MA440 Precalculus Worksheet

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Preface

Those worksheets are developed for the Precalculus course at QCC. Contents in those worksheets are mainly based on the OpenStax Precalculus textbook.

Contents

1.1 Basic Concepts

Definition 1.1.1 A **relation** is a set of ordered pairs. The set of the first components of each ordered pair is called the **domain** and the set of the second components of each ordered pair is called the **range**.

A **function** is a relation that assigns each element in the domain a unique element in the range.

An arbitrary value in the domain is often represented by the lowercase letter x which is called an **independent variable**. An arbitrary output is often represented by the lowercase letter y which is called a **dependent variable**.

Example 1.1.1 Determine if the relation

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\}$$

is a function. Find the domain and the range.

Definition 1.1.2 If a function has x as the independent variable and y as the dependent variable, then we often say that y is a function of x.

Example 1.1.2 Consider items and prices in a grocery store. Is price a function of item? Is item a function of price?

Definition 1.1.3 A function is often named by letters, such as f, F, p, or q. If f is a function of x, then we denote it as $y = f(x)$ which is called the **function notation**. Here $f(x)$ is read as f of x or f at x. The notation $f(x)$ represents the output of the function f for a given input x.

Example 1.1.3 Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

Example 1.1.4 A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ represent?

Example 1.1.5 Using a table to represent the days in the month as the function of month.

Example 1.1.6 Consider the function $f(x) = x^2 + 3x - 4$. Find the values of the following expressions.

(1) $f(2)$ (2) $f(a)$ (3) $f(a+h)$ (4) $\dfrac{f(a+h) - f(a)}{h}$

Example 1.1.7 Consider the function $f(x) = x^2 - 2x$. Find all x values such that $f(x) = 3$.

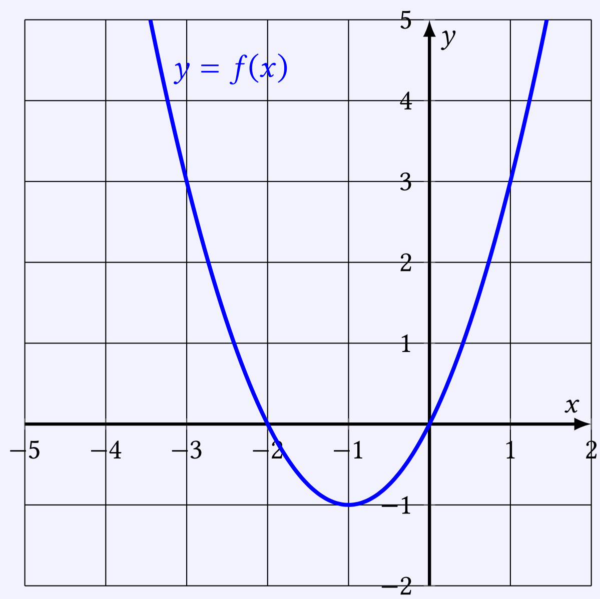
Example 1.1.8 Express the relationship defined by the function $2x - y - 3 = 0$ as a function $y = l(x)$.

Example 1.1.9 Does the equation $x^2 + y^2 = 1$ defines y as a function x. If so, express the relationship as a function $y = f(x)$. If not, under what extra condition does the function $y = f(x)$ exist?

Example 1.1.10 Consider the function $f(x)$ defined by a graph below.

(1) Find $f(-1)$.

(2) Find all x such that $f(x) = 3$.



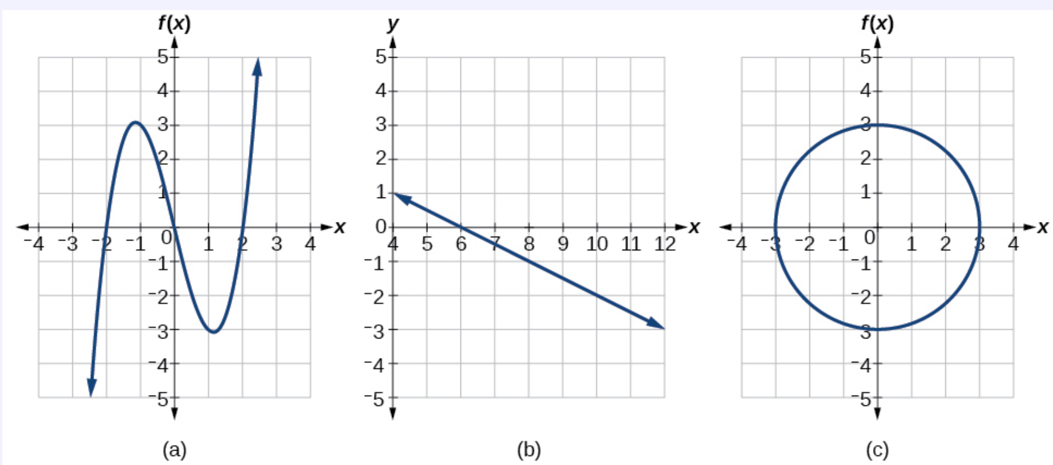
Definition 1.1.4 A function is a **one-to-one function** if each output value corresponds to exactly one input value.

Example 1.1.11 Is the area of a circle a function of its radius? If yes, is the function one-to-one?

How-to *A graph is a function if very vertical line crosses the graph at most once. This method is known as the* ***vertical line test****.*

A function is an one-to-one if very horizontal line crosses the graph at most once. This method is known as the ***horizontal line test****.*

Example 1.1.12 Determine if the graph defines a function. If so, is it a one-to-one function?



Exercises

✍ **Exercise 1.1.1** Consider the function $f(x) = 2x^2 + x - 3$. Find the values of the following expressions.

(1) $f(-1)$ (2) $f(a)$ (3) $f(a+h)$ (4) $\dfrac{f(a+h)-f(a)}{h}$

✍ **Exercise 1.1.2** For the function $f(x) = -4x + 5$, evaluate and simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$.

✍ **Exercise 1.1.3** Consider the function $f(x) = -x^2 - 4x$. Find all x values such that $f(x) = 3$.

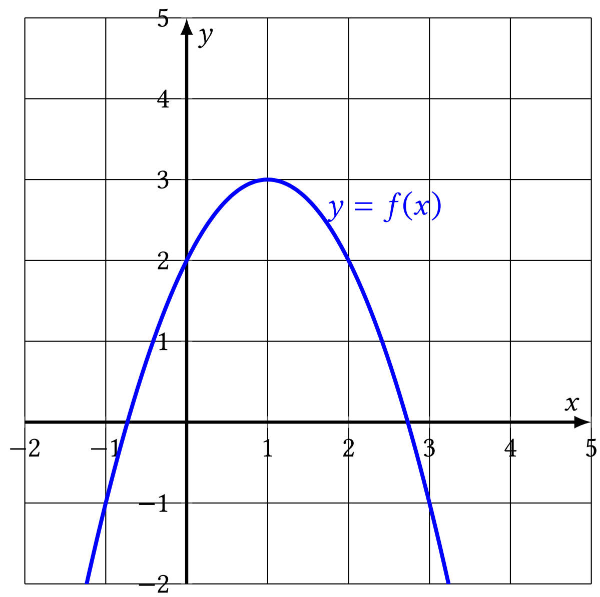
Exercise 1.1.4 Express the relationship defined by the function $3x - 2y - 6 = 0$ as a function $y = l(x)$.

Exercise 1.1.5 If $8x - y^3 = 0$, express y as a function of x.
Is y a one-to-one function of x?

Exercise 1.1.6 Consider the function $f(x)$ defined by a graph below.

(1) Find $f(1)$.

(2) Find all x such that $f(x) = 3$.



1.2 Domains and Ranges

How-to *The domain of a function f consists of possible input values x. Or equivalently, the domain consists of all x values except those that will make the function is undefined.*

The range of a function f consists of all possible output values y. Equivalently, the range consists of y value such that equation $y = f(x)$ has a solution x.

Example 1.2.1 Find the domain of the function

$$f(x) = \frac{x+1}{2-x}.$$

Example 1.2.2 Find the domain of the function

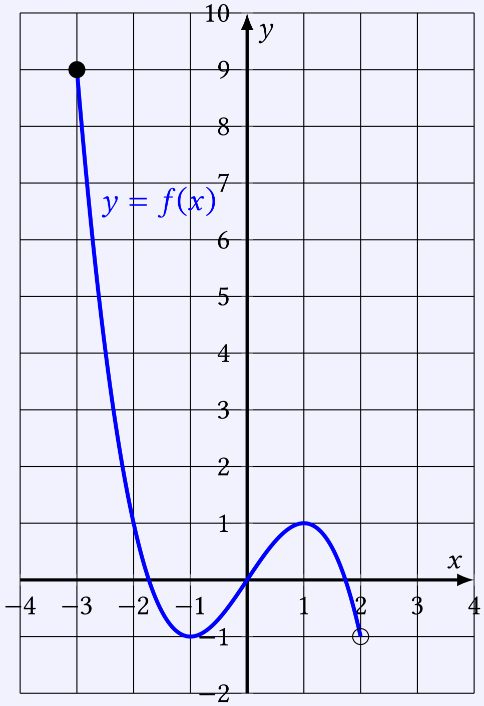
$$f(x) = \sqrt{7-x}.$$

Definition 1.2.1 **Set-builder notation** is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x|$ statement about x$\}$ which is read as, "the set of all x such that the statement about x is true."

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set.

Example 1.2.3 Find the domain of the function $f(x) = \dfrac{\sqrt{x+2}}{x-1}$. Write your answer in set-builder notation and interval notation.

Example 1.2.4 Find the domain and range of the function f whose graph is shown in Figure.



Example 1.2.5 Find the domain and range of the function

$$f(x) = \frac{2}{x+3}.$$

Example 1.2.6 Find the domain and range of the function
$$f(x) = 3\sqrt{x+2}.$$

Example 1.2.7 Consider the piecewise function
$$f(x) = \begin{cases} 2x-3 & \text{if} \quad x \leq -1 \\ -x^2 & \text{if} \quad -1 < x < 1 \\ -2x+4 & \text{if} \quad 1 \leq x. \end{cases}$$

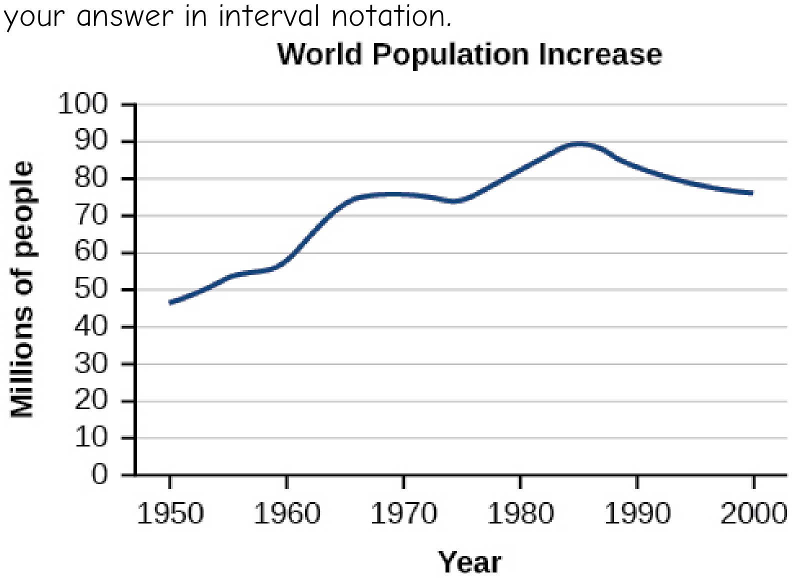
(1) Sketch the graph (2) Find $f(-4)$ (3) Find $f(2)$

Exercises

✍ **Exercise 1.2.1** Find the domain of the function

(1) $f(x) = \dfrac{1+4x}{2x-1}$ (2) $f(x) = \sqrt{5+2x}$ (3) $f(x) = \dfrac{\sqrt{x+1}}{x-1}$ (4) $f(x) = \dfrac{x-2}{x^2+7x-44}$

✍ **Exercise 1.2.2** Estimate the domain and range for the function defined by the graph. Write your answer in interval notation.



✍ **Exercise 1.2.3** Find the domain and range of each of the following functions. Write your answer in set-builder notation and interval notation.

(1) $f(x) = \frac{3}{x-2}$

(2) $f(x) = -2\sqrt{x+4}$

✍ **Exercise 1.2.4** Consider the piecewise function

$$f(x) = \begin{cases} -2x+5 & \text{if} \quad x < -2 \\ x^2 - 1 & \text{if} \quad -2 \leq x \leq 2 \\ 2x - 3 & \text{if} \quad 2 < x. \end{cases}$$

(1) Sketch the graph

(2) Find $f(-4)$

(3) Find $f(2)$

1.3 Rates of Change and Behavior of Graphs

Definition 1.3.1 (Rate of Change) The average rate of change of f over an interval $[a, b]$ is defined as

$$\text{Average Rate Of Change} = \frac{f(b) - f(a)}{b - a}.$$

The average rate of change is the same as the slope of secant line passing through $(a, f(a))$ and $(b, f(b))$.

By taking $x = a$ and $h = b - a$, the average of rate of change is the same the difference quotient of a function f which is defined as

$$\text{Difference Quotient} = \frac{f(x+h) - f(x)}{h}.$$

Example 1.3.1 After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in Table. Find her average speed over the first 6 hours.

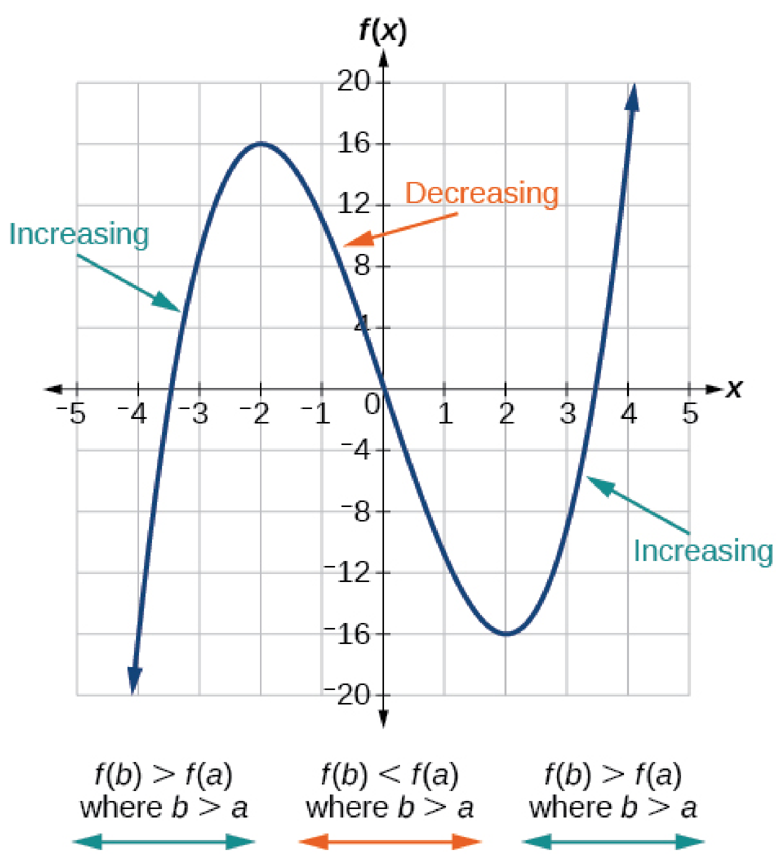
t (hours)	0	1	2	3	4	5	6	7
$D(t)$ (miles)	10	55	90	153	214	240	292	300

Example 1.3.2 Find the average rate of change of $f(x) = x^2 - \dfrac{1}{x}$ over the interval $[2, 4]$.

Example 1.3.3 Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. The answer will be an expression involving a.

Definition 1.3.2 (Increasing and Decreasing) A function f is **increasing** over an interval (a, b) if $f(x_2) > f(x_1)$ for any $x_1 < x_2$ in (a, b). Equivalently, f is increasing over (a, b) if the average rate of change is positive over any subinterval (x_1, x_2) of (a, b).

A function f is **decreasing** over an interval (a, b) if $f(x_2) < f(x_1)$ for any $x_1 < x_2$ in (a, b). Equivalently, f is decreasing over (a, b) if the average rate of change is negative over any subinterval (x_1, x_2) of (a, b).



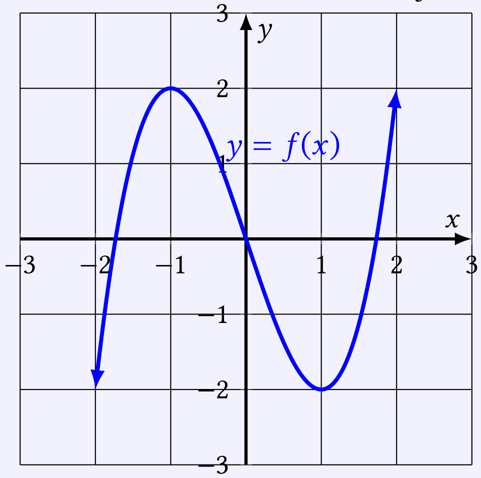
Definition 1.3.3 (Local Maxima and Minima) A function f has a **local maximum** at $x = c$ if $f(c) \geq f(x)$ for any x in a small interval containing c. A small interval containing c is also known as a small neighborhood of c.

A function f has a **local minimum** at $x = c$ if $f(c) \leq f(x)$ for any x in a small interval containing c.

How-to *A function f has a local maximum at $x = c$ if it changes from increasing to decreasing at c in a neighborhood of c.*

A function f has a local minimum at $x = c$ if it changes from decreasing to increasing at c in a neighborhood of c.

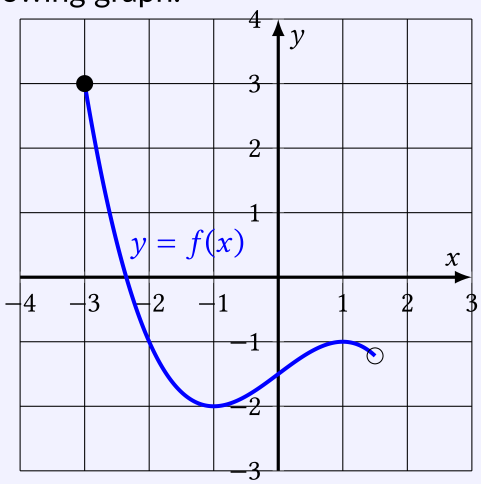
Example 1.3.4 Find the interval of increasing and the interval of decreasing, and the local maxima and local minima of the function f defined by the following graph.



Definition 1.3.4 (Absolute Maxima and Minima) The **absolute maximum** of f at $x = c$ is $f(c)$ where $f(c) \geq f(x)$ for all x in the domain of f.

The **absolute minimum** of f at $x = c$ is $f(c)$ where $f(c) \leq f(x)$ for all x in the domain of f.

Example 1.3.5 Finding the absolute maximum and minimum of the function f defined by the following graph.

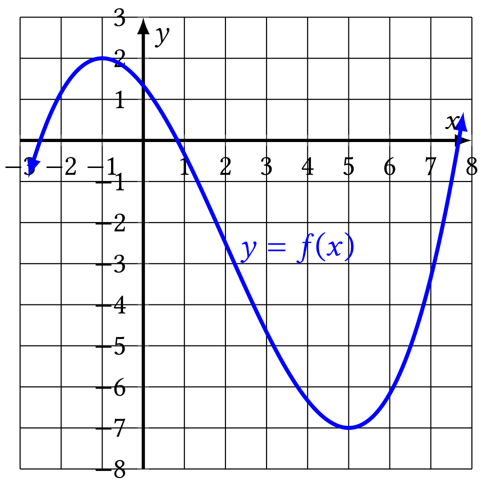


Exercises

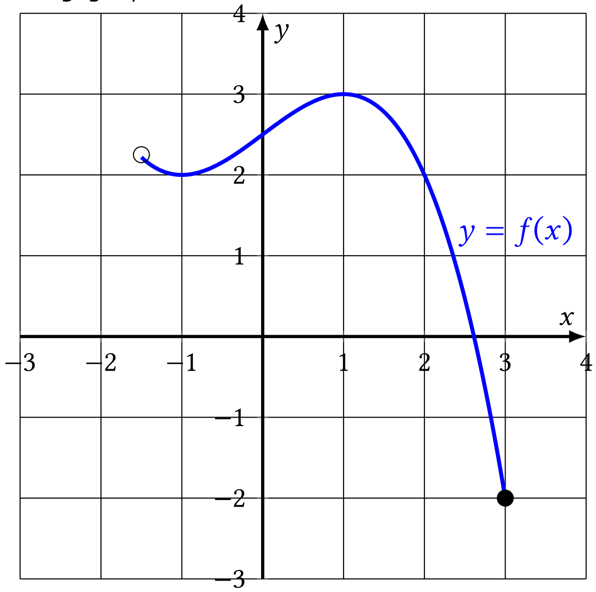
Exercise 1.3.1 The electrostatic force F, measured in newtons, between two charged particles can be related to the distance between the particles d, in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

Exercise 1.3.2 Find the average rate of change of $f(x) = x^2 + 2x - 8$ on the interval $[5, a]$.

Exercise 1.3.3 Find the interval of increasing and the interval of decreasing, and the local maxima and local minima of the function f defined by the following graph.



Exercise 1.3.4 Finding the absolute maximum and minimum of the function f defined by the following graph.



Exercise 1.3.5 Find the interval of increasing and the interval of decreasing, and the local maxima and local minima of the function $f(x) = x^3 - 6x^2 - 15x + 20$ using its graph.

1.4 Combination and Composition of Functions

Definition 1.4.1 (Algebraic Operations of Functions) Let f and g be two functions with domains A and B respectively. We define the linear combination, product, and quotient functions as follows.

Linear combination:	$(af+bg)(x) = af(x)+bg(x)$	with the domain $A \cap B$.
Product:	$(fg)(x) = f(x)g(x)$	with the domain: $A \cap B$.
Quotient:	$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$	with the domain: $A \cap B \cap \{x \mid g(x) \neq 0\}$.

Example 1.4.1 Consider the functions $f(x) = x - 1$ and $g(x) = x^2 - 1$. Find and simplify the functions $(g - f)(x)$ and $\left(\dfrac{g}{f}\right)(x)$, and their domains.

Definition 1.4.2 (Composition of functions) Let f and g be two functions with domains A and B respectively. The **composite function** $f \circ g$ (also called the composition of f and g) is defined as

$$(f \circ g)(x) = f(g(x)) \qquad \text{with the domain:} \quad B \cap \{x \mid g(x) \in A\}.$$

We read the left-hand side as "f composed with g at x," and the right-hand side as "f of g of x."

Example 1.4.2 Consider the functions $f(x) = \sqrt{x-2}$ and $g(x) = x^2 + 1$.

(1) Find and simplify the functions $(f \circ g)(x)$ and $(g \circ f))(x)$. Are they the same function?

(2) Find the domains of $f \circ g$ and $g \circ f$. Are they the same?

Example 1.4.3 Consider $f(t) = t^2 - 4t$ and $h(x) = \sqrt{x+3}$. Evaluate

(1) $\dfrac{f(1)}{g(1)}$ (2) $h(f(-1))$ (3) $(f \circ h)(-1))$ (4) $(f - h)(-1)$

Example 1.4.4 Using the graphs to evaluate the given functions.

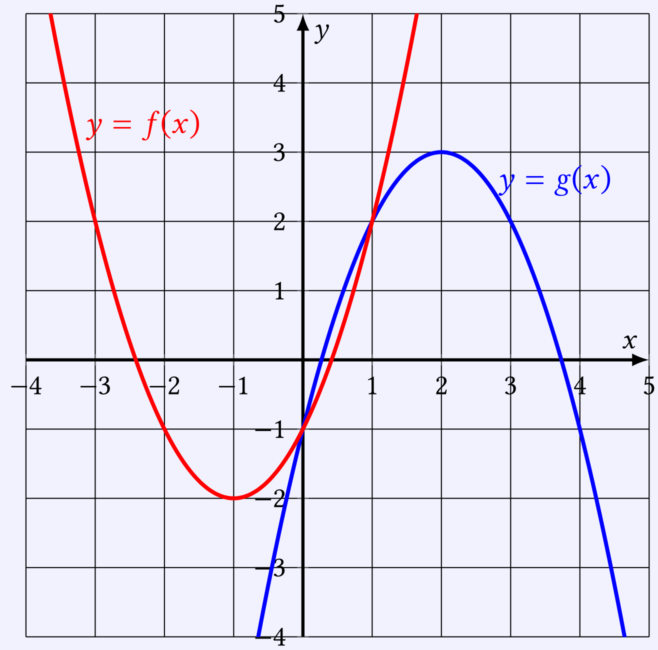
(1) $(f + g)(1)$

(2) $(fg)(1)$

(3) $\left(\dfrac{f}{g}\right)(1)$

(4) $(g \circ f)(-3)$

(5) $f(g(0))$



Example 1.4.5 Consider the function $h(x) = \sqrt{x^2+1}$. Find two functions f and g so that $h(x) = f(g(x))$.

Exercises

✍ **Exercise 1.4.1** Consider the functions $f(x) = x^2 - 1$ and $g(x) = x + 1$.

(1) Find the function $(f - g)(x)$ and its domain.

(2) Find the function $(fg)(x)$ and its domain.

(3) Find $\left(\frac{f}{g}\right)(x)$ and its domain.

(4) Find $(2f - 3g)(1)$.

(5) Find $2fg - \left(\frac{3g}{f}\right)(1)$.

✍ **Exercise 1.4.2** Consider the functions $f(x) = \dfrac{1}{x-2}$ and $g(x) = \sqrt{x+4}$.

(1) Find $f \circ g$ and its domain.

(2) Find $(g \circ f)(3)$.

Exercise 1.4.3 Using the graphs to evaluate the given functions.

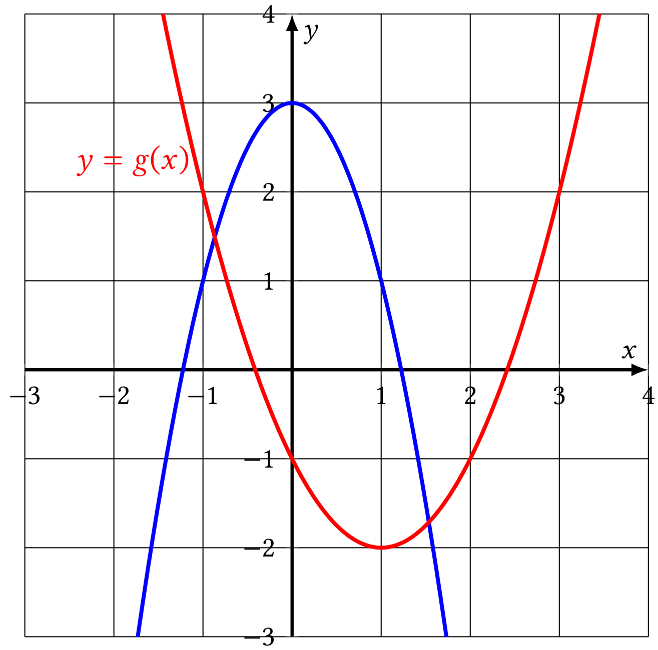
(1) $(f-g)(1)$

(2) $(fg)(0)$

(3) $\left(\dfrac{f}{g}\right)(0)$

(4) $(f\circ g)(2)$

(5) $g(f(0))$



Exercise 1.4.4 Consider the function $h(x) = \sqrt[3]{2x-1}$. Find two functions f and g so that $h(x) = f(g(x))$.

1.5 Transformations

Definition 1.5.1 Given a function $y = f(x)$, the function $y = f(x) + k$, where k is a constant, is a **vertical shift** of the function f.

How-to *Suppose k is positive.*
- *To graph $y = f(x)+k$, shift the graph of $y = f(x)$ upward k units.*
- *To graph $y = f(x)-k$, shift the graph of $y = f(x)$ downward k units.*

Example 1.5.1 Consider the functions $f(x) = x^2$, $g(x) = x^2 - 1$ and $h(x) = x^2 + 2$.
(1) Describe how to get the graph of g from the graph of f.

(2) Describe how to get the graph of h from the graph of f.

(3) Describe how to get the graph of f from the graph of h.

(4) Describe how to get the graph of h from the graph of g.

Definition 1.5.2 Given a function $y = f(x)$, the function $y = f(x - h)$, where h is a constant, is a **horizontal shift** of the function f.

How-to *Suppose h is positive.*
- *To graph $y = f(x-h)$, shift the graph of $y = f(x)$ to the right h units.*
- *To graph $y = f(x+h)$, shift the graph of $y = f(x)$ to the left h units.*

Example 1.5.2 Consider the functions $f(x) = x^2$, $g(x) = (x + 1)^2$ and $h(x) = (x - 2)^2$.
(1) Describe how to get the graph of g from the graph of f.

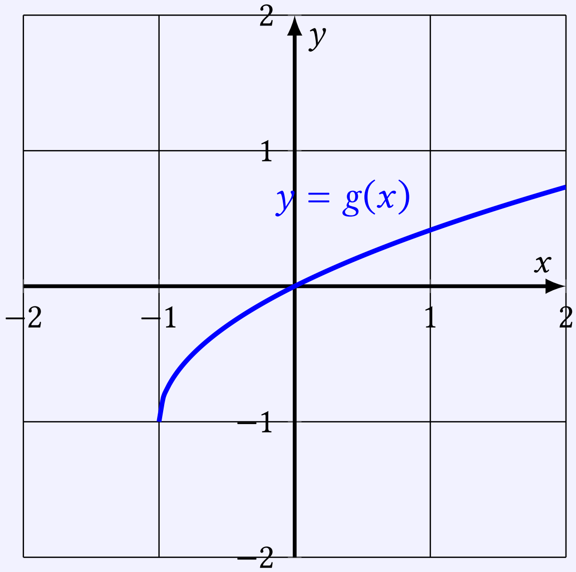
(2) Describe how to get the graph of h from the graph of f.

(3) Describe how to get the graph of f from the graph of h.

(4) Describe how to get the graph of h from the graph of g.

Example 1.5.3 Sketch the graph of $f(x) = |x|$. Then use the graph to sketch the graph of $h(x) = f(x+2) - 1$.

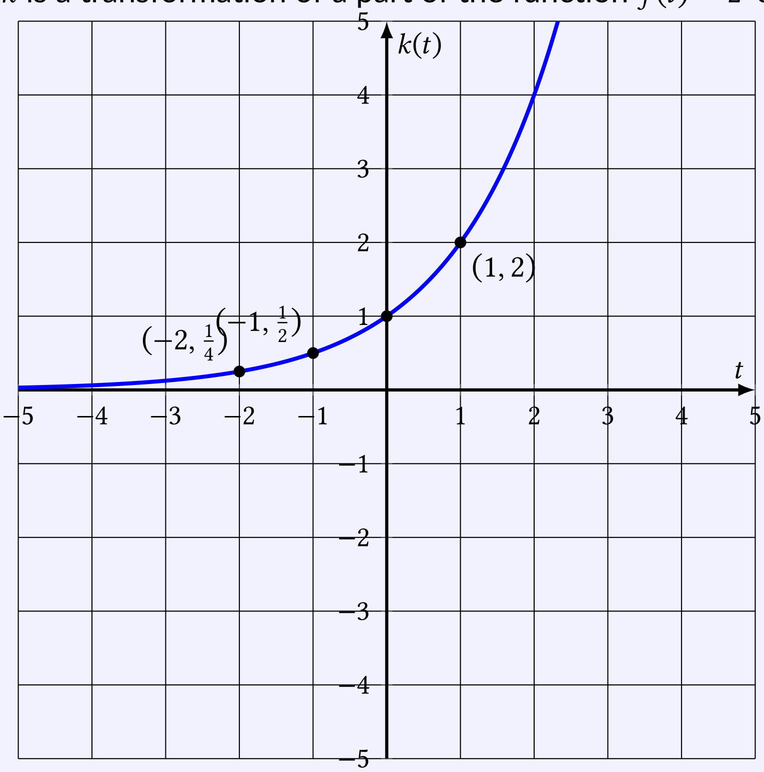
Example 1.5.4 The function $y = g(x)$ shown in the picture is a shift of the square root function $y = \sqrt{x}$. Find $g(x)$.



Definition 1.5.3 Given a function $y = f(x)$, the function $g(x) = -f(x)$ is a **vertical reflection** of the function $y = f(x)$, or a reflection about the x-axis; the function $g(x) = f(-x)$ is a **horizontal reflection** of the function $y = f(x)$ or a reflection about the y-axis.

Example 1.5.5 Reflect the graph of $f(x) = |x - 1|$
(1) first vertically, (2) then horizontally.
Denote the new function by $y = g(x)$. Find $g(x)$.

Example 1.5.6 A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions, and $t > 0$. The function k is a transformation of a part of the function $f(t) = 2^t$ shown below. Sketch the graph of $k(t)$.



Definition 1.5.4 A function is called an **even function** if $f(-x) = f(x)$ for x in the domain of f.
A function is called an **odd function** if $f(-x) = -f(x)$ for x in the domain of f.

Remark The graph of an even function is symmetric about y-axis.
The graph of an odd function is symmetric about the origin. This symmetry is known as a rotation symmetry.

Example 1.5.7 Group the functions according to even, odd, or other.

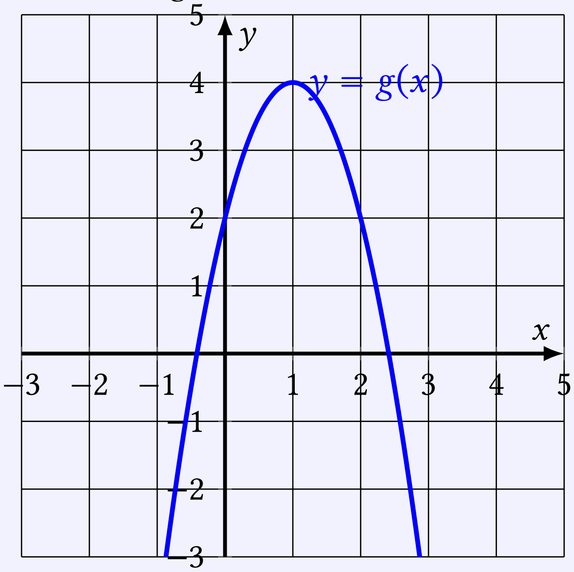
(1) $f(x) = x^2 - 1$ (2) $g(x) = |x - 1|$ (3) $h(x) = x^3 - 2x$ (4) $k(x) = \dfrac{1}{x^2}$.

Definition 1.5.5 Let c be a positive number. The function $g(x) = cf(x)$ is called a **vertical stretch** or **vertical compression** of $y = f(x)$ by a factor of c if $c > 1$ or $0 < c < 1$ respectively.

Remark If $a < 0$, then $g(x) = cf(x)$ is a combination of a vertical stretch or compression with a vertical reflection.

Example 1.5.8 The point $(9, -15)$ is on the graph of $y = f(x)$. Find a point on the graph of $g(x) = \dfrac{1}{3}f(x)$.

Example 1.5.9 The function $y = g(x)$ given in the following graph can be obtained from $f(x) = x^2$ by a combination of shifting, reflecting, and stretching. Describe the transformation and find an equation of g.



Definition 1.5.6 Let c be a positive number. The function $g(x) = f(cx)$ is called a **horizontal stretch** or **horizontal compression** of $y = f(x)$ by a factor of $\frac{1}{c}$ if $0 < c < 1$ or $c > 1$ respectively.

Remark If $c < 0$, then $g(x) = f(cx)$ is a combination of a horizontal stretch or compression with a horizontal reflection.

Example 1.5.10 The function $y = f(x)$ has two x-intercepts $(-2, 0)$ and $(4, 0)$. Determine if the function $g(x) = f(2x)$ has any x-intercepts. If so, find them. Otherwise explain why it has no x-intercept.

Example 1.5.11 Describe how to get the graph of the function $g(x) = 4x^2$ from the graph of the function $f(x)$.

How-to *The graph of the function $g(x) = Af(Bx + C) + D$ can be obtained by the following transformations in the given order.*

(1) *A vertical stretch/compression with the factor $|A|$ followed by a refection about x-axis if $A < 0$.*

(2) *A vertical shift of D units*

(3) *A horizontal shift of C units.*

(4) *A horizontal stretch/compression with the factor $|B|$ followed by a refection about y-axis if $B < 0$.*

Remark Note the horizontal and vertical transformation may be switched.

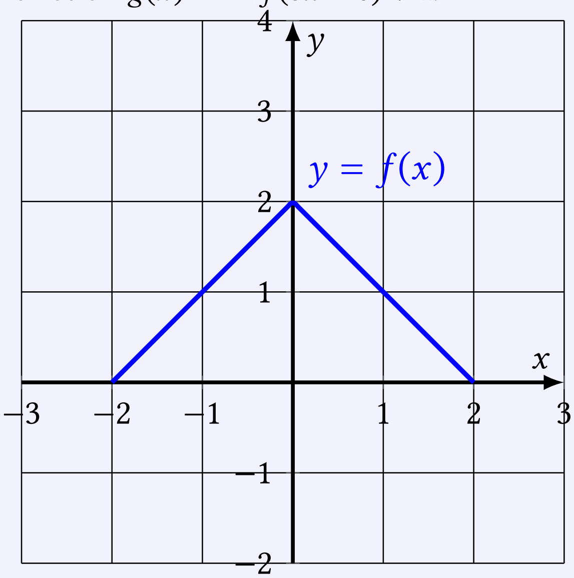
The order of horizontal or vertical transformation depends on how to get the point (x, y) from a point (a, b) on the original function under the substitutions $a = Bx + C$ and $y = Ab + D$.

To get x, one may add $-C$ to both sides first which corresponds to a horizontal shift of $-C$ units, and then multiply by $\frac{1}{B}$ which corresponds to a horizontal stretch/compression by a factor of $\frac{1}{B}$. To get y, one may first multiply b by A which corresponds to a vertical stretch/compression by a factor A and then add D which corresponds to a vertical shift of D units.

Note one may also solve x from $a = Bx + C$ by multiplying $\frac{1}{B}$ first then add $-\frac{C}{B}$ which corresponds to horizontal stretch/compression by a factor $\frac{1}{B}$ followed by a horizontal shift by $-\frac{C}{B}$ units.

Similarly, one may also get y as $y = A(b + \frac{D}{A})$ which leads to a vertical shift of $\frac{D}{A}$ units followed by a vertical stretch/compression by a factor A.

Example 1.5.12 Using the graph of the function $y = f(x)$ given below to sketch the graph of the function $g(x) = -2f(3x - 6) + 4$.



Example 1.5.13 Sketch the graph of the function $g(x) = 2\sqrt{3x-1} - 4$ by a sequence of transformation applied on the graph of $f(x) = \sqrt{x}$.

Example 1.5.14 Find an equation of the function $y = g(x)$ whose graph is obtained from $f(x) = \sqrt{x}$ by the following transformations in the given order.

(1) stretch vertically by a factor of 2

(2) shift downward 2 units

(3) shift 3 units to the left

(4) stretch horizontally by a factor $\frac{1}{2}$.

Exercises

✍ **Exercise 1.5.1** Consider the functions $f(x) = x^2$, $g(x) = (x+1)^2 - 2$ and $h(x) = (x-2)^2 + 1$.

(1) Describe how to get the graph of g from the graph of f.

(2) Describe how to get the graph of h from the graph of g.

✍ **Exercise 1.5.2** Determine if the function is even, odd, or neither.

(1) $f(x) = 1 - x^2$. (2) $g(x) = \sqrt[3]{-x}$. (3) $g(x) = x^4 - x^3$.

✍ **Exercise 1.5.3** Sketch the graph of the function $g(x) = 2|3x - 6| + 4$ by a sequence of transformation applied on the graph of $f(x) = |x|$.

✍ **Exercise 1.5.4** Find an equation of the function $y = g(x)$ whose graph is obtained from $f(x) = \sqrt[3]{x}$ by the following transformations in the given order.

(1) Compress vertically by a factor of $\frac{1}{2}$.

(2) Reflect vertically.

(3) shift downward 2 units.

(4) Compress horizontal by a factor 2.

(5) Shift 3 units to the right.

1.6 Inverse Functions

Definition 1.6.1 Let $y = f(x)$ be a one-to-one function with the domain A. A function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(f(x)) = x$ for all x in A.

The notation f^{-1} is read "f inverse."

Remark

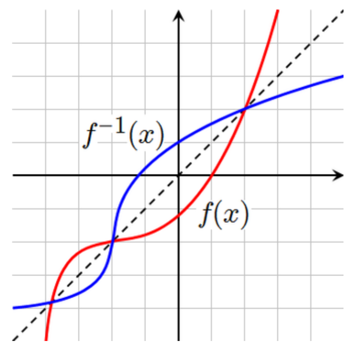
(1) If f is a one-to-one function, then it has a unique inverse function f^{-1}. Here is the proof. Suppose g is also an inverse f. Then $f(g(x)) = x = f(f^{-1}(x))$. Then $g(x) = f^{-1}(f(g(x))) = f^{-1}(f(f^{-1}(x))) = f^{-1}(x)$.

(2) Note that if f^{-1} is the inverse of f, then f is also the inverse of f^{-1} that is $f(f^{-1})(x) = x$ for all x in the domain of f^{-1}.

(3) In general, $f^{-1}(x) \neq f(x)^{-1}$.

(4) The graphs of a one-to-one function f and its inverse f^{-1} are symmetric about the diagonal line $y = x$.

(5) Suppose f has the domain A and the range B, then f^{-1} has the domain B and the range B (and vice verse).



The above graph of f and f^{-1} is taken from Wikipedia.

Example 1.6.1 Let f be a one-to-one function with $f(3) = 4$ and $f(4) = 5$. Find $f^{-1}(4)$.

Example 1.6.2 Let $f(x) = \dfrac{1}{x-1}$ and $g(x) = \dfrac{x+1}{x}$. Determine if g is the inverse function of f.

Example 1.6.3 Consider the function $f(x) = x^2 + 1$ with $x > 0$. Sketch the graph of $y = f^{-1}(x)$ without finding its equation.

How-to *Given a function $y = f(x)$, the inverse function is the solution y of the equation $f(y) = x$. The domain and the range of f and f^{-1} can be obtained from the domains of f and f^{-1}.*

Example 1.6.4 Consider the function $f(x) = 2x - 3$. Find the inverse function f^{-1} and its domain and range.

Example 1.6.5 Consider the function $f(x) = \dfrac{x}{x-1}$. Find the inverse function f^{-1} and its domain and range.

Example 1.6.6 Consider the function $f(x) = 2(x+1)^3 - 1$. Find the inverse function f^{-1} and its domain and range.

Example 1.6.7 Consider the function $f(x) = \sqrt{x-2}$. Find the inverse function f^{-1} and its domain and range.

Example 1.6.8 Find the inverse of each of the following functions if it exists.

Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \dfrac{1}{x}$
Reciprocal squared	Cube Root	Square Root	Absolute Value	
$f(x) = \dfrac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \lvert x \rvert$	

Exercises

✍ **Exercise 1.6.1** Let f be a one-to-one function with $f(-2) = -3$ and $f(-3) = 4$. Find $f^{-1}(-3)$.

✍ **Exercise 1.6.2** Let $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$. Is $g = f^{-1}$?

✍ **Exercise 1.6.3** Consider the function $f(x) = \dfrac{1}{x-1} + 1$. Sketch the graph of f^{-1} without finding its equation.

✍ **Exercise 1.6.4** Consider the function $f(x) = \dfrac{1-x}{x+1}$. Find the inverse function f^{-1} and its domain and range.

✍ **Exercise 1.6.5** Consider the function $f(x) = 3(x-1)^3 + 2$. Find the inverse function f^{-1} and its domain and range.

✍ **Exercise 1.6.6** Consider the function $f(x) = \sqrt{x+1} - 1$. Find the inverse function f^{-1} and its domain and range.

2.1 Quadratic Functions

Definition 2.1.1 A function $f(x) = ax^2 + bx + c$ with $a \neq 0$ is called a **quadratic function**. Its graph is called a **parabola**. By completing the square (let $h = -\dfrac{b}{2a}$ and $k = f(h)$), a quadratic function can written in the **standard form** (or **vertex form**): $f(x) = a(x-h)^2 + k$. The vertical line $x = -\dfrac{b}{2a}$ (or $x = h$) is called the **axis of symmetry**. The **vertex** (h, k) is the intersection of the axis of symmetry and the parabola.

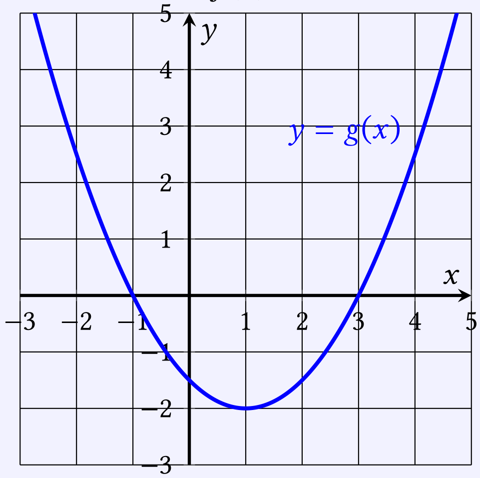
Note *The y-intercept of a quadratic function is $(0, f(0))$. The x-coordinates of x-intercepts are the zeros (or roots) of the function f, that is, the solutions of the equation $f(x) = 0$.*

Example 2.1.1 Find the vertex form of the quadratic function $f(x) = 2x^2 + 4x + 1$ and determine the vertex, axis of symmetry, x-intercepts, and y-intercept of the function.

Note

- *A quadratic function $f(x) = ax^2 + bx + c$ can be obtained from $y = x^2$ by a combination of vertical stretch by a factor $|a|$, a vertical reflection if $a < 0$, a vertical shift of $f\left(-\frac{b}{2a}\right)$ units, and a horizontal shift of $-\frac{b}{2a}$ units.*
- *The domain of a quadratic function is $(-\infty, \infty)$.*
- *If $a > 0$, then the parabola opens upward, the function has an absolute minimum $f\left(-\frac{b}{2a}\right)$, and the domain of the function is $[f\left(-\frac{b}{2a}\right), \infty)$.*
- *If $a < 0$, then the parabola opens downward, the function has an absolute maximum $f\left(-\frac{b}{2a}\right)$, and the domain of the function is $(-\infty, f\left(-\frac{b}{2a}\right)]$.*

Example 2.1.2 Find the vertex form equation for the quadratic function g in figure below as a transformation of $f(x) = x^2$, and then simplify the equation into general form.



Example 2.1.3 Find the domain and range of each function.

(1) $f(x) = 3x^2 + 6x - 5$.

(2) $f(x) = -2x^2 + 4 - 1$.

Example 2.1.4 A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

Example 2.1.5 A local newspaper currently has 84,000 subscribers at a quarterly charge of \$30. Market research has suggested that if the owners raise the price to \$32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Example 2.1.6 A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

(1) When does the ball reach the maximum height?

(2) What is the maximum height of the ball?

(3) When does the ball hit the ground?

Exercise

Exercise 2.1.1 For each of the following functions, (a) $f(x) = x^2 - 4x + 1$,(b) $f(x) = -2x^2 - 4x + 1$,

(1) write the function in vertex form,

(2) find the axis of symmetry,

(3) find the vertex,

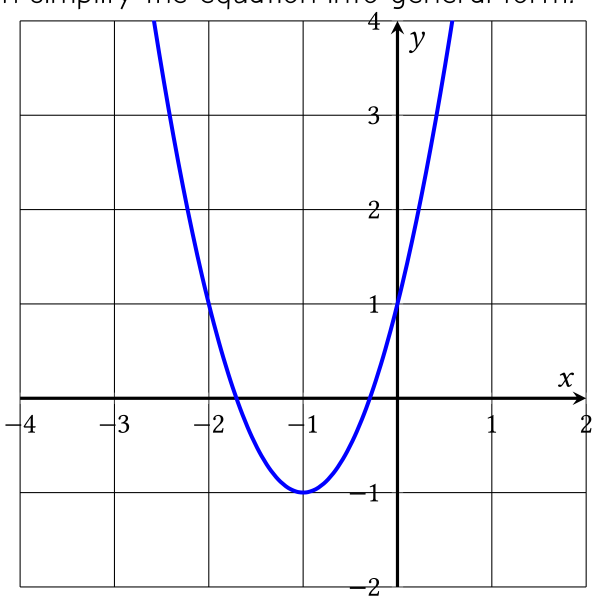
(4) find the y-intercept,

(5) find the x-intercepts if they exist,

(6) find the domain and range,

(7) find the global maximum or minimum if it exist.

Exercise 2.1.2 Find the vertex form equation for the quadratic function f in figure below, and then simplify the equation into general form.



✍ **Exercise 2.1.3** Find the dimensions of the rectangular parking lots producing the greatest area given that 500 feet of fencing will be used to for three sides.

✍ **Exercise 2.1.4** A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$. Find the maximum height the rocket attains.

✍ **Exercise 2.1.5** A soccer stadium holds $62,000$ spectators. With a ticket price of $\$11$, the average attendance has been $26,000$. When the price dropped to $\$9$, the average attendance rose to $31,000$. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

2.2 Power and Polynomial Functions

Definition 2.2.1 A **power function** is a function that can be represented in the form

$$f(x) = kx^p,$$

where k and p are real numbers, and k is known as the **coefficient**.

Example 2.2.1 Determine if the function is a power function.

(1) $f(x) = -2x^3$ (2) $f(x) = \dfrac{1}{x^2}$ (3) $f(x) = \sqrt[3]{x}$ (4) $f(x) = 2^x$ (5) $f(x) = 2x^2 \cdot 3x^5$ (6) $f(x) = \dfrac{x}{x+1}$

Definition 2.2.2 The **end behavior** of a function f is the general direction that the function f approaches as x goes to ∞ or $-\infty$.

We use an arrow $\to$ to describe "goes to" or "approaches to". The notation $x \to \infty$ or $x \to -\infty$ means "x goes to infinity" or "x goes to negative infinity" respectively. The notation $f(x) \to \infty$ or $f(x) \to -\infty$ means "$f(x)$ goes to infinity" or "$f(x)$ goes to negative infinity" respectively.

If $f(x) \to b$ as $x \to \infty$ or $x \to -\infty$, then we say the line $y = b$ is a **horizontal asymptote**.

How-to *To determine the end behavior of a function f, take a large positive number N.*

If $f(N)$ is a large positive number, then $f(x) \to \infty$ as $x \to \infty$.

If $-f(N)$ is a large positive number, then $f(x) \to -\infty$ as $x \to \infty$.

If $f(-N)$ is a large positive number, then $f(x) \to \infty$ as $x \to -\infty$.

If $-f(-N)$ is a large positive number, then $f(x) \to -\infty$ as $x \to -\infty$.

Example 2.2.2 Determine the end behavior(s) of the function.

(1) $f(x) = -2x^3$ (2) $f(x) = \dfrac{1}{x^2}$ (3) $f(x) = \sqrt[3]{x}$

Definition 2.2.3 Let n be a non-negative integer. A **polynomial function** of **degree** n is a function that can be written in the form

$$f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0.$$

- Each a_i is called a **coefficient**.
- Each product $a_i x^i$ is called a **term** of a polynomial function.
- The term $a_n x^n$ is called the **leading term**. The number a_n is called the **leading coefficient**.
- The number a_0 is called the **constant term**.

Note *The end behavior of a polynomial function $f(x) = a_n x^2 + \cdots + a_0$ of degree n is completely determined by the end behavior of the power function $g(x) = a_n x^n$.*

The domain of a polynomial function is $(-\infty, \infty)$. The range of an odd degree polynomial function is also $(-\infty, \infty)$. The range of an even degree polynomial function is either $[y_{min}, \infty)$ if $a_n > 0$ or $(-\infty, y_{max}]$ if $a_n < 0$, where y_{min} (respectively, y_{max}) is the absolute minimum (respectively, maximum) of the function.

Example 2.2.3 Determine the end behavior of the function.

(1) $f(x) = 2x^4 - 3x + 1$ (2) $g(x) = -3x^3 + 2x^2 - x$ (3) $h(x) = -4x^6 - 7x^5 + 10x^4 + 2$

Example 2.2.4 Identify the degree, the leading therm and the end behavior of the polynomial function.

(1) $f(x) = -3x^2(x-1)(x+4)$ (2) $f(x) = 2x^3(1-x)(x+1)$

Definition 2.2.4 If f is a polynomial function, then a number c is called a **zero** of f if $f(c) = 0$.

Proposition 2.2.5 *Let f be a polynomial and c a real number. Then the following are equivalent:*
(1) *c is a zero of f.*

(2) *$x = c$ is a solution of the equation $f(x) = 0$.*

(3) *$x - c$ is a factor of $f(x)$.*

(4) *$(c, 0)$ is an x-intercept of the function of $y = f(x)$.*

Example 2.2.5 Find x-intercepts and the y-intercept of the polynomial function $f(x) = x^3 + 3x^2 - x - 3$.

Example 2.2.6 Find x-intercepts and the y-intercept of the polynomial function $f(x) = x^4 + 2x^2 - 3$.

Definition 2.2.6 A **continuous** function has no breaks in its graph. A **smooth** function is a continuous function whose graph that has no sharp corners.

Note *Polynomial functions are smooth functions.*

Definition 2.2.7 A **turning point** is a point at which the function values change from increasing to decreasing or decreasing to increasing.

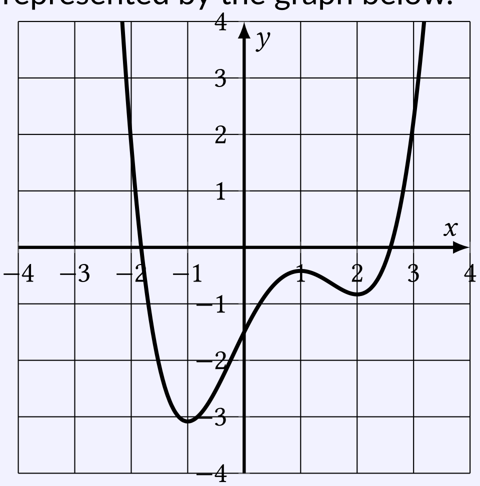
Theorem 2.2.8 (Fundamental Theorem of Algebra) *A degree n polynomial function has at least one complex zero.*

Proposition 2.2.9 *A degree n polynomial function may have at most n real zeros and $n - 1$ turning points.*

[0]For a relatively elementary proof of the Fundamental Theorem of Algebra, please read https://tinyurl.com/tFToA

Example 2.2.7 Consider the polynomial function $f(x) = (x-2)(x+1)(x-4)$. Determine the zeros, the number of turning points, the x-intercepts, and the y-intercept.

Example 2.2.8 What can we conclude about the leading term of the polynomial function $y = f(x)$ represented by the graph below.



Exercises

✍ **Exercise 2.2.1** Find the degree and leading coefficient, and determined the end behavior for the given polynomial.
(1) $f(x) = -2x^4$ (2) $f(x) = 2x^5 - x^3$ (3) $f(x) = -2x(1 - x^2)$ (4) $f(x) = (x^2 - 1)(2x - 1)(x + 2)$

✍ **Exercise 2.2.2** Find x-intercepts (if they exist) and the y-intercept of the polynomial function.
(1) $f(x) = -2x^4 + x^2 + 1$ (2) $f(x) = 2x + x^3 - 3x^5$ (3) $f(x) = x^3 + x^2 - 4x - 4$

2.3 Graphs of Polynomial Functions

Theorem 2.3.1 (Intermediate Value Theorem for Polynomials) *If f is a polynomial function and $f(a)f(b) < 0$, then there exists at least one value c between a and b such that $f(c) = 0$.*

Corollary 2.3.2 *Let f be a polynomial function, a and b real zeros of f. If f has no other zeros between a and b, then either $f(x) > 0$ for all x between a and b or $f(x) < 0$ for all x between a and b.*

Example 2.3.1 Determine if the polynomial function $f(x) = 5x^4 - 2x^3 - 20$ has a zero on the interval $[1, 2]$.

2.4 Dividing of Polynomials

2.5 Zeros of Polynomials

2.6 Rational Functions

2.7 Polynomial and Rational Inequalities