Probability and Statistics 1

Sheet 2

22 January 2024

Every problem sheet in this course has two sections:

1. **Practice makes permanent**: problems that support you to develop a sound knowledge of and capacity to apply the definitions, ideas and methods introduced each week.
2. **The deeper thinking**: problems that provide challenge and intrigue, alongside a richer appreciation of the underlying ideas.

You should attempt all of the problems in the **practice makes permanent** section. Solutions to these problems will be provided at a later date.

The deeper thinking problems are optional - but encouraged! Full solutions to these problems will **not** usually be provided, though I may share hints, ideas and outline solutions for some of these.

Practice makes permanent

Problem 1. Two fair dice are thrown. Let A be the event that the first shows an odd number, B be the event that the second shows an even number, and C be the event that either both are odd or both are even.

Show that A, B, C are pairwise independent but not independent.

Comment: The aim of this question is to give reason to the definition of independence for multiple events (definition 2.3.2 from lectures).

Problem 2. There are a number of socks in a drawer, three of which are red and the rest of which are black. Nada chooses her socks by selecting two at random from the drawer, and puts them on. She is three times more likely to wear socks of different colours than to wear matching red socks. What is the probability that Nada wears matching black socks?

Problem 3. A single card is removed at random from a deck of 52 cards. From the remainder we draw two cards at random and find that they are both spades. What is the probability that the first card removed was also a spade?

Problem 4. Suppose A and B are events and $\mathbb{P}$ a probability measure such that $0 < \mathbb{P}(A) < 1$ and $0 < \mathbb{P}(B) < 1$. In each case below: if the answer is "yes", show this by constructing an example; if the answer is "no", prove it.

(a) If A and B are mutually exclusive, can they be independent?

(b) If A and B are independent, can they be mutually exclusive?

(c) If $A \subset B$, can A and B be independent?

(d) If A and B are independent, can A and $A \cup B$ be independent?

Problem 5. A breath analyzer is used by the police to test whether drivers exceed the legal limit set for the blood alcohol percentage while driving. The policy data analytics team uses a probability model under which

$$\mathbb{P}(A \mid B) = \mathbb{P}(A^c \mid B^c) = p,$$

where A is the event "breath analyzer indicates that legal limit is exceeded" and B "driver's blood alcohol percentage exceeds legal limit". On the night of the winter solstice in the land of the druids, it is estimated that 8% of the drivers exceed the limit, and the policy data analytics team wishes to make this proportion a feature of their model.

(a) Describe in words the meaning of $\mathbb{P}(B^c \mid A)$, and why the police want to keep this probability as small as possible.

(b) Determine $\mathbb{P}(B^c \mid A)$ if $p = 0.95$

(c) How large should p be so that $\mathbb{P}(B \mid A) = 0.9$?

Problem 6. In this question we explore various definitions of independence.

(a) Let A and B be events satisfying $\mathbb{P}(A), \mathbb{P}(B) > 0$, and such that $\mathbb{P}(A \mid B) = \mathbb{P}(A)$. Show that $\mathbb{P}(B \mid A) = \mathbb{P}(B)$.

(b) Show that $\mathbb{P}(A^c \cap B^c) = \mathbb{P}(A^c)\mathbb{P}(B^c)$ is an equivalent condition to $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Checkpoint! Showing X is 'equivalent' to Y usually means showing a double implication: that X $\implies$ Y, and also that Y $\implies$ X. Review your solutions to the problem above to check you have fully proved equivalence.

Comment: If A and B be events satisfying $\mathbb{P}(A), \mathbb{P}(B) > 0$, any of the following conditions are equivalent:

- $\mathbb{P}(A \mid B) = \mathbb{P}(A)$
- $\mathbb{P}(B \mid A) = \mathbb{P}(B)$
- $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$
- Any version of the above three conditions where A is replaced by A^c, or B is replaced by B^c, or both.

The deeper thinking

The two problems this week are very 'wordy', but don't let that put you off! They are both interesting and accessible.

Problem 7. Did UEFA fix the draw?

In the UEFA Euro 2004 playoffs draw, 10 national football teams were matched in pairs. Many people complained that "the draw was not fair" because it seemed that stronger teams had been matched with weaker teams (which is commercially the most interesting).

Let's call such a draw, where the 5 strongest teams are matched against the 5 weakest teams, a 'dream draw'. In this question we will compute the probability of a dream draw - under a probability model built from the assumption that the pairings were chosen at random.

To create a random set of pairings, we assume we have a ranking of the teams from strongest to weakest. Imagine writing down the names of the 10 teams on separate pieces of paper, folding these up and putting them into a hat. We then take out two pieces of paper at a time (shaking up the hat after each pick, and not replacing the pieces of paper already picked) to determine each of the 5 pairings.

Define events D_i as "the ith pair drawn is a dream combination", where a 'dream combination' is a pair of a strong team with a weak team, and $i = 1, \ldots, 5$.

(a) Compute $\mathbb{P}(D_1)$

(b) Compute $\mathbb{P}(D_1 \cap D_2)$

(c) Compute $\mathbb{P}(D_1 \cap D_2 \cap D_3)$

(d) Continue this procedure to obtain the probability of a dream draw.

Problem 8. The generalised Monty Hall Problem

Monty Hall's generalised game show now has N doors. 1 door is hiding a snazzy new super-green electric car. The other $N-1$ doors are hiding goats. Monty knows where the car is hidden but you, the contestant, do not. You are invited to pick a door, and you do so. Monty opens k doors and behind each of them is a goat (he never reveals a car at this stage, and he never opens the door you have picked). Monty now invites you to change your selection.

Show that the probability that you win the car if you switch your choice is

$$\frac{N-1}{N(N-k-1)}.$$

Is this always the best strategy, for any N and any k (where $1 \leq k < N-1$)?