CCM115a Sequences and Series: Assignment 5

You should submit solutions to all of the problems on the list below that are marked with the symbol '†'. The deadline for submission of these solutions is

4pm, Monday 13th November.

Submission is online, via the Keats pages for this course (and full instructions are available there).

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1. Using the definition of convergence, check that the following sequences s_n converge to the given limit ℓ as $n \to \infty$. That is, given $\varepsilon > 0$, find an explicit natural number n_0 (depending on ε) with the property that for all $n \geq n_0$ one has $|s_n - \ell| < \varepsilon$.

   (a) $s_n = \frac{3}{n^3} \to 0$

   (b)† $s_n = \frac{5}{\sqrt{n}} \to 0$

   (c) $s_n = \frac{2}{(-5)^n} \to 0$

   (d) $s_n = \frac{10}{(\log n)^2} \to 0$

   (e)† $s_n = \frac{n}{3n+1} \to \frac{1}{3}$

   (f) $s_n = \frac{2}{\sqrt{n^2+1}} \to 0$

2. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence which converges to a limit ℓ. Let $\{t_n\}_{n=1}^{\infty}$ be the sequence defined by $t_n = 2s_n$ for all n. Without using the Algebra of Limits, prove that $t_n \to 2\ell$ as $n \to \infty$. Argue directly from the definition of convergence.

3. Let $s_n \to \ell$ as $n \to \infty$. Consider the sequence $t_n = s_{n+7}$, $n \geq 1$. Prove that $t_n \to \ell$ as $n \to \infty$.

4. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence which converges to zero. Let $\{t_n\}_{n=1}^{\infty}$ be the sequence defined by
$$t_n = \begin{cases} 1 & \text{if } n \leq 1000, \\ s_n & \text{if } n > 1000. \end{cases}$$
Prove that $t_n \to 0$ as $n \to \infty$.

5.† Let ℓ be a real number and let $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ be sequences such that both $s_n \to \ell$ and $t_n \to \ell$ as $n \to \infty$. Then define $\{r_n\}_{n=1}^{\infty}$ to be the sequence with the property that for each n one has
$$r_n = \begin{cases} s_n & \text{if } n \text{ is odd} \\ t_n & \text{if } n \text{ is even.} \end{cases}$$
Prove that $r_n \to \ell$ as $n \to \infty$.

6. For each of the following sequences, determine whether it is bounded or unbounded. Prove your answer.
   (a) $n^2(-1)^n$
   (b) $\dfrac{(-1)^n n - 2^{-n}}{n}$
   (c) $\sqrt{n}\cos(\frac{\pi n}{2})$
7. For each of the following sequences, determine whether it (i) diverges to $+\infty$ (ii) diverges to $-\infty$ (iii) neither of the above. Prove your answer.
   (a) $(3/5)^n$
   (b) $(5/3)^n$
   (c)† $n^2 - n$
   (d) $(1/5)^n - 3^n$