HW 6 - proofs

Saturday, October 5, 2019 12:17 PM

Problem 1. G is a directed graph

a) Theorem:
prove there is a walk from $u \in V(G)$ to $v \in V(G)$, then there is a path from u to v

Proof:
For a contradiction,

① Let $A = \{ \gamma \mid \gamma$ is a walk from u to v but not a path $\}$

② Then $A \neq \emptyset$

③ Assume $x \in A$ is the shortest walk from u to v, by WOP.

Case 1:

If x does not have repeated vertices, then it is a path, by definition of "path".

Case 2:

Assume x is not a path.

If x is a walk, but not a path, then $\exists$ v, a vertex, $\in X$ that is repeated.

Choose $P \notin A$ st. P contains all the vertices in X, excluding the vertices between the repition of v.

Then P is a path from u to v, while also being a walk where

$|P| < |\gamma|$

Σ ∎

b) Theorem:
If there is a path from $u \in V(G)$ to $v \in V(G)$, then there is a shortest path from u to v.

Proof:
Let $A = \{ |P| \mid P$ is a path from u to v $\}$

Assume $A \neq \emptyset$ and $A \subseteq (\mathbb{N} \cup \{0\})$

By WOP, choose $x \in A$ st x is the smallest element of A.

Then there is a path from u to v where $|P| = x$.

$\therefore$ If there is a path from $u \in V(G)$ to $v \in V(G)$, then there is a shortest path from u to v. ∎

Then ... is a shortest path ... $\square$

c) **Theorem:** If $V(G)$ is finite, then there is a longest path in G.

Proof:

Let $A = \{ |P| \mid P$ is a path from u to $v \}$

~~Assume~~ $A \neq \emptyset$ and A is finite s.t. $A \subseteq \{0, 1, 2, 3, \ldots, n \in \mathbb{N}\}$

By WOP, $m \in A$ is a largest element.

There is a path x in P s.t. $|x| = m$.

$\therefore$ x is the longest path in G. $\square$

d) No; a walk can contain repeated vertices to an infinite extent, so set $A = \{ |\gamma| \mid \gamma$ is a walk in $G \}$ has no upper bound.

Problem 2

Theorem:

For every $n \in \mathbb{N}$, 3 divides $4^n - 1$

Proof:

For induction:

- base case:

$n = 1$
$4^n - 1 = 4 - 1 = 3$
3 is divisible by 3 because $\exists p \in \mathbb{Z}$ s.t. $3 = 3p$.

- inductive hypothesis:

$P(n) =$ " $\forall n \in \mathbb{N}$, 3 divides $4^n - 1$ "

- Inductive step:

Choose $x \in \mathbb{Z}$ such that $4^n - 1 = 3x$.
for $n+1$:

$$
\begin{aligned}
4^{n+1} - 1 &= 4 \cdot (4^n - 1) + 3 \\
&= 4 \cdot 3x + 3 \\
&= 12x + 3 \\
&= 3(4k + 1)
\end{aligned}
$$

$\therefore\ 4^{n+1}-1$ is divisible by 3, by induction

$\square$

Problem 3

a) Theorem:

A non-empty tournament contains a path of ≥ 0.

Proof:

A non-empty set contains at least one vertex, v.

The path from v to v has length 0.

$\therefore$ a non-empty tournament contains a path w/ a length of ≥ 0. $\square$

b) Theorem:

Let $v_1, \ldots, v_k$ be a path in G, and let $u \in V(G)$ be a vertex not in the path.

Then there is a place in the path were u can be inserted, creating a longer path.

Proof:

Case 1: $\exists v \in \{v_1, \ldots, v_k\}$ st $(v, u) \in E(G)$

if $(v, u) \in E(G)$, then $(u, v) \notin E(G)$.

So, the vertex must be inserted at the end of the path, to increase the path length by one.

Case 2: $\exists v \in \{v_1, \ldots v_k\}$ st. $(u, v) \in E(G)$

If $(u, v) \in E(G)$, then $(v, u) \notin E(G)$.

So, the vertex must be inserted at the beginning of the path, to increase the path length by one.

Case 3: $\exists v_e \in \{v_1, \ldots, v_k\}$ st $(u, v_e) \in E(G) \wedge \exists v_f \in \{v_1, \ldots, v_k\}$ st. $(v_f, u) \in E(G)$

Because there is a path from $u \to v_e$ and $v_f \to u$, u can be inserted between v_f and v_e, increasing the path length by 1.

$\therefore$ There is a place in the path where u can be inserted to create a new path w/ length $k+1$.

□

Theorem

c) Any non-empty tournament G has a ranking.

Proof:

For induction:

Base case:

if $V(G) = \{v_1\}$, path of length 0 exists.

if $V(G) = \{v_1, v_2\}$, path of length 1 exists.

$P(n) =$ "for tournament G_n, $V(G_n) = \{v_1 \dots v_n\}$ where ranking exists"

Inductive step:

$$V(G_{n+1}) = \{v_1 \dots v_n, v_{n+1}\} = V(G_n) \cup \{v_{n+1}\}$$

We can insert v_{n+1} to a path to create a longer path according to proof in 3b.

□

d) $P(n) =$ "$\forall$ tournaments, a ranking exists"

Base: $V(G_1) = \{v_1\}$, a ranking exists bc length $= 0$.

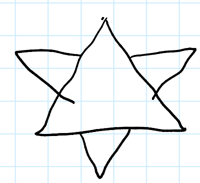
Problem 4.

a) Theorem: $a_n = a_0\left(\frac{8}{5} - \frac{3}{5}\left(\frac{4}{9}\right)^n\right)$

Proof: For induction:

Base:
$$\begin{aligned} a_1 &= a_0\left(\frac{8}{5} - \frac{3}{5}\left(\frac{4}{9}\right)^1\right) \\ &= \frac{\sqrt{3}}{4}\left(\frac{8}{5} - \frac{4}{15}\right) \\ &= \frac{\sqrt{3}}{4}\left(\frac{20}{15}\right) \\ &= \frac{\sqrt{3}}{3} \end{aligned}$$

$$\text{area} = \frac{\sqrt{3}}{4} + 3 \cdot \left(\frac{\frac{\sqrt{3}}{4}}{9}\right) = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} = \frac{4\sqrt{3}}{12} = \frac{\sqrt{3}}{3}$$



$$\begin{aligned} a_2 &= a_0\left(\frac{8}{5} - \frac{3}{5}\left(\frac{4}{9}\right)^2\right) \\ &= \frac{\sqrt{3}}{4}\left(\frac{8}{5} - \frac{16}{135}\right) \\ &= \frac{10\sqrt{3}}{27} \end{aligned}$$

$P(n) =$ "$a_n = a_0\left(8/5 - \frac{3}{5}\left(\frac{4}{9}\right)^n\right)$"

Inductive step:

$$a_n = \frac{8}{5}a_0 - \frac{3a_0}{5}\cdot\left(\frac{4}{9}\right)^n$$

$$a_{n+1} = \frac{8}{5}a_0 - \frac{3a_0}{5}\left(\frac{4}{9}\right)^{n+1}$$

$$= \frac{8}{5}a_0 - \frac{\cancel{3}^{1}a_0}{5}\left(\frac{4}{9}\right)^n \cdot \frac{4}{\cancel{9}_{3}}$$

$$= \underline{a_0\left(\frac{8}{5} - \frac{4}{15}\left(\frac{4}{9}\right)^n\right)}$$

$$a_{n+1} = a_n + a_0\frac{4^n}{3^{2n+1}}$$

$$= \frac{8}{5}a_0 - \frac{3}{5}a_0\left(\frac{4}{9}\right)^n + \frac{a_0}{3}\left(\frac{4}{9}\right)^n$$

$$= \underline{a_0\left(\frac{8}{5} - \frac{4}{15}\left(\frac{4}{9}\right)^n\right)}$$

$$\therefore\ a_n = a_0\left(\frac{8}{5} - \frac{3}{5}\left(\frac{4}{9}\right)^n\right) \quad \square$$

Theorem:

b) $C_n = C_0\left(\frac{4}{3}\right)^n$

Proof: For induction,

Base case:

$$c_1 = c_0\left(\frac{4}{3}\right)^1 = 3\left(\frac{4}{3}\right) = \underline{4}$$

$$\frac{4}{3}\cdot 3 = \underline{4}$$

$P(n)$ = "$C_n = 3\left(\frac{4}{3}\right)^n$"

Assume $P(n)$ is true, then

$$C_{n+1} = 3\left(\frac{4}{3}\right)^{n+1} = 3\left(\frac{4}{3}\right)^n\cdot\frac{4}{3} = \underline{4\left(\frac{4}{3}\right)^n}$$

$$C_{n+1} = C_n + \left(\frac{4}{3}\right)^n = 3\left(\frac{4}{3}\right)^n + \left(\frac{4}{3}\right)^n = \underline{4\left(\frac{4}{3}\right)^n}$$

$$\therefore\ C_n = C_0\left(\frac{4}{3}\right)^n \quad \square$$

c) What happens as n gets larger for a_n & c_n?

$$\lim_{n\to\infty} a_n = a_0\left(\frac{8}{5} - \frac{3}{5}\left(\frac{4}{9}\right)^\infty\right) = \frac{8}{5}a_0$$

$$\lim_{n\to\infty} c_n = c_0\left(\frac{4}{3}\right)^\infty = \infty$$