Exercises to Section 6

Exercises marked with a star * are for submission to your tutor.

Integrability

1. Let f be a bounded monotone function on $[0, 1]$. Prove that

$$\int_0^1 f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f(k/n) = O(1/n), \quad n \to \infty.$$

2. Let f be bounded on $[a, b]$ and such that $|f| \in \mathcal{R}[a, b]$; does it follow that $f \in \mathcal{R}[a, b]$?

3. For an interval Δ, prove the identity

$$\sup_{x,x' \in \Delta} |f(x) - f(x')| = \sup_{x\in\Delta} f(x) - \inf_{x\in\Delta} f(x).$$

   Proceed as follows. Denote $M = \sup_{x\in\Delta} f(x)$, $m = \inf_{x\in\Delta} f(x)$.

   (a) Prove that $\sup_{x,x'\in\Delta} (f(x) - f(x')) \leqslant M - m$.

   (b) Conclude that $\sup_{x,x'\in\Delta} |f(x) - f(x')| \leqslant M - m$.

   (c) Argue that for any $\varepsilon > 0$, there exist $x_1, x_2 \in \Delta$ such that $f(x_1) > M - \varepsilon$ and $f(x_2) < m + \varepsilon$.

   (d) From here prove that $\sup_{x,x'\in\Delta} (f(x) - f(x')) \geqslant M - m - 2\varepsilon$.

   (e) Conclude that $\sup_{x,x'\in\Delta} |f(x) - f(x')| \geqslant M - m$.

4.* Let $f \in \mathcal{R}[a, b]$ and $[c, d] \subset [a, b]$; prove that the restriction $f|_{[c,d]}$ is Riemann integrable on $[c, d]$.

5. Prove the Lemma in the “Oscillatory discontinuities” subsection of the lecture notes. Proceed as follows. Given $\varepsilon > 0$, let P be a partition of $[a + \varepsilon, b - \varepsilon]$ such that

$$U(P, f) - L(P, f) < \varepsilon.$$

   Add the points a, b to the partition P; we obtain a partition P^* of $[a, b]$. Estimate the difference $U(P^*, f) - L(P^*, f)$ by using the previous inequality on $[a + \varepsilon, b - \varepsilon]$ and the boundedness of f on $[a, a + \varepsilon]$ and $[b - \varepsilon, b]$. You should get an inequality of the form

$$U(P^*, f) - L(P^*, f) < C\varepsilon,$$

   where C depends on the upper bound for f. Now use Riemann's criterion.

6. Let $f \in \mathcal{R}[a, b]$; prove that there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions on $[a, b]$ such that

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

   Hint: use piecewise linear interpolation

Challenging exercises

7. Prove that any countable set of point on $\mathbb{R}$ has measure zero.
8. Prove that Thomae's function f_T is integrable on $[0, 1]$, and $\int_0^1 f_T(x)dx = 0$.
9. Let $f \in \mathcal{R}[a, b]$ and let $[c, d] \subset (a, b)$. Prove that f has the following property of *continuity in the mean*:

$$\lim_{h \to 0} \int_c^d |f(x + h) - f(x)|dx = 0.$$