Submission date 7th October
Questions with boxes will be marked for feedback this week
Starred questions are beyond the level of the course

$\boxed{1.}$ Let G be a group and let S be a subset of G.

(a) Prove that if $\mathcal{A}$ is any non-empty set of subgroups of G, then

$$\bigcap_{H\in\mathcal{A}} H$$

is a subgroup of G.

(b) Let $\mathcal{A}_S$ be the set of subgroups $H \subset G$ such that $S \subset H$. Prove that

$$\langle S\rangle := \bigcap_{H\in\mathcal{A}_S} H$$

is the smallest subgroup of G containing S in the sense that

- $S \subset \langle S\rangle$;
- if H is a subgroup of G such that $S \subset H$, then $\langle S\rangle \subset H$.

(c) If G is any group, then what are $\mathcal{A}_\emptyset$ and $\langle\emptyset\rangle$?

(d*) Let $G = \mathrm{GL}_2(\mathbb{R})$ and let

$$S = \left\{ \left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1}\end{smallmatrix}\right) \,\middle|\, r \in \mathbb{R}^\times \right\} \cup \left\{ \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1\end{smallmatrix}\right) \right\}.$$

Prove that $\langle S\rangle = \mathrm{SL}_2(\mathbb{R})$.

2. Let G be a group.

(a) Let $Z(G) = \{\, g \in G \,|\, gh = hg \text{ for all } h \in G \,\}$. Prove that $Z(G)$ is a normal subgroup of G. (The subgroup $Z(G)$ is called the *centre* of G).

(b) Suppose that $h \in G$, and let $Z_G(h) = \{\, g \in G \,|\, gh = hg \,\}$. Prove that $Z_G(h)$ is the largest subgroup H of G such that $h \in Z(H)$. (The subgroup $Z_G(h)$ is called the *centralizer* of h in G.)

(c) Suppose that H is a subgroup of G, and let

$$N_G(H) = \{\, g \in G \,|\, gHg^{-1} = H \,\}.$$

Prove that $N_G(H)$ is the largest subgroup $H' \subset G$ such that H is a normal subgroup of H'. (The subgroup $N_G(H)$ is called the *normalizer* of H in G).

(d) Let $G = \mathrm{GL}_2(\mathbb{R})$ and let $h = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)$. Find $Z(G)$, $Z_G(h)$ and $N_G(\langle h\rangle)$.

3. Suppose that G is a group and $g \in G$ and let $f : \mathbb{Z} \to G$ denote the homomorphism defined by $f(n) = g^n$.

(a) What is $\mathrm{im}(f)$?

(b) Describe $\ker(f)$ in terms of the order of g.

(c) Prove that every cyclic group is isomorphic either to $\mathbb{Z}$, or to $\mathbb{Z}/n\mathbb{Z}$ for some integer $n \geq 1$.

$\boxed{4.}$ Let G be a group, and let $\mathrm{Aut}(G)$ denote the set of automorphisms of G, i.e. the set of isomorphisms $f : G \to G$.

(a) Prove that $\mathrm{Aut}(G)$ is a group under composition.

(b) Suppose that $h \in G$ and define $\varphi_h : G \to G$ by $\varphi_h(g) = hgh^{-1}$. Prove that $\varphi_h \in \mathrm{Aut}(G)$.

(c) Prove that the function $\varphi : G \to \mathrm{Aut}(G)$ defined by $\varphi(h) = \varphi_h$ is a homomorphism such that $\ker(\varphi) = Z(G)$ and $\mathrm{im}(\varphi)$ (denoted $\mathrm{Inn}(G)$) is normal in $\mathrm{Aut}(G)$.

(d) Let $G = \mathbb{Z}/n\mathbb{Z}$. Prove that $\mathrm{Aut}(G)$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$.

5. Suppose that G is a group and N is a normal subgroup of G. Let $\overline{G}$ denote G/N, and let $\mathcal{A} = \mathcal{A}_N$, i.e. the set of subgroups $H \subset G$ such that $N \subset H$. If $g \in G$, then write $\overline{g}$ for $gN \in \overline{G}$, and if $H \in \mathcal{A}$, then write $\overline{H}$ for the subgroup H/N of $\overline{G}$.

(a) Prove that $H \mapsto \overline{H}$ defines a bijection

$$\mathcal{A} \longleftrightarrow \{\text{subgroups of } \overline{G}\}.$$

(b) Prove that if $H \in \mathcal{A}$, then $[G : H] = [\overline{G} : \overline{H}]$.

(c) Prove that if $H \in \mathcal{A}$, then H is normal in G if and only if $\overline{H}$ is normal in $\overline{G}$.

(d) Prove that if $H \in \mathcal{A}$ and H is normal in G, then there is an isomorphism $\overline{G}/\overline{H} \to G/H$ defined by $\overline{g}\overline{H} \mapsto gH$.

6. Let G be a group and let

$$[G,G] = \langle \{\, ghg^{-1}h^{-1} \mid g,h \in G \,\} \rangle.$$

(a) Prove that $[G,G]$ is the smallest normal subgroup N of G such that G/N is abelian. (The subgroup $[G,G]$ is called the *commutator subgroup* of G.)

(b) Determine the commutator subgroups of $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{PGL}_2(\mathbb{R})$. (*Hint: you may use 1d*) here.*)

(c) Prove that $\mathrm{GL}_2(\mathbb{R})$ is not isomorphic to $\mathbb{R}^\times \times \mathrm{PGL}_2(\mathbb{R})$.