CALCULUS 1 TUTORIAL EXERCISES VII

In this tutorial you will work with derivatives of implicit functions, inverse functions, parametric functions, and the mean value theorem.

Tutor's Example[1]

The curve defined as the set of points (x, y) satisfying the equation

$$y^2 = x^3 + x^2$$

is an example of an *elliptic curve*. Find an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and use the result to sketch the curve.

55. Each of the following is an equation that determines y as an implicit function of x. Find in all cases an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

    (a) $x^2 + y^2 = 1$,
    (b) $y^3 + x^3 = 1$,
    (c) $\sinh(x) + \cosh(y) = 1$,
    (d) $xy - e^{x+y} = 2$,
    (e) $\sin(y) + y = x^3$,
    (f) $y^2 + x(x-1)(x+1) = 0$.

56. Each of the following is a pair of equations, which determines x and y in terms of a parameter $t \in \mathbb{R}$, which defines a function $y(x)$ implicitly. Find in all cases an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

    (a) $x = \cos(t),\ y = \sin(t)$,
    (b) $x = \cosh(t),\ y = \sinh(t)$,
    (c) $x = t + \sin(t),\ y = \cos(t)$,
    (d) $x = t^3,\ y = t^2$,
    (e) $x = e^{2t},\ y = \tanh(t)$,
    (f) $x = e^t \cos(t),\ y = e^t \sin(t)$.

57. Consider the function $y(x)$ defined parametrically through

    $$x(t) = e^{t+1} + 1, \qquad y(t) = e^{t^2}$$

    where $t \in \mathbb{R}$ is the curve parameter.

[1] To be shown by the tutor at the start of the tutorial.

(i) Find an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t from this parametric form.

(ii) Find an explicit expression for y in terms of x that no longer involves t.

(iii) Use the result of (ii) to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x, and verify that this agrees with what you got in (i).

58. Find all values c for which the mean value theorem is satisfied for the functions:

(a) $f(x) = 2x^3 + 5x^2 - 23x + 10$ on $[1, 2]$

(b) $f(x) = \sin(x)$ on $[\pi/4, 3\pi/4]$.

(c) $f(x) = \tanh(x)$ on $[0, 1]$ (give your answer as a function of e).

59. Use both the intermediate value theorem and the mean value theorem to prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - 2x^2 + 4x - 8$ has exactly one real root.

60. **Exam-style question**[2] Consider the curve $y(x)$ defined parametrically by

$$\begin{cases} x(t) = a(\cos(t) + \ln(\tan(t/2))) \\ y(t) = a\sin(t) \end{cases}$$

for $t \in (0, \pi/2)$ and some fixed $a > 0$.

(i) [**5 marks**] Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan(t).$$

(ii) [**10 marks**] Sketch $y(x)$, carefully showing features including axis intercepts, asymptotes, maxima/minima, the slope of the function, and the behaviour as t approaches the endpoints of $0, \pi/2$.

(iii) [**8 marks**] The quantity

$$R = \frac{(x'^2 + y'^2)^{3/2}}{|x'y'' - y'x''|}$$

is called the *radius of curvature* of the curve $y(x)$, where x' and y' represent the derivatives of these functions with respect to t. Show that $R = a\cot(t)$.

[2] Some questions on the final exam will be longer with several parts which delve into a topic and build on each other. To help you prepare, each sheet will include such a question.

61. **Advanced Python**[3] Find out how to use `matplotlib` to plot parametric curves. Hence, use Python to plot the so-called *Butterfly curve* given by

$$\begin{cases} x(t) = \sin(t)\left(e^{\cos(t)} - 2\cos(4t) - \sin^5\left(\frac{t}{12}\right)\right) \\ y(t) = \cos(t)\left(e^{\cos(t)} - 2\cos(4t) - \sin^5\left(\frac{t}{12}\right)\right) \end{cases}$$

for $0 \leq t \leq 12\pi$.

62. **A challenging problem**[4] Show that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(uv) = \sum_{k=0}^{n} \binom{n}{k} \frac{\mathrm{d}^k u}{\mathrm{d}x^k} \frac{\mathrm{d}^{n-k} v}{\mathrm{d}x^{n-k}}$$

and use this formula to find expressions constraining $f^{(n)}(0)$ where $f(x) = \tan(x)$ (and $f^{(n)}(x) = \frac{\mathrm{d}^n f}{\mathrm{d}x^n}$). As far as possible, use these relations to find a closed expression for $f^{(n)}(0)$.

[3] Each sheet will contain a Python problem which should complement the programming aspect of the course. In some cases, like this one, you may have to look up unfamiliar functions and use your knowledge of Python to apply them.

[4] All challenging problems in the tutorial problem sets are beyond the scope of the course, and are not examinable, but they are, hopefully interesting and inspiring.