1. The full $A = QR$ factorization presented during the lectures contains more information than necessary to reconstruct A.

   (a) What are the smallest matrices $\tilde{Q}$ and $\tilde{R}$ such that $\tilde{Q}\tilde{R} = A$?

   (b) Use the *reduced* QR factorization obtained in part (a) to find an expression for the matrix $H = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. How many matrices must be inverted? (note: inverting an orthogonal matrix by taking its transpose doesn't count)

2. Give an example of a 2×2 matrix that has no real eigenvectors. Justify your solution without using any mathematics.

3. Let x and y be vectors of m elements. The least squares solution for a best-fit line for a plot of y versus x is

$$\hat{\beta} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y$$

   where

$$X = \begin{bmatrix} | & | \\ 1 & x \\ | & | \end{bmatrix}$$

   (a) Suppose you have a full singular value factorization $X = U\Sigma V^{\mathrm{T}}$ (i.e., the singular value factorization describe in the lectures). Find an expression for $\hat{\beta}$ in terms of U, Σ, and V. Hint: only square matrices are invertible.

   (b) Like the full QR factorization, the full singular value factorization contains more information than necessary. Find a reduced singular value factorization of X and use it to simplify your answer to part (a). Justify all calculations.

4. Let $\tilde{X}$ be an $m \times n$ matrix ($m > n$) whose columns each sum to zero, and let $\tilde{X} = \tilde{Q}\tilde{R}$ be a reduced QR factorization of $\tilde{X}$. The squared *Mahalanobis* distance to the point $\tilde{x}_i^{\mathrm{T}}$ (the i^{th} row of $\tilde{X}$) is

$$d_i^2 = \tilde{x}_i^{\mathrm{T}}\hat{\Sigma}^{-1}\tilde{x}_i$$

   where $\hat{\Sigma} = \frac{1}{m-1}\tilde{X}^{\mathrm{T}}\tilde{X}$ is a covariance matrix. Compute d_i^2 without inverting a matrix.