# Exercises to Section 1

Exercises in red are from the list of the typical exercises for the exam.
Exercises marked with a star $^*$ are for submission to your tutor.

### Sets

1. Prove that any interval $(a, b)$ is open in the sense of the definition given in the notes (i.e. it contains a neighbourhood of each of its points).

### Functions, natural domains

2. Determine the natural domains of the following functions:

   (a) $f(x) = \sqrt{3x - x^3}$;

   (b)$^*$ $f(x) = \sqrt{\frac{1+x}{1-x}}$;

   (c) $f(x) = \sqrt{\cos x}$;

   (d) $f(x) = \frac{\sqrt{x}}{\sin \pi x}$.

### Boundedness

3. Which of the following functions are bounded on the given interval? Sketch the graph and justify your answer.

   (a) $f(x) = \frac{x}{1+x}$ on $[0, \infty)$

   (b) $f(x) = 1/\sqrt{x}$ on $(0, 1)$

   (c) $f(x) = \sqrt{1 + x^2}$ on $(0, 1)$

   (d) $f(x) = \sqrt{1 + x^2}$ on $(1, \infty)$

   (e)$^*$ $f(x) = x \sin x$ on $(1, \infty)$

   (f) $f(x) = \frac{1}{x} \sin(\frac{1}{x})$ on $(0, 1)$

### Limit of a function

4. Write down in the "$\epsilon - \delta$ language" the following definitions and give examples:

   (a) $\lim\limits_{x \to x_0} f(x) = \infty$;

   (b) $\lim\limits_{x \to x_0} f(x) = -\infty$;

   (c) $\lim\limits_{x \to \infty} f(x) = -\infty$.

### $O$ and $o$ notation

5. Determine whether the following relations are true (i) for $x \to 0$; (ii) for $x \to \infty$ and justify your answer:

   (a)$^*$ $2x - x^2 = O(x)$;

   (b) $x \sin \sqrt{|x|} = O(|x|^{3/2})$;

(c) $x \sin(1/x) = O(x)$;

(d) $\log|x| = o(|x|^{\varepsilon})$, for any $\varepsilon > 0$;

(e) $\sqrt{x+\sqrt{x}} = O(\sqrt{x})$.

**Challenging exercises**

6. Let $n$ be an odd natural number, and let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a monic polynomial of degree $n$ (*monic* means that the coefficient in front of the highest power of $x$ equals one). Prove that $\lim\limits_{x\to\infty} P(x) = \infty$ and $\lim\limits_{x\to-\infty} P(x) = -\infty$. What changes here when $n$ is even?

7. Let $A_1, A_2 \subset \mathbb{R}$ be open sets. Prove that $A_1 \cup A_2$ and $A_1 \cap A_2$ are open. Can this be extended to the union and intersection of finitely many open sets? Infinitely many open sets?

8. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers; we denote

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup\{x_j\}_{j=n}^{\infty}, \qquad \liminf_{n\to\infty} x_n := \lim_{n\to\infty} \inf\{x_j\}_{j=n}^{\infty}.$$

In this exercise, we focus on $\limsup$. Let us assume for simplicity that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded.

(a) Prove that the limit in the definition of $\limsup$ always exists. (*Hint:* use a theorem about bounded convergent sequences).

(b) Let $a$ be a limit point of $\{x_n\}_{n=1}^{\infty}$, namely $a = \lim_{k\to\infty} x_{n_k}$. Prove that $a \leqslant \limsup\limits_{n\to\infty} x_n$ by passing to the limit in the inequality

$$x_{n_k} \leqslant \sup\{x_j\}_{j=n_k}^{\infty}.$$

(c) Prove that $\limsup_{n\to\infty} x_n$ is a limit point of $\{x_n\}_{n=1}^{\infty}$.
*Hint:* argue by contradiction.

(d) Conclude that $\limsup_{n\to\infty} x_n$ is the maximal limit point of our sequence. (Similarly, $\liminf_{n\to\infty} x_n$ is the minimal limit point of our sequence.)