Linear Algebra and Geometry I 2023 — Problem Sheet 9

You are encouraged to work with other students on the module. If you are having difficulty with any of the questions or want feedback on specific answers you should ask your tutor in your next tutorial or attend the lecturer's office hours.

1. Consider the following matrices

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 1 & 2 & 1 & 1 & -1 \\ 2 & 4 & 0 & 1 & 0 \end{pmatrix}.$$

(a) Determine the rank and nullity of each of the matrices.

(b) For each matrix, find a basis for its image and kernel.

2. Complete each of the following matrices so they have rank 1.

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{pmatrix}, \quad B = \begin{pmatrix} & 9 & \\ 1 & & \\ 2 & 6 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} a & b \\ c & \end{pmatrix}.$$

3. In each of the following cases, give an example of a linear map with the stated properties (by giving it's matrix) or explain why no such map exists.

(a) An injective linear map $A : \mathbb{R}^3 \to \mathbb{R}^5$.

(b) A surjective linear map $A : \mathbb{R}^4 \to \mathbb{R}^3$ such that $r(A) = n(A)$.

(c) A linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $r(A) = 1$ and $(1, 0, 0) \in \operatorname{Im} A$.

(d) A linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $n(A) = 1$ and $(1, 0, 0) \in \operatorname{Ker} A$.

4. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear map given by

$$Tf(t) = f''(t) + tf'(t), \quad f \in \mathbb{P}_2.$$

(a) Determine whether the function $g(t) = t + 1$ is in the kernel of T.

(b) Show that the function $h(t) = t^2$ is **not** in the image of T.

(c) Explain why this tells you that T is not invertible.

(d) Let $[T]$ denote the matrix of T with respect to the monomial basis of $\mathbb{P}_2$. Is it possible to express $[T]$ as a product of elementary matrices? Justify your answer.

5. Let $A : V \to W$ and $B : U \to V$ be linear maps.

(a) Prove the following statements.

(i) $r(AB) \leq r(A)$. *(Hint: first show that* $\operatorname{Im} AB \subseteq \operatorname{Im} A$.*)*

(ii) $n(AB) \geq n(B)$.

(b) It is **not** true that $n(AB) \geq n(A)$. Find a counterexample to show this.

(c) Is it true that $n(AB) \leq n(A)$? Justify your answer.

6. Let $\mathcal{S}$ be the standard basis in $\mathbb{R}^3$ and let $\mathcal{E}$ and $\mathcal{F}$ be the bases

$$\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{F} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

(a) Determine the transition matrices $[I]_{\mathcal{S}}^{\mathcal{E}}$ and $[I]_{\mathcal{S}}^{\mathcal{F}}$ from $\mathcal{E}$ and $\mathcal{F}$ respectively to the standard basis.

(b) Determine the transition matrix $[I]_{\mathcal{F}}^{\mathcal{E}}$. *(Hint: you might want to first go from $\mathcal{E}$ to $\mathcal{S}$, then from $\mathcal{S}$ to $\mathcal{F}$.)*

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ 2y \\ -2x - y + 3z \end{pmatrix}.$$

Determine the matrix of T with respect to the basis $\mathcal{F}$, i.e. find $[T]_{\mathcal{F}}^{\mathcal{F}}$.

7. Let $\mathcal{S}$ be the standard basis in $\mathbb{R}^2$ and let $\mathcal{E}$ be the basis $\{(2, 1), (-1, 1)\}$.

(a) Let $\mathbf{v} = (-1, 2)$. Find $[\mathbf{v}]_{\mathcal{E}}$.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map defined by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7x - 2y \\ -x + 8y \end{pmatrix}.$$

(i) Find the matrix $[T]_{\mathcal{S}}^{\mathcal{S}}$ of T with respect to the standard basis.

(ii) Find the matrix $[T]_{\mathcal{E}}^{\mathcal{E}}$ of T with respect to the basis $\mathcal{E}$.

(iii) Describe geometrically the transformation T in terms of the basis $\mathcal{E}$.