COMPUTATIONAL FINANCE & RISK MANAGEMENT

UNIVERSITY *of* WASHINGTON

Department of Applied Mathematics

AMATH 460: Mathematical Methods for Quantitative Finance

7.2 Taylor Series

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Outline

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Taylor's Formula for Function of One Variable

- Let $f(x)$ be at least n times differentiable and let a be a real number
- The Taylor polynomial of order n around the point a is

$$\begin{aligned} P_n(x) &= f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a) \\ &= \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) \end{aligned}$$

- Want to use $P_n(x)$ to approximate $f(x)$
- Questions:
  - **Convergence:** does $P_n(x) \to x$ as $n \to \infty$?
  - **Order:** how well does $P_n(x)$ approximate $f(x)$?

Approximation Error

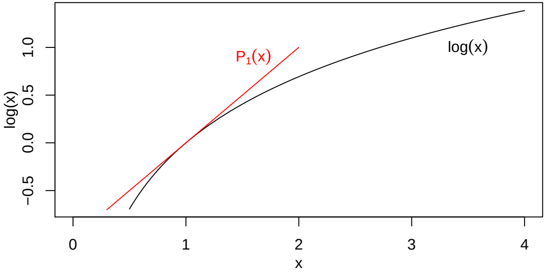
- Difference between $f(x)$ and $P_n(x)$ is called the <u>n^{th} order Taylor approximation error</u>
- Taylor approximation error: derivative form
  - Let $f(x)$ be $n+1$ times differentiable, $f^{(n+1)}$ continuous
  - There is a point c between a and x such that

$$f(x) - P_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

- Taylor approximation error: integral form
  - Let $f(x)$ be $n+1$ times differentiable, $f^{(n+1)}$ continuous

$$f(x) - P_n(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt$$

Example: Linear approximation of $\log(x)$



- Linear approximation of $\log(x)$ around the point $a = 1$
- Taylor polynomial of order 1 for $f(x) = \log(x)$

$$
\begin{aligned}
P_1(x) &= f(a) + \frac{(x-a)}{1!} f'(a) \\
&= 0 + (x-1)\frac{1}{1} \\
&= x - 1
\end{aligned}
$$

Example: Integral Form of Taylor Approximation Error

- What is the Taylor approximation error at the point $x = e$?

$$
\begin{aligned}
f(x) - P_1(x) &= \int_1^x \frac{(x-t)}{1!} f''(t)\,dt \\
f(e) - P_1(e) &= \int_1^e (e-t)\frac{-1}{t^2}\,dt \\
\log(e) - (e-1) &= \int_1^e \left[\frac{1}{t} - \frac{e}{t^2}\right]\,dt \\
2 - e &= \left[\log(t) + \frac{e}{t}\right]\Big|_1^e \\
&= \left[\log(e) + \frac{e}{e}\right] - \left[\log(1) + \frac{e}{1}\right] \\
&= 2 - e \approx -0.718
\end{aligned}
$$

Example: Derivative Form of Taylor Approximation Error

- What is the Taylor approximation error at the point $x = e$?

$$
\begin{aligned}
f(x) - P_1(x) &= \frac{(x-1)^{(1+1)}}{(1+1)!} f''(c) && c \in [1, e] \\
2 - e &= \frac{-(e-1)^2}{2c^2} \\
c &= \sqrt{\frac{(e-1)^2}{2e-4}} && 1 \leq c \approx 1.434 \leq e \approx 2.718
\end{aligned}
$$

- Have to know the approximation error to find c to find the ...
- How is this useful?

Bounding the Taylor Approximation Error

- Know that the true approximation error occurs at $c \in [1, e]$
- Follows that

$$
\begin{aligned}
|\text{error}| &\leq \max_{c \in [1,e]} \left| \frac{(x-1)^2}{2!} f''(c) \right| \\
&\leq \max_{c \in [1,e]} \frac{(e-1)^2}{2c^2} \\
&\leq \frac{1}{2}(e-1)^2 \approx 1.476
\end{aligned}
$$

- Thus $|f(e) - P_1(e)| < 1.477$

Outline

"Big O" Notation

- Consider a degree n polynomial as $x \to \infty$

$$P(x) = \sum_{k=0}^{n} a_k x^k$$

- As $n \to \infty$, the highest order term dominates the others

$$\lim_{x\to\infty} \frac{|P(x)|}{x^n} = \lim_{x\to\infty} \frac{|\sum_{k=0}^{n} a_k x^k|}{x^n} = \lim_{x\to\infty} \left| a_n + \sum_{k=0}^{n-1} \frac{a_k}{x^{n-k}} \right| = |a_n|$$

- "Big O" notation provides a compact way to state the same information

$$P(x) = O(x^n) \qquad \text{as } x \to \infty$$

“Big O” Notation

- Formally
  - Let $f, g : \mathbb{R} \to \mathbb{R}$
  - $f(x) = O(g(x))$ as $x \to \infty$ means there are $C > 0$ and $M > 0$ such that
    $$\left|\frac{f(x)}{g(x)}\right| \le C \quad \text{when} \quad x \ge M$$
  - For finite points: $f(x) = O(g(x))$ as $x \to a$ means there are $C > 0$ and $\delta > 0$ such that
    $$\left|\frac{f(x)}{g(x)}\right| \le C \quad \text{when} \quad |x - a| \le \delta$$
- Example: Taylor polynomial approximation error
  $$f(x) - P_n(x) = O\big((x-a)^{n+1}\big) \qquad \text{as } x \to a$$

"Big O" Notation

- Can also write the Taylor polynomial approximation as

$$\begin{aligned} f(x) - P_n(x) &= O\big((x-a)^{n+1}\big) \\ f(x) &= P_n(x) + O\big((x-a)^{n+1}\big) \\ f(x) &= f(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a) + O\big((x-a)^{n+1}\big) \end{aligned}$$

- Linear approximation is second order

$$f(x) = f(a) + (x-a)f'(a) + O\big((x-a)^2\big) \quad \text{as } x \to a$$

- Quadratic approximation is third order

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(a) + O\big((x-a)^3\big) \quad \text{as } x \to a$$

Outline

Taylor's Formula for Functions of Several Variables

- Let f be a function of n variables $x = (x_1, x_2, \ldots, x_n)$
- Linear approximation of f around the point $a = (a_1, a_2, \ldots, a_n)$

$$f(x) \approx f(a) + \sum_{i=1}^{n} (x_i - a_i) \frac{\partial f}{\partial x_i}(a)$$

- If 2nd order partial derivatives continuous $\Rightarrow$ 2nd order approximation

$$f(x) = f(a) + \sum_{i=1}^{n} (x_i - a_i) \frac{\partial f}{\partial x_i}(a) + O(\|x - a\|^2) \quad \text{as } x \to a$$

- $O(\|x - a\|^2) = \sum_{i=1}^{n} O(|x_i - a_i|^2)$
- Quadratic approximation around a is $O(\|x - a\|^3)$

$$f(x) \approx f(a) + \sum_{i=1}^{n} (x_i - a_i) \frac{\partial f}{\partial x_i}(a) + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{(x_i - a_i)(x_j - a_j)}{2!} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)$$

Taylor's Formula in Matrix Notation

- Let

$$Df(x) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_n} & \cdots & \dfrac{\partial f}{\partial x_n} \end{bmatrix}$$

$$x - a = \begin{bmatrix} x_1 - a_1 \\ x_2 - a_2 \\ \vdots \\ x_n - a_n \end{bmatrix} \qquad D^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

- Linear Taylor approximation

$$f(x) = f(a) + Df(a)(x-a) + O(\|x-a\|^2)$$

- Quadratic Taylor approximation

$$f(x) = f(a) + Df(a)(x-a) + \frac{1}{2!}(x-a)^\intercal D^2 f(a)(x-a) + O(\|x-a\|^3)$$

Example: Functions of Two Variables

- Let f be a function of 2 variables
- Linear approximation of f around the point (a, b)

$$f(x,y) \approx f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b)$$

- Second order approximation since (as $(x,y) \to (a,b)$)

$$\begin{aligned} f(x,y) &= f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\ &\quad + O(|x-a|^2) + O(|y-b|^2) \end{aligned}$$

- In matrix notation

$$f(x,y) = f(a) + Df(a,b)\begin{bmatrix} x-a \\ y-b \end{bmatrix} + O(|x-a|^2) + O(|y-b|^2)$$

Example: Functions of Two Variables (continued)

- Quadratic approximation of f around the point (a, b)

$$
\begin{aligned}
f(x,y) &= f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\
&+ \frac{(x-a)^2}{2!}\frac{\partial^2 f}{\partial x^2}(a,b) + (x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) \\
&+ \frac{(y-b)^2}{2!}\frac{\partial^2 f}{\partial y^2}(a,b) + O(|x-a|^3) + O(|y-b|^3)
\end{aligned}
$$

- Matrix notation

$$
\begin{aligned}
f(x,y) &= f(a,b) + Df(a,b)\begin{bmatrix} x-a \\ y-b \end{bmatrix} \\
&+ \frac{1}{2}\begin{bmatrix} x-a & y-b \end{bmatrix} D^2 f(a,b)\begin{bmatrix} x-a \\ y-b \end{bmatrix} \\
&+O(|x-a|^3) + O(|y-b|^3)
\end{aligned}
$$

Outline

Taylor Series Expansions

- If f is infinitely many times differentiable, can define Taylor series expansion as

$$T(x) = \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a)$$

- A Taylor series expansion is a special case of a power series

$$T(x) = S(x) \equiv \lim_{n\to\infty} \sum_{k=0}^{n} a_k (x-a)^k = \sum_{k=0}^{\infty} a_k (x-a)^k$$

- Power series coefficients $a_k = \frac{f^{(k)}(a)}{k!}$
- Convergence properties for Taylor series inherited from convergence properties of power series

Radius of Convergence

- The radius of convergence is the number $R > 0$ such that

$$S(x) = \sum_{k=0}^{\infty} a_k (x-a)^k < \infty \qquad \forall\, x \in (a-R, a+R)$$

- $S(x)$ infinitely many times differentiable on the interval $(a-R, a+R)$
- $S(x)$ not defined if $x < a - R$ or if $x > a + R$
- If $\lim_{k\to\infty} |a_k|^{1/k}$ exists, then

$$R = \frac{1}{\lim\limits_{k\to\infty} |a_k|^{1/k}}$$

- For Taylor series expansions, if $\lim_{k\to\infty} \frac{k}{|f^{k)}(a)|^{1/k}}$ exists, then

$$R = \frac{1}{e} \lim_{k\to\infty} \frac{k}{|f^{(k)}(a)|^{1/k}}$$

Radius of Convergence

- So far, $T(x) < \infty$ for $x \in (a - R, a + R)$
- Want to know if/where $T(x) = f(x)$
- Theorem: let $0 < r < R$, if

$$\lim_{k \to \infty} \left[\frac{r^k}{k!} \max_{z \in [a-r, a+r]} |f^{(k)}(z)| \right] = 0$$

- Then $T(x) = f(x) \quad \forall \, |x - a| \leq r$

Example

- Taylor series expansion of $f(x) = \log(1+x)$ around the point $a = 0$

$$T(x) = \sum_{k=0}^{\infty} \frac{(x-0)^k}{k!} f^{(k)}(0) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(x-0)^k}{k!} f^{(k)}(0)$$

- $f'(x) = (1+x)^{-1}, \ldots, f^{(k)}(x) = (-1)^{(k+1)}(k-1)!(1+x)^{-k}$
- $f^{(k)}(0) = (-1)^{(k+1)}(k-1)!(1)^{-k} = (-1)^{(k+1)}(k-1)!$
- Taylor series expansion of $f(x) = \log(1+x)$ around $a = 0$

$$\begin{aligned} T(x) &= \sum_{k=1}^{\infty} \frac{(x-0)^k}{k!} f^{(k)}(0) = \sum_{k=1}^{\infty} (-1)^{(k+1)} \frac{x^k}{k} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \end{aligned}$$

Example (continued)

- Find radius of convergence

$$\begin{aligned}\lim_{k\to\infty} \frac{k}{|f^{(k)}(a)|^{1/k}} &= \lim_{k\to\infty} \frac{k}{|(-1)^{(k+1)}(k-1)!|^{1/k}} \\ &= \lim_{k\to\infty} \frac{k}{\left[(k-1)!\right]^{1/k}} \\ &= \text{hmmm} \ldots\end{aligned}$$

- Power series definition

$$\begin{aligned}\lim_{k\to\infty} |a_k|^{1/k} &= \lim_{k\to\infty} \left|\frac{(-1)^{(k+1)}}{k}\right|^{1/k} \\ &= \lim_{k\to\infty} \frac{1}{k}^{1/k} \\ &= \lim_{u\searrow 0} u^u = 1\end{aligned}$$

Example (continued)

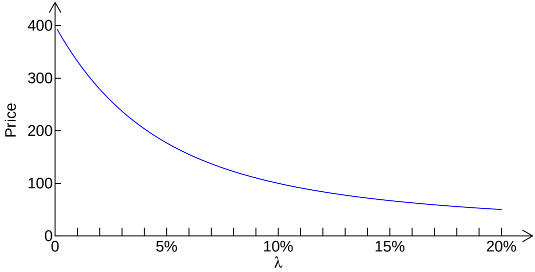
- Radius of convergence: $R = \dfrac{1}{\lim_{k\to\infty} |a_k|^{1/k}} = 1$
- Where does $T(x) = \log(1+x)$? $\quad (0 < r < R = 1)$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{r^n}{n!} \max_{z\in[-r,r]} |f^n(z)| &= \lim_{n\to\infty} \frac{r^n}{n!} \max_{z\in[-r,r]} \left| \frac{(-1)^{n+1}(n-1)!}{(1+z)^n} \right| \\
&= \lim_{n\to\infty} \frac{r^n}{n!} \max_{z\in[-r,r]} \frac{(n-1)!}{|1+z|^n} \\
&= \lim_{n\to\infty} \frac{r^n}{n!} \frac{(n-1)!}{(1-r)^n} \\
&= \lim_{n\to\infty} \frac{1}{n} \left(\frac{r}{1-r} \right)^n = 0 \;\text{ for }\; r \le \frac{1}{2}
\end{aligned}
$$

- $T(x) = \log(1+x)$ for $|x| < \frac{1}{2}$

Outline

Bond Pricing Formula

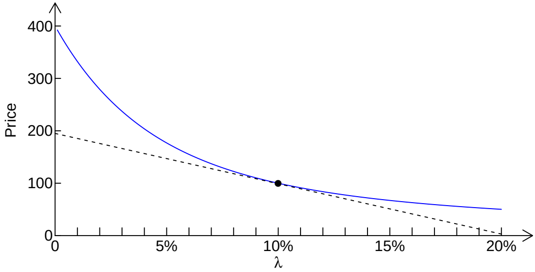


- The price P of a bond is

$$P \;=\; \frac{F}{[1+\lambda]^n} + \sum_{k=1}^{n} \frac{c_k}{1+\lambda^k}$$

where: c_k = coupon payment $\quad n$ = # coupon periods remaining
F = face value $\quad \lambda$ = yield to maturity

Linear Approximation



- The tangent line used for approximation

$$L(\lambda) = P(0.10) + (\lambda - 0.10)\frac{dP}{d\lambda}(0.10) \quad \text{where} \quad \frac{dP}{d\lambda} - -D_M\,P$$

- Said last time: approximation can be improved by adding a quadratic term $\Longrightarrow$ approximate using a degree 2 Taylor polynomial

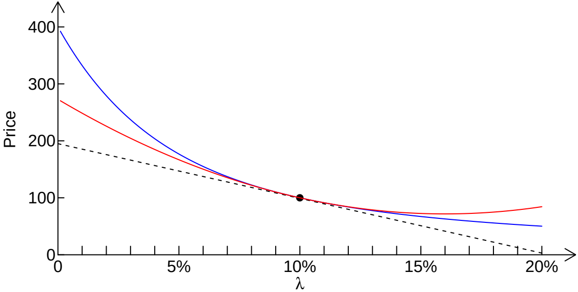
Convexity

- Recall: $PV_k = \dfrac{a_k}{[1+\lambda]^k} \qquad P = \sum_{k=1}^{n} PV_k = \sum_{k=1}^{n} \dfrac{a_k}{[1+\lambda]^k}$

- Convexity: $$\begin{aligned} C &= \frac{1}{P}\frac{d^2P}{d\lambda^2} \\ &= \frac{d^2}{d\lambda^2}\left[\frac{1}{P}\sum_{k=1}^{n} a_k[1+\lambda]^{-k}\right] \\ &= \frac{1}{P}\sum_{k=1}^{n} a_k\,\frac{d^2}{d\lambda^2}[1+\lambda]^{-k} \\ &= \frac{1}{P}\sum_{k=1}^{n} a_k\,k(k+1)[1+\lambda]^{-(k+2)} \\ &= \frac{1}{P[1+\lambda]^2}\sum_{k=1}^{n} k(k+1)\frac{a_k}{[1+\lambda]^k} \end{aligned}$$

Convexity

- Let P_0 and λ_0 be the price and yield of a bond
- Let D_M and C be the modified duration and convexity
- $\Delta P \approx -D_M P \Delta\lambda + \frac{1}{2} PC(\Delta\lambda)^2$
- $$\begin{aligned} P &\approx P_0 + -D_M P(\lambda - \lambda_0) + \frac{1}{2} PC(\lambda - \lambda_0)^2 \\ &\approx P_0 + (\lambda - \lambda_0)\frac{dP}{d\lambda}(P_0) + \frac{(\lambda - \lambda_0)^2}{2}\frac{d^2 P}{d\lambda^2}(P_0) \end{aligned}$$



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