HW 3b - proofs

Thursday, September 12, 2019 2:56 PM

1. Proof of an implication

a. $\sum$ of any 2 odd integers is even

Proof:

1. Let $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$

2a. choose $n \in \mathbb{Z}$ st. $n = 2x+1$ (n is odd).

2b. choose $m \in \mathbb{Z}$ st. $m = 2y+1$ (m is odd).

3. $n+m = (2x+1)+(2y+1) =$
$2x+2y+2 = 2(x+y+1)$

4. $(x+y+1) \in \mathbb{Z}$

$\therefore$, by the definition of "even", the sum of any 2 odd integers is even. $\blacksquare$

b. The difference of any 2 odd integers is even.

Proof:

1. Let $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$

2a. choose $n \in \mathbb{Z}$ st. $n = 2x+1$ (n is odd).

2b. choose $m \in \mathbb{Z}$ st. $m = 2y+1$ (m is odd).

3. $n-m = (2x+1)-(2y+1) = 2(x-y)$

4. $(x-y) \in \mathbb{Z}$

$\therefore$, by the definition of "even", the difference of any 2 odd integers is even. $\blacksquare$

c. The product of any 2 odd integers is odd

Proof:

1. Let $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$

2a. choose $n \in \mathbb{Z}$ st. $n = 2x+1$ (n is odd).

2b. choose $m \in \mathbb{Z}$ st. $m = 2y+1$ (m is odd).

3. $n \times m = (2x+1) \times (2y+1) =$
$2x \cdot 2y + 2x + 2y + 1 = 2(2xy+x+y)+1$

$$2x \cdot 2y + 2x + 2y + 1 = 2(2xy + x + y) + 1$$

4. $(2xy + x + y) \in \mathbb{Z}$

$\therefore$, by the definition of "odd", the product of any 2 odd integers is odd. $\boxed{}$

d. The square of any odd integer is odd.

Proof:

1. Let $x \in \mathbb{Z}$
2. choose $n \in \mathbb{Z}$ st. $n = 2x+1$ (n is odd).
3. $n \times n = (2x+1) \times (2x+1) =$

$$2x \cdot 2x + 2x + 2x + 1 = 2(2x^2 + 2x) + 1$$

4. $(2x^2 + 2x) \in \mathbb{Z}$

$\therefore$, by the definition of "odd", the square of any odd integer is odd. $\boxed{}$

2.

a. The product of 4 consecutive integers is divisible by 4.

proof:

case 1:

1. Let x be an even integer.
2. choose $a, b, c, d \in \mathbb{Z}$ st. $a = x$, $b = x+1$, $c = x+2$, $d = x+3$. (consecutive)

3a. choose $k \in \mathbb{Z}$ st. $x = 2k$ (even)

3b. $a = 2k$, $b = 2k+1$, $c = 2k+2$, $d = 2k+3$.

4. $a \cdot b \cdot c \cdot d = 2k(2k+1)(2k+2)(2k+3)$

$$= 2k(2k+1) \cdot 2(k+1)(2k+3)$$

$$= 4(2k+1)(k^2+k)(2k+3)$$

5. $(2k+1)(k^2+k)(2k+3) \in \mathbb{Z}$
6. To be divisible by 4, $\exists p \in \mathbb{Z}$ (namely $(2k+1)(k^2+k)(2k+3)$) st. $a \cdot b \cdot c \cdot d = 4p$

$\therefore$ If the first of 4 consecutive integers is even, then $a \cdot b \cdot c \cdot d$ is divisible by 4.

case 2:

1. Let x be an odd integer.
2. choose $a, b, c, d \in \mathbb{Z}$ st. $a = x$, $b = x+1$, $c = x+2$, $d = x+3$. (consecutive)

3a. choose $k \in \mathbb{Z}$ st. $x = 2k+1$ (odd)

3b. $a = 2k+1$, $b = 2k+2$, $c = 2k+3$, $d = 2k+4$.

4. $a \cdot b \cdot c \cdot d = (2k+1)(2k+2)(2k+3)(2k+4)$

$$= (2k+1) \cdot 2(k+1)(2k+3) \cdot 2(k+2)$$

$$= 4(2k+1)(k^2+3k+2)(2k+3)$$

5. $(2k+1)(k^3+3k+2)(2k+3) \in \mathbb{Z}$
6. To be divisible by 4, $\exists p \in \mathbb{Z}$ (namely $(2k+1)(k^2+3k+2)(2k+3)$) st. $a \cdot b \cdot c \cdot d = 4p$

$\therefore$ If the first of 4 consecutive integers is odd, then $a \cdot b \cdot c \cdot d$ is divisible by 4. $\boxed{}$

integers is even, then $a \cdot b \cdot c \cdot d$ is divisible by 4.

integers is odd, then $a \cdot b \cdot c \cdot d$ is divisible by 4.

$\therefore$ The product of four consecutive integers is divisible by 4. ∎

b. For all integers m + n, m+n and m−n are both odd or both even

$\forall m \in \mathbb{Z}, n \in \mathbb{Z}, \ (m+n \land m-n \text{ odd}) \lor (m+n \land m-n \text{ even})$

Proof:

Case1:

1. Let m be an even integer and n be an even integer
2. choose $x \in \mathbb{Z}$ st $m = 2x$ + choose $y \in \mathbb{Z}$ st $n = 2y$
3. $m+n = 2x + 2y = 2(x+y)$
4. $m-n = 2x - 2y = 2(x-y)$
5. $(x+y) \in \mathbb{Z}$ and $(x-y) \in \mathbb{Z}$

$\therefore$, by the def of "even", the m+n and m−n of even m + n are both even.

Case2:

1. Let m be an even integer and n be an odd integer
2. choose $x \in \mathbb{Z}$ st $m = 2x$ + choose $y \in \mathbb{Z}$ st $n = 2y+1$
3. $m+n = 2x + 2y + 1 = 2(x+y) + 1$
4. $m-n = 2x - 2y - 1 = 2(x-y) - 1$
5. $(x+y) \in \mathbb{Z}$ and $(x-y) \in \mathbb{Z}$

$\therefore$, by the def of "odd", the m+n + m−n of even m + odd n are both odd.

Case3:

1. Let m be an odd integer and n be an even integer
2. choose $x \in \mathbb{Z}$ st $m = 2x+1$ + choose $y \in \mathbb{Z}$ st $n = 2y$
3. $m+n = 2x+1 + 2y = 2(x+y)+1$
4. $m-n = 2x+1 - 2y = 2(x-y)+1$
5. $(x+y) \in \mathbb{Z}$ and $(x-y) \in \mathbb{Z}$

$\therefore$, by the def of "odd", the m+n and m−n of odd m + even n are both odd.

Case4:

1. Let m be an odd integer and n be an odd integer
2. choose $x \in \mathbb{Z}$ st $m = 2x+1$ + choose $y \in \mathbb{Z}$ st $n = 2y+1$
3. $m+n = 2x+1+2y+1 = 2(x+y+1)$
4. $m-n = 2x+1-2y-1 = 2(x-y)$
5. $(x+y+1) \in \mathbb{Z}$ and $(x-y) \in \mathbb{Z}$

$\therefore$, by the def of "even", the m+n and m−n of odd m + n are both even. ∎

$\therefore$ for all $m + n \in \mathbb{Z}$, m+n + m−n are either both even or odd.