Probability and Statistics 1

Sheet 1

15 January 2024

Every problem sheet in this course has two sections:

1. **Practice makes permanent**: problems that support you to develop a sound knowledge of and capacity to apply the definitions, ideas and methods introduced each week.
2. **The deeper thinking**: problems that provide challenge and intrigue, alongside a richer appreciation of the underlying ideas.

You should attempt all of the problems in the **practice makes permanent** section. Solutions to these problems will be provided at a later date.

The deeper thinking problems are optional - but encouraged! Full solutions to these problems will **not** usually be provided, though I may share hints, ideas and outline solutions for some of these.

Practice makes permanent

Problem 1. We flip a coin three times. For this experiment we choose the sample space

$$\Omega = \{HHH, THH, HTH, HHT, TTH, THT, HTT, TTT\}$$

where T stands for tails and H for Heads.

(a) Write down the set of outcomes corresponding to each of the following events:
A: "we flip tails exactly two times."
B: "we flip tails at least two times."
C: "tails did not appear before a head appeared."
D: "the first flip results in tails."

(b) Write down the set of outcomes corresponding to each of the following events:
A^c, $B \backslash A$, $A \cup (C \cap D)$ and $A \cap D^c$

Problem 2. Given a sample space Ω and two events $E, F \subset \Omega$,

(a) Copy and complete the proof below to show that $(E \cup F)^c = E^c \cap F^c$, filling in the empty boxes as you go:

Let x be an arbitrary element of $(E \cup F)^c$, so $x \in (E \cup F)^c$.
$\implies x \notin E \cup F$
$\implies x \notin E$ and $\boxed{}$
$\implies x \in E^c$ and $\boxed{}$
$\implies x \in \boxed{} \cap \boxed{}$
Hence, $(E \cup F)^c \subseteq E^c \cap F^c$.
Now let x be an arbitrary element of $E^c \cap F^c$, so $x \in E^c \cap F^c$.
$\implies \boxed{}$
$\implies \boxed{}$
$\implies \boxed{}$
Hence, $E^c \cap F^c \subseteq (E \cup F)^c$.
We therefore conclude that $(E \cup F)^c = E^c \cap F^c$.

(b) Construct a similar proof to show that $(E \cap F)^c = E^c \cup F^c$.

Problem 3. A fair coin is flipped 4 times. Describe the appropriate *sample space* and define the *probability measure* when

(a) the outcome of every flip is of interest.

(b) only the total number of tails is of interest.

Problem 4. By drawing a Venn diagram or otherwise:

(a) Let A, B, C be three events and $\mathbb{P}$ a probability measure such that $\mathbb{P}(A \cap B) = \frac{3}{10}$, $\mathbb{P}(B \cap C) = \frac{2}{5}$, $\mathbb{P}(A \cap C) = \frac{1}{5}$ and $\mathbb{P}(A \cap B \cap C) = \frac{1}{10}$. Find the probability that

(i) exactly two of the events A, B, C occur

(ii) less than two of the events A, B, C occur

(b) Let Q, R, S be three events and $\mathbb{P}$ a probability measure such that $\mathbb{P}(Q \cap S) = \mathbb{P}(Q \cap R) = \mathbb{P}(R \cap S) = \frac{1}{3}$ and where it is known that Q, R and S cannot occur simultaneously. Can you determine $\mathbb{P}(Q)$?

Problem 5. The equation $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ (*) established in lectures is often useful to compute the probability of the union of two events.

(a) Draw a Venn diagram to represent three events, A, B and C. Use your diagram to write down an equation for computing the probability of the union of three events. Your equation should start with

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \ldots$$

(b) Prove your equation by a repeated application of (*), or otherwise.

Problem 6. Using induction, prove *Boole's Inequality*:

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i).$$

The deeper thinking

Problem 7. Prove *Bonferroni's inequality*:

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} \mathbb{P}(A_i^C).$$

Problem 8. In this question you will need to know that for two *independent* events A, B, $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. (We will cover independece in lectures next week.)

(a) Prove carefully, for events $A_1, A_2, \ldots, A_n$, that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots$$
$$+(-1)^{n+1}\mathbb{P}\left(\bigcap_{i} A_i\right).$$

This formula is known as the **inclusion-exclusion principle**.

(b) Each year, Santa has the job of delivering n presents to n children. Each present is unique and intended for a specific child.

One year, Santa takes a lazy approach. He delivers each present to one of the n children randomly (and independently of any other present's destination) with the result that some children may receive more than one gift!

By using (a) or otherwise, find an expression for the probability that at least one present is delivered to the right child.

(c) The next year, Santa improves his present-assigning method by visiting the n children in turn, gifting each exactly one present by selecting it uniformly at random from those still remaining in his sack. Find an expression for the probability that, using this system, no child receives the right present.

(d) Find the limits of the expressions in (b) and (c) as n tends to infinity.

You may assume that, for real x,

$$e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!} = \lim_{N\to\infty} \left(1 + \frac{x}{N}\right)^N.$$