CALCULUS 1 TUTORIAL EXERCISES I

In this tutorial you will practise working with injective, surjective and bijective functions.

Tutor's Example[1]

Let $f : A \to B$ be given by

$$f(x) = \frac{\sin x}{x}.$$

Identify the largest domain $A \subset \mathbb{R}$ for which f is well-defined. Sketch the graph of f. If $B = (-\infty, n] \subset \mathbb{R}$ identify the smallest value of n for which f is well-defined.

1. State the maximum domains and minimum ranges in $\mathbb{R}$ for which the following are well-defined functions:

   (a) $f(x) = \dfrac{(x+2)^2}{(x-2)^2}$

   (b) $g(x) = \dfrac{1}{x}$

   (c) $h(x) = \begin{cases} 0 & x < 0 \\ 1/2 & x = 0 \\ 1 & x > 0. \end{cases}$

   The function $h(x)$ is a common definition of the Heaviside step function, named after Oliver Heaviside.

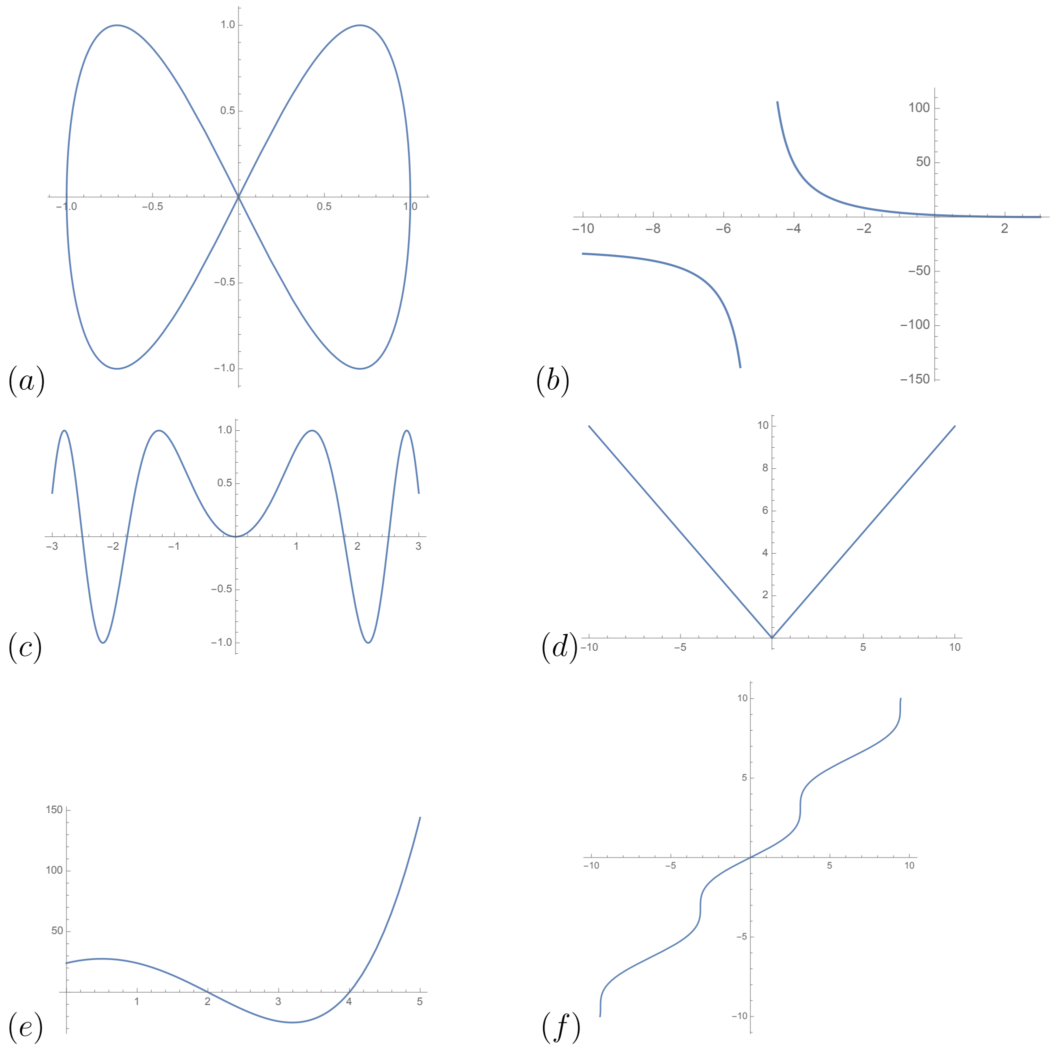
2. For each of the three functions in question 1, sketch the graph and qualitatively describe the slope of each graph—in particular state whether the slope is positive, negative or does not exist.

   Where appropriate, show clearly any intercepts with the axes, any asymptotes, where the function is positive/negative, what happens at discontinuities, and how the function behaves as $x \to \pm\infty$.

3. Which of the following graphs could correspond to well-defined functions? State whether the functions could be bijections. You may assume that the graphs are plotted over the full domain and range in each case (i.e. this means you can assume the range is such that each

[1] To be discussed by the tutor at the start of the tutorial.

function is surjective). In each case where the function is well-defined, make a guess for what the function might be (is it polynomial, trigonometric, rational, etc?).



4. Sketch the graphs of the following functions and for each case state whether the function is injective, surjective, neither, or if the function is not well-defined. *Again consider the aspects of the function mentioned in question 2.*

   (a) $y : [3, \infty) \to \mathbb{R}$ given by $y(x) = x^2 - 6x + 9$.

   (b) $y : \mathbb{R} \to \mathbb{R}$ given by $y(x) = (x - 2)^3$.

   (c) $y : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ given by $y(x) = (x - 2)(x + 1)(x - 3)$.

   (d) $y : \mathbb{R} \to \mathbb{R}_0^+$ given by $y(x) = \sqrt{x^2 + 4x + 4}$.

   (e) $y : \mathbb{R} \to \mathbb{R}$ given by $y(x) = 2^x - 3^x$.

   (f) $y : \mathbb{R} \to \mathbb{R}$ given by $y(x) = x^4$.

(g) $y : \mathbb{R}_0^+ \to [1, \infty)$ given by $y(x) = x + 2^x$.

(h) $y : [3, \infty) \to \mathbb{R}$ given by $y(x) = x^4 + 2x^3 - 13x^2 - 14x + 24$.

5. **Exam-style question**[2] Let $\eta(x)$ (this is the Greek letter *eta*) be a function which is never negative and satisfies

$$\eta(x)^2 = x^3 + ax + b$$

for some real constants a and b. Furthermore, assume that the polynomial function $p(x) = x^3 + ax + b$ has a single real root, x_0.

(a) **[5 marks]** Sketch a graph of $p(x)$ and hence determine the maximum domain of $\eta(x)$. [Note, there may be several different 'shapes' for p depending on the values of a and b, does this change the domain of η?]

(b) **[4 marks]** Now sketch a graph of $\eta(x)$ and deduce its minimum range for the domain you found in part (a).

(c) **[8 marks]** Show that when $b > 0$, $\eta(x)$ is injective if and only if $a \geq 0$.

5. **Advanced Python**[3] Graphs are plotted on Python using the library `matplotlib.pyplot` and the function `plot`. It is also helpful to use the package `numpy` to define mathematical functions. In Python, you can import these using

```
import matplotlib.pyplot as plt
import numpy as np
```

You have to set the x values over which the function is to be plotted, and how many points to plot in that range. To plot 100 between -2 and 2 you can define

```
x = np.linspace( -2, 2, 100 )
```

You must also define the function you want to plot, for example, to plot $f(x) = x^2$ we define

```
def f( var ) :
    return var ** 2
```

Finally, we can ask python to plot $f(x)$ on the specified range, and very importantly, ask python to display that plot.

[2]Some questions on the final exam will be longer with several parts which delve into a topic and build on each other. To help you prepare, each sheet will include such a question.

[3]Each sheet will contain a Python problem which should complement the programming aspect of the course. In some cases, like this one, you may have to look up unfamiliar functions and use your knowledge of Python to apply them.

```
plt.plot( x, f( x ) )
plt.show()
```

Practise using Python to confirm your graph sketches of the functions in problem 1 (some research will be required to plot $h(x)$ in part (c)).

6. **A challenging problem**[4] A set S is countably infinite if it can be put in one-to-one correspondence with the natural numbers $\mathbb{N} := \{1, 2, 3, \ldots\}$, i.e. if a bijection $f : S \to \mathbb{N}$ can be constructed. Discuss whether the following are countably infinite sets:

   (a) The set of positive even integers.

   (b) The set of all integers $\mathbb{Z}$.

   (c) The set of ordered pairs of the non-negative integer numbers, given by

   $$\{(n, m) \,|\, n, m \in \mathbb{Z}_+\}.$$

   N.B. the order of the pair of numbers matters, so that $(1, 2) \neq (2, 1)$;

   $$\mathbb{Z}_+ = \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}.$$

   (d) The set of rational numbers $\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p \in \mathbb{Z},\, q \in \mathbb{N} \right\}$.

   (e) The set of real numbers $\mathbb{R}$.

   Consider the set of all finite lists of natural numbers, L, i.e. the elements of L are lists of positive integers $(n_1, n_2, \ldots, n_N)$ and each $n_i \in \mathbb{N}$, e.g.

   $$(1, 4, 6),\ (4, 1, 6, 6, 6, 4),\ \text{and}\ (7, 52, 23456, 3, 504, 10023)$$

   are elements of L.

   (f) Argue that L is countably infinite and construct a bijection from L to $\mathbb{N}$.

[4] All challenging problems in the tutorial problem sets are beyond the scope of the course, and are not examinable, but they are, hopefully interesting and inspiring.