HW4b - proofs

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Problem 1.

Theorem:

a) for all integers n, if n^2 odd $\to$ n odd

$\forall n \in \mathbb{Z},\ n^2 \text{ odd} \to n \text{ odd}$

$\equiv \forall n \in \mathbb{Z},\ n \text{ even} \to n^2 \text{ even}$

Proof:

To prove the theorem, prove the contrapositive: let n be an even integer.

Choose $k \in \mathbb{Z}$ s.t. $n = 2k$

So, $n^2 = 2k \cdot 2k = 4k^2 = 2(2k^2)$

And $2k^2 \in \mathbb{Z}$

$\therefore$ by the definition of "even", if n is even, then n^2 is even $\square$

b) Theorem:

for all integers n, if n^2 even $\to$ n even

$\forall n \in \mathbb{Z},\ n^2 \text{ even} \to n \text{ even}$

$\equiv \forall n \in \mathbb{Z},\ n \text{ odd} \to n^2 \text{ odd}$

Proof:

To prove the theorem, prove the contrapositive: let n be an odd integer.

Choose $k \in \mathbb{Z}$ s.t. $n = 2k+1$

So, $n^2 = (2k+1)(2k+1) = 4k^2+4k+1 = 2(2k^2+2k)+1$

and $(2k^2+2k) \in \mathbb{Z}$

$\therefore$ by, the definition of "odd", if n is odd, then n^2 is odd $\square$

c) Theorem:

For all integers n, if n^2 is divisible by 3, then n divisible by 3.

Proof: To prove the theorem, prove the contrapositive:
Let n be a number not divisible by 3.

case 1:
choose k st. $n = 3k+1$.
Then, $n^2 = (3k+1)(3k+1) = 9k^2+6k+1 = 3(3k^2+2k)+1$
$\therefore$ n^2 is not divisible by 3.

Case 2:
choose k st. $n = 3k+2$
Then, $n^2 = (3k+2)(3k+2) = 9k^2+12k+4 = 3(3k^2+4k+1)+1$
$\therefore$ n^2 is not divisible by 3.

$\therefore$ if n is not divisible by 3, then n^2 is not divisible by 3. $\square$

d) Theorem:

For all integers m + n, if mn is even then either m is even or n is even.

Proof: To prove the theorem, prove the contrapositive:
let m + n both be odd integers.

choose $k \in \mathbb{Z}$ st. $m = 2k+1$.
choose $j \in \mathbb{Z}$ st. $n = 2j+1$.

$$m \cdot n = (2k+1)\cdot(2j+1) = 2k \cdot 2j + 2k + 2j + 1$$
$$= 2(kj + k + j) + 1$$

$\therefore$ by the definition of "odd", if m & n are odd, $m \cdot n$ is odd. $\square$

Problem 2

Theorem:

a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof:

Let A, B, and C be sets.

(NOTE: if $\exists n \in A$ or if $\exists n \in (B \cap C)$, then $\exists n \in (A \cup (B \cap C))$.)

case 1:

choose x. Assume $x \in A$.
Since $x \in A$, $x \in A \vee x \in B$, which means $x \in (A \cup B)$

since $x \in A$, $x \in A \vee x \in C$ which means $x \in (A \cup C)$

$\therefore$ if $x \in A$, then $x \in ((A \cup B) \wedge (A \cup C))$

case 2:

choose x. Assume $x \in (B \wedge C)$
since $x \in (B \wedge C)$, $x \in B \wedge x \in C$
Because $x \in B$, $x \in (A \cup B)$
Because $x \in C$, $x \in (A \cup C)$

$\therefore$ if $x \in (B \cap C)$, then $x \in ((A \cup B) \wedge (A \cup C))$

Therefore, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $\square$

b) Theorem:

$$(A \cup B)^c = A^c \cap B^c$$

Proof: Let A and B be sets

choose x. Assume $x \in (A \cup B)^c$.

so, $x \notin (A \cup B)$,

(NOTE: $\forall n$, $n \in A \cup B \leftrightarrow n \in A \vee n \in B$ and

$p \leftrightarrow q \equiv \bar{p} \leftrightarrow \bar{q}$)

meaning $\forall x$, $x \notin (A \cup B) \leftrightarrow x \notin A \wedge x \notin B$

so $x \notin A \wedge x \notin B$.

(NOTE: $\forall k$, $k \in A^c \leftrightarrow k \notin A$)

$\therefore$, by the definition provided in the second note,

$(A \cup B)^c = A^c \cap B^c$ $\square$

c) Theorem:

$$(A \subseteq B) \wedge (A \subseteq C) \longrightarrow A \subseteq (B \cap C)$$

Proof: Let A, B, and C be sets.

Choose x. Assume $x \in A$.

(NOTE: If $A \subseteq B$, then $\forall n$, $n \in A \rightarrow n \in B$)

If $x \in A$ and $A \subseteq B$, then $x \in B$,

If $x \in A$ and $A \subseteq C$, then $x \in C$.

so, $x \in (B \cap C)$.

$\therefore$, by the definition provided in the note,

If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq (B \cap C)$

$\square$

d) Theorem:

$$C \subseteq (B - A) \rightarrow A \cap C = \emptyset.$$

Proof:

Let A, B, and C be sets.

choose x. Assume $x \in C$.

(Note: $\forall k,\ k \in A - B \Leftrightarrow k \in A \wedge k \notin B$

and

If $A \subseteq B$, then $\forall n,\ n \in A \rightarrow n \in B$)

So $x \in (B - A)$.
So $x \in B \wedge x \notin A$.

Since $x \notin A$, it won't exist in $A \cap C$ because for a value to exist in $A \cap C$, it must exist in <u>both</u> A and C.

$\therefore A \cap C = \emptyset$

$\square$