Linear Algebra and Geometry II - Sheet 1

Please attempt all of the problems on this sheet. You are welcome to make use of the Lecturers' office hours and the tutorial for asking questions about any of the problems. Solutions will be made available by the start of Week 3.

All matrices are square.

1. Write down what it means for a matrix A to be *similar* to a matrix B.

   Prove that similarity of matrices is an equivalence relation, i.e.,

   (a) every matrix A is similar to itself;

   (b) if A is similar to B, then B is similar to A;

   (c) if A is similar to B and B is similar to C, then A is similar to C.

2. Suppose that A is similar to B.

   (a) Show that $\det(A) = \det(B)$.

   (b) Give an example to show that the converse is not always true, i.e. find two matrices A and B such that $\det(A) = \det(B)$ but A is not similar to B.

   (c) Prove that A^n is similar to B^n for all positive integers n.

3. Suppose that A is an $n \times n$ complex matrix and that $\mathbf{v} \in \mathbb{C}^n$ is an eigenvector for A with eigenvalue λ.

   (a) Write down an equality relating A, λ and $\mathbf{v}$.

   (b) Prove that for every positive integer k, $\mathbf{v}$ is eigenvector for A^k with eigenvalue λ^k.

4. A matrix A is called *nilpotent* if $A^k = 0$ for some $k \in \mathbb{N}$. Prove that if A is nilpotent $n \times n$ complex matrix, then $\sigma(A) = \{0\}$ (or, in other words, 0 is the only eigenvalue of A).

5. Compute the characteristic polynomial and eigenvalues of the following matrices:

   (a)
   $$A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}.$$

   (b)
   $$B = \begin{pmatrix} 2024 & 2023 & 2022 \\ 0 & 2023 & 2022 \\ 0 & 0 & 2022 \end{pmatrix}.$$

(c)

$$C = \begin{pmatrix} 0 & 3 & -2 \\ 2 & -1 & 2 \\ 1 & -3 & 3 \end{pmatrix}.$$

6. Let A be a square matrix. The *trace* of A, denoted $\mathrm{tr}(A)$, is the sum of the diagonal entries of A.

   (a) Find the eigenvalues λ_1, λ_2 of the matrix

   $$A = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}.$$

   (b) Compute $\mathrm{tr}(A)$ and $\det(A)$, and show that $\mathrm{tr}(A) = \lambda_1 + \lambda_2$ and $\det(A) = \lambda_1\lambda_2$.

7. For any choice of $a, b \in \mathbb{R}$ prove that the matrix $A = \begin{pmatrix} a+b & a \\ -b & 0 \end{pmatrix}$ in $M_2(\mathbb{R})$ has eigenvalues a and b.

8. (a) Compute the characteristic polynomial of the 2×2 matrix

   $$A_2 := \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

   (b) Compute the characteristic polynomial of the 3×3 matrix

   $$A_3 := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

   (c) Formulate a conjecture for the formula of the characteristic polynomial of the $j \times j$ matrix

   $$A_j := \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 2 & 2 & \dots & 2 & 2 \\ 0 & 0 & 3 & \dots & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & j-1 & j-1 \\ 0 & 0 & 0 & \dots & 0 & j \end{pmatrix}.$$

   (d) [Optional bonus problem]
   Prove the conjecture formulated in the previous step (hint: use induction).