HW 9 - proofs

Thursday, October 24, 2019 3:13 PM

Problem 1

Suppose that $f: X \to X$ is surjective.

a) prove $f \circ f$ is surjective.

Proof: Take $x \in X$.

Since f is surjective, we can choose $a \in X$ s.t. $f(a) = x$.

Since f is surjective, we can chose $b \in X$ s.t. $f(b) = a$.

$\therefore$ $(f \circ f)(b) = f(f(b)) = f(a) = x$.

$\square$

b) Define $f^{\circ n}$ recursively by $f^{\circ 1} = f$ and $f^{\circ n+1} = f \circ f^{\circ n}$ for $n \geq 2$. Prove that $f^{\circ n}$ is surjective for every $n \in \mathbb{N}$.

Proof:

$P(n) =$ " $f^{\circ n}$ is surjective for every $n \in \mathbb{N}$ "

Base case:

$n=1$: $f^{\circ 1} = f$, which is surjective.

$n=2$: $f^{\circ 2} = f \circ f^{\circ 1} = f \circ f$, which is surjective.

Assume $P(n)$ to be true, solve $P(n+1)$.

(want to show $f^{\circ n+1}$ is surjective for every $n \in \mathbb{N}$)

$f^{n+1} = f \circ f^{\circ n}$

and f is surjective by definition.
and $f^{\circ n}$ is surjective by $P(n)$.

The composition of a surjection to a surjection is a surjection by proof in 1a.

$\square$

Problem 2

If $f: X \to Y$ and $g: Y \to Z$ and $g \circ f$ is surjective, must g be surjective?

$\boxed{\text{Yes!}}$

Proof: Assume $g \circ f$ is surjective.

Let $z \in Z$.

Since $g \circ f$ is surjective, $\exists x \in X$ st

$(g \circ f)(x) = g(f(x)) = z$.

So if we choose $y = f(x) \in Y$,

then $g(y) = z$.

$\square$

Problem 3

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If $f: X \to Y$ and $g: Y \to Z$ are fxn and $g \circ f$ is surjective, must f be surjective?

$\boxed{\text{No}}$

counter-ex:

Let $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 10x$
let $g: \mathbb{Z} \to \mathbb{Z}$ defined by $g(x) = \begin{cases} \frac{x}{10} & \text{if } x \text{ divisible by } 10. \\ 0 & \text{if } x \text{ not divisible by } 10. \end{cases}$

f is not surjective bc values in $\mathbb{Z}$ not divisible by 10 are not in the range so the codomain $\neq$ range for f.

but $(g \circ f)$ is surjective bc the range of f is integer multiples of 10. So the input to g would be a value that is divisible by 10, so:

$$(g \circ f)(x) = g(f(x)) = g(10x) = \frac{10x}{10} = x.$$

if $(g \circ f)(x)$ equals x, then the range of the composition is the range as well, making it surjective w/o f having to be surjective.