & \ldots \end{array}\right]$$

- Let the blocks of A be its columns and the blocks of B be its rows

$$AB = \underbrace{\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}}_{m \times n} \underbrace{\begin{bmatrix} — & b_1 & — \\ & \vdots & \\ — & b_n & — \end{bmatrix}}_{n \times p} = \sum_{i=1}^{n} \underbrace{\begin{bmatrix} \quad a_i b_i \quad \end{bmatrix}}_{m \times p}$$

Outline

Elimination in Practice

- Solve the following system using elimination

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 2 \\ 4x_1 + 9x_2 - 3x_3 &= 8 \\ -2x_1 - 3x_2 + 7x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

- Augment A: the augmented matrix A' is

$$A' = \begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

- Strategy: find the pivot in the first row and eliminate the values below it

Example (continued)

- $E^{(1)} = E_{2,1}(2)$ subtracts twice the first row from the second

$$A^{(1)} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E^{(1)}} \underbrace{\begin{bmatrix} \color{red}{2} & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}}_{A'} = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

- $E^{(2)} = E_{3,1}(-1)$ adds the first row to the third

$$A^{(2)} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{E^{(2)}} \underbrace{\begin{bmatrix} \color{red}{2} & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix}}_{A^{(1)}} = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{bmatrix}$$

- Strategy continued: find the pivot in the second row and eliminate the values below it

Example (continued)

- $E^{(3)} = E_{3,2}(1)$ subtracts the second row from the third

$$A^{(3)} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{E^{(3)}} \underbrace{\begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{bmatrix}}_{A^{(2)}} = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

- Use back substitution to solve

$$\begin{aligned} 4x_3 &= 8 & \Longrightarrow & & & & x_3 &= 2 \\ x_2 + x_3 &= 4 & \Longrightarrow & & x_2 + 2 = 4 & \Longrightarrow & x_2 &= 2 \\ 2x_1 + 4x_2 - 2x_3 &= 2 & \Longrightarrow & & 2x_1 + 8 - 4 = 2 & \Longrightarrow & x_1 &= -1 \end{aligned}$$

- Solution $x = (-1, 2, 2)$ solves original system $Ax = b$

Caveats:

- May have to swap rows during elimination
- The system is singular if there is a row with no pivot

Outline

Inverse Matrices

- A square matrix A is <u>invertible</u> if there exists A^{-1} such that

$$A^{-1}A = I \qquad \text{and} \qquad AA^{-1} = I$$

- The inverse (if it exists) is unique, let $BA = I$ and $AC = I$

$$B(AC) = (BA)C \quad \Longrightarrow \quad BI = IC \quad \Longrightarrow \quad B = C$$

- If A is invertible, the unique solution to $Ax = b$ is

$$\begin{aligned} Ax &= b \\ A^{-1}Ax &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

- If there is a vector $x \neq 0$ such that $Ax = 0$ then A not invertible

$$x = Ix = A^{-1}Ax = A^{-1}(Ax) = A^{-1}0 = 0$$

Inverse Matrices

- A 2×2 matrix is invertible iff $ad - bc \neq 0$

$$[A]^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- The number $ad - bc$ is called the <u>determinant</u> of A
- A matrix is invertible if its determinant is not equal to zero
- A diagonal matrix is invertible when none of the diagonal entries are zero

$$A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \quad \Longrightarrow \quad A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix}$$

Inverse of a Product

- If A and B are invertible then so is the product AB

$$(AB)^{-1} = B^{-1}A^{-1}$$

- Easy to verify

$$\begin{aligned}(AB)^{-1}(AB) &= B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I \\ (AB)(AB)^{-1} &= A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I\end{aligned}$$

- Same idea works for longer matrix products

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Calculation of A^{-1}

- Want to find A^{-1} such that $AA^{-1} = I$
- Let

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{so that} \quad \begin{bmatrix} | & | & | \\ e_1 & e_2 & e_3 \\ | & | & | \end{bmatrix} = I$$

- Let x_1, x_2 and x_3 be the columns of A^{-1}, then

$$AA^{-1} = A\left[x_1\, x_2\, x_3\right] = \left[e_1\, e_2\, e_3\right] = I$$

- Have to solve 3 systems of equations

$$Ax_1 = e_1, \quad Ax_2 = e_2, \quad \text{and} \quad Ax_3 = e_3$$

- Computing A^{-1} three times as much work as solving $Ax = b$
- Worst case:
  - Gauss-Jordan method requires n^3 elimination steps
  - Compare to solving $Ax = b$ which requires $n^3/3$

Singular versus Invertible

- Let A be an $n \times n$ matrix
- With n pivots, can solve the n systems

$$Ax_i = e_i \qquad i = 1, \ldots, n$$

- The solutions x_i are the columns of A^{-1}
- In fact, elimination gives a complete test for A^{-1} to exist: there must be n pivots

Outline

Elimination = Factorization

- Key ideas in linear algebra $\sim$ <u>factorization</u> of matrices
- Look closely at 2×2 case:

$$E_{21}(3)\,A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = U$$

$$E_{21}^{-1}(3)\,U = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = A$$

- Notice $E_{21}^{-1}(3)$ is <u>lower triangular</u> $\Longrightarrow$ call it L
- $A = LU$ $\quad$ L lower triangular $\quad$ U upper triangular
- For a 3×3 matrix:

$$(E_{32}\,E_{31}\,E_{21})A = U \quad \text{becomes} \quad A = (E_{21}^{-1}\,E_{31}^{-1}\,E_{32}^{-1})U = LU$$

(products of lower triangular matrices are lower triangular)

Seems Too Good To Be True . . . But Is

- The strict lower triangular entries of L are the elimination multipliers

$$l_{ij} = \text{multiplier}\left[E_{ij}(m_{ij})\right] = m_{ij}$$

- Recall elimination example:
  1. $E_{21}(2)$: subtract twice the first row from the second
  2. $E_{31}(-1)$: subtract minus the first row from the third
  3. $E_{32}(1)$: subtract the second row from the third

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}^{-1}(2)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{E_{31}^{-1}(-1)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{E_{32}^{-1}(1)} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}}_{L}$$

One Square System = Two Triangular Systems

- Many computer programs solve $Ax = b$ in two steps
  - i. Factor A into L and U
  - ii. Solve: use L, U, and b to find x

$$\text{Solve} \quad Lc = b \quad \text{then solve} \quad Ux = c$$

($Lc = b$ by forward substitution; $Ux = b$ by back substitution)

- Can see that answer is correct by premultiplying $Ux = c$ by L

$$\begin{aligned} &Ux = c \\ &L(Ux) = Lc \\ &(LU)x = b \\ &Ax = b \end{aligned}$$

Example

- Solve system represented in matrix form by

$$\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 21 \end{bmatrix}$$

- Elimination (multiplier $= 2$) step:

$$\begin{bmatrix} 2 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

- Lower triangular system: $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 21 \end{bmatrix} \quad \Longrightarrow \quad c = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$

- Upper triangular system: $\begin{bmatrix} 2 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix} \quad \Longrightarrow \quad x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

LU Factorization

- Elimination factors A into LU
- The upper triangular U has the pivots on its diagonal
- The lower triangular L has ones on its diagonal
- L has the multipliers l_{ij} below the diagonal

Computational Cost of Elimination

- Let A be an $n \times n$ matrix
- Elimination on A requires about $\frac{1}{3}n^3$ multiplications and $\frac{1}{3}n^3$ subtractions

Storage Cost of LU Factorization

- Suppose we factor
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$
into
$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} d_1 & u_{12} & u_{13} \\ 0 & d_2 & u_{23} \\ 0 & 0 & d_3 \end{bmatrix}$$
(d_1, d_2, d_3 are the pivots)

- Can write L and U in the space that initially stored A
$$L \text{ and } U = \begin{bmatrix} d_1 & u_{12} & u_{13} \\ l_{21} & d_2 & u_{23} \\ l_{31} & l_{32} & d_3 \end{bmatrix}$$

Outline

The R Environment for Statistical Computing

What is R?

R is a language and environment for statistical computing and graphics

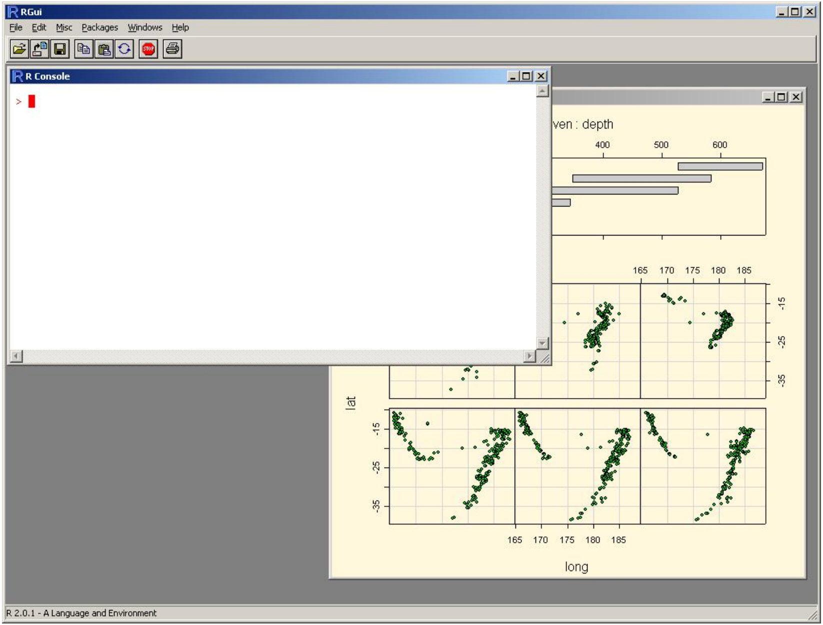
R offers: (among other things)

- a data handling and storage facility
- a suite of operators for calculations on arrays, in particular matrices
- a well-developed, simple and effective programming language includes
  - conditionals
  - loops
  - user-defined recursive functions
  - input and output facilities

- R is free software

`http://www.r-project.org`

The R Application



R Environment for Statistical Computing

R as a calculator

- R commands in the lecture slides look like this

```
> 1 + 1
```

- and the output looks like this

```
[1] 2
```

- When running R, the console will look like this

```
> 1 + 1
[1] 2
```

- Getting help # and commenting your code

```
> help("c") # ?c does the same thing
```

Creating Vectors

- Several ways to create vectors in R, some of the more common:

```
> c(34, 12, 65, 24, 15)
[1] 34 12 65 24 15

> -3:7
[1] -3 -2 -1  0  1  2  3  4  5  6  7

> seq(from = 0, to = 1, by = 0.05)
 [1] 0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40
[10] 0.45 0.50 0.55 0.60 0.65 0.70 0.75 0.80 0.85
[19] 0.90 0.95 1.00
```

- Can save the result of one computation to use an input in another:

```
> x <- c(24, 30, 41, 16, 8)
> x
[1] 24 30 41 16  8
```

Manipulating Vectors

- Use square brackets to access components of a vector

```
> x
[1] 24 30 41 16  8
> x[3]
[1] 41
```

- The argument in the square brackets can be a vector

```
> x[c(1,2,4)]
[1] 24 30 16
```

- Can also use for assignment

```
> x[c(1,2,4)] <- -1
> x
[1] -1 -1 41 -1  8
```

Vector Arithmetic

- Let x and y be vectors of equal length

```
> x <- c(6, 12, 4, 5, 14, 2, 16, 20)
> y <- 1:8
```

- Use + to add vectors (+, −, $*$, / are component-wise functions)

```
> x + y
[1]  7 14  7  9 19  8 23 28
```

- Many functions work component-wise

```
> log(x)
[1] 1.792 2.485 1.386 1.609 2.639 0.693 2.773 2.996
```

- Can scale and shift a vector

```
> 2*x - 3
[1]  9 21  5  7 25  1 29 37
```

Creating Matrices

- Can use the `matrix` function to shape a vector into a matrix

```
> x <- 1:16
> matrix(x, 4, 4)
     [,1] [,2] [,3] [,4]
[1,]    1    5    9   13
[2,]    2    6   10   14
[3,]    3    7   11   15
[4,]    4    8   12   16
```

- Alternatively, can fill in row-by-row

```
> matrix(x, 4, 4, byrow = TRUE)
     [,1] [,2] [,3] [,4]
[1,]    1    2    3    4
[2,]    5    6    7    8
[3,]    9   10   11   12
[4,]   13   14   15   16
```

Manipulating Matrices

- Create a 3×3 matrix A

```
> A <- matrix(1:9, 3, 3)
> A
     [,1] [,2] [,3]
[1,]    1    4    7
[2,]    2    5    8
[3,]    3    6    9
```

- Use square brackets with 2 arguments (row, column) to access entries of a matrix

```
> A[2, 3]
[1] 8
```

Manipulating Matrices

- Can select multiple rows and/or columns

```
> A[1:2, 2:3]
     [,1] [,2]
[1,]    4    7
[2,]    5    8
```

- Leave an argument empty to select all

```
> A[1:2, ]
     [,1] [,2] [,3]
[1,]    1    4    7
[2,]    2    5    8
```

- Use the `t` function to transpose a matrix

```
> t(A)
     [,1] [,2] [,3]
[1,]    1    2    3
[2,]    4    5    6
[3,]    7    8    9
```

Dot Products

Warning R always considers * to be component-wise multiplication

- Let x and y be vectors containing n components

```
> x <- 4:1
> y <- 1:4
> x * y
[1] 4 6 6 4
```

- For the dot product of two vectors, use the %*% function

```
> x %*% y
     [,1]
[1,]   20
```

- Sanity check

```
> sum(x * y)
[1] 20
```

Matrix-Vector and Matrix-Matrix Multiplication

- Let `x` a be vector of n components
- Let `A` be an $n \times n$ matrix and `B` be an $n \times p$ matrix $(p \neq n)$

- The operation
  ```
  > x %*% A
  ```
  treats `x` as a row vector so the dimensions are conformable

- The operation
  ```
  > A %*% x
  ```
  treats `x` as a column vector

- The operation
  ```
  > A %*% B
  ```
  gives the matrix product AB

- The operation
  ```
  > B %*% A
  ```
  causes an error because the dimensions are not conformable

Solving Systems of Equations

- Recall the system ...

$$x = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \qquad \text{solves} \qquad \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

- Can solve in R using the `solve` function

```
> A <- matrix(c(2, 4, -2, 4, 9, -3, -2, -3, 7), 3, 3)
> b <- c(2, 8, 10)

> solve(A, b)

[1] -1  2  2
```

COMPUTATIONAL FINANCE & RISK MANAGEMENT

UNIVERSITY *of* WASHINGTON

Department of Applied Mathematics

http://computational-finance.uw.edu