1 Limits of Functions

1.1 Intuitive definition

Limits:

We say that the limit of $f(x)$ as x approaches a is L if $f(x)$ can get arbitrarily close to L when x approaches to a.

$$\lim_{x \to a} f(x) = L$$

Limit from the left:

We say that L is the left limit of $f(x)$ if $f(x)$ can get arbitrarily close to L when x approaches to a from the left.

$$\lim_{x \to a^-} f(x) = L.$$

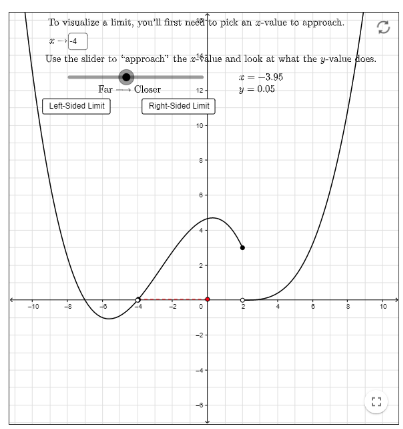
Limit from the right:

We say that L is the right limit of $f(x)$ if $f(x)$ can get arbitrarily close to L when x approaches to a from the right.

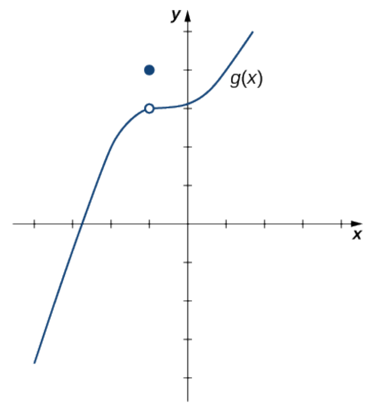
$$\lim_{x \to a^+} f(x) = L.$$

1.2 Determine limits from graphs

Example 1.1. Using the graph of the function f shown below to determine the limit $\lim\limits_{x \to -4} f(x)$ and the limit $\lim\limits_{x \to 2} f(x)$.



Example 1.2. Using the graph of the function g to evaluate the limit $\lim_{x \to -1} g(x)$.



1.3 Determine limits numerically using tables

Numerically, we say that $\lim_{x \to a} f(x) = L$ if for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$.

Example 1.3. Use a table to determine the limit $\lim_{x \to 1} x$.

Example 1.4. Use a table to determine the limit $\lim_{x \to 1} c$, where c is a constant number.

Example 1.5. Use a table to determine the limit $\lim_{x \to 0} \frac{\sin x}{x}$.

Example 1.6. Use a table to determine the limit $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$.

1.4 One-sided limits and their relations with limits

Example 1.7. For the function $f(x) = \begin{cases} x+1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$, determine each of the following limits.

(1) $\lim_{x \to 2^-} f(x)$

(2) $\lim_{x \to 2^+} f(x)$

(3) $\lim_{x \to 2} f(x)$

Example 1.8. For the function $f(x) = \begin{cases} x^2 + 1, & \text{if } x < 1 \\ 3x - 1, & \text{if } x \geq 1 \end{cases}$, determine each of the following limits.

(1) $\lim_{x \to 1^-} f(x)$

(2) $\lim_{x \to 1^+} f(x)$

(3) $\lim_{x \to 1} f(x)$

Theorem 1.1. *Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. Then,*

$$\lim_{x \to a} f(x) = L$$

if and only if $\lim_{x \to a^-} f(x) = L$ *and* $\lim_{x \to a^+} f(x) = L$.

1.5 Infinite limits

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

(1) If the values of $f(x)$ can be arbitrarily large as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is positive infinity and we write

$$\lim_{x \to a^-} f(x) = +\infty.$$

(2) If the values of $f(x)$ can be arbitrarily large as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is negative infinity and we write

$$\lim_{x \to a^-} f(x) = -\infty.$$

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

(1) If the values of $f(x)$ can be arbitrarily large as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the left is positive infinity and we write

$$\lim_{x \to a^+} f(x) = +\infty.$$

(2) If the values of $-f(x)$ can be arbitrarily large as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the left is negative infinity and we write

$$\lim_{x \to a^+} f(x) = -\infty.$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a

(1) If the values of $f(x)$ can be arbitrarily large as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is positive infinity and we write

$$\lim_{x \to a} f(x) = +\infty.$$

(2) If the values of $-f(x)$ can be arbitrarily large as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is negative infinity and we write

$$\lim_{x \to a} f(x) = -\infty.$$

Example 1.9. Evaluate each of the following limits, if possible.

(1) $\lim\limits_{x \to 0^-} \frac{1}{x}$

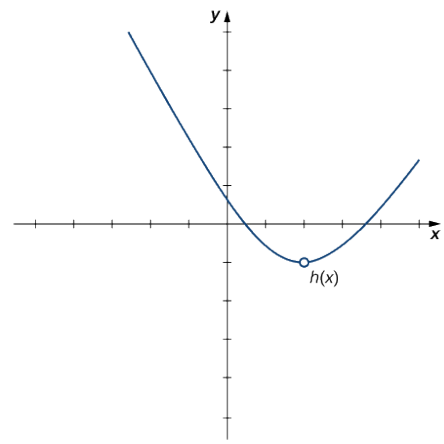
(2) $\lim\limits_{x \to 0^+} \frac{1}{x}$

(3) $\lim\limits_{x \to 0} \frac{1}{x}$

1.6 Practice

Exercise 1.1. Use the graph of $h(x)$ shown below to determine the limit $\lim\limits_{x \to 2} h(x)$.

Exercise 1.2. Determine the limit $\lim\limits_{x \to 1} \dfrac{\frac{1}{x} - 1}{x - 1}$.



Exercise 1.3. Determine the limit $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$.

Exercise 1.4. Determine the limits

(1) $\lim_{x \to 1^-} \frac{|x-1|}{x-1}$.

(2) $\lim_{x \to 1^+} \frac{|x-1|}{x-1}$.

(3) $\lim\limits_{x \to 1^+} \dfrac{|x-1|}{x-1}$.

Exercise 1.5. Evaluate each of the following limits, if possible.

(1) $\lim\limits_{x \to 0^-} \frac{1}{x^2}$

(2) $\lim\limits_{x \to 0^+} \frac{1}{x^2}$

(3) $\lim\limits_{x \to 0} \frac{1}{x^2}$

2 Limit Laws

2.1 Limits under arithmetical operations

Basic Limit Results

For any real number a and any constant c,

(1) $\lim_{x\to a} x = a$

(2) $\lim_{x\to a} c = c$

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

- **Sum law for limits**:

$$\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$$

- **Difference law for limits**:

$$\lim_{x\to a}(f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M$$

- **Constant multiple law for limits**:

$$\lim_{x\to a} cf(x) = c \cdot \lim_{x\to a} f(x) = cL$$

- **Product law for limits**:

$$\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M$$

- **Quotient law for limits**:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$$

  for $M \neq 0$.

- **Power law for limits**:

$$\lim_{x\to a} \left(f(x)\right)^n = \left(\lim_{x\to a} f(x)\right)^n = L^n$$

  for every positive integer n.

- **Root law for limits**:

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$$

for all L if n is odd and for $L \geq 0$ if n is even.

Theorem 2.1. *Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then,*

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$$

when $q(a) \neq 0$.

Example 2.1. Evaluate $\lim_{x \to 2} \frac{2x^2-3x+1}{x^3+4}$

Example 2.2. Evaluate $\lim_{x \to 1} \sqrt{\frac{5x^2-1}{x^3+1}}$

The following observation allows us to evaluate many limits of the type that the function f is undefined at a but a function g equivalent to f away from a is well defined at a.

Theorem 2.2. *If for all $x \neq a$, $f(x) = g(x)$ over some open interval containing a, then*

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

Example 2.3. Evaluate $\lim_{x \to -3} \frac{\frac{1}{x+2} + 1}{x+3}$.

Example 2.4. Evaluate $\lim\limits_{x \to -1} \dfrac{\sqrt{x+2}-1}{x+1}$.

Example 2.5. Evaluate $\lim\limits_{x \to 3} \left(\dfrac{1}{x-3} - \dfrac{4}{x^2-2x-3} \right)$.

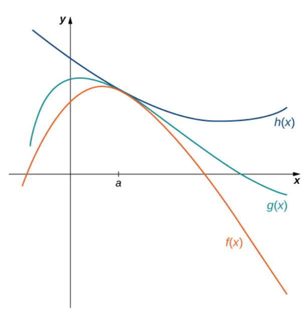
Theorem 2.3 (Squeeze Theorem)**.** *Let $f(x)$, $g(x)$, and $h(x)$ be defined for all $x \neq a$ over an open interval containing a. If*

$$f(x) \leq g(x) \leq h(x)$$

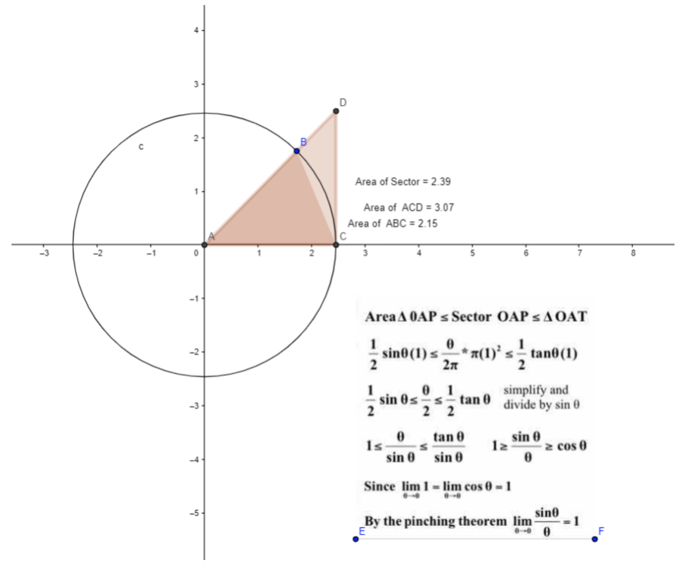
for all $x \neq a$ in an open interval containing a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

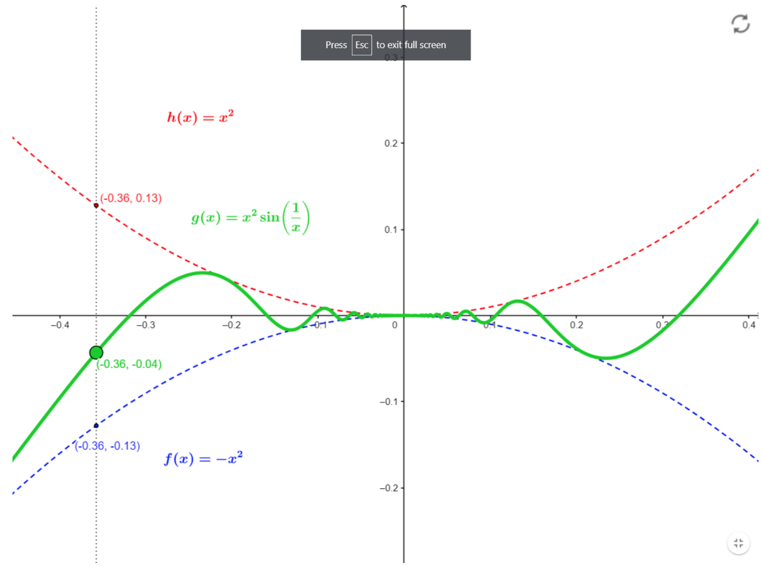
where L is a real number, then $\lim\limits_{x \to a} g(x) = L$.



Example 2.6. Evaluate the limit $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$



Example 2.7. Use the squeeze theorem to evaluate $\lim_{x\to 0} x^2 \sin\frac{1}{x}$.



Example 2.8. Suppose that $\lim_{x \to a} |f(x) - 2| = 0$. Evaluate the limit $\lim_{x \to a} f(x)$.

2.2 Limits of indeterminate forms

Some limits of indeterminate forms, such as $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, and $\infty - \infty$, can be evaluated using algebraic methods. Here are some examples.

Example 2.9. Evaluate the limit $\lim_{x \to 0} \dfrac{1 - \cos x}{x}$

Proposition 2.4. *Let f be function defined near* 0 *and b be number. If* $\lim_{x\to 0}\dfrac{f(x)}{x} = L$ *and* $b \neq 0$, *then* $\lim_{x\to 0}\dfrac{f(bx)}{x} = bL$.

Example 2.10. Evaluate the limit $\lim_{\theta\to 0}\frac{\sin(3\theta)}{5\theta}$

Example 2.11. Evaluate $\lim_{x\to 2}\left(\dfrac{1}{x^2-2x}-\dfrac{1}{2x-4}\right)$.

2.3 Apply limit laws to infinite limits

Limit laws mostly work for infinite limits. However, one should be very careful when apply a limit laws with infinite limits.

Example 2.12. Evaluate $\lim_{x\to 2^-}\dfrac{x-3}{x^2-2x}$.

2.4 Practice

Exercise 2.1. Evaluate $\lim_{x \to -1} \sqrt{\frac{8x^2-9x+10}{x^2+2}}$

Exercise 2.2. Evaluate $\lim_{x \to 3} \dfrac{x^2-9}{x-3}$.

Exercise 2.3. Evaluate $\lim_{x \to 49} \dfrac{\sqrt{x}-7}{x-49}$.

Exercise 2.4. Evaluate $\lim_{x \to 0} |x| \sin\left(\frac{1}{x}\right)$

Exercise 2.5. Evaluate $\lim_{x\to 1}\left(\frac{1}{x^2-x}-\frac{1}{2x^2-3x+1}\right)$.

Exercise 2.6. Evaluate the limit $\lim_{t\to 0}\frac{\cos(3t)-1}{\sin t}$

Exercise 2.7. Evaluate $\lim_{x\to 1}\frac{x+2}{(x-1)^2}$.

3 Continuity

3.1 Continuous at a point

Definition 3.1. A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied:

(1) $f(a)$ is defined

(2) $\lim_{x \to a} f(x)$ exists

(3) $\lim_{x \to a} f(x) = f(a)$

A function is discontinuous at a point a if it fails to be continuous at a.

Example 3.1. Using the definition, determine whether the function $f(x) = \dfrac{x^2 - 4}{x - 2}$ is continuous at $x = 2$. Justify the conclusion.

Example 3.2. Determining Continuity at a Point, Condition 2 Using the definition, determine whether the function $f(x) = \begin{cases} -x^2 + 4, & \text{if } x \le 3 \\ 4x - 8, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$. Justify the conclusion.

Example 3.3. Determining Continuity at a Point, Condition 3** Using the definition, determine whether the function $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

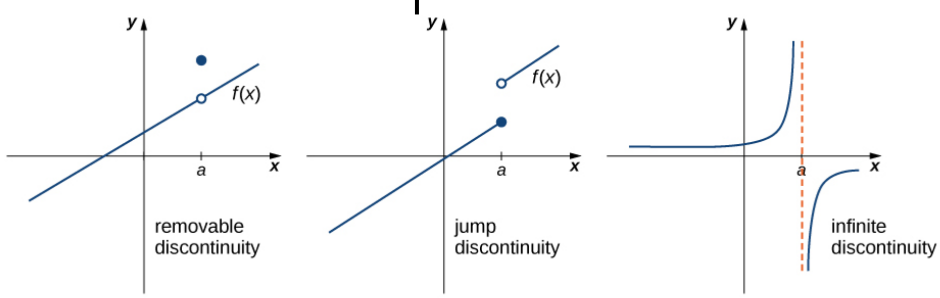
3.2 Discontinuities

Definition 3.2. If $f(x)$ is discontinuous at a, then

(1) f has a removable discontinuity at a if $\lim_{x \to a} f(x)$ exists. (Note: When we state that $\lim_{x \to a} f(x)$ exists, we mean that $\lim_{x \to a} f(x) = L$, where L is a real number.)

(2) f has a jump discontinuity at a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$. (Note: When we state that $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)

(3) f has an infinite discontinuity at a if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.



(a) (b) (c)

Example 3.4. In Example, we showed that $f(x) = \dfrac{x^2 - 4}{x - 2}$ is discontinuous at $x = 2$. Classify this discontinuity as removable, jump, or infinite.

3.3 Continuity from the right and from the left

Definition 3.3. A function $f(x)$ is said to be continuous from the right at a if $\lim_{x \to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be continuous from the left at a if $\lim_{x \to a^-} f(x) = f(a)$

3.4 Continuity over an interval

Definition 3.4. A function is continuous over an open interval if it is continuous at every point in the interval. A function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous from the right at a and is continuous from the left at b. Analogously, a function $f(x)$ is continuous over an interval of the form $(a, b]$ if it is continuous over (a, b) and is continuous from the left at b. Continuity over other types of intervals are defined in a similar fashion.

Theorem 3.5 (Rules of continuity)**.** *Let f and g be two functions continuous over an interval I. Then*

(1) *the linear combination $af + bg$ is continuous over the interval I;*

(2) *the multiplication $f \cdot g$ is continuous over the interval I;*

(3) *the quotient $\frac{f}{g}$ is continuous over the intersection of I and domain of g.*

Corollary 3.6. *Polynomials and rational functions are continuous over their domains.*

Example 3.5. State the interval(s) over which the function $f(x) = \dfrac{x-1}{x^2+2x}$ is continuous.

Example 3.6. State the interval(s) over which the function $f(x) = \sqrt{4-x^2}$ is continuous.

3.5 Continuity of Composite Functions

Theorem 3.7. *If $f(x)$ is continuous at L and $\lim_{x\to a} g(x) = L$, then $\lim_{x\to a} f\big(g(x)\big) = f\big(\lim_{x\to a} g(x)\big) = f(L)$. In particular, if g is continuous at a, then $f \circ g$ is continuous at a.*

Example 3.7. Evaluate the limit $\lim_{x\to 0} \frac{\sin^2 x}{x^2}$.

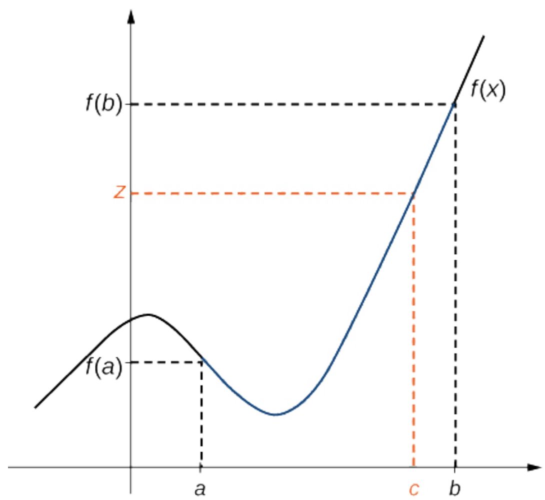
3.6 Continuity of Trigonometric Functions

Theorem 3.8. *Trigonometric functions are continuous over their entire domains.*

Example 3.8. Evaluate the limit $\lim_{x \to 0} \cos(3x)$.

3.7 The Intermediate Value Theorem

Theorem 3.9 (Intermediate Value Theorem)**.** *Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$.*



Example 3.9. Show that $f(x) = x - \cos x$ has at least one zero.

3.8 Practice

Exercise 3.1. Using the definition, determine whether the function $f(x) = \begin{cases} 2x+1, & \text{if } x < 1 \\ 2, & \text{if } x = 1 \\ -x+4, & \text{if } x > 1 \end{cases}$ is continuous at $x = 1$. If the function is not continuous at 1, indicate the condition for continuity at a point that fails to hold.

Exercise 3.2. Classify this discontinuity as removable, jump, or infinite.

(1) Determine whether $f(x) = \dfrac{x+2}{x+1}$ is continuous at -1. If the function is discontinuous at -1, classify the discontinuity as removable, jump, or infinite.

(2) Determine whether $f(x) = \begin{cases} x^2, & \text{if } x \neq 1 \\ 3, & \text{if } x = 1 \end{cases}$ is continuous at 1. If f is not continuous at 1, classify the discontinuity as removable, jump, or infinite.

(3) Determine whether the function $f(x) = \begin{cases} -x+1, & \text{if } x \leq 2 \\ x^2-3, & \text{if } x > 2 \end{cases}$

is continuous at $x = 2$. If f is not continuous at 1, classify the discontinuity as removable, jump, or infinite.

Exercise 3.3. State the interval(s) over which the function $f(x) = \frac{x-1}{\sqrt{x+3}}$ is continuous.

Exercise 3.4. Evaluate the limit $\lim_{x \to \frac{\pi}{3}} \tan(2\pi - 3x)$.

Exercise 3.5. Show that $f(x) = x^3 - x^2 - 3x + 1$ has a zero over the interval $[0, 1]$.

Exercise 3.6. For $f(x) = 1/x$, $f(-1) = -1 < 0$ and $f(1) = 1 > 0$. Can we conclude that $f(x)$ has a zero in the interval $[-1, 1]$?

4 Defining the Derivative

4.1 Difference Quotient

Let f be a function defined on an interval I containing a. If $x \neq a$ is in I, then the *difference quotient.* is

$$Q = \frac{f(x) - f(a)}{x - a}.$$

If $h \neq 0$ is chosen so that $a + h$ is in I, then

$$Q = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$

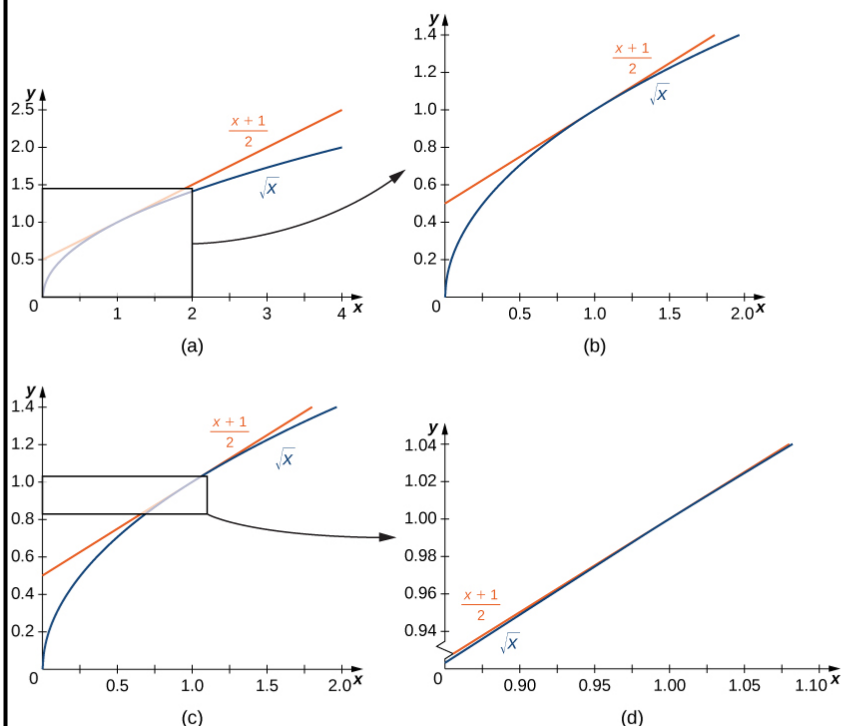
is a difference quotient with increment h.

The slope of a tangent line is the limit of a difference quotient

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Equivalently,

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



Smooth curves are locally linear

Example 4.1. Find the equation of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

4.2 Derivative

Definition 4.1. Let f be a function defined in an open interval containing a. The **derivative of the function** $f(x)$ **at** a, denoted by $f'(a)$, is defined by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided this limit exists.

Alternatively, we may also define the derivative of $f(x)$ at a as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

The function f is said **differentiable** at a if the limit $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists.

Example 4.2. For $f(x) = x^2 + 3x + 2$, find $f'(1)$ using the definition of derivative.

Example 4.3. Find the line tangent to $f(x) = \sqrt{x}$ at $x = 4$.

Example 4.4. The following limit defines the derivative of a function f at some number a. Find the function f and a.

$$\lim_{h\to 0} \frac{2(2+h)^2 - 8}{h}$$

Example 4.5. Find the derivative of the following function at $x = 0$ if it exists.

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

4.3 Instantaneous Rate of Change

Definition 4.2. The **instantaneous rate of change** of a function $f(x)$ at a value a is its derivative $f'(a)$.

The **average velocity** is a rate of change

$$v_{ave} = \frac{s(t) - s(a)}{t - a}.$$

The **instantaneous velocity** is the limit of the average velocity

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

Example 4.6. A rock is dropped from a height of 64 feet. Its height above ground at time t seconds later is given by $s(t) = -16t^2 + 64$, $0 \le t \le 3$. Find its instantaneous velocity 1 second after it dropped.

4.4 Practice

Exercise 4.1. Find the slope of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4$.

Exercise 4.2. Find the tangent line to $f(x) = \sin(x)$ at $x = 0$.

Exercise 4.3. The following limit defines the derivative of a function f at some number a. Find the function f and a.

$$\lim_{h \to 0} \frac{4\sqrt[3]{8 + h} - 8}{h}$$

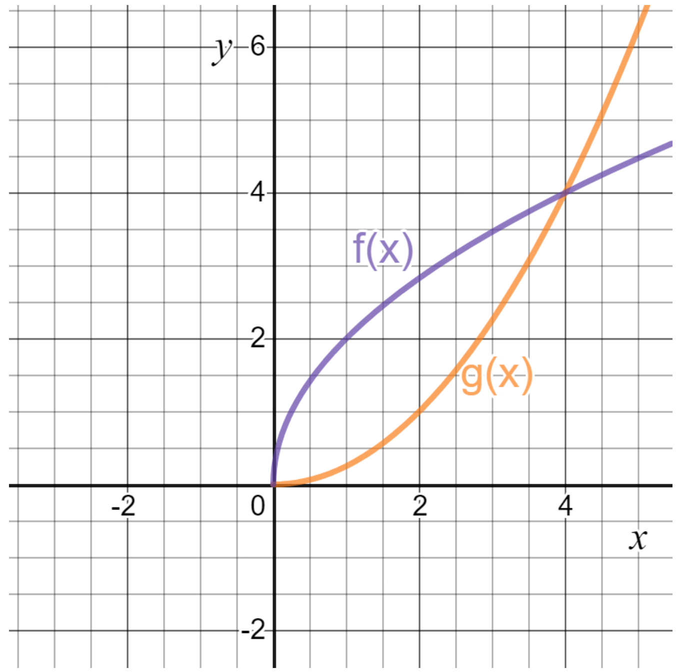
Exercise 4.4. Determine whether the derivative of the following function at $x = 0$ exists.

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Exercise 4.5. A coffee shop determines that the daily profit on scones obtained by charging s dollars per scone is $P(s) = -20s^2 + 150s - 10$. The coffee shop currently charges \$3.25 per scone. Find $P'(3.25)$, the rate of change of profit when the price is \$3.25 and decide whether or not the coffee shop should consider raising or lowering its prices on scones.

Exercise 4.6. Two particles traveling along straight lines side by side from start at the same time. The graphs of their position functions $f(t)$ and $g(t)$ are given below.

At time $t = 4$, which particle travels slower? Why?


y
6
4
2
-2
-2
0
2
4
x
f(x)
g(x)

5 Derivative Functions

5.1 Derivative Function - Definition

Definition 5.1. Let f be a function. The **derivative function**, denoted by f', is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

A function is said to be **differentiable over an open set** U if it is differentiable at every point in U.

A function is called a **differential function** if it is differentiable over its domain. At boundary points of the domain, the differentiability is taken to be the left or right differentiability.

Example 5.1. Determine whether the function $f(x) = x^3$ is differentiable and find the derivative function if it is differentiable.

Example 5.2. Determine whether the function $f(x) = \sqrt{x}$ is differentiable and find the derivative function if it is differentiable.

5.2 Notations for Derivatives and Differentiation

When a function is given in the form $y = f(x)$, we also use y' to denote the derivative function. The notations f' and y' are known as Lagrange's "prime" notation.

As the derivative function is essentially (the limit) a difference quotient, we also use $\frac{\mathrm{d}y}{\mathrm{dx}}$ (or $\mathrm{d}y/\mathrm{dx}$) to denote the derivative functions. The notation $\frac{\mathrm{d}y}{\mathrm{dx}}$ was introduced by Leibniz and called Leibniz's notation.

Remark. The notation $\mathrm{d}x$ and $\mathrm{d}y$ may be considered as variables and are called **differentials**. Indeed, $\mathrm{d}x = x - a$ is the difference, by $\mathrm{d}y = f'(a)(x-a)$ is only an approximation of $f(x) - f(a)$.

The process of calculating a derivative is known as **differentiation**. We view the prime $'$ or better the notation $\frac{\mathrm{d}}{\mathrm{d}x}$ as an operator and call it the **differential operator**.

Sometimes, a differential operator is also denoted as D or D_x to indicate the independent variable x. Those notations are known as Euler's notation.

5.3 Differentiability implies continuity

Theorem 5.2. *If the function f is differentiable at a, then f is continuous at a.*

Remark. (1) If a function is not continuous, it cannot be differentiable.

(2) Not every continuous function is differentiable.

Example 5.3. (Continuous but not differentiable functions)

(1) The function $f(x) = |x|$ is continuous but failed to be differentiable at 0. Its graph has a sharp corner at 0..

(2) The function $f(x) = \sqrt[3]{x}$ also fails to be differentiable at 0. Because it has a vertical tangent line at 0.

(3) The function

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = \end{cases}$$

is continuous but fails to be differentiable at a point in more complicated ways.

Example 5.4. Find values of a and b that make the following function differential.

$$f(x) = \begin{cases} ax+ & x \geq 1 \\ x^2 - 3 & x < 1 \end{cases}$$

5.4 Higher derivatives

Higher derivatives are defined as repeated differentiations of functions.

The **second derivative** $f''(x) = (f'(x))'$ is defined the derivative of the first derivative of f.

The **third derivative** is defined as $f'''(x) = (f''(x))'$.

The n-**th derivative** is defined recursively as $f^{(n)}(x) = (f^{(n-1)}(x))'$.

Example 5.5. Find the second derivative f'' of the function $f(x) = 2x^2 - 3x + 1$.

Example 5.6. The position of a particle along a coordinate axis at time t (in seconds) is given by $s(t) = 3t^2 - 4t + 1$ (in meters). Find the function that describes its acceleration at time t.

5.5 Practice

Exercise 5.1. Determine whether the function $f(x) = x^2 - 3x$ is differentiable and find the derivative function if it is differentiable.

Exercise 5.2. Determine whether the function $f(x) = \sqrt[3]{x}$ is differentiable and find the derivative function if it is differentiable.

Exercise 5.3. Determine whether the function $f(x) = \frac{1}{x+1}$ is differentiable and find the derivative function if it is differentiable.

Exercise 5.4. Find values of a and b that make the following function differentiable at $x = 3$.

$$f(x) = \begin{cases} ax + b, & \text{if } x < 3 \\ x^2, & \text{if } x \geq \end{cases}$$

Exercise 5.5. The position of a particle along a coordinate axis at time t (in seconds) is given by $s(t) = 2t^3 - 3t^2 + t$ (in meters). Find the function that describes its acceleration at time t.

6 Rules of Derivatives

6.1 Basic Rules

The Constant Rule: Let c be a constant number. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0.$$

The Power Rule: Let n be a positive integer. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}.$$

Example 6.1. Let $f(x) = x^4$. Find $f'(x)$.

6.2 Linear Combination Rules

Let f and g be differentiable functions. Let a and b be two constant numbers. Then

$$(af + bf)'(x) = af'(x) + bg'(x).$$

Example 6.2. Let $f(x) = 4x^5 - 3x^2 + 7$. Find $f'(x)$

6.3 The Product and Quotient Rules

Product Rule: Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x))\\ =&\frac{\mathrm{d}}{\mathrm{d}x}(f(x))\cdot g(x)+\frac{\mathrm{d}}{\mathrm{d}x}(g(x))\cdot f(x).\end{aligned}$$

Example 6.3. Find the derivative of the function $f(x) = (x^3 - 2x + 1)(x^2 - 3x + 5)$.

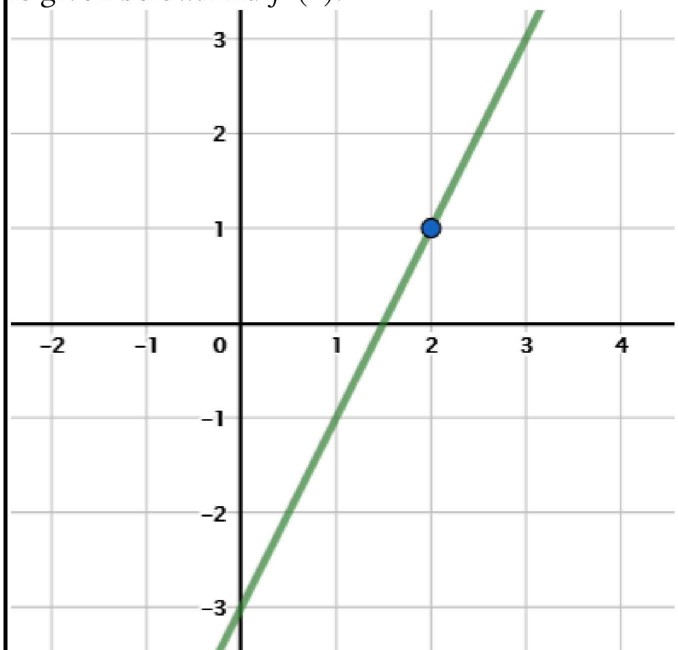
Example 6.4. For $p(x) = f(x)g(x)$, use the product rule to find $p'(2)$ if $f(2) = 3,\ f'(2) = -4,\ g(2) = 1$, and $g'(2) = 6$.

The Quotient Rule: Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{\mathrm{d}}{\mathrm{d}x}(f(x))\cdot g(x)-\frac{\mathrm{d}}{\mathrm{d}x}(g(x))\cdot f(x)}{\big(g(x)\big)^2}.$$

Example 6.5. Find the derivative of the function $f(x) = \dfrac{x^2 - 3x}{3x - 5}$.

Example 6.6. Let $f = \dfrac{g}{x^2+1}$ where the graph of the function g is given below.Find $f'(2)$.



6.4 Extended Power Rule

Let k is a real number. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^k) = kx^{k-1}.$$

If k is a negative integer, the formula follows from the quotient rule. The proof of the general cases has to be postponed after logarithmic and exponential functions in Calculus II.

Example 6.7. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^{-5})$.

Example 6.8. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{1}{\sqrt[3]{x}}$.

6.5 More Examples and Practice

Example 6.9. Let $f(x) = 3x^2h(x) + \dfrac{g(x)}{x}$. Find $f'(x)$ in terms of h, g, h' and g'.

Example 6.10. Determine the values of x for which $f(x) = x^3 - 7x^2 + 8x + 1$ has a horizontal tangent line.

Example 6.11. Find equations of the normal line to the curve $y = 4\sqrt{x} - x$ at $(1, 3)$.

6.6 Practice

Exercise 6.1. Let $f(x) = x^{11}$. Find $f'(x)$.

Exercise 6.2. Let $g(x) = 5x^{10} - 5x^3 - 9$. Find $g'(x)$.

Exercise 6.3. Find the equation of the line tangent to the graph of $f(x) = x^2 - 4x + 6$ at $x = 1$.

Exercise 6.4. Let $h(x) = f(x)(x^3 - 3x^2 - 2x)$ where f is differentiable, $f(1) = 2$ and $f'(1) = 3$. Find $h'(x)$.

Exercise 6.5. Find $h'(x)$ for $h(x) = \dfrac{1}{x^5}$.

Exercise 6.6. Let f and g be differentiable functions such that $f(3) = 2$, $f'(3) = -1$, $g(3) = -2$ and $g'(3) = 1$. Find the $h'(3)$ where $h(x) = \dfrac{f(x) - 2}{g(x)}$.

Exercise 6.7. Find the derivative of the function $f(x) = \dfrac{1}{\sqrt[7]{x^3}}$.

Exercise 6.8. Find $g'(x)$ for $g(x) = \frac{3x^2+5x-7}{\sqrt{x}}$.

Exercise 6.9. Let $h(x) = \dfrac{2x^3 f(x)}{3x+g(x)}$. Find $h'(x)$ in terms of f, g, f' and g'.

Exercise 6.10. Determine the values of x for which $f(x) = 2x^3 - 5x^2 - x + 7$ has a tangent line parallel to $y = 3x - 1$.

Exercise 6.11. Suppose that f and g are both differentiable functions with $f(4) = 3$, $g(4) = 2$, $f'(4) = 9$ and $g'(4) = 5$. Find $h'(4)$ where

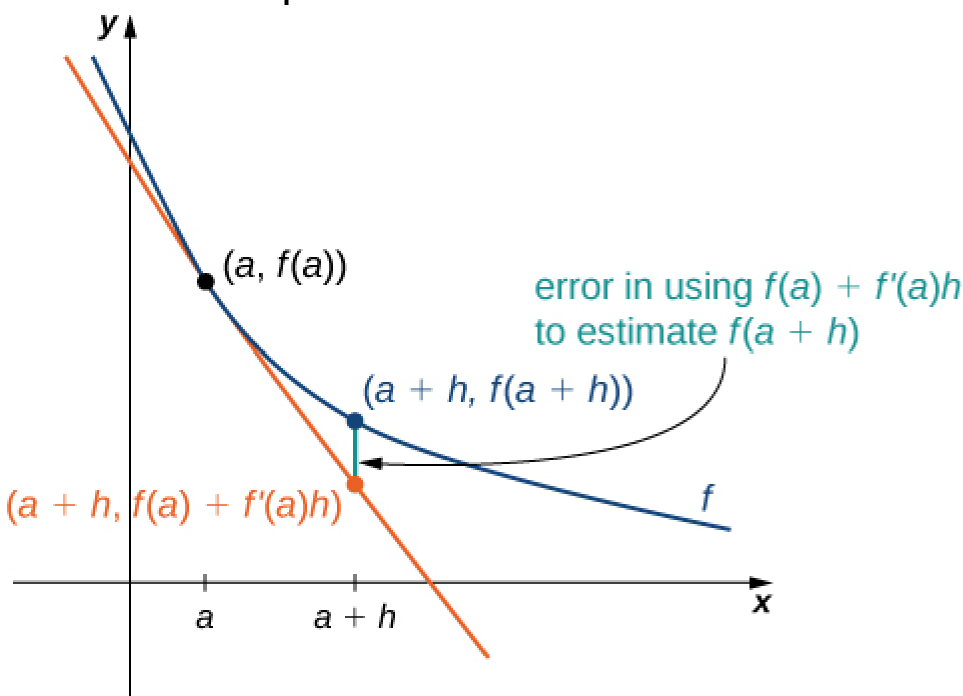
$$h(x) = \frac{2}{\sqrt{x}} - \frac{f(x)-1}{g(x)}.$$

7 Derivatives as Rates of Change

If $f(x)$ is a function defined on a small interval $[a, a+h]$, then the amount of change of $f(x)$ over the interval $f(a+h) - f(a)$ is approximately $f'(a)h$ if f is differentiable at a. This is because

$$\lim_{h\to 0} h \cdot \left(\frac{f(a+h) - f(a)}{h} - f'(a) \right) = 0$$

which means $f(a+h) - f(a) - f'(a)h$ can be arbitrarily small as long as h is small enough.



7.1 Motion Along a Line

Definition: Suppose an object is moving along a coordinate line. Let $s(t)$ be a function giving the position of the object at time t.

- The **displacement** of the object over the time interval from t to $t + \Delta t$ is $\Delta s = s(t + \Delta t) + s(t)$, where Δt is an increment of time.
- The **average velocity** of the object over a time interval from t to $t + \Delta t$ is $v_{ave} = \dfrac{\Delta s}{\Delta t}$.

- The **velocity (instantaneous velocity)** of the object at time t is given by $v(t) = \lim_{\Delta t \to 0} \dfrac{\Delta s}{\Delta t} = s'(t)$.
- The **speed** of the object at time t is given by $|v(t)|$.
- The **acceleration** of the object at t is given by $a(t) = v'(t) = s''(t)$.

Example 7.1. A ball is dropped from a height of 64 feet. Its height above ground (in feet) t seconds later is given by $s(t) = -16t^2 + 64$.

(1) What is the velocity of the ball when it hits the ground?

(2) When is the object at rest?

Example 7.2. A particle moves along a coordinate axis. Its position at time t is given by $s(t) = t^3 - 9t^2 + 24t + 4$.

(1) Find $v(t)$ and $a(t)$ and use these values to answer the following questions.

(2) On what time intervals is the particle moving from left to right?

(3) On what time intervals is the particle moving from right to left?

7.2 Population Change

Definition 7.1. If $P(t)$ is the number of entities present in a population, then the **population growth rate** of $P(t)$ is defined to be $P'(t)$.

Example 7.3. The population of a city is tripling every 5 years. If its current population is 10,000, what will be its approximate population 2 years from now?

7.3 Derivative in Economics

- If $C(x)$ is the cost of producing x items, then the **marginal cost** $MC(x)$ is $MC(x) = C'(x)$.
- If $R(x)$ is the revenue obtained from selling x items, then the **marginal revenue** $MR(x)$ is $MR(x) = R'(x)$.
- If $P(x) = R(x) - C(x)$ is the profit obtained from selling x items, then the **marginal profit** $MP(x)$ is defined to be $MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x)$.

Example 7.4. Suppose that the profit obtained from the sale of x fish-fry dinners is given by $P(x) = -0.03x^2 + 8x - 50$. Use the marginal profit function to estimate the profit from the sale of the 101st fish-fry dinner.

7.4 Practice

Exercise 7.1. A particle moves along a coordinate axis. Its position at time t is given by $s(t) = t^2 - 5t + 1$. Is the particle moving from right to left or from left to right at time $t = 3$?

Exercise 7.2. A particle moves along a coordinate axis. Its position at time t is given by $s(t) = t^3 - 8t$.

(1) Determine the direction the particle is traveling when $s(t) = 0$.

(2) On what time intervals is the particle moving from right to left?

(3) Determine the direction the train is traveling when $a(t) =$

0?

Exercise 7.3. The current population of a mosquito colony is known to be 3,000; that is, $P(0) = 3,000$. If $P'(0) = 100$, estimate the size of the population in 3 days, where t is measured in days.

Exercise 7.4. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 20 + 100\sqrt{x} + 0.01x^2$.

(1) Find the marginal cost function.

(2) Find $C'(100)$ and explain it meaning.

(3) What is the difference between $C'(100)$ and the additional cost for producing the 1001st commodity?

8 Derivatives of Trigonometric Functions

8.1 The Derivative of the Sine Function

Recall the formula of sum of angles

$$\sin(x+h) = \sin x \cos h + \cos x \sin h$$

$$\cos(x+h) = \cos x \cos h - \sin x \sin h.$$

Then for any x, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h} \\
&= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x.
\end{aligned}$$

Example 8.1. Find the derivative of $f(x) = x^2 \sin x$.

8.2 Derivatives of Other Trigonometric Functions

Using the sum angle formula for cosine and limit laws of quotients, we obtain the following formulas of derivative of trigonometric functions.

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$
$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

Example 8.2. Find the derivative of $f(x) = \cos x \sin x$.

Example 8.3. Find the derivative of $f(x) = \dfrac{\cos x - 1}{\sin x + 1}$.

Example 8.4. Find the derivative of $f(x) = 2x \tan x - 3 \cot x$.

Example 8.5. Find the derivative of $f(x) = \sec^2 x - x \csc x$.

Example 8.6. Find the 59-th derivative $\dfrac{\mathrm{d}^{59}}{\mathrm{d}x^{59}}(\sin x)$.

8.3 Practice

Exercise 8.1. Find the derivative of $f(x) = \dfrac{x^3 + x}{\sin x}$.

Exercise 8.2. Find the derivative of $f(x) = \dfrac{\sin x + \cos x}{\sin x - \cos x}$.

Exercise 8.3. Find the derivatives of the function $f(x) = 3\tan x + \sqrt{x^5}$.

Exercise 8.4. Find the derivative of $f(x) = (x+\tan x)(\sec x+x^2)$.

Exercise 8.5. Find the 3-th derivative $\frac{\mathrm{d}^3}{\mathrm{d}x^3}(\tan x)$.

9 The Chain Rule

If $y = f(u)$ and $u = g(x)$ are differentiable functions, intuitively, using Leibniz's notation, you may find

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}.$$

This identity is indeed true and called the Chain Rule which is one of the most important of the differentiation rules.

Theorem 9.1. *If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and*

$$F'(x) = f'(g(x)) \cdot g'(x).$$

In the chain rule, $f'(g(x))$ mean the "output" of the derivative function f' for the "input" $g(x)$.

The theorem can be proved using the error function of an estimation. Let $\varepsilon(t) = \frac{f(u+t)-f(u)}{t} - f'(u)$ and $k = g(x+h) - g(x)$. Then

$$\begin{aligned}\frac{f(g(x+h)) - f(g(x))}{h} &= \frac{f(g(x)+k) - f(g(x))}{h} \\ &= \frac{k(\varepsilon(k) + f'(g(x))}{h}.\end{aligned}$$

Taking limits will give the chain rule formula.

Example 9.1. (General Power Rule): Let f be a function differentiable at x and $h(x) = (f(x))^n$. Find $h'(x)$.

Example 9.2. Find the derivative of $f(x) = \dfrac{1}{(x+1)^3}$.

Example 9.3. Find the derivative of $f(x) = \sin(\frac{\pi}{2} - x)$.

Example 9.4. Find the derivative of $f(x) = \tan(\cos x + 1)$.

Example 9.5. Find the derivative of $f(x) = \dfrac{x}{(2x+1)^3}$.

Example 9.6. Find the derivative $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{\sqrt{1+\sin(x^2+1)}}\right)$.

Example 9.7. Find all points on the curve $y = \cos x - \cos^2 x$ at which the tangent line is horizontal.

9.1 Practice

Exercise 9.1. Find the derivative of $y = \sqrt{x^2 + 5}$.

Exercise 9.2. Find the derivative $\dfrac{\mathrm{d}}{\mathrm{d}x} \cos(u + 1)$ where $u = x^3$.

Exercise 9.3. Find the derivatives of the function $f(x) = \sqrt{\frac{(x+1)(x-2)}{(x-1)(x+2)}}$.

Exercise 9.4. Find all points on the curve $y = \sqrt{4x+1}$ at which the tangent line is parallel to the line $2x - y = 1$.

10 Implicit Differentiation

As an application of the chain rule, the technique of **implicit differentiation** allows us to find the derivative of an implicitly defined function without ever solving for the function explicitly.

Problem-Solving Strategy: Implicit Differentiation

(1) Take the derivative of (or differentiate)both sides of the equation. Keep in mind that y is a function of x.

(2) Solve for y' (or $\frac{\mathrm{d}y}{\mathrm{d}x}$) from the resulting equation.

Example 10.1. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ where y is a function of x defined by the equation $x^3 + y^2 = 1$.

Example 10.2. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ where y is a function of x defined by the equation $2\sin x - 1 = \cos y$.

Example 10.3. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the function y implicitly defined by the equation $x^3 \sin y + \tan y = 3x + 2$.

Example 10.4. Find $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ given that y is implicitly defined by $x^2 - y^2 = 15$.

Example 10.5. Find the equations of lines tangent to the curve $2xy = \dfrac{x}{y} + \dfrac{y}{x}$ at the point (1,1).

Example 10.6. The number of cars produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $40x^{\frac{1}{3}}y^{\frac{2}{3}} = 480$.

(1) Find a formula in terms of x and y for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(2) Find the value of of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point (27,8).

Example 10.7. Find all points (x, y) on the graph of the equation $2x^2+8y^2 = x+4y$ such that $x > 0$, $y > 0$ and at where the tangent line is

(1) horizontal

(2) vertical

10.1 Practice

Exercise 10.1. Find y' where y is a function of x defined implicitly by the equation $x^2 - xy + y^2 = 1$.

Exercise 10.2. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for y defined implicitly by the equation $x^2 \cos y + y^2 = 3x + 1$.

Exercise 10.3. Find the $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ where y is a function defined implicitly by the equation $x^3 + y^2 = 3x + 1$.

Exercise 10.4. Find the equation of the line tangent to the graph of $y^3 + x^3 - 3xy = 0$ at the point $(\frac{3}{2}, \frac{3}{2})$.

Exercise 10.5. Find all points (x, y) on the graph of the equation $2x^2+8y^2 = x+4y$ such that $x > 0, y > 0$ and at where the tangent line is

(1) horizontal

(2) vertical

11 Related Rates

In many real-world applications, related variables are changing with respect to a same variable. For example, the volume V and height h of water in a cylindrical tank change with respect to time t. Such two rates of change, for example, $\frac{\mathrm{d}V}{\mathrm{d}t}$ and $\frac{\mathrm{d}h}{\mathrm{d}t}$ are called **related rates**.

To solve a related rate problem, the following strategy will be helpful.

Problem-Solving Strategy: Solving a Related-Rates Problem

(1) Assign symbols to all variables involved in the problem. Draw a figure if applicable.

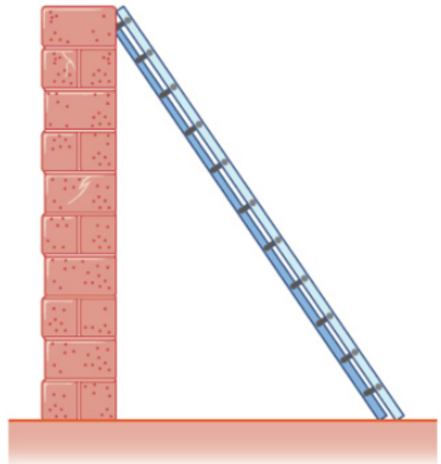
(2) State, in terms of the variables, the information that is given and the rate to be determined.

(3) Find an equation relating the variables introduced in step 1.

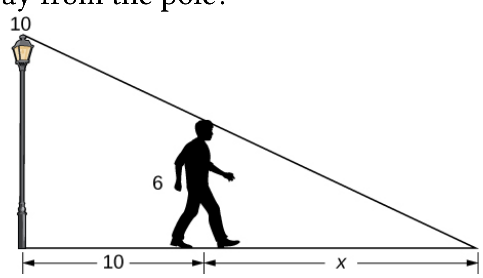
(4) Using the chain rule, differentiate both sides of the equation found in step 3 with respect to the independent variable. This new equation will relate the derivatives.

(5) Substitute all known values into the equation from step 4, then solve for the unknown rate of change.

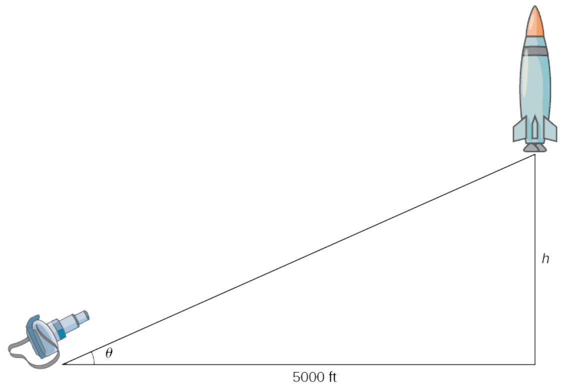
Example 11.1. A 13-ft ladder is leaning against a wall. If the bottom of the ladder slides away from the wall at a rate 2 ft/min, how fast is the top of the ladder slides down the wall when the bottom of the ladder is 12 ft from the wall.



Example 11.2. A 6-ft-tall person walks away from a 10-ft lamppost at a constant rate of 3 ft/sec. What is the rate that the tip of the shadow moves away from the pole when the person is 10 ft away from the pole?

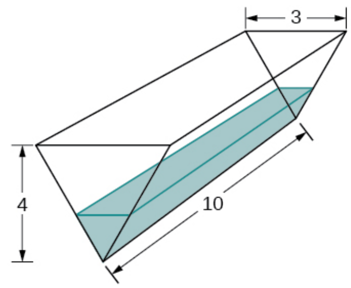


Example 11.3. A rocket is launched so that it rises vertically. A camera is positioned 5000 ft from the launch pad. When the rocket is 1000 ft above the launch pad, its velocity is 600 ft/sec.



Example 11.4. The altitude of a triangle is increasing at a rate of 1.5 centimeters/minute while the area of the triangle is increasing at a rate of 5 square centimeters/minute. At what rate is the base of the triangle changing when the altitude is 11 centimeters and the area is 85 square centimeters?

Example 11.5. A trough has ends shaped like isosceles triangles, with width 3 m and height 4 m, and the trough is 10 m long. Water is being pumped into the trough at a rate of 5 m^3/min. At what rate does the height of the water change when the water is 1 m deep?



Example 11.6. Find the rate at which the surface area of the water changes when the water is 10 ft high if the cone leaks water at a rate of 10 ft^3/min.

11.1 Practice

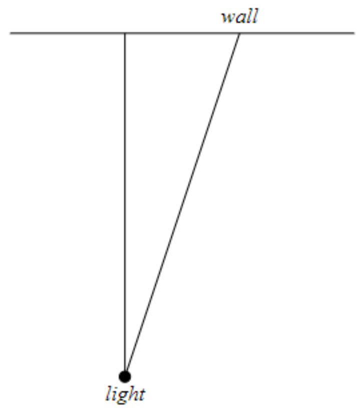
Exercise 11.1. The volume of a cube decreases at a rate of 10 m/sec. Find the rate at which the side of the cube changes when the side of the cube is 2 m.

Exercise 11.2. An airplane is flying overhead at a constant elevation of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man. If the plane is flying at the rate of 600 ft/sec, at what rate is the distance between the man and the plane increasing when the plane passes over the radio tower?

Exercise 11.3. Two airplanes are flying in the air at the same height: airplane A is flying east at 250 mi/h and airplane B is flying north at 300 mi/h. If they are both heading to the same airport, located 30 miles east of airplane A and 40 miles north of airplane B, at what rate is the distance between the airplanes changing?

Exercise 11.4. A circle is inside a square. The radius of the circle is decreasing at a rate of 4 meters per minute and the sides of the square are increasing at a rate of 2 meters per minute. When the radius is 4 meters, and the sides are 20 meters, then how fast is the AREA outside the circle but inside the square changing?

Exercise 11.5. A rotating light is located 14 feet from a wall. The light completes one rotation every 5 seconds. Find the rate at which the light projected onto the wall is moving along the wall when the light's angle is 15 degrees from perpendicular to the wall.



Exercise 11.6. A cylinder is leaking water but you are unable to determine at what rate. The cylinder has a height of 2 m and a radius of 2 m. Find the rate at which the water is leaking out of the cylinder if the rate at which the height is decreasing is 10 cm/min when the height is 1 m.

12 Linearization

Let f be a function differentiable at $x = a$. Then for a number x near a, the value of the function $f(x)$ can be approximated using the tangent line $y = f'(a)(x - a) + f(a)$.

That is

$$f(x) \approx f'(a)(x - a) + f(a)$$

We call the function $L(x) := f'(a)(x-a)+f(a)$ the ***linearization*** or ***linear approximation*** of f at a.

Example 12.1. Find the linearization of $f(x) = \sqrt{x}$ at $x = 1$ and estimate $f(1.01)$. `https://www.desmos.com/calculator/urjragsxae`

Example 12.2. Find the linearization of $f(x) = \cos x$ at $x = 60^\circ$ and estimate $\cos(61^\circ)$.

Example 12.3. Estimate the value of $\sqrt[3]{0.98}$ using local linear approximation for the function $f(x) = \sqrt[3]{1+x}$.

12.1 Differentials

Sometimes, we only need to know the relative change. The linear approximation provides an estimate of the relative change in the dependent variable y using the relative change of the independent variable x. Suppose that $y = f(x)$. We denote by $\mathrm{d}x$ the change in x which can be assigned with any value, and define

$$\mathrm{d}y = f'(x)\mathrm{d}x.$$

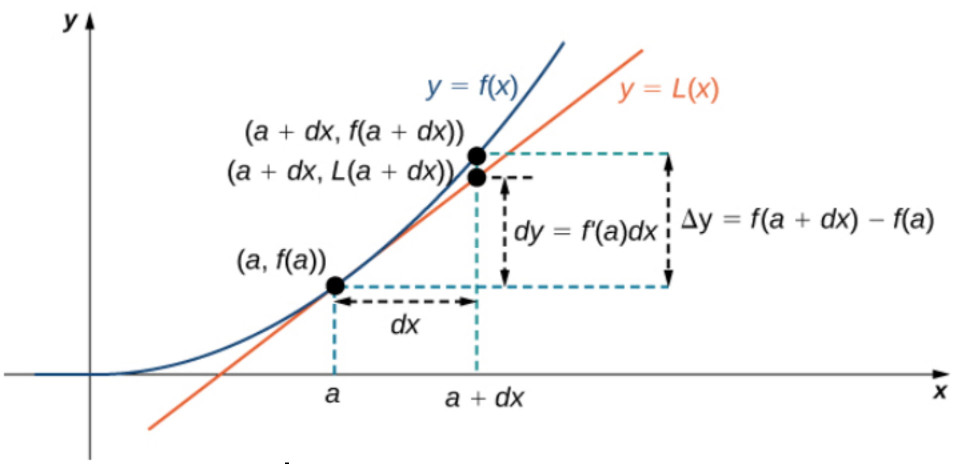
Then $\mathrm{d}y$ can be viewed as a function of $\mathrm{d}x$. We call $\mathrm{d}x$ and $\mathrm{d}y$ ***differentials***.

Given that f is differentiable at a and $\mathrm{d}x$ is small, the value $f(a + \mathrm{d}x)$ is approximately

$$f(a + \mathrm{d}x) \approx L(a + \mathrm{d}x) = f(a) + f'(a)(a + \mathrm{d}x - a).$$

Then the actual change $\Delta y = f(a + \mathrm{d}x) - f(a)$ is approximately

$$\Delta y = f(a + \mathrm{d}x) - f(a) \approx L(a + \mathrm{d}x) - f(a) = f'(a)\mathrm{d}x = dy.$$



Example 12.4. Find the differential $\mathrm{d}y$ for the function $y = x^3 - \frac{1}{x}$ and evaluate for $x = 1$ and $\mathrm{d}x = 0.01$.

Example 12.5. Let $y = tan(5x + 2)$. Using the differential to estimate Δy when $x = 1$ and $\mathrm{d}x = 0.4$.

12.2 Applications and Measurement Errors

In application, due errors in measurement, if a quantity is calculated based on a measurement, it is likely subject to an error too. This type of error is known as a **propagated error**. Suppose the quantity is determined by a function f of the measurement x. If the measurement has an error $\mathrm{d}x$, then the propagated error $f(a + \mathrm{d}x) - f(a)$ is approximately

$$\Delta y = f(a + \mathrm{d}x) - f(a) \approx f'(a)\mathrm{d}x = dy.$$

Remark.

Remark. When $f'(x)$ is continuous, if we don't know a, by continuity, we may the measured value $a + \mathrm{d}x$, to estimate the propagated error

$$\Delta y \approx dy \approx f'(a + \mathrm{d}x)\mathrm{d}x.$$

Example 12.6. Suppose the side length of a cube is measured to be 2 cm with an accuracy of 0.1 cm.

(1) Use differentials to estimate the error in the computed volume of the cube.

(2) Compute the volume of the cube if the side length is (a) 1.9 cm and (b) 2.1 cm to compare the estimated error with the

actual potential error.

Given an absolute error Δq for a particular quantity, we define the relative error as $\frac{\Delta q}{q}$, where q is the actual value of the quantity.

Example 12.7. An astronaut using a camera measures the radius of Earth as 4000 mi with an error of ± 80 mi. Let's use differentials to estimate the relative and percentage error of using this radius measurement to calculate the volume of Earth, assuming the planet is a perfect sphere.

12.3 Practice

Exercise 12.1. Find the linearization of $f(x) = \sqrt[3]{x}$ at $x = 8$ and estimate $f(8.03)$.

Exercise 12.2. Find the linearization of $f(x) = \sec x$ at $x = 45°$ and estimate $f(44°)$.

Exercise 12.3. Find dy for $y = \sqrt[n]{x+1}$ and evaluate when $x = 0$ and d$x = 0.01$, where n is a positive integer.

Exercise 12.4. Find dy for $y = \sin\left(\frac{\pi x + \pi}{2}\right)$ and evaluate it when $x = 0$ and d$x = 0.01$).

Exercise 12.5. Estimate the value of $\frac{6}{\sqrt{1-x}}$ at $x = 0.01$ using linear approximation.

Exercise 12.6. The circumference of a sphere was measured to be 73 cm with a possible error of 0.5 cm.

(1) Use linear approximation to estimate the maximum error in the calculated surface area.

(2) Estimate the relative error in the calculated surface area.

Exercise 12.7. Let $y = \sin(5x)$.

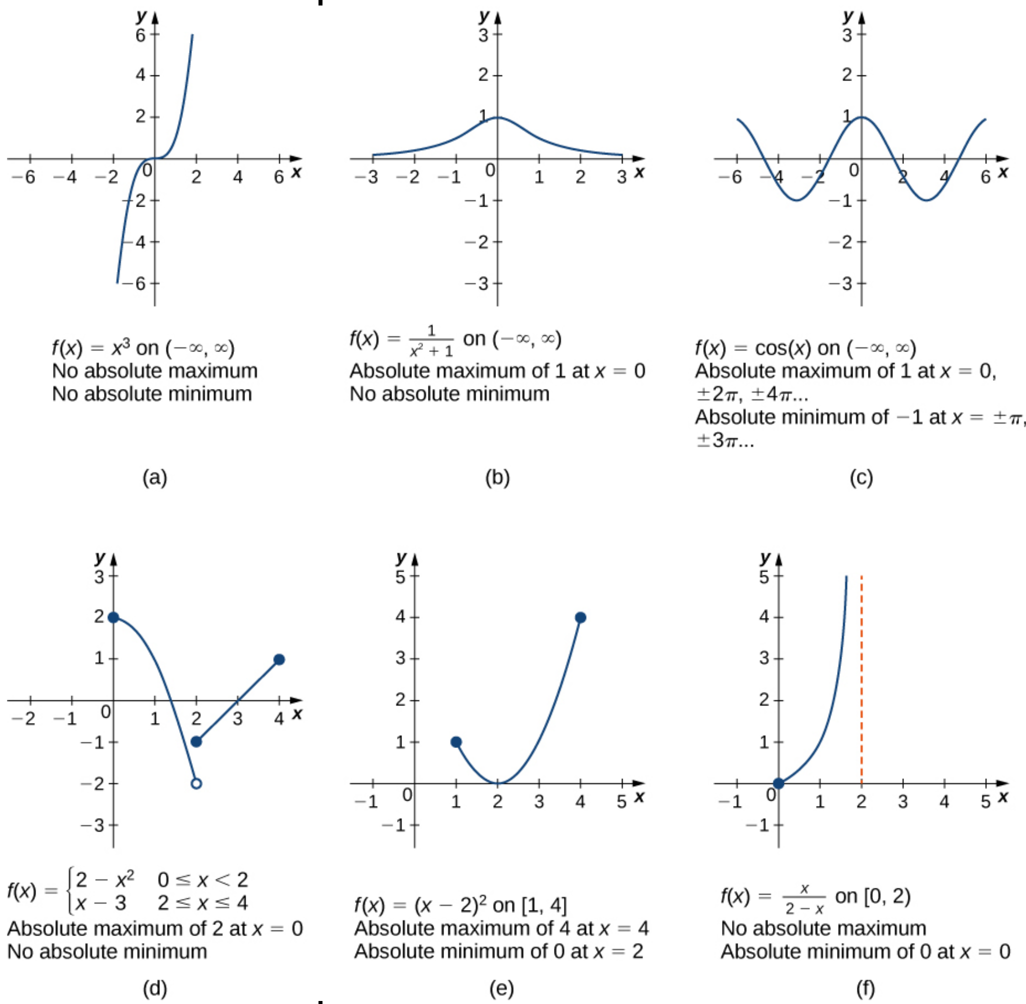
(1) Estimate $\sin(0.5)$ using linear approximation.

(2) Find the percentage error

Exercise 12.8. Use linear approximation to estimate the amount of paint in cubic centimeters needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with a diameter of 40 meters.

13 Maxima and Minima

Definition 13.1. Let f be a function defined over an interval I and c a number in I. We say f has an ***absolute maximum*** on I at c if $f(c) \geq f(x)$ for all x in I. We say f has an ***absolute minimum*** on I at c if $f(c) \leq f(x)$ for all x in I. If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has an absolute extremum on I at c.



Theorem 13.2. ***(Extreme Value Theorem):***

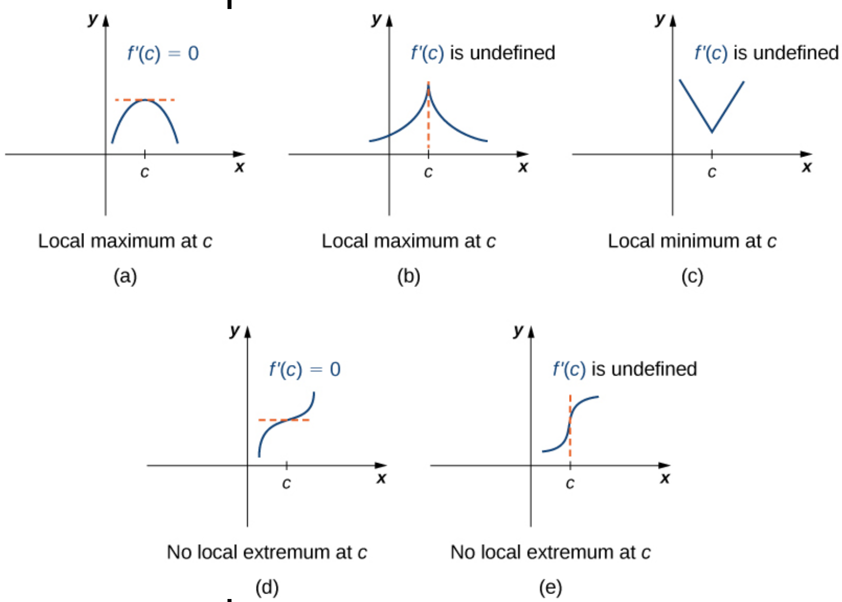
If f is a continuous function over the closed, bounded interval $[a, b]$, then there is a point in $[a, b]$ at which f has an absolute *maximum over $[a, b]$ and there is a point in $[a, b]$ at which f has an absolute minimum over $[a, b]$.*

Definition 13.3. A function f has a **local maximum** at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all x in I. A function f has a **local minimum** at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all x in I. A function f has a **local extremum** at c if f has a local maximum at c or f has a local minimum at c.

A local extremum is also known as a relative extremum.

Theorem 13.4. ***(Fermat's Theorem):*** *If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.*

Definition 13.5. Let c be an interior point in the domain of f. We say that c is a critical point of f if $f'(c) = 0$ or $f'(c)$ is undefined.



Example 13.1. For each of the following functions, find all critical points and determine whether the function has a local extremum at each of the critical points.

(1) $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 - 4x$

(2) $f(x) = \frac{x}{x^2-1}$

13.1 Where to locate absolute extrema

The absolute maximum of a function f over I and the absolute minimum of f over a closed interval I must occur at endpoints of I or at critical points of f in I.

Example 13.2. For each of the following functions, find points where the function has the absolute maximum or absolute minimum over the given interval.

(1) $f(x) = -x^2 - 2x - 3$ over $[0, 4]$.

(2) $f(x) = x^2 - 3x^{2/3}$ 2. over $[0, 2]$.

Example 13.3. Find the absolute maximum and absolute minimum of the function $f(x) = \sin(x) + \cos(x)$.

Example 13.4. Find the absolute maximum and absolute minimum of the function
$$f(x) = |x+1| + |x-1| \quad \text{over} \quad [-3, 2].$$

Example 13.5. Find the absolute maximum and absolute minimum of the function
$$f(x) = x\sqrt{4-x^2}.$$

13.2 Practice

Exercise 13.1. Find the critical values of the function
$$f(x) = 3\sqrt{x} + x^2.$$

Exercise 13.2. Find the critical values of the function
$$f(x) = |x^2 - 2x - 8|.$$

Exercise 13.3. Find the absolute maximum and absolute minimum of the function

$$f(x) = 4x - x^2 \quad \text{over} \quad [-3, 6].$$

Exercise 13.4. Find the absolute maximum and absolute minimum of the function

$$g(x) = 3x + 6\sin x \quad \text{over} \quad [0, \pi].$$

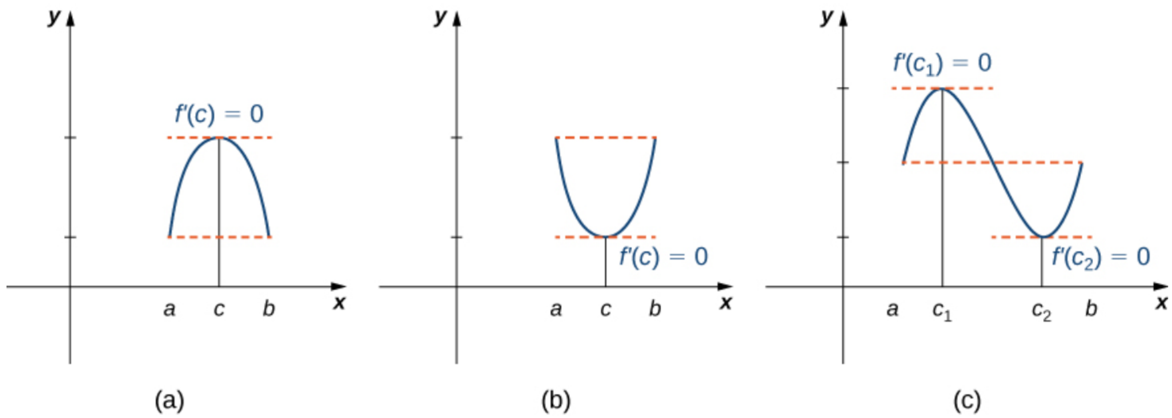
Exercise 13.5. Find the absolute maximum and absolute minimum of the function

$$h(x) = x^2\sqrt{9x - x^2}.$$

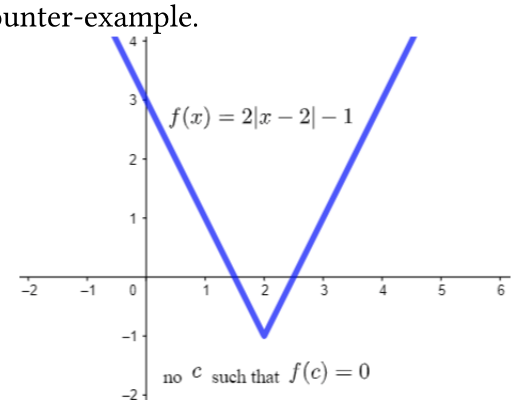
14 Mean Value Theorem

14.1 Rolle's Theorem

Let f be a function continuous over a closed interval $[a, b]$ and differentiable over the open interval (a, b). If $f(a) = f(b)$, then exists at least one value c in (a, b) such that $f'(c) = 0$.



Warning: Differentiability is important. The following is a counter-example.



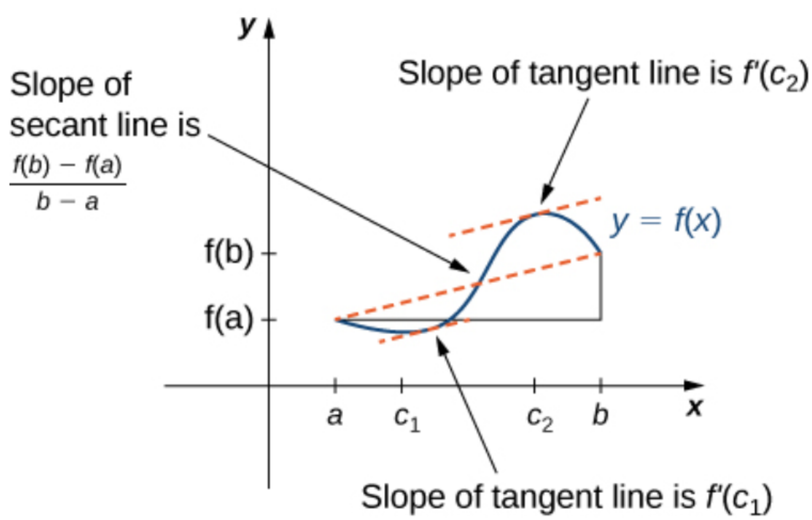
Example 14.1. Verify that the function $f(x) = x^2 + 4x - 3$ satisfies the criteria stated in Rolle's theorem and find all values c in the interval $(-3, 3)$ at where $f'(c) = 0$.

Example 14.2. Verify that the function $f(x) = \sin x$ satisfies the criteria stated in Rolle's theorem and find all values c in the interval $(-\pi, \pi)$ at where $f'(c) = 0$.

14.2 Mean Value Theorem

Theorem 14.1. *Let f be a function continuous over a closed interval $[a, b]$ and differentiable over the open interval (a, b). Then exists at least one value c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



To prove the theorem, let $g(x) = f(x) - [\frac{f(b)-f(a)}{b-a}(x-a) + f(a)]$, then apply Rolle's theorem.

Example 14.3. Verify the function $f(x) = \sqrt{x}$ defined over the interval $[0, 9]$ satisfies the condition of the Mean Value Theorem, and show that there exists a value c in $(0, 4)$ such that $f'(c)$ is equal to the slope of the secant line passing through $(0, f(0))$ and $(9, f(9))$. Find these values c guaranteed by the Mean Value Theorem.

Example 14.4. Suppose that $f(0) = -1$ and $f'(x) > 2$ for all values of x in $[-2, 5]$. How large can $f(x)$ possibly be?

14.3 Corollaries of the Mean Value Theorem

Corollary 14.2. *Let f be a function differentiable over an interval (a, b). If $f'(x) = 0$ for all x in (a, b), then $f(x) = c$ for all x in (a, b), where c is a constant.*

Corollary 14.3. *Let f and g be two functions differentiable over an interval (a, b). If $f'(x) = g'(x)$ for all x in (a, b), then $f(x) = g(x) + c$ for all x in (a, b), where c is a constant.*

Corollary 14.4. *Let f be a function continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b).*

- *If $f'(x) > 0$ for all x in (a, b), then f is increasing over (a, b).*

- *If $f'(x) < 0$ for all x in (a, b), then f is decreasing over (a, b).*

Example 14.5. Show that the equation $2x - \sin x = 0$ has exactly one real root.

14.4 Practice

Exercise 14.1. Verify that the function $f(x) = \cos^2 x$ satisfies the criteria stated in Rolle's theorem and find all values c in the interval $(-\pi, \pi)$ at where $f'(c) = 0$.

Exercise 14.2. Suppose that $f(-1) = 2$ and $f'(x) < 3$ for all values of x less that 6. How small can $f(x)$ possibly be?

Exercise 14.3. Show that the equation $x^2 - 2 + \cos x = 0$ has exactly one real root.

15 Monotonicity and Concavity

15.1 Monotonicity Test

Proposition 15.1. ***(Increasing/Decreasing Test)*** *Let f be a function differentiable on an interval I.*

(1) *If $f'(x) > 0$ on an interval I, then f is increasing on that interval I.*

(2) *If $f'(x) < 0$ on an interval I, then f is decreasing on that interval I.*

Example 15.1. Show that $f(x) = x^2 - 2x$ is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$.

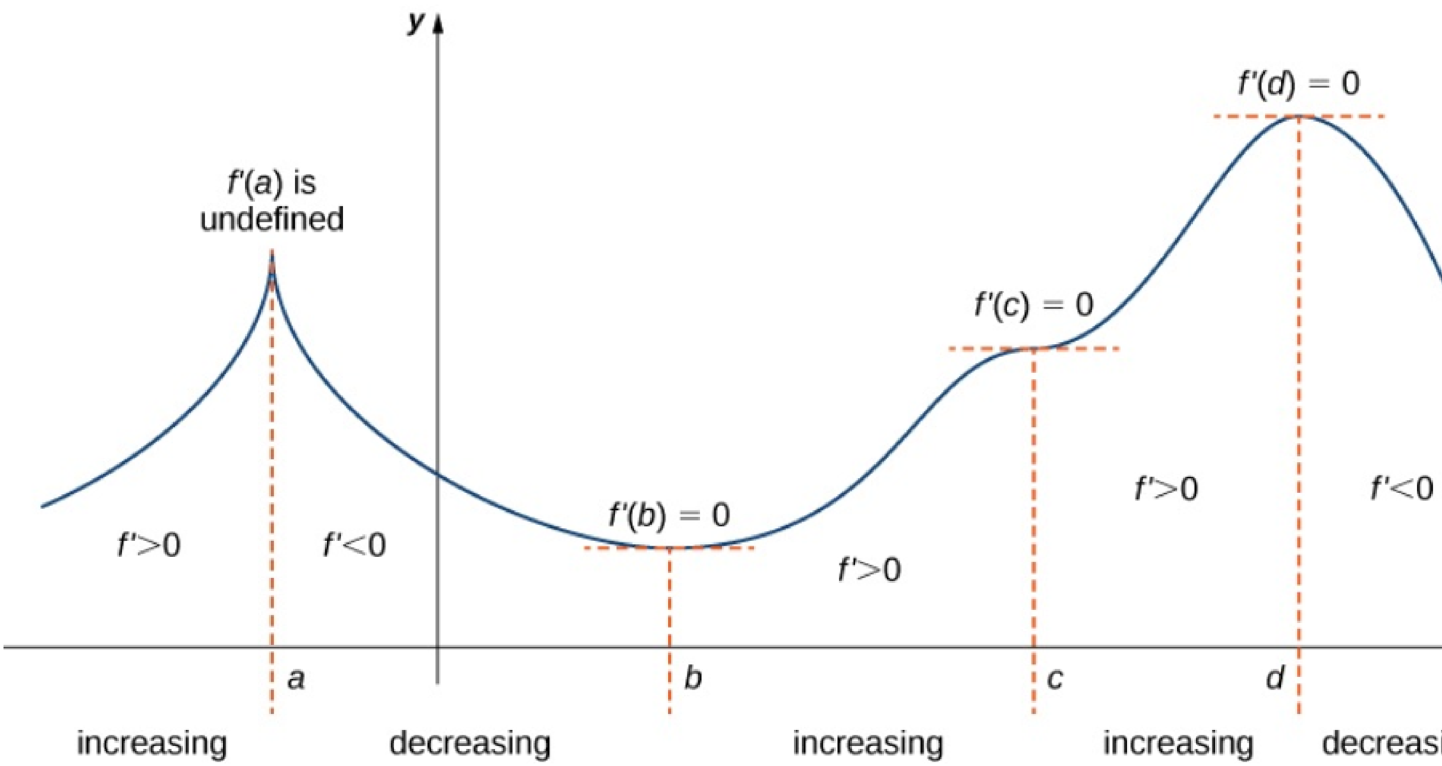
Example 15.2. Find intervals where $f(x) = x - 2\cos x$ is increasing or decreasing.

Proposition 15.2. ***(First Derivative Test for a Local Extremum)*** *Let f be a function continuous on (a, b) and differentiable on $(a, b) \setminus \{c\}$.*

(1) *If $f'(x)$ chance from positive to negative when x moves to the right passing c, then f has a local maximum at c.*

(2) *If $f'(x)$ chance from negative to positive when x moves to the right passing c, then f has a local minimum at c.*

(3) *If $f'(x)$ has the same sign on both sides of c, then f does not have a local extremum at c.*

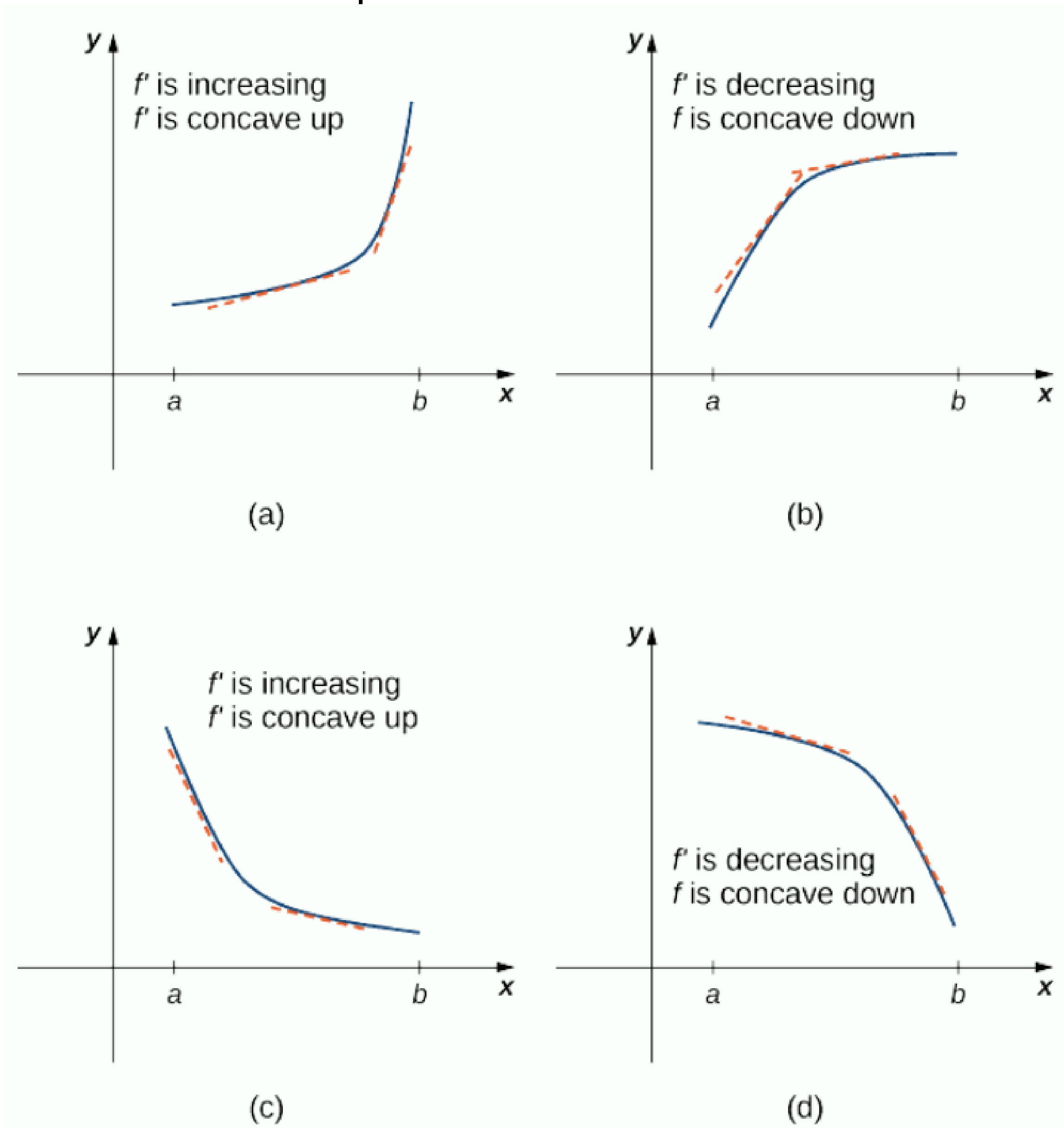


Example 15.3. Use the first derivative test to find the location of all local extrema for $f(x) = x^3 - 3x^2 - 9x - 1$.

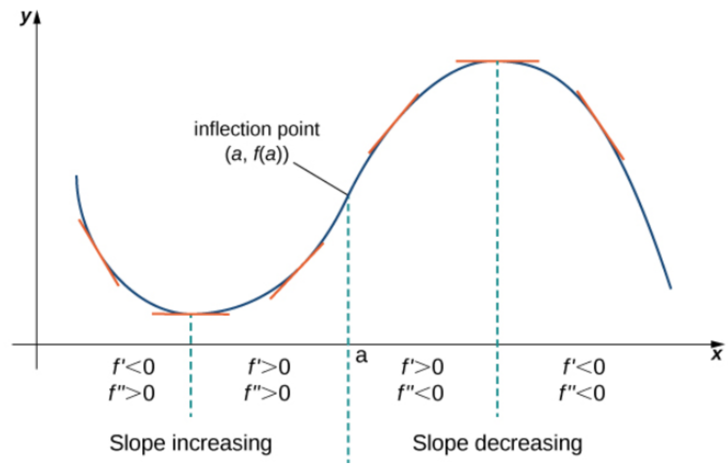
Example 15.4. Use the first derivative test to find the location of all local extrema for $f(x) = 5x^{1/3} - x^{5/3}$.

15.2 Concavity Test

Definition 15.3. If the graph of f lies above all of its tangents on an interval I, then it is called ***concave upward*** on I. If the graph of f lies below all of its tangents on I, it is called ***concave downward*** on I.



A point P on a curve $y = f(x)$ is called an ***inflection point*** if f is continuous at P and the curve changes the direction of concavity at P.



Proposition 15.4. ***(Concavity Test)*** *Let f be a function on (a, b).*

(1) *If $f''(x) > 0$ on (a, b), then f is concave upward on (a, b).*

(2) *If $f''(x) < 0$ on (a, b), then f is concave downward on (a, b).*

Example 15.5. For the function $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f.

Proposition 15.5. ***(Second Derivative Test)*** *Let f be a function defined on (a, b). Assume that there is a number c in (a, b) such that $f'(c) = 0$ and $f''(c)$ exists.*

(1) *If $f''(c) < 0$, then f has local maximum value at c.*

(2) *If $f''(c) > 0$, then f has local minimum value at c.*

Example 15.6. Consider the function $f(x) = x^3 - \frac{3x^2}{2} - 18x$. Use the second derivative test to determine local extrema of f.

Example 15.7. Let $f(x) = x^3 - 6x^2$.

(1) Find the intervals where f is increasing and where it is decreasing.

(2) Find the local extrema if they exists.

(3) Find the interval where f is concave upward and where it is concave downward.

(4) Find the inflection points if they exits.

15.3 Practice

Exercise 15.1. Determine intervals where $f(x) = \sin x + \sin^3 x$, where $0 < x < \pi$, is increasing or decreasing.

Exercise 15.2. Use the first derivative test to find the location of all local extrema for $f(x) = x + x^2 - x^3$.

Exercise 15.3. For the function $f(x) = x + \sin(2x)$ over $[\frac{\pi}{2}, \frac{\pi}{2}]$,

(1) determine all intervals where f is concave up or concave down;

(2) list all inflection points for f.

Exercise 15.4. Consider the function $f(x) = \sin(\pi x) - \cos(\pi x)$ over $x = [-1, 1]$. Determine

(1) intervals where f is increasing or decreasing,

(2) local minima and maxima of f,

(3) intervals where f is concave up and concave down, and

(4) the inflection points of f.

Exercise 15.5. Consider the function $f(x) = \frac{1}{1-x}$, where $x \neq 1$. Determine

(1) intervals where f is increasing or decreasing,

(2) local minima and maxima of f,

(3) intervals where f is concave up and concave down, and

(4) the inflection points of f.

16 Curve Sketching

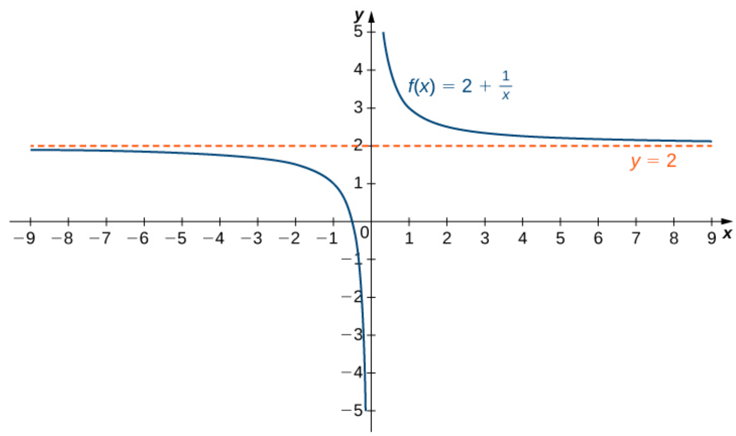
16.1 Limits at Infinity

Definition 16.1. Let f be a function defined on an interval (a, ∞). If the values of $f(x)$ becomes arbitrarily close to L as x becomes sufficiently large, we say the function f has a **limit at infinity** and write $\lim_{x\to\infty} f(x) = L$.

Let f be a function defined on an interval $(-\infty, a)$. If the value of $f(x)$ becomes arbitrarily close to L for as $-x$ becomes sufficiently large, we say that the function f has a **limit at negative infinity** and write $\lim_{x\to-\infty} f(x) = L$.

Remark. For limits at infinity, limit laws still work.

Example 16.1. Consider the function $f(x) = 2+\frac{1}{x}$. Find $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.



Theorem 16.2. *Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ and $q(x) = b_mx^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0$ are two polynomials. Then*

$$\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \text{if } n < m \\ \frac{a_n}{b_m} & \text{if } n = m \end{cases}$$

When $n > m$, the limit at infinity is an infinite limit.

Recall: A function f has an infinite limit at a if $\lim_{x\to a} f(x) = \infty$ or $-\infty$, where a can be a finite number, infinity or negative infinity.

Example 16.2. Find the limits $\lim_{x\to\infty} \frac{x^4-4x^3+1}{2-2x^2-7x^4}$.

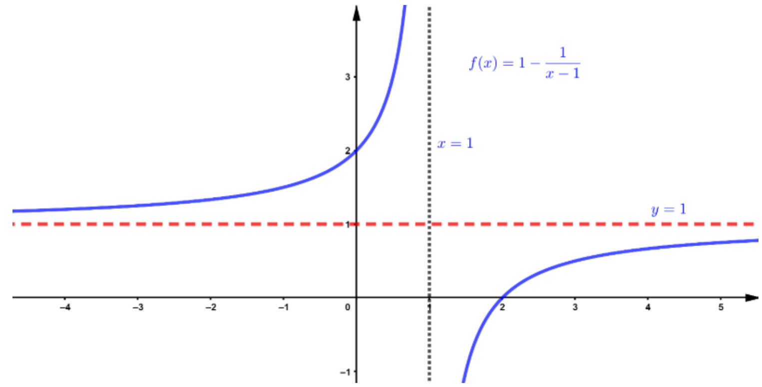
Example 16.3. Find the limit $\lim_{x\to-\infty} = \frac{x^2+\cos(x^3)}{2-2x^3}$.

16.2 Asymptotes

Definition 16.3. Let f be a function defined for x or $-x$ sufficiently large. If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, we say the line $y = L$ is a *horizontal asymptote* of f.

Let f be a function defined near $x = a$. If $\lim_{x\to a^-} f(x) = \infty$ or $-\infty$ or $\lim_{x\to a^+} f(x) = \infty$ or $-\infty$, we say the line $x = a$ is a *vertical asymptote* of f.

Example 16.4. Consider the function $f(x) = 1 - \frac{1}{x-1}$. Find all horizontal and vertical asymptotes.



Example 16.5. Determine the horizontal asymptote(s) and the vertical asymptote(s) for the function f if they exist.

(1) $f(x) = 3 - \frac{5}{x^2}$

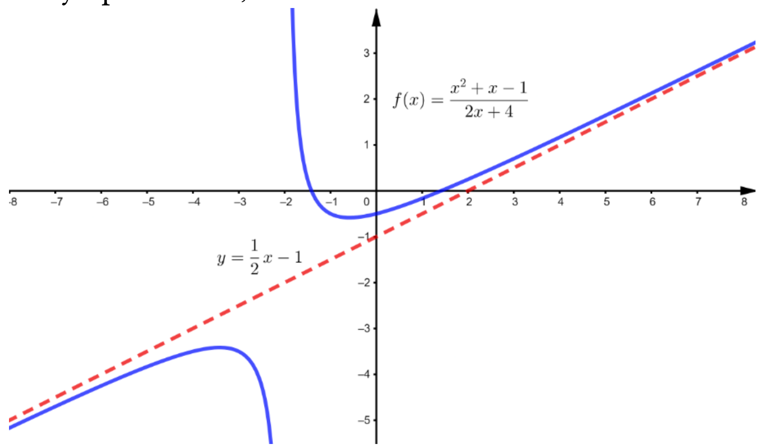
(2) $f(x) = \frac{\sin x}{x}$

16.3 Slant Asymptote

Definition 16.4. A line $y = mx + b$ is a **slant asymptote** of a function f if

$$\lim_{x\to\infty} [f(x) - (mx + b)] = 0 \text{ or } \lim_{x\to\infty} [f(x) - (mx + b)] = 0.$$

Example 16.6. Determine if the function $f(x) = \frac{x^2+x-1}{2x+4}$ has a slant asymptote. If so, find it.



16.4 A guideline to curve sketching

(1) Find the domain of f.

(2) Solve for x from $f(x) = 0$ to find the x-intercepts and find the y-intercept $(0, f(0))$.

(3) Determine if the function has any symmetry:

(a) Is f an even function, i.e. $f(-x) = f(x)$ for all x?

(b) Is f an odd function, i.e. $f(-x) = -f(x)$ for all x?

(c) Is f a periodic function, i.e. there is a constant p such that $f(x+p) = f(x)$ for all x?

(4) Evaluate $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$ to determine the end behavior of f. Find horizontal asymptotes if they exist.

(5) Find vertical asymptotes if they exist.

(6) Find slant asymptotes, i.e. $y = mx+b$ such that $\lim\limits_{x\to\pm\infty} (f(x)-(mx+b)) = 0$. Note that $m = \lim\limits_{x\to\infty} \frac{f(x)}{x}$ and $b = \lim\limits_{x\to\infty} (f(x) - mx)$.

(7) Calculate f' and find all critical values if they exist. Determine the intervals of increasing and decreasing.

(8) Determine local extrema if they exist.

(9) Calculate f''. Determine the intervals of concave up and concave down. Determine inflection points if they exist.

(10) Sketch the curve using the above information.

Example 16.7. Sketch a graph of $f(x) = (x-1)^2(x+1)$.

Example 16.8. Sketch the graph of $f(x) = \frac{x+2}{x^2+5x+4}$.

16.5 Practice

Exercise 16.1. Find the limits $\lim_{x \to -\infty} \frac{x}{x-2}$.

Exercise 16.2. Find the limit $\lim_{x \to -\infty} \frac{\sqrt{4x^2-1}}{x+2}$.

Exercise 16.3. Find the limit $\lim_{x \to \infty} \frac{\sqrt{x^5+1}}{x^2-\sqrt{x}+1}$.

Exercise 16.4. Determine the horizontal asymptote(s) and the vertical asymptote(s) for the function f if they exist.

(1) $f(x) = \frac{x}{1-x^2}$

(2) $f(x) = \frac{x \sin x}{x^2-1}$

Exercise 16.5. Determine if the function $f(x) = \frac{x^2+7x+6}{x+2}$ has a slant asymptote. If so, find it.

Exercise 16.6. Sketch the graph of $f(x) = x^3 - 3x^2 + 4$.

Exercise 16.7. Sketch the graph of $f(x) = \frac{2x+3}{x^2+8x+12}$.

Exercise 16.8. Sketch the graph of $f(x) = \sqrt{x^2 - 5x + 4}$.

17 Optimization Problems

17.1 Optimization Problem Solving Strategy

(1) Understand the Problem and represent known and unknown quantities using variables and expressions. In this step, it is useful to draw a diagram and identify variables and expressions on the diagram.

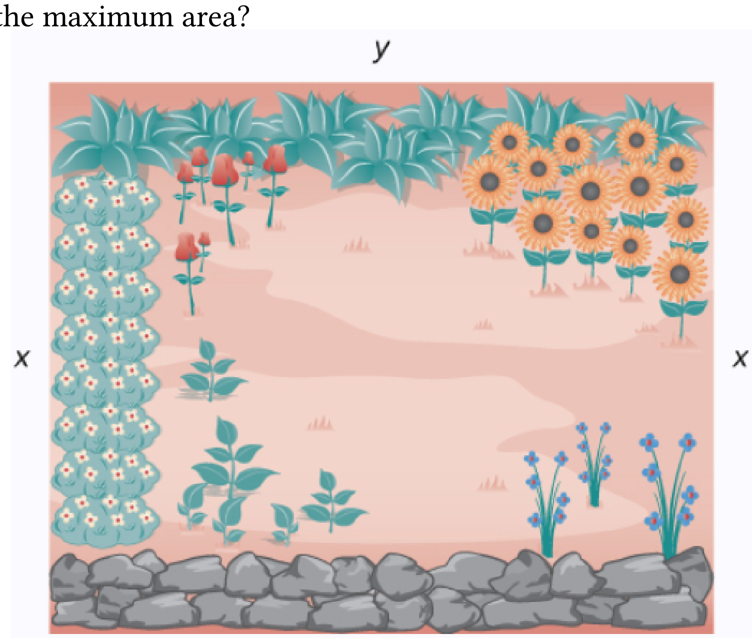
(2) Write equations relating the variables.

(3) Determine which quantity is to be maximized or minimized. Find the range of values of the other variables if possible.

(4) Express the quantity to be maximized or minimized as an explicitly define function of other variables.

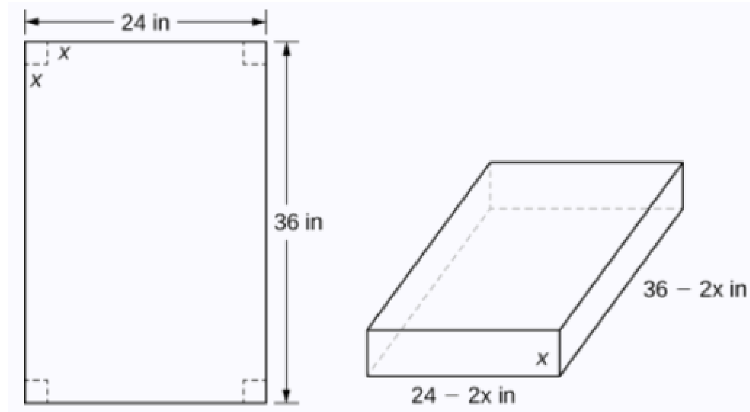
(5) Locate the maximum or minimum value of the function.

Example 17.1. A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides. Given 100 ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

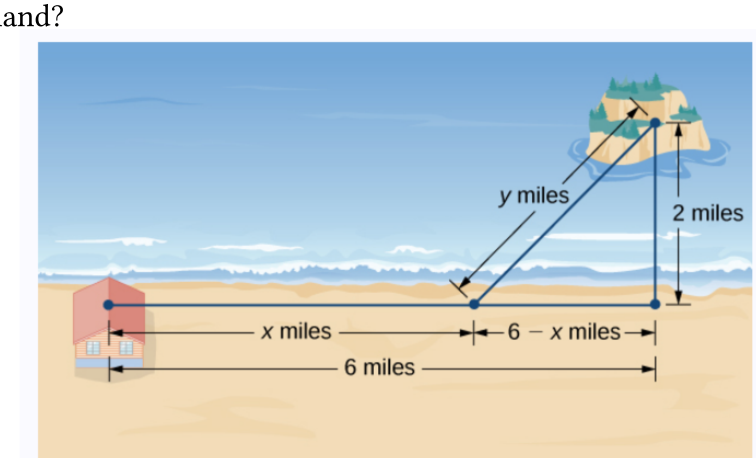


Example 17.2. An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square

should be cut out of each corner to get a box with the maximum volume?



Example 17.3. An island is 2 mi due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 mi west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 mph and swims at a rate of 3 mph. How far should the visitor run before swimming to minimize the time it takes to reach the island?

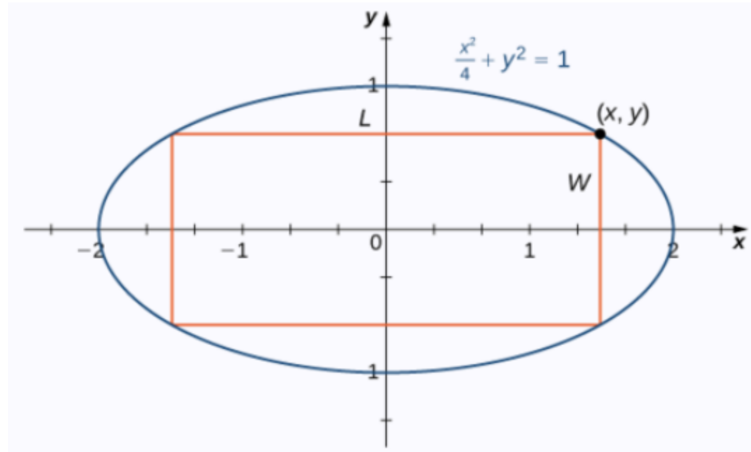


Example 17.4. Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where $50 \leq p \leq 200$, the number of cars n they rent per day can be modeled by the linear function $n(p) = 1000 - 5p$. If they charge \$50 per day or less, they will rent all their cars. If they charge \$200 per day or more, they will not rent any cars. Assuming the owners plan to charge customers between \$50 per day and 200 per day to rent a car, how much should they charge to maximize their revenue?

Example 17.5. A rectangle is to be inscribed in the ellipse

$$\frac{x^2}{4} + y^2 = 1$$

What should the dimensions of the rectangle be to maximize its area? What is the maximum area?



17.2 First Derivative Test for Absolute Extreme Values

Theorem 17.1. *Suppose that c is a critical number of a continuous function f defined on an interval.*

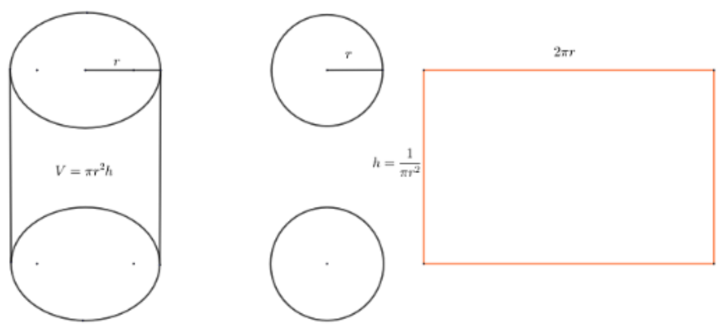
(1) *If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.*

(2) *If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then*

$f(c)$ is the absolute minimum value of f.

Example 17.6. A closed box with a square base is to contain 180 cubic feet. The bottom costs \$4 per square foot, the top costs \$1 per square foot, and the sides cost \$3 per square foot. Find the dimensions that will minimize the cost.

Example 17.7. A cylindrical can is to be made to hold 1 L of tomato sauce. Find the dimensions that will minimize the cost of the metal to manufacture the can.



A picture of cylindrical can

17.3 Practice

Exercise 17.1. To carry a suitcase on an airplane, the length + width + height of the box must be less than or equal to 62 in.

(1) Assuming the height is fixed, show that the maximum volume is $V = h(31 - (\frac{1}{2})h)^2$.

(2) What height allows you to have the largest volume?

Exercise 17.2. Two poles are connected by a wire that is also connected to the ground. The first pole is 20 ft tall and the second pole is 10 ft tall. There is a distance of 30 ft between the two poles. Where should the wire be anchored to the ground to minimize the amount of wire needed?

Exercise 17.3. For every x pizzas sold, a pizzeria would make a revenue of $R(x) = 12x$. It is known that the cost of the pizzeria to make x pizzas is $C(x) = 2x + x^2$. How many pizzas sold would maximize the profit?

Exercise 17.4. At where is the parabola $y = x^2$ closest to point $(0, 3)$?

Exercise 17.5. A top-opened cylinder has the volume 16π. Find the dimensions of the cylinder that has the least amount of surface area.

18 Antiderivatives

Definition 18.1. A function F is an antiderivative of the function f if
$$F'(x) = f(x)$$
for all x in the domain of f.

18.1 The most general form of an antiderivative

Let F be an antiderivative of f over an interval I. Then, by the mean value theorem, for each constant C, the function $F(x) + C$ is also an antiderivative of f over I; if G is an antiderivative of f over I, there is a constant C for which $G(x) = F(x) + C$ over I. In other words, the most general form of the antiderivative of f over I is $F(x) + C$.

Example 18.1. For each of the following functions, find all antiderivatives.

(1) $f(x) = 3x^2$

(2) $f(x) = \dfrac{1}{x}$

(3) $f(x) = \cos x$

18.2 Rules of Antiderivatives

Function	An antiderivative
$af + bg$	$aF + bG$
x^n, $n \neq -1$	$\frac{x^{n+1}}{n+1}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$

In the above table, F and G are antiderivative of f and g respectively, a and b are arbitrary constants.

Example 18.2. Find the most general antiderivative for each functions.

(1) $(5x^3 - 7x^2 + 3x + 4)$

(2) $\dfrac{x^2 + 4\sqrt[3]{x}}{x}$

(3) $\dfrac{4}{1 + x^2}$

(4) $\tan x \cos x$

18.3 Antiderivative with initial value

The most general antiderivative defines a family of curves. If a point is given, then there will be specific antiderivative. The problem of finding a function y that satisfies a differential equation $\frac{dy}{dx} = f(x)$ with the additional condition $y(x_0) = y_0$ is known as an initial-value problem. The condition $ y(x_0)=y_0$ is known as an initial condition.

Example 18.3. Solve the initial-value problem

$$\frac{dy}{dx} = \sin x, \quad \text{with} \quad y(0) = 5.$$

Example 18.4. Find an equation for the function y that satisfies the following conditions

$$\frac{dy}{dx} = 3x^{-2}, \quad \text{and} \quad y(1) = 2.$$

Example 18.5. Find an equation for the function f that satisfies the following conditions

$$f''(x) = \sin x, \quad f'(0) = 1, \text{and} \quad f(\pi) = 0.$$

18.4 Application

Example 18.6. A car is traveling at the rate of 88 ft/s when the brakes are applied. The car begins decelerating at a constant rate of 15 ft/s^2.

(1) How many seconds elapse before the car stops?

(2) How far does the car travel during that time?

18.5 Practice

Exercise 18.1. Find the antiderivative $F(x)$ of each function $f(x)$.

(1) $f(x) = 5x^4 + 4x^5$

(2) $f(x) = sec^2(x) + 1$

(3) $f(x) = 2\sin x - \cos x$

(4) $f(x) = \frac{x+1}{\sqrt{x}}$

Exercise 18.2. Solve the initial value problem.

(1) $f'(x) = x^{-3}, \quad f(1) = 1$

(2) $f'(x) = \sqrt{x} + x^2, \quad f(0) = 2$

(3) $f''(x) = x^2 + 2, \quad f'(0) = 1, \quad f(1) = 2$

Exercise 18.3. A car is being driven at a rate of 40 mph when the brakes are applied. The car decelerates at a constant rate of 10 ft/sec2. How long before the car stops?

19 Approximating Area

19.1 What is an area and how to measure it

The area of a shape can be measured by comparing the shape to squares of a fixed size, say 1×1 sized square. For a regular shape such as a rectangle, the area can be easily measured by dividing the rectangle into squares and taking the sum. This simple idea can be generalized to measure irregular areas approximately. The accurate measurement of an irregular shape needs the limit.

For areas under curve f, we take the following approach.

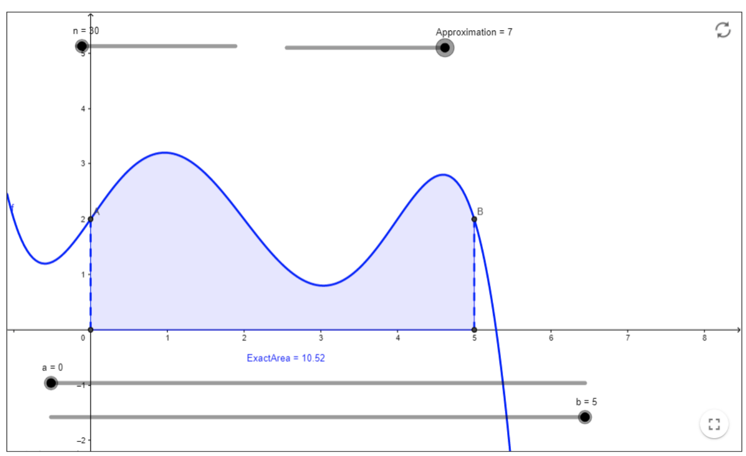
(1) Dividing the interval $[a, b]$ into n subintervals of equal width, $\Delta x = \dfrac{b-a}{n}$. Let $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = a, x_n = b$, and $x_i = x_0 + i\Delta x$ be the boundary points of those subintervals.

(2) The area under the curve over a subinterval $[x_{i-1}, x_i]$ can by estimated by $f(x_i^*)\Delta x$, where x_i^* is a point in the interval $[x_{i-1}, x_i]$, and $i = 1, 2, \ldots, n$.

(3) The area under the curve over $[a, b]$ is then approximately

$$f(x_1^*)\Delta x + f(x_2^*) + \cdots + f(x_{n-1}^*)\Delta x + f(x_n^*)\Delta x.$$

Approximate the Area Under the Curve on the Interval [a,b].



19.2 Sigma Notation

For convenience, we use sigma notation $\sum_{i=1}^{n} s_n$ to write sums of a sequence of n values s_1,s_2,. . ., s_n. In the sigma notation, the letter i is called **the index**, 1 is the starting value and n is the ending value in the sequence.

For example,

$$\sum_{i=1}^{n} f(x_i^*)\Delta x := f(x_1^*)\Delta x + f(x_2^*) + \cdots + f(x_{n-1}^*)\Delta x + f(x_n^*)\Delta x.$$

$$\sum_{n=1}^{100} n := 1 + 2 + 3 + \cdots + 100.$$

19.3 Properties of Sigma Notation

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers m, with $1 \le m \le n$.

(1) $\displaystyle\sum_{i=1}^{n} c = nc$

(2) $\displaystyle\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

(3) $\displaystyle\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

(4) $\displaystyle\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

(5) $\displaystyle\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$

19.4 Sum of powers of a sequence of integers

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Example 19.1. Find the sum of the values of $2n - 1$ for $n = 1, 2, \ldots, 100$.

Example 19.2. Find the sum of the values of $n^2 - n + 1$ for $n = 1, 2, \ldots, 100$.

19.5 Approximating Areas

Definition 19.1. A set of points $P = x_i$ for $i = 0, 1, 2, \ldots, n$ with $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1]$,$[x_1, x_2]$,…,$[x_{n-1}, x_n]$ is called a **partition** of $[a, b]$. If the subintervals all have the same width, the set of points forms a **regular partition** of the interval $[a, b]$.

Definition 19.2. Let f be a continuous function. The sum

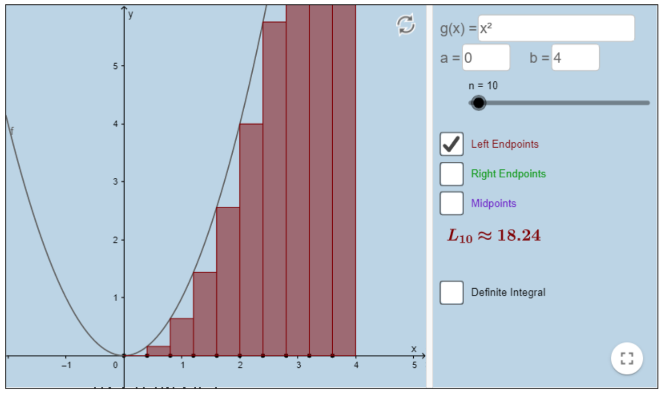
$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x,$$

where $x_0 = a$, $\Delta = \frac{b-a}{n}$ and $x_i = a + (i - 1)\Delta x$, is called a **left-endpoint approximation** of the area under f over the interval $[a, b]$.

Definition 19.3. Let f be a continuous function. The sum

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_1)\Delta x + \cdots + f(x_n)\Delta x,$$

where $x_0 = a$, $\Delta = \frac{b-a}{n}$ and $x_i = a + (i-1)\Delta x$, is called a **right-endpoint approximation** of the area under f over the interval $[a, b]$.



Example 19.3. Find the left-endpoint and right-endpoint approximation of the area under the curve $f(x) = x^2$ over the interval $[0, 2]$ using a partition of $n = 10$ subintervals.

19.6 Riemann Sum

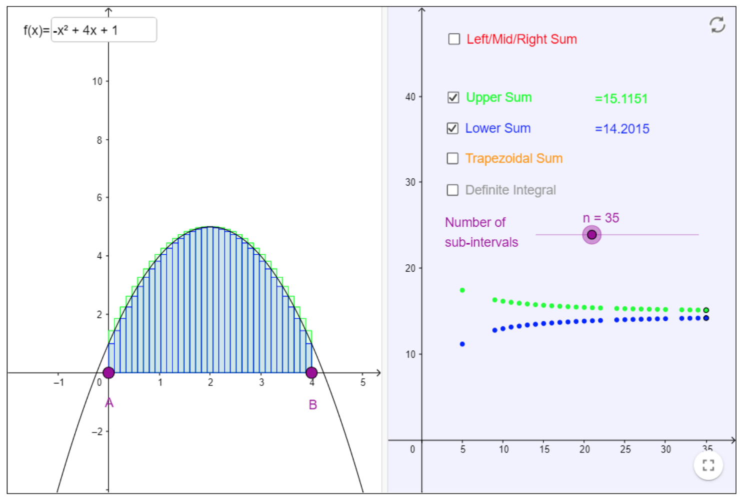
Definition 19.4. Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be a regular partition of $[a, b]$. Let Δx be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1}, x_i]$. A **Riemann sum** is defined for $f(x)$ as

$$RM_n := \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

Definition 19.5. A function is integrable if $\lim_{n\to\infty} R_n = A$ for any partition of $[a, b]$.

Fact: All functions which are continuous except at finitely many removable or jumping discontinuities are integral.

A proof of this fact uses the **upper sum** and **lower sum** where the sample points are where the function has max and min value respectively. in the subinterval. Because of the continuity, the difference between the maximum value and minimum value of f can be make arbitrarily small by restricting to sufficiently small intervals. That will make the difference between the upper and lower Riemann sums neglectable.



Fact: For an integrable function f, the area under the curve f over $[a, b]$ is the limit of any Riemann sum. The midpoint is also frequently used to calculate the Riemann sum.

Example 19.4. Find the lower sum and the upper sum for $f(x) = 1 - x^2$ on $[1, 2]$ using a regular partition of $n = 4$ subintervals.

19.7 Definite Integral

Let $f(x)$ be an integrable function, particularly, a continuous function, defined over an interval $[a, b]$. The **definite integral** of f from a to b is defined as

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

The numbers a and b are called the upper and lower limits of integration. The function $f(x)$ is called the integrand. The differential dx is called the variable of integration.

Like the index in a sigma notation, the variable of integration is a dummy variable, which mean you may use any other letter instead of x to write the integral.

Fact: The definite integral $\int_a^b f(x)\mathrm{d}x$ is the signed area under the curve f over the interval $[a, b]$.

Example 19.5. Find the definite integral $\int_0^2 x\mathrm{d}x$ using a Riemann sum and a graph.

Example 19.6. Find the definite integral $\int_0^1 \sqrt{1-x^2}\mathrm{d}x$ using a graph.

19.8 Practice

Exercise 19.1. Find the sum of the values of $3n + 2$ for $n = 1, 2, \ldots, 100$.

Exercise 19.2. Find the lower sum and the upper sum for $f(x) = x^2 - 2x$ on $[0, 2]$ using a regular partition of $n = 5$ subintervals.

Exercise 19.3. Find the definite integral $\int_0^2 (2x + 1)\mathrm{d}x$ using a Riemann sum and a graph.

Exercise 19.4. Find the definite integral $\int_{-2}^0 \sqrt{4 - x^2}\mathrm{d}x$ using a graph.

20 Definite Integrals

20.1 Definite Integrals

Definition 20.1. Let $f(x)$ be an integrable function, particularly, a continuous function, defined over an interval $[a, b]$. The **definite integral** of f from a to b is defined as

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

The symbol $\int$ is called an integral sign. The numbers a and b are called the **upper and lower limits of integration**. The function $f(x)$ is called the **integrand**. The **differential** dx is called the **variable of integration**.

Like the index in a sigma notation, the variable of integration is a dummy variable, which mean you may use any other letter instead of x to write the integral.

Fact: The definite integral $\int_a^b f(x)\mathrm{d}x$ is the signed area under the curve f over the interval $[a, b]$.

Example 20.1. Find the definite integral $\int_a^b c\mathrm{d}x$ using a Riemann sum and a graph.

Example 20.2. Find the definite integral $\int_a^b x\mathrm{d}x$ using a Riemann sum and a graph.

Example 20.3. Find the definite integral $\int_a^b x^2 \mathrm{d}x$ using a Riemann sum.

Example 20.4. Find the definite integral $\int_0^1 \sqrt{1-x^2}\mathrm{d}x$ using a graph.

Example 20.5. Express the limit as an integral and evaluate

$$\lim_{x\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{i^2}{n^2}$$

20.2 Signed Area

Definition 20.2. (Signed Area) Let $f(x)$ be an integrable function defined on an interval $[a, b]$. Let A_1 represent the area between $f(x)$ and the x-axis that lies above the axis and let A_2 represent the area between $f(x)$ and the x-axis that lies below the axis. Then, the (net) signed area between $f(x)$ and the x-axis is given by

$$\int_a^b f(x)\,dx = A_1 - A_2.$$

The total area between $f(x)$ and the x-axis is given by

$$\int_a^b |f(x)|\,dx = A_1 + A_2.$$

Example 20.6. Find the total area between the function $f(x) = x - 2$ and the x-axis over the interval $[0, 3]$.

20.3 Rules of Definite Integrals

(1)

$$\int_a^a f(x)\,\mathrm{d}x = 0$$

If the limits of integration are the same, the integral is just a line and contains no area.

(2)

$$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$$

If the limits are reversed, then place a negative sign in front of the integral.

(3)

$$\int_a^b [f(x) + g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

The integral of a sum is the sum of the integrals.

(4)

$$\int_a^b [f(x) - g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x$$

The integral of a difference is the difference of the integrals

(5)

$$\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$$

for constant c. The integral of the product of a constant and a function is equal to the constant multiplied by the integral of the function.

(6)
$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$
Although this formula normally applies when c is between a and b, the formula holds for all values of a, b, and c, provided $f(x)$ is integrable on the largest interval.

Example 20.7. If it is known that $\displaystyle\int_1^5 f(x)\,\mathrm{d}x = -3$ and $\displaystyle\int_2^5 f(x)\,\mathrm{d}x = 4$, find the value of $\displaystyle\int_1^2 f(x)\,\mathrm{d}x$.

Theorem 20.3. ***Comparison Theorem:***

(1) *If $f(x) \geq 0$ for $a \leq x \leq b$, then*
$$\int_a^b f(x)\,\mathrm{d}x \geq 0.$$
(2) *If $f(x) \geq g(x)$ for $a \leq x \leq b$, then*
$$\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x.$$
(3) *If m and M are constants such that $m \leq f(x) \leq M$ for $a \leq x \leq b$, then*
$$m(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a).$$

Example 20.8. Determine the sign of $\int_0^1 (\sqrt{1+x^2} - \sqrt{1+x^4})\mathrm{d}x$

20.4 Practice

Exercise 20.1. Using geometry to evaluate the integral

$$\int_{-3}^{3} (3 - |x|)\, dx.$$

Exercise 20.2. Express the limit as integrals

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sin^2(2\pi x_i^*)\Delta x \quad \text{over} \quad [0, 1].$$

Exercise 20.3. Using integral to evaluate the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{i-1}{n}$$

Exercise 20.4. Suppose that $\int_0^4 f(x)\, dx = 5$ and $\int_0^2 f(x)\, dx = -3$, and $\int_0^4 g(x)\, dx = -1$ and $\int_0^2 g(x)\, dx = 2$ Compute the following integrals.

(1) $\displaystyle\int_0^4 (3f(x) - 4g(x))\, dx$

(2) $\displaystyle\int_2^4 (2f(x) + 3g(x))\, dx$

Exercise 20.5. Show that $\displaystyle\int_{-\pi/4}^{\pi/4} \cos t\, dt \geq \pi\sqrt{2}/4$.

21 Fundamental Theorem of Calculus

Theorem 21.1. ***(Fundamental Theorem of Calculus I)*** *Suppose that $f(x)$ is continuous on the interval $[a, b]$. If $F(x)$ is any antiderivative of $f(x)$, then*

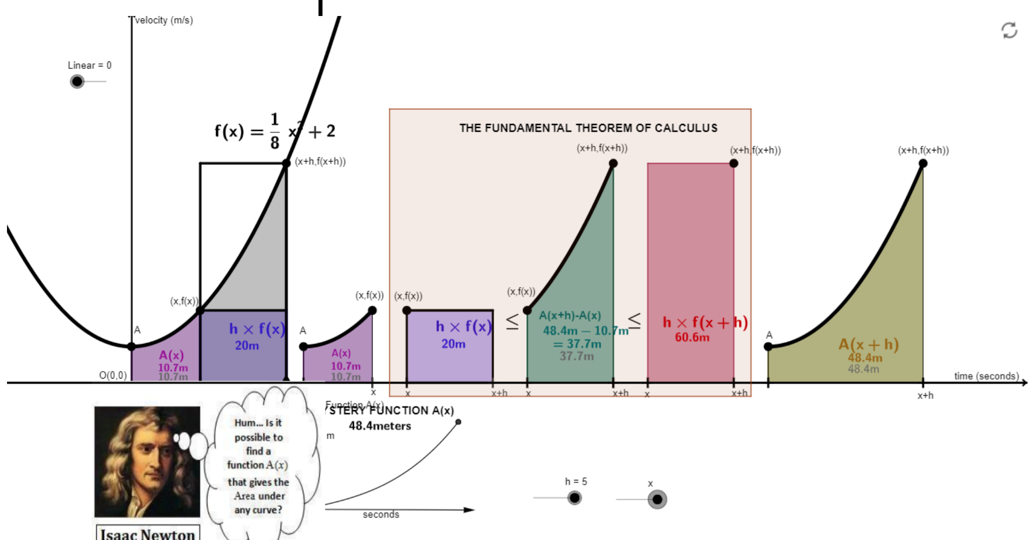
$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

Theorem 21.2. ***(Fundamental Theorem of Calculus II)*** *Suppose that $f(x)$ is continuous on the interval $[a, b]$ and let*

$$F(x) = \int_a^x f(t)\,dt.$$

Then $F'(x) = f(x)$.

Proof. **Idea of Proof:** FTC I follows from FTC II by taking $x = b$. FTC II is following the continuity, the comparison theorem and the squeeze theorem. □



Example 21.1. Evaluate the integral

$$\int_1^3 (x^2 + 3x)\,\mathrm{d}x$$

Example 21.2. Evaluate the integral

$$\int_0^{\pi} \sin x \, \mathrm{d}x$$

Example 21.3. Find the derivative of

$$G(x) = \int_1^{x} (t^2 - 3t) \, dt$$

Example 21.4. Find the derivative of

$$G(x) = \int_1^{x^2} \cos(3t) \, dt$$

Example 21.5. Find the derivative of

$$G(x) = \int_x^1 \tan(t^2)\,dt$$

21.1 Practice

Exercise 21.1. Evaluate the definite integral.

(1) $\displaystyle\int_{-1}^{2} (x^2 - 3x)\,dx$

(2) $\displaystyle\int_{-2}^{3} (t+2)(t-3)dt$

Exercise 21.2. Find the derivative of the function.

(1) $f(x) = \dfrac{d}{dx}\displaystyle\int_3^x \sqrt{9 - y^2}\,dy$

(2) $f(x) = \dfrac{d}{dx}\displaystyle\int_0^{x^2} \sqrt{1 - t^2}\,dt$

(3) $f(x) = \dfrac{d}{dx}\displaystyle\int_{\cos x}^{1} \sqrt{1-t^2}\,dt$

22 Indefinite Integrals and Net Changes

Definition 22.1. Let F be an antiderivative of the function f, that is $F'(x) = f(x)$. We call the most general anti-derivative of f is the indefinite integral and denoted as

$$\int f(x)\,dx = F(x) + c,$$

where c is any constant.

Formulas of Indefinite Integrals:

- $\displaystyle\int (af(x) + bg(x))\ dx = a\int f(x)\,dx + b\int g(x)\,dx$
- $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, as long as $n \neq -1$
- $\displaystyle\int \cos(x)\,dx = \sin(x) + C$
- $\displaystyle\int \sin(x)\,dx = -\cos(x) + C$
- $\displaystyle\int \sec^2(x)\,dx = \tan(x) + C$
- $\displaystyle\int \sec(x)\tan(x)\,dx = \sec(x) + C$
- $\displaystyle\int \csc^2(x)\,dx = -\cot(x) + C$
- $\displaystyle\int \csc(x)\cot(x)\,dx = -\csc(x) + C$

Example 22.1. Evaluate the following indefinite integral.

(1) $\displaystyle\int (x^4 + 3x - 2)\,dx$

(2) $\displaystyle\int \sqrt{x}(x^2 - 1)\,dx$

(3) $\displaystyle\int (\sin x - \sec^2 x)\,dx$

Definition 22.2. The **net change** of a quantity is the integral of the rate of change of that quantity.

$$\text{Net change} = \int_a^b r(t)\,dt$$

where $r(t)$ is the rate of change function.

Note the **total change** is

$$\text{Net change} = \int_a^b |r(t)|\,dt$$

where $r(t)$ is the rate of change function.

Example 22.2. Water is flowing into a tank at a rate of $r(t) = 3t^2 - 2t$ ft^3/min. How much water flows into the tank over the time interval 2 min. to 4 min.?

In Physics, the net and total changes are known as the displacement and distance. The distinction is between velocity and speed.

Displacement = Net change in position = integral of velocity.

Distance traveled Total = change in position = integral of speed.

Example 22.3. A particle moves along a straight line. The velocity is observed as a linear function $v(t) = 2t - 6$ m/s from time $t = 0$ to time $t = 3$.

(1) Find the net displacement of the particle.

(2) Find the distance the particle traveled.

Definition 22.3. Suppose that f is continuous over $[a, b]$. The **average value of the function** of f over $[a, b]$ is defined as

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example 22.4. Let f be a continuous function over $[a, b]$. Show that there is number c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example 22.5. Find the average of the function $f(x) = x^2$ over $[-2, 2]$.

22.1 Integration of Even and Odd Functions

For continuous even functions f, that is $f(-x) = f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

For continuous odd functions f, that is $f(-x) = -f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Example 22.6. Evaluate the following integrals.

(1) $\int_{-2}^{2} (3x^4 + 1)\,dx$

(2) $\int_{\pi}^{-\pi} 6\sin x\,dx$

22.2 Practice

Exercise 22.1. Evaluate the indefinite integral

(1) $\int (\sqrt{x} - \frac{1}{\sqrt{x}})\,dx.$

(2) $\int (\sin x - \cos x)\,dx$

Exercise 22.2. Suppose that a particle moves along a straight line with velocity $v(t) = 4 - 2t$, where $0 \leq t \leq 2$ (in meters per second).

Find the displacement at time t and the total distance traveled up to $t = 2$.

Exercise 22.3. Find the average of the function

$$f(x) = \sqrt{x} \qquad \text{over} \qquad [0, 4].$$

Exercise 22.4. Evaluate the integral

$$\int_{-2}^{2} x\sqrt{x^4 + 1}\, dx.$$

23 Substitution Method

Recall rules of derivatives.

(1) Chain rule: $(f(g(x)))' = f'(g(x))g'(x)$.

(2) Product Rule: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.

Reverse those rules, we will get techniques of integration.

Theorem 23.1. ***(Substitution Method for Indefinite Integrals)*** *Let $u = g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,*

$$\int f(g(x))g'(x)\mathrm{d}x = \int f(u)\,)u = F(u) + C = F(g(x)) + C.$$

Example 23.1. Find the indefinite integral

$$\int 2x(x^2+5)^7\mathrm{d}x.$$

Example 23.2. Find the indefinite integral

$$\int \sqrt{2x+1}\mathrm{d}x.$$

Example 23.3. Find the indefinite integral

$$\int 8(x+1)\sqrt[3]{x^2+2x}\mathrm{d}x.$$

Example 23.4. Find the indefinite integral
$$\int \frac{1}{\sqrt{x}(\sqrt{x}+1)} \mathrm{d}x.$$

Example 23.5. Find the indefinite integral
$$\int \frac{\sin x}{\cos^5 x} \mathrm{d}x.$$

Example 23.6. Find the indefinite integral
$$\int \cos x (2 \sin x + 3)^3 \mathrm{d}x.$$

Example 23.7. Find the indefinite integral
$$\int \frac{x+1}{\sqrt{x-1}}\mathrm{d}x.$$

Theorem 23.2. ***Substitution for Definite Integral***

Let $u = g(x)$ and let g' be continuous over an interval $[a, b]$, and let f be continuous over the range of $u = g(x)$. Then,
$$\int_a^b f(g(x))g'(x)\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\mathrm{d}u.$$

Example 23.8. Evaluate the definite integral
$$\int_0^1 x\sin\left(\frac{\pi x^2}{2}\right)\mathrm{d}x.$$

Example 23.9. Evaluate the definite integral
$$\int_1^2 \cos(3x-2)\mathrm{d}x.$$

Example 23.10. Evaluate the definite integral
$$\int_{-2}^1 \frac{x}{\sqrt{x^2+1}}\mathrm{d}x.$$

Example 23.11. Evaluate the definite integral

$$\int_{\pi/4}^{3\pi/4} \sin^2 t \cos t \mathrm{d}t.$$

23.1 Practice

Exercise 23.1. Evaluate the following integrals.

(1) $\displaystyle\int x\sqrt{x+1}\mathrm{d}x$

(2) $\displaystyle\int \frac{x}{(4x^2+9)^2}\mathrm{d}x$

(3) $\displaystyle\int (x-2)^7\mathrm{d}x$

(4) $\int \cos^3\theta \sin\theta \mathrm{d}\theta$

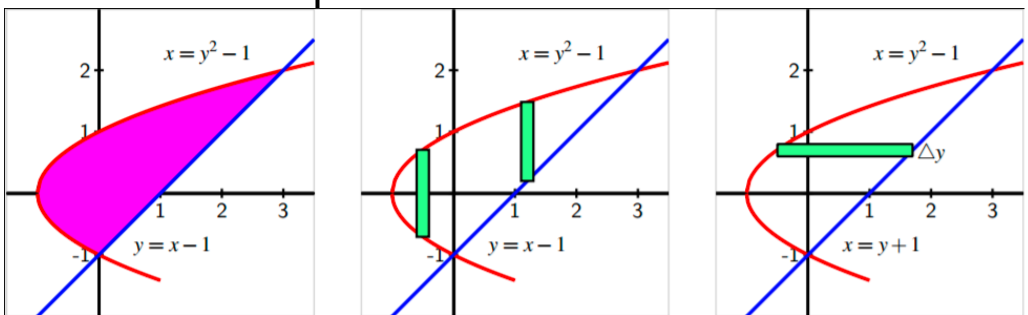
Exercise 23.2. (1) $\int_0^1 x\sqrt{1-x^2}\mathrm{d}x$

(2) $\int_0^{\pi/4} \sec^2\theta \tan\theta \mathrm{d}\theta$

(3) $\int_0^{\pi/4} \frac{\sin\theta}{\cos^4\theta}\mathrm{d}\theta$

24 Area Between Curves

The idea of finding area under an curve can be generalized to finding areas between curves. That is to slice the region enclosed by the curves, represent the area of a slice using the functions and a differential, and then take the sum which can be expressed as an integral.



24.1 Summarize the idea, we get the following strategy

(1) Sketch the curves and the enclosed region.

(2) Determine the boundaries of the region: find intersection points if they are on the boundary.

(3) Draw and label the dimension of a representative slice.

(4) State the area of the representative slice:

- if vertical slices are used, the area of a slice can be represented by $|U(x) - L(x)|\,dx$, where $U(x)$ and $L(x)$ are the upper and lower curves respectively;
- if horizontal slices are used, the area of a slice can be represented by $|R(y) - L(y)|\,dy$, where $R(y)$ and $L(y)$ are the right and left curves respectively;

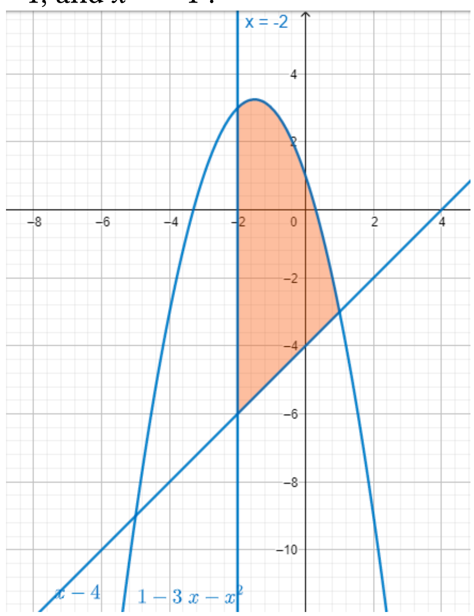
(5) Write a definite integral to represent area:

- if vertical slides: $\displaystyle\int_{x_L}^{x_R} |U(x) - L(x)|\,dx$, where x_L and x_R are the left and right boundaries of the region.
- if horizontal slides: $\displaystyle\int_{y_L}^{y_U} |R(y) - L(y)|\,dy$

Example 24.1. Find the area between $f(x) = -x^2 + 4x$ and

$g(x) = x^2 - 6x + 5$ over the interval $0 \le x \le 1$.

Example 24.2. Find the area enclosed by $y = x - 4$, $y = -x^2 - 3x - 1$, and $x = -1$.



Example 24.3. Find the area between $y = \sin x \cos x$ and $y = \sin x$, $-\pi/2 \le x \le \pi$.

Example 24.4. Find the area between $x = y^2 - y - 5$ and $x = y + 3$.

Example 24.5. Evaluate the integral by interpreting it as the area between two curves

$$\int_0^3 |\sqrt{x+5} - x|\, dx$$

24.2 Practice

Exercise 24.1. Find the area between $y = x^2 + 2$ and $y = 2x + 5$.

Exercise 24.2. Find the area enclosed by $y = \cos\theta$, $y = 0.5$, $x = 0$, and $x = \pi$.

Exercise 24.3. Find the area between $y = |x|$ and $y = x^2$.

Exercise 24.4. Find the area between $y = x$ and $x = y^2$.

Exercise 24.5. Find the area between $x = -3 + y^2$ and $x = y - y^2$.

Exercise 24.6. Find the area between $y = \sin x$ and $y = \cos x$ over $[-\pi, \pi]$.