

MA440 Precalculus Worksheet

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Preface

Those worksheets are developed for the Precalculus course at QCC. Contents in those worksheets are mainly based on the [OpenStax Precalculus](#) textbook.

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1.1 Basic Concepts

Definition 1.1.1 A **relation** is a set of ordered pairs. The set of the first components of each ordered pair is called the **domain** and the set of the second components of each ordered pair is called the **range**.

A **function** is a relation that assigns each element in the domain a unique element in the range.

An arbitrary value in the domain is often represented by the lowercase letter x which is called an **independent variable**. An arbitrary output is often represented by the lowercase letter y which is called a **dependent variable**.

Example 1.1.1 Determine if the relation

$$\{(1, 2), (2, 4), (3, 6), (4, 8), (5, 10)\}$$

is a function. Find the domain and the range.

Definition 1.1.2 If a function has x as the independent variable and y as the dependent variable, then we often say that y is a function of x .

Example 1.1.2 Consider items and prices in a grocery store. Is price a function of item? Is item a function of price?

Definition 1.1.3 A function is often named by letters, such as f , F , p , or q . If f is a function of x , then we denote it as $y = f(x)$ which is called the **function notation**. Here $f(x)$ is read as f of x or f at x . The notation $f(x)$ represents the output of the function f for a given input x .

Example 1.1.3 Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

Example 1.1.4 A function $N = f(y)$ gives the number of police officers, N , in a town in year y . What does $f(2005) = 300$ represent?

Example 1.1.5 Using a table to represent the days in the month as the function of month.

Example 1.1.6 Consider the function $f(x) = x^2 + 3x - 4$. Find the values of the following expressions.

(1) $f(2)$

(2) $f(a)$

(3) $f(a + h)$

(4) $\frac{f(a + h) - f(a)}{h}$

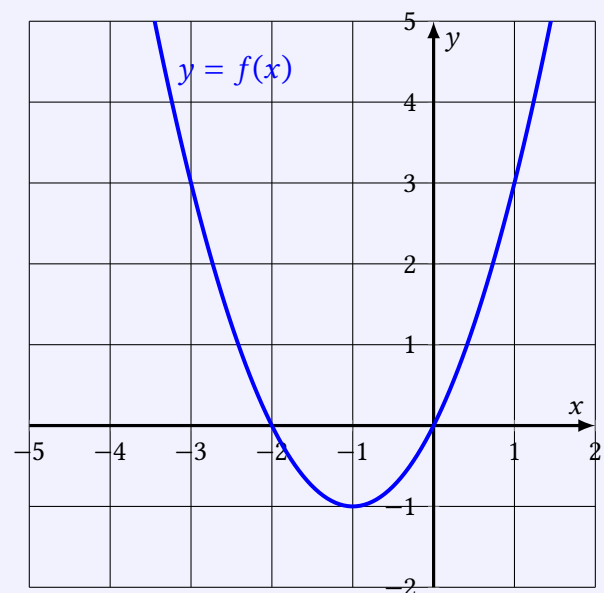
Example 1.1.7 Consider the function $f(x) = x^2 - 2x$. Find all x values such that $f(x) = 3$.

Example 1.1.8 Express the relationship defined by the function $2x - y - 3 = 0$ as a function $y = l(x)$.

Example 1.1.9 Does the equation $x^2 + y^2 = 1$ defines y as a function x . If so, express the relationship as a function $y = f(x)$. If not, under what extra condition does the function $y = f(x)$ exist?

Example 1.1.10 Consider the function $f(x)$ defined by a graph below.

- (1) Find $f(-1)$.
- (2) Find all x such that $f(x) = 3$.



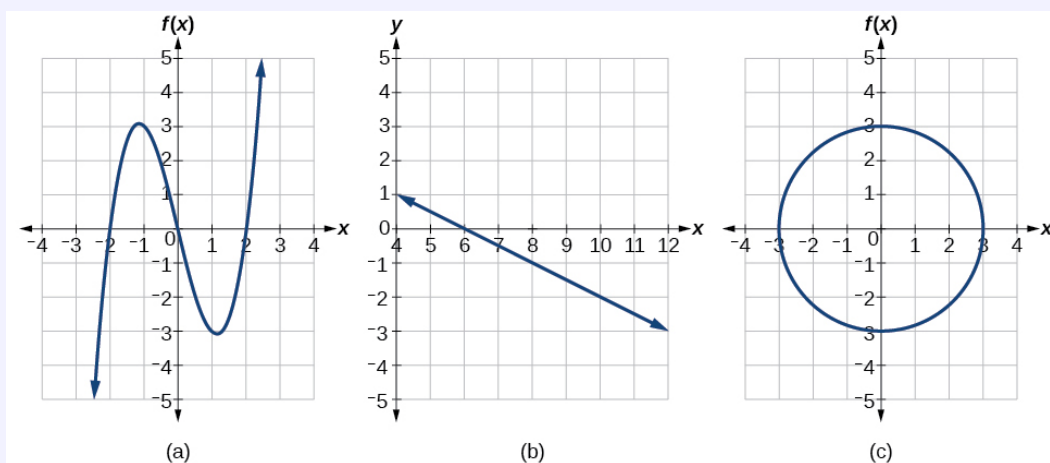
Definition 1.1.4 A function is a **one-to-one function** if each output value corresponds to exactly one input value.

Example 1.1.11 Is the area of a circle a function of its radius? If yes, is the function one-to-one?


How-to A graph is a function if every vertical line crosses the graph at most once. This method is known as the **vertical line test**.

A function is an one-to-one if every horizontal line crosses the graph at most once. This method is known as the **horizontal line test**.

Example 1.1.12 Determine if the graph defines a function. If so, is it a one-to-one function?



Exercises


 **Exercise 1.1.1** Consider the function $f(x) = 2x^2 + x - 3$. Find the values of the following expressions.


(1) $f(-1)$


(2) $f(a)$


(3) $f(a + h)$

(4) $\frac{f(a + h) - f(a)}{h}$

 **Exercise 1.1.2** For the function $f(x) = -4x + 5$, evaluate and simplify the difference quotient $\frac{f(x + h) - f(x)}{h}$.

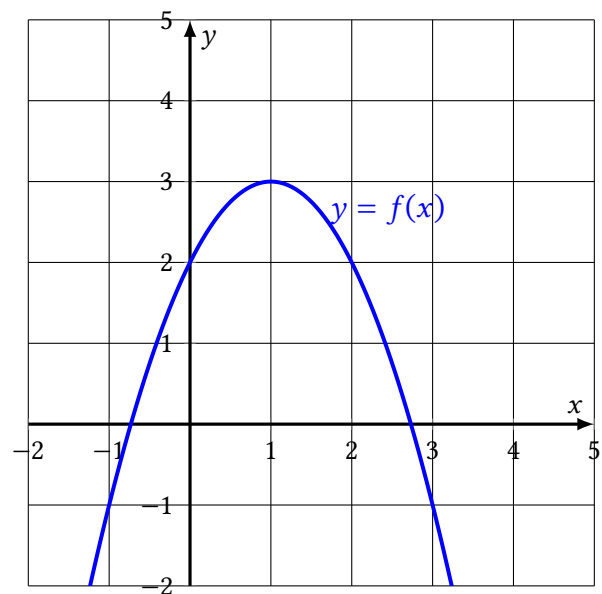
 **Exercise 1.1.3** Consider the function $f(x) = -x^2 - 4x$. Find all x values such that $f(x) = 3$.

 **Exercise 1.1.4** Express the relationship defined by the function $3x - 2y - 6 = 0$ as a function $y = l(x)$.

 **Exercise 1.1.5** If $8x - y^3 = 0$, express y as a function of x .
Is y a one-to-one function of x ?

 **Exercise 1.1.6** Consider the function $f(x)$ defined by a graph below.

- (1) Find $f(1)$.
- (2) Find all x such that $f(x) = 3$.



1.2 Domains and Ranges

How-to The domain of a function f consists of possible input values x . Or equivalently, the domain consists of all x values except those that will make the function is undefined.

The range of a function f consists of all possible output values y . Equivalently, the range consists of y value such that equation $y = f(x)$ has a solution x .

Example 1.2.1 Find the domain of the function

$$f(x) = \frac{x+1}{2-x}.$$

Example 1.2.2 Find the domain of the function

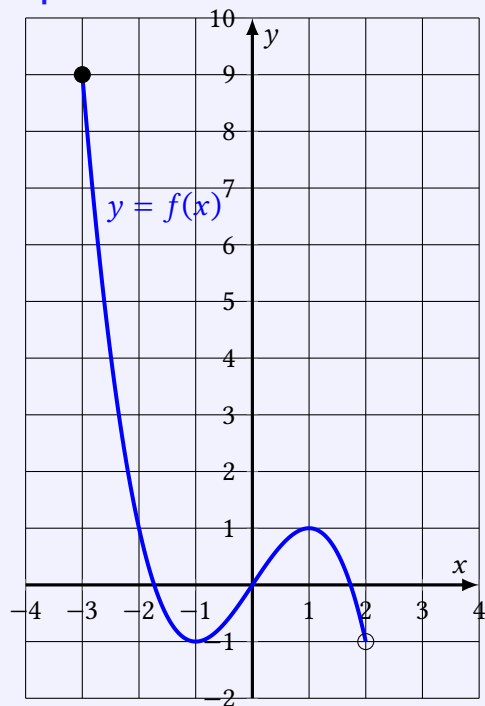
$$f(x) = \sqrt{7-x}.$$

Definition 1.2.1 Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x \mid \text{statement about } x\}$ which is read as, "the set of all x such that the statement about x is true."

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set.

Example 1.2.3 Find the domain of the function $f(x) = \frac{\sqrt{x+2}}{x-1}$. Write your answer in set-builder notation and interval notation.

Example 1.2.4 Find the domain and range of the function f whose graph is shown in Figure.



Example 1.2.5 Find the domain and range of the function

$$f(x) = \frac{2}{x+3}.$$

Example 1.2.6 Find the domain and range of the function

$$f(x) = 3\sqrt{x+2}.$$

Example 1.2.7 Consider the piecewise function


$$f(x) = \begin{cases} 2x - 3 & \text{if } x \leq -1 \\ -x^2 & \text{if } -1 < x < 1 \\ -2x + 4 & \text{if } 1 \leq x. \end{cases}$$

(1) Sketch the graph


(2) Find $f(-4)$

(3) Find $f(2)$

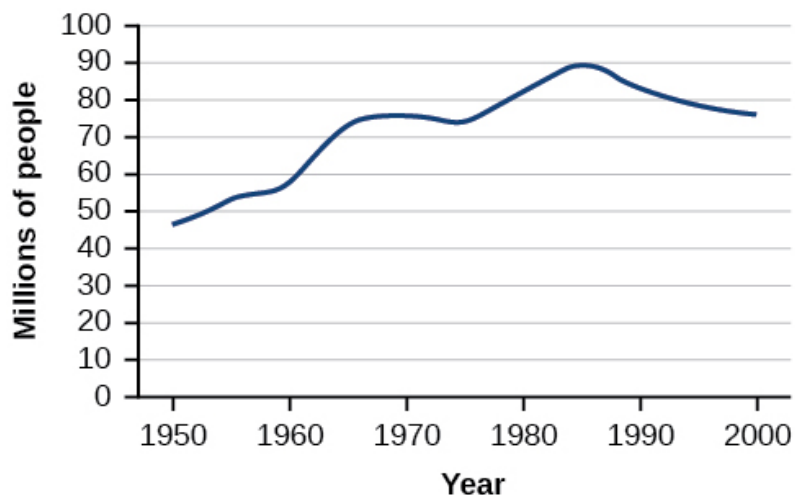
Exercises


 **Exercise 1.2.1** Find the domain of the function

(1) $f(x) = \frac{1+4x}{2x-1}$ (2) $f(x) = \sqrt{5+2x}$ (3) $f(x) = \frac{\sqrt{x+1}}{x-1}$ (4) $f(x) = \frac{x-2}{x^2+7x-44}$

 **Exercise 1.2.2** Estimate the domain and range for the function defined by the graph. Write your answer in interval notation.

World Population Increase



 **Exercise 1.2.3** Find the domain and range of each of the following functions. Write your answer in set-builder notation and interval notation.

(1) $f(x) = \frac{3}{x-2}$

(2) $f(x) = -2\sqrt{x+4}$

 **Exercise 1.2.4** Consider the piecewise function

$$f(x) = \begin{cases} -2x + 5 & \text{if } x < -2 \\ x^2 - 1 & \text{if } -2 \leq x \leq 2 \\ 2x - 3 & \text{if } 2 < x. \end{cases}$$

(1) Sketch the graph

(2) Find $f(-4)$

(3) Find $f(2)$

1.3 Rates of Change and Behavior of Graphs

Definition 1.3.1 (Rate of Change) The average rate of change of f over an interval $[a, b]$ is defined as

$$\text{Average Rate Of Change} = \frac{f(b) - f(a)}{b - a}.$$

The average rate of change is the same as the slope of secant line passing through $(a, f(a))$ and $(b, f(b))$.

By taking $x = a$ and $h = b - a$, the average of rate of change is the same the difference quotient of a function f which is defined as

$$\text{Difference Quotient} = \frac{f(x + h) - f(x)}{h}.$$

Example 1.3.1 After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in Table. Find her average speed over the first 6 hours.

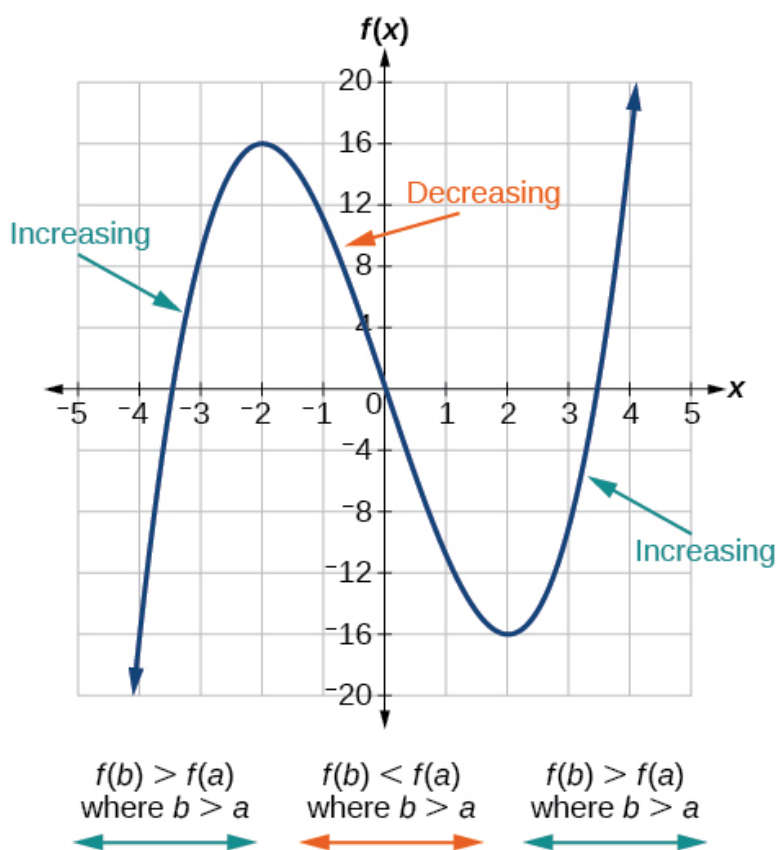
t (hours)	0	1	2	3	4	5	6	7
$D(t)$ (miles)	10	55	90	153	214	240	292	300

Example 1.3.2 Find the average rate of change of $f(x) = x^2 - \frac{1}{x}$ over the interval $[2, 4]$.

Example 1.3.3 Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. The answer will be an expression involving a .

Definition 1.3.2 (Increasing and Decreasing) A function f is **increasing** over an interval (a, b) if $f(x_2) > f(x_1)$ for any $x_1 < x_2$ in (a, b) . Equivalently, f is increasing over (a, b) if the average rate of change is positive over any subinterval (x_1, x_2) of (a, b) .

A function f is **decreasing** over an interval (a, b) if $f(x_2) < f(x_1)$ for any $x_1 < x_2$ in (a, b) . Equivalently, f is decreasing over (a, b) if the average rate of change is negative over any subinterval (x_1, x_2) of (a, b) .



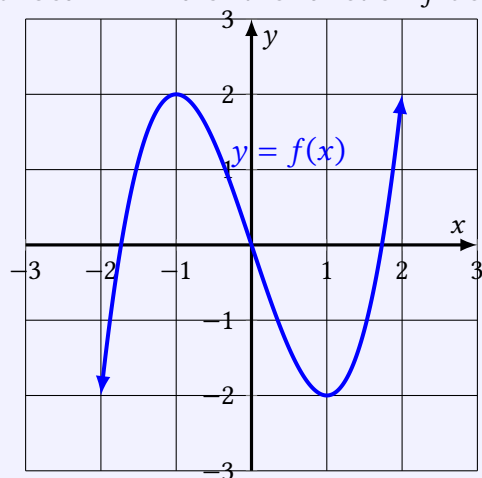
Definition 1.3.3 (Local Maxima and Minima) A function f has a **local maximum** at $x = c$ if $f(c) \geq f(x)$ for any x in a small interval containing c . A small interval containing c is also known as a small neighborhood of c .

A function f has a **local minimum** at $x = c$ if $f(c) \leq f(x)$ for any x in a small interval containing c .

How-to A function f has a local maximum at $x = c$ if it changes from increasing to decreasing at c in a neighborhood of c .

A function f has a local minimum at $x = c$ if it changes from decreasing to increasing at c in a neighborhood of c .

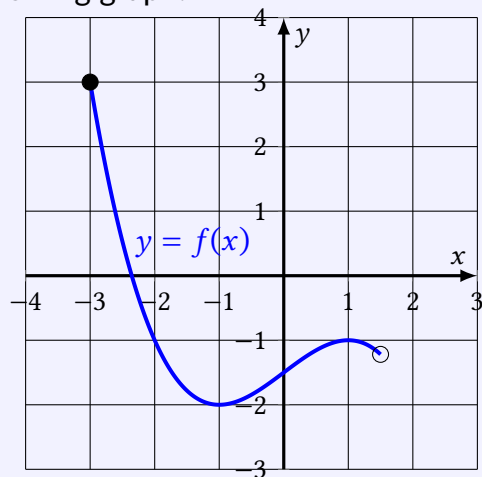
Example 1.3.4 Find the interval of increasing and the interval of decreasing, and the local maxima and local minima of the function f defined by the following graph.




Definition 1.3.4 (Absolute Maxima and Minima) The **absolute maximum** of f at $x = c$ is $f(c)$ where $f(c) \geq f(x)$ for all x in the domain of f .


The **absolute minimum** of f at $x = c$ is $f(c)$ where $f(c) \leq f(x)$ for all x in the domain of f .


Example 1.3.5 Finding the absolute maximum and minimum of the function f defined by the following graph.

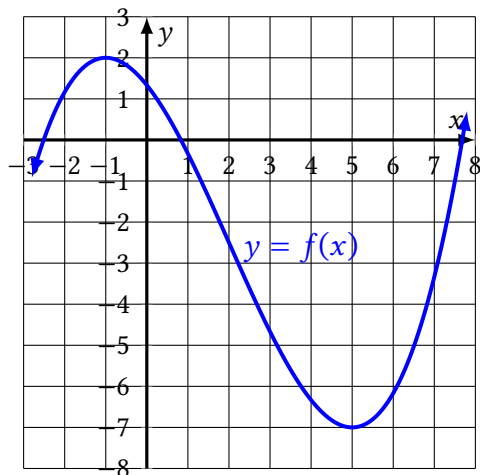


Exercises

 **Exercise 1.3.1** The electrostatic force F , measured in newtons, between two charged particles can be related to the distance between the particles d , in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

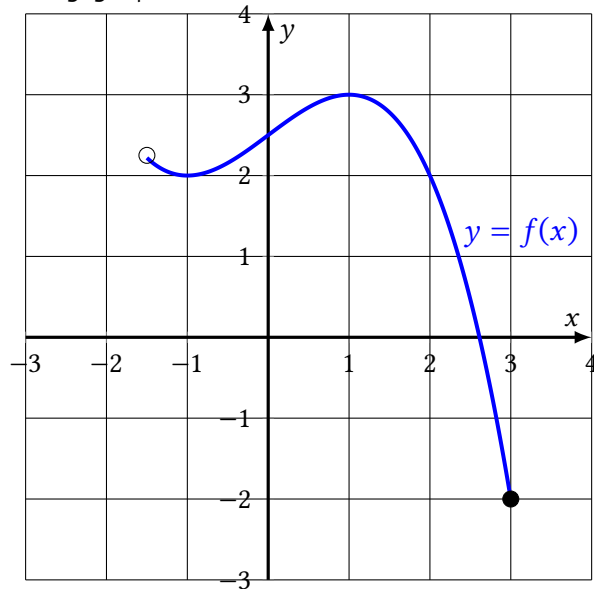
 **Exercise 1.3.2** Find the average rate of change of $f(x) = x^2 + 2x - 8$ on the interval $[5, a]$.

 **Exercise 1.3.3** Find the interval of increasing and the interval of decreasing, and the local maxima and local minima of the function f defined by the following graph.





Exercise 1.3.4 Finding the absolute maximum and minimum of the function f defined by the following graph.



Exercise 1.3.5 Find the interval of increasing and the interval of decreasing, and the local maxima and local minima of the function $f(x) = x^3 - 6x^2 - 15x + 20$ using its graph.

1.4 Combination and Composition of Functions

Definition 1.4.1 (Algebraic Operations of Functions) Let f and g be two functions with domains A and B respectively. We define the linear combination, product, and quotient functions as follows.

Linear combination:	$(af + bg)(x) = af(x) + bg(x)$	with the domain $A \cap B$.
Product:	$(fg)(x) = f(x)g(x)$	with the domain: $A \cap B$.
Quotient:	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	with the domain: $A \cap B \cap \{x \mid g(x) \neq 0\}$.

Example 1.4.1 Consider the functions $f(x) = x - 1$ and $g(x) = x^2 - 1$. Find and simplify the functions $(g - f)(x)$ and $\left(\frac{g}{f}\right)(x)$, and their domains.

Definition 1.4.2 (Composition of functions) Let f and g be two functions with domains A and B respectively. The **composite function** $f \circ g$ (also called the composition of f and g) is defined as

$$(f \circ g)(x) = f(g(x)) \quad \text{with the domain: } B \cap \{x \mid g(x) \in A\}.$$

We read the left-hand side as “ f composed with g at x ,” and the right-hand side as “ f of g of x .”

Example 1.4.2 Consider the functions $f(x) = \sqrt{x - 2}$ and $g(x) = x^2 + 1$.

- (1) Find and simplify the functions $(f \circ g)(x)$ and $(g \circ f)(x)$. Are they the same function?
- (2) Find the domains of $f \circ g$ and $g \circ f$. Are they the same?

Example 1.4.3 Consider $f(t) = t^2 - 4t$ and $h(x) = \sqrt{x + 3}$. Evaluate

(1) $\frac{f(1)}{g(1)}$

(2) $h(f(-1))$

(3) $(f \circ h)(-1)$

(4) $(f - h)(-1)$

Example 1.4.4 Using the graphs to evaluate the given functions.

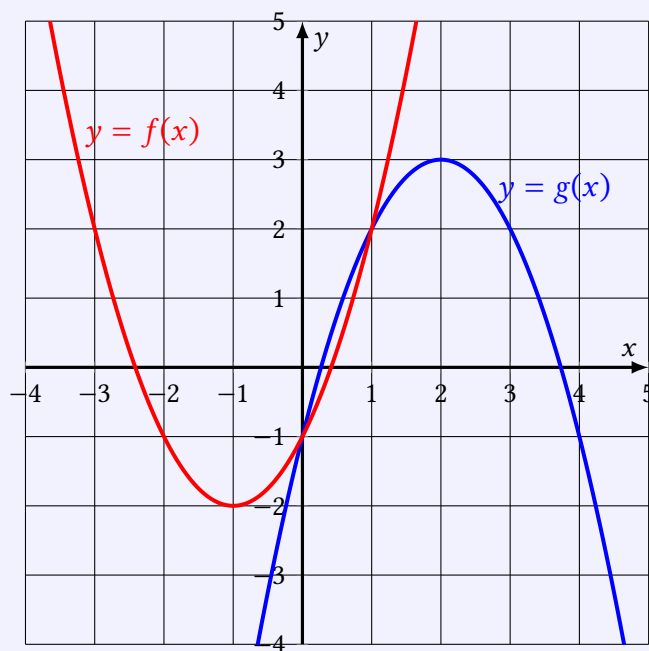
(1) $(f + g)(1)$

(2) $(fg)(1)$

(3) $\left(\frac{f}{g}\right)(1)$

(4) $(g \circ f)(-3)$

(5) $f(g(0))$



Example 1.4.5 Consider the function $h(x) = \sqrt{x^2 + 1}$. Find two functions f and g so that $h(x) = f(g(x))$.

Exercises



Exercise 1.4.1 Consider the functions $f(x) = x^2 - 1$ and $g(x) = x + 1$.

(1) Find the function $(f - g)(x)$ and its domain.

(2) Find the function $(fg)(x)$ and its domain.

(3) Find $\left(\frac{f}{g}\right)(x)$ and its domain.

(4) Find $(2f - 3g)(1)$.

(5) Find $2fg - \left(\frac{3g}{f}\right)(1)$.



Exercise 1.4.2 Consider the functions $f(x) = \frac{1}{x - 2}$ and $g(x) = \sqrt{x + 4}$.

(1) Find $f \circ g$ and its domain.

(2) Find $(g \circ f)(3)$.

 **Exercise 1.4.3** Using the graphs to evaluate the given functions.

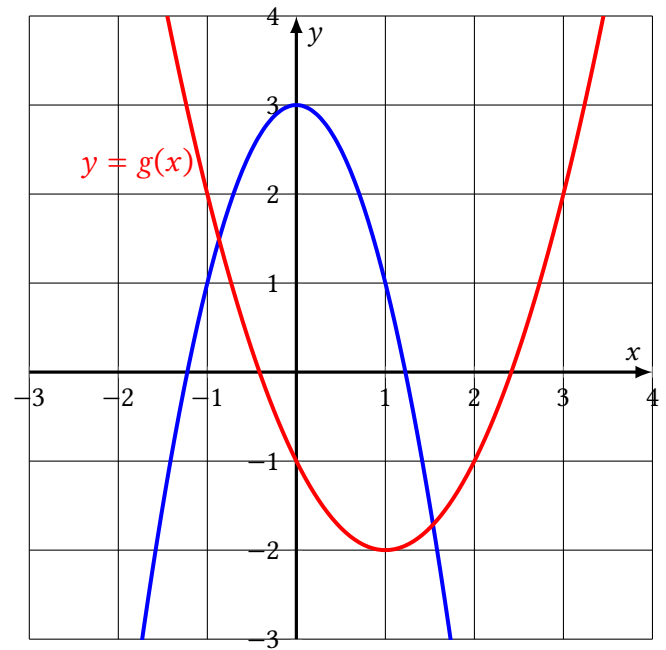
(1) $(f - g)(1)$


(2) $(fg)(0)$

(3) $\left(\frac{f}{g}\right)(0)$

(4) $(f \circ g)(2)$

(5) $g(f(0))$



 **Exercise 1.4.4** Consider the function $h(x) = \sqrt[3]{2x-1}$. Find two functions f and g so that $h(x) = f(g(x))$.

1.5 Transformations

Definition 1.5.1 Given a function $y = f(x)$, the function $y = f(x) + k$, where k is a constant, is a **vertical shift** of the function f .

How-to Suppose k is positive.

- To graph $y = f(x) + k$, shift the graph of $y = f(x)$ **upward** k units.
- To graph $y = f(x) - k$, shift the graph of $y = f(x)$ **downward** k units.

Example 1.5.1 Consider the functions $f(x) = x^2$, $g(x) = x^2 - 1$ and $h(x) = x^2 + 2$.

- (1) Describe how to get the graph of g from the graph of f .
- (2) Describe how to get the graph of h from the graph of f .
- (3) Describe how to get the graph of f from the graph of h .
- (4) Describe how to get the graph of h from the graph of g .

Definition 1.5.2 Given a function $y = f(x)$, the function $y = f(x - h)$, where h is a constant, is a **horizontal shift** of the function f .

How-to Suppose h is positive.

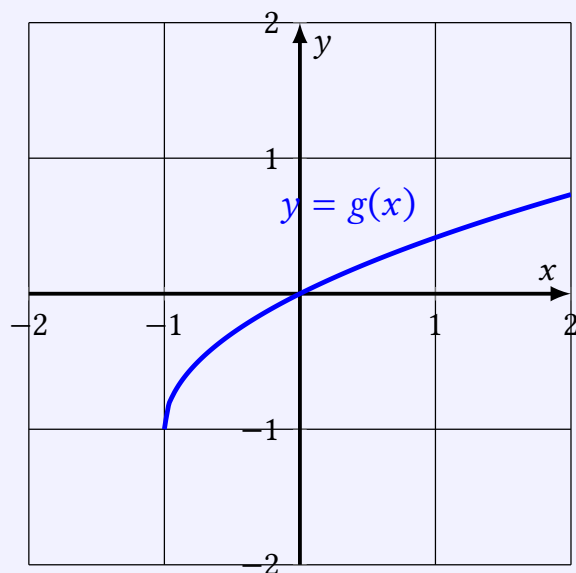
- To graph $y = f(x - h)$, shift the graph of $y = f(x)$ to the **right** h units.
- To graph $y = f(x + h)$, shift the graph of $y = f(x)$ to the **left** h units.

Example 1.5.2 Consider the functions $f(x) = x^2$, $g(x) = (x + 1)^2$ and $h(x) = (x - 2)^2$.

- (1) Describe how to get the graph of g from the graph of f .
- (2) Describe how to get the graph of h from the graph of f .
- (3) Describe how to get the graph of f from the graph of h .
- (4) Describe how to get the graph of h from the graph of g .

Example 1.5.3 Sketch the graph of $f(x) = |x|$. Then use the graph to sketch the graph of $h(x) = f(x + 2) - 1$.

Example 1.5.4 The function $y = g(x)$ shown in the picture is a shift of the square root function $y = \sqrt{x}$. Find $g(x)$.



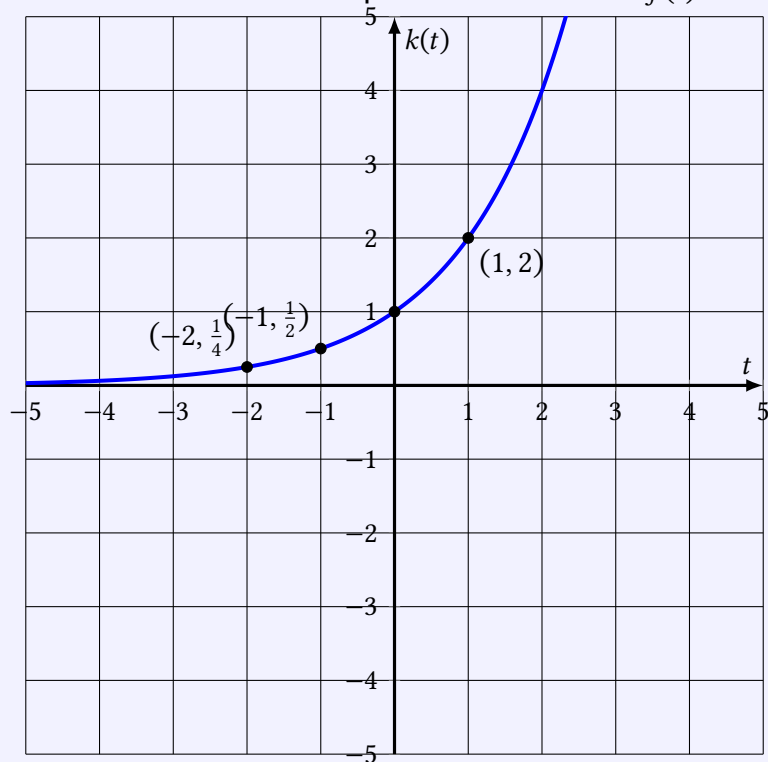
Definition 1.5.3 Given a function $y = f(x)$, the function $g(x) = -f(x)$ is a **vertical reflection** of the function $y = f(x)$, or a reflection about the x -axis; the function $g(x) = f(-x)$ is a **horizontal reflection** of the function $y = f(x)$ or a reflection about the y -axis.

Example 1.5.5 Reflect the graph of $f(x) = |x - 1|$

(1) first vertically, (2) then horizontally.

Denote the new function by $y = g(x)$. Find $g(x)$.

Example 1.5.6 A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions, and $t > 0$. The function k is a transformation of a part of the function $f(t) = 2^t$ shown below. Sketch the graph of $k(t)$.



Definition 1.5.4 A function is called an **even function** if $f(-x) = f(x)$ for x in the domain of f .
A function is called an **odd function** if $f(-x) = -f(x)$ for x in the domain of f .

Remark The graph of an even function is symmetric about y -axis.

The graph of an odd function is symmetric about the origin. This symmetry is known as a rotation symmetry.

Example 1.5.7 Group the functions according to even, odd, or other.

(1) $f(x) = x^2 - 1$

(2) $g(x) = |x - 1|$

(3) $h(x) = x^3 - 2x$

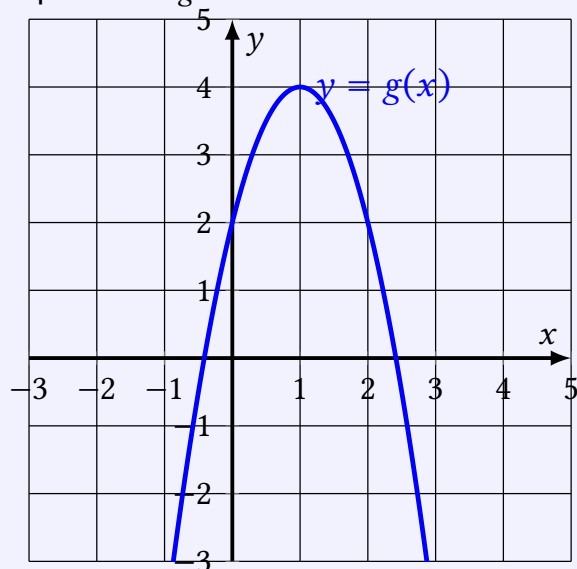
(4) $k(x) = \frac{1}{x^2}$.

Definition 1.5.5 Let c be a positive number. The function $g(x) = cf(x)$ is called a **vertical stretch** or **vertical compression** of $y = f(x)$ by a factor of c if $c > 1$ or $0 < c < 1$ respectively.

Remark If $a < 0$, then $g(x) = cf(x)$ is a combination of a vertical stretch or compression with a vertical reflection.

Example 1.5.8 The point $(9, -15)$ is on the graph of $y = f(x)$. Find a point on the graph of $g(x) = \frac{1}{3}f(x)$.

Example 1.5.9 The function $y = g(x)$ given in the following graph can be obtained from $f(x) = x^2$ by a combination of shifting, reflecting, and stretching. Describe the transformation and find an equation of g .



Definition 1.5.6 Let c be a positive number. The function $g(x) = f(cx)$ is called a **horizontal stretch** or **horizontal compression** of $y = f(x)$ by a factor of $\frac{1}{c}$ if $0 < c < 1$ or $c > 1$ respectively.

Remark If $c < 0$, then $g(x) = f(cx)$ is a combination of a horizontal stretch or compression with a horizontal reflection.

Example 1.5.10 The function $y = f(x)$ has two x -intercepts $(-2, 0)$ and $(4, 0)$. Determine if the function $g(x) = f(2x)$ has any x -intercepts. If so, find them. Otherwise explain why it has no x -intercept.

Example 1.5.11 Describe how to get the graph of the function $g(x) = 4x^2$ from the graph of the function $f(x)$.

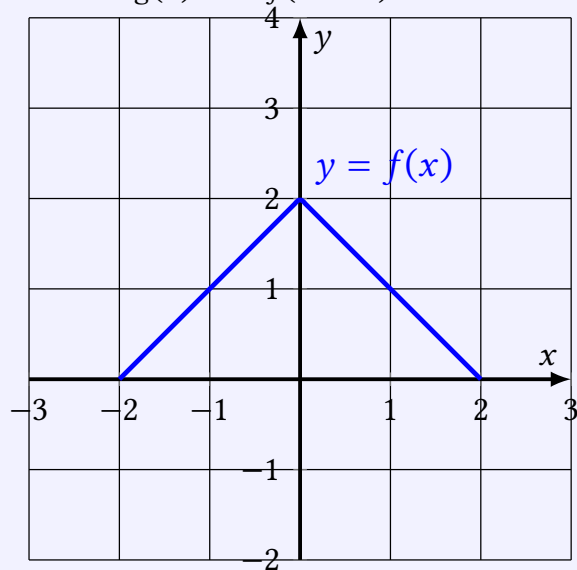
How-to The graph of the function $g(x) = Af(Bx + C) + D$ can be obtained by the following transformations in the given order.

- (1) A vertical stretch/compression with the factor $|A|$ followed by a reflection about x -axis if $A < 0$.
- (2) A vertical shift of D units
- (3) A horizontal shift of C units.
- (4) A horizontal stretch/compression with the factor $|B|$ followed by a reflection about y -axis if $B < 0$.

Remark A vertical shift of $\frac{D}{A}$ units may be done before the vertical stretch/compression. Because g can also be written as $g(x) = A \left(f(BX + C) + \frac{D}{A} \right)$

A horizontal shift of $\frac{C}{B}$ units may be done after the horizontal stretch/compression. Because g can also be written as $g(x) = Af \left(B \left(X + \frac{C}{B} \right) \right) + D$.

Example 1.5.12 Using the graph of the function $y = f(x)$ given below to sketch the graph of the function $g(x) = -2f(3x - 6) + 4$.





Example 1.5.13 Sketch the graph of the function $g(x) = 2\sqrt{3x - 1} - 4$ by a sequence of transformation applied on the graph of $f(x) = \sqrt{x}$.


Example 1.5.14 Find an equation of the function $y = g(x)$ whose graph is obtained from $f(x) = \sqrt{x}$ by the following transformations in the given order.


- (1) stretch vertically by a factor of 2
- (2) shift downward 2 units
- (3) shift 3 units to the left
- (4) stretch vertically by a factor $\frac{1}{2}$.

Exercises

-  **Exercise 1.5.1** Consider the functions $f(x) = x^2$, $g(x) = (x + 1)^2 - 2$ and $h(x) = (x - 2)^2 + 1$.
- (1) Describe how to get the graph of g from the graph of f .
 - (2) Describe how to get the graph of h from the graph of g .

-  **Exercise 1.5.2** Determine if the function is even, odd, or neither.
- (1) $f(x) = 1 - x^2$.
 - (2) $g(x) = \sqrt[3]{-x}$.
 - (3) $g(x) = x^4 - x^3$.

 **Exercise 1.5.3** Sketch the graph of the function $g(x) = 2|3x - 6| + 4$ by a sequence of transformation applied on the graph of $f(x) = |x|$.

 **Exercise 1.5.4** Find an equation of the function $y = g(x)$ whose graph is obtained from $f(x) = \sqrt[3]{x}$ by the following transformations in the given order.

- (1) Compress vertically by a factor of $\frac{1}{2}$.
- (2) Reflect vertically.
- (3) shift downward 2 units.
- (4) Compress horizontal by a factor 2.
- (5) Shift 3 units to the right.

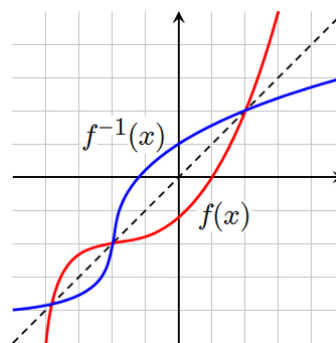
1.6 Inverse Functions

Definition 1.6.1 Let $y = f(x)$ be a one-to-one function with the domain A . A function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(f(x)) = x$ for all x in A .

The notation f^{-1} is read “ f inverse.”

Remark

- (1) If f is a one-to-one function, then it has a unique inverse function f^{-1} . Here is the proof. Suppose g is also an inverse f . Then $f(g(x)) = x = f(f^{-1}(x))$. Then $g(x) = f^{-1}(f(g(x))) = f^{-1}(f(f^{-1}(x))) = f^{-1}(x)$.
- (2) Note that if f^{-1} is the inverse of f , then f is also the inverse of f^{-1} that is $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} .
- (3) In general, $f^{-1}(x) \neq f(x)^{-1}$.
- (4) The graphs of a one-to-one function f and its inverse f^{-1} are symmetric about the diagonal line $y = x$.
- (5) Suppose f has the domain A and the range B , then f^{-1} has the domain B and the range A (and vice versa).



The above graph of f and f^{-1} is taken from [Wikipedia](#).

Example 1.6.1 Let f be a one-to-one function with $f(3) = 4$ and $f(4) = 5$. Find $f^{-1}(4)$.

Example 1.6.2 Let $f(x) = \frac{1}{x-1}$ and $g(x) = \frac{x+1}{x}$. Determine if g is the inverse function of f .

Example 1.6.3 Consider the function $f(x) = x^2 + 1$ with $x > 0$. Sketch the graph of $y = f^{-1}(x)$ without finding its equation.

How-to *Given a function $y = f(x)$, the inverse function is the solution y of the equation $f(y) = x$. The domain and the range of f and f^{-1} can be obtained from the domains of f and f^{-1} .*

Example 1.6.4 Consider the function $f(x) = 2x - 3$. Find the inverse function f^{-1} and its domain and range.

Example 1.6.5 Consider the function $f(x) = \frac{x}{x-1}$. Find the inverse function f^{-1} and its domain and range.

Example 1.6.6 Consider the function $f(x) = 2(x + 1)^3 - 1$. Find the inverse function f^{-1} and its domain and range.


Example 1.6.7 Consider the function $f(x) = \sqrt{x - 2}$. Find the inverse function f^{-1} and its domain and range.


Example 1.6.8 Find the inverse of each of the following functions if it exists.


Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$
Reciprocal squared	Cube Root	Square Root	Absolute Value	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = x $	


Exercises


 **Exercise 1.6.1** Let f be a one-to-one function with $f(-2) = -3$ and $f(-3) = 4$. Find $f^{-1}(-3)$.

 **Exercise 1.6.2** Let $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x + 1}$. Is $g = f^{-1}$?

 **Exercise 1.6.3** Consider the function $f(x) = \frac{1}{x-1} + 1$. Sketch the graph of f^{-1} without finding its equation.

 **Exercise 1.6.4** Consider the function $f(x) = \frac{1-x}{x+1}$. Find the inverse function f^{-1} and its domain and range.

 **Exercise 1.6.5** Consider the function $f(x) = 3(x-1)^3 + 2$. Find the inverse function f^{-1} and its domain and range.

 **Exercise 1.6.6** Consider the function $f(x) = \sqrt{x+1} - 1$. Find the inverse function f^{-1} and its domain and range.

2.1 Quadratic Functions

Definition 2.1.1 A function $f(x) = ax^2 + bx + c$ with $a \neq 0$ is called a **quadratic function**. Its graph is called a **parabola**. By completing the square (let $h = -\frac{b}{2a}$ and $k = f(h)$), a quadratic function can be written in the **standard form** (or **vertex form**): $f(x) = a(x - h)^2 + k$. The vertical line $x = -\frac{b}{2a}$ (or $x = h$) is called the **axis of symmetry**. The **vertex** (h, k) is the intersection of the axis of symmetry and the parabola.

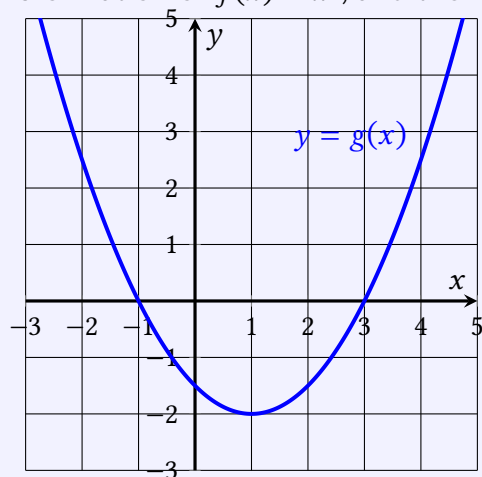
Note The y -intercept of a quadratic function is $(0, f(0))$. The x -coordinates of x -intercepts are the zeros (or roots) of the function f , that is the solutions of the equation $f(x) = 0$.

Example 2.1.1 Find the vertex form of the quadratic function $f(x) = 2x^2 + 4x + 1$ and determine the vertex, axis of symmetry, x -intercepts, and y -intercept of the function.

Note

- A quadratic function $f(x) = ax^2 + bx + c$ can be obtained from $y = x^2$ by a combination of vertical stretch by a factor $|a|$, a vertical reflection if $a < 0$, a vertical shift of $f(-\frac{b}{2a})$ units, and a horizontal shift of $-\frac{b}{2a}$ units.
- The domain of a quadratic function is $(-\infty, \infty)$.
- If $a > 0$, then the parabola opens upward, the function has an absolute minimum $f(-\frac{b}{2a})$, and the domain of the function is $[f(-\frac{b}{2a}), \infty)$.
- If $a < 0$, then the parabola opens downward, the function has an absolute maximum $f(-\frac{b}{2a})$, and the domain of the function is $(-\infty, f(-\frac{b}{2a})]$.

Example 2.1.2 Find the vertex form equation for the quadratic function g in figure below as a transformation of $f(x) = x^2$, and then simplify the equation into general form.



Example 2.1.3 Find the domain and range of each function.

(1) $f(x) = 3x^2 + 6x - 5$.

(2) $f(x) = -2x^2 + 4 - 1$.

Example 2.1.4 A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

Example 2.1.5 A local newspaper currently has 84,000 subscribers at a quarterly charge of \$30. Market research has suggested that if the owners raise the price to \$32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?


Example 2.1.6 A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

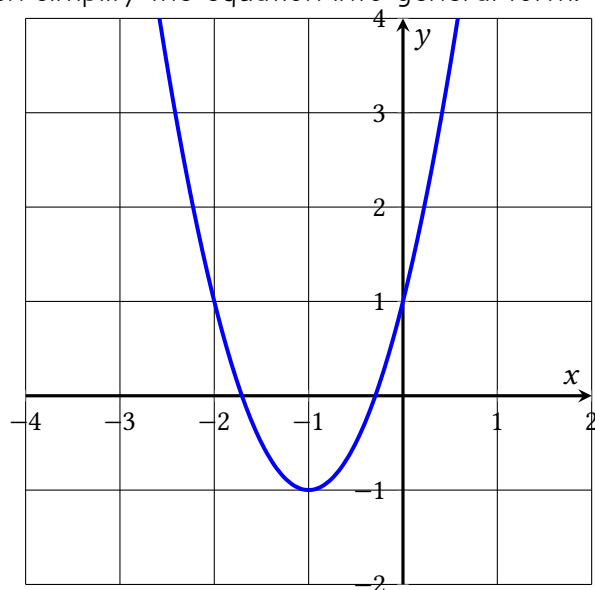
- (1) When does the ball reach the maximum height?
- (2) What is the maximum height of the ball?
- (3) When does the ball hit the ground?


Exercise


 **Exercise 2.1.1** For each of the following functions, (a) $f(x) = x^2 - 4x + 1$, (b) $f(x) = -2x^2 - 4x + 1$,


- (1) write the function in vertex form,
- (2) find the axis of symmetry,
- (3) find the vertex,
- (4) find the y -intercept,
- (5) find the x -intercepts if they exist,
- (6) find the domain and range,
- (7) find the global maximum or minimum if it exist.

 **Exercise 2.1.2** Find the vertex form equation for the quadratic function f in figure below, and then simplify the equation into general form.



 **Exercise 2.1.3** Find the dimensions of the rectangular parking lots producing the greatest area given that 500 feet of fencing will be used to for three sides.

 **Exercise 2.1.4** A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$. Find the maximum height the rocket attains.

 **Exercise 2.1.5** A soccer stadium holds 62,000 spectators. With a ticket price of \$11, the average attendance has been 26,000. When the price dropped to \$9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

2.2 Power and Polynomial Functions

Definition 2.2.1 A **power function** is a function that can be represented in the form

$$f(x) = kx^p,$$

where k and p are real numbers, and k is known as the **coefficient**.

Example 2.2.1 Determine if the function is a power function.

(1) $f(x) = -2x^3$ (2) $f(x) = \frac{1}{x^2}$ (3) $f(x) = \sqrt[3]{x}$ (4) $f(x) = 2^x$ (5) $f(x) = 2x^2 \cdot 3x^5$ (6) $f(x) = \frac{x}{x+1}$

Definition 2.2.2 The **end behavior** of a function f is the general direction that the function f approaches as x goes to ∞ or $-\infty$.

We use an arrow \rightarrow to describe “goes to” or “approaches to”. The notation $x \rightarrow \infty$ or $x \rightarrow -\infty$ means “ x goes to infinity” or “ x goes to negative infinity” respectively. The notation $f(x) \rightarrow \infty$ or $f(x) \rightarrow -\infty$ means “ $f(x)$ goes to infinity” or “ $f(x)$ goes to negative infinity” respectively.

If $f(x) \rightarrow b$ as $x \rightarrow \infty$ or $x \rightarrow -\infty$, then we say the line $y = b$ is a **horizontal asymptote**.

How-to To determine the end behavior of a function f , take a large positive number N .

If $f(N)$ is a large positive number, then $f(x) \rightarrow \infty$ as $x \rightarrow \infty$.

If $-f(N)$ is a large positive number, then $f(x) \rightarrow -\infty$ as $x \rightarrow \infty$.

If $f(-N)$ is a large positive number, then $f(x) \rightarrow \infty$ as $x \rightarrow -\infty$.

If $-f(-N)$ is a large positive number, then $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$.

Example 2.2.2 Determine the end behavior(s) of the function.

(1) $f(x) = -2x^3$

(2) $f(x) = \frac{1}{x^2}$

(3) $f(x) = \sqrt[3]{x}$

Definition 2.2.3 Let n be a non-negative integer. A **polynomial function of degree n** is a function that can be written in the form

$$f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0.$$

- Each a_i is a **coefficient**.
- Each product $a_i x^i$ is a **term** of a polynomial function.
- The term $a_n x^n$ is called the **leading term**. The number a_n is called the **leading coefficient**.
- The number a_0 is called the **constant term**.

Note The end behavior of a polynomial function $f(x) = a_n x^n + \cdots + a_0$ of degree n is completely determined by the end behavior of the power function $g(x) = a_n x^n$.

The domain of a polynomial function is $(-\infty, \infty)$. The range of an odd degree polynomial function is also $(-\infty, \infty)$. The range of an even degree polynomial function is either $[y_{\min}, \infty)$ if $a_n > 0$ or $(-\infty, y_{\max}]$ if $a_n < 0$, where y_{\min} (respectively, y_{\max}) is the absolute minimum (respectively, maximum) of the function.

Example 2.2.3 Determine the end behavior of the function.

(1) $f(x) = 2x^4 - 3x + 1$ (2) $g(x) = -3x^3 + 2x^2 - x$ (3) $h(x) = -4x^6 - 7x^5 + 10x^4 + 2$

Example 2.2.4 Identify the degree, the leading term and the end behavior of the polynomial function.

(1) $f(x) = -3x^2(x - 1)(x + 4)$

(2) $f(x) = 2x^3(1 - x)(x + 1)$

Definition 2.2.4 If f is a polynomial function, then a number c is called a **zero** of f if $f(c) = 0$.

Proposition 2.2.5 Let f be a polynomial and c a real number. Then the following are equivalent:

- (1) c is a zero of f .
- (2) $x = c$ is a solution of the equation $f(x) = 0$.
- (3) $x - c$ is a factor of $f(x)$.
- (4) $(c, 0)$ is an x -intercept of the function of $y = f(x)$.

Example 2.2.5 Find x -intercepts and the y -intercept of the polynomial function $f(x) = x^3 + 3x^2 - x - 3$.

Example 2.2.6 Find x -intercepts and the y -intercept of the polynomial function $f(x) = x^4 + 2x^2 - 3$.

Definition 2.2.6 A **continuous** function has no breaks in its graph. A **smooth** function is a continuous function whose graph that has no sharp corners.

Note Polynomial functions are smooth functions.

Theorem 2.2.7 (Intermediate Value Theorem for Polynomials) If f is a polynomial function and $f(a)f(b) < 0$, then there exists at least one value c between a and b such that $f(c) = 0$.

Corollary 2.2.8 Let f be a polynomial function, a and b real zeros of f . If f has no other zeros between a and b , then either $f(x) > 0$ for all x between a and b or $f(x) < 0$ for all x between a and b .

Example 2.2.7 Determine if the polynomial function $f(x) = 5x^4 - 2x^3 - 20$ has a zero on the interval $[1, 2]$.

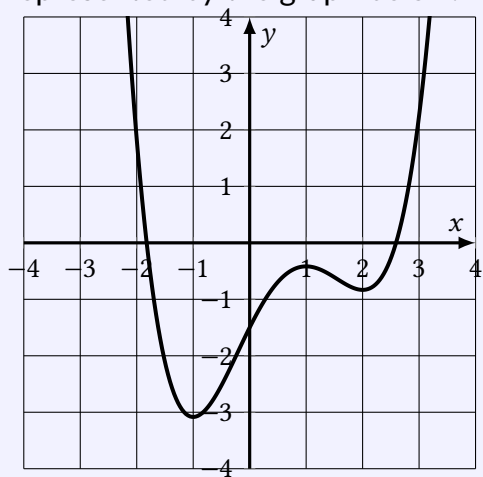
Definition 2.2.9 A **turning point** is a point at which the function values change from increasing to decreasing or decreasing to increasing.

Theorem 2.2.10 (Fundamental Theorem of Algebra) A degree n polynomial function has at least one complex zero.

Proposition 2.2.11 A degree n polynomial function may have at most n real zeros and $n - 1$ turning points.

Example 2.2.8 Consider the polynomial function $f(x) = (x-2)(x+1)(x-4)$. Determine the zeros, the number of turning points, the x -intercepts, and the y -intercept.

Example 2.2.9 What can we conclude about the leading term of the polynomial function $y = f(x)$ represented by the graph below.



2.3 Graphs of Polynomial Functions

2.4 Dividing of Polynomials

2.5 Zeros of Polynomials

2.6 Rational Functions

2.7 Polynomial and Rational Inequalities