Lecture Notes on Differential Equations

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These lecture notes are based on the open source textbook Elementary Differential Equations with Boundary Value Problems by William F. Trench, the open source textbook Differential Equations for Engineers by Daniel An, lecture notes on Ordinary Differential Equations by Zhixian Zhu and other online resources.

Contents

Week 1: Introduction

8/26–9/1

1.1 Motivating Examples

1.1.1 What is a Differential Equation

Definition 1.1

A **differential equation** *is an equation that contains one or more derivatives of an unknown function.*
A differential equation is an **ordinary differential equation** *(ODE) if it involves an unknown function of only one variable.* ♣

In this course, we will consider only ordinary differential equations and simply call them differential equations.

Example 1.1 Which one is a differential equation?

1. $y' = y + x$ is a differential equation. The unknown function is y depending on the variable x and the derivative of y is involved in the equation.
2. $\sin(x) = 1$ is not a differential equation. There is no unknown *function*.
3. $y = 5 + x$ is not a differential equation. There is a function y in the equation but the derivative is not involved.

✍ **Exercise 1.1** Determine whether the followings are differential equations.

1. $y' = y$.
2. $(\sin x)' = \cos x$.
3. $g''(x) - 2g'(x) + g(x) = x^2$

Solution

1. *$y' = y$ is a differential equation. The unknown function is y depending on the variable x and the derivative of y is involved in the equation.*
2. *$(\sin(x))' = \cos x$ is not a differential equation. There is no unknown function.*
3. *$g''(x) - 2g'(x) + g(x) = x^2$ is a differential equation. The unknown function is $g(x)$ depending on the variable x and the derivatives of $g(x)$ are involved in the equation.*

■

From the above equations and exercise, you see that ordinary differential equations are

usually in the form
$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}).$$

1.1.2 Where to Find Them

In physical and many real life problems, we want to study relations between changing quantities. To apply mathematical methods to such a problem, we need to formulate the problem using mathematical concepts and construct a mathematical model to describe it. The process of developing a mathematical model is known as mathematical modeling.

Comparing rates of change often helps to model relations by sufficiently simple mathematical equations. Those equations often involve functions and their derivatives. They are called differential equations. The focus of this course is to study how to solve differential equations.

A good the mathematical model should have, but not limited to, the following qualities:

1. It's sufficiently simple so that the mathematical problem can be solved.
2. It should fit the actual situation sufficiently well so that it can be used to make predictions that are verifiable by experimental data.

Now let's see some examples of mathematical models involving differential equations. You will learn how to solve the various types of differential equations in those examples later.

In this course, we will use the prime notation y' and the Leibniz notation $\frac{\mathrm{d}y}{\mathrm{d}t}$ interchangeably.

Example 1.2 Population Growth and Decay The number P of members of a population (people in a given country, bacteria in a laboratory culture, etc.) at any given time t can be modeled using differential equations. In most models, it is assumed that the differential equation takes the form
$$P'(t) = a(P)P(t), \tag{1.1}$$
where a is a continuous function of the population $P(t)$ that represents the relative rate of change per unit time, known as the growth rate.

In the Malthusian model, the growth rate a is assumed to be a constant r, and the equation 1.1 becomes
$$P'(t) = rP(t). \tag{1.2}$$
From Calculus, we know that the equation 1.2 has a solution $P(t) = P(0)e^{rt}$.

The Malthusian model has the limitation. Suppose that we are modeling the population of a country. Starting from a time $t = 0$, as time goes, the population might either be 0 if $a < 0$ or infinity if $a > 0$ which is not reasonable. Indeed, the population breaks the country's limit of resources, the model will no longer be valid. Because of the limitation of space and resources, the relative population growth rate should decrease as the population increase.

Another model that reflects the above mentioned phenomenon is the Verhulst Model:

$$P'(t) = rP(t)(1 - \alpha P(t)), \tag{1.3}$$

where r is the growth rate and $1/\alpha$ is the carrying capacity. As long as P is relatively small compared to $1/\alpha$, in other words, αP is approximately 0, the growth is approximately exponential because the ratio $\frac{P'(t)}{P(t)}$ is approximately r. However as P increases, the growth rate decrease.

Equation 1.3 is known as the logistic equation. It can be re-written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\ln(P(t)\Big) - \frac{\mathrm{d}}{\mathrm{d}t}\Big(\ln(1 - \alpha P(t)\Big) = r.$$

Integrating both sides implies that the logistic equations has the solution

$$P(t) = \frac{P(0)}{\alpha P(0) + (1 - \alpha P(0))e^{-rt}}.$$

Note that $\lim\limits_{t\to\infty} P(t) = \frac{1}{\alpha}$ and $\frac{1}{\alpha}$ is independent of $P(0)$.

The following figure shows the solutions of the logistic equation for various P_0.

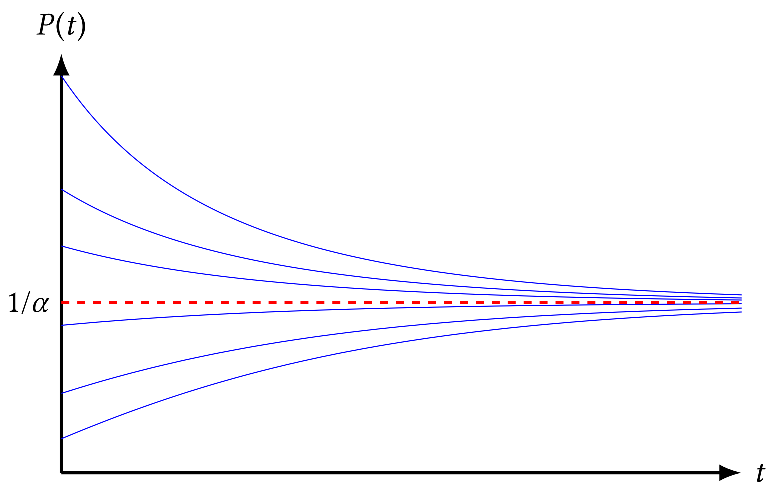

Figure 1.1: Solutions to a logistic equation

Example 1.3 Newton's Law of Cooling According to Newton's law of cooling, the temperature of a body changes at a rate directly proportional to the difference in the temperatures between the temperature of the body and the temperature of its surroundings. If T_m is the temperature of the surrounding and $T = T(t)$ is the temperature of the body at time t, then

$$T'(t) = -k(T(t) - T_m(t)), \tag{1.4}$$

where k is a positive constant and the minus sign indicates that heat will transfer from hot to cold objects.

When the surrounding temperature $T_m(t) = T_m$ is constant, the equation 1.4 has a solution

$$T(t) = T_m + (T_0 - T_m)e^{-kt},$$

where $T_0 = T(0)$ is the initial temperature of the body. As you can imagine, the temperature will be approximately balanced as time goes. A mathematical explanation is that $\lim_{t\to\infty} T(t) = T_m$.

The following figure shows typical graphs of the function $T(t)$ with various values of T_0 and a fixe T_m.

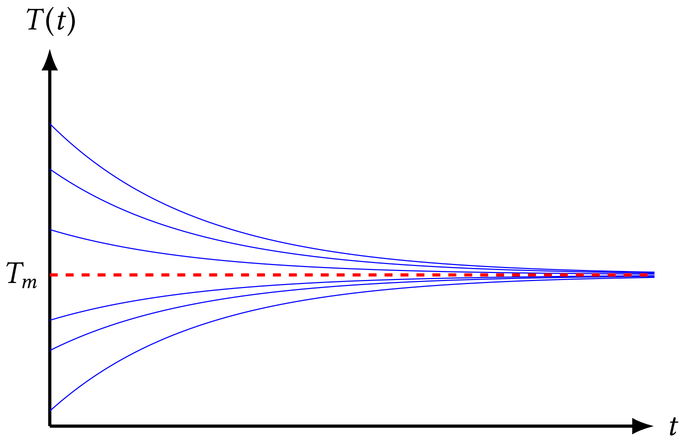

Figure 1.2: Functions from Newton's Law of Cooling

Assuming that the surrounding remains at constant temperature seems reasonable in some situations such as cooling a cup of hot coffee in a room. In this case, the heat transferred to the room won't increase its temperature much. However, if we cool down the cup of hot coffee in a small pot of water, you will find the water temperature increases as time goes. In this situation, to model the temperature changes more accurate, we must consider the heat exchanges. Suppose energy is conserved, that is, the total heat in the object and its surrounding remain constant.

In physics, it is known that the heat transfer is directly proportional the change of temperature. Then we have an extra equation in addition to Equation 1.4:

$$a(T(t) - T_0) = a_m(T_m(t) - T_{m_0}),$$

where $T_{m_0} = T_m(0)$ is the initial temperature of the surrounding. Solving $T_m(t)$ and plug in

Equation 1.4, we get

$$\begin{aligned} T'(t) &= -k(T(t) - T_m(t)) \\ &= -k\left(\left(1+\frac{a}{a_m}\right)T(t) - \left(\frac{a}{a_m}T_0 + T_{m_0}\right)\right) \\ &= -k\left(1+\frac{a}{a_m}\right)\left(T(t) - \frac{aT_0 + a_m T_{m_0}}{a + a_m}\right) \end{aligned}$$

Again, the equation can be rewritten in the form $\ln(F(t)) = C$ form (Can you find the $F(t)$ and C?). Using Calculus, you will find that it has a solution

$$T(t) = \frac{aT_0 + a_m T_{m_0}}{a + a_m} + \frac{a_m(T_0 - T_{m_0})}{a + a_m} e^{-k(1+\frac{a}{a_m})t}.$$

Remark In the above examples, the differential equations can be re-written in the form

$$\frac{F'(t)}{F(t)} = C, \tag{1.5}$$

where $F(t)$ is a function and C is a constant. Note that

$$\frac{F'(t)}{F(t)} = \Big(\ln(F(t))\Big)'.$$

Integrate both sides of Equation 1.5, you will find that $F(t) = F(0)e^{cT}$.

Example 1.4 Newton's Second Law of Motion For an object with a constant mass m, Newton's second law of motion states that the force F acting on the object and the instantaneous acceleration a of an object are related by the equation $F = ma$.

In many applications, there are multiple forces that may act on the object.

Assume that the motion of an object with the mass $m = 1$ is moving along a vertical line above the surface of the Earth. Let y be the displacement of the object from some reference point above the surface. The following type of forces normally act:

- The gravity $g(y)$ that depends only on the position y, where $g(y) < 0$.
- The atmospheric resistance $-r(y, y')y'$ that depends on the position and velocity of the object, where r is a nonnegative function. The $-y'$ "outside" of the function is used to indicate that the resistive force is always in the direction opposite to the velocity y'.
- The force $f = f(t)$ from other external sources (such as a towline from a helicopter) which depends only on t.

In this case, Newton's second law implies that

$$y'' = -r(y, y')y' - g(y) + f(t),$$

which is usually rewritten as

$$y'' + r(y, y')y' + g(y) = f(t).$$

Since the second order derivative of y occurs in this equation and no more higher order derivative, we say that this equations is a second order differential equation.

Example 1.5 Interacting Species: Competition Let $P = P(t)$ and $Q = Q(t)$ be the populations of two species at time t. Assume that each population would grow exponentially by the Malthusian model if the other did not exist; that is, in the absence of competition we would have

$$P' = aP \quad \text{and} \quad Q' = bQ \tag{1.6}$$

where a and b are positive constants.

To model the effect of competition, one way is to assume that the growth rate per individual of each population is reduced by an amount proportional to the other population, so Equation 1.6 is replaced by

$$\begin{aligned} P' &= aP - \alpha Q \\ Q' &= -\beta P + bQ, \end{aligned}$$

where α and β are positive constants. The relation between the populations of the competing species can be described by the following figure. The arrows indicate direction of rates of

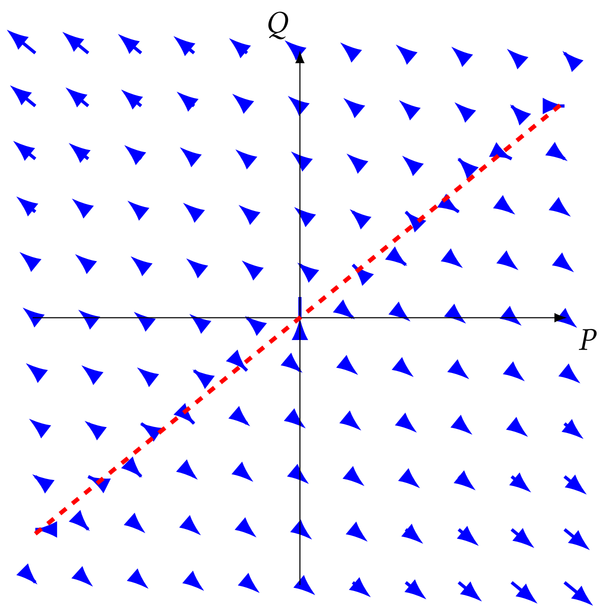

Figure 1.3: A model for populations of competing species

change of populations with increasing t. The dashed line L through the origin depends only on a, b, α and β. If (P_0, Q_0) is above L, then the species with population P will extinct, but if (P_0, Q_0) is below L, the species with population Q will extinct.

In the example, we are dealing with a homogeneous system of differential equations with constant coefficients. The slope of the dashed line is related to a eigenvector of the coefficient matrix of the system. For more information, you may read Section 10.4 of Trench's book.

1.2 Basic Concepts

1.2.1 What is the Order

Definition 1.2

*The **order** of a differential equation is the order of the highest derivative that it contains.* ♣

Example 1.6 What's the order of the differential equation?

1. $y' - x^2 = 0$ is a first order differential equation.
2. $y' + xy^2 = y^3$ is also a first order differential equation.
3. $y'' - xy' + y = xy^2$ is a second order differential equation.
4. $xy^{(4)} + y^2 = \sin x$ is a fourth order differential equation.

✍ **Exercise 1.2** Determine the order of each differential equation.

1. $y' = x$.
2. $y'' \cdot y' + y = \cos x$.

Solution

1. *$y' = x$ is a first order differential equation.*
2. *$y'' \cdot y' + y = \cos x$ is a second order differential equation.*

■

1.2.2 What is a Solution

Definition 1.3

*A **solution** of a differential equation is a* function *that satisfies the differential equation on some open interval.* ♣

Example 1.7 Is the function a solution?

1. $y = e^x$ is a solution of $y' - y = 0$ on the interval $(-\infty, \infty)$.

If $y = e^x$, then $y' = e^x$ and

$$y' - y = e^x - e^x = 0.$$

In fact, for any number c, $y = ce^x$ is a solution.

2. $y = \sin(x)$ is a solution of $y'' + y = 0$.
   If $y = \sin(x)$, then $y' = \cos(x)$ and $y'' = -\sin(x)$ and

$$y'' + y = -\sin(x) + \sin(x) = 0.$$

   So y is a solution of $y' - y = 0$ on the interval $(-\infty, \infty)$
3. $y = \sin(x)$ is not a solution of $y' - y = 0$.
   If $y = \sin(x)$, then $y' = \cos(x)$ and

$$y' - y = \sin(x) - \cos(x) \neq 0.$$

Example 1.8 Show that for any constants c_1 and c_2 the function

$$y = c_1 \sin x + c_2 \cos x$$

is a solution of

$$y'' + y = 0$$

on $(-\infty, \infty)$.

Solution

Differentiating function twice yields

$$y' = c_1 \cos x - c_2 \sin x,$$

$$y'' = -c_1 \sin x - c_2 \cos x.$$

Therefore y is a solution of the differential equation on $(-\infty, \infty)$. ■

Exercise 1.3 Determine whether the following functions are solutions of $y' = y^2$

1. $y = -\frac{1}{x}$.
2. $y = x$.

Solution *If $y = -\frac{1}{x}$, then $y' = \frac{1}{x^2}$ and $y^2 = \left(-\frac{1}{x}\right)^2 = \frac{1}{x^2}$. So, $y = -\frac{1}{x}$ is a solution.*

If $y = x$, then $y' = 1$. But $y^2 = x^2 \neq 1 = y'$. Thus, $y = x$ is not a solution. ■

1.2.3 What is an Initial Value Problem

We've seen that a differential equation may have an infinite family of solutions. To specifies one solution, extra constants will be needed. Those constants are initial conditions.

Definition 1.4

An **initial value problem** *(IVP for short) for an n-th order differential equation requires y and its first $n-1$ derivatives to have specific values at a point. Such a specific value is called an* **initial condition**.

A **solution of an initial value problem** *is a function which satisfies both the differential equation and initial conditions.* ♣

Example 1.9 The following is an initial value problem for a first order differential equation.

$$\begin{cases} y' = y. \\ y(0) = 3. \end{cases}$$

The initial condition is $y(0) = 3$.

To solve this initial value problem, we first find the general solution and then determine the parameter.

Note that the equation is equivalent to $(\ln y)' = 1$. Integrating both sides yields the general solution $y = ce^x$. The initial condition to determine the value of the constant $c = y(0) = 3$ Hence $y = 3e^x$ is the solution of the initial value problem.

✍ **Exercise 1.4** Consider the ordinary differential equation $y' - 2y + e^x = 0$.

1. Show that $y = ce^{2x} + e^x$ is the general solution.
2. Solve the initial value problem

$$\begin{cases} y' - 2y + e^x = 0 \\ y(0) = 2. \end{cases}$$

Solution

1. *If $y = ce^{2x} + e^x$, then $y' = 2ce^{2x} + e^x$ and*

$$y' - 2y + e^x = 2ce^{2x} + e^x - 2(ce^{2x} + e^x) + e^x = 0.$$

Hence, $y = ce^{2x} + e^x$ is the general solution.

2. *We use the initial condition to determine the choice of c. For $y = ce^{2x} + e^x$ we have $y(0) = c + 1$. We match it with the initial condition and we have*

$$c + 1 = 2$$

or simply $c = 1$. Hence $y = e^{2x} + e^x$ is the solution of the initial value problem.

■

1.2.4 What is a General Solution

In the above example, the solution is a particular solution of the differential equation.

Like antiderivatives, without any constraint, a differential equation may have a family of solutions. Such a family parametrizes all solutions of the differential equation.

Definition 1.5

A **general solution** *consists of all solutions of a differential equation.* ♣

Example 1.10 Show that $y = c_1 e^x + c_2 e^{-x}$ is the general solution of

$$y'' - y = 0$$

on $(-\infty, \infty)$, where c_1 and c_2 can be any numbers.

Solution *Differentiating the function y yields*

$$y' = c_1 e^x - c_2 e^{-x},$$

$$y'' = c_1 e^x + c_2 e^{-x}.$$

Then

$$y'' - y = c_1 e^x + c_2 e^{-x} - (c_1 e^x + c_2 e^{-x}) = 0.$$

So the function y is a solution. The fact that any solution can be written in this form is equivalent to the uniqueness of a solution to the initial value problem with the general initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$. The proof of the uniqueness is above the level of this course. However, one can also argue using Calculus. The differential equation can be re-written as follows

$$\begin{aligned} (y' - y)' + (y' - y) &= 0 \\ \frac{(y' - y)'}{y' - y} &= -1 \\ \left(\ln(y' - y)\right)' &= -1 \end{aligned}$$

Integrating the equation $\left(\ln(y' - y)\right)' = -1$ yields the general solution $y' - y = ce^{-x}$. Similarly, the equation may be re-written as

$$\left(\ln(y - c_2 e^{-x})\right)' = 1,$$

where $c_2 = -\frac{c}{2}$. Integrating the equation yields that $y = c_1 e^x + c_2 e^{-x}$. ■

✍ **Exercise 1.5** Find the general solution to the differential equation

$$y''' = \sin x.$$

Solution *Integrating the differential equation repeatedly yields*

$$y'' = -\cos x + c_1,$$

$$y' = -\sin x + c_1 x + c_2,$$

$$y = \cos x + c_1 x + c_2 x + c_3,$$

where c_1, c_2 and c_3 are arbitrary constants. Since y'' and y' are both general solutions by Rolle's theorem, so is y. ■

1.2.5 Solving Differential Equations by Direct Integration

To solve a general differential equation is usually difficult. In this course, we will only focus on some special types of differential equations. For example, some differential equations can be solved by simply integrating both sides using techniques from Calculus. We call them ***direct integration problem***.

Example 1.11 Solve the initial value problem.

$$y' = \frac{1}{x^2 + x}, \qquad y(1) = 0.$$

Solution *Using partial fraction expansion, the differential equation can be written as*

$$y' = \frac{1}{x} - \frac{1}{x+1}.$$

Integrating both sides yields

$$\begin{aligned} y &= \int \frac{3}{x^3 + x}\, \mathrm{d}x \\ &= \int \left(\frac{1}{x} - \frac{1}{x+1}\right) \mathrm{d}x \\ &= \ln|x| - \ln(|x+1|) + c \end{aligned}$$

Now using the initial condition $y(1) = 0$ to determine C:

$$0 = \ln 1 - \ln 2 + c$$

Since $\ln 1 = 0$, we get $c = \ln 2$. The solution of this initial value problem is

$$y = \ln|x| - \ln(|x+1|) + \ln 2.$$

Note that the domain of the solution function is $(-\infty, -1) \cup (-1, 0) \cup (0, \infty)$. ■

In order to solve direct integral problems, you will need to review integrations of some elementary functions and various methods of integrations such as integration by parts, integration by substitution, partial fraction expansion.

✍ **Exercise 1.6** Solve the initial value problem.

$$y' = xe^x, \qquad y(0) = 1.$$

Solution *Using integration by parts, one will find that*

$$y = xe^x - e^x + c.$$

The initial condition $y(0) = 1$ *implies that* $c = 2$. *So the solution of this initial value problem is* $y = xe^x - e^x + 2$. ■

1.2.6 Integral Curve*

Definition 1.6

The graph of a solution of a differential equation is called a **solution curve**.
A curve C *is called an* **integral curve** *of a differential equation if every solution curve is a part of it.* ♣

From the definition, we see that any solution curve of a differential equation is an integral curve, but an integral curve need not be a solution curve.

Example 1.12 For any positive constant a, consider the circle defined by $x^2 + y^2 = a^2$. Verify the circle is an integral curve of $y' = -\frac{x}{y}$ but not a solution curve.

Solution *The circe is made of two solution curves defined by* $y = \sqrt{a^2 - x^2}$ *and* $y = -\sqrt{a^2 - x^2}$. *But the circle is not a function. Note that the differential equation can be re-written as* $yy' = x$. *Integrate both sides, you will see that all solutions of* $y' = -\frac{x}{y}$ *satisfy the equation* $x^2 + y^2 = a^2$ *for some* a. *Thus, the circle is an integral curve.* ■

✍ **Exercise 1.7** Verify that the cuspidal curve defined by $y^2 = x^3 + c$ is an integral curve for $y' = -\frac{3x^2}{2y}$ but not a solution curve.

Solution *The equation* $y' = -\frac{3x^2}{2y}$ *has two solution curves defined by* $y = \sqrt{x^3 + c}$ *and* $y = -\sqrt{x^3 + c}$. *Apply the same method as in the above example, one can verify that the equation* $y^2 = x^3 + c$ *defines an integral curve.* ■

1.2.7 Direction Fields for First Order Equations

When finding an explicit formula for the solution of a differential equation is impossible or the formula is too complicated, we may use graphical or numerical methods to investigate how the solution behaves.

In this section, we consider a graphical method for first order differential equations $y' = f(x, y)$.

The idea is to sketch the integral curve using the first derivative y'. To be specific, the slope y'_0 of an integral curve of the equation $y' = f(x, y)$. through a given point (x_0, y_0) is given by the number $f(x_0, y_0)$. Through the point (x_0, y_0), we may draw a small line segment with the slope y'_0.

Definition 1.7

Let f be a function defined on a set $R \in \mathbb{R}^2$. The graph consists of line segments through every point (x, y) in R with the slope $f(x, y)$ is called the **direction filed** *(also called the* **slope field***) for $y' = f(x, y)$ in R.* ♣

In practice, we can't actually draw line segments through very points in R if R is an infinite set. Instead, we select a finite subset of R and draw line segments through points in it. For example, we may create a rectangular grid and draw line segments through grid points.

Example 1.13 Consider the differential equation $y' = -\frac{x}{y}$.

Given a grid, the direction field (Figure 1.4 can be constructed by calculating y' at grid points and drawing a short line segment with the slope y' and trough (x, y).

From the direction field, you may guess what does an integral curve look like (see Figure 1.5). It's a circe (see Example 1.12)!

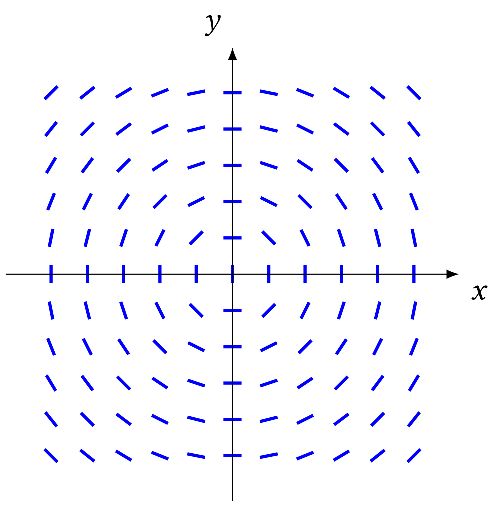

Figure 1.4: The direction field for $y' = -\frac{x}{y}$.

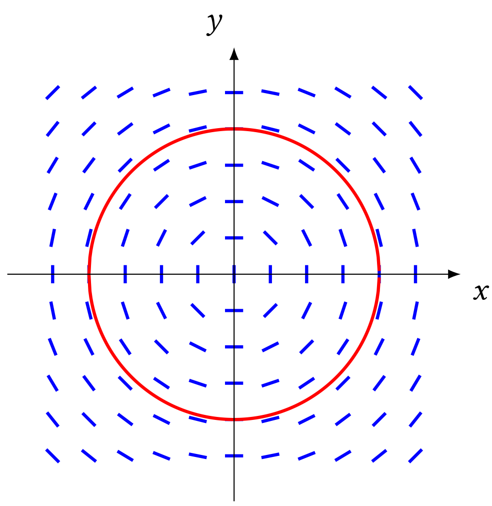

Figure 1.5: The direction field with an integral curve for $y' = -\frac{x}{y}$.

✍ **Exercise 1.8** Consider the differential equation $y' = \frac{x}{y}$. Sketch the direction field and guess a solution.

Solution *The direction field for* $y' = \frac{x}{y}$ *suggests a solution curve is a piece of hyperbola. The general solutions of the differential equations satisfy the equation* $y^2 - x^2 = c$ *(see Figure 1.5). If* $c > 0$*, then an integral curve will look like the red curve in Figure 1.5. If* $c < 0$*, then an integral curve will look like the green curve in Figure 1.5.*

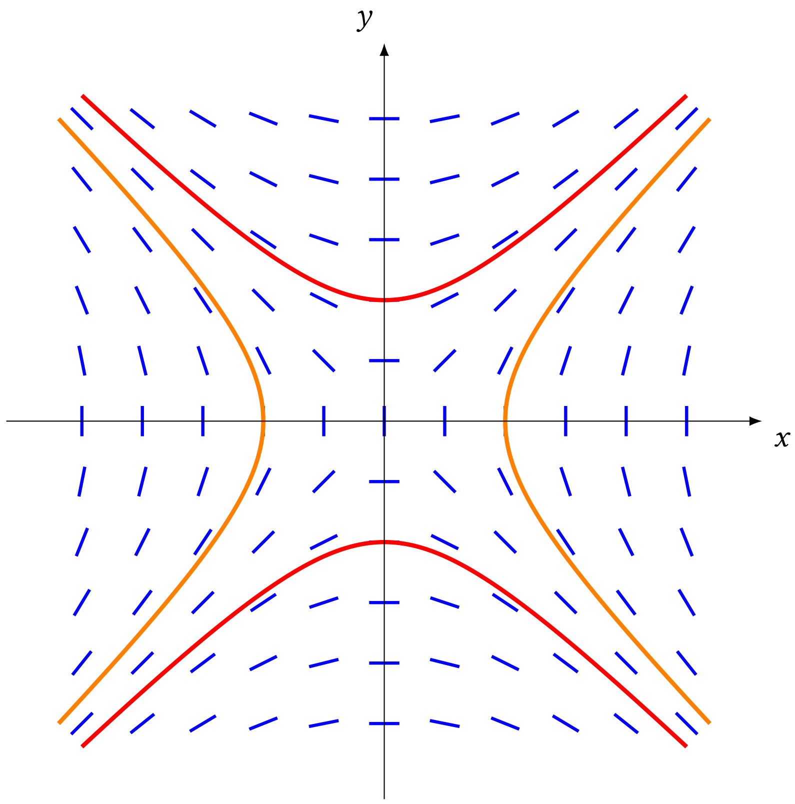

Figure 1.6: The direction field with an integral curve for $y' = \frac{x}{y}$.

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Week 2: Separable and Linear First Order Equations

9/2–9/19

A first order differential equation can always be written in the ***standard form***

$$y' = f(x, y).$$

Sometimes, it might be easier to consider the ***differential form*** of the equation.

$$M(x, y)\,\mathrm{d}\,x + N(x, y)\,\mathrm{d}\,y = 0.$$

Like direction integration, different type of differential equations will be solved using different techniques.

In the coming weeks, you will learn how to solve following types of first order differential equations.

1. Separable equation
$$A(x)\,\mathrm{d}\,x + B(y)\,\mathrm{d}\,y = 0.$$
or
$$y' = \frac{F(x)}{G(y)}.$$
2. Linear first order equations
$$y' + p(x)y = q(x).$$
3. Bernoulli equations
$$y' + p(x)y = q(x)y^n.$$
4. Homogeneous equations $y' = f(x, y)$ if $f(tx, ty) = f(x, y)$.
5. Exact equations
$$M(x, y)\,\mathrm{d}\,x + N(x, y)\,\mathrm{d}\,y = 0$$
and
$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}.$$
6. Other nonlinear first order equations that can be solved by substitution.

2.1 Separable Equations

Definition 2.1

A first order differential equation is said to be **separable** *if it can be written as*

$$h(y)y' = g(x) \tag{2.1}$$

Rewriting a separable differential equation in this form is called **separation of variables**. ♣

2.1.1 How to solve separable equations

Suppose that $H(y)$ is an antiderivative of $h(y)$, i.e $\frac{d}{dy}H(y) = h(y)$. Then $\Big(H(y(x))\Big)' = h(y(x))y'(x)$ and Equation 2.1 can be written as

$$\Big(H(y(x))\Big)' = g(x)$$

Suppose $G(x)$ is an antiderviative of $g(x)$. Integrating both sides of the above equation yields

$$H(y(x)) = G(x) + c. \tag{2.2}$$

It can be checked that a solution of Equation 2.1 must satisfy Equation 2.2.

Equation 2.2 is called an ***implicit solution*** of $h(y)y' = g(x)$.

In conclusion, to solve a separable equation, it suffices to find antiderivatives using direct integrations.

From Calculus, we know that continuous functions have an antiderivatives. Therefore, as long as $h(y)$ and $g(x)$ are continuous, the differential equation 2.1 will have an implicit solution. The existence of a solution function follows from a result from advance Calculus called the implicit function theorem. Moreover, the solution of an initial value problem for Equation 2.1 is unique.

If the constant c in Equation 2.2 satisfies the initial condition, then we say the implicit solution is an ***implicit solution of the initial value problem*** for Equation 2.1

Example 2.1 Solve the equation

$$y' = x(1 + y^2).$$

Solution *Separating variables yields*

$$\frac{y'}{1 + y^2} = x.$$

Integrating leads to

$$\tan^{-1} y = \frac{x^2}{2} + c.$$

Therefore, the general solution of the equation is

$$y = \tan\left(\frac{x^2}{2} + c\right).$$

■

Exercise 2.1 Solve the equation

$$y' = \frac{\sin x}{y}$$

Solution *Rewrite the equation as*

$$yy' = \sin x.$$

Integrating gives

$$\frac{1}{2}y^2 = -\cos x + c.$$

Hence, the general solutions are

$$y = \pm\sqrt{-2\cos x + 2c}.$$

■

2.1.2 Constant solutions

In the above example, dividing $1 + y^2$ is an equivalent transformation because it is always non-zero. In general, re-writing the function $y' = g(x)p(y)$ into $\frac{y'}{p(y)} = g(x)$ may lose some constant solutions.

Example 2.2 Solve the differential equation

$$y' = 2xy^2$$

Solution *It can be check that the function $y = 0$ is a solution. Suppose that y is a solution that is not identically zero. Then there must be intervals on which y is never zero. Over this interval, we can separate variables, which yields*

$$\frac{y'}{y^2} = 2x.$$

Integrating both sides leads to

$$-\frac{1}{y} = x^2 + c$$

Therefore, $y = -\frac{1}{x^2+c}$ is the general solution of the equation that is not identically zero. ■

Remark If a first order separable is in the form

$$y' = g(x)p(y).$$

Before rewrite the differential equation in the form

$$\frac{y'}{p(y)} = g(x),$$

we need to check for values of y that make $g(y) = 0$. The equation $g(y) = 0$ often leads to constant solutions.

Exercise 2.2 Find the general solution of

$$y' = e^x y^2.$$

Solution *Setting $y^2 = 0$ gives a constant solution $y = 0$.*

Now suppose that y is not a constant solution.

Rewrite the equation as

$$\frac{y'}{y^2} = e^x.$$

Integrating yields

$$-\frac{1}{y} = e^x + c$$

Therefore, the general solution is

$$y = \frac{1}{-c - e^x}.$$

■

2.1.3 Initial value problems

Example 2.3 Solve the initial value problem

$$y' = \frac{\cos x}{y}, \qquad y(0) = 3.$$

Solution *We first find the general solution. The equation is the same as*

$$yy' = \cos x.$$

Integrating both sides yields

$$\frac{1}{2}y^2 = \sin x + c$$

or

$$y = \pm\sqrt{2c + 2\sin x}.$$

The initial condition leads to

$$y(0) = \sqrt{2c} = 3$$

Hence, $2c = 9$ *and the solution of the initial value problem is*

$$y = \sqrt{9 + 2\sin x}.$$

■

Exercise 2.3 Solve the initial value problem

$$y' = \frac{x^3 + 2}{y^3 + 2}, \qquad y(0) = 1.$$

Solution *Rewrite the equation as*

$$(y^3 + 2)y' = x^3 + 2$$

Integrating both sides yields

$$\frac{1}{4}y^4 + 2y = \frac{1}{4}x^4 + 2x + c.$$

The initial condition implies that

$$c = \frac{1}{4}1^4 + 2 \cdot 1 = \frac{9}{4}.$$

Therefore, the solution of the initial value problem is

$$\frac{1}{4}y^4 + 2y = \frac{1}{4}x^4 + 2x + \frac{9}{4}.$$

■

2.1.4 Autonomous Equations*

Definition 2.2

A differential equation is called autonomous if it can be written as

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = f(x).$$

♣

Autonomous differential equations are separable. The general solution satisfies

$$\int \frac{1}{f(x)}\,\mathrm{d}\,x = t + c.$$

In other words, an autonomous differential equation always has a general implicit solution.

Those equations have many applications.

Example 2.4 The population $P = P(t)$ of a species satisfies the logistic equation

$$P' = aP(1 - \alpha P)$$

and $P(0) = P_0 > 0$, where a and α are constants. Find P for $t > 0$.

Solution *The equation can be re-written as*

$$\frac{P'}{P(1 - \alpha P)} = a.$$

Applying the partial fraction decomposition to left hand side yields

$$\left(\frac{P'}{P} + \frac{\alpha P'}{1 - \alpha P}\right) = a,$$

or equivalently

$$\Big(\ln(P) - \ln(1 - \alpha P)\Big)' = a.$$

Therefore,

$$\frac{P}{1 - \alpha P} = ce^{at}. \tag{2.3}$$

By the initial condition $P(0) = P_0$,

$$c = \frac{P_0}{1 - \alpha P_0}.$$

Solving for P *from Equation 2.3 leads to*

$$\begin{aligned} P &= \frac{P_0 e^{at}}{1 - \alpha P_0 + \alpha P_0 e^{rt}} \\ &= \frac{P_0}{\alpha P_0 + (1 - \alpha P_0)e^{-rt}} \end{aligned}$$

■

✍ **Exercise 2.4** A frozen pizza, initially at $32^\circ F$ is put into an oven that is pre-heated to $400^\circ F$. The pizza warmed up to $50^\circ F$ in 2 minutes. Find how long would it take to reach $200^\circ F$ by using Newton's cooling law $T'(t) = -k(T(t) - T_m(t))$, where $T(t)$ is the temperature of the pizza after t minutes and $T_m(t)$ is the ambient temperature,

Solution *Since the oven temperature is constantly* $400^\circ F$, *the Newton cooling law implies*

$$\frac{dT}{dt} = k(400 - T)$$

Dividing by $(400 - T)$ *and integrating yields*

$$-\ln(400 - T) = kt + C$$

Solving for T *gives the general solution*

$$T = 400 - e^{-kt-C},$$

or equivalent

$$T = 400 - ce^{-kt}.$$

Since the initial temperature of the pizza was 32, that is $T(0) = 32$*, plugging it into the function* $T(t)$ *and solving for* c *gives* $c = 368$*. Since it takes 2 minutes for the temperature to reach* $50^\circ F$*, that is* $T(2) = 50$*, the value* k *must satisfy the equation*

$$50 = 400 - 368e^{-2k}.$$

Solving for k *yields*

$$k = -\frac{1}{2}\ln\frac{350}{368} = 0.02507$$

Thus the temperature of the pizza at time t *is:*

$$T(t) = 400 - 368e^{-0.02507t}.$$

To reach $200^\circ F$*, the time that it takes should satisfies*

$$200 = 400 - 368e^{-0.02507t}.$$

Solving this for t *implies that it take 24.3 minutes for the pizza to reach* $200^\circ F$ ■

2.1.5 More examples

Example 2.5 Solve the differential equation

$$y' = -2xy + 3x$$

Solution *Factoring out* x *from the right-hand side separates variables*

$$y' = -2x(y - 3).$$

Setting $y - 3 = 0$ *leads to a constant solution* $y = 3$.

Suppose that y *is not identically 3. Then the equation may be re-written as*

$$\frac{y'}{y-3} = -2x.$$

Integrating both sides implies

$$\ln(|y-3|) = x^2 + c_1.$$

Solve for y *leads to the general solutions*

$$y = \pm c e^{x^2} + 3,$$

where $c > 0$.

Note that the constant solution may includes in the general solutions $y = \pm c e^{x^2} + 3$ *by allowing* $c = 0$. ■

Example 2.6 Solve the initial value problem

$$y' = e^{x+2y}, \qquad y(0) = 1.$$

Solution *The equation can be re-written as*

$$y' = e^x e^{2y}$$

Since $e^{2y} > 0$ *for any* y, *dividing by* e^{2y} *leads to the following equivalent differential equation*

$$e^{-2y} y' = e^x$$

Direct integration yields

$$-\frac{1}{2} e^{-2y} = e^x + c.$$

Solving for y *gives the general solution*

$$y = -\frac{1}{2} \ln(-2e^x - 2c).$$

The initial condition $y(0) = 1$ *implies that*

$$1 = -\frac{1}{2} \ln(-2 - 2c).$$

Therefore, $c = -\frac{1}{2} e^{-2} - 1$. *Then the solution of the initial value problem is*

$$y = -\frac{1}{2} \ln(-2e^x + e^{-2} + 2).$$

■

2.2 Linear First Order Equations

Definition 2.3

A first order differential equation is called **linear** *if it can be written as*

$$y' + p(x)y = f(x).$$

A first order differential equation that cannot be written like this is **nonlinear***.*
A linear first order differential equation is said to be homogeneous if $f(x)$ is identically 0. ♣

Example 2.7 Determine whether the equation is linear, homogeneous linear or not.

1. $y' - 2y = -e^x$.
2. $x^2y' + e^x y = 0$.
3. $xy' + y^2 = 0$.

Solution

1. *$y' - 2y = -e^x$ is a linear first order differential equation because the highest order and exponents of y are both 1.*
2. *$x^2y' + e^x y = 0$ is also a linear first order differential equation. Moreover, it is homogeneous because, it has no constant term, i.e. it is in the form $y' + p(x)y = 0$.*
3. *$xy' + y^2 = 0$ is not linear because the highest exponent of y is 2.*

■

Note that a homogeneous linear first order differential equation $y' + p(x)y = 0$ is as special separable differential equation. It always has $y = 0$ as a solution. We call it the trivial solution.

If $p(x) = 0$, the the equation becomes $y' = f(x)$ which can be solve by direct integration if $f(x)$ is continuous.

How can we solve a general linear non-homogeneous differential equation? There are two methods both uses the product rule in some ways.

2.2.1 The integrating factor method

Let's first see an example.

Example 2.8 Solve the equation

$$y' + \frac{2}{x}y = \frac{e^x}{x^2}.$$

Solution *Let's first clear the denominator by multiplying x^2 to both sides which yields*

$$2xy + x^2y' = e^x.$$

Now, notice that the left side is same as $\left(x^2 y\right)'$ by the product rule. Therefore, the equation can be re-written as

$$\left(x^2 y\right)' = e^x$$

Integrating both sides leads to

$$x^2 y = e^x + c.$$

Solving for y gives the general solution

$$y = \frac{1}{x^2}e^{3x} + \frac{C}{x^2}.$$

■

It seems that we are lucky that the left hand side becomes the derivative of a product. But it also suggests that we may look for a multiplier so that the left hand side will be the derivative of a product. The existence a multiplier should be clear once we find it.

Let's suppose that there is a function $r(x)$ such that

$$r(x)y' + r(x)p(x)y = \Big(r(x)y\Big)'.$$

Apply the product rule to $r(x)y$ and compare both sides of the above equation, we see that $r(x)$ must satisfy the separable differential equation

$$r'(x) = r(x)p(x).$$

Solving this separable differential equation, we get $r(x) = e^{\int p(x)\,\mathrm{d}x}$ which is called the ***integrating factor*** for $y' + p(x)y = f(x)$. After finding $r(x)$, then the linear first order differential equation becomes

$$\Big(r(x)y\Big)' = r(x)f(x)$$

which has a solution

$$y = e^{-\int p(x)\,\mathrm{d}x}\left(\int e^{\int p(x)\,\mathrm{d}x} f(x)\,\mathrm{d}x + c\right).$$

This method is called the ***integrating factor method***.

Remark Note that it doesn't matter which antiderivative we take when computing the integrating factor. Because, it will eventually alter the constant c by a factor.

In the following, we will abuse the notation $\int p(x)\,\mathrm{d}x$ and set it equals a specific antiderivative.

Example 2.9 Find the general solution of

$$y' + 2y = x^3 e^{-2x}.$$

Solution *The integrating factor $r(x)$ can be taken as*

$$r(x) = e^{\int 2\,dx} = e^{2x}.$$

Multiplying both sides of the equation by e^{2x} transforms it to

$$e^{2x}y' + 2e^{2x}y = x^3,$$

or equivalently

$$\left(e^{2x}y\right)' = x^3.$$

Integrating both sides yields

$$ye^{2x} = \frac{1}{4}x^4 + c.$$

So $y = \frac{1}{4}e^{-2x}(x^4 + c)$ is the general solution of the equation. ■

✍ **Exercise 2.5** Find the general solution of

$$y' + y = 1.$$

Solution *The integrating factor is*

$$r(x) = e^{\int dx} = e^x.$$

Multiplying both sides with the integrating factor e^x leads to

$$e^x y' + e^x y = e^x$$

which is the same as

$$(e^x y)' = e^x.$$

Integrating both side gives the equation

$$e^x y == e^x + c.$$

Hence, the general solution of the equation is

$$y = 1 + ce^{-x}.$$

■

Example 2.10 Solve the initial value problem.

$$y' + 4y = e^{-4x} \qquad y(1) = 3.$$

Solution *Since $p(x) = 4$, the integral factor is $\int p(x)\,\mathrm{d}x = 4x$. Multiplying the equation with e^{4x} leads to*

$$e^{4x}y' + 4e^{4x}y = 1$$

which is the same as

$$(e^{4x}y)' = 1$$

Integrating both sides yields

$$e^{4x}y = x + c.$$

So the general solution is

$$y = xe^{-4x} + ce^{-4x}.$$

The initial condition means when x=1, $y = 4$. Plugging the point into the function y produces an equation of c

$$e^4 \cdot 3 = 1 + c,$$

which implies that

$$c = 3e^4 - 1.$$

Hence, the solution to the initial value problem is

$$y = xe^{-4x} + (3e^4 - 1)e^{-4x}.$$

■

✍ **Exercise 2.6** Solve the initial value problem

$$y' - 3y = e^x, \qquad y(0) = 0.$$

Solution *Since $p(x) = -3$, $\int p(x)\,\mathrm{d}x = -3x$ and the integrating factor is e^{-3x}. Multiplying both sides with e^{2x} yields*

$$e^{-3x}y' - 3e^{-3x}y = e^{-2x}$$

The product rule and chain rule together implies that

$$e^{-3x}y' + 2e^{-3x}y = (e^{-3x}y)'.$$

Hence, the original equation can be transformed into

$$(e^{-3x}y)' = e^{-2x}.$$

Integrating both sides yields

$$e^{-3x}y = -\frac{1}{2}e^{-2x} + c.$$

Solving for y gives the general solution

$$y = -\frac{1}{2}e^x + ce^{3x}.$$

Since $y(0) = 0$, c satisfies

$$0 = -\frac{1}{2} + c,$$

or equivalently $c = \frac{1}{2}$.

Therefore, the solution to the initial value problem is

$$y = -\frac{1}{2}e^x + \frac{1}{2}e^{3x}.$$

■

2.2.2 The method of variation of parameters

If you read Trench's book, you will find there is another method called *variation of parameters*. The idea is to solve the ***complimentary*** linear homogeneous equation $y' + p(x)y = 0$ first to get a solution y_o. Then find a "parameter" $u(x)$ such that $y = uy_o$ is a solution of the original linear non-homogeneous equation. By plugging uy_o into the non-homogeneous linear equation and comparing both sides, you will find that $u' = \frac{f(x)}{y_o}$ which can be solved by direct integration.

A possible motivation of the idea is as follows. Let y_p be a particular solution for $y' + p(x)y = f(x)$. Then $y = y_o + y_p$ is also a solution. Whatever the y_p is, $y = y_o(1 + \frac{y_p}{y_o})$ is a solution. Here $u = 1 + \frac{y_p}{y_o}$ is the parameter that varies.

Both the integrating factor method and then variation of parameters work equally well for first order equations. For higher order differential equations, they have there own advantages and limitations. The integrating factor method can also be used for exact equations which includes all linear first order equations. The method of variation of parameters can be used for some nonlinear first order or linear higher order equations.

Example 2.11 Find the general solution of

$$y' + 2y = 4x.$$

Solution *First, we solve the complimentary linear homogeneous equation*

$$y' + 2y = 0.$$

Note that the constant solution $y = 0$ won't be a solution to the original equation. So we assume *that y is not identically zero and hence dividing y is legitimate. The linear homogeneous equation can be re-written as*

$$\frac{y'}{y} = -2x.$$

Integrating both sides implies $y = e^{-2x}$. Again, you can check that adding a constant won't change the general solution.

Now, we want to find a function u such that ue^{-2x} is a solution of the original equation.

Suppose that ue^{-2x} is a solution of the original equation. Then the function u must satisfy the following equation

$$u'e^{-2x} - 2ue^{-2x} + 2u^{-2x} = 4x,$$

or equivalently

$$u' = 4xe^{2x}.$$

Integrating using integration by parts yields

$$u = 2xe^{2x} + 2e^{2x} + c.$$

Therefore, the general solution of the original equation is

$$y = ue^{-2x} = 2x + 2 + ce^{-2x}.$$

■

✍ **Exercise 2.7** Find the general solution of

$$y' + 2y = x^3e^{-2x}.$$

using the method of variation of parameters.

Solution *Solving the homogeneous linear equation $y' + 2y = 0$ produces a solution $y = e^{-2x}$. Suppose that $y = ue^{-2x}$ is a solution of the equation $y' + 2y = x^3e^{-2x}$. Then the function u satisfy the following differential equation.*

$$u'e^{-2x} - 2ue^{-2x} + 2ue^{-2x} = x^3e^{-2x},$$

or equivalently

$$u' = x^3.$$

Therefore,

$$u = \frac{x^4}{4} + c,$$

and

$$y = ue^{-2x} = \frac{1}{4}e^{-2x}\left(x^4 + c\right)$$

is the general solution. ■

2.3 Change of Variables

Some non-linear first order differential equations may be solve by changing variable. In this section, we will look into two of such types.

2.3.1 Bernoulli Equations

Definition 2.4

A **Bernoulli first order** *differential equation is a differential equation of the form*

$$y' + p(x)y = q(x)y^n.$$

♣

On way to solve this type of equation is to use the method of variation of parameters.

Example 2.12 Solve the Bernoulli equation

$$y' - y = xy^2.$$

Solution *Solving the complimentary homogeneous equation $y' - y = 0$ yields a solution $y_o = e^x$. We look for solutions of the original equation in the form $y = ue^x$. Plugging $y = ue^x$ into the Bernoulli equation yields*

$$u'e^x = xu^2e^{2x},$$

or equivalently

$$u' = xu^2e^x$$

which is a separable equation. Separating variables and integrating both sides gives

$$\begin{aligned}\frac{u'}{u^2} &= xe^x\\ -\left(\frac{1}{u}\right)' &= xe^x\\ -\frac{1}{u} &= xe^x - e^x + c.\end{aligned}$$

Hence,

$$u = -\frac{1}{(x-1)e^x + c}$$

and

$$y = -\frac{1}{x - 1 + ce^{-x}}.$$

■

Another way is to use a substitution to reduce the equation to a linear equation. Indeed, suppose y is not identically zero. Then dividing y^n from both sides yields

$$y' y^{-n} + p(x)y^{-(n-1)} = q(x).$$

Notice that $y' y^{-n} = \left(y^{-(n-1)}\right)'$. Let $v = y^{-(n-1)}$, then the equation becomes

$$v' + p(x)z = q(x)$$

which can be solved by multiple methods.

Example 2.13 Find the solution of the initial value problem

$$y' + 2y = y^3 \qquad y(0) = 2.$$

Solution *Clearly $y = 0$ is a trivial solution of the differential equation but not a solution of the initial value problem. Assume that y is not identically zero. Since $n = 3$, let $v = y^{-2}$. Dividing y^3 from both sides of the equations implies that z satisfies the equation.*

$$v' - 4v = -2.$$

According to the method of integrating factor, the integrating factor is

$$r(x) = e^{\int (-4)\,\mathrm{d}x} = e^{-4x}.$$

Multiplying with $r(x)$ and integrating both sides yields

$$\begin{aligned} v' e^{-4x} - 4ve^{-4x} &= -2e^{-4x} \\ \left(ve^{-4x}\right)' &= -2e^{-4x} \\ ve^{-4x} &= \frac{1}{2}e^{-4x} + c \\ v &= \frac{1}{2} + ce^{4x}. \end{aligned}$$

As $v = y^{-2}$, then the general solutions are

$$y = \frac{\pm 1}{\sqrt{v}} = \frac{\pm 1}{\sqrt{\frac{1}{2} + ce^{4x}}}.$$

Since $y(0) = 2$, the constant c satisfies

$$\frac{\pm 1}{\sqrt{\frac{1}{2} + ce^{4x}}} = 2.$$

Because square root is nonnegative, only the equation

$$\frac{1}{\sqrt{\frac{1}{2} + ce^{4x}}} = 2$$

may have a solution. Solving for c gives

$$c = -\frac{1}{4}.$$

Therefore, the solution of the initial value problem is

$$y = \frac{1}{\sqrt{\frac{1}{2} - \frac{1}{4}e^{4x}}}.$$

■

Remark Note that in this exercise if we change the initial condition to $y(0) = -2$ the solution will be $y = -\frac{1}{\sqrt{\frac{1}{2} - \frac{1}{4}e^{4x}}}$. This suggests that the solution of an initial value problem may depend on the initial condition.

✍ **Exercise 2.8** Solve the initial value problem

$$y' - y = \frac{1}{y}, \qquad y(0) = 1.$$

Solution *Multiplying both sides with y and applying the substitution $v = y^2$ implies that v satisfies*

$$v' - 2v = 2$$

The integrating factor is

$$r(x) = e^{-2x}$$

and the equation for v is equivalent to

$$(ve^{-2x})' = 2e^{-2x}$$

whose general solution is

$$v = e^{2x} \int 2e^{-2x}\, \mathrm{d}x = e^{2x}(c - e^{-2x}) = ce^{2x} - 1.$$

Hence

$$y = \pm\sqrt{v} = \pm\sqrt{ce^{2x} - 1}.$$

Since $y(0) = 1$, *then* $c = 2$ *and* $y = \sqrt{2e^{2x} - 1}$. ■

Week 3: Substitution Methods and Exact Equations

9/20–9/26

3.1 Substitution Methods

3.1.1 Homogeneous Equations

Definition 3.1

A function f is said to be **homogeneous of degree** m *if* $f(tx, ty) = t^m f(x, y)$ *for any nonzero constant* t.

Example 3.1 The functions $z = \frac{y}{x}$ and $z = \frac{x}{y}$ are homogeneous of degree 0.

The function $z = \frac{xy}{x^2+y^2}$ is also homogeneous of degree 0.

But the function $z = x^2 + xy - y^2$ is homogeneous of degree 2.

From the example, you may conjecture that a homogeneous functions of degree 0 is a function of the single variable $v = xy$. Indeed, the following theorem confirms that.

Theorem 3.1

A function $f(x, y)$ is homogeneous of degree zero if and only if it depends on $\frac{y}{x}$ only.

Proof If $f(x, y)$ is homogeneous of degree zero, then

$$f(x, y) = f\left(\frac{1}{x} \cdot x, \frac{1}{x} \cdot y\right) = f\left(1, \frac{y}{x}\right)$$

where the right hand side is a function depends only on $\frac{y}{x}$.

Conversely, if $f(x, y) = g\left(\frac{y}{x}\right)$, where g is a single variable function, then

$$f(tx, ty) = g\left(\frac{ty}{tx}\right) = g\left(\frac{y}{x}\right) = f(x, y).$$

■

Definition 3.2

The differential equation $\frac{dy}{dx} = f(x, y)$ is called a **homogeneous first order differential equation** *if f is homogeneous of degree* 0.

To solve a homogeneous first order differential equation, we substitute y by xv, where $v = \frac{y}{x}$. Then

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = v + x\frac{\mathrm{d}\,v}{\mathrm{d}\,x}.$$

By Theorem 3.1, the homogeneous first order differential equation $\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = f(x, y)$ can be reduced to the following separable equation

$$x\frac{\mathrm{d}\,v}{\mathrm{d}\,x} = q(v) - v,$$

where q is a single variable function such that $f(x, y) = q(\frac{x}{y})$.

Example 3.2 Solve

$$y' = e^{-\frac{y}{x}} + \frac{y}{x}.$$

Solution

Step 1: *Since $\frac{ty}{tx} = \frac{y}{x}$ for any nonzero number t, the function $z = e^{-\frac{y}{x}} + \frac{y}{x}$ is homogeneous of degree 0 and the differential equation is a homogeneous first order equation.*

Step 2: *Consider the new unknown function $v = \frac{y}{x}$. The function z can be re-written as*

$$z = e^{-\frac{y}{x}} + \frac{y}{x} = e^{-v} + v.$$

Step 3: *Differentiating the equation $y = xv$ with respect to x using the product rule implies that*

$$y' = v + xv'.$$

Step 4: *Then the original equation can be transformed into*

$$\begin{aligned} v + xv' &= e^{-v} + v \\ xv' &= e^{-v} \end{aligned}$$

Step 5: *Note that the equation $xv' = e^{-v}$ is a separable equation which can be re-written as*

$$e^{v}v' = \frac{1}{x}$$

Integrating both side yields

$$e^{v} = \ln|x| + c.$$

Therefore,

$$v = \ln(\ln|x| + c),$$

Step 6: *Substituting v by $\frac{x}{y}$ and solving for y yields the general solution*

$$y = x\ln(\ln|x| + c).$$

■

✍ **Exercise 3.1** Find the general solution of

$$y' = \frac{y + x}{x}$$

Solution

Step 1: The equation is homogeneous of first order. Because

$$\frac{(ty) + (tx)}{tx} = \frac{t(y + x)}{tx} = \frac{y + x}{x}.$$

Step 2: Set $y = xv$. The right-hand side function can be rewritten as the follows

$$\frac{y + x}{x} = \frac{xv + x}{x} = v + 1.$$

Step 3: Applying the product rule to $y = xv$ with respect to x yields

$$y' = v + xv'.$$

Step 4: The original equation is transformed into the following separable equation

$$\begin{aligned} v + xv' &= v + 1 \\ xv' &= 1. \end{aligned}$$

Step 5: Solving the resulting equation yields

$$\begin{aligned} v' &= \frac{1}{x} \\ v &= \ln|x| + c \end{aligned}$$

Step 6: Solving for y gives the general solution

$$y = x(\ln|x| + c).$$

■

✍ **Exercise 3.2** Solve

$$y' = \frac{xy - y^2}{x^2}.$$

Solution

Step 1: The equation is homogeneous of first order because

$$\frac{(tx)(ty)-(ty)^2}{(tx)^2}=\frac{t^2(xy-y^2)}{t^2x^2}=\frac{xy-y^2}{x^2}.$$

Step 2: Set $y=xv$. Rewriting the function by substitution implies

$$\frac{xy-y^2}{x^2}=\frac{x(xv)-(xv)^2}{x^2}=\frac{x^2v-x^2v^2}{x^2}=v-v^2.$$

Step 3: Differentiating $y=xv$ with respect to x gives

$$y'=v+xv'.$$

Step 4: The original equation can be transformed into

$$v+xv'=v-v^2.$$

Step 5: Solving the equation for v yields

$$\begin{aligned} v+xv' &= v-v^2\\ xv' &= -v^2\\ -\frac{v'}{v^2} &= \frac{1}{x}\\ \left(\frac{1}{v}\right)' &= \frac{1}{x}\\ \frac{1}{v} &= \ln|x|+c\\ v &= \frac{1}{\ln|x|+c}. \end{aligned}$$

Step 6: Substituting $v=\frac{x}{y}$ and solving for y gives the general solution.

$$y=\frac{x}{\ln|x|+c}.$$

■

3.1.2 Linear substitution*

For a first order differential equation $F(x,y,y')=0$, sometimes, a substitution $y=u(x,v)$, where $v=v(x,y)$ is a function that is linear in y, may reduce the equation into a new equation that is much easier to solve.

Here are a few classes of equations that can be reduced to separable equations using a linear substitution.

If the equation is in the form $y' = f(ax + by + c)$, then the substitution $v = ax + by + c$ reduces the equation into a separable equation $v' = f(v) - a$.

Example 3.3 Solve the equation

$$y' = (2x + y - 3)^2.$$

Solution *Let $v = 2x + y - 3$. Differentiating it with respect to x yields*

$$v' = 2 + y',$$

or equivalently

$$y' = v' - 2.$$

Then substituting $2x + y - 3$ by the function v yields

$$v' - 2 = v^2$$

or equivalently,

$$\frac{v'}{v^2 + 2} = 1.$$

Integrating both sides yields

$$\begin{aligned}
\frac{1}{\sqrt{2}} \arctan(\frac{v}{\sqrt{2}}) &= x + C \\
\arctan(\frac{v}{\sqrt{2}} &= x\sqrt{2} + C\sqrt{2} \\
\frac{v}{\sqrt{2}} &= \tan(x\sqrt{2} + C\sqrt{(2)}) \\
v &= \sqrt{2}\tan(x\sqrt{2} + C\sqrt{2}).
\end{aligned}$$

Replacing $C\sqrt{2}$ by c and v by $x + y - 3$, and solving for y gives the general solution

$$y = \sqrt{2}\tan(x\sqrt{2} + c) - 2x + 3.$$

■

If the equation is in the form $xy' = yF(xy)$, then the substitution $v = xy$ reduces the equation into a separable equation $v' = \frac{v}{x}(F(v) + 1)$.

Example 3.4 Solve the equation

$$xy' = xy^2 - y.$$

Solution *Let $v = xy$. Then $xy' = v' - y$. Substituting $y = \frac{v}{x}$ yields*

$$\begin{aligned} v' - y &= vy - y \\ v' &= vy \\ v' &= \frac{v^2}{x} \\ \frac{v'}{v^2} &= \frac{1}{x}. \end{aligned}$$

Integrating both sides gives a solution

$$-\frac{1}{v} = \ln(|x|) + c.$$

Replacing v by xy implies

$$y = -\frac{1}{x(\ln|x| + c)}.$$

■

3.2 Exact Equations

Using implicit differentiation, one can show that $F(x, y) = c$ (with c as a constant) is an implicit solution of the differential equation

$$\frac{\partial}{\partial x}F(x, y)\,\mathrm{d}x + \frac{\partial}{\partial y}F(x, y)\,\mathrm{d}y = 0.$$

This observation suggests an approach to solve so-called exact equations.

Definition 3.3

A first order differential equation

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$$

is said to be **exact** *if there is a function $F(x, y)$ such that*

$$\begin{aligned} \frac{\partial}{\partial x}F(x, y) &= \frac{\partial}{\partial x}F(x, y) = M(x, y) \\ \frac{\partial}{\partial y}F(x, y) &= \frac{\partial}{\partial y}F(x, y) = N(x, y). \end{aligned}$$

♣

The above definition not only defines exact functions but also gives solutions. However, it is not a very practical criterion. Given an equation of the form

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0,$$

how do we know it is exact? Knowing the equation is exact, how do we find F?

The first question is answered by the following theorem. The proof of the theorem will also answer the second question.

Theorem 3.2 (Exactness Condition)

Suppose that M and N are continuous and have partial derivatives $M_y = \frac{\partial M}{\partial y}$ and $N_x = \frac{\partial N}{\partial x}$ on an open rectangle R. Then the equation

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$$

is exact if and only if

$$N_x(x,y) = M_y(x,y).$$

on R.

Proof If the equation is exact, then there exists a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$. Therefore, by the fact that

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}F = \frac{\partial}{\partial y}\frac{\partial}{\partial x}F,$$

we know that $N_x(x,y) = M_y(x,y)$.

Conversely, suppose that $N_x(x,y) = M_y(x,y)$. We can find a F such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$. Integrating both sides of the equation $\frac{\partial F}{\partial x} = M$ yields that

$$F(x,y) = \int M\,\mathrm{d}x + g(y),$$

where g is a single variable function. Because $\frac{\partial F}{\partial y} = N$. The function g must satisfies the equation

$$\begin{aligned}\frac{\partial F}{\partial y} &= \frac{\partial}{\partial y}\left(\int M\,\mathrm{d}x + g(y)\right)\\ N &= \frac{\partial}{\partial y}\int M\,\mathrm{d}x + g'(y)\end{aligned}$$

. This yields

$$g(y) = N - \frac{\partial}{\partial y}\int M\,\mathrm{d}x.$$

Because

$$\frac{\partial}{\partial y}\int M\,\mathrm{d}x = \int \frac{\partial M}{\partial y}\,\mathrm{d}x.$$

Then

$$\frac{\partial F}{\partial y} = \int \frac{\partial M}{\partial y}\,\mathrm{d}x = \int \frac{\partial N}{\partial x}\,\mathrm{d}x = N(x,y).$$

This shows that

$$M\,\mathrm{d}x + N\,\mathrm{d}x = \mathrm{d}F = \frac{\partial F}{\partial x}\,\mathrm{d}x + \frac{\partial F}{\partial y}\,\mathrm{d}y.$$

Hence, $M\,\mathrm{d}x + N\,\mathrm{d}x = 0$ is exact. ■

The definition of exact equations suggests the following approach to find F such that $F(x,y) = c$ is the general solution.

Step 1: Integrating both sides of the equation $\frac{\partial F}{\partial x} = M$ yields

$$F(x,y) = \int M\,\mathrm{d}x + g(y),$$

where g is a singular variable function.

Step 2: The equation $\frac{\partial F}{\partial y} = N$ together with $F(x,y) = \int M\,\mathrm{d}x + g(y)$ implies that g satisfies the following equation to obtain a function

$$\frac{\partial}{\partial y}\int M\,\mathrm{d}x + g'(y) = N.$$

Solve for g.

Step 3: The equation $F(x,y) = c$ gives the general solution of $M\,\mathrm{d}x + N\,\mathrm{d}y = 0$.

Example 3.5 Check whether the equation

$$(\sin x + y)\,\mathrm{d}x + (e^y + x)\,\mathrm{d}y = 0$$

is exact.

Solution *The coefficient function of* $\mathrm{d}x$ *is* $M(x,y) = \sin x + y$. *Taking its partial derivative with respect to* y *yields*

$$\frac{\partial}{\partial y}M(x,y) = 1.$$

The coefficient function of $\mathrm{d}x$ *is* $N(x,y) = e^y + x$. *Taking its partial derivative with respect to* x *yields*

$$\frac{\partial}{\partial x}N(x,y) = 1.$$

Because $\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)$. *By Theorem 3.2, the equation is exact.* ■

✍ **Exercise 3.3** Check whether the equation

$$(\sin x + xy)\,\mathrm{d}x + (e^y + xy)\,\mathrm{d}y = 0$$

is exact.

Solution *Because*

$$\begin{aligned}\frac{\partial}{\partial y}M(x,y) &= \frac{\partial}{\partial y}(\sin x + xy) &&= x,\\ \frac{\partial}{\partial x}N(x,y) &= \frac{\partial}{\partial x}(e^y + xy) &&= y,\end{aligned}$$

*and they are not equal. The equation is **NOT** exact.* ■

The inverse direction will become clear after we answer the second question: How to find $F(x, y)$?

The definition of exactness suggests that F is a solution for both $\frac{\partial}{\partial x}F = M$ and $\frac{\partial}{\partial F} = N$. So the idea to solve is to integrate on the those two equations for F and plug in the other to determined the undetermined single variable function.

Example 3.6 Solve

$$(4x^3y^3 + 3x^2)\,\mathrm{d}\,x + (3x^4y^2 + 6y^2)\,\mathrm{d}\,y = 0.$$

Solution

Step 1: Check exactness.

Because

$$M(x, y) = 4x^3y^3 + 3x^2, N(x, y) = 3x^4y^2 + 6y^2.$$

Then

$$\frac{\partial}{\partial y}M(x, y) = \frac{\partial}{\partial x}N(x, y) = 12x^3y^2$$

for all (x, y). Therefore, by the exactness condition theorem, there's a function F such that

$$F_x(x, y) = M(x, y) = 4x^3y^3 + 3x^2$$

and

$$F_y(x, y) = N(x, y) = 3x^4y^2 + 6y^2$$

for all (x, y).

Step 2: Integrate F_x or F_y.

To find F, we integrate Equation $F_x(x, y) = 4x^3y^3 + 3x^2$ with respect to x to obtain

$$F(x, y) = x^4y^3 + x^3 + g(y),$$

where $g(y)$ is the "constant term" of integration with respect to x.

Step 3: Differentiate F.

To determine g so that F also satisfies the equation $F_y(x, y) = 3x^4y^2 + 6y^2$, assume that g is differentiable and differentiate $F(x, y) = x^4y^3 + x^3 + g(y)$ with respect to y. That gives

$$F_y(x, y) = 3x^4y^2 + g'(y).$$

Step 4: Determine the equation for g'.

Comparing this equation with the equation $F_y(x, y) = 3x^4y^2 + 6y^2$ shows that

$$g'(y) = 6y^2.$$

Step 5: Integrate g'.
Integrating this equation with respect to y yields

$$g(y) = 2y^3.$$

Note that here we take the constant of integration to be zero because adding a constant won't change the general implicit solution $F(x, y) = c$.

Step 6: Find the general implicit solution.
Substituting g in $F(x, y)$ using this equation yields

$$F(x, y) = x^4y^3 + x^3 + 2y^3 + C.$$

Now Theorem 3.2 implies that

$$x^4y^3 + x^3 + 2y^3 = c$$

is an implicit solution of Equation.

Step 7: Find the explicit solution(s).
Solving this for y yields the explicit solution

$$y = \left(\frac{c - x^3}{2} + x^4\right)^{1/3}.$$

■

✍ **Exercise 3.4** Consider the equation

$$(2x + y)\,\mathrm{d}x + (2y + x)\,\mathrm{d}y = 0.$$

1. Verify the equation is exact,
2. Find the general solution.

Solution *Here $M(x, y) = 2x + y$ and $N(x) = 2y + x$. Taking partial derivatives yields that*

$$\frac{\partial}{\partial y}M = 1 = \frac{\partial}{\partial x}N.$$

Hence, by the exactness condition theorem, the equation is exact.

To find $F(x, y)$, integrating $M = 2x + y$ with respect to x to get

$$\int M(x, y)\,\mathrm{d}x = x^2 + yx. = \int (2x + y)\,\mathrm{d}x = x^2 + yx.$$

Note that y is treated as a number in the above integral since the left hand side is partial derivative of x.

Hence

$$F(x, y) = x^2 + xy + g(y),$$

where $g(y)$ *is the constant with respect to* x*.*

Differentiating F *with respect to* y *and equating with* $N(x, y)$ *yields*

$$\frac{\partial}{\partial y}F = x + g'(y) = 2y + x.$$

Hence

$$g'(y) = 2y,$$

and a particular solution

$$g(y) = y^2.$$

Therefore, F *can be taken to be* $F(x, y) = x^2 + xy + y^2$ *and the general solution is*

$$x^2 + xy + y^2 = c.$$

■

Week 4: Integrating Factors and Applications

9/27–9/30

4.1 Integrating Factors for Non-exact Equations

Suppose

$$M(x, y)\,\mathrm{d}\,x + N(x, y)\,\mathrm{d}\,y = 0 \tag{4.1}$$

is not exact. How to solve the equation? If it is a separable or linear first order, we already know how to solve. One of the method is to use integrating factors. It seems reasonable to multiplying a non-zero factor to both sides of Equation (4.1) to get an exact equation.

Definition 4.1

A function $\mu(x, y)$ is called an **integrating factor** *of Equation* (4.1) *if the following equation is exact after multiplying it to both sides of Equation* (4.1)

$$\mu(x, y)M(x, y)\,\mathrm{d}\,x + \mu(x, y)N(x, y)\,\mathrm{d}\,y = 0.$$

♣

Remark It is possible to lose or gain solutions when multiplying by an integrating factor. In general, when using integrating factors, you should check whether any solution to $\mu(x, y) = 0$ is in fact a solution to the original differential equation (4.1).

Do integrating factors always exist?

This question is hard to answer in general. However, if we assume Equation (4.1) has a general solution F, then the integrating factor exists and must be the ratio function

$$\mu(x, y) = \frac{\frac{\partial}{\partial x}F}{M} = \frac{\frac{\partial}{\partial y}F}{N}.$$

How to construct integrating factors?

By Theorem 3.2, a nonzero function $\mu(x, y)$ is an integrating factor if and only if it satisfies the equation

$$\frac{\partial}{\partial y}\Big(\mu(x, y)M(x, y)\Big) = \frac{\partial}{\partial x}\Big(\mu(x, y)N(x, y)\Big),$$

or equivalently

$$\frac{\partial}{\partial y}\mu M - \frac{\partial}{\partial x}\mu N = \left(\frac{\partial}{\partial x}N - \frac{\partial}{\partial y}M\right)\mu. \tag{4.2}$$

Equation 4.2 in general is even harder than the original equation to solve. Nevertheless, there are a few situations where Equation 4.2 can be solved relatively easily.

Clearly, if either $\frac{\partial}{\partial x}\mu = 0$ or $\frac{\partial}{\partial y}\mu = 0$, or equivalently, $\mu(x, y) = q(y)$ or $\mu(x, y) = p(x)$, then the above Equation 4.2 can be solved easily. That leads to the following proposition.

Proposition 4.1 (Constructing Integrating Factor)

Given a differential equation $M(x, y)\,\mathrm{d}\,x + N(x, y)\,\mathrm{d}\,y = 0$, suppose that M, N, $\frac{\partial}{\partial x}M$ and $\frac{\partial}{\partial y}N$ are continuous. Then

Type I: *If*

$$\frac{1}{N(x, y)}\left(\frac{\partial}{\partial_M} - \frac{\partial}{\partial_N}\right) = p(x)$$

is a function only in x, then we take integrating factor as a function of x,

$$\mu(x, y) = e^{\int p(x)\,\mathrm{d}\,x};$$

Type II: *If*

$$\frac{1}{M(x, y)}\left(\frac{\partial}{\partial_M} - \frac{\partial}{\partial_N}\right) = q(y)$$

is a function only in y, then we take integrating factor as a function of y,

$$\mu(x, y) = e^{\int -q(y)\,\mathrm{d}\,y}.$$

Remark There are also some not so-obvious-cases that integrating factors can be constructed.

If $u(x, y) = f(w(x, y))$, where f is a single variable function, then an integrating factor is

$$\mu(x, y) = \int e^{\frac{\frac{\partial}{\partial y}M - \frac{\partial}{\partial x}N}{N\frac{\partial w}{\partial_x} - M\frac{\partial w}{\partial_y}\,\mathrm{d}\,w}}.$$

If M and N are both homogeneous functions, then an integrating factor is

$$\mu(x, y) = \frac{1}{xM(x, y) + yN(x, y)}.$$

If $M = yp(xy)$, $N = xq(xy)$, and $p(xy) \neq q(xy)$, then an integrating factor is

$$\mu(x, y) = \frac{1}{xM - yN}.$$

For more information, see the section on Integrating Factors by Vladimir Dobrushkin.

Example 4.1 Consider the equation

$$(2x^2 + yx)\,\mathrm{d}\,x + (2xy + x^2)\,\mathrm{d}\,y = 0.$$

1. Show that the equation is not exact;

2. Show that $\dfrac{\frac{\partial}{\partial y}M(x,y) - \frac{\partial}{\partial x}N(x,y)}{N(x,y)}$ depends only on x;
3. Reduce the equation to an exact equation.

Solution

1. *Because* $M(x,y) = 2x^2 + xy$ *and* $N(x,y) = 2xy + x^2$. *The partial derivatives are*

$$\begin{aligned}\frac{\partial}{\partial y}M &= x,\\ \frac{\partial}{\partial x}N &= 2y + 2x.\end{aligned}$$

   Hence $\frac{\partial}{\partial y}M \neq \frac{\partial}{\partial x}N$, *and the equation is not exact.*
2. *We compute*

$$\frac{\frac{\partial}{\partial y}M(x,y) - \frac{\partial}{\partial x}N(x,y)}{N(x,y)} = \frac{x - (2y + 2x)}{2xy + x^2} = \frac{-x - 2y}{x(2y + x)} = -\frac{1}{x},$$

   which depends only on x.
3. *By Proposition 4.1, We find an integrating factor of type I,*

$$\mu = e^{\int \frac{\frac{\partial}{\partial y}M(x,y) - \frac{\partial}{\partial x}N(x,y)}{N(x,y)}\,\mathrm{d}x} = e^{\int -\frac{1}{x}\,\mathrm{d}x} = e^{-\ln x} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

   Multiplying the equation by the integrating factor μ *yields*

$$\begin{aligned}\frac{1}{x}((2x^2 + yx)\,\mathrm{d}x + (2xy + x^2)\,\mathrm{d}y) &= 0\\ (2x + y)\,\mathrm{d}x + (2y + x)\,\mathrm{d}y &= 0.\end{aligned}$$

   This equation is exact because

$$\frac{\partial}{\partial y}(2x + y) = 1 = \frac{\partial}{\partial x}(2y + x).$$

■

Example 4.2 Consider the equation

$$y\,\mathrm{d}x + (y - x)\,\mathrm{d}y = 0.$$

1. Show that the equation is not exact, and $\dfrac{\frac{\partial}{\partial y}M(x,y) - \frac{\partial}{\partial x}N(x,y)}{M(x,y)}$ depends only on y.
2. Reduce the equation to an exact equation.
3. Solve the equation.

Solution

1. *Because $M(x, y) = y$, $N(x, y) = y - x$ and*

$$\frac{\partial}{\partial y} M(x, y) = 1$$
$$\frac{\partial}{\partial x} N(x, y) = -1$$

The equation is not exact but

$$\frac{\frac{\partial}{\partial y} M(x, y) - \frac{\partial}{\partial x} N(x, y)}{M(x, y)} = \frac{2}{y}$$

depends only on y.

2. *By Proposition 4.1, we look for an integrating factor of type II,*

$$I = e^{\int -\frac{\frac{\partial}{\partial y} M(x,y) - \frac{\partial}{\partial x} N(x,y)}{M(x,y)} \, \mathrm{d}\, y} = e^{-\int \frac{2}{y} \, \mathrm{d}\, x} = e^{-2 \ln y} = (e^{\ln(x)})^{-2} = y^{-2} = \frac{1}{y^2}.$$

Multiplying μ to the non-exact equation yields

$$\frac{1}{y^2}(y \, \mathrm{d}\, x + (y - x) \, \mathrm{d}\, y) = 0$$
$$\frac{1}{y} \, \mathrm{d}\, x + \frac{y - x}{y^2} \, \mathrm{d}\, y = 0$$

This is an exact equation because

$$\frac{\partial}{\partial y}\left(\frac{1}{y}\right) = -\frac{1}{y^2}$$
$$\frac{\partial}{\partial x}\left(\frac{y - x}{y^2}\right) = -\frac{1}{y^2}$$

3. *We look for F such that*

$$\frac{\partial}{\partial x} F \; \mathrm{d}\, x + \frac{\partial}{\partial y} F \; \mathrm{d}\, y = \frac{1}{y} \, \mathrm{d}\, x + \frac{y - x}{y^2} \, \mathrm{d}\, y.$$

Equivalently,

$$\begin{cases} \frac{\partial}{\partial x} F = \frac{1}{y} \\ \frac{\partial}{\partial y} F = \frac{y-x}{y^2} \end{cases}$$

Integrating the first equation with respect to x implies

$$F = \int \frac{1}{y} \, \mathrm{d}\, x = \frac{x}{y} + h(y).$$

Since F has to be a solution solves the second equation and

$$\frac{\partial}{\partial y}F = \frac{\partial}{\partial y}\left(\frac{x}{y} + h(y)\right) = -\frac{x}{y^2} + h'(y),$$

the function $h(y)$ must satisfy the equation

$$-\frac{x}{y^2} + h'(y) = \frac{y-x}{y^2},$$

or equivalently,

$$h'(y) = \frac{1}{y}.$$

Therefore, a solution is $h = \ln y$. Hence,

$$F = \frac{x}{y} + \ln y$$

and the general solution is

$$F = \frac{x}{y} + \ln y = c.$$

■

Exercise 4.1 Consider the equation

$$-y\,\mathrm{d}\,x + x\,\mathrm{d}\,y = 0.$$

1. Show that the equation is not exact, and $\dfrac{\frac{\partial}{\partial y}M(x,y) - \frac{\partial}{\partial x}N(x,y)}{N(x,y)}$ depends only on x.
2. Reduce the equation to an exact equation.
3. Solve the equation.

Solution

1. *Because $M(x,y) = -y$, $N(x,y) = x$ and*

$$\frac{\partial}{\partial y}M(x,y) = -1$$
$$\frac{\partial}{\partial x}N(x,y) = 1$$

The equation is not exact but

$$\frac{\frac{\partial}{\partial y}M(x,y) - \frac{\partial}{\partial x}N(x,y)}{N(x,y)} = \frac{-2}{x}$$

depends only on x.

2. *By Proposition 4.1, we look for an integrating factor of type I,*

$$I = e^{\int \frac{\frac{\partial}{\partial y}M(x,y) - \frac{\partial}{\partial x}N(x,y)}{N(x,y)}\,\mathrm{d}\,x} = e^{\int \frac{-2}{x}\,\mathrm{d}\,x} = e^{-2\ln x} = (e^{\ln(x)})^{-2} = x^{-2} = \frac{1}{x^2}.$$

Multiplying the equation by μ implies

$$\frac{1}{x^2}(-y\,\mathrm{d}\,x + x\,\mathrm{d}\,y) = 0$$

which is

$$-\frac{y}{x^2}\,\mathrm{d}\,x + \frac{1}{x}\,\mathrm{d}\,y = 0$$

This is an exact equation because

$$\frac{\partial}{\partial y}(-\frac{y}{x^2}) = -\frac{1}{x^2}$$
$$\frac{\partial}{\partial x}(\frac{1}{x}) = -\frac{1}{x^2}$$

3. *We look for F such that*

$$\frac{\partial}{\partial x}F\ \ \mathrm{d}\,x + \frac{\partial}{\partial y}F\ \ \mathrm{d}\,y = -\frac{y}{x^2}\,\mathrm{d}\,x + \frac{1}{x}\,\mathrm{d}\,y,$$

Equivalently,

$$\begin{cases} \frac{\partial}{\partial x}F = & -\frac{y}{x^2} \\ \frac{\partial}{\partial y}F = & \frac{1}{x} \end{cases}$$

Integrating the first equation with respect to x implies

$$F = \int -\frac{y}{x^2}\,\mathrm{d}\,x = \frac{y}{x} + h(y)$$

Since F has to satisfy the second equation and

$$\frac{\partial}{\partial y}F = \frac{\partial}{\partial y}(\frac{y}{x} + h(y)) = \frac{1}{x} + h'(y)$$

the function $h(y)$ satisfies the equation

$$\frac{1}{x} + h'(y) = \frac{1}{x}$$

or

$$h'(y) = 0$$

and $h = 0$. Hence,

$$F = \frac{y}{x}$$

and the general solution is

$$F = \frac{y}{x} = c$$

Equivalently,

$$y = cx.$$

■

✍ **Exercise 4.2** Consider the equation

$$(x - y)\,\mathrm{d}\,x + x\,\mathrm{d}\,y = 0.$$

1. Show the equation is not exact but $\dfrac{\frac{\partial}{\partial y}M(x,y) - \frac{\partial}{\partial x}N(x,y)}{N(x,y)}$ depends only on x.
2. Reduce the equation to an exact equation.
3. Solve the equation

Solution

1. $M = x - y,\ N = x$ *and*

$$\frac{\partial}{\partial y}M = -1$$
$$\frac{\partial}{\partial x}N = 1$$

The equation is not exact but

$$\frac{\frac{\partial}{\partial y}M(x,y) - \frac{\partial}{\partial x}N(x,y)}{N(x,y)} = \frac{-2}{x}$$

depends only on x.

2. *By Proposition 4.1, we look for an integrating factor of type I,*

$$\mu = e^{\int \frac{\frac{\partial}{\partial y}M(x,y) - \frac{\partial}{\partial x}N(x,y)}{N(x,y)}\,\mathrm{d}\,x} = e^{\int \frac{-2}{x}\,\mathrm{d}\,x} = \frac{1}{x^2}.$$

Multiplying the equation by μ implies

$$\frac{1}{x^2}[(x - y)\,\mathrm{d}\,x + x\,\mathrm{d}\,y] = 0$$

which is

$$\frac{x - y}{x^2}\,\mathrm{d}\,x + \frac{1}{x}\,\mathrm{d}\,y = 0$$

This is an exact equation because

$$\frac{\partial}{\partial y}\left(\frac{x-y}{x^2}\right)=\frac{-1}{x^2}$$
$$\frac{\partial}{\partial x}\left(\frac{1}{x}\right)=\frac{-1}{x^2}$$

3. *We look for F such that*

$$\frac{\partial}{\partial x}F=\frac{x-y}{x^2}$$
$$\frac{\partial}{\partial y}F=\frac{1}{x}$$

Integrating the first equation with respect to x implies

$$F=\int\frac{x-y}{x^2}\,\mathrm{d}x=\int\frac{1}{x}-\frac{y}{x^2}\,\mathrm{d}x=\ln x+\frac{y}{x}+h(y)$$

Since F has to satisfy the second equation and

$$\frac{\partial}{\partial y}F=\frac{\partial}{\partial y}(\ln x+\frac{y}{x}+h(y))=\frac{1}{x}+h'(y),$$

the function $h(y)$ satisfies the equation

$$\frac{1}{x}+h'(y)=\frac{1}{x}$$

or

$$h'(y)=0$$

and $h=0$. Hence,

$$F=\ln x+\frac{y}{x}$$

and the general solution is

$$\ln x+\frac{y}{x}=c,$$

or

$$y=cx-x\ln x.$$

■

4.2 Existence and Uniqueness*

Solving differential equations can be very complicated. It is impossible to find useful formulas for the solutions of most differential equations. However, knowing the existence and uniqueness can helps us looking for solutions.

Theorem 4.1 (Picard's Existence and Uniqueness)

1. *If f is a function continuous on an open rectangle $R : \{a < x < b, c < y < d\}$ that contains (x_0, y_0), then the initial value problem*

$$y' = f(x, y), \quad y(x_0) = y_0$$

has at least one solution on some open subinterval of (a, b) that contains x_0.

2. *If both f and f_y are continuous on R then the initial value problem*

$$y' = f(x, y), \quad y(x_0) = y_0$$

has a unique solution on some open subinterval of (a, b) that contains x_0

A key to the proof of this theorem is the fact that that a continuously differential function $y(x)$ is a solution of the differential equation if and only if it satisfies the following integral equation:

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,\mathrm{d}\,t.$$

To prove the theorem, the French mathematician Éile Picard used a sequence of approximations with $y_0(x) = y_0$ and

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,\mathrm{d}\,t.$$

This approximation is known as Picard's method of successive approximations. An interactive demonstration can be found on GeoGebra: Picard's Method of Successive Approximations.

The theorem can be proved by showing that $y_n(x)$ converges uniformly to a function $y(x)$ which is a solution. We refer the reader to (Simmons 2016, Chapter 13) for a proof.

Example 4.3 Consider the initial value problem

$$y' = 3y^{2/3}, \quad y(0) = 0.$$

Show that both $y = 0$ and $y = x^3$ are solutions.

Does it contradict Picard's Existence and Uniqueness Theorem?

Solution *It is clear that $y = 0$ is a solution. The function $y = x^3$ is also a solution because $y(0) = 0^3 = 0$ and*

$$y' = 3x^2 = 2(x^3)^{\frac{2}{3}} = 3y^{\frac{2}{3}}.$$

This example does not contradict the uniqueness of the Theorem. Because the partial derivative $f_y = 2y^{-\frac{1}{3}}$ is not continuous along the line $(x, 0)$. ■

Example 4.4 Consider the initial value problem

$$y' = 3xy^{\frac{1}{3}}, \quad y(x_0) = y_0.$$

1. For what points (x_0, y_0) does Picard's Existence and Uniqueness Theorem imply that this initial value problem has a solution?
2. For what points (x_0, y_0) does Picard's Existence and Uniqueness Theorem imply that this initial value problem has a unique solution on some open interval that contains x_0?

Solution *Because $f(x, y) = 3xy^{\frac{1}{3}}$ is continuous for all points (x, y). The theorem implies that the initial value problem has a solution.*

The partial derivative

$$f_y = \frac{\partial}{\partial y}\left(3xy^{\frac{1}{3}}\right) = 3x\frac{\partial}{\partial y}y^{\frac{1}{3}} = xy^{-\frac{2}{3}}$$

is undefined when $y = 0$. For any point (x, y) such that $y \neq 0$, f_y is continuous. Therefore, by the theorem, the initial value problem has a unique solution for $y \neq 0$.

Indeed, when $y_0 = 0$, the initial value problem has a trivial solution $y = 0$ and an implicit solution $y^{\frac{2}{3}} = x^2 - x_0^2$. If $y_0 \neq 0$, only $y^{\frac{2}{3}} = x^2 - x_0^2$ is a solution. ■

Exercise 4.3 Find all (x_0, y_0) for which Picard's Existence and Uniqueness Theorem implies that the initial value problem

$$y' = \frac{x}{y}, \quad y(x_0) = y_0$$

has 1. a solution and 2. a unique solution on some open interval that contains x_0.

Solution *Since $f(x, y) = \frac{x}{y}$ is continuous only when $y \neq 0$. The theorem implies that the initial value problem has a solution when $y_0 \neq 0$.*

The partial derivative

$$f_y = \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = x\frac{\partial}{\partial y}\left(\frac{1}{y}\right) = -\frac{x}{y^2}$$

is discontinuous again when $y = 0$. Therefore, by the theorem, the initial value problem has a unique solution for $y \neq 0$. ■

4.3 Applications of First Order Differential Equations

Some examples of applications of differential equations are mentioned in the introduction. In this section, we will discuss a few more examples.

4.3.1 Exponential Growth

When modeling population growth, Malthus's exponential model is frequently used:

$$P' = rP,$$

where r is the constant.

As a separable equation, the general solution of this model is

$$P(t) = ce^{rt}.$$

With an initial condition $P(t_0) = P_0$, the solution is

$$P(t) = P_0 e^{r(t-t_0)}.$$

In this model, normally, $P > 0$. Therefore, if $a > 0$, then P is increasing without upper bound. If $a < 0$, then P is decreasing with the lower bound 0.

Example 4.5 A bacteria culture starts with 10 bacteria and grows to 90 bacteria after 2 hour. Assume it grows at a rate proportional to its size.

1. Express the population after t hours as a function of t.
2. What is the population after 9 hours?
3. How long it will take for the population to reach 2500?

Solution

1. *Since the growth rate is constantly proportional to its size, the population satisfies the exponential model*

   $$P'(t) = rP(t).$$

   Solving it yields the general solution

   $$P(t) = e^{rt+c_1} = e^{c_1}e^{rt} = ce^{rt},$$

   where c is a constant to be determined by an initial condition.
   In this model, both c and a are to be determined. In the statement of the question, the sentence "A bacteria culture starts with 10 bacteria and grows to 90 bacteria after 2 hour" means

   $$P(0) = 10 \quad \text{and} \quad P(2) = 90.$$

Those two conditions implies that a and c satisfy the following system of equations

$$\begin{cases} N(0) = c \cdot e^0 = 10 \\ N(2) = c \cdot e^{2a} = 90 \end{cases}$$

The first equation implies that $c = 10$. Plugging it in to second equation yields

$$\begin{aligned} e^{2a} &= 9 \\ 2a &= \ln 9 \\ a &= \frac{1}{2}\ln 9 \\ a &= \ln 3. \end{aligned}$$

Hence the population function is

$$P(t) = 10e^{t\ln 3}.$$

2. *The population after 9 hours is*

$$P(9) = 10e^{9\ln 3} \approx 196830.$$

3. *The time that it will takes the culture to 2500 satisfies the equation*

$$N(t) = 2500.$$

Solving the equation yields

$$\begin{aligned} 10e^{t\ln 3} &= 2500 \\ e^{t\ln 3} &= 250 \\ t\ln 3 &= \ln(250) \\ t &= \frac{\ln(250)}{\ln 3} \\ t &\approx 5. \end{aligned}$$

So it takes about 5 hours for the bacteria culture to grow to 2500.

■

In the previous population model, when $a > 0$, the population grows exponentially without a limit. In reality, the growth is limited by the environment capacity $\frac{1}{\alpha}$. A refined model is Verhulst's logistic population model,

$$P' = rP(1 - \alpha P),$$

where a is the growth rate when the capacity has no or minimal impact on the growth, and $\frac{1}{\alpha}$

is capacity, i.e. the limit of the population in the environment.

Note that the equation is also an separable (indeed, autonomous) equation. The equation can be solved using partial fraction decomposition:

$$\begin{aligned} P' &= rP(1-\alpha P) \\ \frac{P'}{P(1-\alpha P)} &= r \\ \frac{P'}{P} + \frac{\alpha P'}{1-\alpha P} &= r \\ (\ln(P))' - (\ln(1-\alpha P))' &= r \\ \left(\ln\left(\frac{P}{1-\alpha P}\right)\right)' &= r \\ \frac{P}{1-\alpha P} &= e^{rt} \\ P &= e^{rt}(1-\alpha P) \\ P + e^{rt}\alpha P &= ce^{rt} \\ P &= \frac{ce^{rt}}{1+c\alpha e^{rt}}. \end{aligned}$$

Note that the limit of $P(t)$ as $t \longrightarrow \infty$ is nothing but the capacity $\frac{1}{\alpha}$.

Example 4.6 One hundred rabbits were released in a forest. . It is observed that the population after t years develops according to Verhulst's logistic population model. The carrying capacity is estimated to be $10,000$.

Suppose the growth rate is $r = 2$. What is the population size after 5 years.

Solution *Since the carrying capacity is* $\frac{1}{\alpha} = 10,000$, *the value* α *is* 0.00001. *The logistic model is then*

$$P'(t) = 2P(t)(1 - 0.00001P(t)).$$

The discussion of the logistic model above shows that the general solution is

$$P(t) = \frac{ce^{2t}}{1 + 0.00001ce^{2t}}.$$

Since $P(0) = 100$, *solving for* c *yields* $c \approx 100$.

So the population of rabbits after t *years is*

$$P(t) = \frac{100e^{2t}}{1 + 0.001e^{2t}}.$$

Therefore, after 10 years the population will be

$$P(15) = \frac{100e^{10}}{1 + 0.001e^{10}} \approx 95657.$$

■

✍ **Exercise 4.4** A bread dough increases in volume at a rate proportional to the volume V present. Suppose the initial volume is V_0. After 2 hours, the volume increases to $1.5V_0$. How long will it take for the volume to increase to $2V_0$?

Solution *Suppose the proportional factor is k. Then $V'(t) = kV(t)$ after t hours. Since the initial condition is $V(0) = V_0$, it follows that $V(t) = V_0 e^{-kt}$.*

Because $V(2) = 1.5V_0$. The constant k satisfies the equation

$$1.5V_0 = V_0 e^{2k}.$$

Solving the equation yields $k = \frac{1}{2}\ln(1.5)$.

The time needed for the volume to $2V_0$ satisfies the equation

$$V_0 e^{t\frac{\ln(1.5)}{2}} = 2V_0.$$

Solving for t from the equation gives

$$t = \frac{2\ln 2}{\ln(1.5)} \approx 3.4.$$

■

4.3.2 Exponential decay

If a quantity decreases at a rate proportional to its current value, then we say it is subject to exponential decay. Suppose the quantity is $N(t)$ after a time t and the rate of decreasing, called the *exponential decay rate*, is $k > 0$. Then the quantity $N(t)$ satisfies the following equation

$$\frac{dN}{dt} = -kN.$$

Solving the equation yields that

$$N(t) = N_0 \cdot e^{-kt},$$

where $N_0 = N(0)$ is the initial quantity.

Exponential decay applies in a wide variety of situations, particularly in natural science.

For example, the quantity of radioactive material decays exponentially.

In a radioactive decay model, the decay rate k can be determined by the *half-life*, that is the time required to decay the quantity to one half of its initial value. Suppose the half-life of a radioactive material is τ. Then the exponential decay rate k satisfies the following equation

$$N_0 e^{-k\tau} = \frac{1}{2} N_0.$$

Solving for k implies that

$$k = \frac{\ln 2}{\tau}$$

and the exponential decay model becomes

$$N(t) = N_0 e^{-\ln 2 \cdot \frac{t}{\tau}} = N_0 2^{-\frac{t}{\tau}}.$$

Example 4.7 A radioactive substance has a half-life of 40 days. Suppose its mass is now 300 g (grams).

After how long will the amount present be 200 g.

Solution *Applying the radioactive decay model with $\tau = 40$ and $N_0 = 300$ implies that*

$$N(t) = 300 \cdot 2^{\frac{t}{40}}.$$

When $N(t) = 200$, the time t satisfies the equation

$$200 = 300 \cdot 2^{-\frac{t}{40}}.$$

Solving the equation for t yields

$$\begin{aligned} 200 &= 300 \cdot 2^{-\frac{t}{40}} \\ \frac{2}{3} &= 2^{-\frac{t}{40}} \\ \ln\left(\frac{2}{3}\right) &= \ln 2 \cdot \left(-\frac{t}{40}\right) \\ t &= \frac{40(\ln 3 - \ln 2)}{\ln 2} \\ t &\approx 23. \end{aligned}$$

So it takes about 23 days for the mass decrease to 200 g. ■

Example 4.8 Mixed Growth and Decay A radioactive substance has a half-life of 100 days. Suppose its mass is now 500 mg and additional amounts are added at the rate of 4 mg per day. Suppose the mass after t days is $Q(t)$.

1. Find a formula for the rate of change $Q'(t)$ of the mass in terms of t.
2. Find $Q(t)$ in terms of t.

Solution

1. *The rate of change composes of two rates: the rate of decreasing and rate of increasing. By exponential decay model, the The rate of decreasing is $-kQ(t)$, where $k = \frac{\ln 2}{100}$ is the exponential decay rate determined by the half-life.*
   The rate of increasing is the fixed rate 5 mg/day.
   Therefore, the rate of change is
   $$Q'(t) = -\frac{\ln(2)}{100}Q(t) + 5.$$
2. *Since the right hand side of the above equation is a linear expression of $Q(t)$, a linear substitution $v = -\frac{\ln 2}{100}Q(t) + 5$ will reduce the differential equation to the separable equation*
   $$v' = -\frac{\ln 2}{100}v.$$
   Therefore,
   $$v = ce^{-\frac{\ln 2}{100}t}$$
   and
   $$Q(t) = -\frac{100}{\ln 2}\left(ce^{-\frac{\ln 2}{100}t} - 5\right).$$
   The initial condition $Q(0) = 500$ implies
   $$\begin{aligned} -\frac{100}{\ln 2}(c-5) &= 500 \\ (c-5) &= -5\ln 2 \\ c &= -5\ln 2 + 5. \end{aligned}$$
   Therefore,
   $$Q(t) = \frac{(500\ln 2 - 500)2^{-\frac{t}{100}} + 500}{\ln 2}.$$

■

✍ **Exercise 4.5** Living cells maintain a consistent level of carbon-14. However, when the cell dies, carbon-14 start decaying exponentially at a constant rate. It is known that the half-life of carbon-14 is about 5570 years.

An archaeologist investigating the site of an ancient village finds a burial ground where the amount of carbon-14 present in individual remains is about 63% of the amount present in live individuals. Estimate the age of the village.

Solution *Let $Q = Q(t)$ be the quantity of carbon-14 remained in individuals t years after death, and let Q_0 be the quantity that would be present in live individuals. Since carbon-14* *decays exponentially with half-life 5570 years, its exponential decay rate is*

$$k = \frac{\ln 2}{5570}.$$

Therefore,

$$Q = Q_0 e^{-t(\ln 2)/5570}.$$

Since the quantity remained in individuals now is $Q(t) = 0.63Q_0$, the age t satisfies the equation

$$Q_0 e^{-t(\ln 2)/5570} = 0.63Q_0$$

Solving the equation for t yields

$$t = -5570\frac{\ln(0.63)}{\ln 2} \approx 3713.$$

Therefore, the village is about 3713 years old. ■

4.3.3 Newton's law of cooling

Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and the temperature of its surrounding.

Let $T(t)$ be the temperature of the object and $T_m(t)$ the temperature of the surrounding medium at the time t. Then Newton's law of cooling can be state as a differential equation:

$$T'(t) = -k(T(t) - T_m(t)).$$

Here k is a positive constant, called the temperature decay constant. The reason that k is positive is because the temperature of the object must decrease if $T(t) > T_m(t)$, or increase if $T(t) < T_m(t)$.

For simplicity, assume that the medium is maintained at a constant temperature T_m. This model apply to many situations but not all. For example, to cool down a cup of hot coffee in the room temperature, the change of room temperature is neglectful. However, to cool down a cup of hot coffee is a small pot of cold water, the temperature of the water will change accordingly. In the later case, the heat transfer law in thermodynamics will be needed.

In this section, the surrounding temperature will be assumed to be constant. In this case, note that $T'(t) = (T(t) - T_m)'$, then Newton's law of cooling can be re-formulated as

$$(T(t) - T_m)' = -k(T(t) - T_m).$$

That is, $T - T_m$ decays exponentially, with decay constant k. Therefore,

$$T(t) - T_m = (T_0 - T_m)e^{-kt}$$

or equivalently

$$T(t) = (T_0 - T_m)e^{-kt} + T_m,$$

where T_0 is the initial temperature of the object.

Example 4.9 A ceramic insulator is baked at 400°C and cooled in a room in which the temperature is 25°C. After 4 minutes the temperature of the insulator is 200°C. What is its temperature after 8 minutes?

Solution *Here $T_0 = 400$ and $T_m = 25$, so the temperature function of the ceramic insulator is*

$$T = 25 + 375e^{-kt}.$$

Since $T(4) = 200$, which determines k, then

$$200 = 25 + 375e^{-4k}.$$

Solving for k yields

$$k = -\frac{1}{4}\ln\left(\frac{7}{15}\right).$$

Substituting this into $T(t)$ yields

$$T = 25 + 375e^{-\frac{t}{4}\ln\left(\frac{7}{15}\right)}.$$

Therefore, the temperature of the insulator after 8 minutes is

$$\begin{aligned} T(8) &= 25 + 375e^{-2\ln\left(\frac{7}{15}\right)} \\ &= 25 + 375\left(\frac{7}{15}\right)^2 \approx 107^{\circ}\text{ C.} \end{aligned}$$

■

✍ **Exercise 4.6** A metal bar at a temperature of 200°F is placed in a room at a constant temperature of 50°F. If after 20 minutes, the temperature of the bar is 90°F, find

1. Find the formula of temperature as a function of t.
2. Find the time it will take the bar to reach a temperature of 60°F;
3. Find the limit of the temperature as time goes to infinity.

Solution

1. *Since* $T_m = 50$ *and* $T_0 = 200$, *the temperature function is*

$$T(t) = 50 + 150e^{-kt}.$$

Because $T(20) = 90$, *then* k *is determined by the equation*

$$90 = 50 + 150 \cdot e^{-20k}.$$

Solving the equation yields

$$\begin{aligned} 40 &= 150 \cdot e^{-20k} \\ e^{-20k} &= \frac{4}{15} \\ -20k &= \ln\left(\frac{4}{15}\right) \\ k &= -\frac{1}{20}\ln\left(\frac{4}{15}\right). \end{aligned}$$

Hence

$$T(t) = 50 + 150e^{\frac{1}{20}\ln(\frac{4}{15})t}.$$

2. *The time that it will take the bar to* 60° *satisfies the equation*

$$50 + 150 \cdot e^{-kt} = 60,$$

where $k = -\frac{1}{20}\ln\left(\frac{4}{15}\right)$.
Solving the equation yields

$$\begin{aligned} 150 \cdot e^{-kt} &= 10 \\ e^{-kt} &= \frac{1}{15} \\ -kt &= \ln(\frac{1}{15}) \\ t &= -\frac{1}{k}\ln(\frac{1}{15}) \\ t &= -\frac{1}{-\frac{1}{20}\ln(\frac{4}{15})}\ln(\frac{1}{15}) \\ t &= 20 \cdot \frac{\ln(1/15)}{\ln(4/15)} \\ t &\approx 41. \end{aligned}$$

So it takes about 41 minutes for the bar to cool down to 60°*F.*

3. *When* t *goes to infinity, because* k *is positive, the limit* $\lim_{t\to\infty} e^{-kt} = 0$. *Hence*

$$\lim_{t\to\infty} T(t) = 50 + 150 \cdot \lim_{t\to\infty} e^{\frac{1}{20}\ln(\frac{4}{15})t} = 50.$$

The answer is the room temperature.

■

4.3.3.1 Mixing Problems

Example 4.10 A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains 1/2 pound of salt per gallon is poured into the tank at the rate of 4 gal/min and the mixture is drained from the tank at the same rate.

1. Find a differential equation for the quantity $Q(t)$ of salt in the tank at time $t > 0$, and solve the equation to determine $Q(t)$.
2. Find $\lim\limits_{t\to\infty} Q(t)$.

Solution *To find a differential equation for the quantity of salt Q, because the given information is about the rate of change of the quantity of salt Q', we will find an equation for Q'. is the rate of change of the quantity of salt in the tank changes with respect to time; The rate of change composes of two parts*

$$Q' = \text{ in-rate} - \text{ out-rate.}$$

The in-rate is

$$\left(\frac{1}{2}\ \text{lb/gal}\right) \times (4\ \text{ gal/min}) = 2\ \text{ lb/min.}$$

Since the concentration changes, the determine the out-rate, we need know the concentration at t. Because the in-flow and out-flow rates are the same, the volume of the mixture is constant which is 600 gal. Therefore, the concentration at time t is $\frac{Q(t)}{600}$, and the out-rate is then

$$(\text{ concentration}) \times (\text{ rate of flow out}) = \frac{Q(t)}{00} \times 4 = \frac{Q(t)}{150}.$$

Therefore,

$$Q' = 2 - \frac{Q}{150},$$

which is a first order linear equation that can be rewritten as

$$Q' + \frac{Q}{150} = 2.$$

The integrating factor is

$$r(t) = e^{\int \frac{1}{150}\,\mathrm{d}t} = e^{\frac{t}{150}}.$$

Multiplying the integrating factor to both sides and integrating both sides yields

$$\begin{aligned} Q(t)e^{\frac{t}{150}} &= \int 2e^{\frac{t}{150}}\,\mathrm{d}t \\ Q(t)e^{\frac{t}{150}} &= 300e^{\frac{t}{150}} + c \\ Q(t) &= 300 + ce^{-\frac{t}{150}}. \end{aligned}$$

Since $Q(0) = 40$, $c = -260$ *and*

$$Q = 300 - 260e^{-t/150}.$$

The limit of $Q(t)$ *as* $t \longrightarrow \infty$ *is*

$$\lim_{t\longrightarrow\infty} Q(t) = \lim_{t\longrightarrow\infty} \left(300 - 260e^{-t/150}\right) = 300.$$

This is intuitively reasonable. Because, the incoming solution contains 1/2 pound of salt per gallon and there are always 600 gallons of water in the tank. ■

Example 4.11 A 500-liter tank initially contains 10 g of salt dissolved in 200 liters of water. Starting at $t_0 = 0$, water that contains 1/4 g of salt per liter is poured into the tank at the rate of 4 liters/min and the mixture is drained from the tank at the rate of 2 liters/min. Find a differential equation for the quantity $Q(t)$ of salt in the tank at time t prior to the time when the tank overflows and find the concentration $K(t)$ (g/liter) of salt in the tank at any such time.

Solution *The difference between this example and the above equation is that the volume of the mixture changes in this example. Let* $W(t)$ *of solution in the tank at any time* t *prior to overflow. Since* $W(0) = 200$, *in-flow rate 4 liters/min and the out-flow rate 2 liters/min, the net flow rate is then 2 liters/min. Therefore, the volume is*

$$W(t) = 200 + \int_0^t 2\,\mathrm{d}x = 200 + 2t.$$

Since the volume of tank is 500 liter, and $W(150) = 500$, *this formula is valid for when the time* t *is from* 0 *to* 150 *min, i.e.* $0 \le t \le 150$.

Now let $Q(t)$ *be the number of grams of salt in the tank at time* t, *where* $0 \le t \le 150$. *As in above example*

$$\begin{aligned} Q' &= \text{rate in} - \text{ rate out} \\ &= \left(\frac{1}{4}\text{ g/liter}\right) \times (4\text{ liters/min}) + Q(t) \cdot \frac{2}{2t+200}\text{ g/min} \\ &= 1\text{ g/min} + Q(t) \cdot \frac{1}{t+100}\text{ g/min} \\ &= \left(1 + \frac{Q(t)}{t+100}\right)\text{ g/min.} \end{aligned}$$

Therefore, the salt flow rate is

$$Q' = 1 - \frac{Q}{t+100},$$

or

$$Q' + \frac{1}{t+100}Q = 1.$$

The integrating factor is

$$r(x) = e^{\int \frac{1}{t+100}\,\mathrm{d}t} = t + 100.$$

Multiplying the equation by $r(x)$ and integrating both sides yields

$$Q(t) = \frac{1}{t+100}\int (t+100)\,\mathrm{d}t = \frac{t+100}{2} + \frac{c}{t+100}.$$

Since $Q(0) = 10$, solving for c implies

$$c = -4000.$$

Hence,

$$Q(t) = \frac{t+100}{2} - \frac{4000}{t+100}.$$

Now let $K(t)$ be the concentration of salt at time t. Then

$$K(t) = \frac{Q(t)}{W(t)} = \frac{1}{4} - \frac{2000}{(t+100)^2}.$$

■

4.3.4 Motion Under Gravity in a Resisting Medium*

In this section, we will consider an object with constant mass m moving vertically in a resisting medium near Earth's surface under a force $F(t)$. We will take the upward direction to be positive. Let $y = y(t)$ be the displacement of the object from a reference point above the ground at time t. Let $v = v(t)$ and $a = a(t)$ be the velocity and acceleration of the object at time t. Then $a(t) = v'(t)$ and $v(t) = y'(t)$.

Newton's second law of motion asserts that the force F equals the product of the mass m and the acceleration a:

$$F(t) = ma(t).$$

When an object moves vertically in a resisting medium, two main forces are the gravitational force $-mg$ and the medium resistive force. Here, g is the acceleration due to gravity.

Example 4.12 An object with mass m and initial velocity $v(0) = v_0$ moves under constant gravitational force mg through a medium that exerts a resistance with magnitude proportional to the speed $|v|$ of the object.

1. Find the velocity of the object as a function of t.
2. Find the limit $\lim\limits_{t\to\infty} v(t)$.

Solution

Taking the upward as the positive direction, the total force acting on the object is

$$F = -mg + F_1,$$

where $-mg$ is the force due to gravity and F_1 is the resisting force of the medium. Since, the resisting force F_1 has the magnitude proportional to $|v|$. There is a positive constant k such that $|F_1| = mk|v|$. Because the resistance force is always opposite the direction of the velocity. If the object is moving downward, that is $v \le 0$, the resisting force is upward. So

$$F_1 = mk(-v) = -mkv > 0.$$

If the object is moving upward, that is $v \ge 0$, the resisting force is downward. So

$$F_1 = -mkv < 0$$

Therefore, the total force F is

$$F = -mg - mkv,$$

regardless of the sign of the velocity.

From Newton's second law of motion,

$$F = ma = mv'.$$

So

$$mv' = -mg - mkv,$$

or equivalently,

$$v' = -kv - g.$$

This equation can be solved using multiple methods. Because the coefficients are constants, we may use the linear substitution $v = y - \frac{g}{k}$. Plugging it into the above equation yields

$$y' = -ky,$$

which has the general solution $y = ce^{-kt}$. Hence,

$$v = ce^{-kt} - \frac{g}{k}.$$

Since $v(0) = v_0$, the constant c is determined by

$$c - \frac{g}{k} = v_0.$$

Therefore, $c = v_0 + \frac{g}{k}$.

So the velocity function is

$$v = \left(v_0 + \frac{g}{k}\right) e^{-kt} - \frac{g}{k}$$

Letting $t \longrightarrow \infty$ here shows that the limit of $v(t)$ is

$$\lim_{t \longrightarrow \infty} v(t) = -\frac{g}{k}.$$

■

We see that under reasonable assumptions on the resisting force, the velocity approaches a limit as $t \longrightarrow \infty$, which is called the *terminal velocity*. You will find an object reaches its terminal velocity when its acceleration is 0.

Example 4.13 A 10 kg mass is given an initial velocity $v(0) = v_0 \leq 0$ near Earth's surface. The only forces acting on it are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the resistance is 8 Newton (N for short) when the speed is 2 meters/second (m/s for short).

1. Find the velocity of the object as a function of t.
2. Find the terminal velocity.

Solution *Since the object is falling, the resistance is in the upward direction which is assumed to be the positive direction. Hence,*

$$mv' = -mg + kv^2,$$

where k is a constant. Since the magnitude of the resistance is 8 N when $v = 2$ m/s, that is,

$$k(2^2) = 8.$$

So $k = 2$ N-s^2/ m^2, where $N - s$ means the product of N and s. Since $m = 10$ kg and $g = 9.8$ m/ s^2, the velocity satisfies the following equation.

$$10v' = -98 + 2v^2 = 2(v^2 - 49).$$

Note that the solutions $v = -7$ or $v = 7$ of the equation $v^2 - 49 = 0$ are solutions to the above differential equation. Since $v_0 \leq 0$ and the object is falling, the velocity v has a negative direction. So $v = 7$ can not be a solution under the assumption that $v_0 \leq 0$. Moreover, $v = -7$ is a solution only when $v_0 = -7$.

Now assume $v_0 \neq -7$. Separate variables implies

$$\left(\frac{1}{v^2 - 49}\right) v' = \frac{1}{5}.$$

Integrating both sides using partial fraction decomposition yields

$$
\begin{aligned}
\left(\frac{1}{v^2-49}\right) v' &= \frac{1}{5}\\
\left(\frac{1}{(v-7)(v+7)}\right) &= \frac{1}{5}\\
\frac{1}{14}\left(\frac{1}{v-7}-\frac{1}{v+7}\right) v' &= \frac{1}{5}\\
\frac{v'}{v-7}-\frac{v'}{v+7} &= \frac{14}{5}\\
\int \frac{v'}{v-7}\,\mathrm{d}\,t - \int \frac{v'}{v+7}\,\mathrm{d}\,t &= \int \frac{14}{5}\,\mathrm{d}\,t\\
\int \frac{1}{v-7}\,\mathrm{d}(v-7) - \int \frac{1}{v+7}\,\mathrm{d}(v+7) &= \frac{14t}{5}+c\\
\ln|v-7| - \ln|v+7| &= \frac{14t}{5}+c\\
\ln\left(\left|\frac{v-7}{v+7}\right|\right) &= \frac{14t}{5}+c\\
\left|\frac{v-7}{v+7}\right| &= ce^{\frac{14t}{5}}
\end{aligned}
$$

Since $v(0) = v_0$,

$$c = \left|\frac{v_0-7}{v_0+7}\right|$$

If $v_0 \le -7$, *then* $v'(0) > 0$ *and* $v(t)$ *will be increasing until* $v' = 0$. *So* $v_0 \le v(t) \le -7$. *Then*

$$\left|\frac{v-7}{v+7}\right| = \frac{v-7}{v+7}$$

and

$$c = \frac{v_0-7}{v_0+7}.$$

So

$$\frac{v-7}{v+7} = e^{\frac{14t}{15}} \cdot \frac{v_0-7}{v_0+7}.$$

If $v_0 \ge -7$, *then* $v'(0) < 0$ *and* $v(t)$ *will be decreasing until* $v' = 0$. *So* $-7 \le v(t) \le v_0$. *Then*

$$\left|\frac{v-7}{v+7}\right| = -\frac{v-7}{v+7}$$

and

$$c = -\frac{v_0-7}{v_0+7}.$$

Again

$$\frac{v-7}{v+7} = e^{\frac{14t}{15}} \cdot \frac{v_0-7}{v_0+7}.$$

Solving for v yields

$$v = -7\frac{(v_0 - 7)e^{\frac{14t}{5}} + v_0 + 7}{(v_0 - 7)e^{\frac{14t}{5}} - v_0 - 7}.$$

Since $v_0 \le 0$, v is defined and negative for all $t > 0$. The terminal velocity is

$$\lim_{t\to\infty} v(t) = -7 \text{ m/s},$$

independent of v_0. ■

In examples above, we have been using international metric system for force. There are other metric systems (see Wiki page on Newton_(unit) for details). One of them is the British metric system which uses *foot* (lb) for *length*, *pound-force* (lb) for *force*, and *slug* (sl) for *mass*.

Exercise 4.7 A 320-lb object is given an initial upward velocity of 20 ft/s near the surface of Earth. The atmosphere resists the motion with a force of 2 lb for each ft/s of speed. Assuming that the only other force acting on the object is the gravity $g = 32$ ft/ s^2.

Find its velocity v as a function of t, and find its terminal velocity.

Solution *Note that here 320-lb means the gravitational force is 320 lb. Since $mg = 320$ and $g = 32$, $m = 320/32 = 10$ lb-s^2/ ft. Suppose the velocity function is $v = v(t)$ ft/s after t Then the atmospheric resistance is $-2v$ lb.*

By Newton's second law of motion,

$$10v' = -320 - 2v,$$

equivalently,

$$5v' = -160 - v,$$

Solving the equation by a linear substitution $y = v + 1600$ yields

$$\begin{aligned}
5v' &= -\,160 - v\\
5y' &= -\,y\\
\frac{y'}{y} &= -\,5\\
\int \frac{y'}{y}\,\mathrm{d}\,t &= \int -5\,\mathrm{d}\,t\\
\ln|y| &= -\,5t + c\\
|y| &= ce^{-5t}\\
|v + 160| &= ce^{-5t}.
\end{aligned}$$

Since $v(0) = 20$ and $v'(0) < 0$, the velocity v will decrease. Note that $v' = 0$ when $v = -160$.

Then $-160 \le v(t) \le 20$, *and* $|v + 160| = v + 160$. *Therefore,*

$$v = -160 + ce^{-5t}.$$

Since the initial velocity is $v(0) = 20$, *the constant* c *satisfies the equation*

$$20 = -160 + c.$$

Thus, $c = 180$ *and the velocity function is*

$$v = -160 + 180e^{-t/10} \quad \text{ft/ s}.$$

The terminal velocity is

$$\lim_{t \to \infty} v(t) = \lim_{t \to \infty} -160 + 180e^{-t/10} = -160 \quad \text{ft/ s}.$$

■

4.3.5 Orthogonal Trajectories*

We've seen that general solutions to first order differential equations are in the form

$$F(x, y, c) = 0,$$

where c is a constant taking varies real values. The graph of those solutions are known as one-parameter families of curves. We will simply call them families of curves.

Two curves C_1 and C_2 are said to be ***orthogonal*** at a point of intersection (x_0, y_0) if they have perpendicular tangents at (x_0, y_0).

A curve is said to be an ***orthogonal trajectory*** of a given family of curves if it is orthogonal to every curve in the family.

Suppose $F(x, y, c) = 0$ is a family of integral curves of the differential equation

$$y' = -f(x, y).$$

From the definition, an orthogonal trajectory of $F(x, y, c) = 0$ has perpendicular tangents. So slopes of tangent lines of an orthogonal trajectory are determined by

$$y' = \frac{1}{f(x, y)}.$$

Therefore, families of integral curves of the differential equations $y' = f(x, y)$ and $y' =$

$-\frac{1}{f(x,y)}$ are orthogonal trajectories to each other.

That suggests a method for finding orthogonal trajectories of a family of curves:

Step 1: Find a differential equation $y' = f(x, y)$ without c for the given family $F(x, y, c)$.

Step 2: Solve the differential equation $y' = -\dfrac{1}{f(x, y)}$ to find the family of orthogonal trajectories.

Example 4.14 Find the family of orthogonal trajectories to the family of curves defined by

$$y = cx^2, \quad c \neq 0.$$

Solution *Taking derivative implies*

$$y' = 2cx.$$

Since $y = cx^2$, solving for c yields $c = \frac{y}{x^2}$. So the above differential equation can be re-written as

$$y' = \frac{2y}{x}.$$

Therefore, orthogonal trajectories satisfy the equation

$$y' = -\frac{x}{2y}.$$

Solving the equation yields that the ellipse defined by

$$y^2 + \frac{1}{2}x^2 = c,$$

is a family of orthogonal trajectories. ■

Exercise 4.8 Find the orthogonal trajectories of the family of hyperbolas

$$xy = c \quad c \neq 0.$$

Solution *Differentiating the equation implicitly with respect to x yields*

$$y + xy' = 0.$$

So

$$y' = -\frac{y}{x}.$$

Thus, the integral curves of

$$y' = \frac{x}{y}$$

are orthogonal trajectories of the given family. Separating variables and solving equations

yields

$$\begin{aligned} y' &= \frac{x}{y} \\ y'y &= x \\ \int y'y \, dx &= \int x \, dx \\ y^2 &= x^2 + k \\ y^2 - x^2 &= k. \end{aligned}$$

So an orthogonal trajectory is either hyperbola (if $k \neq 0$), or the union of the lines $y = x$ and $y = -x$ (if $k = 0$). ■

4.A Integrating factor methods for linear first order equations revisited

4.A.1 The integrating factor method for linear first order equations

Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x).$$

Recall an integrating factor for this equation is

$$r(x) = \int e^{\int p(x)\mathrm{d}x}\mathrm{d}x.$$

Multiplying the equation by $r(x)$ yields

$$\begin{aligned} r(x)\frac{\mathrm{d}y}{\mathrm{d}x} + r(x)p(x)y &= r(x)q(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}(r(x)y) &= r(x)q(x) \\ r(x)y &= \int r(x)q(x)\mathrm{d}x \\ y &= \frac{\int r(x)q(x)\mathrm{d}x}{r(x)} \\ y &= \frac{\int (q(x)e^{\int p(x)\mathrm{d}x})\mathrm{d}x}{e^{\int p(x)\mathrm{d}x}}. \end{aligned}$$

4.A.2 Linear first order equations viewed as non-exact equations

This integrating method can be considered as a special case of the integrating method discussed today. Note that the differential equation can be rewritten as

$$(p(x)y - q(x))\mathrm{d}x + \mathrm{d}y = 0.$$

Let $M(x, y) = p(x)y - q(x)$ and $N(x, y) = 1$. Then

$$\frac{\partial M}{\partial y} = p(x) \quad \text{and} \quad \frac{\partial N}{\partial x} = 0.$$

Therefore,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = p(x).$$

From the proposition discussed today, an integrating factor for the equation $(p(x)y - q(x))\mathrm{d}x + \mathrm{d}y$ is

$$\mu(x) = e^{\int p(x)\mathrm{d}x}.$$

From the above calculation, you may notice that $r(x) = \mu(x)$.

Now, to solve the equation $(p(x)y - q(x))\mathrm{d}x + \mathrm{d}y = 0$, multiplying both sides by $\mu(x)$ yields an exact equation

$$e^{\int p(x)\mathrm{d}x}(p(x)y - q(x))\mathrm{d}x + e^{\int p(x)\mathrm{d}x}\mathrm{d}y = 0.$$

The expected solution is an implicitly function defined by $F(x, y) = c$, that is F is a function such that

$$\frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y = \left[e^{\int p(x)\mathrm{d}x}(p(x)y - q(x))\right]\mathrm{d}x + e^{\int p(x)\mathrm{d}x}\mathrm{d}y.$$

To find F, we take the following steps:

1. Integrate $e^{\int p(x)\mathrm{d}x}\mathrm{d}y$. Denote the resulting function by $F(x, y)$. Then

$$F(x, y) = ye^{\int p(x)\mathrm{d}x} + g(x).$$

The reason to add a $g(x)$ is because $\frac{\partial}{\partial y}g(x) = 0$.

2. Solve for g from the equation

$$\frac{\partial F}{\partial x} = e^{\int p(x)\mathrm{d}x}(p(x)y - q(x)).$$

Note that

$$\begin{aligned}\frac{\partial F}{\partial x} =& y\frac{\partial}{\partial x}e^{\int p(x)\mathrm{d}x} + \frac{\partial}{\partial x}g(x)\\ =& yp(x)e^{\int p(x)\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}g(x).\end{aligned}$$

Then the function $g(x)$ satisfies the following equation

$$yp(x)e^{\int p(x)\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}g(x) = e^{\int p(x)\mathrm{d}x}(p(x)y - q(x)),$$

or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}x}g(x) = -q(x)e^{\int p(x)\mathrm{d}x}.$$

Therefore,

$$g(x) = \int \left(-q(x)e^{\int p(x)\mathrm{d}x}\right) \mathrm{d}x,$$

and

$$F(x,y) = ye^{\int p(x)\mathrm{d}x} - \int \left(q(x)e^{\int p(x)\mathrm{d}x}\right) \mathrm{d}x.$$

Set $F(x,y) = 0$ and solve for y, we find a solution

$$y = \frac{\int \left(q(x)e^{\int p(x)\mathrm{d}x}\right) \mathrm{d}x}{e^{\int p(x)\mathrm{d}x}},$$

which is the seem as the solution obtained using the first method.

Remark Instead of treating the whole equation as an non-exact equation and solve it, one can also find an integrating factor $\mu(x)$ of $yq(x)\mathrm{d}x + \mathrm{d}y$ and find F so that

$$\frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y q(x)\mathrm{d}x + \mathrm{d}y.$$

Then the equation $(yq(x) - q(x))\mathrm{d}x + \mathrm{d}y = 0$ can be expressed as

$$\mathrm{d}F(x,y) = \mu(x)q(x)\mathrm{d}x.$$

Then integrating both sides yields the solution.

Week 5: Linear Second Order Equations

10/4–10/7

5.1 Homogeneous Linear Second Order Equations

5.1.1 Basic concepts

The standard form of a second order equation is

$$y''(x) = f(x, y(x), y'(x)).$$

We often write it simply as

$$y'' = f(x, y, y')$$

when it is understood that x is the variable and y is the unknown.

For a 2nd order equation, the initial value problem is of the form

$$\begin{cases} y'' = f(x, y, y'). \\ y(x_0) = y_0. \\ y'(x_0) = y_1 \end{cases}$$

That is, in additional to the equation, the values of the function and its first derivative are given at a point.

A second order differential equation is said to be ***linear*** if it can be written in the following form:

$$y'' + p(x)y' + q(x)y = f(x).$$

We say that a linear second order equation is ***homogeneous*** if $f \equiv 0$, that is, the equation can be written in the following form.

$$y'' + p(x)y' + q(x)y = 0.$$

Example 5.1 Determine the whether the following equations are linear and/or homogeneous:

1. $2xy'' + x^2y' - (\sin x)y = x^3$;
2. $yy'' + xy' = x^2$;
3. $2e^x y'' + y = 0$.

Solution

1. *Linear but non-homogeneous;*
2. *Nonlinear;*
3. *Linear and homogeneous.*

■

5.1.2 Existence and uniqueness of solutions

The good thing about linear equations is the existence and uniqueness question has a nice and simple answer.

Theorem 5.1

Suppose p and q are continuous on an open interval (a, b), let x_0 be any point in (a, b), and let k_0 and k_1 be arbitrary real numbers. Then the initial value problem

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on (a, b).

The proof of this theorem also relies on Picard's method of successive approximations. We refer the reader to (Simmons 2016, Chapter 13) for a proof.

Example 5.2 Consider the equation

$$y'' - y = 0.$$

1. Find the general solution.
2. Given that $y(0) = 1$ and $y'(0) = 3$, find the specific solution to this initial value problem.
3. Determine if this example verifies the existence and uniqueness theorem.

Solution

1. *The equation can be solved using substitutions. First let $z = y' + y$, then*

$$\begin{aligned} y'' - y &= 0 \\ z' - y' - y &= 0 \qquad \text{where } z = y' + y \\ z' - z &= 0 \\ z' &= z \\ \frac{z'}{z} &= 1 \\ z &= 2c_1 e^x \end{aligned}$$

Here, we use $2c_1$ so that in the next substitution no fraction will appear. Now, let $u = y - c_1 e^x$, *then*

$$\begin{aligned} y' + y &= 2c_1 e^x \\ u' &= -u \qquad \text{where } u = y - c_1 e^x \\ \frac{u'}{u} &= -1 \\ u &= c_2 e^{-x} \\ y - c_1 e^x &= c_2 e^{-x} \\ y &= c_1 e^x + c_2 e^{-x} \end{aligned}$$

So the general solution is $y = c_1 e^x + c_2 e^{-x}$.

2. *The constants* c_1 *and* c_2 *are determined by the initial conditions* $y(0) = 1$ *and* $y'(0) = 3$. *Setting* $x = 0$ *in the functions* y *and* y' *yields*

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 3. \end{aligned}$$

Solving the system of equations yields $c_1 = 2$ *and* $c_2 = -1$. *Therefore,* $y = 2e^x - e^{-x}$ *is the unique solution of the equation* $y'' - y = 0$ *on* $(-\infty, \infty)$.

3. *Because the coefficient of* y' *and* y *are constant, hence continuous on* $(-\infty, \infty)$. *The theorem asserts that there is an unique solution on* $(-\infty, \infty)$ *which is also confirmed by the above solution.*

■

Note that in the above example, both $y = e^x$ and $y = e^{-x}$ are solution of the equation $y'' - y = 0$. This observation suggests an approach of solving linear second order equations.

5.1.3 The general solution

If you know linear algebra, you will find some similarity between solutions of linear systems and differential equations.

Theorem 5.2

Let y_1 *and* y_2 *be two solutions of the homogeneous and linear second order equation*

$$y'' + p(x)y' + q(x)y = 0.$$

Then, for any real numbers c_1 *and* c_2, *the linear combination* $y = c_1 y_1 + c_2 y_2$ *is also a solution.* ♥

The theorem indeed follows from the fact that differentiations are linear.

Proof By the linearity of differentiation, the first and second derivative of $y = c_1y_1 + c_2y_2$ are

$$\begin{aligned} y' &= c_1y_1' + c_2y_2' \\ y'' &= c_1y_1'' + c_2y_2''. \end{aligned}$$

Since y_1 and y_2 are solutions, the following equalities hold true

$$y_1'' + p(x)y_1' + q(x)y_1 = 0, \; y_2'' + p(x)y_2' + q(x)y_2 = 0.$$

Therefore,

$$\begin{aligned} & y'' + p(x)y' + q(x)y \\ =& c_1y_1'' + c_2y_2'' + p(x)(c_1y_1' + c_2y_2') + q(x)(c_1y_1 + c_2y_2) \\ =& c_1(y_1'' + p(x)y_1' + q(x)y_1) + c_2(y_2'' + p(x)y_2' + q(x)y_2) \\ =& 0, \end{aligned}$$

that is, $y = c_1y_1 + c_2y_2$ is a solution too. ■

Bases on the above theorem, you may wonder whether every solution of $y'' + p(x)y' + q(x)y = 0$ can be written in the form $y = c_1y_1 + c_2y_2$. The answer is affirmative if y_1 and y_2 are linearly independent. Two functions y_1 and y_2 are said to be ***linearly independent*** if there is no constant number c such that $y_1 = cy_2$ or $y_2 = cy_1$.

Theorem 5.3

If y_1 and y_2 are two linearly independent solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0,$$

then the general solution of the above equation is of the form

$$y = c_1y_1 + c_2y_2$$

For a complete proof, the reader may read Section 15 in the book “Differential equations with applications and historical notes" by George F. Simmons. The following is a rough idea.

By the uniqueness of the solution (Theorem 5.1), to show that any solution y can be written as $c_1y_1 + c_2y_2$, it suffices to show that, for some x_0, the following system of equations for c_1 and c_2 is solvable

$$\begin{cases} c_1y_1(x_0) + c_2y_2(x_0) = y(x_0) \\ c_1y_1'(x_0) + c_2y_2'(x_0) = y'(x_0). \end{cases}$$

From linear algebra, or college algebra, we know that the system is solvable if the determinant

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} := y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0) \tag{5.1}$$

is nonzero.

To show that the linear independence implies the nonzero of the determinant, we will need to study the following function.

The function of x defined by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

is called the ***Wronskian determinant*** or simply ***Wronskian*** of y_1 and y_2.

Wronskians has the following properties.

Lemma 5.1

Suppose p and q are continuous on (a, b), let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b).
Let x_0 be any point in (a, b). Then

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} p(t)\,\mathrm{d}t}, \quad a < x < b,$$

which is called Abel's formula.
Therefore, the Wronskian W is either identically zero or never zeros on (a, b).

Proof Let y_1 and y_2 be two solutions of the homogeneous and linear second order equation. Then

$$y_1'' + p(x)y_1' + q(x)y_1 = 0, \tag{5.2}$$

$$y_2'' + p(x)y_2' + q(x)y_2 = 0. \tag{5.3}$$

Multiplying equation 5.2 by y_2 and equation 5.3 by y_1 and taking the difference of the resulting equations yields

$$y_1''y_2 - y_2''y_1 + p(x)y_1'y_2 - y_2'y_1 = 0.$$

Note that

$$\begin{aligned} &(y_1'y_2 - y_2'y_1)' \\ =&(y_1''y_2 + y_1'y_2') - (y_1''y_2 + y_1'y_2') \\ =&y_1''y_2 - y_2''y_1. \end{aligned}$$

Let $W = y_1y_2' - y_2y_1'$. Then

$$-W' - p(x)W = 0.$$

Solving the equation yields that the general solution is

$$W = e^{\int -p(x)\,\mathrm{d}x}.$$

If $W(x_0)$ is given, then a specific solution is

$$W = W(x_0)e^{\int_{x_0}^{x} -p(t)\,\mathrm{d}t}.$$

Since $e^{\int_{x_0}^{x} p(t)\,\mathrm{d}t}$ in nonzero for all x, $W(x)$ is identically zero if $W(x_0) = 0$, or $W(x)$ is never zero if $W(x_0) \neq 0$. ■

The above lemma also implies the following useful result.

Corollary 5.1

The following are equivalent

1. *$W(x)$ is identically zero.*
2. *$W(x_0)$ is zero for some x_0.*

With the above properties of Wronskian, one can show that linear dependence is equivalent to that $W(x) \equiv 0$.

Lemma 5.2

If y_1 and y_2 are two solutions of equation $y'' + p(x)y' + q(x)y = 0$, then they are linearly dependent if and only if their Wronskian $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$ is identically zero.

Theorem 5.3 follows from the above lemma. Because linear independence implies that $W(x_0) \neq 0$ for some x_0, which implies the system of equations (5.1) has a nontrivial solution.

Example 5.3 Consider the equation

$$y'' - 3y' + 2y = 0$$

Determine whether

$$y = c_1 e^x + c_2 e^{2x}$$

is the general solution, where c_1 and c_2 are arbitrary constants.

Solution *To see if $y = c_1 e^x + c_2 e^{2x}$ is the general solution, it suffices to check that $y_1 = e^x$ and $y_2 = e^{2x}$ are two linearly independent solutions. Since*

$$(e^x)'' - 3(e^x)' + 2e^x = e^x - 3e^x + 2e^x = 0$$

and

$$(e^{2x})'' - 3(e^{2x})' + 2e^{2x} = 4e^{2x} - 6e^{2x} + 2e^{2x} = 0,$$

both y_1 and y_2 are solutions.

As there is no constant number c such that the equations $e^x = ce^{2x}$ and $ce^x = e^{2x}$ hold true for all x, the functions y_1 and y_2 are linearly independent.

Therefore, $y = c_1 e^x + c_2 e^{2x}$ is the general solution. ■

Note the equation $y'' - 3y' + 2y = 0$ again can be solve using substitutions (try it as an exercise).

Example 5.4 Verify Abel's formula for the following differential equations and the corresponding solutions.

$$y'' - y = 0; \quad y_1 = e^x, \ y_2 = e^{-x}.$$

Solution *Since $p \equiv 0$, the integral $\int p(x)\,\mathrm{d}\,x$ is a constant. A direct calculation shows that*

$$W(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x(-e^{-x}) - e^x e^{-x} = -2$$

for all x. This verifies Abel's formula. ■

Exercise 5.1 Verify Abel's formula for the following differential equations and the corresponding solutions,

$$y'' + y = 0; \quad y_1 = \cos x, \ y_2 = \sin x.$$

Solution *Since $p \equiv 0$, we can verify Abel's formula by showing that W is constant. A direct calculation shows that*

$$\begin{aligned} W(x) &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\ &= \cos^2 x - \sin x(-\sin x) \\ &= 1 \end{aligned}$$

for all x. ■

5.1.4 The reduction of order method*

Lemma 5.1 also provides a way of finding the another solution y_2 given a solution y_1.

Proposition 5.1

Suppose y_1 is a solution to $y'' + p(x)y' + q(x)y = 0$. Then a function y_2 is a solution if and only if

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)\,\mathrm{d}\,x}}{\big(y_1(x)\big)^2}\,\mathrm{d}\,x.$$

Proof Let y_1 is a solution. If y_2 is another solution, then

$$y_1 y_2' - y_1' y_2 = W = e^{\int p(x)\,\mathrm{d}x}.$$

Note that $y_1 y_2' - y_1' y_2 = y_1^2 \left(\frac{y_2}{y_1}\right)'$. This equation can be solved by a linear substitution $z = \frac{y_2}{y_1}$. Indeed, since $z' = \frac{y_2' y_1 - y_2 y_1'}{y_1^2}$, the equation can be rewritten as

$$y_1^2 z' = e^{\int p(x)\,\mathrm{d}x}.$$

Dividing both sides by y_1^2 and integrating directly yields

$$\frac{y_2}{y_1} = z = \int \left(\frac{e^{-\int p(x)\,\mathrm{d}x}}{y_1^2}\right) \mathrm{d}x.$$

Hence,

$$y_2(x) = y_1 \int \frac{e^{-\int p(x)\,\mathrm{d}x}}{\left(y_1\right)^2}\,\mathrm{d}x.$$

Conversely, by the Fundamental Theorem of Calculus,

$$y_2' = y_1' \int \frac{e^{-\int p(x)\,\mathrm{d}x}}{\left(y_1\right)^2}\,\mathrm{d}x + \frac{e^{-\int p(x)\,\mathrm{d}x}}{y_1}$$

and

$$\begin{aligned} y_2'' =& y_1'' \int \frac{e^{-\int p(x)\,\mathrm{d}x}}{\left(y_1\right)^2}\,\mathrm{d}x + y_1' \frac{e^{-\int p(x)\,\mathrm{d}x}}{\left(y_1\right)^2} + \frac{-p(x)e^{-\int p(x)\,\mathrm{d}x} y_1 - y_1' e^{-\int p(x)\,\mathrm{d}x}}{y_1^2}. \\ =& y_1'' \int \frac{e^{-\int p(x)\,\mathrm{d}x}}{\left(y_1\right)^2}\,\mathrm{d}x - \frac{p(x)e^{-\int p(x)\,\mathrm{d}x}}{y_1}. \end{aligned}$$

Therefore,

$$\begin{aligned} & y_2'' + p(x)y_2' + y_2 \\ =& y_1'' \int \frac{e^{-\int p(x)\,\mathrm{d}x}}{\left(y_1\right)^2}\,\mathrm{d}x - \frac{p(x)e^{-\int p(x)\,\mathrm{d}x}}{y_1} \\ & + p(x)\left(y_1' \int \frac{e^{-\int p(x)\,\mathrm{d}x}}{\left(y_1\right)^2}\,\mathrm{d}x + \frac{e^{-\int p(x)\,\mathrm{d}x}}{y_1}\right) + q(x)y_1 \int \frac{e^{-\int p(x)\,\mathrm{d}x}}{\left(y_1\right)^2}\,\mathrm{d}x \\ =& y_1'' \int \frac{e^{-\int p(x)\,\mathrm{d}x}}{\left(y_1\right)^2}\,\mathrm{d}x + p(x)y_1' \int \frac{e^{-\int p(x)\,\mathrm{d}x}}{\left(y_1\right)^2}\,\mathrm{d}x + q(x)y_1 \int \frac{e^{-\int p(x)\,\mathrm{d}x}}{\left(y_1\right)^2}\,\mathrm{d}x \\ =& \left(y_1'' + p(x)y_1' + q(x)y_1\right) \int \frac{e^{-\int p(x)\,\mathrm{d}x}}{\left(y_1\right)^2}\,\mathrm{d}x \\ =& 0. \end{aligned}$$

So y_2 is also a solution. ■

Example 5.5 Consider the equation

$$x^2y'' + xy' - y = 0.$$

1. Guess a solution and find another solution that is linearly independent to it.
2. Find the general solution.

Solution *Let $y_1 = x$. Then it is a solution. Rewrite the equation so that the leading coefficient is 1:*

$$y'' + \frac{y'}{x} - \frac{y}{x^2} = 0.$$

So $p(x) = \frac{1}{x}$. By the proposition, another solution can be calculated as follows

$$\begin{aligned} y_2 =& y_1 \int \left(\frac{e^{-\int p(x)\,\mathrm{d}x}}{y_1^2} \right) \mathrm{d}x \\ =& x \int \left(\frac{e^{-\int \frac{1}{x}\,\mathrm{d}x}}{x^2} \right) \mathrm{d}x \\ =& x \int \left(\frac{x^{-1}}{x^2} \right) \mathrm{d}x \\ =& x \int \left(\frac{1}{x^3} \right) \mathrm{d}x \\ =& x \left(-\frac{1}{2x^2} \right) \\ =& -\frac{1}{2x}. \end{aligned}$$

It's not so hard to check that x and $\frac{1}{2x}$ are linearly independent.

Therefore, the function $y = c_1x + \frac{c_2}{x}$ is the general solution. ■

✍ **Exercise 5.2** Consider the equation

$$xy'' - y' = 0.$$

1. Guess a solution and find another solution that is linearly independent to it.
2. Find the general solution.

Solution *Clearly, the constant function $y_1 = 1$ is a solution. Rewrite the equation so that the leading coefficient is 1:*

$$y'' - \frac{y'}{x} = 0.$$

So $p(x) = -\frac{1}{x}$. By the proposition, another solution can be calculated as follows

$$\begin{aligned}
y_2 =& y_1 \int \left(\frac{e^{-\int p(x)\,\mathrm{d}x}}{y_1^2} \right) \mathrm{d}x \\
=& \int e^{\int \frac{1}{x}\,\mathrm{d}x}\,\mathrm{d}x \\
=& \int e^{\ln x}\,\mathrm{d}x \\
=& \int x\,\mathrm{d}x \\
=& \frac{x^2}{2}
\end{aligned}$$

If there are numbers such that $c_1 y_1 + c_2 y_2 =$, then $c_1 + c_2 \frac{x^2}{2} = 0$ for all x, which can only hold true if $c_1 = c_2 = 0$. Therefore, y_1 and y_2 are linearly independent.

Therefore, the function $y = c_1 + c_2 x^2$ is the general solution. ■

5.2 Constant Coefficient Homogeneous Equations

A linear second order differential equation is said to be a ***constant coefficient equation*** if it can be written as

$$y'' + ay' + by = f(x),$$

where a and b are constant real numbers.

In this section, we consider the homogeneous constant coefficient equation

$$y'' + ay' + by = 0.$$

How to solve this type of equations? Recall, by Theorem 5.3, if y_1 and y_2 are two solutions not proportional to each other, then $y = c_1 y_1 + c_2 y_2$ is the general solution. The question is how to find two linearly independent solutions.

Since a and b are constants, a solution function has to have the same "degree" as its derivatives. We know such a function, $y = e^{rx}$. Plugging it into the equation $y'' + ay' + by = 0$ yields

$$\begin{aligned}
(e^{rx})'' + a(e^{rx})' + be^{rx} =& 0 \\
r(e^{rx})' + are^{rx} + be^{rx} =& 0 \\
r^2 e^{rx} + are^{rx} + be^{rx} =& 0 \\
r^2 + ar + b =& 0
\end{aligned}$$

So r is a solution of the quadratic equation $x^2 + ax + b = 0$.

We call $p(r) = r^2 + ar + b$ the ***characteristic polynomial*** of the equation $y'' + ay' + by = 0$. The quadratic equation $p(r) = 0$ is called the ***characteristic equation***.

Example 5.6 Consider the equation

$$y'' - y' - 2y = 0.$$

Find the general solution.

Solution *We expect that $y = e^{rx}$ to be a solution for some r. Since*

$$\begin{aligned} y' &= re^{rx} \\ y'' &= r^2 e^{rx} \end{aligned}$$

Then

$$y'' - y' - 2y = r^2 e^{rx} - re^{rx} - 2e^{rx} = (r^2 - r - 2)e^{rx}.$$

Therefore, $y = e^{rx}$ is a solution if and only if

$$(r^2 - r - 2)e^{rx} = 0.$$

Equivalently,

$$r^2 - r - 2 = 0.$$

Solving the equation by the factorization

$$r^{-} r - 2r = (r + 1)(r - 2)$$

yields, $r = -1$ or $r = 2$. Therefore, $y = e^{-x}$ and $y = e^{2x}$ are both solutions. Moreover, if there are numbers d_1 and d_2 such that

$$d_1 e^{-x} + d_2 e^{2x} = 0$$

Then

$$d_1 + d_2 e^{3x} = 0$$

equivalently, $d_1 = 0$ and $d_2 = 0$. Therefore, the two solutions are linearly independent. Hence the general solution is

$$y = c_1 e^{-x} + c_2 e^{2x}.$$

■

The methods used in this example works well if r_1 and r_2 are two distinct real root. If $r_1 = r_2$, then the method of reduction of order will be needed. If r_1 and r_2 are two conjugate complex solutions, the Euler's formula and the fact that $a + ib = 0$ if and only if $a = 0$ and $b = 0$ will be need.

The equation can also be solved using decomposition and substitution as follows. Let r_1 and r_2 be two roots of the characteristic equation. Then

$$y'' + ay' + by = (y' - r_1 y)' + r_2(y' - r_1 y).$$

Using this decomposition, we can reduced the equation $y''+ay'+by = 0$ to first order equations by the substitution $z = y' - r_1 y$.

Theorem 5.4

Let $p(r) = r^2 + ar + b$ be the **characteristic polynomial** *of*

$$y'' + ay' + by = 0.$$

1. *If $p(r) = 0$ has distinct real roots r_1 and r_2, then the general solution of the equation is*

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

2. *If $p(r) = 0$ has a repeated root r then the general solution of the equation is*

$$y = e^{rx}(c_1 + c_2 x).$$

3. *If $p(r) = 0$ has complex conjugate roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, where $\beta > 0$, then the general solution of the equation is*

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x).$$

Proof Let r_1 and r_2 be two roots of the characteristic equation

$$r^2 + ar + b = 0.$$

Then

$$\begin{aligned} y'' + ay' + by =& y'' - (r_1 + r_2)y' + r_1 r_2 y \\ =&(y'' - r_1 y') - r_2(y' - r_1 y) \\ =&(y' - r_1 y)' - r_2(y' - r_1 y) \end{aligned}$$

Therefore, the equation $y'' + ay' + by$ can be reduced to first order differential equations by the substitution $z = y' - r_1 y$:

$$\begin{aligned} y'' + ay' + by &= 0 \\ (y' - r_1 y)' - r_2(y' - r_1 y) &= 0 \\ z' - r_2 z &= 0 \\ \frac{z'}{z} &= r_2 \\ z &= c_1 e^{r_2 x}. \end{aligned}$$

Thus,

$$y' - r_1 y = c_1 e^{r_2 x}.$$

Since the coefficients of y' and y are 1 and $-r_1$ respectively, the integrating factor is

$$r(x) = e^{\int (-r_1)\,\mathrm{d}x} = e^{-r_1 x}.$$

Therefore,

$$\begin{aligned} y' - r_1 y &= c_1 e^{r_2 x} \\ e^{-r_1 x} y' - r_1 e^{-r_1 x} y &= c_1 e^{-r_1 x} e^{r_2 x} \\ (y e^{-r_1 x})' &= c_1 e^{-r_1 x} e^{r_2 x} \\ y e^{-r_1 x} &= c_1 \int e^{(r_2 - r_1)x}\,\mathrm{d}x \\ y &= c_1 e^{r_1 x} \int e^{(r_2 - r_1)x}\,\mathrm{d}x \end{aligned}$$

If $r_1 = r_2 = r$, then

$$\int e^{(r_2 - r_1)x}\,\mathrm{d}x = \int \mathrm{d}x = x + c.$$

Therefore, the general solution is

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

If $r_1 \neq r_2$, then

$$\int e^{(r_2 - r_1)x}\,\mathrm{d}x = \frac{e^{(r_2 - r_1)x}}{r_2 - r_1} + c$$

Therefore, the general solution is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

If r_1 and r_2, are real numbers, then $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is a real-valued function and is the general solution.

If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ are complex numbers, then $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is a solution but it is a complex-valued function. We want to find a real-valued function from it.

Euler's formula says

$$e^{ix} = \cos x + i \sin x \qquad e^{-ix} = \cos x - i \sin x.$$

Then

$$\begin{aligned}
& c_1 e^{r_1 x} + c_2 e^{r_2 x} \\
=& c_1 e^{\alpha x + i\beta x} + c_2 e^{\alpha x - i\beta x} \\
=& c_1 e^{\alpha x}(\cos(\beta x) + i \sin(\beta x)) + c_2 e^{\alpha x}(\cos(\beta x) - i \sin(\beta x)) \\
=& c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x}(\cos(\beta x) + i\,(c_1 e^{\alpha x} \sin(\beta x) - c_2 e^{\alpha x} \sin(\beta x)) \\
=& (c_1 + c_2) e^{\alpha x} \cos(\beta x) + i(c_1 - c_2) e^{\alpha x} \sin(\beta x)
\end{aligned}$$

Note that a complex-valued function $w(x) = u(x) + iv(x)$ is a solution of $y'' + ay' + by = 0$ if and only if both $u(x)$ and $v(x)$ are solutions. Therefore, by Theorem 5.3, the real-valued linear combination

$$y = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x)$$

is the general solution. ■

Example 5.7 Solve the initial value problem

$$y'' + 6y' + 5y = 0, \quad y(0) = 3, \ y'(0) = -1.$$

Solution *The characteristic equation of the differential equation is*

$$r^2 + 6r + 5 = 0.$$

Solving it by factoring yields two distinct real roots: $r_1 = -1$ *and* $r_2 = -5$. *By Theorem 5.4, the general solution of the differential equation is*

$$y = c_1 e^{-x} + c_2 e^{-5x}.$$

Since y *satisfies the initial conditions* $y(0) = 3$ *and* $y'(0) = -1$, *and the first derivative of* y *is*

$$y' = -c_1 e^{-x} - 5c_2 e^{-5x},$$

the constants c_1 *and* c_2 *satisfy the following system of equations*

$$\begin{aligned}
c_1 + c_2 &= 3 \\
-c_1 - 5c_2 &= -1.
\end{aligned}$$

The solution of this system is $c_1 = \frac{7}{2}, c_2 = -\frac{1}{2}$. *Therefore, the solution of the initial value problem is*

$$y = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}.$$

■

Example 5.8 Find the general solution of

$$y'' - 2y' + y = 0.$$

Solution *The characteristic equation is*

$$r^2 - 2r + 1 = 0$$

which is the same as

$$(r-1)^2 = 0$$

Hence, $r = 1$ *is the repeated root. By Theorem 5.4, the general solution is*

$$y = (c_1 + c_2 x)e^x.$$

■

Example 5.9 Solve the initial value problem

$$y'' - 2y' + 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 2.$$

Solution *The characteristic equation is*

$$r^2 - 2r + 2 = 0$$

which is the same as

$$(r-1)^2 + 1 = 0$$

Hence, $r = 1 \pm \mathrm{i}$ *are the solutions. By Theorem 5.4, the general solution is*

$$y = e^x(c_1 \cos x + c_2 \sin x).$$

Since $y(0) = 1$, $y'(0) = 2$ *and*

$$y' = e^x(c_1 \cos x + c_2 \sin x) + e^x(c_2 \cos x - c_1 \sin x),$$

the constants c_1 and c_2 satisfy the system of equations

$$\begin{cases} c_1 = 1 \\ c_1 + c_2 = 2 \end{cases}$$

Therefore, $c_1 = 1$ and $c_2 = 1$, and the solution of the initial value problem is

$$y = e^x(\cos x + \sin x).$$

■

✍ **Exercise 5.3** Find the general solution of the differential equation

$$y'' - 3y' + 2y = 0.$$

Solution *Solving the characteristic equation*

$$r^2 - 3r + 2 = 0$$

yields $r_1 = 1$ and $r_2 = 2$. By Theorem 5.4, the general solution is

$$y = c_1 e^x + c_2 e^{2x}.$$

■

✍ **Exercise 5.4** Solve the initial value problem

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = -1.$$

Solution *Solving the characteristic equation*

$$r^2 - 4r + 4 = 0$$

yields a double root $r = 2$. By Theorem 5.4, the general solution is

$$y = (c_1 + c_2 x)e^{2x}.$$

Since $y(0) = 1$, $y'(0) = -1$, and $y' = c_2 e^{2x} + 2(c_1 + c_2 x)e^{2x}$, the constants c_1 and c_2 satisfy

$$\begin{cases} c_1 = 1 \\ 2c_1 + c_2 = -1. \end{cases}$$

Therefore, $c_1 = 1$, $c_2 = -3$, and the solution of the initial value problem is

$$y = (1 - 3x)e^{2x}.$$

■

✍ **Exercise 5.5** Find the general solution of the equation

$$y'' + 4y = 0.$$

Solution *Solving the characteristic equation*

$$r^2 + 4 = 0$$

yields two complex roots $r_1 = 2\mathrm{i}$ and $r_2 = -2\mathrm{i}$. By Theorem 5.4, the general solution is

$$y = c_1 \cos(2x) + c_2 \sin(2x).$$

■

5.3 Non-Homogeneous Linear Equations

In this section, we consider the nonhomogeneous linear second order equation

$$y'' + p(x)y' + q(x)y = f(x).$$

We will assume that p, q, and f are continuous on a interval (a, b).

Like first order equation, to find the general solution, it is necessary to find the general solution of the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0,$$

which is called the ***complementary equation***.

5.3.1 The form of the general solution

If we have two linearly independent solutions of the complementary equation, which is also known as a fundamental set, and a particular solution of the nonhomogeneous equation, then the general solution of the nonhomogeneous equation is a linear combination of those solutions.

Theorem 5.5

Suppose p, q, and f are continuous on (a, b). Let y_p be a particular solution of

$$y'' + p(x)y' + q(x)y = f(x) \tag{5.4}$$

on (a, b), and let $\{y_1, y_2\}$ be a fundamental set of solutions of the complementary equation

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b). Then y is a solution of (5.4) on (a, b) if and only if

$$y = y_p + c_1 y_1 + c_2 y_2,$$

where c_1 and c_2 are constants.

Proof Suppose that $y = y_p + c_1 y_1 + c_2 y_2$. Since y_1 and y_2 are solutions of the complementary equation, and y_p is a solution of Equation (5.4), we see that

$$\begin{aligned} y_1'' + p(x)y_1' + q(x)y_1 &= 0 \\ y_2'' + p(x)y_2' + q(x)y_2 &= 0 \\ y_p'' + p(x)y_p' + q(x)y_p &= f(x) \end{aligned}$$

Then

$$\begin{aligned} &y'' + p(x)y' + q(x)y \\ =&(y_p + c_1 y_2 + y_2)'' + p(x)(y_p + c_1 y_2 + y_2)' + q(x)(y_p + c_1 y_2 + y_2) \\ =&(y_p'' + p(x)y_p' + q(x)y_p) + c_1(y_1'' + p(x)y_1' + q(x)y_1) + c_2(y_2'' + p(x)y_2' + q(x)y_2) \\ =&f(x). \end{aligned}$$

Therefore, y is a solution of Equation (5.4).

Conversely, suppose that y is a solution of Equation (5.4). Then

$$y'' + p(x)y' + q(x)y = f(x)$$

and

$$\begin{aligned} &(y - y_p)'' + p(x)(y - y_p)' + q(x)(y - y_p) \\ =&(y'' + p(x)y' + q(x)y) - (y_p'' + p(x)y_p' + q(x)y_p) \\ =&f(x) - f(x) \\ =&0. \end{aligned}$$

Therefore, $y - y_p$ is a solution of the complementary equation. So there exists constants c_1 and c_2 such that

$$y - y_p = c_1 y_1 + c_2 y_2,$$

or

$$y = y_p + c_1 y_1 + c_2 y_2.$$

Thus, y is a solution of Equation (5.4) if and only if $y = y_p + c_1 y_2 + c_2 y_2$. ■

Example 5.10 Find the general solution of the equation

$$y'' - y' - 6y = 12.$$

Solution *We first solve the complementary equation*

$$y'' - y' - 6y = 0.$$

Its characteristic equation is

$$r^2 - r - 6 = 0$$

which has two solutions $r_1 = -2$ and $r_2 = 3$. Therefore, the fundamental set is $\{e^{-2x}, e^{3x}\}$.

Note that $y = 2$ is a particular solution. Then the general solution of the equation is

$$y = 2 + c_1 e^{-2x} + c_2 e^{3x}.$$

■

Example 5.11 Find the general solution of

$$y'' - 3y' + 2y = x^2 + 1$$

Solution *The complementary homogenous equation is*

$$y'' - 3y' + 2y = 0.$$

Since its characteristic equation

$$r^2 - 3r + 2 = 0$$

has two real roots $r_1 = 1$ and $r_2 = 2$, the general solution to the complementary equation is

$$y_h = c_1 e^x + c_2 e^{2x}.$$

Next we need to find a particular solution of

$$y'' - 3y' + 2y = x^2 + 1.$$

Since the right hand side is a degree 2 polynomial and taking derivatives decreases degrees, we may *assume that a particular solution is a quadratic function*

$$y_p = ax^2 + bx + c.$$

Since

$$\begin{aligned} y_p' &= 2ax + b \\ y_p'' &= 2a, \end{aligned}$$

the undetermined coefficients a, b, and c satisfy the following equation for any value of x

$$\begin{aligned} y_p'' - 3y_p' + 2y_p &= x^2 + 1 \\ 2a - 6ax - 3b + 2ax^2 + 2bx + 2c &= x^2 + 1 \\ 2ax^2 + (2b - 6a)x + (2a - 3b + 2c) &= x^2 + 1 \end{aligned}$$

By comparing coefficients of powers of x, we get a system of linear equations

$$\begin{cases} 2a & = 1 \\ 2b - 6a & = 0 \\ 2a - 3b + 2c & = 1. \end{cases}$$

Solving this system implies that $a = \frac{1}{2}$, $b = \frac{3}{2}$ and $c = \frac{9}{4}$. Hence

$$y_p = \frac{2x^2 + 6x + 9}{4}$$

is a particular solution.

Therefore, the general solution is

$$y = \frac{2x^2 + 6x + 9}{4} + c_1 e^x + c_2 e^{2x}$$

■

The method used to find a particular solution in the above example is known as the method of undetermined coefficients. We will revisit this method in the next section.

Exercise 5.6 Find the general solution of the equation

$$y'' - y = x.$$

Solution *The complementary equation is*

$$y'' - y = 0$$

whose general solution is $y_h = c_1 e^{-x} + c_2 e^{x}$.

We may assume that $y_p = ax + b$ *is a particular solution. Then* a *and* b *satisfies the following equation for all* x

$$-(ax + b) = x.$$

Therefore, $a = -1$, $b = 0$, *and the particular solution is* $y_p = -x$.

Hence, the general solution of the equation is

$$y = -x + c_1 e^{-x} + c_2 e^{x}.$$

■

Week 6: Linear Second Order Equations II

10/12–10/21

6.1 The Method of Undetermined Coefficients

In this section, we will study how to solve some linear second order constant coefficient nonhomogeneous equations

$$y'' + ay' + by = f(x).$$

Recall that the general solution of the equation can be written as

$$y = c_1 y_1 + c_2 y_2 + y_p,$$

where y_1 and y_2 are non-proportional solutions of the complementary equation $y'' + ay' + by = 0$, and y_p is a particular solution of the nonhomogeneous equation.

We already knew how to find a general solution $c_1 y_1 + c_2 y_2$ of the complementary equation. A particular solution may be found by guessing. The method of undetermined coefficients provides an approach to find a particular solution.

Because the derivative a polynomial function is still a polynomial, the derivative of an exponential function is still an exponential, and the derivative of a trigonometric function is still a trigonometric function, we often expect a same type of function as a particular solution.

6.1.1 Non-linear with basic functions

6.1.1.1 General cases

- If $f(x)$ is a polynomial, then a particular solution is often a polynomial of the same degree.
- If $f(x) = Pe^{\alpha x}$, then a particular solution is often of the form $Ae^{\alpha x}$.
- If $f(x) = P \sin \beta x + Q \cos \beta x$, then a particular solution is often of the form: $A \sin \beta x + B \cos \beta x$.

Example 6.1 Find the general solution of the equation

$$y'' - y = x^2$$

Solution *We first find the general solution of the complementary equation* $y'' - y = 0$. *Since the* *characteristic equation*

$$r^2 - 1 = 0$$

has two distinct solutions $r_1 = -1$ *and* $r_2 = 1$. *The general solution of the complementary equation is*

$$y_h = c_1 e^{-x} + c_2 e^x.$$

Since the differentiations of a function remains of the same type, and the right-hand side is an polynomial function, we expect a particular solution

$$y_p = ax^2 + bx + c.$$

Plugging it into the equation yields

$$\begin{aligned}(ax^2 + bx + c)'' - (ax^2 + bx + c) &= x^2\\ 2a - (ax^2 + bx + c) &= x^2\\ -ax^2 - bx + (a - c) &= x^2.\end{aligned}$$

Comparing coefficients of powers of x, *we see that* $a = -1$, $b = 0$ *and* $c = a = -1$.

Therefore, a particular solution is $y_p = -x^2 - 1$ *and the general solution is*

$$y = c_1 e^{-x} + c_2 e^x - x^2 - 1.$$

■

Example 6.2 Find the general solution of the equation

$$y'' - y' - 2y = 7e^{3x}$$

Solution *We first find the general solution of the complementary equation* $y'' - y' - 2y$. *Since the characteristic equation*

$$r^2 - r - 2 = 0$$

has two distinct solutions $r_1 = -1$ *and* $r_2 = 2$. *The general solution of the complementary equation is*

$$y_h = c_1 e^{-x} + c_2 e^{2x}.$$

Since the differentiations of a function remains of the same type, and the right-hand side is an exponential function, we expect a particular solution

$$y_p = ce^{3x}.$$

Plugging it into the equation yields

$$\begin{aligned}
(ce^{3x})'' - (ce^{3x}) + ce^{3x} =& 7e^{3x}\\
(3ce^{3x})' - 3ce^{3x} + ce^{3x} =& 7e^{3x}\\
9ce^{3x} - 2ce^{3x} =& 7e^{3x}\\
7ce^{3x} =& 7e^{3x}\\
c =& 1.
\end{aligned}$$

Therefore, a particular solution is $y_p = e^{3x}$ *and the general solution is*

$$y = c_1e^{-x} + c_2e^{2x} + e^{3x}.$$

■

Example 6.3 Find the general solution of the equation

$$y'' - 3y' + 2y = \sin x.$$

Solution *Since the characteristic equation* $r^2 - 3r + 2$ *has two solutions* $r_1 = 1$ *and* $r_2 = 2$. *Then the general solution of the complementary equation is*

$$y_h = c_1e^{-x} + c_2e^{x}.$$

Since the right-hand side is $\sin x$ *whose higher derivatives are either* $\sin x$ *or* $\cos x$, *we expect a particular solution*

$$y_p = A\cos x + B\sin x.$$

Since $(\sin x)'' = -\sin x$ *and* $(\cos x)'' = -\cos x$ *Plugging it into the equation yields*

$$\begin{aligned}
(A\cos x + B\sin x)'' - 3(A\cos x + B\sin x)' + 2(A\cos x + B\sin x) &= \sin x\\
(-A\cos x - B\sin x) - 3(-A\sin x + B\cos x) + 2(A\cos x + B\sin x) &= \sin x\\
(3A + B)\sin x + (A - 3B)\cos x &= \sin x.
\end{aligned}$$

Then A *and* B *satisfy the following system of equations*

$$\begin{cases} 3A + B = 1\\ A - 3B = 0. \end{cases}$$

Solving the equation by elimination method yields

$$A = \frac{3}{10} \qquad \text{and} \qquad B = \frac{1}{10}.$$

Therefore, a particular solution is

$$y_p = \frac{3}{10}\cos x + \frac{1}{10}\sin x$$

and the general solution is

$$y = c_1 e^{-x} + c_2 e^x + \frac{3}{10}\cos x + \frac{1}{10}\sin x.$$

■

6.1.1.2 Exceptional cases

In the above examples, the right-hand side function contains no factor which is a solution of the complementary equation. In some exceptional cases, we can adjust the particular solution to be used.

Example 6.4 Find a particular solution y_p of

$$y'' + 2y' - 3y = 4e^x.$$

Solution *Since the characteristic equation $r^2 + 2r - 3 = 0$ has two distinct root, the complementary equation has a general solution*

$$y_h = c_1 e^x + c_2 e^{-3x}.$$

Since e^x is a solution, plugging Ae^x into the equation will give

$$0 = e^x.$$

So Ae^x can not be a particular solution for any A.

Remember, when the characteristic equation has a repeated root, an extra solution can be taking in the form xe^{rx}. Here, we can try

$$y_p = Axe^x.$$

Plugging y_p into the equation yields

$$\begin{aligned}
(Axe^x)'' + 2(Axe^x)' - 3Axe^x &= 4e^x \\
A(e^x + xe^x)' + 2A(e^x + xe^x) - 3Axe^x &= 4e^x \\
A(e^x + e^x + xe^x) + 2A(e^x + xe^x) - 3Axe^x &= 4e^x \\
4Ae^x &= e^x \\
A &= 1
\end{aligned}$$

Therefore, a particular solution is $y_p = xe^x$, *and the general solution is*

$$y = c_1 e^x + c_2 e^{-3x} + xe^x.$$

■

Indeed, there is no surprise that the xe^{rx} terms disappear. The n-th derivatives has exactly one term with a factor x, the term is $r^n xe^{rx}$. Because r is a root of the equation $r^2 + ar + b = 0$, then the sum of those terms equals zero.

What if xe^{rx} is also a solution? Well, we raise the power of x by 1.

Example 6.5 Find the general solution of the equation

$$y'' - 2y' + y = 2e^x.$$

Solution *The characteristic equation* $r^2 - 2r + 1 = 0$ *has a repeated root* $r = 1$. *Then the complementary equation has a general solution*

$$y_h = c_1 e^x + c_2 xe^x.$$

Since both e^x *ane* xe^x *are solutions, we try* $y_p = Ax^2 e^x$.

Plugging $y_p = Ax^2 e^x$ *into the equation yields*

$$\begin{aligned}
(Ax^2e^x)'' - 2(Ax^2e^x)' + Ax^2e^x &= 2e^x \\
A(2xe^x + x^2e^x)' - 2A(2xe^x + x^2e^x) + Ax^2e^x &= 2e^x \\
A(2e^x + 2xe^x + 2xe^x + x^2e^x)' - 2A(2xe^x + x^2e^x) + Ax^2e^x &= 2e^x \\
A(x^2e^x - 2x^2e^x + x^2e^x) + A(4xe^x - 4xe^x) + 2Ae^x &= 2e^x \\
2Ae^x &= 2e^x \\
A &= 1
\end{aligned}$$

Therefore, a particular solution is $y_p = 2x^2e^x$, *and the general solution is*

$$y = c_1 e^x + c_2 xe^x + x^2 e^x.$$

■

When the characteristic equation has complex roots, the methods shown in the above examples still work.

Example 6.6 Find a particular solution y_p of

$$y'' + y = 2\sin x$$

Solution *Since the characteristic equation is $r^2+1 = 0$ which has two conjugate roots $\pm\mathbb{i}$, the general solution of the complementary equation is $y_h = c_1\cos x + c_2\sin x$. Taking $y_p = A\sin x + B\cos x$ won't work because plugging it into the left-hand side of the equation yields*

$$y_p'' + y_p = 0.$$

However, the right-hand sides is $2\sin x$. So $A\sin x + B\cos x$ can not be a solution.

Let's try

$$y_p = x(A\sin x + B\cos x).$$

Differentiating y_p yields

$$\begin{aligned} y_p' =& Ax\cos x + A\sin x - Bx\sin x + B\cos x \\ y_p'' =& - Ax\sin x + 2A\cos x - Bx\cos x - 2B\sin x, \end{aligned}$$

and

$$\begin{aligned} &y_p'' + y_p \\ =& - Ax\sin x + 2A\cos x - Bx\cos x - 2B\sin x + Ax\sin x + Bx\cos x \\ =& 2A\cos x - 2B\sin x \end{aligned}$$

Hence, $y_p = Ax\sin x + Bx\cos x$ is a solution if

$$2A\cos x - 2B\sin x = 2\sin x.$$

Equivalently,

$$\begin{cases} 2A & = 0 \\ -2B & = 2 \end{cases}$$

Solving this system of equations yields $A = 0$ and $B = -1$. Therefore, a particular solution is

$$y_p = -x\cos x.$$

The general solutions is

$$y = c_1\cos x + c_2\sin x - x\cos x.$$

■

You will find that a particular solution of $y'' + ay' = f(x)$ should also have a higher degree than the polynomial $f(x)$. This is because, if y_p has the same degree as of f, then the left-hand side will have the degree 1 less due to the differentiation.

Example 6.7 Find a general solution of the equation

$$y'' + y' = 2x$$

Solution *The characteristic equation $r^2 + r = 0$ has two distinct roots $r_1 = 0$ and $r_2 = 1$. So the general solution of the complementary equation is*

$$y_h = c_1 + c_2 e^x.$$

We are looking for a polynomial as a particular solution. Since differentiating polynomials decrease it degree by 1 and there is no y-term in the equation, a particular solution should be 1 degree bigger than $f(x) = 2x$. Indeed, one will find that taking $y_p = ax + b$ won't work.

Let's try

$$y_p = x(ax + b) = ax^2 + bx.$$

The first and second derivatives are

$$y_p' = 2ax + b \qquad \text{and} \qquad y_p'' = 2a.$$

Then

$$y_p'' + y_p' = 2a + 2ax + b = 2ax + (2a + b).$$

Hence, $y_p = ax^2 + bx$ is a solution given that

$$2ax + (2a + b) = 2x.$$

So a and b satisfy

$$\begin{cases} 2a = 1 \\ 2a + b = 0 \end{cases}$$

Solving the system yields $a = 1$ and $b = -2$ and

$$y_p = x^2 - 2x$$

is a particular solution. Therefore, the general solution is

$$y = c_1 + c_2 e^x + x^2 - 2x.$$

■

✍ **Exercise 6.1** Find the general solution of

$$y'' - 4y = 10e^{3x}.$$

Solution *Since the equation $r^2 - 4 = 0$ has two distinct roots $r_1 = -2$ and $r_2 = 2$, the general solution of the complementary equation $y'' - 4y = 0$ is*

$$y_h = c_1 e^{-2x} + c_2 e^{2x}.$$

We expect a particular solution of the form

$$y_p = ce^{3x}.$$

Differentiating the function implies $y_p' = 3ce^{3x}$, $y_p'' = 9ce^{3x}$, and

$$y'' - 4y = 5ce^{3x}.$$

Therefore,

$$5ce^{3x} = 10e^{3x}$$

which implies $c = 2$. A particular solution is $y_p = 2e^{3x}$ and the general solution is

$$y = 2e^{3x} + c_1 e^{2x} + c_2 e^{-2x}.$$

■

✍ **Exercise 6.2** Find a general solution of the equation

$$y'' - 7y' + 12y = 2e^{4x}.$$

Solution *Since the characteristic polynomial $r^2 - 7r + 12 = 0$ has two distinct real solution $r_1 = 3$ and $r_2 = 4$, the complementary equation has the general solution*

$$y_h = c_1 e^{3x} + c_2 e^{4x}.$$

Because e^{4x} is a solution of the complementary equation and right-hand side is $2e^{4x}$. We may take a particular solution $y_p = Axe^{4x}$.

Plugging y_p into the equation yields

$$\begin{aligned}
(Axe^{4x})'' - 7(Axe^{4x})' + 12Axe^{4x} &= 2e^{4x}\\
A(e^{4x} + 4xe^{4x})' - 7A(e^{4x} + 4xe^{4x}) + 12Axe^{4x} &= 2e^{4x}\\
A(e^{4x} + 4e^{4x} + 16xe^{4x}) - 7A(e^{4x} + 4xe^{4x}) + 12Axe^{4x} &= 2e^{4x}\\
-2Ae^{4x} &= 2e^{4x}\\
A &= -1
\end{aligned}$$

Therefore, a particular solution is

$$y_p = -xe^{4x}$$

and the general solution is

$$y = c_1e^{3x} + c_2e^{4x} - xe^{4x}.$$

■

6.1.2 The Principle of Superposition

When the function $f(x)$ in the equation $y'' + p(x)y' + q(x)y = f(x)$ is a sum of several functions, we can solve the equation by break it into several equation with few terms on the right-hand side. Indeed, Theorem 5.5 is such an application: if y_h is a solution of $y'' + p(x)y' + q(x)y = 0$ and y_p is a solution of the equation $y'' + p(x)y' + q(x)y = f(x)$, then $y_p + y_h$ is a solution of the equation $y'' + p(x)y' + q(x)y = 0 + f(x)$. In general, we have the principle of superposition which has analogous in linear algebra.

Theorem 6.1 (Principle of Superposition)

Suppose y_{p_1} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x)$$

and y_{p_2} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x).$$

Then

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x).$$

♥

Proof Since

$$\begin{aligned} y_{p_1}'' + p(x)y_{p_1}' + q(x)y_{p_1} &= f_1(x) \\ y_{p_2}'' + p(x)y_{p_2}' + q(x)y_{p_2} &= f_2(x), \end{aligned}$$

taking the sum of those two equations yields

$$(y_{p_1} + y_{p_2})'' + p(x)(y_{p_1} + y_{p_2})' + q(x)(y_{p_1} + y_{p_2}) = f_1(x) + f_2(x).$$

Therefore, $y_p = y_{p_1} + y_{p_2}$ is a solution of the equation $y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$. ■

Example 6.8 Find a particular solution y_p of

$$y'' + y' + y = \cos x + x + 1.$$

Solution *To find a particular solution, we may first find particular solutions for*

$$y'' + y' + y = \cos x$$

and

$$y'' + y' + y = x + 1.$$

For the equation $y'' + y' + y = \cos(x)$, since the derivation of $\cos x$ is $-\sin x$, we may assume a solution is $y_{p_1} = a\sin x + b\cos x$. Then a and b satisfy the equation

$$(-a\sin x - b\cos x) + (a\cos x - b\sin x) + (a\sin x + b\cos x) = \cos x$$

for all x. Hence, $a = 1$, $b = 0$ and $y_{p_1} = \sin x$ is a particular solution.

For the equation $y'' + y' + y = x$, we may assume a solution is $y_{p_2} = cx + d$. Then c and d satisfy

$$c + cx + d = x + 1$$

for all x. Hence, $c = 1$, $d = 0$, and $y_{p_2} = x$ is a solution.

Therefore, by Theorem 6.1, a particular solution of the original equation is

$$y = y_{p_1} + y_{p_2} = \sin x + x.$$

■

✍ **Exercise 6.3** Find a particular solution y_p of

$$y'' - y' + y = e^x + x.$$

Solution *To find a particular solution, we may first find particular solutions for*

$$y'' - y' + y = e^x$$

and

$$y'' - y' + y = x.$$

For the equation $y'' - y' + y = e^x$, we may assume a solution is $y_{p_1} = ce^x$. Then c satisfies the equation

$$ce^x - ce^x + ce^x = e^x$$

for all x. Hence, $c = 1$ and $y_{p_1} = e^x$ is a particular solution.

For the equation $y'' - y' + y = x$, we may assume a solution is $y_{p_2} = ax + b$. Then a and b satisfy

$$-a + ax + b = x$$

for all x. Hence, $a = 1$, $b = 1$, and $y_{p_2} = x + 1$ is a solution.

Therefore, by Theorem 6.1, a particular solution of the original equation is

$$y = y_{p_1} + y_{p_2} = e^x + x + 1.$$

■

6.1.3 Nonlinear with Product Functions

When the function $f(x)$ is a product of those basic functions, we can expect a same type of function as a particular solution.

Example 6.9 Find a general solution of the equation

$$y'' + 2y = e^x \sin x.$$

Solution *The general solution of the complementary equation is $y_h = c_1 + c2e^{-2x}$*

We try $y_p = Ae^x \cos x + Be^x \sin x$. The derivatives of y_p are

$$\begin{aligned} y_p' =& Ae^x \cos x - Ae^x \sin x + Be^x \sin x + Be^x \cos x \\ =& (A + B)e^x \cos x + (B - A)e^x \sin x \end{aligned}$$

and

$$y_p'' = 2Be^x \cos x - 2Ae^x \sin x.$$

Plugging y_p into the left-hand side of the equation implies

$$\begin{aligned}
&y_p'' + 2y_p \\
=&2Be^x \cos x - 2Ae^x \sin x + 2Ae^x \cos x + 2Be^x \sin x \\
=&(2B + 2A)e^x \cos x + (2B - 2A)e^x \sin x
\end{aligned}$$

Hence, y_p is a solution if

$$(2B + 2A)e^x \cos x + (2B - 2A)e^x \sin x = e^x \sin x$$

or if

$$\begin{cases} 2B + 2A = 0 \\ 2B - 2A = 1. \end{cases}$$

Solving the system yields $A = -\frac{1}{4}$ and $B = \frac{1}{4}$. Hence, a particular solution is

$$y_p = -\frac{1}{4}e^x \cos x + \frac{1}{4}e^x \sin x.$$

The general solution is

$$y = c_1 + c_2 e^{-2x} - \frac{1}{4}e^x \cos x + \frac{1}{4}e^x \sin x.$$

■

Example 6.10 Find the general solution of the equation

$$y'' - y = x \sin x$$

Solution *The general solution of the complementary equation is*

$$y_h = c_1 + c_2 e^{-x}.$$

For a particular solution, we try

$$y_p = Ax \sin x + Bx \cos x + C \sin x + D \cos x.$$

The derivatives of y_p are

$$\begin{aligned}
y_p' =&Ax \cos x + A \sin x - Bx \sin x + B \cos x + C \cos x - D \sin x \\
=&Ax \cos x + (A - D) \sin x - Bx \sin x + (B + C) \cos x,
\end{aligned}$$

and

$$\begin{aligned} y_p' =& A\cos x - Ax\sin x + (A-D)\cos x - B\sin x - Bx\cos x - (B+C)\sin x \\ =& -Ax\sin x + (2A-D)\cos x - Bx\cos x - (2B+C)\sin x, \end{aligned}$$

Therefore,

$$\begin{aligned} & y_p'' - y_p \\ =& -Ax\sin x + (2A-D)\cos x - Bx\cos x - (2B+C)\sin x \\ & -(Ax\cos x + (A-D)\sin x - Bx\sin x + (B+C)\cos x) \\ =& -2Ax\sin x + (-A-B-2C+D)\sin x + (2A-B-C-D)\cos x \end{aligned}$$

Hence, y_p is a solution if

$$-2Ax\sin x + (-A-B-2C+D)\sin x + (2A-B-C-D)\cos x = x\sin x$$

or if

$$\begin{cases} A = 1 \\ B = 0 \\ 2A + D = 0 \\ -2B + C = 0 \end{cases}$$

Consequently, $A = 1, B = 0, C = 0, D = -2$ *and*

$$y_p = x\sin x - 2\cos x$$

is a particular solution.

The general solution is

$$y = c_1 + c_2 e^x$$

■

Exercise 6.4 Find a particular solution of

$$y'' - 3y' + 2y = e^{3x}(2x+1).$$

Solution *We can try a particular solution* $y_p = e^{3x}(ax+b)$. *Differentiating* y_p *yields*

$$\begin{aligned} y_p' =& 3e^{3x}(ax+b) + ae^{3x} \\ =& 3axe^{3x} + (a+3b)e^{3x} \end{aligned}$$

$$\begin{aligned} y_p'' =& 3ae^{3x} + 9axe^{3x} + 3(a+3b)e^{3x} \\ =& 9axe^{3x} + (6a+9b)e^{3x}. \end{aligned}$$

Plugging y_p into the equation implies

$$\begin{aligned}
9axe^{3x} + (6a+9b)e^{3x} - 3(3axe^{3x} + (a+3b)e^{3x}) + 2(axe^{3x} + be^{3x}) &= e^{3x}(2x+1) \\
9axe^{3x} + (6a+9b)e^{3x} - 9axe^{3x} - (3a+9b)e^{3x}) + 2axe^{3x} + 2be^{3x} &= e^{3x}(2x+1) \\
(2ax + (3a+2b))e^{3x} &= e^{3x}(2x+1) \\
2ax + (3a+2b) &= (2x+1)
\end{aligned}$$

So y_p is a solution if

$$\begin{cases} 2a = & 2 \\ 3a + 2b = & 1 \end{cases}$$

Solving the system yields $a = 1$ and $b = -1$.

Therefore, a particular solution is

$$y_p = e^{3x}(x-1).$$

■

More generally, the method of undetermined coefficients can be applied to

$$y'' + ay' + by = P(x)e^{\alpha x}$$

and

$$y'' + ay' + by = e^{\alpha x}(P(x)\cos(\beta x) + Q(x)\sin(\beta x)).$$

Theorem 6.2

Consider the equation

$$y'' + ay' + by = P(x)e^{\alpha x}$$

where G a polynomial of degree k. Let $A(x) = a_k x^k + \cdots + a_1 x + a_0$ be a polynomial of degree k.

- *If $e^{\alpha x}$ is not a solution of the complementary equation, then a particular root is*

$$y_p = A(x)e^{\alpha x}.$$

- *If $e^{\alpha x}$ is a solution but $xe^{\alpha x}$ is not a solution of the complementary equation, then a particular solution is*

$$y_p = xA(x)e^{\alpha x}.$$

- *If $e^{\alpha x}$ and $xe^{\alpha x}$ are both solutions of the complementary equation, then a particular solution is*

$$y_p = x^2 A(x)e^{\alpha x}.$$

♥

Theorem 6.3

Consider the equation

$$y'' + ay' + by = e^{\alpha x}\left(P(x)\cos\beta x + Q(x)\sin\beta x\right)$$

with $P(x)$ and $Q(x)$ polynomials such that the larger degree is k.

- *If $\alpha + \mathrm{i}\beta$ is not a root of the characteristic polynomial $p(r) = r^2 + ar + b$, then a particular solution is*
$$y_p = e^{\alpha x}\left(A(x)\cos\beta x + B(x)\sin\beta x\right),$$
where $A(x) = a_k x^k + \cdots + a_1 x + a_0$ and $B(x) = b_k x^k + \cdots + b_1 x + b_0$.
- *If $\alpha + \mathrm{i}\beta$ is not a root of the characteristic polynomial, then a particular solution is*
$$y_p = e^{\alpha x}\left(A(x)\cos\beta x + B(x)\sin\beta x\right),$$
where $A(x) = a_{k+1} x^{k+1} + \cdots + a_1 x + a_0$ and $B(x) = b_{k+1} x^{k+1} + \cdots + b_1 x + b_0$.

Using substitution $y_p = u(x)e^{\alpha x}$, where α is allowed to be a complex number, the proof of the theorem can be reduced to the case that $f(x)$ is a polynomial. For example, the equation $y'' + ay + by = P(x)e^{\alpha x}$ can be reduced to $u'' + (a + 2r)u' + (r^2 + ar + b)u = P(x)$. Further more, by the principle of superposition, we may assume that $f(x) = a_n x^n$. We can even take $a_n = \frac{1}{n!}$. Taking derivatives of both sides n times, we get an equations.

$$y^{(n+2)} + ay^{(n+1)} + by^{(n)} = 1.$$

Let $u = y^n$, then

$$u'' + au' + bu = 1.$$

If $b \neq 0$, then a particular solution is $u_p = 1$. If $b = 0$ but $a \neq 0$, then a particular solution is $u_p = \frac{x}{a}$. If $a = 0$ and $b = 0$, then a particular solution is $u_p = \frac{x^2}{2}$. Then a particular solution y_p such that $y_p^n = u_p$ is a polynomial of n terms with degree n, $n + 1$ or $n + 2$.

Two useful references can be found at the following webpages: https://mathoverflow.net/questions/124694/reference-for-a-nice-proof-of-undetermined-coefficients and http://www.math.utah.edu/~gustafso/undetermined-coeff.pdf.

6.2 Variation of Parameters

The method of variation of parameters for first order differential equations can also be applied to second order equations.

Consider the equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Assume that $y_h = c_1y_1 + c_2y_2$ is a general solutions of

$$y'' + p(x)y' + q(x)y = 0.$$

We assume that the equation

$$y'' + p(x)y' + q(x)y = f(x)$$

has a solution y in the form

$$y = v_1y_1 + v_2y_2,$$

where v_1 and v_2 are undetermined functions of x. Computing the derivative of y yields

$$y' = (v_1'y_1 + v_1y_1') + (v_2'y_2 + v_2y_2').$$

The second derivative is

$$y'' = (v_1'y_1)' + v_1'y_1' + v_1y_1'') + ((v_2'y_2)' + v_2'y_2 + v_2y_2'').$$

Plugging $y_1v_1 + y_2v_2$ into the left hand side of the nonhomogeneous equation yields

$$\begin{aligned}
&y'' + p(x)y' + q(x)y \\
=&((v_1'y_1)' + v_1'y_1' + v_1y_1'')) + ((v_2'y_2)' + v_2'y_2' + v_2y_2'') \\
&+ p(x)(v_1'y_1 + v_1y_1') + p(x)(v_2'y_2 + v_2y_2') \\
&+ q(x)(v_1y_1 + v_2y_2) \\
=&((v_1'y_1)' + p(x)v_1'y_1 + v_1'y_1') \\
&+ (v_2'y_2)' + p(x)v_2'y_2 + v_2'y_2' \\
=&(v_1'y_1 + v_2'y_2)' + p(x)(v_1'y_1 + v_2'y_2) + (v_1'y_1' + v_2'y_2').
\end{aligned}$$

6.2.1 Variation of Parameter

Compare with the right hand side of the nonhomogeneous equation, we may assume that v_1 and v_2 satisfy the following system of equations.

$$\begin{cases} v_1'y_1 + v_2'y_2 &= 0 \quad (1) \\ v_1'y_1' + v_2'y_2' &= f(x) \quad (2) \end{cases}$$

Solving for v_1' and v_2' yields

$$\begin{cases} v_1' = \dfrac{-y_2f(x)}{W(y_1, y_2)} \\ v_2' = \dfrac{y_1f(x)}{W(y_1, y_2)}, \end{cases} \tag{6.1}$$

where $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$ is the Wronskian of y_1 and y_2.

Therefore,

$$\begin{cases} v_1' = \int \left(\dfrac{-y_2 f(x)}{W(y_1, y_2)} \right) \mathrm{d}\,x \\ v_2' = \int \left(\dfrac{y_1 f(x)}{W(y_2, y_1)} \right) \mathrm{d}\,x, \end{cases}$$

and a particular solution of $y'' + p(x)y' + q(x) = f(x)$ is

$$y = y_1 \int \left(\frac{-y_2 f(x)}{W(y_1, y_2)} \right) \mathrm{d}\,x + y_2 \int \left(\frac{y_1 f(x)}{W(y_2, y_1)} \right) \mathrm{d}\,x.$$

The method of variation of parameters can be generalized to higher order equations.

Example 6.11 Find a particular solution of the equation

$$y'' - 3y' + 2y = \frac{e^{3x}}{1 + e^x}.$$

Solution *The characteristic polynomial of the complementary equation is*

$$r^2 - 3r + 2 = (r - 1)(r - 2) = 0$$

which has two distinct real roots, $r_1 = 1$ *and* $r_2 = 2$*. So the complementary equation has two linearly independent solutions* $y_1 = e^x$ *and* $y_2 = e^{2x}$*.*

A particular solution of non-homogeneous equations can be taken in the form

$$y_p = v_1 e^x + v_2 e^{2x},$$

where v_1 *and* v_2 *are functions of* x *that satisfy the following system of equations*

$$\begin{aligned} v_1' e^x + v_2' e^{2x} &= 0 \\ v_1' e^x + 2v_2' e^{2x} &= \frac{e^{3x}}{1 + e^x}. \end{aligned}$$

Subtracting the first equation from the second one implies

$$v_2' e^{2x} = \frac{e^{3x}}{1 + e^x}$$

So

$$\begin{aligned} v_2' &= \frac{e^x}{1+e^x} \\ v_2 &= \int \left(\frac{e^x}{1+e^x}\right) \mathrm{d}\,x \\ v_2 &= \ln(1+e^x) \end{aligned}$$

The first equation together with v_2 yields

$$\begin{aligned} v_1' &= -\,v_2' e^x \\ v_1' &= -\,\frac{e^{2x}}{1+e^x} \\ v_1 &= -\int \left(\frac{e^{2x}}{1+e^x}\right) \mathrm{d}\,x \\ v_1 &= -\int \left(e^x - \frac{e^x}{1+e^x}\right) \mathrm{d}\,x \\ v_1 &= -\,e^x + \ln(1+e^x). \end{aligned}$$

Therefore

$$\begin{aligned} y_p &= v_1 e^x + v_2 e^{2x} \\ &= (e^x + \ln(1+e^x))e^x + \ln(1+e^x)e^{2x}. \end{aligned}$$

■

Example 6.12 Find a particular solution of the equation

$$x(y'' + y') - 2(y' + y) = x^2$$

Solution *The complementary equation $x(y'' + y') - 2(y' + y) = 0$ can be solved by substitution. Plugging $u = y' + y$ into the complementary equation and solving for u yields*

$$\begin{aligned} x(y'' + y') - 2(y' + y) &= 0 \\ xu' - 2u &= 0 \\ \frac{u'}{u} &= \frac{2}{x} \\ \int \frac{u'}{u}\,\mathrm{d}\,x &= \int \frac{2}{x}\,\mathrm{d}\,x \\ \ln(u) &= \ln(x^2) + c \\ u &= c_1 x^2 \end{aligned}$$

Then

$$y' + y = c_1 x^2.$$

The integrating factor is $r(x) = e^x$. Therefore, applying the method of integration by parts implies

$$\begin{aligned} y_h =& \frac{1}{e^x} c_1 \int \left(x^2 e^x)\right) \mathrm{d}\, x \\ =& \frac{1}{e^x}(c_1 e^x(x^2 - 2x + 2) + c_2) \\ =& c_1(x^2 - 2x + 2) + c_2 e^{-x} \end{aligned}$$

Let $y_1 = x^2 - 2x + 2$ and $y_2 = e^{-x}$. Both of them are solutions of the complementary equation. Since $y_1' = 2x - 2$, $y_2' = -e^{-x}$ and the Wronskian is

$$\begin{aligned} W(y_1, y_2) =& y_1 y_2' - y_1' y_2 \\ =& -(x^2 - 2x + 2)e^{-x} - (2x - 2)e^{-x} \\ =& (-x^2 - 4x + 2)e^{-x} \neq 0, \end{aligned}$$

they are linearly independent.

A particular solution of non-homogeneous equations can be taken in the form

$$y_p = v_1(x^2 - 2x + 2) + v_2 e^{-x},$$

where v_1 and v_2 are functions of x that satisfy the following system of equations

$$\begin{aligned} v_1'(x^2 - 2x + 2) + v_2' e^{-x} =& 0 \\ v_1'(2x - 2) - v_2' e^{-x} =& x^2. \end{aligned}$$

The sum of the two equations yields

$$\begin{aligned} x^2 v_1' =& x^2 \\ v_1' =& 1 \\ v_1 =& x \end{aligned}$$

Then v_2 satisfies

$$\begin{aligned} v_2' e^{-x} =& - v_1'(x^2 - 2x + 2) \\ v_2' =& -(x^2 - 2x + 2)e^x \\ v_2 =& - \int \left((x^2 - 2x + 2)e^x\right) \mathrm{d}\, x \\ v_2 =& - \int \left((x^2 - 2x + 2)e^x\right) \mathrm{d}\, x \\ v_2 =& - e^{-x}(x^2 - 4x + 6). \end{aligned}$$

Therefore,

Therefore,

$$y_p = x(x^2 - 2x + 2) - e^{-x} e^x (x^2 - 4x + 6) = x^3 - 3x^2 + 6x - 6$$

is a particular solution. ■

✍ **Exercise 6.5** Find a particular solution of

$$x^2y'' - xy' + y = x^3,$$

using variation of parameter. It is known that $y_1 = x$ and $y_2 = x\ln x$ are linearly independent solutions of the complementary equation.

Solution *We first write the equation in standard form:*

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = x$$

Since $y_1 = x$ and $y_2 = x\ln x$ are linearly independent solutions of the complementary equation, a particular solution of the equation can be written as

$$y_p = v_1x + v_2x\ln x,$$

where v_1 and v_2 are functions of x.

We will find v_1 and v_2 under the following conditions:

$$\begin{cases} v_1'x + v_2'x\ln x & = 0 \quad (1) \\ v_1' + v_2'(1+\ln x) & = x \quad (2) \end{cases}$$

The first derivatives v_1' and v_2' can be solved by the elimination method. Subtracting the product of Equation (1) with $(1+\ln x)$ from the product of Equation (2) with $x\ln x$ implies

$$v_1' = -x\ln x$$

Applying integration by parts yields

$$v_1 = -\frac{1}{2}x^2\ln x + \frac{x^2}{4}.$$

Plugging v_1' into Equation (1) implies

$$v_2' = x.$$

Then

$$v_2 = \frac{1}{2}x^2.$$

Therefore, a particular solution is

$$y_p = (-\frac{1}{2}x^2 \ln x + \frac{x^2}{4})x + \frac{1}{2}x^3 \ln x = \frac{x^3}{4}.$$

■

✍ **Exercise 6.6** Find the general solution to

$$y'' - y = e^x$$

using variation of parameter.

Solution *The complementary homogenous equation*

$$y'' - y = 0,$$

has the general solution is

$$y = c_1 e^x + c_2 e^{-x}.$$

We look for a particular solution of the form

$$y_p = v_1(x)e^x + v_2(x)e^{-x}.$$

Consider the following system of equations of v_1 and v_2:

$$\begin{cases} v_1' e^x + v_2' e^{-x} & = 0 \\ v_1' e^x + (-1) v_2' e^{-x} & = e^x \end{cases}$$

Solving for v_1' and v_2' yields

$$v_1' = \frac{1}{2} \qquad \text{and} \qquad v_2' = -\frac{1}{2}e^{2x}.$$

Direct integrations implies that

$$v_1 = \frac{1}{2}x \qquad \text{and} \qquad v_2 = -\frac{1}{4}e^{2x}.$$

Therefore, a particular solution is

$$y_p = \frac{1}{2}xe^x + (-\frac{1}{4}e^{2x})e^{-x} = \frac{1}{2}xe^x - \frac{1}{4}e^x.$$

By Theorem 5.5, the general solution of the original equation is

$$y = \frac{1}{2}xe^x - \frac{1}{4}e^x + c_1 e^x + c_2 e^{-x}.$$

■

6.2.2 Reduction of Order

You may wonder if there is another way to find v_1 and v_2 other than using the system of equation (6.1). The answer is yes. We may take $v_2 = 0$, then v_1 will be solution of the following equation

$$(v_1' y_1)' + p(x) v_1' y_1 + v_1' y_1' = f(x).$$

Setting $z = v_1'$. The equation can be reduce to a linear first order equation of z:

$$z' y_1 + z(p(x) y_1 + 2y_1') = f(x),$$

or equivalently (assuming $y_1 \not\equiv 0$)

$$z' + z(p(x) + 2\ln(y_1)) = \frac{f(x)}{y_1}.$$

By the method of integrating factor,

$$z = e^{-\int (p(x)+2\ln(y_1))\,\mathrm{d}\,x} \int \left(\left(\frac{f(x)}{y_1} \right) e^{\int (p(x)+2\ln(y_1))\,\mathrm{d}\,x} \right) \mathrm{d}\,x$$

Hence,

$$v_1 = \int \left(e^{-\int (p(x)+2\ln(y_1))\,\mathrm{d}\,x} \int \left(\left(\frac{f(x)}{y_1} \right) e^{\int (p(x)+2\ln(y_1))\,\mathrm{d}\,x} \right) \mathrm{d}\,x \right) \mathrm{d}\,x.$$

To summarize, we may use the variation of parameter with a particular solution of the complementary equation of a liner second order equation to find a particular solution.

However, this method only works for second order equations.

Example 6.13 Find a particular solution of the equation

$$y'' + y = \tan x.$$

Solution *The complementary equation* $y'' + y = 0$ *has a solution* $y = \cos x$. *Assume* $y = u \cos x$ *is a solution of* $y'' + y = tanx$. *Then, the function* u *of* x *satisfies the following equation*

$$\begin{aligned}
(u \cos x)'' + u \cos x &= \tan x \\
(u' \cos x - u \sin x)' + u \cos x &= \tan x \\
(u'' \cos x - u' \sin x) - (u' \sin x + u \cos x) + u \cos x &= \tan x \\
u'' \cos x - 2u' \sin x &= \tan x
\end{aligned}$$

Let $z = u'$, then z satisfies

$$z' \cos x - 2z \sin x = \tan x,$$

or

$$z' - 2z\frac{\sin x}{\cos x} = \frac{\sin x}{\cos^2 x}.$$

As a linear first order equation, it can be solved by the method of integrating factor. Since the coefficient of z is $-\frac{2\sin x}{\cos x}$, an integrating factor is

$$r(x) = e^{\int \frac{-2\sin x}{\cos x}\,\mathrm{d}\,x} = \cos^2 x.$$

Multiplying both sides of the equation of z yields

$$\begin{aligned}
(z\cos^2 x)' &= \cos^2 x \cdot \frac{\sin x}{\cos^2 x} \\
z\cos^2 x &= \int \sin x\,\mathrm{d}\,x \\
z\cos^2 x &= -\cos x \\
z &= -\frac{1}{\cos x}
\end{aligned}$$

Therefore,

$$\begin{aligned}
u' &= z \\
u &= \int \left(-\frac{1}{\cos x}\right)\mathrm{d}\,x \\
u &= -\int \frac{1}{\cos^2 x}\,\mathrm{d}(\sin x) \\
u &= -\int \frac{1}{1-\sin^2 x}\,\mathrm{d}(\sin x) \\
u &= -\frac{1}{2}\int \left(\frac{1}{1-\sin x} + \frac{1}{1+\sin x}\right)\mathrm{d}(\sin x) \\
u &= -\frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right)
\end{aligned}$$

Thus, the original equation has a solution

$$y = -\frac{1}{2}\cos x \ln\left(\frac{1+\sin x}{1-\sin x}\right).$$

■

Exercise 6.7 Find a particular solution of the equation

$$x^2 y'' - xy' + y = x^3.$$

Solution *The complementary equation*

$$x^2y'' - xy' + y = 0$$

has a solution $y = x$. *This can be found by guessing, or by a decomposition similar to the proof of Theorem 5.4.*

Suppose a particular solution is of the form

$$y = xu.$$

Then

$$y' = xu' + u$$

and

$$y'' = xu'' + 2u'.$$

Hence,

$$x^2y'' - xy' + y = x^2(xu'' + 2u') - x(xu' + u) + xu = x^3u'' + x^2u'$$

Therefore, $y = xu$ *is a solution if*

$$x^3u'' + x^2u' = x^3,$$

or

$$u'' + \frac{1}{x}u' = 1.$$

To solve this second order equation for u, *set* $v = u'$. *The equation become a first order linear equation of* v

$$v' + \frac{1}{x}v = 1.$$

The integrating factor is

$$r(x) = e^{\int \frac{1}{x}\,\mathrm{d}x} = e^{\ln x} = x.$$

Then

$$v = \frac{1}{x}\int x\,\mathrm{d}x = \frac{x}{2} + c_1\frac{1}{x}$$

Therefore,

$$\begin{aligned} u' &= \frac{x}{2} + c_1\frac{1}{x} \\ u &= \frac{x^2}{4} + c_1\ln x + c_2 \end{aligned}$$

and

$$y = xu = \frac{x^3}{4} + c_1x\ln x + c_2x.$$

■

Week 7: Applications of Linear Second Order Equations

10/25–10/28

7.1 Vibrating Springs

We consider the motion of an object with mass attached to an end of a spring with negligible mass that is vertically attached to a fixed object.

The spring–mass system is in ***equilibrium*** when the object is at rest and the forces acting on it sum to zero. The position of the object in this case is the ***equilibrium position***.

Denote by y the displacement of the object from its equilibrium position at time t, measured positive upward.

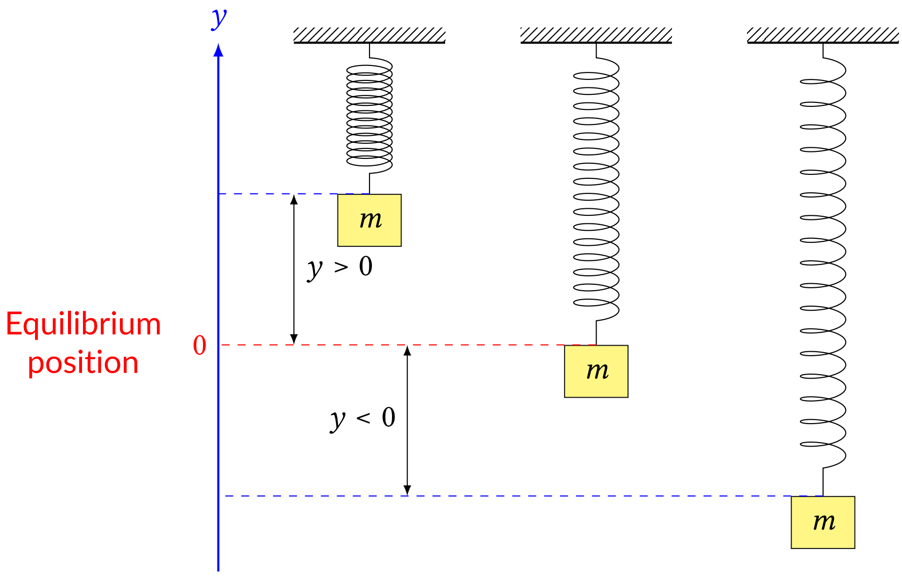

Figure 7.1: Spring-mass system

Hooke's Law - Wikipedia says that if the length of the spring is changed by an amount ΔL from its natural length, if it is then the spring exerts a force F_s whose magnitude that is proportional to ΔL, that is $|F_s| = k\Delta L$, where k is a positive number called the spring constant. Since we take upward as the positive direction, the force is $F_s = k\Delta L$ if the spring is stretched or $F_s = -k\Delta L$ if the spring is compressed.,

Besides Earth's gravitational force and the force of the spring, there can be other forces. The system may have a damping force $F_d = -cy'$ that resists the motion with a force proportional to the velocity of the object. It may be due to resistance or friction. We say that the

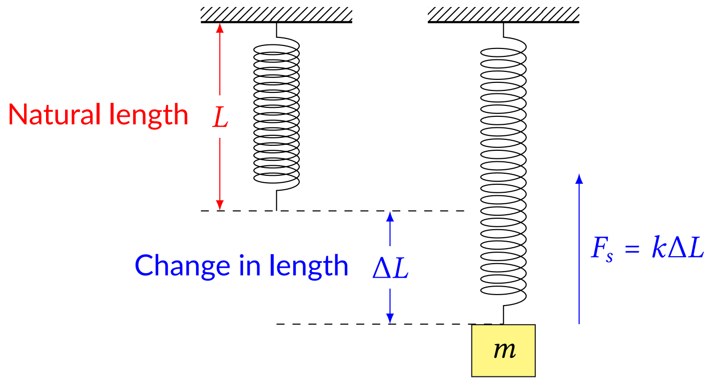

Figure 7.2: Hooke's Law of Spring

motion is ***undamped*** if $c = 0$, or ***damped*** if $c > 0$.

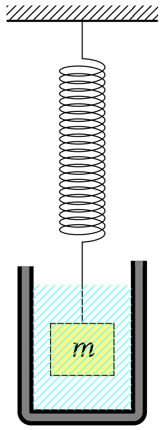

Figure 7.3: A spring system with damping

It may have an external force F, other than the force due to gravity, that may vary with t, but is independent of displacement and velocity. We say that the motion is ***free*** if $F \equiv 0$, or ***forced*** if $F \not\equiv 0$.

By Newton's second law of motion, we have

$$ma = -mg + F_d + F_s + F = -mg - cy' + F_s + F,$$

where $a = y''$ is the acceleration.

Let Δl be the change of length when the system reaches equilibrium. Then $mg = k\Delta l$. When the displacement of the object is y, the change in length of the spring is $|y - \Delta l|$ and $F_s = k(\Delta l - y)$.

Therefore,

$$-mg + F_s = -ky.$$

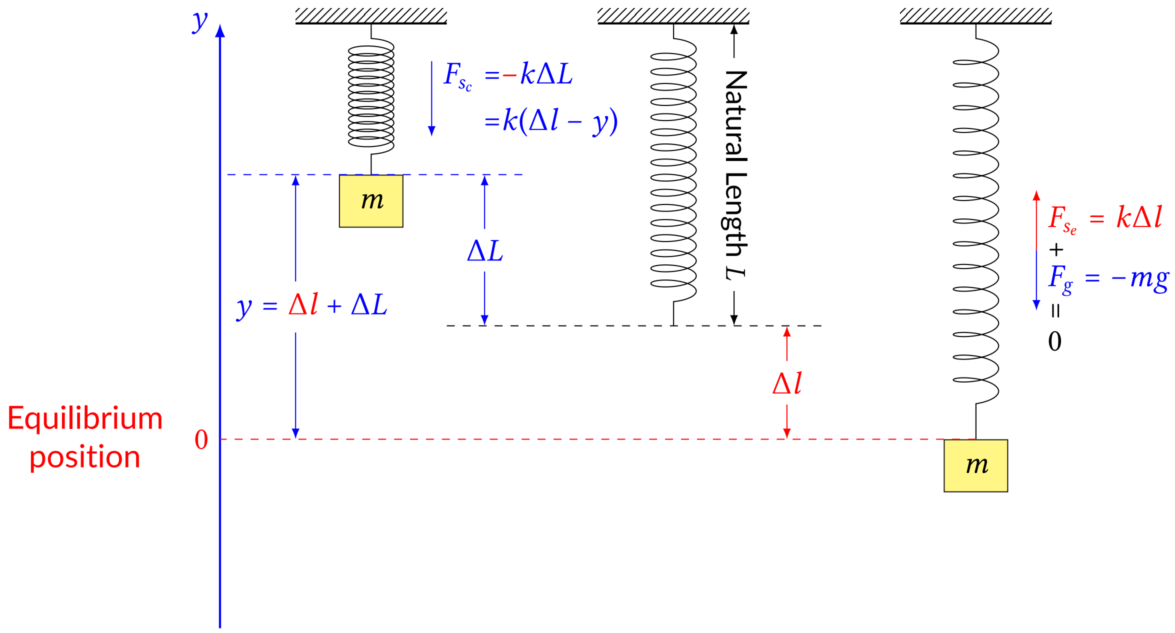

Figure 7.4: Relation between displacement and change in length

So the displacement y of a object attached to a spring satisfies the following equation

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{F(t)}{m}. \tag{7.1}$$

Note that for a horizontal spring motion, Equation (7.1) is still valid.

7.1.1 Simple harmonic motion

Assume there is no damping force or other external force, that is, $F_d \equiv 0$ and $F \equiv 0$. Then, by Newton's second law of motion, the displacement y satisfies the following equation

$$y'' + \frac{k}{m}y = 0.$$

This motion is known as the ***simple harmonic motion***.

Let $\omega = \sqrt{\frac{k}{m}}$. Then the general solution of the equation $my' + ky = 0$ is

$$y = e^{-\frac{c}{2m}}C_1\cos(\omega t) + e^{-\frac{c}{2m}}C_2\sin(\omega t).$$

If further, we let $R = \sqrt{\left(e^{-\frac{c}{2m}}C_1\right)^2 + \left(e^{-\frac{c}{2m}}C_2\right)^2}$ and ϕ the angle in $(-\pi, \pi)$ such that $\cos\phi = \frac{e^{-\frac{c}{2m}}C_1}{R}$ and $\sin\phi = -\frac{e^{-\frac{c}{2m}}C_2}{R}$, then the general solution can be written as

$$y = R\cos(\omega t + \phi).$$

Here, the angle ϕ is known as the ***phase angle***, R is the amplitude of the oscillation and the solution $y = R\cos(\omega t + \phi)$ is called the ***amplitude-phase form*** of the displacement. The constant ω is the ***frequency*** of the motion. If the time t is measured in seconds, then, the frequency is measured in cycle per second or, in international unit, Hertz (Hz for short). Because the ***period*** T of the amplitude-phase form displacement is

$$T = \frac{2\pi}{\omega}.$$

Example 7.1 An object stretches a spring $5\,\mathrm{m}$ in equilibrium.

1. Find the displacement of the object after t seconds if it is initially displaced $18\,\mathrm{m}$ above equilibrium and given a downward velocity of $7\,\mathrm{m/s}$.
2. Find the frequency, amplitude, the phase angle of this motion, and the amplitude-phase form of the displacement.

Solution

1. *The motion of the object is undamped and free, that is $F_d \equiv 0$ and $F \equiv 0$. So the displacement y away from the equilibrium position satisfies the differential equation*

$$y'' + \frac{k}{m}y = 0,$$

where m is the mass and k is the spring constant.
Since the spring stretches $5\,\mathrm{m}$ to reach the equilibrium position, the spring constant k is determined by the equation

$$mg = k \cdot 6.$$

Since the units for the displacement is in meters, we take the gravitational acceleration to be $g = 9.8\,\mathrm{m/s^2}$ and then

$$\frac{k}{m} = \frac{9.8}{5} = \frac{49m}{25}.$$

Therefore, the displacement y satisfies

$$\begin{aligned} my'' + \frac{49m}{25}y &= 0 \\ y'' + \frac{49}{25}y &= 0. \end{aligned}$$

The associated characteristic equation $r^2 + \frac{49}{25} = 0$ has two complex solutions $r = \pm\frac{7}{5}$. Therefore, the general solution is

$$y = e^{-\frac{c}{2m}}C_1\frac{7}{5}\cos\left(\frac{7}{5}t\right) + e^{-\frac{c}{2m}}C_2\frac{7}{5}\sin\left(\frac{7}{5}t\right).$$

Because the object is initially displaced $18\,\mathrm{m}$ above equilibrium and given a downward velocity *of* $1\ \mathrm{m/s}$. *In terms of mathematical equalities, we have*

$$y(0) = 18 \qquad \textit{and} \qquad y'(0) = -7.$$

Note that

$$y' = -e^{-\frac{c}{2m}} C_1 \sin\left(\frac{7}{5}t\right) + e^{-\frac{c}{2m}} C_2 \cos\left(\frac{7}{5}t\right)$$

Then $e^{-\frac{c}{2m}} C_1 = 18$, $e^{-\frac{c}{2m}} C_2 = -5$, *and the displacement* y *at time* t *is*

$$y = 18\cos\left(\frac{7}{5}t\right) - 5\sin\left(\frac{7}{5}t\right) \text{ m.}$$

2. *The frequency is* $\omega = \frac{7}{5}\mathrm{rad/s}$.
   The amplitude is

$$R = \sqrt{18^2 + (-5)^2} = \sqrt{349}\mathrm{rad/s}.$$

Since

$$\cos\phi = \frac{e^{-\frac{c}{2m}} C_1}{R} = \frac{18}{\sqrt{18^2 + (-5)^2}} = \frac{18}{\sqrt{349}}$$

$$\sin\phi = \frac{-e^{-\frac{c}{2m}} C_2}{R} = -\frac{-5}{\sqrt{18^2 + (-5)^2}} = \frac{5}{\sqrt{349}}$$

$$\tan\phi = \frac{-e^{-\frac{c}{2m}} C_2}{e^{-\frac{c}{2m}} C_1} = \frac{5}{18},$$

the angle ϕ *is in the first quadrant, and*

$$\phi = \arctan\left(\frac{5}{18}\right) \text{ rad.}$$

The amplitude-phase form of the displacement is about

$$y = \sqrt{349}\cos\left(\frac{7}{5}t + \arctan\left(\frac{5}{18}\right)\right) \text{ m.}$$

■

✍ **Exercise 7.1** The natural length of a spring is $1\,\mathrm{m}$. An object is attached to it and the length of the spring increases to $1.2\,\mathrm{m}$ when the object is in equilibrium. Then the object is initially displaced downward $0.4\,\mathrm{m}$ and given an upward velocity of $2.1\,\mathrm{m/s}$. Find the amplitude-phase form of the displacement for $t > 0$.

Solution *Since the gravitational acceleration is* $g = 9.8\,\mathrm{m/s^2}$ *and the change in length at the equilibrium position is* $\Delta l = 1.2 - 1 = 0.2$ m, *the ratio of the spring constant and the mass is*

$$\frac{k}{m} = \frac{g}{\Delta l} = \frac{9.8}{0.2} = 49,$$

and the frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{49} = 7\,\mathrm{Hz}.$$

Therefore, the displacement y satisfies the initial value problem

$$y'' + 49y = 0, \quad y(0) = -0.4, \quad y'(0) = 1.4.$$

The general solution of the differential equation is

$$y = e^{-\frac{c}{2m}} C_1 \cos(\omega t) + e^{-\frac{c}{2m}} C_2 \sin(\omega t) = e^{-\frac{c}{2m}} C_1 \cos(7t) + e^{-\frac{c}{2m}} C_2 \sin(7t).$$

So

$$y' = -7e^{-\frac{c}{2m}} C_1 \sin(7t) + 7e^{-\frac{c}{2m}} C_2 \cos(7t).$$

Substituting the initial conditions into y and y' yields $e^{-\frac{c}{2m}} C_1 = -0.4$ and $e^{-\frac{c}{2m}} C_2 = 0.3$. Hence, the displacement is

$$y = -0.4\cos(7t) + 0.3\sin(7t)$$

The amplitude is

$$R = \sqrt{\left(e^{-\frac{c}{2m}} C_1\right)^2 + \left(e^{-\frac{c}{2m}} C_2\right)^2} = \sqrt{(-0.4)^2 + 0.3^2} = 0.5$$

The phase angle is determined by

$$\cos\phi = \frac{e^{-\frac{c}{2m}} C_1}{R} = \frac{-0.4}{0.5} = -\frac{4}{5} \quad \textit{and} \quad \sin\phi = -\frac{e^{-\frac{c}{2m}} C_2}{R} = -\frac{0.3}{0.5} = -\frac{3}{5}.$$

Therefore, ϕ is in the third quadrant and

$$\phi = -\arccos\left(-\frac{4}{5}\right) \textit{rad}.$$

So the amplitude-phase form of the displacement is

$$y = 0.5\cos(7t - 0.64).$$

■

7.2 Free Vibrations with Damping

Assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion, that is

$$F_d = -cy',$$

where c is a positive constant, called the damping constant. Then the displacement satisfies the equation

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0.$$

The solution of this equation depends on the value of c, or more precisely, the value of $\sqrt{c^2 - 4km}$. The reason is that the associated characteristic equation

$$r^2 + \frac{c}{m}r + \frac{k}{m} = 0,$$

whose solutions $r = \frac{-c\pm\sqrt{c^2-4km}}{2m}$ may be two complex roots, a double root, or two distinct real roots.

Underdamping

The motion is said to be ***underdamped***, if $c^2 < 4mk$. In this case, the solutions are complex roots

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = \frac{-c \pm \mathrm{i}\sqrt{4mk - c^2}}{2m}.$$

Let $\omega_1 = \frac{\sqrt{4mk-c^2}}{2m}$. Then the general solution is

$$y = e^{-\frac{ct}{2m}}(C_1 \cos(\omega_1 t_+ C_2 \sin(\omega_1 t)).$$

Again, we may derive an amplitude-phase form of the displacement

$$y = Re^{-\frac{ct}{2m}} \cos(\omega_1 t + \phi),$$

where

$$R = \sqrt{C_1^2 + C_2^2}, \qquad \cos\phi = \frac{C_1}{R}, \qquad \sin\phi = -\frac{C_2}{R}, \qquad -\phi < \phi < \pi.$$

In the amplitude-phase form of the displacement, the factor $Re^{-\frac{ct}{2m}}$ is called the ***time–varying amplitude*** of the motion, the quantity ω_1 is called the ***frequency***, and $T = \frac{2\pi}{\omega_1}$ is called the ***quasi–period*** of the displacement.

Example 7.2 An object with mass $2\,\mathrm{kg}$ is hanging at one end of a spring with the spring constant $k = 50\,\mathrm{kg/s^2}$. The spring is subject to a damping force with the damping constant $c = 12\,\mathrm{kg/s}$. Suppose the object is initially displaced downward $4\,\mathrm{m}$ with a downward velocity

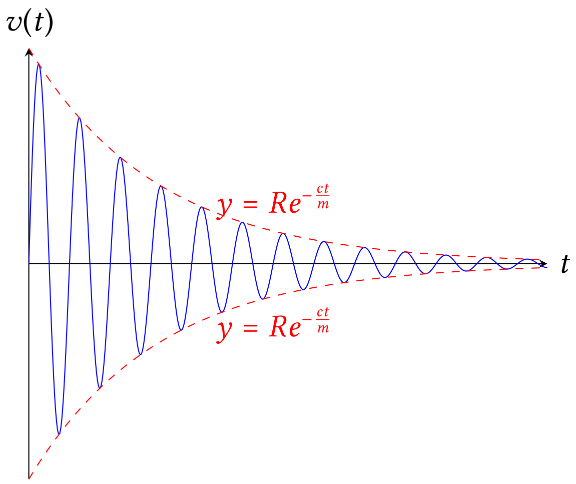

Figure 7.5: Graph of the displacement of an undamped motion

$8\,\mathrm{m/s}$. Find the displacement of the object.

Solution *If the time t is measured in seconds, then from the given information, the displacement y, measured in meters, is the solution of the following initial value problem*

$$2y'' + 12y' + 50y = 0, \qquad y(0) = -4, \qquad y'(0) = -8,$$

or equivalently

$$y'' + 6y' + 25y, \qquad y(0) = -4, \qquad y'(0) = -8.$$

The characteristic equation

$$r^2 + 6r + 25 = 0$$

has two complex solutions

$$r = \frac{-6 \pm \sqrt{6^2 - 25^2}}{2 \cdot 2} = -3 \pm 4\mathrm{i}.$$

Therefore, the general solution is

$$y = e^{-3t}(C_1 \cos(4t) + C_2 \sin(4t)).$$

Since

$$y' = e^{-3t}((-3C_1 + 4C_2)\cos(4t) + (-4C_1 - 3C_2)\sin(4t)),$$

plugging the initial conditions into y and y' yields

$$C_1 = -4 \qquad \text{and} \qquad C_2 = -5.$$

So the displacement y is given by

$$y = -e^{-5t}(4\cos(4t) + 5\sin(4t)).$$

■

Overdamping

The motion is said to be ***overdamped*** if $c^2 > 4mk$. In this case, the solutions of the characteristic equation are distinct real roots

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$$

The general solution is

$$y = C_1 e^{\frac{\left(-c+\sqrt{c^2-4mk}\right)t}{2m}} + C_2 e^{\frac{\left(-c-\sqrt{c^2-4mk}\right)t}{2m}}.$$

Example 7.3 An object with mass $2\,\mathrm{kg}$ is hanging at one end of a spring with the spring constant $k = 50\,\mathrm{kg/s^2}$. The spring is subject to a damping force with the damping constant $c = 52\,\mathrm{kg/s}$. Suppose the object is initially displaced $6\,\mathrm{m}$ above the equilibrium position with a downward velocity $30\,\mathrm{m/s}$. Find the displacement of the object.

Solution *If the time t is measured in seconds, then from the given information, the displacement y, measured in meters, is the solution of the following initial value problem*

$$2y'' + 52y' + 50y = 0, \qquad y(0) = 6, \qquad y'(0) = -30$$

or equivalently

$$y'' + 26y' + 25y, \qquad y(0) = 6, \qquad y'(0) = -30$$

The characteristic equation

$$r^2 + 26r + 25 = 0$$

has two real solutions

$$r_1 = -1 \qquad \textit{and} \qquad r_2 = -25.$$

Therefore, the general solution is

$$y = C_1 e^{-t} + C_2 e^{-25t}.$$

Since

$$y' = -C_1 e^{-t} - 25C_2 e^{-25t},$$

plugging the initial conditions into y and y' yields

$$C_1 = 5 \qquad \textit{and} \qquad C_2 = 1.$$

So the displacement y is given by

$$y = 5e^{-t} + e^{-25t}.$$

■

Critically Damping

The motion is said to be ***critically damped*** if $c = \sqrt{4mk}$. In this case $r_1 = r_2 = -\frac{c}{2m}$ and the general solution is

$$y = e^{-\frac{ct}{2m}}(c_1 + c_2 t).$$

Example 7.4 An object with mass $2\,\mathrm{kg}$ is hanging at one end of a spring with the spring constant $k = 50\,\mathrm{kg/s^2}$. The spring is subject to a damping force with the damping constant $c = 20\,\mathrm{kg/s}$. Suppose the object is initially displaced $1\,\mathrm{m}$ below the equilibrium position with a upward velocity $3\,\mathrm{m/s}$. Find the displacement of the object.

Solution *If the time t is measured in seconds, then from the given information, the displacement y, measured in meters, is the solution of the following initial value problem*

$$2y'' + 20y' + 50y = 0, \qquad y(0) = -1, \qquad y'(0) = 3$$

or equivalently

$$y'' + 10y' + 25y, \qquad y(0) = -1, \qquad y'(0) = 3$$

The characteristic equation

$$r^2 + 10r + 25 = 0$$

has a repeated root

$$r = -5.$$

Therefore, the general solution is

$$y = e^{-5t}(C_1 + C_2 t).$$

Since

$$y' = e^{-5t}((-5C_1 + C_2) - 5C_2 t),$$

plugging the initial conditions into y and y' yields

$$C_1 = -1 \qquad \text{and} \qquad C_2 = -2.$$

So the displacement y is given by

$$y = -e^{-5t} - 2te^{-5t}.$$

■

7.3 Forced vibrations

Suppose that the motion of the spring is affected by an external force $F(t)$ depending on the time. Then the displacement y satisfies the equation

$$my'' + cy' + ky = F(t),$$

which is a non-homogeneous equation.

Forced vibration without damping

In many mechanical problems, a device is subjected to periodic external forces $F(t)$ but not damping, for example, $F(t) = F_0 \cos(\omega_0 t)$. In this case, the displacement of the object satisfies the equation

$$y'' + \frac{k}{m} y = \frac{F_0}{m} \cos(\omega_0 t).$$

If $\omega_0 \neq \omega = \sqrt{\frac{k}{m}}$, solving this equation using the method of undetermined coefficients and using the identity $k = m\omega^2$ yields the general solution

$$y = e^{-\frac{c}{2m}} C_1 \cos(\omega t) + e^{-\frac{c}{2m}} C_2 \sin(\omega t) + \frac{F_0 \cos(\omega_0 t)}{m(\omega^2 - \omega_0^2)}.$$

If $\omega_0 = \omega$, the general solution is

$$\begin{aligned} y =& e^{-\frac{c}{2m}} C_1 \cos(\omega t) + e^{-\frac{c}{2m}} C_2 \sin(\omega t) + \frac{F_0 t \sin(\omega t)}{2m\omega} \\ =& e^{-\frac{c}{2m}} C_1 \cos(\omega t) + \left(e^{-\frac{c}{2m}} C_2 + \frac{F_0 t}{2m\omega} \right) \sin(\omega t). \end{aligned}$$

In this case, the amplitude increases as time goes. This phenomenon is known as the ***resonance***.

Example 7.5 A $2\,\mathrm{kg}$ object is attached to a spring with constant $k = 275\,\mathrm{kg/s}$ and subjected to an external force $F(t) = 32\cos 8t\,\mathrm{kg} - \mathrm{m/s}^2$. The object begins at rest in its equilibrium position. Find its displacement for $t > 0$, with $y(t)$ measured positive upward.

Solution *From the given information, the displacement satisfies*

$$2y'' + 128y = 32\cos(8t), \qquad y(0) = 0, \qquad y'(0) = 0,$$

equivalently,

$$y'' + 64y = 16\cos(8t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solving the equation by the method of undetermined coefficients yields the general solution to the *homogeneous equation is*

$$y(t) = C_1\cos(8t) + C_2\sin(8t) + t\sin(8t).$$

Since $y(0) = 0$ *and* $y'(0) = 0$*, the undetermined coefficients are*

$$C_1 = 0 \qquad C_2 = 0.$$

Therefore, the displacement is

$$y = t\sin(8t).$$

■

Forced vibrations with damping

Suppose the damping constant c is nonzero and the external force $F(t) = F_0\cos(\omega_0 t)$. The general solution of the equation

$$my'' + cy' + ky = F_0\cos(\omega_0 t)$$

is in the form

$$y = y_h + y_p,$$

where y_h is the general solution of the complementary equation and y_p is a particular solution in the form

$$y_p = A\cos(\omega_0 t) + B\sin(\omega_0 t),$$

where A and B can be determined by plugging y_p in the equation.

Differentiating y_p yields

$$y_p' = -A\omega\sin(\omega_0 t) + B\omega\cos(\omega_0 t).y_p' = \quad -A\omega^2\cos(\omega_0 t) - B\omega^2\sin(\omega_0 t).$$

Plugging them into $my'' + cy' + ky$ yields

$$\begin{aligned} &my'' + cy' + ky \\ =&(-mA\omega_0^2 + cB\omega_0 + kA)\cos(\omega_0 t) + (-mB\omega_0^2 - cA\omega_0 + kB)\sin(\omega_0 t) \end{aligned}$$

So y_p is a particular solution if

$$\begin{cases} (k - m\omega_0^2)A + cB\omega_0 &= F_0 \\ -cA\omega_0 + (k - m\omega_0^2)B &= 0. \end{cases}$$

Solving A and B yields

$$A = \frac{(k - m\omega_0^2)F_0}{(k - m\omega_0^2)^2 + c^2\omega_0^2}$$
$$B = \frac{(c\omega_0)F_0}{(k - m\omega_0^2)^2 + c^2\omega_0^2}.$$

Therefore,

$$y_p = \frac{F_0}{(k - m\omega_0^2)^2 + c^2\omega_0^2}\left[(k - m\omega_0^2)\cos\omega_0 t + c\omega_0 \sin\omega_0 t\right]$$

which can be written in the amplitude-phase form as

$$y_p = R\cos(\omega_0 t + \phi),$$

where

$$R = \frac{F_0}{\sqrt{(k - m\omega_0^2)^2 + c^2\omega_0^2}}$$

and the phase angle ϕ is determined by

$$\cos\phi = \frac{k - m\omega_0^2}{\sqrt{(k - m\omega_0^2)^2 + c^2\omega_0^2}} \quad \text{and} \quad \sin\phi = -\frac{c\omega_0}{\sqrt{(k - m\omega_0^2)^2 + c^2\omega_0^2}}.$$

When the motion is underdamped, that is, $c^2 < 4km$, the general form of the displacement is

$$y = e^{-\frac{ct}{2m}}(C_1\cos(\omega_1 t) + C_2\sin(\omega_1 t)) + R\cos(\omega_0 t + \phi),$$

where $\omega_1 = \frac{4km - c^2}{2m}$.

When the motion is overdamped, that is, $c^2 > 4km$, the general form of the displacement is

$$y = C_1 e^{\frac{\left(-c+\sqrt{c^2-4mk}\right)t}{2m}} + C_2 e^{\frac{\left(-c-\sqrt{c^2-4mk}\right)t}{2m}}) + R\cos(\omega_0 t + \phi).$$

When the motion is critically damped, that is, $c^2 = 4km$, the general form of the displacement is

$$y = e^{-\frac{ct}{2m}}(C_1 + C_2 t)) + R\cos(\omega_0 t + \phi).$$

Since the exponential summand approaches to 0 as t goes to the infinity, for large t, the displacement y is closely approximated by the particular solution y_p:

$$y \approx y_p = R\cos(\omega_0 t + \phi).$$

For this reason, we say that y_h is the ***transient component*** of the solution y, and y_p is the ***steady state component*** of y. Thus, for large t the motion is like simple harmonic motion at the frequency of the external force.

An interesting question is to find the value of ω_0 so that the amplitude of the steady state component is maximal. Let

$$\rho(x) = (k - mx)^2 + c^2 x = m^2 x^2 + (c^2 - 2km)x + k^2, \qquad x \geq 0.$$

Then R reaches its maximum R_{max} when $\rho(\omega_0^2)$ attains its minimum.

If $c > \sqrt{2km}$, then the minimum of $\rho(\omega_0^2)$ is at $\omega_0 = 0$ and

$$R_{\text{max}} = \frac{F_0}{k}.$$

If $c < \sqrt{2km}$, then the minimum of $\rho(\omega_0^2)$ is at

$$\omega = \sqrt{\frac{2km - c^2}{2m^2}}$$

and

$$R_{\text{max}} = \frac{2mF_0}{c\sqrt{4km - c^2}}.$$

Example 7.6 A $1\,\text{kg}$ object is attached to a spring with spring constant $k = 8\,\text{kg/s}$, and subjected to an damping force with the constant $c = 4\,\text{kg/s}$ and an external force $F(t) = 3\cos(5t)\,\text{kg}\,\text{m/s}^2$. Find the general solution and also the steady periodic solution

Solution *The displacement y satisfies the equation*

$$y'' + 4y' + 8y = 3\cos(5t).$$

Solving the characteristic equation

$$\begin{aligned} r^2 + 4r + 8 &= 0 \\ (r+2)^2 + 4 &= 0 \end{aligned}$$

yields

$$r = -2 \pm 2i.$$

Then the general solution of the complementary equation is

$$y_h = e^{-2t}(C_1 \cos 2t + C_2 \sin 2t)$$

A particular solution can be taken in the form

$$y_p = A\cos 5t + B\sin 5t,$$

where A and B can be determined by the method of undetermined coefficients. Differentiating y_p

yields

$$\begin{aligned} y_p' &= 5B\cos 5t - 5A\sin 5t \\ y_p'' &= -25A\cos 5t + -25B\sin 5t. \end{aligned}$$

Hence

$$y_p'' + 4y_p' + 8y_p = (-17A + 20B)\cos 5t + (-17B - 20A)\sin 5t.$$

For y_p to be a solution, the above expression must equal to $3\cos(5t)$, which implies

$$\begin{cases} -17A + 20B &= 3 \\ 20A + 17B &= 0 \end{cases}$$

Solving this system of equations yields

$$A = \frac{-51}{689} \qquad \text{and} \qquad B = \frac{60}{689}.$$

So the general solution is:

$$y = y_h + y_p = e^{-2t}(C_1\cos 2t + C_2\sin 2t) - \frac{51}{689}\cos 5t + \frac{60}{689}\sin 5t.$$

The steady periodic solution is

$$y = -\frac{51}{689}\cos 5t + \frac{60}{689}\sin 5t.$$

■

Week 8: Series Solutions

11/01–11/04

Most functions seen in calculus belongs to a class known as the elementary functions. The class of elementary functions consists of polynomials, rational functions, radical functions, trigonometric functions, inverse trigonometric functions, exponential functions, logarithmic functions and all others that can be constructed from those by adding, subtracting, multiplying, dividing, or composing. In applications, many second order differential equations cannot be solved in terms of elementary functions. For example, Airy's Equation

$$y'' - xy = 0$$

does not have any solution that is an elementary function. How do we know that? This question turns out to be complicated. A proof employs differential Galois theory. Interested reader many find an argument in the paper An algorithm for solving second order linear homogeneous differential equations by Kovacic.

One approach to solve equations that have no elementary solutions is to use power series. We can then define special functions using power series and study their properties.

8.1 Review of Power Series

Convergence

A ***power series*** in $(x - x_0)$ is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots .$$

A power series in $(x - x_0)$ is said to converge at x if the limit

$$\lim_{m\to\infty} \sum_{n=0}^{m} (x - x_0)$$

exists.

Clearly, the power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges at 0. If it also converges at another point, then either it converges for all x, or converges over an interval $(x_0 - R, x_0 + R)$ for some positive number R and diverges over the interval $(-\infty, x_0 - R) \cup (x_0 + R, \infty)$.

Theorem 8.1

For any power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

exactly one of the following statements holds true:

1. *The power series converges only for $x = x_0$.*
2. *The power series converges for all values of x.*
3. *There's a positive number R such that the power series converges if $|x - x_0| < R$ and diverges if $|x - x_0| > R$.*

The number R in the third, case 3., is called the ***radius of convergence*** of the power series. For convenience, we set $R = 0$ in the first case and $R = \infty$ in the second case. The interval $(x_0 - R, x_0 + R)$ is called the interval of convergence.

In calculus, several methods of finding the radius of convergence are given. One of them is the following theorem.

Theorem 8.2

Suppose there's an integer N such that $a_n \neq 0$ if $n \geq N$ and

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L,$$

where $0 \leq L \leq \infty$. Then the radius of convergence of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is $R = \frac{1}{L}$, where the the convention that $R = 0$ if $L = \infty$, or $R = \infty$ if $L = 0$ is used.

Example 8.1 Find the radius of convergence of the series

$$\sum_{n=10}^{\infty}(-1)^n\frac{x^n}{n!}.$$

Solution *When n goes to infinity, the limit $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ is*

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{(n+1)!}}{\frac{(-1)^n}{n!}}\right| \\ &= \lim_{n\to\infty}\frac{1}{n+1} \\ &=0.\end{aligned}$$

Therefore, the radius of convergence is $R = \infty$. ■

✍ **Exercise 8.1** Find the radius of convergence of the series

$$\sum_{n=5}^{\infty}(-1)^{n+1}\frac{x^n}{2^n}.$$

Solution *When n goes to infinity, the limit $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ is*

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}}{2^{n+1}}}{\frac{(-1)^{n+1}}{2^n}}\right|\\ &= \lim_{n\to\infty}\frac{1}{2}\\ &=\frac{1}{2}.\end{aligned}$$

Therefore, the radius of convergence is $R = \frac{1}{\frac{1}{2}} = 2$. ■

Differentiations of power series

For a continuous function f that has derivatives of all orders for $|x| < R$, the Taylor expansion

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)(x)}}{n!}x^n$$

is a power series known as the Maclaurin series of f. For example,

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}, \quad -\infty < x < \infty;$$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty;$$

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \quad -\infty < x < \infty;$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n \quad -1 < x < 1.$$

Backwards, suppose the power series $\sum_{n=0}^{\infty}a_n(x-x_0)^n$ has a positive radius of convergence R. Then we can define a function

$$f(x) = \sum_{n=0}^{\infty}a_n(x-x_0)^n$$

on its open interval of convergence $(x_0 - R, x_0 + R)$. Such a function is called an analytic function and has very good properties.

Definition 8.1

A function f is **analytic** *at x_0 if*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

and the series is convergent to $f(x)$ for all x in an open interval containing x_0.
A function is analytic on an open interval if it is analytic at all points in the interval.

The set of analytic functions is closed under addition and multiplication.

Theorem 8.3

Suppose the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ has a positive radius R and a function f is defined by

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n.$$

Then

1. *f has derivatives of all orders in the open interval $(x_0 - R, x_0 + R)$,*
2. *successive derivatives of f can be obtained by repeatedly differentiating the power series term by term, that is*

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \\ f''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}, \\ &\vdots \\ f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (x - x_0)^{n-k}, \end{aligned}$$

3. *all of these series have the same radius of convergence R.*

As a corollary, we see that if f is a function defined by a power series, then the Taylor series of the function is exactly the power series.

Corollary 8.1

Let f be a function defined by

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

where the power series has a positive radius of convergence R. Then

$$a_n = \frac{f^{(n)}(x_0)}{n!},$$

that is, the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is the Taylor series of f about x_0.

Example 8.2 Consider the power series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ which converges for all real number x. Show that the derivative of the power series defines a function which is $\cos x$.

Solution *The derivative of a power series is the infinite sum of derivatives of terms. So*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) &= \sum_{n=0}^{\infty}(-1)^n \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{2n+1}}{(2n+1)!}\right) \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\end{aligned}$$

Comparing with the Maclaurin series of $\cos x$, we see that the derivative is $\cos x$. Indeed, the function defined by the power series is noting but $\sin x$. ■

✍ **Exercise 8.2** Consider the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ which converges for all real number x. Show that the second derivative of the power series defines a function which is e^x.

Solution *The first derivative is*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) &= \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^n}{n!}\right) \\ &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!}\end{aligned}$$

which is the power series itself. So is the second derivative. Comparing with the Maclaurin series of e^x, we see that the power series, its first derivative and the second derivative are the same function e^x. ■

Shifting indices

You probably know that the summation index is called a dummy index, and can changed to any other name. For a power series, whose n-th term is not a multiple of $(x-x_0)^n$, one can use a substitution and rename the index to shift exponent to n. For example

$$\begin{aligned}f'(x) &= \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1} \\ &= \sum_{k=0}^{\infty}(k+1)a_{k+1}(x-x_0)^k, \quad \text{substituting} k = n-1 \\ &= \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-x_0)^n, \quad \text{renaming}\end{aligned}$$

In general, for any integer k, the power series

$$\sum_{n=n_0}^{\infty} a_n(x-x_0)^{n-k}$$

can be rewritten as

$$\sum_{n=n_0-k}^{\infty} a_{n+k}(x-x_0)^n,$$

that is, replacing n by $n+k$ in the general term and n_0 by n_0-k in the lower limit of summation leaves the series unchanged.

Example 8.3 Given that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

write the function xy as a power series in which the general term is a constant multiple of x^n.

Solution *Since*

$$xy(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$

replacing n by $n-1$ and 0 by $0-(-1)=1$ yields

$$xy(x) = \sum_{n=1}^{\infty} a_{n-1} x^n.$$

■

✍ **Exercise 8.3** Given that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

write the function y'' as a power series in which the general term is a constant multiple of x^n.

Solution *Since*

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2},$$

replacing n by $n+2$ and 2 by $2-2=0$ yields

$$y''(x) = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

■

Linear combination of power series

From calculus, particular, the section on Riemann sums, we know that the linear combination of infinite sums is the infinite sums of linear combinations. More precisely, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with the convergence radius R_1 and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ with the convergence radius R_2, then

$$Af(x) \pm Bg(x) = \sum_{n \to 0}^{\infty}(Aa_n + Bb_n)x^n$$

which has the convergence radius $R = \min\{R_1, R_2\}$.

Note that power series can also be multiplied like polynomials:

$$f(x)g(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n.$$

Let's end this section by finding a power series solution for the Airy equation $y'' - xy = 0$.

Example 8.4 Find the first 3 terms of the power series solution for the Airy equation $y''-xy = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 1$.

Solution *Suppose that $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is a solution. Using the linear combination property, and the technique of shifting indices, we see that*

$$\begin{aligned}
y'' - xy &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1} x^n \\
&=(0+2)(0+1)a_{0+2}x^0 + \sum_{n=1}((n+2)(n+1)a_{n+2} - a_{n-1})x^n \\
&=2a_2 + \sum_{n=1}((n+2)(n+1)a_{n+2} - a_{n-1})x^n.
\end{aligned}$$

If $y(x)$ is solution of the Airy equation, then

$$\begin{aligned}
2a_2 &=0 \\
(n+2)(n+1)a_{n+2} - a_{n-1} &=0 \qquad \text{for } n = 1, 2, \ldots.
\end{aligned}$$

Since $y(0) = 0$ and $y'(0) = 1$, we see that $a_0 = 0$ and $a_1 = 1$. Note that $a_2 = 0$ too. Then a_n can be *solve recursively using the second formula. For example,*

$$a_3 = a_{1+2} = \frac{a_{1-1}}{(1+2)(1+1)} = \frac{a_0}{3\cdot 2} = 0,$$

$$a_4 = a_{2+2} = \frac{a_{2-1}}{(2+2)(2+1)} = \frac{a_1}{4\cdot 3} = \frac{1}{12},$$

$$a_5 = a_{3+2} = \frac{a_{3-1}}{(3+2)(3+1)} = \frac{a_2}{5\cdot 4} = 0.$$

$$a_6 = a_{4+2} = \frac{a_{4-1}}{(4+2)(4+1)} = \frac{a_3}{6\cdot 5} = 0.$$

$$a_7 = a_{5+2} = \frac{a_{5-1}}{(5+2)(5+1)} = \frac{a_4}{7\cdot 6} = \frac{1}{7\cdot 6\cdot 4\cdots 3} = \frac{1}{45360}.$$

Indeed, the power series solution of the Airy equation has the form

$$y(x) = x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \frac{1}{45360}x^{10} + \cdots + \frac{1}{\prod_{j=1}^{k} 3j\cdot(3j+1)}x^{3k+1} + \cdots.$$

■

Remark When the recurrence relation is in the form

$$a_{n+2} = c_1 a_n + c_2 a_{n-1},$$

where c_1 and c_2 are constant, the solution, known as the generating function of the sequence a_n, may be expressed as a rational function. See the wiki page Recurrence relation for more information.

Exercise 8.4 Find the first 4 terms of the power series solution for the initial value problem

$$y'' - xy' = 0, y(0) = 1, y'(0) = 1.$$

Solution *Suppose that* $y(x) = \sum_{n=0}^{\infty} a_n x^n$ *is a solution. Then*

$$\begin{aligned} y'' - xy' &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=1}^{\infty} na_n x^{n-1} \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} na_n x^n \\ &= 2a_2 + \sum_{n=1}^{\infty}((n+2)(n+1)a_{n+2} - na_n)x^n \end{aligned}$$

If $y(x)$ is solution of the Airy equation, then

$$\begin{aligned} 2a_2 &= 0 \\ (n+2)(n+1)a_{n+2} - na_n &= 0 \qquad \text{for } n = 1, 2, \ldots. \end{aligned}$$

Since $y(0) = 1$ and $y'(0) = 1$, we see that $a_0 = 1$ and $a_1 = 1$. Note that $a_2 = 0$ too. Then

$$a_3 = \frac{a_1}{6} = \frac{1}{6},$$

$$a_4 = \frac{2a_2}{12} = 0.$$

$$a_5 = \frac{3a_3}{20} = \frac{1}{40}.$$

So the power series solution has the form

$$y(x) = 1 + x + \frac{x^3}{6} + \frac{x^5}{40} + \cdots.$$

■

8.2 Series Solutions Near an Ordinary Point

In this section, we will study the homogeneous linear second order equation of the form

$$P_0 y'' + P_1 y' + P_2 y = 0, \tag{8.1}$$

where the coefficient functions $P_0(x)$, $P_1(x)$ and $P_2(x)$ are polynomials with no common factor and P_0 is not identically zero.

A point x_0 is called an ***ordinary point*** of Equation 8.1 if $P_0(x_0) \neq 0$, otherwise, it is called a ***singular point***.

Example 8.5 Find ordinary points of Legendre's equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

Solution *Solving the equation $P_0(x) = 0$ will gives singular points. In this case, the equation*

$$1 - x^2 = 0$$

has two solutions $x = 1$ and $x = -1$ which are singular points of Legendre's equation. All other points are ordinary point of the equation. ■

Since polynomials are analytic functions, it can be shown that the rational function $\frac{P_1}{P_0}$ and

$\frac{P_2}{P_0}$ are analytic at any ordinary point. Near an ordinary point, we can rewrite Equation 8.1 as

$$y'' + \frac{P_1}{P_0}y' + \frac{P_2}{P_0}y = 0$$

which is called the normalized equation. Since $\frac{P_1}{P_0}$ and $\frac{P_2}{P_0}$ are analytic, using properties of power series, we can find a power series solution $y(x)$ in $x - x_0$ which is valid near x_0.

Theorem 8.4

Let x_0 be an ordinary point of the equation

$$P_0 y'' + P_1 y' + P_2 y = 0$$

and let a_0 and a_1 be arbitrary constants. Then there exists a unique solution $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ near x_0 such that $y(x_0) = a_0$ and $y'(x_0) = a_1$. Moreover, if the power series expansions of $\frac{P_1(x)}{P_0(x)}$ and $\frac{P_2(x)}{P_0(x)}$ converge on an open interval $(x_0 - R, x_0 + R)$, then the power series solution also converges on the same interval.

The first part of this theorem can be proved by solving recursive formula of a_n. The proof of the same convergence interval is a little bit involved. As we mainly focus on solving the equation, we will not discuss the proof of the theorem. We refer the reader to (Simmons 2016, Section 28) for a proof in more general setting.

This theorem, together with the existence and uniqueness of the solution of a linear second order equation, implies that every solution of Equation 8.1 can be represented by a power series. We can such a solution a ***power series solution***.

The basic idea to find a power series solution is similar to the undetermined coefficient method. To simplify notations in calculation, we define a differential operator

$$L = P_0 \frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + P_1 \frac{\mathrm{d}}{\mathrm{d}\,x} + P_2$$

which acts on y by

$$Ly = P_0 \frac{\mathrm{d}^2}{\mathrm{d}\,x^2} y + P_1 \frac{\mathrm{d}}{\mathrm{d}\,x} y + P_2 y = P_0 y'' + P_1 y' + P2y.$$

Suppose $y(x) = \sum\limits_{n=0}^{\infty} a_n(x - x_0)^n$ is a power series solution satisfies the initial conditions $y(0) = a_0$ and $y'(0) = a_1$. Then

$$Ly = \sum_{n=0}^{\infty} b_n(x - x_0)^n,$$

where b_n are expressions in terms of coefficients of P_0, P_1, and P_2, and $a_0, a_1, \dots, a_{n+N}$ for some positive integer N. Then y is a solution if and only if $b_n = 0$ for all $n \geq 0$. The coefficients $a_2, a_3, \dots,$ can be determined recursively using relations $b_n = 0$.

You will find in calculations of power series, the product of a sequence frequent appear. To simplify calculation, we denote the product of a sequence $a_m, a_{m+1}, a_{m+2}, \dots, a_n$, where $n > m$, by

$$\prod_{k=m}^{n} a_k = a_m \cdots a_{m+1} \cdots \cdots \cdot a_n.$$

For convenience, we define

$$\prod_{k=m}^{n} a_k = 1 \qquad \text{if } \ n < m.$$

Example 8.6 Find the power series in x for the general solution of

$$(1 + 2x^2)y'' + 6xy' + 2y = 0.$$

Solution

Let

$$Ly = (1 + 2x^2)y'' + 6xy' + 2y = y'' + 2x^2y'' + 6xy' + 2y.$$

If

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \textit{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

So

$$\begin{aligned}
Ly &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 6x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2 \sum_{n=2}^{\infty} n(n-1)a_n x^n + 6 \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + 2 \sum_{n=0}^{\infty} n(n-1)a_n x^n + 6 \sum_{n=0}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + (2n(n-1) + 6n + 2)a_n\right] x^n \\
&= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + 2(n+1)^2 a_n\right] x^n
\end{aligned}$$

If y is a solution, then $Ly = 0$ which implies that coefficients of the power series expression of y satisfy the recurrence relation

$$a_{n+2} = -2\frac{n+1}{n+2}a_n, \qquad n \geq 0.$$

Since the indices on the left and right differ by two, we write the recurrence relation separately for

$n = 2m$ and $n = 2m+1$. Then

$$a_{2m+2} = -2\frac{2m+1}{2m+2}a_{2m} = -\frac{2m+1}{m+1}a_{2m}, \qquad m \geq 0,$$

and

$$a_{2m+3} = -2\frac{2m+2}{2m+3}a_{2m+1} = -4\frac{m+1}{2m+3}a_{2m+1}, \qquad m \geq 0.$$

Computing the coefficients of even powers of x from the recurrence relation yields

$$\begin{aligned}
a_2 &= -\frac{1}{1}a_0,\\
a_4 &= -\frac{3}{2}a_2 = \left(-\frac{3}{2}\right)\left(-\frac{1}{1}\right)a_0 = \frac{1\cdot 3}{1\cdot 2}a_0,\\
a_6 &= -\frac{5}{3}a_4 = -\frac{5}{3}\left(\frac{1\cdot 3}{1\cdot 2}\right)a_0 = -\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}a_0,\\
a_8 &= -\frac{7}{4}a_6 = -\frac{7}{4}\left(-\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\right)a_0 = \frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 2\cdot 3\cdot 4}a_0.
\end{aligned}$$

In general,

$$a_{2m} = (-1)^m\frac{\prod_{k=1}^m(2k-1)}{m!}a_0, \qquad m \geq 0.$$

Computing the coefficients of odd powers of x yields

$$\begin{aligned}
a_3 &= -4\cdot\frac{1}{3}a_1,\\
a_5 &= -4\cdot\frac{2}{5}a_3 = -4\cdot\frac{2}{5}\left(-4\frac{1}{3}\right)a_1 = 4^2\frac{1\cdot 2}{3\cdot 5}a_1,\\
a_7 &= -4\cdot\frac{3}{7}a_5 = -4\cdot\frac{3}{7}\left(4^2\frac{1\cdot 2}{3\cdot 5}\right)a_1 = -4^3\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}a_1,\\
a_9 &= -4\cdot\frac{4}{9}a_7 = -4\cdot\frac{4}{9}\left(4^3\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)a_1 = 4^4\frac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9}a_1.
\end{aligned}$$

In general,

$$a_{2m+1} = \frac{(-1)^m 4^m m!}{\prod_{k=1}^m(2k+1)}a_1, \qquad m \geq 0.$$

Therefore, the power series form of the general solution is

$$y = a_0\sum_{m=0}^{\infty}(-1)^m\frac{\prod_{j=1}^m(2j-1)}{m!}x^{2m} + a_1\sum_{m=0}^{\infty}(-1)^m\frac{4^m m!}{\prod_{j=1}^m(2j+1)}x^{2m+1}.$$

■

Using the method shown in the above example, we find the power series solution of the more general equation

$$(1+\alpha(x-x_0)^2)y'' + \beta(x-x_0)y' + \gamma y = 0.$$

Theorem 8.5

The coefficients $\{a_n\}$ *in any solution* $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ *of the equation*

$$\left(1 + \alpha (x - x_0)^2\right) y'' + \beta (x - x_0) y' + \gamma y = 0$$

satisfy the recurrence relation

$$a_{n+2} = -\frac{p(n)}{(n+2)(n+1)} a_n, \qquad n \geq 0,$$

where

$$p(n) = \alpha n(n-1) + \beta n + \gamma.$$

Moreover, the coefficients of the even and odd powers of $x - x_0$ *can be computed separately as*

$$a_{2m+2} = -\frac{p(2m)}{(2m+2)(2m+1)} a_{2m}, \quad m \geq 0$$

$$a_{2m+3} = -\frac{p(2m+1)}{(2m+3)(2m+2)} a_{2m+1}, \quad m \geq 0$$

where a_0 *and* a_1 *are arbitrary.*

Example 8.7 Compute $a_0, a_1, \ldots, a_7$ in the power series solution $y = \sum\limits_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$(1 + 2x^2) y'' + 10 x y' + 8 y = 0, \quad y(0) = 2, \quad y'(0) = -3.$$

Solution *Since* $\alpha = 2$, $\beta = 10$, *and* $\gamma = 8$ *in the equation, we have*

$$p(n) = 2n(n-1) + 10n + 8 = 2(n+2)^2.$$

Therefore,

$$a_{n+2} = -2\frac{(n+2)^2}{(n+2)(n+1)} a_n = -2\frac{n+2}{n+1} a_n, \quad n \geq 0.$$

For $n = 2m$, we have

$$a_{2m+2} = -2\frac{(2m+2)}{2m+1} a_{2m} = -4\frac{m+1}{2m+1} a_{2m}, \quad m \geq 0$$

For $n = 2m+1$, we have

$$a_{2m+3} = -2\frac{2m+3}{2m+2} a_{2m+1} = -\frac{2m+3}{m+1} a_{2m+1}, \quad m \geq 0.$$

Since $a_0 = y(0) = 2$, we have

$$a_2 = -4 \cdot \frac{1}{1} 2 = -8,$$
$$a_4 = -4 \cdot \frac{2}{3}(-8) = \frac{64}{3},$$
$$a_6 = -4 \cdot \frac{3}{5}\left(\frac{64}{3}\right) = -\frac{256}{5}.$$

Since $a_1 = y'(0) = -3$, we have

$$a_3 = -\frac{3}{1}(-3) = 9,$$
$$a_5 = -\frac{5}{2} 9 = -\frac{45}{2},$$
$$a_7 = -\frac{7}{3}\left(-\frac{45}{2}\right) = \frac{105}{2}.$$

Therefore, the solution in power series form is

$$y = 2 - 3x - 8x^2 + 9x^3 + \frac{64}{3}x^4 - \frac{45}{2}x^5 - \frac{256}{5}x^6 + \frac{105}{2}x^7 + \cdots.$$

■

✍ **Exercise 8.5** Let x_0 be an arbitrary real number. Find the power series in $(x - x_0)$ for the general solution of

$$y'' + y = 0.$$

Solution *Let*

$$Ly = y'' + y.$$

Suppose

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2}.$$

So

$$\begin{aligned} Ly &= \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2} + \sum_{n=0}^{\infty} a_n(x - x_0)^n \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x - x_0)^n + \sum_{n=0}^{\infty} a_n(x - x_0)^n \\ &= \sum_{n=0}^{\infty}((n+2)(n+1)a_{n+2} + a_n). \end{aligned}$$

Therefore $Ly = 0$ if and only if

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}, \qquad n \geq 0,$$

where a_0 and a_1 are arbitrary.

Since the indices on the left and right sides of the recurrence relation differ by two, we write the recurrence relation separately for n even $n = 2m$ and n odd $n = 2m+1$, where $m = 0, 1, 2, \ldots$. Then

$$a_{2m+2} = \frac{(-1)a_{2m}}{(2m+2)(2m+1)}, \qquad m \geq 0$$

and

$$a_{2m+3} = \frac{-a_{2m+1}}{(2m+3)(2m+2)}, \qquad m \geq 0.$$

Computing the coefficients of the even powers of $x - x_0$ using the recurrence relation yields

$$\begin{aligned}
a_2 &= -\frac{a_0}{2\cdot 1}\\
a_4 &= -\frac{a_2}{4\cdot 3} = -\frac{1}{4\cdot 3}\left(-\frac{a_0}{2\cdot 1}\right) = \frac{a_0}{4\cdot 3\cdot 2\cdot 1},\\
a_6 &= -\frac{a_4}{6\cdot 5} = -\frac{1}{6\cdot 5}\left(\frac{a_0}{4\cdot 3\cdot 2\cdot 1}\right) = -\frac{a_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1},
\end{aligned}$$

and, in general,

$$a_{2m} = (-1)^m \frac{a_0}{(2m)!}, \qquad m \geq 0.$$

Similarly, computing the coefficients of the odd powers of $x - x_0$ yields

$$\begin{aligned}
a_3 &= -\frac{a_1}{3\cdot 2}\\
a_5 &= -\frac{a_3}{5\cdot 4} = -\frac{1}{5\cdot 4}\left(-\frac{a_1}{3\cdot 2}\right) = \frac{a_1}{5\cdot 4\cdot 3\cdot 2},\\
a_7 &= -\frac{a_5}{7\cdot 6} = -\frac{1}{7\cdot 6}\left(\frac{a_1}{5\cdot 4\cdot 3\cdot 2}\right) = -\frac{a_1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2},
\end{aligned}$$

and, in general,

$$a_{2m+1} = \frac{(-1)^m a_1}{(2m+1)!} \qquad m \geq 0.$$

Therefore, the general solution can be written as

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{(x-x_0)^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{(x-x_0)^{2m+1}}{(2m+1)!}.$$

Recall from calculus that

$$\sum_{m=0}^{\infty} (-1)^m \frac{(x-x_0)^{2m}}{(2m)!} = \cos(x - x_0) \quad \text{and} \quad \sum_{m=0}^{\infty} (-1)^m \frac{(x-x_0)^{2m+1}}{(2m+1)!} = \sin(x - x_0).$$

Then the solution is indeed,

$$y = a_0 \cos(x - x_0) + a_1 \sin(x - x_0).$$

■

✍ **Exercise 8.6** Find the coefficients $a_0, a_1, \ldots, a_7$ of the power series solution of the initial value problem

$$y'' + xy' + y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution *Let* $Ly = y'' + xy' + y$ *and* $y = \sum_{n=0}^{\infty} a_n x^n$. *Then*

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Therefore,

$$\begin{aligned}
Ly &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)a_n]x^n
\end{aligned}$$

If y *is a solution, then* $Ly = 0$ *which yields the recurrence relation*

$$a_{n+2} = -\frac{a_n}{n+2}.$$

When $n = 2m$ *is even,*

$$a_{2m} = -\frac{a_{2m-2}}{2m}, \qquad m \geq 1.$$

When $n = 2m+1$ *is odd, then*

$$a_{2m+1} = -\frac{a_{2m-1}}{2m+1}, \qquad m \geq 1.$$

Since $a_0 = y(0) = 1$*, we see that*

$$\begin{aligned} a_2 &= -\frac{a_0}{2} = -\frac{1}{2}, \\ a_4 &= -\frac{a_2}{4} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}, \\ a_6 &= -\frac{a_4}{6} = -\frac{1}{8} \cdot \frac{1}{6} = -\frac{1}{48}. \end{aligned}$$

Since $a_1 = y'(0) = 1$*, we see that*

$$\begin{aligned} a_3 &= -\frac{a_1}{3} = -\frac{1}{3}, \\ a_5 &= -\frac{a_3}{5} = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}, \\ a_7 &= -\frac{a_5}{7} = -\frac{1}{15} \cdot \frac{1}{7} = -\frac{1}{105}. \end{aligned}$$

Therefore, the power series solution is in the form

$$y = 1 + x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4 + \frac{1}{15}x^5 - \frac{1}{48}x^6 - \frac{1}{105}x^7 + \cdots.$$

■

8.3 Method of Frobenius and Euler Equations

Note that differentiating a power series increases the lowest exponent. At a regular point, $\frac{1}{P_0}$ is also analytic. Hence, the equation

$$y'' + \frac{P_1}{P_0}y' + \frac{P_2}{P_0}y = 0$$

may have a power series solution.

At a singular point, the rational function $\frac{1}{P_0}$ is no longer analytic, that is, it may not have a power series expression. For example, in the hypergeometric equation

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0,$$

where a, b, and c are constants, the coefficient $P_0 = x(1-x)$, then

$$\frac{1}{P_0} = \frac{1}{x(1-x)} = x^{-1}\frac{1}{1-x} = x^{-1}\sum_{n=0}^{\infty} x^n.$$

If $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is a solution, then the lowest exponent of y'' may be strictly greater than those of the other two terms. As a consequence, at $x = 0$, there may be no power series solution.

Here is another concrete example, the equation

$$y'' + \frac{2}{x}y' - \frac{2}{x^2}y = 0$$

has two solutions, $y_1 = x$, and $y_2 = x^{-2}$, where $y_2 = x^{-2}$ does not have a power series expression $\sum_{n=0}^{\infty} a_n x^n$.

Mathematicians don't give up, when one method does not work, they look for other methods. When the equation $y'' + \frac{P_1}{P_0}y' + \frac{P_2}{P_0}y = 0$ is not too singular at a singular point x_0, in the sense that $P_0 = (x - x_0)^2 A(x)$, where $A(x)$ is a polynomial and $A(x_0) \neq 0$, German mathematician, Ferdinand Frobenius developed the method of finding series solution in the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n.$$

Definition 8.2

Let P_0, P_1, and P_2 be polynomials with no common factor and suppose $P_0(x_0) = 0$. Then x_0 is a **regular singular** *point of the equation*

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

if

$$\frac{(x - x_0)P_1}{P_0} \quad \text{and} \quad \frac{(x - x_0)^2 P_2}{P_0}$$

are analytic at x_0.
Otherwise, x_0 is called an **irregular singular** *point of the equation.* ♣

Example 8.8 Bessel's equation

The equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

has the singular point $x_0 = 0$. Determine if $x_0 = 0$ is a regular singular point.

Solution *Since*

$$\frac{xP_1}{P_0} = \frac{x^2}{x^2} = 1 \quad \text{and} \quad \frac{x^2 P_2}{P_0} = \frac{x^2(x^2 - \nu^2)}{x^2} = (x^2 - \nu^2)$$

are analytic at 0, the point $x_0 = 0$ is regular singular. ■

✍ **Exercise 8.7 Legendre's equation**

The equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

has the singular points $x_0 = \pm 1$. Determine if $x_0 = \pm 1$ are regular singular points.

Solution *Since*

$$\frac{(x \pm 1)P_1}{P_0} = \frac{-(x \pm 1)2x}{x^2 - 1} = \frac{-2x}{x \mp 1}$$

and

$$\frac{(x \pm 1)^2 P_2}{P_0} = \frac{\alpha(\alpha + 1)(x \pm 1)^2}{x^2 - 1} = \frac{\alpha(\alpha + 1)(x \pm 1)}{x \mp 1}$$

are both analytic at ± 1, the points $x_0 = \pm 1$ are regular singular points of the equation. ■

At this stage, the only second order linear equation we can solve completely near a singular point is the Euler equation.

Definition 8.3 (Euler Equation)

An **Euler equation** *is an equation that can be written in the form*

$$ax^2 y'' + bxy' + cy = 0,$$

where a, b, and c are real constants and $a \neq 0$. ♣

From the existence theorem (Theorem 5.1), we know that Euler equation has solutions defined on $(0, \infty)$ and $(-\infty, 0)$. Since the two intervals are symmetric, by a substitution $t = -x$ when $x < 0$, we may and will restrict ourself to the interval $(0, \infty)$.

The normalized form equation

$$y'' + \frac{p}{x} y' + \frac{q}{x^2} y = 0,$$

where $p = \frac{b}{a}$ and $q = \frac{c}{a}$, suggest that a solution is in the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

We can then determine r and a_n's by plugging the series into the equation. Indeed, for Euler equation, we can take $a_0 = 1$ and $a_n = 0$ for $n = 1, 2, 3, \ldots$.

Differentiating y yields

$$y' = \sum_{n=0} (n + r) a_n x^{n+r-1},$$

$$y'' = \sum_{n=0} (n + r)(n + r - 1) a_n x^{n+r-2},$$

and

$$\begin{aligned}
&y'' + \frac{p}{x}y' + \frac{q}{x^2} \\
=&y'' + px^{-1}y' + qx^{-2} \\
=&\sum_{n=0}(n+r)(n+r-1)a_n x^{n+r-2} + p\sum_{n=0}(n+r)a_n x^{n+r-2} + q\sum_{n=0}^{\infty} a_n x^{n+r-2} \\
=&\sum_{n=0}[(n+r)(n+r-1) + p(n+r) + q]a_n x^{n+r-2}.
\end{aligned}$$

Therefore, y is a solution if

$$[(n+r)(n+r-1) + p(n+r) + q]a_n = 0 \qquad \text{for } n = 0, 1, 2, \ldots.$$

A sufficient condition is that

$$r(r-1) + pr + q = 0 \quad \text{and } a_n = 0 \text{for} n = 1, 2, \ldots.$$

The equation

$$r(r-1) + pr + q = 0$$

or equivalently,

$$ar(r-1) + br + c = 0$$

is called the indicial equation of the differential equation

$$ax^2y'' + bxy' + cy = 0.$$

Example 8.9 Find the general solution of

$$x^2y'' - xy' - 8y = 0$$

on $(0, \infty)$.

Solution *The indicial equation is*

$$r(r-1) - r - 8 = 0.$$

Equivalently,

$$r^2 - 2r - 8 = 0$$

Solving the equation yields $r = -2$ *or* $r = 4$. *Then* $y_1 = x^4$ *and* $y_2 = x^{-2}$ *are solutions of the equation on* $(0, \infty)$.

Because the Wronskian is

$$W(y_1, y_2) = y_1y_2' - y_1'y_2 = x^4 \cdot (x^{-2})' - (x^4)' \cdot x^{-2} = -6x \not\equiv 0.$$

Therefore, y_1 and y_2 are linearly independent and the general solution on $(0, \infty)$ is

$$y = c_1 x^4 + \frac{c_2}{x^2}.$$

■

Exercise 8.8 Find the general solution of

$$6x^2 y'' + 5xy' - y = 0$$

on $(0, \infty)$.

Solution *The indicial equation is*

$$6r(r-1) + 5r - 1 = 0.$$

Solving the equation yields

$$r = \frac{1}{2} \qquad \text{or} \qquad r = -\frac{1}{3}.$$

Therefore, the general solution is

$$y = c_1 x^{1/2} + c_2 x^{-1/3}.$$

■

When the indicial equation has a repeated solution or complex solutions, to find the general solution, we can using the Wronskian (see Proposition 5.1) and Euler's formula

$$x^{i\omega} = e^{i\omega \ln x} = \cos(\omega \ln x) + i \sin(\omega \ln x).$$

The general result is summarized in the following theorem.

Theorem 8.6

Suppose the roots of the indicial equation

$$ar(r-1) + br + c = 0$$

are r_1 and r_2. Then the general solution of the Euler equation

$$ax^2 y'' + bxy' + cy = 0$$

on $(0, \infty)$ is

- $$y = c_1 x^{r_1} + c_2 x^{r_2}$$

if r_1 and r_2 are distinct real numbers;

-
$$y = x^r(c_1 + c_2 \ln x)$$
if $r_1 = r_2 = r$;

-
$$y = x^\lambda \left[c_1 \cos(\omega \ln x) + c_2 \sin(\omega \ln x)\right]$$
if $r_1, r_2 = \lambda \pm i\omega$ with $\omega > 0$.

Here we present a shorter proof using the substitution $z = \ln x$.

Proof Let $z = \ln x$, or equivalently $x = e^z$. Let $u(z) = y(e^z)$. By the chain rule, we get

$$u'(z) = y'(e^z)e^z = xy'(x),$$

$$u''(z) = (y'(e^z)e^z)' = y''(e^z)(e^z)^2 + y'(e^z)e^z = x^2y''(x) + xy'(x)$$

Therefore,

$$ax^2y''(x) + bxy'(x) + cy(x) = au''(z) + (b-a)u'(z) + cu(z).$$

The equation

$$au'' + (b-a)u' + cu = 0$$

has the characteristic equation

$$ar^2 + (b-a)r + c = ar(r-1) + br + c = 0.$$

Because r_1 and r_2 are solutions of this characteristic equation.

If r_1 and r_2 are two distinct real root, then the general solution of the Euler equation is

$$y(x) = u(z) = c_1e^{r_1z} + c_2e^{r_2z} = c_1x^{r_1} + c_2x^{r_2}.$$

If $r_1 = r_2 = r$, then the general solution of the Euler equation is

$$y(x) = u(z) = e^{rz}(c_1 + c_2z) = x^r(c_1 + c_2 \ln x).$$

If $r_1, r_2 = \lambda + i\omega$, then the general solution of the Euler equation is

$$y(x) = u(z) = e^{\alpha z}(c_1 \cos(\omega z) + c_2 \sin(\omega z)) = x^\alpha(c_1 \cos(\omega \ln x) + c_2 \sin(\omega \ln x)).$$

■

Example 8.10 Find the general solution of

$$x^2y'' - 5xy' + 9y = 0$$

on $(0, \infty)$.

Solution *The indicial equation is*

$$r(r-1) - 5r + 9 = 0$$

which has a repeated root $r = 3$.

Therefore, the general solution of the equation on $(0, \infty)$ *is*

$$y = x^3(c_1 + c_2 \ln x).$$

■

Example 8.11 Find the general solution of

$$x^2y'' + 3xy' + 2y = 0$$

on $(0, \infty)$.

Solution *The indicial equation is*

$$r(r-1) + 3r + 2 = 0$$

which has two complex solutions $r_1, r_2 = -1 \pm \mathrm{i}$

Therefore, the general solution is

$$y = \frac{1}{x}\left[c_1 \cos(\ln x) + c_2 \sin(\ln x)\right].$$

■

✍ **Exercise 8.9** Find the general solution of

$$x^2y'' + 5xy' + 4y = 0$$

on $(0, \infty)$.

Solution *The indicial equation is*

$$r(r-1) + 5r + 4 = 0$$

which has a repeated root $r = -2$.

Therefore, the general solution of the equation on $(0, \infty)$ *is*

$$y = x^{-2}(c_1 + c_2 \ln x).$$

■

✍ **Exercise 8.10** Find the general solution of

$$x^2 y'' + xy' + 4y = 0$$

on $(0, \infty)$.

Solution *The indicial equation is*

$$r(r-1) + r + 4 = 0$$

which has two complex solutions $r_1, r_2 = \pm 2\mathrm{i}$

Therefore, the general solution is

$$y = [c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)].$$

■

Week 9: Introduction to Laplace Transforms

11/08–11/11

9.1 Laplace Transform

Using infinite series to solve differential equations provides an new idea. Transforming a solution into a series can reduce the differential equation into algebraic equations of coefficients which are much easier to solve. From calculus, we know that power series are closely related to improper integrals. Indeed, the improper integrals $\int_0^\infty a(t)x^t \,\mathrm{d}\,t = \int_0^\infty a(t)e^{-st} \,\mathrm{d}\,t$ may be considered a continuous analogue of the power series $\sum_{n=0}^{\infty} a_n x^n$, where $-s = \ln x$ and t plays the role of n. The interested reader may watch the youtube video by Professor Arthur Mattuck for a detailed explanation. This observation suggests an idea of applying an integral transformation to reduce a differential equation to algebraic equations. One of such integral transformations is the Laplace transformation which transform function $f(t)$ in real variable is another function depending on a new variable s that can be a complex number in general.

Definition of the Laplace Transform

Definition 9.1

Let $f(t)$ be a function in a real varaiable t. The Laplace transform $\mathcal{L}f$ of f is a function of s defined by

$$\mathcal{L}(f)(s) \ := \int_0^\infty f(t)e^{-st}\mathrm{d}t,$$

whose domain consists of all values of s such that the improper integral converges. ♣

In the definition, the value s may be taken to be complex. For simplicity, we assume that $s > 0$. We often denote $\mathcal{L}(f)(s)$ by $F(s)$.

By the definition of the improper integral,

$$F(s) = \int_0^\infty f(t)e^{-st} \,\mathrm{d}\,t = \lim_{N\to\infty} \int_0^N f(t)e^{-st} \,\mathrm{d}\,t.$$

Example 9.1 Find the Laplace transform for the function $f(t) = 1$.

Solution *By a substitution $u = -st$, the improper integral is*

$$\mathcal{L}(1) = \int_0^\infty e^{-st} \,\mathrm{d}\,t = \lim_{N\to\infty} \int_0^N e^{-st} \,\mathrm{d}\,t = \lim_{N\to\infty} \int_0^{-sN} \frac{1}{-s} e^u \,\mathrm{d}\,u = -\frac{1}{s} \lim_{N\to\infty} \left(e^{-sN} - e^0\right) = \frac{1}{s}$$

if $s > 0$. ■

Example 9.2 Find the Laplace transform for the function $f(t) = t^n$.

Solution *Applying integration by parts iteratedly, the indefinite integral $\int t^n e^{-st}\, \mathrm{d}\, t$ is*

$$\begin{aligned}
\mathcal{L}(t^n) &= \int t^n e^{-st}\, \mathrm{d}\, t \\
&= -\frac{1}{s} t^n e^{-st} + \frac{1}{s}\int n t^{n-1} e^{-st}\, \mathrm{d}\, t \\
&= -\frac{1}{s} t^n e^{-st} - \frac{1}{s^2} n t^{n-1} e^{-st} + \frac{1}{s^2}\int n(n-1) t^{n-2} e^{-st}\, \mathrm{d}\, t \\
&\ \vdots \\
&= -\frac{1}{s} t^n e^{-st} - \frac{1}{s^2} n t^{n-1} e^{-st} - \cdots - \frac{n \cdot \ \cdots \ \cdot 2}{s^n} t e^{-st} - \frac{n!}{s^{n+1}} e^{-st}.
\end{aligned}$$

Since $\lim\limits_{N\to\infty} \frac{t^k}{e^{-st}} = 0$ for any finite k, the improper integral is

$$\int_0^\infty t^n e^{-st}\, \mathrm{d}\, t = \lim_{N\to\infty}\int_0^N t^n e^{-st}\, \mathrm{d}\, t = \frac{n!}{s^{n+1}}$$

if $s > 0$. ■

Exercise 9.1 Find the Laplace transform for the function $f(t) = e^{rt}$.

Solution *By the definition,*

$$\mathcal{L}(e^{rt}) = \int_0^\infty e^{rt} e^{-st}\, \mathrm{d}\, t = \int_0^\infty e^{(r-s)t}\, \mathrm{d}\, t.$$

From the first example, we know that

$$\int_0^\infty e^{(r-s)t}\, \mathrm{d}\, t = \frac{1}{s-r}$$

if $s > r$. ■

Exercise 9.2 Find the Laplace transforms of $f(t) = \sin(\omega t)$ and $g(t) = \cos(\omega t)$ where ω is a constant.

Solution *Let $F(s) = \mathcal{L}(f)(s)$ and $G(s) = \mathcal{L}(g)(s)$. Assume $s > 0$. Applying integration by parts yields*

$$F(s) = -\frac{e^{-st}}{s}\sin(\omega t)\Big|_0^\infty + \frac{\omega}{s}\int_0^\infty e^{-st}\cos(\omega t)\,\mathrm{d}\,t = \frac{\omega}{s}G(s).$$

Similarly,

$$G(s) = -\frac{e^{-st}\cos(\omega t)}{s}\Big|_0^\infty - \frac{\omega}{s}\int_0^\infty e^{-st}\sin(\omega t)\,\mathrm{d}\,t = \frac{1}{s} - \frac{\omega}{s}F(s).$$

Therefore,

$$G(s) = \frac{1}{s} - \frac{\omega^2}{s^2}G(s).$$

Solving for G yields

$$G(s) = \frac{s}{s^2 + \omega^2}, \quad s > 0.$$

Therefore,

$$F(s) = \frac{\omega}{s} \cdot \frac{s}{s^2 + \omega^2} = \frac{\omega}{s^2 + \omega^2}, \quad s > 0.$$

■

Linearity of the Laplace Transform

Because integration is a linear operator, that is, integration of linear combination is the linear combination of integrations. The Laplace transform is also a linear operator.

Theorem 9.1

Suppose $\mathcal{L}(f_i)$ is defined for $s > s_i$, $1 \le i \le 2$. Let s_0 be the largest of the numbers s_1 and s_2, and let c_1 and c_2 constants. Then

$$\mathcal{L}(c_1 f_1 + c_2 f_2) = c_1\mathcal{L}(f_1) + c_2\mathcal{L}(f_2) \qquad \text{for } s > s_0.$$

Proof By the linear combination law of integration, we get

$$\mathcal{L}(c_1 f_1 + c_2 f_2) = \int_0^\infty (c_1 f_1 + c_2 f_2)e^{-st}\,\mathrm{d}\,t = c_1\int_0^\infty f_1 e^{-st}\,\mathrm{d}\,t + c_2\int_0^\infty f_2 e^{-st}\,\mathrm{d}\,t = c_1\mathcal{L}(f_1) + c_2\mathcal{L}(f_2).$$

The equalities hold if $s \ge s_0$. Because both improper integral converge if s is no less than $s_0 = \max\{s_1, s_2\}$. ■

Recall the the hyperbolic trigonometric functions $\sinh x$ and $\cosh x$ are

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x \frac{e^x + e^{-x}}{2}.$$

Example 9.3 Find the $\mathcal{L}(\sinh(bt))(s)$.

Solution *By the linearity and definitions, we get*

$$\mathcal{L}(\sinh(bt))(s) = \mathcal{L}\left(\frac{e^{bt} - e^{-bt}}{2}\right) = \frac{1}{2}\left(\mathcal{L}(e^{bt}) - \mathcal{L}(e^{-bt})\right) = \frac{1}{2}\left(\frac{1}{s-b} - \frac{1}{s+b}\right) = \frac{b}{s^2 - b^2}.$$

if $s > b$. ■

✍ **Exercise 9.3** Find the $\mathcal{L}(\cosh(bt))$.

Solution *By the linearity and definitions, we get*

$$\mathcal{L}(\cosh(bt))(s) = \mathcal{L}\left(\frac{e^{bt}+e^{-bt}}{2}\right) = \frac{1}{2}\left(\mathcal{L}(e^{bt})+\mathcal{L}(e^{-bt})\right) = \frac{1}{2}\left(\frac{1}{s-b}+\frac{1}{s+b}\right) = \frac{s}{s^2-b^2}.$$

if $s > b$.

s-shifting property

From the definition of Laplace transform, it is not hard to see that the Laplace transform of the product $e^{rt}f(t)$ is a shift of the Laplace transform of f.

Theorem 9.2 (s-Shifting Property)

Suppose the Laplace transform

$$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}\,t$$

is defined for $s > s_0$. *Then* $\mathcal{L}(f)(s-r)$ *is the Laplace transform of* $e^{rt}f(t)$ *for* $s > s_0 + r$.

Proof By the definition,

$$\int_0^\infty e^{rt}f(t)e^{-st}\,\mathrm{d}\,t = \int_0^\infty f(t)e^{-(s-r)t}\,\mathrm{d}\,t = \mathcal{L}(f)(s-r).$$

Example 9.4 Find

$$\mathcal{L}(t^n e^{rt}).$$

Solution *By the shifting theorem,*

$$\mathcal{L}(t^n e^{rt})(s) = \mathcal{L}(t^n)(s-r) = \frac{n!}{(s-r)^{n+1}}.$$

Exercise 9.4 Find

$$\mathcal{L}(sin(\omega t)e^{rt}) \qquad \text{and} \qquad \mathcal{L}(cos(\omega t)e^{rt})$$

Solution *By the shifting theorem,*

$$\mathcal{L}(sin(\omega t)e^{rt})(s) = \mathcal{L}(sin(\omega t))(s-r) = \frac{\omega}{(s-r)^2+\omega^2},$$

$$\mathcal{L}(cos(\omega t)e^{rt})(s) = \mathcal{L}(cos(\omega t))(s-r) = \frac{s}{(s-r)^2+\omega^2}.$$

Remark Using the identities

$$\cos t = \frac{e^{it} + e^{-it}}{2} \qquad \text{and} \qquad \sin t = \frac{e^{it} - e^{-it}}{2}$$

and the linearity theorem, one can also find the Laplace transforms of $\cos(\omega t)$ and $\sin(\omega t)$ using the shifting theorem.

Change of scale

By a linear substitution $u = at$, one can find the Laplace transform $\mathcal{L}(f(at))$ in terms of $\mathcal{L}(f(t))$.

Theorem 9.3 (Change of Scale)

Suppose the Laplace transform

$$\mathcal{L}(f(t))(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}\,t$$

is defined for $s > s_0$. Then

$$\mathcal{L}(f(at))(s) = \frac{1}{a}\mathcal{L}(f)\left(\frac{s}{a}\right)$$

for $s > as_0$.

Proof Let $u = at$. Then

$$\mathcal{L}(f(at))(s) = \int_0^\infty f(at)e^{-st}\,\mathrm{d}\,t = \frac{1}{s}\int_0^\infty f(u)e^{-\frac{su}{a}}\,\mathrm{d}\,u = \frac{1}{s}\mathcal{L}(f)\left(\frac{s}{a}\right).$$

■

Transform of derivatives

The most important property of Laplace transform for differential equations may be the relation between $\mathcal{L}(f')$ and $\mathcal{L}(f)$.

Theorem 9.4 (Transform of the first derivative)

Suppose the Laplace transform

$$\mathcal{L}(f(t))(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}\,t$$

is defined for $s > s_0$. Then

$$\mathcal{L}(f'(t))(s) = s\mathcal{L}(f)(s) - y(0)$$

for $s > s_0$.

Proof The definition implies

$$\mathcal{L}(f'(t))(s) = \int_0^\infty f'(t)e^{-st}\,\mathrm{d}\,t = -f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,\mathrm{d}\,t = s\mathcal{L}(f)(s) - f(0)$$

for $s > s_0$.

Note in the third equality, we used the fact that $\lim_{N\to\infty} f(t)e^{-st} = 0$ if $\int_0^\infty f(t)e^{-st}\,\mathrm{d}\,t$ converges. ■

Example 9.5 Suppose the Laplace transform $\mathcal{L}(f)(s)$ is defined for $s > s_0$. Find the Laplace transform $\mathcal{L}(f'')$ in terms of $\mathcal{L}(f)$, $f'(0)$ and $f(0)$.

Solution *Applying the above theorem repeatedly yields*

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0) = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

■

9.2 Existence and Additional Properties*

Existence of Laplace transforms

Not every function has a Laplace transform. For example,

$$\int_0^\infty e^{-st}e^{t^2}\,dt$$

diverges for all s because $\lim_{t\to\infty} e^{-st}e^{t^2} = \infty$. Since definite integrals define for functions that are continuous with at most jumping or removable discontinuities, Laplace transform may be defined for piece-wise continuous function.

Definition 9.2

A function f is **piecewise continuous** *on a finite closed interval $[a, b]$ if $f(a+)$ and $f(b-)$ are finite and f is continuous on the open interval (a, b) except possibly at finitely many points where f may have jump discontinuities or removable discontinuities.*
A function f is **piecewise continuous** *on the infinite interval $[a, \infty)$ if it is piecewise continuous on $[a, b]$ for every $b > a$.* ♣

Example 9.6 The ***unit step function*** or ***Heaviside step function*** defined as follows

$$u(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geq 0. \end{cases}$$

is piecewise continuous over the infinite interval $(-\infty, \infty)$

However, piecewise continuity alone does not guarantee the existence of Laplace transform. An addition sufficient condition is that the function grows slower that e^{-st} for some $s = s_0$ as t goes to infinity.

Definition 9.3

A function f is of **exponential order** s_0 *if there are constants M and t_0 such that*

$$|f(t)| \le Me^{s_0 t}, \quad t \ge t_0.$$

When value of s_0 is irrelevant, we simply say that f is of exponential order.

Theorem 9.5 (Existence of Laplace Transform)

If f is piecewise continuous on $[0, \infty)$ and of exponential order s_0, then $\mathcal{L}(f)$ is defined for $s > s_0$.

Proof Since f is piecewise continuous on $[0, \infty)$, the integral $\int_0^N f(t)e^{-st}\,\mathrm{d}\,t$ is defined for all N, so is the improper integral $\int_0^\infty f(t)e^{-st}\,\mathrm{d}\,t$ as long as it converges.

Since f is of exponential order s_0, we know that

$$|f(t)| \le Me^{s_0 t}.$$

Note that the improper integral

$$\int_0^\infty Me^{s_0 t}e^{-st}\,\mathrm{d}\,t = M\int_0^\infty e^{-(s-s_0)t}\,\mathrm{d}\,t = \frac{M}{s-s_0}.$$

for $s > s_0$. By the comparison test, $|F(s)| = \int_0^\infty |f(t)|e^{-st}\,\mathrm{d}\,t$ converges for $s > s_0$ and so is $\int_0^\infty f(t)e^{-st}\,\mathrm{d}\,t$.

Therefore, as long as f is piecewise continuous and of exponential order, the Laplace transform of f exists. ■

From the proof, we see that $\lim_{s\to\infty} F(s) = 0$. Indeed, as long as the Laplace transform of f exists, it always true that $\lim_{s\to\infty} F(s) = 0$.

Corollary 9.1

Suppose that the Laplace transform $F(s)$ of $f(t)$ exists for $s > s_0$. Then

$$\lim_{s\to\infty} F(s) = 0.$$

Proof It suffice to show that $F(s)$ cam be expressed as a Laplace transform of a bounded function.

Let $g(t) = \int_0^t e^{-s_1 x} f(x)\,\mathrm{d}\,x$, where $s_1 > s_0$. Then $g'(t) = e^{-s_1 t}f(t)$ and $g(t)$ is bounded because

$|g(t)| < |F(s_1)|$. By integrating by parts,

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}\,t \\
&= \int_0^\infty e^{-(s-s_1)t} e^{-s_1 t} f(t)\,\mathrm{d}\,t \\
&= \int_0^\infty e^{-(s-s_1)t}\,\mathrm{d}\,g(t) \\
&= e^{-(s-s_1)t} g(t)\Big|_0^\infty + (s-s_1)\int_0^\infty e^{-(s-s_1)t} g(t)\,\mathrm{d}\,t \\
&= (s-s_1)\int_0^\infty e^{-(s-s_1)t} g(t)\,\mathrm{d}\,t.
\end{aligned}
$$

■

From the corollary, we know that s^n, $\sin s$, $\cos s$, e^s and $\ln s$ can not be Laplace transforms.

Remark The identity $F(s) = (s-s_1)\int_0^\infty e^{-(s-s_1)t} g(t)\,\mathrm{d}\,t$ can also be used to show that $F(s)$ is analytic (see, for example, Doetsch 1974, Chapter 6).

Example 9.7 Find the Laplace transform of the delayed unit step function $u_a(t) = u(t-a)$ with $a > 0$, that is

$$
u_a(t) = \begin{cases} 0 & \text{for } \; t < a, \\ 1 & \text{for } \; t \geq a. \end{cases}
$$

Solution *The improper integral can be calculated directly:*

$$
\int_0^\infty u_a(t) e^{-st}\,\mathrm{d}\,t = \int_0^a 0 \cdot e^{-st}\,\mathrm{d}\,t + \int_a^\infty 1 \cdot e^{-st}\,\mathrm{d}\,t = -\frac{1}{s} e^{-st}\Big|_a^\infty = \frac{1}{s e^{sa}}.
$$

■

Transforms of piecewise functions

Using the unit step function, a piecewise function

$$
f(t) = \begin{cases} f_1(t) & 0 \leq t < t_1 \\ f_2(t) & t_1 \leq t < t_2 \\ \vdots & \\ f_n(t) & t_{n-1} \leq t \end{cases}
$$

can be expresses as

$$
f(t) = (u(t-t_1) - u(t)) f_1(t) + (u(t-t_1) - u(t-t_1)) f_2(t) + \cdots + u(t-t_{n-1}) f_n(t).
$$

Theorem 9.6 (t-Shifting Theorem)

Let f be a function such that $\mathcal{L}(f(t+b))(s)$ exist for $s > s_0$. Then the Laplace transform of the product function $u(t-a)f(t-a)$, where u is the unit step function, is

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}\mathcal{L}(f(t))(s).$$

Proof By the definition of the unit step function,

$$\begin{aligned}\int_0^\infty u(t-a)f(t-a)e^{-st}\,\mathrm{d}\,t = &\int_a^\infty f(t-a)e^{-st}\,\mathrm{d}\,t\\ =&e^{-as}\int_a^\infty f(t-a)e^{-s(t-a)}\,\mathrm{d}(t-a)\\ =&e^{-as}\int_0^\infty f(x)e^{-sx}\,\mathrm{d}\,x\\ =&e^{-as}\mathcal{L}(f(t))(s).\end{aligned}$$

■

Example 9.8 Find the Laplace transform

$$\mathcal{L}(u(t-a)f(t+b))(s).$$

Solution *Let $g(t) = f(t+a+b)$. Then*

$$\mathcal{L}(u(t-a)f(t+b))(s) = \mathcal{L}(u(t-a)g(t-a))(s) = e^{-as}\mathcal{L}(g(t))(s) = e^{-as}\mathcal{L}(f(t+a+b))(s).$$

■

Example 9.9 Find the Laplace transform

$$\mathcal{L}(u(t-1)t^2)(s).$$

Solution *Let $g(t) = (t+1)^2 = t^2+2t+1$. Then $g(t-1) = t^2$ and*

$$\mathcal{L}(u(t-1)t^2)(s) = e^{-s}\mathcal{L}(t^2+2t+1)(s) = e^{-s}\left(\frac{2}{s^3}+\frac{1}{s^2}+\frac{1}{s}\right).$$

■

Example 9.10 Find the Laplace transform of the function

$$f(t) = \begin{cases} t+2, & 0 \le t < 2,\\ 3t, & t \ge 2.\end{cases}$$

Solution *Using the unit step function u, the function f can be written as*

$$f(t) = u(t)(t+2) - u(t-2)(t+2) + 3tu(t-2) = u(t)(t+2) + u(t-2)(2t-2).$$

Then

$$\begin{aligned}\mathcal{L}(f(t))(s) =& \mathcal{L}(u(t)(t+2))(s) + \mathcal{L}(u(t-2)(2t-2))(s)\\ =& \mathcal{L}(t+2)(s) + e^{-2s}\mathcal{L}(2(t+2)-2)(s)\\ =& \mathcal{L}(t+2)(s) + 2e^{-2s}\mathcal{L}(t+1)(s)\\ =& \left(\frac{1}{s^2}+\frac{2}{s}\right) + 2e^{-2s}\left(\frac{1}{s^2}+\frac{1}{s}\right).\end{aligned}$$

■

Transforms of integrals

For some differential equations, computing the Laplace transform of an integral may be necessary.

Theorem 9.7

Let f be piecewise continuous function of exponential order s_0 and $g(t) = \int_0^t f(x)\,\mathrm{d}x$. Then

$$\mathcal{L}(g)(s) = \frac{1}{s}\mathcal{L}(f)(s).$$

Proof Note that $g'(t) = f(t)$ and $g(0) = 0$ By the existence theorem and the theorem of transform of derivative, we get

$$\mathcal{L}(f)(s) = s\mathcal{L}(g)(s) - g(0) = s\mathcal{L}(g)(s)$$

which implies

$$\mathcal{L}(g)(s) = \frac{1}{s}\mathcal{L}(f)(s).$$

■

Tables of Laplace Transforms

$\mathcal{L}\left(t^n\right)(s) = \dfrac{n!}{s^{n+1}}$	$\mathcal{L}\left(e^{at}t^n\right)(s) = \dfrac{1}{(s-a)^{n+1}}$
$\mathcal{L}\left(e^{at}\cos(bt)\right)(s) = \dfrac{s-a}{(s-a)^2+b^2}$	$\mathcal{L}\left(e^{at}\sin(bt)\right)(s) = \dfrac{b}{(s-a)^2+b^2}$
$\mathcal{L}\left(e^{at}\cosh(bt)\right)(s) = \dfrac{s-a}{(s-a)^2-b^2}$	$\mathcal{L}\left(e^{at}\sinh(bt)\right)(s) = \dfrac{b}{s^2-b^2}$

Table 9.1: Table of Basic Laplace Transforms

Linearity	$\mathcal{L}(c_1 f_1 + c_2 f_2) = c_1\mathcal{L}(f_1) + c_2\mathcal{L}(f_2)$
s-shifting	$\mathcal{L}(e^{at} f(t))(s) = \mathcal{L}(f(t))(s-a)$
t-shifting	$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}\mathcal{L}(f(t))(s)$
change of scale	$\mathcal{L}(f(at))(s) = \frac{1}{a}\mathcal{L}(f(t))\left(\frac{t}{a}\right)$
Transform of f'	$\mathcal{L}(f')(s) = \mathcal{L}(f)(s) - f(0)$
Transform of f''	$\mathcal{L}(f'')(s) = \mathcal{L}(f)(s) - f'(0)s - f(0)$
Transform of integral	$\mathcal{L}(\int_0^t f(x)\,\mathrm{d}\,x)(s) = \dfrac{\mathcal{L}(f)(s)}{s}$

Table 9.2: Table of Rules of Laplace Transform

Derivatives of Transforms

Since Laplace transforms are analytic, the derivatives exists.

Theorem 9.8

Let f be a piecewise continuous function of exponential order r. Then

$$\frac{\mathrm{d}}{\mathrm{d}\,s}\mathcal{L}(f)(s) = \mathcal{L}\left(-tf(t)\right)(s), \quad \text{for } s > r$$

Proof Since f is of exponential order, the improper integral converges absolutely. Hence, the derivative and the integral are interchangeable:

$$\begin{aligned} F'(s) &= \frac{\mathrm{d}}{\mathrm{d}\,s}\int_0^\infty f(t)e^{-st}\,\mathrm{d}\,t \\ &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\,s} f(t)e^{-st}\,\mathrm{d}\,t \\ &= \int_0^\infty -tf(t)e^{-st}\,\mathrm{d}\,t \\ &= \mathcal{L}(-tf(t))(s). \end{aligned}$$

■

The above theorem holds true without the assumption that f is of exponential order. In this cae, one needs to express $F(s)$ in terms of a transform of a bounded function $g(t) = \int_0^t f(x)e^{-s_1 x}\,\mathrm{d}\,x$. We refer the interested read to (Doetsch 1974, Chapter 6) for this generality.

Example 9.11 Find the Laplace transform

$$\mathcal{L}(t\cos t).$$

Solution *Applying the theorem of derivative of transform to $f(t) = -\cos t$ yields*

$$\mathcal{L}(t\cos t) = \frac{\mathrm{d}}{\mathrm{d}\,s}\mathcal{L}(-\cos t) = -\frac{\mathrm{d}}{\mathrm{d}\,s}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}.$$

■

Integrals of Transforms

The integral of a Laplace transform can be calculated using the derivative.

Theorem 9.9

Let f be a piecewise continuous function of exponential order r such that $\lim_{t\to 0^+}\frac{f(t)}{t}$ exists. Then

$$\int_s^\infty \mathcal{L}(f)(x)\,\mathrm{d}\,x = \mathcal{L}\left(\frac{f(t)}{t}\right)(s).$$

Proof Applying the theorem of derivative of transform to $\mathcal{L}\left(\frac{f(t)}{t}\right)(s)$ yields

$$\frac{\mathrm{d}}{\mathrm{d}\,s}\mathcal{L}\left(\frac{f(t)}{t}\right)(s) = \mathcal{L}\left(-t\left(\frac{f(t)}{t}\right)\right)(s) = -\mathcal{L}(f)(s).$$

Then by the fundamental theorem of calculus,

$$\mathcal{L}\left(\frac{f(t)}{t}\right)(s) = -\int_a^s \mathcal{L}(f)(x)\,\mathrm{d}\,x.$$

Since $\lim_{s\to\infty}\mathcal{L}(f)(s) = 0$, taking $a = \infty$ yields the equality

$$\mathcal{L}\left(\frac{f(t)}{t}\right)(s) = \int_s^\infty \mathcal{L}(f)(x)\,\mathrm{d}\,x.$$

■

Example 9.12 Find the Laplace transform

$$\mathcal{L}\left(\frac{\sin t}{t}\right).$$

Solution

$$\mathcal{L}\left(\frac{\sin t}{t}\right) = \int_s^\infty \mathcal{L}(\sin t)(s)\,\mathrm{d}\,s = \int_s^\infty \frac{1}{s^2+1}\,\mathrm{d}\,s = \frac{\pi}{2} - \tan^{-1}(s) = \cot^{-1}(s) = \tan^{-1}\left(\frac{1}{s}\right).$$

■

9.3 Inverse Laplace Transforms

In order to solve differential equations using Laplace transforms, inverse Laplace transforms must exist and be unique.

Definition 9.4

Given a function $F(s)$, the **inverse Laplace transform** *of F, denoted by $\mathcal{L}^{-1}(F)$, is a function f such that $\mathcal{L}(f)(s) = F(s)$.* ♣

For example, $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t$ because $\mathcal{L}(e^t)(s) = \frac{1}{s-1}$.

The uniqueness in general may not be true because two functions with different discontinuities may have the same Laplace transform. For example, $\mathcal{L}(1)(s) = \mathcal{L}(u)(s)$, where u is the unit step function.

Fortunately, discontinuities are the only differences that two functions can have if they have the same Laplace transform. This result is known as Lerch's theorem.

Theorem 9.10 (Lerch's Theorem)

Let f and g be two functions having the same Laplace transform: $\mathcal{L}(f)(s) = \mathcal{L}(g)(s)$. If f and g are piecewise continuous, then $f(t) = g(t)$ for any t where f and g are continuous. In particular, if f and g are continuous, then $f = g$. ♥

We refer the reader to (Doetsch 1974, Chapter 5) for a proof.

Properties of inverse Laplace transforms

Like calculating antiderivatives, some inverse Laplace tranforms can be calculated using those basic Laplace transforms and properties of inverse Laplace transforms.

By the uniqueness theorem and linearity of Laplace transform, we see that the inverse Laplace transform is also a linear operator.

Theorem 9.11

Suppose the inverse Laplace transforms for $F(s)$ and $G(s)$ exist. Then

$$\mathcal{L}^{-1}(c_1F_1(s) + c_2F_2(s)) = c_1\mathcal{L}^{-1}(F_1(s)) + c_2\mathcal{L}^{-1}(F_2(s)).$$ ♥

Example 9.13 Find the inverse transform

$$\mathcal{L}^{-1}\left(\frac{1}{s-1} + \frac{1}{s-2}\right)$$

Solution

$$\mathcal{L}^{-1}\left(\frac{1}{s-1}+\frac{1}{s-2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)+\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{t}+e^{2t}$$

■

Exercise 9.5 Find the inverse transform

$$\mathcal{L}^{-1}\left(\frac{1}{s}+\frac{2}{s^2}\right)$$

Solution

$$\mathcal{L}^{-1}\left(\frac{1}{s}+\frac{2}{s^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\right)+\mathcal{L}^{-1}\left(\frac{2}{s^2}\right) = 1+2t.$$

■

Theorem 9.12 (Shifting Inverse Transforms)

If $f(t)=\mathcal{L}^{-1}(F(s))$, then

1. $$\mathcal{L}^{-1}(F(s-a)) = e^{at}f(t),$$

2. $$\mathcal{L}^{-1}(e^{-as}F(s)) = u(t-a)f(t-a).$$

♥

Example 9.14 Find the inverse transform

$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right).$$

Solution

$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) = e^{-t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = te^{-t}.$$

■

Example 9.15 Find

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+2s+2}\right).$$

Solution

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{s}{s^2+2s+2}\right) &= \mathcal{L}^{-1}\left(\frac{s}{(s+1)^2+1}\right)\\
&= \mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1}\right)-\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right)\\
&= e^{-t}\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)-e^{-t}\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)\\
&= e^{-t}\cos(st)-e^{-t}\sin(st).
\end{aligned}$$

■

The method of partial fractional decomposition

Using the Laplace transform to solve differential equations often requires finding the inverse transform of a rational function

$$F(s) = \frac{P(s)}{Q(s)}.$$

By the fundamental theorem of algebra, a rational expression $\frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials and $\deg P(s) < \deg Q(s)$, can always be written as the sum of rational expressions whose denominator are powers of linear or irreducible quadratic polynomials.

Theorem 9.13

Let P and Q are nonzero polynomials. Assume that $\deg P < \deg Q$. Write Q as a produce of powers of distinct irreducible polynomials of degree at most two:

$$Q(s) = \prod_{i}^{n} p_i^{n_i}.$$

Then there are unique polynomials a_{ij} with $\deg a_{ij} < \deg p_i$ such that

$$\frac{P}{Q} = \sum_{i}^{n} \sum_{j}^{n_i} \frac{a_{ij}}{p_i^j}.$$

♥

Example 9.16 Find the inverse Laplace transform of

$$F(s) = \frac{s}{s^2 - s - 2}.$$

Solution *Factoring the denominator yeilds*

$$F(s) = \frac{s}{(s+1)(s-2)}.$$

By the partial fraction decomposition theorem, F has an expression

$$F(s) = \frac{A}{s+1} + \frac{B}{s-2}.$$

Note that

$$A = (s+1)F(s) - \frac{B(s+1)}{s-2} = \frac{s}{s-2} - \frac{B(s+1)}{s-2},$$

$$B = (s-2)F(s) - \frac{A(s-2)}{s+1)} = \frac{s}{s+1} - \frac{A(s-2)}{s+1)}.$$

Because the equities should hold true for all s. Taking $s = -1$ implies

$$A = \frac{-1}{-1-2} = \frac{1}{3}.$$

Taking $s = 2$ implies

$$B = \frac{2}{2+1} = \frac{2}{3}.$$

Therefore,

$$F(s) = \frac{1}{3}\frac{1}{s+1} + \frac{2}{3}\frac{1}{s-2}.$$

By linearity,

$$\mathcal{L}^{-1}(F(s)) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \frac{2}{3}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = \frac{1}{3}e^{-t} + \frac{2}{3}e^{2t}.$$

■

Example 9.17 Find the inverse transform of

$$F(s) = \frac{1}{s(s-1)^2}.$$

Solution *The function F can be decomposed at*

$$F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}.$$

Applying the evaluation method as in the above example yields

$$A = sF(s)\Big|_{s=0} = \frac{1}{(s-1)^2}\Big|_{s=0} = 1,$$

$$C = (s-1)^2F(s)\Big|_{s=1} = \frac{1}{s}\Big|_{s=1} = 1.$$

Subtracting $\frac{C}{(s-1)^2}$ from the decomposition and then applying the evaluation method yields

$$B = \left((s-1)F(s) - \frac{1}{s-1}\right)\Big|_{s=1} = \frac{1-s}{s(s-1)}\Big|_{s=1} = -\frac{1}{s}\Big|_{s=1} = -1.$$

Therefore,

$$F(s) = \frac{1}{s} - \frac{1}{s-1} + \frac{1}{(s-1)^2}.$$

The inverse transform is

$$\mathcal{L}^{-1}(F(s)) = 1 - e^t - te^t.$$

■

Example 9.18 Find the inverse Laplace transform of

$$F(s) = \frac{s^2 + 1}{s(s^2 + 2s + 2)}$$

Solution *The function F can be written as the decomposition*

$$F(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 2}.$$

Then

$$A = sF(s)\Big|_{s=0} = \frac{s^2 + 1}{s^2 + 2s + 2}\Big|_{s=0} = \frac{1}{2}.$$

Note that $s^2 + 2s + 2 = (s + 1)^2 + 1$ which has two roots $s = -1 \pm \mathrm{i}$. Then

$$B(-1 + \mathrm{i}) + C = (s^2 + 2s + 2)F(s)\Big|_{s=-1+\mathrm{i}} = \frac{s^2 + 1}{s}\Big|_{s=-1+\mathrm{i}} = -\frac{3}{2} + \frac{1}{2}\mathrm{i} \tag{1}$$

$$B(-1 - \mathrm{i}) + C = (s^2 + 2s + 2)F(s)\Big|_{s=-1-\mathrm{i}} = \frac{s^2 + 1}{s}\Big|_{s=-1-\mathrm{i}} = -\frac{3}{2} + \frac{1}{2}\mathrm{i}. \tag{2}$$

Solving the liner system implies

$$B = \frac{1}{2} \qquad C = -1.$$

Therefore,

$$F(s) = \frac{1}{2} \cdot \frac{1}{s} + \frac{1}{2}\frac{s - 2}{(s + 1)^2 + 1}.$$

To get the inverse transform using the table, the second term needs to be rewritten as

$$\frac{1}{2}\frac{s - 2}{(s + 1)^2 + 1} = \frac{1}{2}\frac{s + 1}{(s + 1)^2 + 1} - \frac{3}{2} \cdot \frac{1}{(s + 1)^2 + 1}.$$

Then

$$\begin{aligned}
\mathcal{L}^{-1}(F(s)) =& \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{s + 1}{(s + 1)^2 + 1}\right) - \frac{3}{2}\mathcal{L}^{-1}\left(\frac{1}{(s + 1)^2 + 1}\right) \\
=& \frac{1}{2} + \frac{1}{2}e^{-t}\cos(t) - \frac{3}{2}e^{-t}\sin(t).
\end{aligned}$$

■

Week 10: Solving Differential Equations using Laplace Transforms

11/15–11/18

10.1 Solving IVP using Laplace Transforms

Consider the initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_1.$$

Applying the Laplace transform to both sides yields

$$\begin{aligned}
\mathcal{L}(ay'' + by' + cy)(s) &= \mathcal{L}(f)(s) \\
a\mathcal{L}(y'')(s) + b\mathcal{L}(y')(s) + c\mathcal{L}(y)(s) &= \mathcal{L}(f)(s) \\
a[s^2\mathcal{L}(y)(s) - y_0 s - y_1] + b[s\mathcal{L}(y)(s) - y(0)] + c\mathcal{L}(y)(s) &= \mathcal{L}(f)(s) \\
(as^2 + bs + c)\mathcal{L}(y)(s) - [ay_0 s - ay_1 - by_0] &= \mathcal{L}(f)(s) \\
\mathcal{L}(y)(s) &= \frac{\mathcal{L}(f)(s) + ay_0 s + ay_1 + by_0}{as^2 + bs + c}.
\end{aligned}$$

Applying the Inverse Laplace transform yields

$$y(t) = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(f)(s) + ay_0 s + ay_1 + by_0}{as^2 + bs + c}\right).$$

Example 10.1 Use the Laplace transform to solve the initial value problem

$$y'' - 6y' + 5y = 3e^{2t}, \quad y(0) = 2, \quad y'(0) = 3.$$

Solution *Using the formulas*

$$\mathcal{L}(y')(s) = s\mathcal{L}(y) - y(0),$$

$$\mathcal{L}(y'')(s) = s^2\mathcal{L}(y) - sy(0) - y'(0),$$

and applying the Laplace transform to both sides yields

$$(s^2 - 6s + 5)\mathcal{L}(y) = \frac{3}{s-2} + 2(s-6) + 3.$$

Therefore,

$$\mathcal{L}(y) = \frac{3}{(s-2)(s-1)(s-5)} + \frac{2s-9}{(s-1)(s-5)}.$$

Applying the partial fractional decomposition method to the right hand side yields

$$\frac{3}{(s-2)(s-1)(s-5)} + \frac{2s-9}{(s-1)(s-5)} = \frac{5}{2}\cdot\frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{2}\cdot\frac{1}{s-5}.$$

Therefore,

$$\begin{aligned} y =& \mathcal{L}^{-1}\left(\frac{5}{2}\cdot\frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{2}\cdot\frac{1}{s-5}\right) \\ =& \frac{5e^t}{2} - e^{2t} + \frac{e^{5t}}{2}. \end{aligned}$$

■

Example 10.2 Solve the initial value problem

$$y'' + 2y' + 2y = 3, \quad y(0) = 1, \quad y'(0) = 1.$$

Solution *Applying the Laplace transform to the equation and solve for $\mathcal{L}(y)$ implies*

$$\mathcal{L}(y) = \frac{3}{s(s^2+2s+2)} + \frac{s+3}{s^2+2s+2}.$$

The partial fractional decomposition is

$$\frac{3}{s(s^2+2s+2)} + \frac{s+1}{s^2+2s+2} = \frac{3}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s+1}{(s+1)^2+1} - \frac{3}{2}\cdot\frac{1}{(s+1)^2+1}.$$

Therefore,

$$\begin{aligned} y =& \mathcal{L}^{-1}\left(\frac{3}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s+1}{(s+1)^2+1} + \frac{1}{2}\cdot\frac{1}{(s+1)^2+1}\right) \\ =& \frac{2}{3} - \frac{1}{2}e^{-t}\cos t + \frac{1}{2}e^{-t}\sin t. \end{aligned}$$

■

✍ **Exercise 10.1** Solve the initial value problem

$$y'' + y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution *Applying the Laplace transform to the equation yields*

$$\mathcal{L}(y) = \frac{1}{s^2+1}.$$

Then

$$y = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t.$$

■

✍ **Exercise 10.2** Solve the initial value problem

$$y'' + 25y = 9e^t, \quad y(0) = 0, \quad y'(0) = 2.$$

Solution *Applying the Laplace transform to the equation yields*

$$\mathcal{L}(y) = \frac{1}{s^2 + 25} \cdot \left(\frac{9}{s-1} + 2 \right).$$

The partial fractional decomposition of the right hand side is

$$\frac{9}{(s-1)(s^2+25)} + \frac{2}{s^+25} = -\frac{9}{26} \cdot \frac{s}{s^2+25} + \frac{43}{130} \cdot \frac{5}{s^2+25} + \frac{9}{26} \cdot \frac{1}{s-1}.$$

Then

$$\begin{aligned} y =& \mathcal{L}^{-1}\left(-\frac{9}{26} \cdot \frac{s}{s^2+25} + \frac{43}{5} \cdot \frac{5}{s^2+25} + \frac{9}{26} \cdot \frac{1}{s-1} \right) \\ =& -\frac{9}{26}\cos(5t) + \frac{43}{130}\sin(5t) + \frac{9}{26}e^t. \end{aligned}$$

■

10.2 Convolutions*

Let $F(s) = \mathcal{L}(f(t))(s)$ and $G(s) = \mathcal{L}(g(t))(s)$. The inverse transform $\mathcal{L}^{-1}(F(s)G(s))$ can be calculated by an integral of f and g.

Theorem 10.1

Let $F(s) = \mathcal{L}(f(t))(s)$ and $G(s) = \mathcal{L}(g(t))(s)$. Then

$$\mathcal{L}^{-1}(F(s)G(s)) = \int_0^t f(t-x)g(x)\mathrm{d}x.$$

♥

Proof Since changing the dummy variable of integration won't change the integral, the function $F(s)G(s)$ is determined by

$$F(s)G(s) = \int_0^\infty e^{-sx} f(x)\mathrm{d}x \int_0^\infty e^{-st} g(y)\mathrm{d}y.$$

By Fubini's theorem, the order of integration in this case is interchangeable. Together with change of coordinates, it implies that

$$
\begin{aligned}
\int_0^\infty e^{-sx}f(x)\mathrm{d}x \int_0^\infty e^{-sy}g(y)\mathrm{d}y &= \int_0^\infty \int_0^\infty e^{-s(x+y)}f(x)g(y)\mathrm{d}x\mathrm{d}y \\
&= \int_0^\infty \left(\int_0^\infty e^{-s(x+y)}f(x)\mathrm{d}x\right) g(y)\mathrm{d}y \\
&= \int_0^\infty \left(\int_y^\infty e^{-su}f(u-y)\mathrm{d}u\right) g(y)\mathrm{d}y \qquad \text{where } u = x+y \\
&= \int_0^\infty \int_y^\infty \left(e^{-sx}f(x-y)g(y)\right)\mathrm{d}x\mathrm{d}y \\
&= \int_0^\infty \int_0^x \left(e^{-sx}f(x-y)g(y)\right)\mathrm{d}y\mathrm{d}x \quad \text{change of coordinates} \\
&= \int_0^\infty e^{-s(x)}\left(\int_0^x f(x-y)g(y)\mathrm{d}y\right)\mathrm{d}x \\
&= \mathcal{L}\left(\int_0^y f(x-y)g(y)\mathrm{d}x\right)(s).
\end{aligned}
$$

That completes the proof. ■

This theorem motivates the definition of convolution.

Definition 10.1

The **convolution** $f * g$ *of two functions* f *and* g *is defined by*

$$(f * g)(t) = \int_0^t f(x)g(t-x)\mathrm{d}x.$$

♣

Applying a substitution $u = t - x$ and changing the dummy variable u back to x implies

$$(f * g)(t) = \int_0^t f(x)g(t-x)\mathrm{d}x = -\int_t^0 f(t-u)g(u)\mathrm{d}u = \int_0^t g(x)f(t-x)\mathrm{d}x = (g * f)(t).$$

Example 10.3 Let $f(t) = e^{at}$ and $g(t) = e^{bt}$. Find $\mathcal{L}(f * g)$.

Solution *By the theorem of convolution,*

$$\mathcal{L}(f * g) = \mathcal{L}(e^{at})\mathcal{L}(e^{bt}) = \frac{1}{(s-a)(s-b)}.$$

■

Example 10.4 Find $\mathcal{L}^{-1}\left(\frac{1}{s(s-1)^2}\right)$.

Solution *From the table of Laplace transform* $\frac{1}{s} = \mathcal{L}(1)(s)$ *and* $\frac{1}{s^2} = \mathcal{L}(t)(s)$*. By the* s*-shifting theorem,*

$$\frac{1}{(s-1)^2} = \mathcal{L}(t)(s-1) = \mathcal{L}(te^t)(s).$$

Therefore, by the theorem of convolution,

$$\mathcal{L}^{-1}\left(\frac{1}{s(s-1)^2}\right) = (1) * (te^t) = \int_0^t 1 * xe^x \mathrm{d}x = xe^x - e^x + 1.$$

■

✍ **Exercise 10.3** Find $\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}\right)$.

Solution *Note that* $\frac{1}{s^2} = \mathcal{L}(t)(s)$ *and* $\mathcal{L}(\sin(t))(s)$. *Therefore,*

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}\right) = \int_0^t (t-x)\sin x \mathrm{d}x = ((x-t)\cos x)|_0^t - \sin x|_0^t = t - \sin t.$$

■

10.3 Dirac Delta Function*

In real life, there are situations that the systems acted by sudden external forces. The external forces may or may not apply for a period. For example, an electrical surge caused by a lightning strike. In general terms, there is an impulse force whose magnitude can be infinitely large.

In Mathematics, an idea to understand an impulse force is to approximate the force function by a piecewise continuous function.

Consider the function unit step function

$$\delta_h(t) = \begin{cases} \frac{1}{h} & \text{for } 0 < t \leq h \\ 0 & \text{otherwise,} \end{cases}$$

which represents a force of a magnitude $\frac{1}{h}$ during the time period 0 to h.

The limit of this function $\delta_h(t)$ as h goes to 0, denoted as $\delta(t)$, is called the ***Dirac delta function*** (or simply ***delta function***, or ***unit impulse function***). That is

$$\delta(t) = \lim_{h\to 0} \delta_h(t) = \begin{cases} \infty & t = 0 \\ 0 & \text{otherwise.} \end{cases}$$

For any real number x, we define

$$\delta(t-x) = \lim_{h\to 0} \delta_h(t-x) = \begin{cases} \infty & t = x \\ 0 & \text{otherwise.} \end{cases}$$

Remark In defining $\delta(t)$ as a limit, the shape of the impulse sequence $\delta_h(t)$ turns out irrelevant. For example, Dirac delta function $\delta(t)$ can also be defined as the limit of the sequence of Gaussian functions:

$$\delta(t) = \lim_{h \to 0} \frac{1}{h\sqrt{\pi}} e^{-\frac{x}{h^2}}.$$

The interested reader is referred to (Bracewell 1999, Chapter 5) for more details.

The Dirac delta function is not a function in the traditional sense as $\delta(0) = \infty$. It is a generalized function in the sense that it is function from the space of functions to the real line.

Note that

$$\int_{-\infty}^{\infty} \delta_h(t)\,\mathrm{d}\,t = \int_0^h \delta_h(t)\,\mathrm{d}\,t = 1$$

which yields

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}\,t = 1.$$

Note that $\int_{-\infty}^{\infty} \lim_{h \to 0} \delta_h(t) f(t)\,\mathrm{d}\,t = 0$ for any function. The integral is not meaningful if it is understood in this way. So the integral $\int_{-\infty}^{t} \delta(x-a)\,\mathrm{d}\,x$ really should be understood as $\lim_{b \to 0} \int_{-\infty}^{t} \delta_b(x-a)\,\mathrm{d}\,x$. After all, the function $\delta(t)$ is not a real function, the meaning of the integration involving $\delta(t)$ should be generalized too. In the rest of the section, we shall interpret operations involving Dirac delta function, such as multiplication, integration, and differentiation in the way of performing the operation first and then taking the limit. This interpretation of operations involving $\delta(t)$ is compatible with other approaches of Dirac delta function. One of such approaches is to define the Dirac delta function as the derivative of the unit step function in the following sense:

$$\delta(t-a) = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\int_{-\infty}^{t} \delta(x-a)\,\mathrm{d}\,x\right) = \frac{\mathrm{d}}{\mathrm{d}\,t}(u(t-a)) = u'(t-a).$$

This integration property together with $\delta(t) = 0$ for $t \neq 0$ gives the formal characterization of Dirac delta function, that is, the Dirac delta function is a generalized function (or distribution) with the following properties:

1. $\delta(t) = 0$ if $t \neq 0$, and
2. $\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}\,t = 1$, where the integral may be understand as the Riemann–Stieltjes integral.

An important property which highlights the viewpoint of $\delta(t)$ being a function over the space of functions.

Theorem 10.2 (Sifting Property)

For any function f continuous near x,

$$\int_a^b \delta(t-x)f(t)\,\mathrm{d}\,t = \begin{cases} f(x) & \text{for } x \in (a,b) \\ 0 & \text{for } x \notin [a,b]. \end{cases}$$

Proof If $x \notin [a,b]$, then $\delta(t-x) = 0$ on $[a,b]$. So is the integral. Suppose that x is in (a,b). Using the fundamental theorem of calculus, we get

$$\begin{aligned}\int_a^\delta (t-x)f(t)\,\mathrm{d}\,t &= \lim_{h\to 0}\int_a^b \delta_h(t-x)f(t)\,\mathrm{d}\,t \\ &= \lim_{h\to 0}\int_x^{x+h}\frac{1}{h}f(t)\,\mathrm{d}\,t \\ &= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,\mathrm{d}\,t}{h} \\ &= f(x).\end{aligned}$$

■

As a consequence, for any $a > 0$, the Laplace transform of $\delta(t-a)$ is

$$\mathcal{L}(\delta(t-a))(s) = \int_0^\infty \delta(t-a)e^{-st}\,\mathrm{d}\,t = e^{-as}.$$

In particular,

$$\mathcal{L}(\delta(t))(s) = 1.$$

The Dirac delta function can be used to solve the following initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = 0, \quad y'(0) = 0,$$

where $a > 0$. Applying the Laplace transform implies

$$\mathcal{L}(y)(s) = \frac{\mathcal{L}(f)(s)}{as^2 + bs + c}.$$

Write the inverse transform of $\frac{1}{as^+ bs+c}$ as

$$w(t) = \mathcal{L}^{-1}\left(\frac{1}{as^+ bs + c}\right),$$

which is called the ***unit impulse response*** (or ***weight function***). Then applying the theorem of convolution yields

$$y = \mathcal{L}^{-1}\left(\mathcal{L}(f(t))(s)\cdot\frac{1}{as^2 + bs + c}\right) = f(t) * w(t) = \int_0^t f(t-x)w(x)\,\mathrm{d}\,x.$$

Because of the commutativity of convolution, the solution y can also be calculated by

$$y = \int_0^t f(x)w(t-x)\,\mathrm{d}\,x.$$

In particular, if $f(t) = \delta(t)$, then the function $w(t)$ is nothing but the solution of the initial value problem

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

This is the reason why it is called the unit impulse response.

Example 10.5 Solve the initial value problem

$$y'' + y' - 2y = 3e^{5t}, \quad y(0) = 0, y'(0) = 0.$$

Solution *The unit impulse function is determined by*

$$\begin{aligned} w(t) =& \mathcal{L}^{-1}\left(\frac{1}{s^2+s-2}\right) \\ =& \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s-1} - \frac{1}{s+2}\right) \\ =& \frac{1}{3}(e^t - e^{-2t}). \end{aligned}$$

Then the solution of the equation is

$$\begin{aligned} y =& \int_0^t (e^x - e^{-2x})e^{5(t-x)}\,\mathrm{d}\,x \\ =& e^{5t}\int_0^t (e^{-4x} - e^{-7x})\,\mathrm{d}\,x \\ =& e^{5t}\left(-\frac{e^{-4x}}{4} + \frac{e^{-7x}}{7}\right)\Big|_0^t \\ =& e^{5t}\left(-\frac{e^{-4t}}{4} + \frac{e^{-7t}}{7} - \frac{3}{28}\right) \\ =& \frac{e^{-2t}}{7} - \frac{e^t}{4} - \frac{3e^{5t}}{28}. \end{aligned}$$

■

Week 11: Linear System of Equations

11/22–11/30

11.1 Basics of Linear Algebra

A system of linear equations (or simply a linear system) consists of m linear equations in n variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n &= b_m. \end{aligned} \tag{11.1}$$

Linear algebra provides an operational way to solve system of linear equations. Here, the operators are matrix operations.

Matrices

An $m \times n$ matrix A is a number array of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

The element in the i-th row and j-th column is called the ij-th element (or entry).

For simplicity, a matrix is also denoted by its ij-th element, for example, the matrix A can be denote by $A = \left(a_{ij}\right)$.

An $m \times 1$ matrix is called a column vector. A $1 \times n$ matrix is called a row vector.

Example 11.1 The matrix

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

is a column vector.

Two matrices $A = \left(a_{ij}\right)$ and $B = \left(b_{ij}\right)$ are said to be equal if $a_{ij} = b_{ij}$ for $1 \le i \le m$ and

$1 \leq j \leq n$.

A matrix whose element are all equal to 0 is denoted by 0.

Using matrices, the linear system 11.1 can be written as

$$AX = 0.$$

The scalar product of a constant k and a matrix $A = \left(a_{ij}\right)$ is defined as

$$kA = \left(ka_{ij}\right).$$

The sum of two matrices $A = \left(a_{ij}\right)$ and $B = \left(b_{ij}\right)$ is defined as

$$A + B = \left(a_{ij} + b_{ij}\right)$$

The produce AB of two matrices $A = \left(a_{ij}\right)$ and $B = \left(b_{ij}\right)$ is defined if the number of columns of A equals the number of rows of B. If A is a $m \times p$ matrix and B is an $p \times n$ matrix, then the product AB is an $m \times n$ matrix defined as

$$AB = \left(\sum_{k=1}^{n} a_{ik} b_{kj}\right).$$

Note that in general $AB \neq BA$.

A square matrix is a matrix with the same number of rows and column. The number of rows or columns of a square matrix is called the order of the matrix.

The diagonal elements of a square matrix $A = \left(a_{ij}\right)$ are the elements a_{ij} with $i = j$.

A diagonal matrix is a square matrix whose non-diagonal elements are all equal to 0.

An identity matrix, denoted by I, is a diagonal matrix whose diagonal elements are all equal to 1. It is a straightforward result that $IA = A = AI$ for A and I have the same order.

Determinant

The determinant $\left(A\right)$ of a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is defined as

$$\left(A\right) = a_{11}a_{22} - a_{12}a_{21}.$$

Geometrically, the determinant represents the signed area of the parallelograms generated by the column vectors of the matrix. In higher dimension, it represents the signed volume of the parallelepiped.

The determinant $\left(A_{ij}\right)$ of the $(n-1)\times(n-1)$ matrix A_{ij} obtained from a $n \times n$ matrix A by removing the p-th row and the k-th column is called the ij-th minor of A.

The number $C_i j = (-1)^{i+j}\left(A_{ij}\right)$ is called the cofactor of the element a_{ij} of A.

The determinant $\left(A\right)$ of a $n \times n$ matrix $A\left(a_{ij}\right)$ is defined as

$$\left(A\right) = \sum_{k=1}^{n} a_{1k}C_{1k}.$$

It is a fundamental result in linear algebra that

$$\left(A\right) = \sum_{k=1}^{n} a_{ij}C_{ij}.$$

A square matrix A is said to be nonsingular if the determinant is nonzero. Otherwise, it is said to be singular.

Example 11.2 Find the determinant of the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}.$$

Solution *From the definition, the determinant is*

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = 1\cdot 1 - 2\cdot 3 = 1 - 6 = -5.$$

■

Inverse Matrices

A square matrix B is called an inverse of a square matrix A if it satisfies both

$$AB = I \qquad \text{and} BA = I.$$

A square matrix is called invertible if it has an inverse.

A main result about matrix inverse is the following theorem.

Theorem 11.1

Let A be a square matrix.

1. *A is invertible if and only if A is non-singular.*
2. *if A is invertible, then its inverse is unique, which is usually denoted by A_{-1}.*

Example 11.3 Determine if the following matrix is invertible:

$$A = \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}.$$

Solution *The determinant of the matrix A is*

$$\det \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} = 3 \cdot 4 - 2 \cdot 6 = 0.$$

So the matrix is not invertible.

Here is another way to see that it not invertible. Note that

$$\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If the matrix A is invertible, then multiplying the inverse A_{-1} to both sides yields

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which is a contradiction. ■

This example highlights a more general result.

Theorem 11.2

The determinant of a matrix with two proportional row or columns is equal to 0.

Finding the inverse matrix of a given matrix is essentially to solve linear systems.

Example 11.4 Find the inverse matrix of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Solution *Since the determinant* $\left(A\right) = 1$, *this the inverse matrix* A_{-1} *exists. Write*

$$A_{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

From the definition, the elements a, b, c *and* d *satisfies the following equations*

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Simplifying the product yields

$$\begin{pmatrix} c & d \\ -a & -b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, $a = 0$, $b = -1$, $c = 1$ *and* $d = 0$. *It can be checked that*

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the inverse matrix of A *is*

$$A_{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

■

Applying the method in the above example to the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \det\left(A\right) \neq 0$$

yields

$$A^{-1} = \frac{1}{\left(A\right)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

The transpose of a matrix A, denoted by A_T, is the matrix whose ij-th element is the ji-th element of A.

For a higher order square matrix, there is also a formula for the inverse.

Theorem 11.3 (Cofactor formula)

The inverse matrix of $A = \left(a_{ij}\right)$ *is*

$$A_{-1} = \frac{1}{\left(A\right)} \left(C_{ij}\right)^{T},$$

where C^{ij} *is the cofactor of the* ij*-th element of* A*.*

Be aware of the transpose to the matrix $\left(C_{ij}\right)$.

The cofactor formula is better for theoretical applications. In practice, to find the inverse matrix, the Gauss-Jordan elimination method is frequently used. The idea is to solve n equations together by elimination using row operations:

1. interchanging two rows;
2. multiplying a row by a nonzero number;
3. adding a scalar multiple of one row to another row.

Example 11.5 Find the inverse of the matrix

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}.$$

Solution *First form the matrix*

$$\begin{pmatrix} 2 & 5 & & 1 & 0 \\ 1 & 3 & & 0 & 1 \end{pmatrix}.$$

The goal is two use row operations to change the left square to the identity matrix.

$$\begin{pmatrix} 2 & 5 & | & 1 & 0 \\ 1 & 3 & | & 0 & 1 \end{pmatrix} \xrightarrow{\text{switch rows}} \begin{pmatrix} 1 & 3 & | & 0 & 1 \\ 2 & 5 & | & 1 & 0 \end{pmatrix} \xrightarrow{\text{2row1-row2}} \begin{pmatrix} 1 & 3 & | & 0 & 1 \\ 0 & 1 & | & -1 & 2 \end{pmatrix} \xrightarrow{\text{row1-3row2}} \begin{pmatrix} 1 & 0 & | & 3 & -5 \\ 0 & 1 & | & -1 & 2 \end{pmatrix}$$

So the inverse matrix is

$$\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix},$$

which is exactly the one given by the cofactor formula. ■

The reason that this method works is that each of the row operations can be viewed as multiply the matrix from the left by a so-called elementary matrix.

The cofactor formula can be used to deduce the Crammer's rule for linear systems.

Theorem 11.4 (Crammer's rule)

Let $AX = B$ be a linear system, where X and B are column vectors and X is the vector of unknowns. If A is invertible, then

$$X = \left(\frac{\left(A(i|B)\right)}{\left(A\right)}\right),$$

where $A(i|B)$ is the matrix obtained from A by replacing the i-th column by the column vector B.

Example 11.6 Solve the linear system

$$\begin{aligned} x_1 - 2x_2 &= 1 \\ x_1 + x_2 &= 4 \end{aligned}$$

Solution *One can solve this linear system using the Gauss elimination method. Here, let's try the Crammer's rule. Write*

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Then

$$A(1|B) = \begin{pmatrix} 1 & -2 \\ 4 & 1 \end{pmatrix} \qquad A(2|B) = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}.$$

Calculating the determinants of A, $A(1|B)$ and $A(2|B)$ yields

$$\det\left(A\right) = 3, \qquad \det\left(A(1|B)\right) = 9, \qquad \det\left(A(2|B)\right) = 3$$

Therefore,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 9 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Eigenvectors and Eigenvalues

Let A be a square matrix. A non-zero vector $\vec{v}$ is called an eigenvector of A with the eigenvalue λ if they satisfy

$$A\vec{v} = \lambda\vec{v}.$$

A constant λ is an eigenvalue with the non-zero eigenvector $\vec{v}$ if and only if

$$(A - \lambda I)\vec{v} = 0$$

if and only if

$$\left(A - \lambda I\right) = 0.$$

The polynomial $p(x) = \left(A - xI\right)$ is called the characteristic polynomial of the matrix A. A eigenvalue λ is of multiplicity k if $p(x) = (x - \lambda)^k q(x)$ where $q(x)$ is a polynomial had no more factor $x - \lambda$.

Theorem 11.5

Let A be a square matrix. A constant λ is an eigenvalue of A if and only if it is a root of the characteristic polynomial of A.

For a 2×2 matrix A, denote by $\mathrm{tr}(A)$ the sum of diagonal elements of A, which is called the trace of A, and $\det(A)$ the determinant of A, then the characteristic polynomial of A is

$$p(x) = x^2 - \mathrm{tr}(A)x + \det(A).$$

There is also a trick to find an eigenvector of a 2×2 matrix A. Suppose that λ is an eigenvalue of A. If

$$A - \lambda I = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then the vector

$$\begin{pmatrix} -b \\ a \end{pmatrix}$$

is an eigenvector. The reason is that the determinant of $A - \lambda I$ is zero which implies that (c, d) is a multiple of (a, b).

Example 11.7 Find the eigenvalues and their corresponding eigenvectors of the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}.$$

Solution *The determinant of the matrix is*

$$\det \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} = -4.$$

The trace of the matrix is

$$1 + 2 = 3.$$

So the characteristic polynomial of the matrix is

$$p(x) = x^2 - 3x - 4.$$

Solving the equation $p(x) = 0$ yields

$$x = -1 \qquad \text{or} \qquad x = 4.$$

Therefore, the eigenvalues are -1 and 4.

Since

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

An eigenvector of the eigenvalue -1 can be taken to be

$$\vec{v} = \begin{pmatrix} 2 \\ -2 \end{pmatrix},$$

or

$$\vec{v} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

A similar calculation implies that

$$\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

is an eigenvector of the eigenvalue 4. ■

The usefulness of eigenvalues and eigenvectors is highlighted by the following theorem.

Theorem 11.6 (Diagonolization Theorem)

Suppose the eigenvalues $\lambda_1, \ldots, \lambda_n$ of a $n \times n$ matrix A are distinct. Let $\vec{v}_i$ be the eigenvector of λ_i. Write $P = (\vec{v}_1 \cdots \vec{v}_n)$. Then

$$PAP^{-1} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & \cdots & \lambda_n \end{pmatrix}.$$

When the characteristic polynomial of the matrix A has a repeated root λ, more vectors other than eigenvectors will be needed to form an invertible square matrix P.

A vector $\vec{v}$ is called a generalized eigenvector associated to the eigenvalue λ of the matrix A if

$$(A - \lambda I)^k \vec{v} = 0 \qquad \text{for some } k.$$

Theorem 11.7 (Jordan Normal Form)

For a square matrix A, there exist a matrix M consists of independent generalized eigenvectors such that

$$MAM^{-1} = \begin{pmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & \cdots & J_r \end{pmatrix},$$

where J_i is a $k \times k$ square matrix in the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix},$$

and λ_i is an eigenvalue of order k.

Example 11.8 Find the generalized eigenvectors of the matrix

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

Solution *The trace is $2+0 = 2$ and the determinant is $2\times 0-(-1)\times 1 = 1$. Therefore, the characteristic polynomial is $p(x) = x^2 - 2x + 1$ which has a repeated root $x = 1$.*

Subtracting the identity matrix from the given matrix yields

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

So the eigenvector associate to the eigenvalue is

$$\vec{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

A generalized vector in this equation is the a vector $\vec{v}$ that satisfies the following linear system

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Solving the linear system yields that

$$\vec{v} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

For 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the characteristic polynomial $p(x) = x^2 - (a+d)x + \det(A)$ has a repeated root if and only if $x^2 - (a+d)x + \det(A) = \left(x - \frac{a+d}{2}\right)^2$. It follows that the repeated root is $\lambda = \frac{a+d}{2}$ and $\left(\frac{a-d}{2}\right)^2 = -bc$. Then

$$A - \lambda I = \begin{pmatrix} \frac{a-d}{2} & b \\ c & \frac{d-a}{2} \end{pmatrix}.$$

Then

$$\vec{u} = \begin{pmatrix} -b \\ \frac{a-d}{2} \end{pmatrix}$$

is an eigenvector.

Solving the linear system

$$\begin{pmatrix} \frac{a-d}{2} & b \\ c & \frac{d-a}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -b \\ \frac{a-d}{2} \end{pmatrix}$$

yields that the column vector

$$\vec{v} = \vec{v} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

is a generalized eigenvector.

Calculus of Matrices

If the elements of a matrix A are functions of a variable t, then A is called a matrix of functions of t.

A matrix of functions $A(x) = \left(a_{ij}(x)\right)$ is said to be continuous at t_0 if all $a_{ij}(x)$ are continuous at t_0. It is differential at t_0 if all $a_{ij}(x)$ are differential at t_0. We write the derivate of $A(x)$ as

$$\frac{\mathrm{d}\,A}{\mathrm{d}\,t}(x) = A'(x) = \left(a'_{ij}(x)\right).$$

Applying the chain rule to the product of matrix yields

$$(A(x)B(x))' = A'(x)B(x) + A(x)B'(x).$$

Similarly, the integral of a matrix of functions $A(x) = \left(a_{ij}(x)\right)$ is defined as

$$\int_a^b A(x)\,\mathrm{d}\,t = \left(\int_a^b a_{ij}(x)\,\mathrm{d}\,t\right).$$

The Laplace transform of a matrix function $M(t)$ can be defined similarly as

$$\mathcal{L}(M(t))(s) = \int_0^{\infty} M(t)e^{-st}\,\mathrm{d}\,t.$$

Example 11.9 Let

$$A(x) = \begin{pmatrix} t & e^t \\ \sin t & \cos t \end{pmatrix}.$$

Find $A'(x)$, $\int_0^1 A(x)\,\mathrm{d}\,t$ and $\mathcal{L}A$.

Solution *From the definition, direct calculations shows that*

$$A'(x) = \begin{pmatrix} (x)' & (e^t)' \\ (\sin t)' & (\cos t)' \end{pmatrix} = \begin{pmatrix} 1 & e^t \\ \cos t & -\sin t \end{pmatrix},$$

$$\int_0^1 A(x)\,\mathrm{d}\,t = \begin{pmatrix} \int_0^1 t\,\mathrm{d}\,t & \int_0^1 e^t\,\mathrm{d}\,t \\ \int_0^1 \sin t\,\mathrm{d}\,t & \int_0^1 \cos t\,\mathrm{d}\,t \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & e-1 \\ \cos 1 - 1 & -1 \end{pmatrix},$$

and

$$\mathcal{L}(A)s = \begin{pmatrix} \mathcal{L}t(s) & \mathcal{L}e^t \\ \mathcal{L}\sin t & \mathcal{L}\cos t \end{pmatrix} = \begin{pmatrix} \frac{1}{s^2} & \frac{1}{s-1} \\ \frac{s}{s^2+1} & \frac{1}{s^2+1} \end{pmatrix}.$$

■

11.2 Linear Systems

A system of first-order differential equations of n unknown functions of a single independent variable x is a system of differential equations that can be written as

$$\begin{aligned} y_1' &= f_1(x, y_1, y_2, \dots, y_n) \\ y_2' &= f_2(x, y_1, y_2, \dots, y_n) \\ &\vdots \\ y_n' &= f_n(x, y_1, y_2, \dots, y_n) \end{aligned}$$

A system of first-order differential equations is called a ***linear system*** if it can be written as

$$\begin{aligned} y_1' &= a_{11}(x)y_1 + a_{12}(x)y_2 + \cdots + a_{1n}(x)y_n + f_1(x) \\ y_2' &= a_{21}(x)y_1 + a_{22}(x)y_2 + \cdots + a_{2n}(x)y_n + f_2(x) \\ &\vdots \\ y_n' &= a_{n1}(x)y_1 + a_{n2}(x)y_2 + \cdots + a_{nn}(x)y_n + f_n(x). \end{aligned}$$

One mathematical reason for studying systems is that an n-th order differential equation

$$y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$$

can always be turned into a linear system. Set

$$y_1 = y, y_2 = y', \cdots, y_n = y^{(n)}.$$

Then the n-th order differential equation is equivalent to the linear system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_n' &= f(x, y_1, y_2, \dots, y_n). \end{aligned}$$

In terms of matrices of functions, a linear system can be written as

$$\vec{y}' = A(x)\vec{y} + \vec{f}(x),$$

which suggests the following analogue of the existence and uniqueness theorem of the linear first-order differential equations.

Theorem 11.8 (Existence and uniqueness)

In the linear system $\vec{y}' = A(x)\vec{(y)} + \vec{f}(x)$, suppose the coefficient matrix $A(x)$ and the vector function $\vec{f}(x)$ are continuous on (a, b), let x_0 be in (a, b), and let $\vec{k}$ be an arbitrary constant n-vector. Then the initial value problem

$$\vec{y}' = A(x)\vec{y} + \vec{f}(x), \quad \vec{y}(x_0) = \vec{k}$$

has a unique solution on (a, b).

For convenience and clarity, we focus on linear systems of equations in two unknown functions:

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}' = \begin{pmatrix} a_1(x) & b_1(x) \\ a_2(x) & b_2(x) \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} + \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}. \tag{11.2}$$

A linear system is called a ***homogeneous linear system*** if

$$\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = 0$$

Example 11.10 Consider the following linear system

$$
\begin{aligned}
y_1' &= 2y_1 + 4y_2 \\
y_2' &= 4y_1 + 2y_2.
\end{aligned}
$$

Rewrite the linear system into matrix form and verify that the vector of functions

$$\vec{y} = c_1 \begin{pmatrix} e^{6t} \\ e^{6t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$$

is a solution.

Solution *The linear system can be written in matrix form as follows*

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The derivative of the vector function $\vec{y}$ is

$$\vec{y}' = c_1 \begin{pmatrix} 6e^{6t} \\ 6e^{6t} \end{pmatrix} + c_2 \begin{pmatrix} -2e^{-2t} \\ 2e^{-2t} \end{pmatrix}$$

Plugging the vector function into the right hand side of the linear system yields

$$
\begin{aligned}
&\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \left(c_1 \begin{pmatrix} e^{6t} \\ e^{6t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix} \right) \\
=& c_1 \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} e^{6t} \\ e^{6t} \end{pmatrix} + c_2 \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix} \\
=& c_1 \begin{pmatrix} 6e^{6t} \\ 6e^{6t} \end{pmatrix} + c_2 \begin{pmatrix} -2e^{-2t} \\ 2e^{-2t} \end{pmatrix} \\
=& c_1 \begin{pmatrix} 6e^{6t} \\ 6e^{6t} \end{pmatrix} + c_2 \begin{pmatrix} 2e^{-2t} \\ -2e^{-2t} \end{pmatrix}
\end{aligned}
$$

Therefore, the vector function $\vec{y}$ is a solution. ■

Theorems on linear systems

Before discussing how to solve linear systems, we first discuss the structure of general solutions.

Theorem 11.9

If $\vec{u}$ and $\vec{v}$ are two solutions of the homogeneous linear system $\vec{y}' = A\vec{y}$ on $[a, b]$, then

$$c_1\vec{u} + c_2\vec{v}$$

is also a solution on $[a, b]$ for any constants c_1 and c_2.

Proof Since $\vec{u}$ and $\vec{v}$ are solutions,

$$\vec{u}' = A\vec{u} \qquad \text{and} \qquad \vec{v}' = A\vec{v}.$$

Taking the linear combination yields

$$\begin{aligned}(c_1\vec{u} + c_2\vec{v})' =& c_1\vec{u}' + c_2\vec{v}' \\ =& c_1A\vec{u} + c_2A\vec{v} \\ =& A(c_1\vec{u} + c_2\vec{v}).\end{aligned}$$

So $c_1\vec{u} + c_2\vec{v}$ is also a solution. ■

Comparing with linear second-order equations which can be converted to a linear system, you may wondering if the general solution is a linear combination of two linearly independent solution. The answer is yes.

Given two vector functions $\vec{u}$ and $\vec{v}$, the Wronskian $W(x)$ of them is defined to be the determinant of the matrix $\begin{pmatrix}\vec{u} & \vec{v}\end{pmatrix}$.

Example 11.11 Find the Wronskian of the vector functions

$$\vec{u} = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}, \qquad \vec{v} = \begin{pmatrix} 2e^{t} \\ -e^{t} \end{pmatrix}.$$

Solution *From the definition, the Wronskian is*

$$\det\begin{pmatrix} e^{2t} & 2e^{t} \\ e^{2t} & -e^{t} \end{pmatrix} = -e^{2t}e^{t} - 2e^{2t}e^{t} = -3e^{3t}.$$

■

Theorem 11.10

If two solutions $\vec{u}$ and $\vec{v}$ of the linear system $\vec{y}' = A\vec{y}$ have a non-vanishing Wronskian on $[a, b]$, then $c_1\vec{u} + c_2\vec{v}$ is a general solution of the linear system.

The proof is similar to that of the second order differential equations.

Proof By the uniqueness theorem, it sufficiently to show that c_1 and c_2 can be chosen to satisfy arbitrary conditions $y_1(x_0) = a$ and $y_2(x_0) = b$. If the Wronskian does not vanish on $[a, b]$, then the coefficient matrix of the system of equations

$$\begin{aligned} c_1 u_1(x_0) + c_2 v_1(x_0) &= a \\ c_1 u_2(x_0) + c_2 v_2(x_0) &= b \end{aligned}$$

is invertible and hence

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} u_1(x_0) & v_1(x_0) \\ u_2(x_0) & v_2(x_0) \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix}.$$

This proves the theorem. ■

Now you may expect more analog properties between Wronskian for linear second-order differential equation and Wronskian for linear systems.

Theorem 11.11

The Wroskian $W(x)$ of two solutions $\vec{u}$ and $\vec{v}$ on $[a, b]$ of the homogeneous linear system $\vec{y}' = A\vec{y}$ is given by

$$W(x) = W(x_0) e^{\int_{x_0}^{x} \operatorname{tr} A(x)\, \mathrm{d}t},$$

where, $\operatorname{tr} A(x)$ is the sum of the diagonal elements. In particular, the Wronskian is either identically zero or nonwhere zero.

Proof Suppose

$$u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, \qquad v(x) = \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}, \qquad A(x) = \begin{pmatrix} a_1(x) & b_1(x) \\ a_2(x) & b_2(x) \end{pmatrix}.$$

Then

$$\begin{pmatrix} u_1'(x) & v_1'(x) \\ u_2'(x) & v_2'(x) \end{pmatrix} = \begin{pmatrix} a_1(x) & b_1(x) \\ a_2(x) & b_2(x) \end{pmatrix} \begin{pmatrix} u_1(x) & v_1(x) \\ u_2(x) & v_2(x) \end{pmatrix},$$

Applying Jacobi's formula yields

$$\begin{aligned} W'(x) &= W(x) \operatorname{tr} \left(\begin{pmatrix} u_1(x) & v_1(x) \\ u_2(x) & v_2(x) \end{pmatrix}^{-1} \begin{pmatrix} a_1(x) & b_1(x) \\ a_2(x) & b_2(x) \end{pmatrix} \begin{pmatrix} u_1(x) & v_1(x) \\ u_2(x) & v_2(x) \end{pmatrix} \right) \\ &= W(x) \operatorname{tr} \left(\begin{pmatrix} a_1(x) & b_1(x) \\ a_2(x) & b_2(x) \end{pmatrix} \begin{pmatrix} u_1(x) & v_1(x) \\ u_2(x) & v_2(x) \end{pmatrix} \begin{pmatrix} u_1(x) & v_1(x) \\ u_2(x) & v_2(x) \end{pmatrix}^{-1} \right) \\ &= W(x) \operatorname{tr}(A(x)). \end{aligned}$$

Solving this separable equation implies,

$$W(x) = W(x_0) e^{\int_{x_0}^{x} \operatorname{tr} A(t)\, \mathrm{d}t}.$$

Therefore, $W(x) \equiv 0$ if and only if $W(x_0) = 0$ for some x_0 which implies the second statement. ■

It follows directly from the above two theorems that the linear combination of two solutions is a general solution if and only if their Wronskian is non-zero.

Theorem 11.12

The linear combination of two solutions $\vec{u}$ and $\vec{v}$ on $[a, b]$ of the homogeneous linear system $\vec{y}' = A\vec{y}$ is a general solution if and only if the Wronskian is non-zero.

Conceptually, there are equivalent characterization of solutions that form a general solution.

Two solutions $\vec{u}$ and $\vec{v}$ of $\vec{y}' = A\vec{y}$ are called ***linearly independent*** if the only constants c_1 and c_2 such that

$$c_1\vec{u} + c_2\vec{v} = 0$$

are zero.

A set of solutions $\{\vec{u}, \vec{v}\}$ of $\vec{y}' = A\vec{y}$ is called ***fundamental set*** if every solution can be written as the linear combination of those solutions.

The matrix $\left(\vec{u} \mid \vec{v}\right)$ is called a ***fundamental matrix*** for $\vec{y}' = A\vec{y}$ if the column vectors $\vec{u}$ and $\vec{v}$ are linearly independent.

Example 11.12 Consider the linear system

$$\vec{y}' = A\vec{y},$$

where

$$A = \begin{pmatrix} -4 & -3 \\ 6 & 5 \end{pmatrix}.$$

1. Verify that the vector functions

$$\vec{u} = \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix}$$

   are solutions of the linear system on $(-\infty, \infty)$.
2. Compute the Wronskian of $\vec{u}$ and $\vec{v}$.
3. Find the general solution.

Solution *For the first two question, it is convenient to work with the matrix*

$$F(t) = \begin{pmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{pmatrix}.$$

The derivative of $F(t)$ is

$$F'(t) = \begin{pmatrix} -2e^{2t} & e^{-t} \\ 4e^{2t} & -e^{-t} \end{pmatrix}.$$

The product $A(t)F(t)$ is

$$A(t)F(t) = \begin{pmatrix} -4 & -3 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{pmatrix} = \begin{pmatrix} -2e^{2t} & e^{-t} \\ 4e^{2t} & -e^{-t} \end{pmatrix}$$

Therefore, $F'(t) = A(t)F(t)$ which verifies that $\vec{u}$ and $\vec{v}$ are solutions.

The Wronskian of $\vec{u}$ and $\vec{v}$ is

$$W(t) = \begin{pmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{pmatrix} = -e^{2t}e^{-t} - 2e^{2t}(-e^{-t}) = e^{t}.$$

Since $W(t) \not\equiv 0$, the solutions $\vec{u}$ and $\vec{v}$ are linearly independent and the linear combination

$$c_1 \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix}$$

is the general solution. ■

We now present the theorem about solutions of non-homogeneous linear systems.

Theorem 11.13

Let y_p be a particular solution of the linear system the linear system $\vec{y}' = A\vec{y} + \vec{f}$ and y_h be the general solution of the homogeneous linear system $\vec{y}' = A\vec{y}$. Then $\vec{y}_h + \vec{y}_p$ is the general solution of $\vec{y}' = A\vec{y} + \vec{f}$. ♥

Proof The statement follows from the fact that the difference of two solutions of $\vec{y}' = A\vec{y} + \vec{f}$ is a solution of $\vec{y}' = A\vec{y}$. ■

11.3 Constant Coefficient Homogeneous Linear Systems

Assume the matrix A of the homogeneous linear system $\vec{y}' = A\vec{y}$ is a matrix of constant functions. If you recall that the equation $y' = cy$ has the exponential solution $y = y(0)e^{ct}$, then it is natural to ask where $\vec{y} = e^{At}\vec{y(0)}$ is a solution of the linear system. This is indeed true if we make sense of e^{At}. Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. The matrix exponential e^{At} is defined as

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}.$$

Theorem 11.14

The linear system $\vec{y}' = A\vec{y}$ with $\vec{y(0)} = \vec{y}_0$ has the solution

$$\vec{y} = e^{At}\vec{y}_0.$$

The question is how to calculate e^{At}. In a special case, say A is a diagonal matrix:

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where $\lambda_1 \neq \lambda_2$, the power A^n is also diagonal and

$$A^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix},$$

which leads to the fact that

$$e^{At} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1(0)e^{\lambda_1 t} \\ y_2(0)e^{\lambda_2 t} \end{pmatrix}$$

is the solution.

For general cases, there are results from linear algebra that can be used to calculate e^{At}. For a constant coefficient homogeneous linear system of two unknown functions, we have the following results.

Theorem 11.15

Consider the constant coefficient homogeneous linear system of two unknown functions $\vec{y}' = A\vec{y}$.

Case 1: *If A has two distinct real roots λ_1 and λ_2, then the general solution is*

$$y = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2,$$

where $\vec{v}_1$ and $\vec{v}_2$ are associated eigenvectors of λ_1 and λ_2 respectively.

Case 2: *If A has two distinct complex eigenvalues $\lambda_1, \lambda_2 = \alpha + i\beta$, then the general solution is*

$$\vec{y} = c_1 e^{\alpha t}(\cos(\beta t)\vec{u} - \sin(\beta t)\vec{v}) + c_2 e^{\alpha t}(\cos(\beta t)\vec{v} + \sin(\beta t)\vec{u})$$

where $\vec{u} \pm \mathrm{i}\vec{v}$ are eigenvectors of $\alpha \pm \mathrm{i}\beta$ respectively.

Case 3: *If A has a repeated eigenvalue λ, then the general solution is*

$$\vec{y} = c_1 e^{\lambda t}\vec{u} + c_2(e^{\lambda t}\vec{v} + te^{\lambda t}\vec{u}),$$

where $\vec{u}$ is an eigenvector of λ and $\vec{v}$ is a vector satisfying $(A - \lambda I)\vec{v} = \vec{u}$.

Example 11.13 Solve the linear system

$$\vec{y}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\vec{y}$$

Solution *The characteristic polynomial is*

$$p(x) = x^2 - \mathrm{tr}(A)x + \det(A) = x^2 - 4x + 3$$

which has two roots $x = 1$ and $x = 3$.

Since

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

an associated eigenvector is

$$\vec{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - 3 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

an associated eigenvector is

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, the general solution is

$$\vec{y}' = c_1 e^{t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

■

Example 11.14 Solve the linear system

$$\vec{y}' = \begin{pmatrix} 4 & 6 \\ -3 & -2 \end{pmatrix}\vec{y}$$

Solution *The characteristic polynomial is*

$$p(x) = x^2 - \operatorname{tr}(A)x + \det(A) = x^2 - 2x + 10 = (x-1)^2 + 9$$

which has two complex roots $x = 1 \pm 3\mathrm{i}$.

Since

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - (1+3\mathrm{i}) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-3\mathrm{i} & 1 \\ 1 & 1-3\mathrm{i} \end{pmatrix},$$

an associated eigenvector is

$$\vec{u} = \begin{pmatrix} -1 \\ 1-3\mathrm{i} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mathrm{i}\begin{pmatrix} 0 \\ -3 \end{pmatrix}.$$

Therefore, the general solution is

$$\vec{y}' = c_1 e^t \left(\cos(3t) \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \sin(3t) \begin{pmatrix} 0 \\ -3 \end{pmatrix} \right) + c_2 e^t \left(\sin(3t) \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \cos(3t) \begin{pmatrix} 0 \\ -3 \end{pmatrix} \right).$$

■

Example 11.15 Solve the linear system

$$\vec{y}' = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} \vec{y}$$

Solution *The characteristic polynomial is*

$$p(x) = x^2 - \operatorname{tr}(A)x + \det(A) = x^2 + 2x + 1$$

which has a repeated roots $x = -1$.

Since

$$\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} - (-1) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},$$

an associated eigenvector is

$$\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solving the linear system

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

yields

$$\vec{v} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Therefore, the general solution is

$$\vec{y}' = c_1 e^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right).$$

■

11.4 Variation of Parameters for Nonhomogeneous Linear Systems*

For nonhomogeneous linear system $\vec{y}' = A\vec{y} + \vec{f}$, the method of variation of parameters can be used to find a particular solution. Suppose

$$Y = \left(\vec{y}_1 \mid \vec{y}_2 \right)$$

be a fundamental matrix of the complementary linear system $\vec{y}' = A\vec{y}$, a particular solution $\vec{y}_p$ can be expected in the form

$$\vec{y}_p = Y\vec{u},$$

where $\vec{u}$ is a column vector of functions to be determined. Note that

$$Y' = AY.$$

Plugging $\vec{y}_p$ into $\vec{y}' = A\vec{y} + \vec{f}$ and simplifying the equation yields

$$Y\vec{u}' = \vec{f}.$$

It follows that

$$\vec{u} = \int Y^{-1}\vec{f} \, \mathrm{d}\, t.$$

Example 11.16 Consider the linear system

$$\vec{y}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{y} + \begin{pmatrix} 2e^{4t} \\ e^{4t} \end{pmatrix}.$$

1. Find a particular solution of the system.
2. Find the general solution.

Solution *Solving the complementary homogeneous linear system*

$$\vec{y}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{y}$$

yields a fundamental matrix

$$Y = \begin{pmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t}. \end{pmatrix}$$

Since the determinant of Y is

$$\det(Y) = 2e^{2t},$$

the inverse matrix of Y is

$$Y^{-1} = \frac{1}{2e^{2t}} \begin{pmatrix} e^{3t} & -e^{3t} \\ e^{-t} & e^{-t} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{t} & -e^{t} \\ e^{-3t} & e^{-3t} \end{pmatrix}.$$

Therefore,

$$Y^{-1}\vec{f} = \frac{1}{2} \begin{pmatrix} e^{t} & -e^{t} \\ e^{-3t} & e^{-3t} \end{pmatrix} \begin{pmatrix} 2e^{4t} \\ e^{4t} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{5t} \\ 3e^{t} \end{pmatrix}$$

and the parameter vector is

$$\vec{u} = \frac{1}{2} \begin{pmatrix} \int e^{5t}\, \mathrm{d}\, t \\ \int 3e^{t}\, \mathrm{d}\, t \end{pmatrix} = \begin{pmatrix} \frac{1}{10} e^{5t} \\ \frac{3}{2} e^{t} \end{pmatrix}.$$

It follows that a particular solution is

$$\vec{y}_p = Y\vec{u} = \begin{pmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix} \begin{pmatrix} \frac{1}{10} e^{5t} \\ \frac{3}{2} e^{t} \end{pmatrix} = \begin{pmatrix} \frac{8e^{4t}}{5} \\ \frac{7e^{4t}}{4} \end{pmatrix}.$$

Therefore, the general solution is

$$\vec{y} = Y\vec{c} + \vec{y}_p = \begin{pmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \frac{8e^{4t}}{5} \\ \frac{7e^{4t}}{4} \end{pmatrix}.$$

■

11.5 Solving Linear System by Laplace Transform

Recall that the linear system $\vec{y}' = A\vec{y}$ has the solution $\vec{y} = e^{At}$. One way to find e^{At} is the use the Jordan normal form of A. Another method is to use Laplace transform.

Recall that the Laplace transform of the first derivative of a function y is

$$\mathcal{L}(y') = s\mathcal{L}(y) - y(0).$$

Therefore,

$$\begin{aligned}
\mathcal{L}((e^{At})')(s) =& \mathcal{L}(e^{At}) - e^{A\cdot 0}\\
\mathcal{L}(Ae^{At})(s) =& \mathcal{L}(e^{At}) - I\\
A\mathcal{L}(e^{At})(s) =& \mathcal{L}(e^{At}) - I\\
(A - sI)\mathcal{L}(e^{At}) =& -I\\
\mathcal{L}(e^{At}) =& -(A - sI)^{-1}\\
\mathcal{L}(e^{At}) =& (sI - A)^{-1}.
\end{aligned}$$

The matrix $sI - A$ is known as the characteristic matrix of the matrix A.

Example 11.17 Find the general solution of the linear system

$$\vec{y}' = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \vec{y}$$

Solution *The characteristic matrix is*

$$s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} s-1 & 1 \\ 0 & s-1 \end{pmatrix}.$$

The determinant of the characteristic matrix is $(s-1)^2$. *So the inverse matrix is*

$$\begin{pmatrix} s-1 & 1 \\ 0 & s-1 \end{pmatrix} = \begin{pmatrix} \frac{1}{s-1} & 0 \\ -\frac{1}{(s-1)^2} & \frac{1}{s-1} \end{pmatrix}.$$

Therefore, the fundamental matrix e^{At} *is*

$$\mathcal{L}^{-1}\left(\begin{pmatrix} \frac{1}{s-1} & 0 \\ -\frac{1}{(s-1)^2} & \frac{1}{s-1} \end{pmatrix}\right) = \begin{pmatrix} e^t & 0 \\ -te^t & e^t. \end{pmatrix}$$

The general solution is

$$\vec{y} = \begin{pmatrix} e^t & 0 \\ -te^t & e^t. \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

■

Week 12: Bounded Value Problems and Fourier Series

12/6–12/09

12.1 Introduction to Bounded Value Problems

In studying heat conduction, or wave propagation, the following type of equation will be employed

$$y'' + \lambda y = 0, \qquad y(0) = 0, \qquad y(L) = 0,$$

where λ is a real number and $L > 0$.

More generally, the conditions may be given as

$$ay(x_0) + by'(x_0) = \alpha \qquad cy(x_1) + dy'(x_1) = \beta,$$

which are called ***boundary conditions***. A differential equation with boundary conditions is called ***bounded value problems***.

Not like linear second-order differential equations with constant coefficients and initial conditions, with boundary conditions, the existence and number of solutions vary.

Example 12.1 Solve $y'' + y = 0$ with each of the following boundary conditions

1. $y(0) = 1$ and $y(\pi/2) = 0$.
2. $y(0) = 0$ and $y(\pi) = 1$.
3. $y(0) = 0$ and $y(\pi) = 0$.

Solution *Since the characteristic polynomial of the equation is $p(r) = r^2 + 1$ which has two roots $r = \pm i$. Then the general solution of the equation is*

$$y = c_1 \cos x + c_2 \sin x.$$

1. *The boundary condition $y(0) = 0$ implies that $c_1 = 1$. The boundary condition $y(\pi/2) = 0$ implies that $c_2 = 0$. So this boundary value problem has a unique solution $y = \cos x$.*
2. *The boundary condition $y(0) = 0$ implies that $c_1 = 0$. But the boundary condition $y(\pi) = 1$ implies that $c_1 = -1$. That's a contradiction, which means this boundary value problem has no solution.*
3. *The boundary conditions implies that $c_1 = 0$ and c_2 can be any numbers. So this boundary value problem has infinitely many solutions in the form $y = c \sin x$.*

■

The existence and number of solutions also depend on the value of λ.

Example 12.2 Solve $y'' + \frac{1}{4}y = 0$ with boundary conditions $y(0) = 0$ and $y(\pi) = 0$.

Solution *Since the characteristic polynomial of the equation is $p(r) = r^2 + \frac{1}{4}$ which has two roots $r = \pm\frac{1}{2}i$. Then the general solution of the equation is*

$$y = c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right).$$

The boundary condition $y(0) = 0$ implies that $c_1 = 0$. The boundary condition $y(\pi) = 1$ implies that $c_2 = 0$. So this boundary value problem has only a trivial solution $y = 0$. ■

The boundary value problem

$$y'' + \lambda y = f(x), \qquad ay(0) + by'(0) = \alpha \qquad cy(L) + dy'(L) = \beta$$

is called a ***nonhomogeneous boundary value problem*** over $[0, L]$ if either $f(x)$, α or β is nonzero. The associated ***homogeneous boundary value problem*** is given by setting $f = 0$, $\alpha = 0$ and $\beta = 0$.

Like initial value problems, solutions of nonhomogeneous boundary value problems are sums of a particular solution and solutions of the associated homogeneous boundary problem.

Lemma 12.1

Let u be a solution of the nonhomogeneous boundary value problem

$$y'' + \lambda y = f(x), \qquad ay(0) + by'(0) = \alpha \qquad cy(L) + dy'(L) = \beta.$$

Then any solution y of this nonhomogeneous boundary problem is given by

$$y = u + z$$

for some solution z of the associated homogeneous boundary problem

$$y'' + \lambda y = 0, \qquad ay(0) + by'(0) = 0 \qquad cy(L) + dy'(L) = 0.$$

♥

Proof It sufficiently to show that $y - u$ is a solution of the associated boundary value problem. But that can be verified directly. ■

To find all solutions of a boundary value problem, the key is to study the associated homogeneous boundary value problem.

The following property of homogeneous boundary value problem can be verified directly.

Theorem 12.1 (Superposition Principle)

Let u_1 and u_2 be two solutions of the homogeneous boundary value problem

$$y'' + \lambda y = 0, \qquad ay(0) + by'(0) = 0 \qquad cy(L) + dy'(L) = 0.$$

Then the linear combination

$$c_1 u_2 + c_2 u_2$$

is also a solution.

12.2 Eigenvalues and Eigenfunctions

In this section, we study homogeneous boundary value problems.

Consider the boundary value problem

$$y'' + \lambda y = 0, \qquad ay(0) + by'(0) = 0 \qquad cy(L) + dy'(L) = 0.$$

A value of λ such that the boundary value problem has a nontrivial solution is called an ***eigenvalue of the boundary value problem***, and the nontrivial solutions are called λ-***eigenfunctions***, or ***eigenfunctions associated to*** λ.

Example 12.3 Find eigenvalues and eigenfunctions of the boundary value problem

$$y'' + \lambda y = 0, \qquad y(0) = 0 \qquad y(L) = 0.$$

Solution *If $\lambda = 0$, then the general solution of the equation $y'' + \lambda y = 0$ is $y = c_1 + c_2 x$. The boundary conditions imply $c_1 = 0$ and $c_2 = 0$. So $\lambda = 0$ is not an eigenvalue.*

If $\lambda < 0$, then equation $y'' + \lambda y = 0$ has the general solution

$$y = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}.$$

Note that the general solution can also be written as a linear combination $y = c_1 \cosh(\sqrt{-\lambda}x) + c_2 \sinh(\sqrt{-\lambda}x)$ of the hyperbolic functions:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2},$$

which will be convenient to determine c_1 and c_2 from the boundary conditions.

Since $\cosh(x) > 0$ for any x and $\sinh(x) = 0$ if and only if $x = 0$, then the boundary condition $y(0) = 0$ implies that $c_1 = 0$ and the boundary condition $y(L) = 0$ then implies $c_2 = 0$. So a negative λ is not an eigenvalue.

If $\lambda > 0$, the the general solution is

$$y = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

The boundary conditions imply c_1 and c_2 satisfy the system

$$\begin{cases} c_1 = & 0 \\ c_2 \sin(\sqrt{\lambda}L) = & 0 \end{cases}$$

So λ is an eigenvalue if $\sqrt{\lambda}L = n\pi$,or equivalently $\lambda = \frac{n^2\pi^2}{L^2}$, where $n \neq 0$.

In this case, the function

$$y_n = \sin\left(\frac{n\pi x}{L}\right)$$

is an eigenfunction associated to the eigenvalue $\lambda = \frac{n^2\pi^2}{L^2}$. ■

Exercise 12.1 Find eigenvalues and eigenfunctions of the boundary value problem

$$y'' + \lambda y = 0, \qquad y'(0) = 0 \qquad y'(L) = 0.$$

Solution *If $\lambda = 0$, then the general solution of the equation $y'' + \lambda y = 0$ is $y = c_1 + c_2 x$. The boundary conditions imply $c_2 = 0$. So $\lambda = 0$ is an eigenvalue with an eigenfunction $y = 1$.*

If $\lambda < 0$, then equation $y'' + \lambda y = 0$ has the general solution

$$y = c_1 \cosh(\sqrt{-\lambda x}) + c_2 \sinh(\sqrt{-\lambda x}).$$

Since $(\cosh x)' = -\sinh x$ and $(\sinh x) = \cosh x$, the boundary conditions $y'(0) = 0$ and $y'(L) = 0$ imply that $c_1 = 0$ and $c_2 = 0$. So a negative λ is not an eigenvalue.

If $\lambda > 0$, the the general solution is

$$y = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

The boundary conditions imply c_1 and c_2 satisfy the system

$$\begin{cases} c_2 = & 0 \\ c_1 \sin(\sqrt{\lambda}L) = & 0 \end{cases}$$

So λ is an eigenvalue if $\sqrt{\lambda}L = n\pi$,or equivalently $\lambda = \frac{n^2\pi^2}{L^2}$, where $n \neq 0$.

In this case, the function

$$y_n = \cos\left(\frac{n\pi x}{L}\right)$$

is an eigenfunction associated to the eigenvalue $\lambda = \frac{n^2\pi^2}{L^2}$. ■

Those eigenfunctions has very interesting properties.

Two integrable functions f and g are said to be ***orthogonal*** on an interval $[a, b]$ if

$$\int_a^b f(x)g(x)\,\mathrm{d}\,x = 0.$$

More generally, a collection of integrable functions $\phi_1, \phi_2, ..., \phi_n, ...,$ are orthogonal on $[a, b]$ if they are mutually orthogonal.

Example 12.4 The eigenfunctions $\sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, \ldots$, are orthogonal to each other over $[0, L]$.

Solution *Using the product to sum identity*

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right),$$

the integral is calculated by

$$\begin{aligned}
&\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}\,x \\
&= \frac{1}{2}\left(\int_0^L \cos\left(\frac{(m-n)\pi x}{L}\right)\mathrm{d}\,x - \int_0^L \cos\left(\frac{(m+n)\pi x}{L}\right)\mathrm{d}\,x\right) \\
&= \frac{1}{2}\left(\frac{L}{(m+n)\pi}\sin\left(\frac{(m+n)\pi x}{L}\right)\Big|_0^L - \frac{L}{(m-n)\pi}\sin\left(\frac{(m-n)\pi x}{L}\right)\Big|_0^L\right) \\
&= \frac{1}{2}\left(\frac{L}{(m+n)\pi}\sin((m+n)\pi) - \sin((m-n)\pi)\right) \\
&= 0.
\end{aligned}$$

Therefore, those eigenfunctions are mutually orthogonal. ■

✍ **Exercise 12.2** Show that the functions

$$1,\ \cos\left(\frac{\pi x}{L}\right),\ \sin\left(\frac{\pi x}{L}\right),\ \cos\left(\frac{2\pi x}{L}\right),\ \sin\left(\frac{2\pi x}{L}\right), \ldots, \cos\left(\frac{n\pi x}{L}\right),\ \sin\left(\frac{n\pi x}{L}\right), \ldots$$

are orthogonal on $[-L, L]$.

Solution *From the definition of orthogonality, we need to show that*

$$\int_{-L}^L f(x)g(x)\,\mathrm{d}\,x = 0,$$

where f and g are two distinct functions in the given set of functions.

For any integer $r \neq 0$, because $\sin$ is an odd function, calculating using the substitution $t = -x$ yields

$$\int_{-L}^{L} \sin\left(\frac{r\pi x}{L}\right) \mathrm{d}\,x = -\int_{-L}^{L} \sin\left(\frac{r\pi t}{L}\right) \mathrm{d}\,t$$

which implies

$$\int_{-L}^{L} \sin\left(\frac{r\pi x}{L}\right) \mathrm{d}\,x = 0.$$

Similarly,

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}\,x = 0,$$

because the integrand is an odd function.

Calculating using the substitution $t = \frac{r\pi x}{L}$ yields

$$\int_{-L}^{L} \cos\left(\frac{r\pi x}{L}\right) \mathrm{d}\,x = \frac{L}{r\pi} \int_{-r\pi}^{r\pi} \cos t \,\mathrm{d}\,t = \frac{L}{r\pi} \sin t\Big|_{-r\pi}^{r\pi} = 0.$$

Therefore, 1 is orthogonal with any function in the set, and sine and cosine functions in the set are also orthogonal.

It remains to check orthogonality among sine functions and among cosine functions.

The trigonometric identities,

$$\sin\alpha \sin\beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta)),$$

$$\cos\alpha \cos\beta = \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta)),$$

and the following result

$$\int_{-L}^{L} \cos\left(\frac{r\pi x}{L}\right) \mathrm{d}\,x = 0,$$

together implies the orthogonality among sine function or among cosine function.

Therefore, the set of functions are orthogonal. ■

12.3 Introduction to Fourier Series

The orthogonality of eigenfunctions has great applications in solving partial differential equations. Eigenfunctions can be used to express a function f as a sum of linear combination of them if the function f is good enough.

Theorem 12.2

Suppose the functions ϕ_1, ϕ_2, ϕ_3, ..., are orthogonal on $[a, b]$ and

$$\int_a^b \phi_n^2(x)\,\mathrm{d}x \neq 0, \qquad n = 1, 2, 3, \ldots .$$

Let c_1, c_2, c_3,... be constants such that the partial sums

$$f_N(x) = \sum_{m=1}^{N} c_m \phi_m(x)$$

satisfy the inequalities

$$|f_N(x)| \leq M, \qquad a \leq x \leq b, \qquad N = 1, 2, 3, \ldots$$

for some constant $M < \infty$. Suppose also that the series

$$f(x) = \sum_{m=1}^{\infty} c_m \phi_m(x)$$

converges and is integrable on $[a, b]$. Then

$$c_n = \frac{\int_a^b f(x)\phi_n(x)\,\mathrm{d}x}{\int_a^b \phi_n^2(x)\,\mathrm{d}x}, \qquad n = 1, 2, 3, \ldots .$$

♥

Suppose ϕ_1, ϕ_2,..., ϕ_n, ..., are orthogonal on $[a, b]$ and $\displaystyle\int_a^b \phi_n^2(x)\,\mathrm{d}x \neq 0$, $n = 1, 2, 3, \ldots$. Let f be integrable on $[a, b]$, and define

$$c_n = \frac{\int_a^b f(x)\phi_n(x)\,\mathrm{d}x}{\int_a^b \phi_n^2(x)\,\mathrm{d}x}, \qquad n = 1, 2, 3, \ldots .$$

Then the infinite series

$$\sum_{n=1}^{\infty} c_n \phi_n(x)$$

is called the ***Fourier expansion*** of f in terms of the orthogonal set $\{\phi_n\}_{n=1}^{\infty}$, and c_1, c_2,..., c_n,... are called the ***Fourier coefficients*** of f with respect to $\{\phi_n\}_{n=1}^{\infty}$.

Due to the fact that the Fourier expansion of a function may diverges every where, we indicate the relationship between f and its Fourier expansion by

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x), \qquad a \leq x \leq b.$$

Recall that the set of functions

$$1,\ \cos\left(\frac{\pi x}{L}\right),\ \sin\left(\frac{\pi x}{L}\right),\ \cos\left(\frac{2\pi x}{L}\right),\ \sin\left(\frac{2\pi x}{L}\right),\dots,\cos\left(\frac{n\pi x}{L}\right),\ \sin\left(\frac{n\pi x}{L}\right),\dots$$

are orthogonal on $[-L,L]$.

For any piecewise continuous function on $[-L,L]$, the Fourier expansion

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right\}$$

is called the Fourier series of f, where a_n and b_n, for $n = 0,1,2,\dots$, are given by the Euler-Fourier formulas

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}\,x \qquad \text{and} \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}\,x.$$

Example 12.5 Find the Fourier series of $f(x) = |x|$ on $[-\pi,\pi]$.

Solution

Since f is even and $\sin(nx)$ is odd, the number $b_n = 0$ for all n and the Fourier series can be written as

$$F(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx).$$

Since $\cos(nx)$ is also even, applying the methods of substitutions and integration by parts implies

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\,\mathrm{d}\,x = \frac{2}{\pi}\int_0^{\pi} x\,\mathrm{d}\,x = \pi.$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos(nx)\,\mathrm{d}\,x\\
&= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,\mathrm{d}\,x\\
&= \frac{2}{n\pi}\left(x\sin(nx)\Big|_0^{\pi} - \int_0^{\pi}\sin(nx)\,\mathrm{d}\,x\right)\\
&= \frac{2}{n^2\pi}\cos(nx)\Big|_0^{\pi}\\
&= \frac{2((-1)^n-1)}{n^2\pi}\\
&= \begin{cases} 0 & \text{if } n = 2(k+1)\\ -2 & \text{if } n = 2k+1\end{cases} \qquad k = 0,1,2,\dots.
\end{aligned}$$

Therefore

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos[(2k+1)x].$$

■

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