COMPUTATIONAL FINANCE & RISK MANAGEMENT
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AMATH 460: Mathematical Methods for Quantitative Finance

7.1 Lagrange's Method

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Outline

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Investment Portfolios

- Portfolio of n assets
- Let w_i be the proportion of the portfolio invested in asset i
- Have constraint

$$\sum_{i=1}^{n} w_i = 1$$

- Can take long and short positions $\Longrightarrow$ no constraints on individual w_i
- Let μ_i be the expected rate of return on asset i
- Let σ_i^2 be the risk of asset i
- Let ρ_{ij} be the correlation between assets i and j
- Expected rate of return and risk of the portfolio:

$$\begin{aligned} \text{Expected Return} &= \sum_{i=1}^{n} w_i \mu_i \\ \text{Risk} &= \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{1 \le i < j \le n} w_i w_j \sigma_i \sigma_j \rho_{ij} \end{aligned}$$

Investment Portfolios: Matrix Notation

- Let $w = (w_1, \ldots, w_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$
- The expected rate of return can be written in matrix notation as

$$\text{Return} = \sum_{i=1}^{n} w_i \mu_i = w^{\mathsf{T}} \mu$$

- The risk can be written as

$$\text{Risk} = w^{\mathsf{T}} \Sigma w$$

- Σ is the covariance matrix of the n assets

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \cdots & \sigma_1\sigma_n\rho_{1n} \\ \sigma_2\sigma_1\rho_{21} & \sigma_2^2 & \cdots & \sigma_2\sigma_n \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n\sigma_1\rho_{n1} & \sigma_n\sigma_2\rho_{n2} & \cdots & \sigma_n^2 \end{bmatrix}$$

Optimal Investment Portfolios

- Given μ, Σ, and investor selected w, can compute
  - portfolio return
  - portfolio risk

Two notions of optimality

- For a target expected return, choose w to minimize portfolio risk
- For a target level of risk, choose w to maximize expected return

- Both notions are constrained optimization problems that can be solved using Lagrange multipliers

- Minimum variance optimization

n asset case

$$\begin{aligned} \text{minimize:} \quad & w^\mathsf{T} \Sigma w \\ \text{subject to:} \quad & e^\mathsf{T} w = 1 \\ & \mu^\mathsf{T} w = \mu_P \end{aligned}$$

2 asset case

$$\begin{aligned} \text{minimize:} \quad & \sigma_1^2 w_1^2 + 2\rho\sigma_1\sigma_2 w_1 w_2 + \sigma_2^2 w_2^2 \\ \text{subject to:} \quad & w_1 + w_2 = 1 \\ & \mu_1 w_1 + \mu_2 w_2 = \mu_P \end{aligned}$$

- Maximum expected return optimization

n asset case

$$\begin{aligned} \text{maximize:} \quad & \mu^\mathsf{T} w \\ \text{subject to:} \quad & e^\mathsf{T} w = 1 \\ & w^\mathsf{T} \Sigma w = \sigma_P^2 \end{aligned}$$

2 asset case

$$\begin{aligned} \text{maximize:} \quad & \mu_1 w_1 + \mu_2 w_2 \\ \text{subject to:} \quad & w_1 + w_2 = 1 \\ & \sigma_1^2 w_1^2 + 2\rho\sigma_1\sigma_2 w_1 w_2 + \sigma_2^2 w_2^2 = \sigma_P^2 \end{aligned}$$

Outline

Relative Extrema of Single Variable Functions

- A local minimum (maximum) of a function f is a point x_0 where

$$f(x_0) \leq (\geq) f(x) \qquad \forall x \in (x_0 - \epsilon, x_0 + \epsilon)$$

  for some $\epsilon > 0$

- A local extrema is a point that is a local minimum or maximum

- If f is twice differentiable and f'' is continuous
  - Any local extremum is a critical point of f: $f'(x_0) = 0$
  - Can classify critical points using second derivative test

$$\begin{array}{ll} f'(x_0) < 0 & \text{local maximum} \\ f'(x_0) > 0 & \text{local minimum} \\ f'(x_0) = 0 & \text{anything possible} \end{array}$$

Relative Extrema of Functions of n Variables

- A local minimum (maximum) of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a point $x_0 \in \mathbb{R}^n$ where

$$f(x_0) \leq (\geq)\, f(x) \qquad \forall x \;:\; \|x - x_0\| < \epsilon$$

- Every local extremum is a critical point: $Df(x_0) = 0$
- If f is twice differentiable and has continuous second order partial derivatives
  - $D^2 f(x_0)$ is a symmetric matrix with real eigenvalues
  - Second order conditions

All eigenvalues of $D^2 f(x_0) > 0$	local minimum
All eigenvalues of $D^2 f(x_0) < 0$	local maximum
$D^2 f(x_0)$ has $\pm$ eigenvalues	saddle point
$D^2 f(x_0)$ singular	anything can happen

Finding Extrema: Functions of 2 Variables

- Find the local extrema of $f(x, y) = x^2 + xy + y^2$

$$Df(x,y) = \begin{bmatrix} 2x+y & x+2y \end{bmatrix} \qquad Df(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad (0,0) \text{ is a critical point}$$

$$D^2 f(x,y) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

- Can use R to compute the eigenvalues

```
> A <- matrix(c(2, 1, 1, 2), 2, 2)
> eigen(A)$values
[1] 3 1
```

- Since both eigenvalues are greater than 0 $\Longrightarrow$ $(0,0)$ a local minimum

Finding Extrema: Functions of 2 Variables (Take 2)

- Find the local extrema of $f(x,y) = -x^2 - xy - y^2$

$$Df(x,y) = \begin{bmatrix} -2x - y & -x - 2y \end{bmatrix} \qquad Df(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad (0,0) \text{ is a critical point}$$

$$D^2 f(x,y) = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$$

- Can use R to compute the eigenvalues

```
> A <- matrix(-c(2, 1, 1, 2), 2, 2)
> eigen(A)$values
[1] -1 -3
```

- Since both eigenvalues are less than 0 $\Longrightarrow$ $(0,0)$ a local maximum

Finding Extrema: Functions of 2 Variables

- Find the local extrema of $f(x,y) = x^2 + 3xy + y^2$

$$Df(x,y) = \begin{bmatrix} 2x + 3y & 3x + 2y \end{bmatrix} \qquad Df(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad (0,0) \text{ is a critical point}$$

$$D^2 f(x,y) = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

- Can use R to compute the eigenvalues

```
> A <- matrix(c(2, 3, 3, 2), 2, 2)
> eigen(A)$values
[1]  5 -1
```

- One positive and one negative eigenvalue $\Longrightarrow$ $(0,0)$ a saddle point

Finding Extrema: Functions of 2 Variables

- Find the local extrema of $f(x, y) = 2xy - (1 - y^2)^{\frac{3}{2}}$
- First order condition

$$Df(x,y) = \begin{bmatrix} 2y & 2x + 3y\sqrt{1-y^2} \end{bmatrix}$$

$$Df(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix} \implies (0,0) \text{ is a critical point}$$

- Second order condition

$$D^2f(x,y) = \begin{bmatrix} 0 & 2 \\ 2 & \frac{3-6y^2}{\sqrt{1-y^2}} \end{bmatrix} \qquad D^2f(0,0) = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$$

- Compute the eigenvalues of the Hessian at the critical point

```
> eigen(matrix(c(0, 3, 3, 2), 2, 2))$values
[1]  4.162278 -2.162278
```

- One positive and one negative eigenvalue $\implies (0,0)$ a saddle point

Outline

Lagrange's Method

- Problem:

$$\begin{aligned} \text{maximize:} \quad & f(x_1, x_2, \ldots, x_n) \\ \text{subject to:} \quad & g_1(x_1, x_2, \ldots, x_n) = 0 \\ & g_2(x_1, x_2, \ldots, x_n) = 0 \\ & \vdots \\ & g_m(x_1, x_2, \ldots, x_n) = 0 \end{aligned} \tag{1}$$

- 18th-century mathematician Joseph Louis Lagrange proposed the following method for the solution
- Form the function

$$F(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = f(x_1, \ldots, x_n) + \sum_{i=1}^{m} \lambda_i g_i(x_1, x_2, \ldots, x_n)$$

- Optimal value for problem (1) occurs at one of the critical points of F

- **Terminology:**
  - The function $F(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m)$ is called the Lagrangian
  - The column vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ is called the Lagrange multipliers vector

- **Necessary Condition:**
  - Let $x = (x_1, x_2, \ldots, x_n)$
  - Let $g(x) = (g_1(x), g_2(x), \ldots, g_m(x))$ be a vector-valued function of the constraints
  - The gradient $D(g(x))$ must have full rank at any point where the constraint $g(x) = 0$ is satisfied, that is

$$\text{rank}(Dg(x)) = m \quad \forall\, x \;\text{ where }\; g(x) = 0$$

Partial Derivatives of the Lagrangian

- $D\,F(x,\lambda)$ has $n+m$ variables, compute gradient in 2 parts

$$D\,F(x,\lambda) = \begin{bmatrix} D_x F(x,\lambda) & D_\lambda F(x,\lambda) \end{bmatrix}$$

- Recall Lagrangian:

$$F(x,\lambda) = f(x_1,\ldots,x_n) + \sum_{i=1}^{m} \lambda_i g_i(x_1,x_2,\ldots,x_n)$$

- The partial derivatives are

$$\frac{\partial F}{\partial x_j} = \frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} \qquad\qquad \frac{\partial F}{\partial \lambda_i} = g_i(x)$$

- Gradient of f: $\quad Df(x) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \ldots & \dfrac{\partial f}{\partial x_n} \end{bmatrix}$

Partial Derivatives of the Lagrangian

- Gradient of $g(x)$:

$$Dg(x) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix}$$

- Can express sum in second term in matrix notation

$$\sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} = \lambda^{\mathsf{T}} \left[Dg(x) \right]_j$$

- It follows that

$$D\,F(x, \lambda) = \left[Df(x) + \lambda^{\mathsf{T}} Dg(x) \quad (g(x))^{\mathsf{T}} \right]$$

Outline

Example

- Want to

$$\begin{aligned} \text{max/min:} \quad & 4x_2 - 2x_3 \\ \text{subject to:} \quad & 2x_1 - x_2 - x_3 = 0 \\ & x_1^2 + x_2^2 - 13 = 0 \end{aligned}$$

- Start by writing down the Lagrangian

$$\begin{aligned} F(x, \lambda) &= f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x) \\ &= 4x_2 - 2x_3 + \lambda_1(2x_1 - x_2 - x_3) + \lambda_2(x_1^2 + x_2^2 - 13) \end{aligned}$$

- Check necessary condition:

$$Dg(x) = \begin{bmatrix} 2 & -1 & -1 \\ 2x_1 & 2x_2 & 0 \end{bmatrix}$$

Derivatives of the Lagrangian

- The Lagrangian

$$F(x, \lambda) = 4x_2 - 2x_3 + \lambda_1(2x_1 - x_2 - x_3) + \lambda_2(x_1^2 + x_2^2 - 13)$$

- Gradient of the Lagrangian

$$D\,F(x, \lambda) = \begin{bmatrix} 2\lambda_1 + 2\lambda_2 x_1 \\ 4 - \lambda_1 + 2\lambda_2 x_2 \\ -2 - \lambda_1 \\ 2x_1 - x_2 - x_3 \\ x_1^2 + x_2^2 - 13 \end{bmatrix}^{\mathsf{T}}$$

- Set $D\,F(x, \lambda) = 0$ and solve for x and λ get $\lambda_1 = -2$ for free

$$\begin{aligned} 2\lambda_1 + 2\lambda_2 x_1 &\overset{\text{set}}{=} 0 \\ 4 - \lambda_1 + 2\lambda_2 x_2 &\overset{\text{set}}{=} 0 \\ 2x_1 - x_2 - x_3 &\overset{\text{set}}{=} 0 \\ x_1^2 + x_2^2 - 13 &\overset{\text{set}}{=} 0 \end{aligned}$$

Example (continued)

- A little algebra gives

$$x_1 = \frac{2}{\lambda_2} \qquad x_2 = \frac{-3}{\lambda_2} \qquad x_3 = \frac{7}{\lambda_2}$$

- Also know that

$$x_1^2 + x_2^2 = 13 \quad \Longrightarrow \quad \left(\frac{2}{\lambda_2}\right)^2 + \left(\frac{-3}{\lambda_2}\right)^2 = \frac{13}{\lambda_2^2} = 13 \quad \Longrightarrow \quad \lambda_2 = \pm 1$$

- The critical points are
  - $\lambda = (-2, -1),\ x = (-2, 3, -7),\ f(x) = 26$
  - $\lambda = (-2, 1),\quad x = (2, -3, 7),\quad f(x) = -26$

Outline

Minimum Variance Portfolio

- Recall: minimum variance portfolio optimization

$$\begin{aligned} \text{minimize:} \quad & w^\mathsf{T} \Sigma w \\ \text{subject to:} \quad & e^\mathsf{T} w = 1 \\ & \mu^\mathsf{T} w = \mu_P \end{aligned}$$

- Lagrange's method setup

$$\begin{aligned} f(w) &= w^\mathsf{T} \Sigma w \\ g(w) &= \begin{bmatrix} g_1(w) \\ g_2(w) \end{bmatrix} = \begin{bmatrix} \mu^\mathsf{T} w - \mu_P = 0 \\ e^\mathsf{T} w - \ 1 \ \ = 0 \end{bmatrix} \end{aligned}$$

- First, check necessary condition

$$Dg(x) = \begin{bmatrix} \mu^\mathsf{T} \\ e^\mathsf{T} \end{bmatrix}$$

Derivative of a Quadratic Form

- Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$
- Let $f(x) = x^\mathsf{T} A x = a x_1^2 + 2b x_1 x_2 + c x_2^2$
- Then $Df(x) = \begin{bmatrix} 2ax_1 + 2bx_2 & 2bx_1 + 2cx_2 \end{bmatrix} = 2x^\mathsf{T} A$

- In general, let A be an $n \times n$ symmetric matrix
- The derivative (gradient) of the quadratic form $f(x) = x^\mathsf{T} A x$ is

$$Df(x) = 2x^\mathsf{T} A$$

Minimum Variance Portfolio

- The Lagrangian

$$F(y, \lambda) = w^{\mathsf{T}}\Sigma w + \lambda_1 \left[e^{\mathsf{T}} w - 1 \right] + \lambda_2 \left[\mu^{\mathsf{T}} w - \mu_P \right]$$

- Gradient of the Lagrangian

$$\begin{aligned} D\,F(w, \lambda) &= \begin{bmatrix} Df(w) + \lambda^{\mathsf{T}}(Dg(w)) & (g(w))^{\mathsf{T}} \end{bmatrix} \\ &= \begin{bmatrix} 2w^{\mathsf{T}}\Sigma + \lambda_1 e^{\mathsf{T}} + \lambda_2 \mu^{\mathsf{T}} & e^{\mathsf{T}} w - 1 & \mu^{\mathsf{T}} w - \mu_P \end{bmatrix} \end{aligned}$$

- Find the critical point by solving the linear system

$$\begin{bmatrix} 2\Sigma & e & \mu \\ e^{\mathsf{T}} & 0 & 0 \\ \mu^{\mathsf{T}} & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \mu_P \end{bmatrix}$$

Minimum Variance Portfolio

- Further reading:
  - Second order conditions, e.g., Theorem 9.2 and Corollary 9.1 in PFME

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