HW 5 – proofs

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Problem 2:

a) **Theorem:** every integer $n \geq 8$ can be written in the form $n = 3k + 5l$ for some non-negative k & l

Proof:

Let $x \in \mathbb{N}$ st. $x = 8$
$y \in \mathbb{N}$ st $y = 9$
$z \in \mathbb{N}$ st $z = 10$

Then, $\exists\, k, l \in \mathbb{N}$ st $x = 3k + 5l$
namely, $k = l = 1$, so

$x = 3(1) + 5(1) = 8.$

Then, $\exists\, k, l \in \mathbb{N}$ st. $y = 3k + 5l$
namely, $k = 3$ and $l = 0$.

$y = 3(3) + 5(0) = 9.$

Then, $\exists\, k, l \in \mathbb{N}$ st. $z = 3k + 5l$
namely, $k = 0$ and $l = 2$.

$z = 3(0) + 5(2) = 10.$

Case 1: Let n be a multiple of 3.

Choose $p \in \mathbb{N}$ st. $p \geq 3$ and $n = 3p$

Then, $n = 3k + 5l$ if $k = p$ and $l = 0$.

$\therefore$ if n is a multiple of 3, then it can be expressed in the

$\therefore$ if n is a multiple of 3, then it can be expressed in the form $n = 3k + 5l$

Case 2: Let n be a multiple of 3, plus 1.

Choose $p \in \mathbb{N}$ st. $p \geq 3$ and $n = 3p+1$

Then, $n = 3k + 5l$ if

$k = p-3$ and $l = 2$

$\therefore$ if n is 1 plus a multiple of 3 that is greater than 9, then it can be expressed in the form $n = 3k + 5l$

Case 3: Let n be a multiple of 3, plus 2.

Choose $p \in \mathbb{N}$ st. $p \geq 3$ and $n = 3p + 2$

Then, $n = 3k + 5l$ if

$k = p-1$ and $l = 1$

$\therefore$ if n is 2 plus a multiple of 3 that is greater than 9, then it can be expressed in the form $n = 3k + 5l$

$\therefore$ all integers ≥ 8 can be expressed in the form $n = 3k + 5l$ where k, l are non-negative integers $\square$

Problem 3:

Theorem:

for $n \geq 1$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

Proof:

For a contradiction, let

$$C = \{ x \in \mathbb{N} \mid \sum_{i=0}^{x} 2^i \neq 2^{x+1} - 1 \}$$

and assume $C \neq \emptyset$

By WOP, C has a smallest element.

choose $n \in C$ st. n is the smallest element of C

$n-1 \notin C$ since n itself is the smallest element of C.

This means $n-1$ fulfills

$$\sum_{i=0}^{x} 2^i = 2^{x+1} - 1 \text{ if } x = n-1,$$

bc $n-1$ is not in C.

But if $x = n+1$ then

$$\begin{aligned} 1+2+4+\dots+2^n+2^{n+1} &= (1+2+4+\dots+2^n)+2^{n+1} \\ &= 2^{n+1}-1+2^{n+1} \\ &= 2^{n+2}-1 \end{aligned}$$

so for $x = (n-1)+1 = n$,

$$\sum_{i=0}^{x} 2^i = 2^{(n-1)+1} - 1 = 2^n - 1 \quad \lightning$$

$\therefore$ for all $n \geq 1$,

$$1+2+4+\dots+2^n = 2^{n+1} - 1$$

$\square$

Problem 4:

Theorem:

$\forall n \in \mathbb{Z} \neq 0,\ d \in \mathbb{Z} \neq 0,\ \exists q \in \mathbb{Z}$ st. $0 \leq n - dq < d$

Proof:

Let $S = \{ n - dq \mid q \in \mathbb{Z} \wedge n - dq \geq 0 \}$
and assume $S \neq \emptyset$.

By WOP, S has a smallest element.

Choose $r \in S$ be the smallest element of S.

choose r st. $r \geq d$, $r = n - dq$

Because $n = dq + r$, $n - d(q+1) = (n - dq) - d = r - d \geq 0$

$\Rightarrow$ As a result there are integers q and r w/ $0 \leq r < d$.

$\therefore$ for every integer n and every integer $d \neq 0$, n can be written in the form $n = dq + r$ where $q, r \in \mathbb{Z}$ and $0 \leq r < d$. $\square$