Probability and Statistics 1

Participation Homework 1

Submit by 16:00 on 12 February 2024

Participation homeworks function as follows:

- There are 3 participation homeworks for this module (set in weeks 4, 7 and 10).
- Each participation homework will contain questions related to the previous three weeks of lectures.
- 10% of the total marks available on this course are for participation homeworks.
- Each of the three participation homeworks is equally weighted, so accounts for $3\frac{1}{3}\%$ of the overall module mark.
- Students who make reasonable attempts at the questions in a participation homework will gain full participation marks for that homework. Students who do not will gain no participation marks for that homework.
- In other words, the participation mark for each participation homework is either a "0" or a "1" (Bernoulli style!).
- You are permitted to work in groups on the problems, but you must submit your own solutions in order to earn the participation marks.
- Participation homeworks are marked by your tutorial tutor. They will provide written feedback as well as assessing your participation mark. Participation homeworks provide the only opportunity in the course for you to receive individual, formative feedback on your work.

Problem 1. A probability model is defined as follows:

- Sample space: $\Omega = \{1, 2, \dots, n\}$
- Event space: $\mathcal{E}$ = all possible subsets of Ω
- Probability measure: for $A \in \mathcal{E}$, $\mathbb{P}(A) = \frac{|A|}{n}$, where $|A|$ stands for "the number of elements in A".

Show that the above model meets the three axioms of probability (these are set out in Definition 1.2.1 of the lecture notes).

Problem 2. Every morning I either drink tea or coffee, and I either eat weetabix or porridge. I drink tea with probability 1/3. If I drink tea, I eat weetabix with probability 4/5. If I drink coffee, I eat porridge with probability 5/7.

(a) Draw a tree diagram to represent this situation.

(b) Given that I ate porridge on Tuesday morning, find the probability that I drank tea that same morning.

Problem 3. We say that X has a discrete uniform distribution on $\{1, 2, \ldots, n\}$ if

$$\mathbb{P}(X = k) = \frac{1}{n} \text{ for } k = 1, 2, \ldots, n$$

in other words if X can take any of the values $1, 2, \ldots, n$ with equal probability (and can take no other values).

(a) Prove that $\mathbb{E}(X) = \frac{n+1}{2}$

For the rest of this question you may assume that $\mathrm{Var}(X) = \frac{n^2-1}{12}$.

(b) In a game of Dungeons & Dragons, I roll a fair 12-sided die to determine how much damage I do to an enemy. Damage is a numeric quantity here represented by D. The damage I do is determined by the formula $D = 3X + 5$, where X is the score shown on the die. Find the expectation and variance of D.

(c) The discrete random variable Y has uniform distribution over $2, 5, 8, 11, \ldots, 32, 35$. Find the expectation and variance of Y.

Extension question

The following problem does not count towards the assessment of your participation mark, but attempts will be reviewed by your tutor.

(i) If X has a Geometric distribution with parameter p, show that

$$\mathbb{P}(X > m + n | X > m) = \mathbb{P}(X > n)$$

for $m, n = 0, 1, 2, \ldots$.

We say that X has the 'memoryless property' since the above shows that if we have not seen a success for the first m trials (i.e. $X > m$), then the chance of waiting another n trials until the first success (i.e. $X > m + n$) is the same as the chance of waiting for n trials until the first success from the start ($X > n$).

(ii) Show that the geometric distribution is the only distribution over the positive integers with the memoryless property.

[Hint: Use induction to show that $\mathbb{P}(X > m + 1) = (\mathbb{P}(X > 1))^{m+1}$, then use this formula to deduce an equation for $\mathbb{P}(X = m)$]