Introduction to Abstract Algebra: Sheet 1

For discussion in Week 1 Skills Sessions and Week 2 Tutorials

Please complete all core problems. The additional practice problems are not required: try them if you're interested and have time. The problems marked with (SS) will be discussed in the Skills Session for Week 1. Please attempt these *before* your Skills Session. Some of the other problems will be discussed in your tutorial for Week 2. Please attempt these problems *before* your tutorial.

Core problems

Exercise 1. Consider the following subsets of $\mathbb{Z}$:

$$A = \{3n \mid n \in \mathbb{Z}\},\ B = \{\, a \in \mathbb{Z} \mid -6 < a \leq 6 \,\},\ C = \{\, -3, 2, 4, 9, 31 \,\}.$$

List all elements of the following sets:

(i) $B \cup (A \cap C)$

(ii) $(B \cup A) \cap C$

(iii) $(B \setminus A) \cap C$

Exercise 2 (SS)**.** Let A, B, and C be as defined in Exercise 1. List the elements of the set $(A \cap B) \times (A \cap C)$

Exercise 3. Let A, B and C be subsets of a set X. Prove that

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C).$$

Exercise 4 (SS)**.** Let X and Y be sets. Let A_1 and A_2 be subsets of X. Prove that

$$(A_1 \times Y) \cap (A_2 \times Y) = (A_1 \cap A_2) \times Y$$

Exercise 5 (SS)**.** Let $f : \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $f((x, y)) = x + y$. Show that f is surjective.

Exercise 6. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Show that the function $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $f(x) = (x^2, g(x))$ is injective.

Exercise 7. Let A, B, C be sets. Suppose $f : A \to B$ and $g : B \to C$ are surjective functions. Show that $g \circ f$ is surjective.

Exercise 8. Prove by induction that the sum of the first n odd positive squares is $\frac{1}{3}(4n^3 - n)$, i.e. that

$$1 + 9 + 25 + \cdots + (2n - 1)^2 = \frac{1}{3}(4n^3 - n).$$

Additional practice

Exercise 9. Find a function $f : \mathbb{Z} \to \mathbb{Z}$ such that

(i) f is injective but not surjective.

(ii) f is surjective but not injective.

Exercise 10. Let X and Y be sets. Let $A_1, A_2 \subset X$ and $B_1, B_2 \subset Y$. For the following statements about subsets of $X \times Y$, give either a proof or a counterexample.

(i) $(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$.

(ii) $(A_1 \times B_1) \cup (A_2 \times B_2) = (A_1 \cup A_2) \times (B_1 \cup B_2)$.

Exercise 11. Let X be a finite set with n elements.

(i) How many subsets does X have?

(ii) How many functions $X \to \{0, 1\}$ exist?

(iii) Think about how and why your answer to (i) relates to your answer to (ii).

Exercise 12. Let A be a set with n elements and let B be a set with m elements. Find a formula for the number of injective functions $f : A \to B$.