Linear Algebra and Geometry II - Sheet 2

Please attempt all of the problems on this sheet. You are welcome to make use of the Lecturers' office hours and the tutorial for asking questions about any of the problems. Solutions will be made available in Week 4.

Participation Mark: Please submit your solution to Problems 2 and 5, marked with a blue triangle (△), by using the submission link on Keats. They are due on Monday 5th February 2024 at 4pm.

All matrices are square.

1. Suppose that the A is an $n \times n$ matrix with *real* entries. Prove that if λ is an eigenvalue of A then also its complex conjugate $\bar{\lambda}$ is an eigenvalue of A, and moreover if $\mathbf{v}$ is an eigenvector for λ then $\bar{\mathbf{v}}$, the coordinate-wise complex conjugate,* is an eigenvector for $\bar{\lambda}$.

2. △ For each of the following linear operators on $\mathbb{R}^2$ determine whether or not it is diagonalisable. You must give a reason for your answer.

   (a) $\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}$

3. Consider the vector space V over $\mathbb{R}$ consisting of polynomials in one variable that have degree at most three. Thus
$$V = \{f(t) = at^3 + bt^2 + ct + d \mid a, b, c, d \in \mathbb{R}\}.$$
   Determine the eigenvalues and eigenvectors of the linear operator $D : V \to V$ given by differentiation
$$D(f) := \frac{d}{dt} f.$$
   Is this linear operator diagonalizable?

4. Let A be a 5×5 nonzero matrix satisfying $A^5 = \mathbf{0}$. Prove that A cannot be diagonalised. [Hint: you may wish to try a proof by contradiction.]

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*explicitly: if $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ then $\bar{\mathbf{v}} = \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_n \end{pmatrix}$.

5. $\triangle$ For each of the following matrices A, compute the characteristic polynomial and determine whether A is diagonalisable (over the complex numbers). Find an invertible matrix R such that $R^{-1}AR$ is diagonal if A is diagonalisable.

   (a) $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 0 & 3 & -2 \\ 2 & -1 & 2 \\ 1 & -3 & 3 \end{pmatrix}$

6. Give an example of a 3×3 matrix that cannot be diagonalized. Justify your answer.

7. [Optional bonus problem] Prove that if A is an $n \times n$-matrix, then the polynomial

$$f(x) = \det(xI - A)$$

   has highest order term x^n, in other words it is a *monic* polynomial of degree n.

   Remark. Note that a *monic* polynomial is one for which the coefficient of the highest order term is 1, so that in the degree n case, the highest order term is precisely x^n. Note also that the polynomial f almost agrees with the characteristic polynomial, $p_A(x) = \det(A - xI)$. The two are the same up to a sign: namely $f(x) = (-1)^n p_A(x)$. This is because $f(x)$ is the determinant of the matrix $M = xI - A$ while $p_A(x)$ is the determinant of $-M$ (where all n columns of M have been multiplied by -1).