4CCM115a: Sequences and Series

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The Greek Alphabet:

A	α	*alpha*
B	β	*beta*
Γ	γ	*gamma*
Δ	δ	*delta*
E	ε	*epsilon*
Z	ζ	*zeta*
H	η	*eta*
Θ	θ	*theta*
I	ι	*iota*
K	κ	*kappa*
Λ	λ	*lambda*
M	μ	*mu*
N	ν	*nu*
Ξ	ξ	*xi*
O	o	*omicron*
Π	π	*pi*
P	ϱ	*rho*
Σ	σ	*sigma*
T	τ	*tau*
Υ	υ	*upsilon*
Φ	ϕ	*phi*
X	χ	*chi*
Ψ	ψ	*psi*
Ω	ω	*omega*

0 Notation

0.1 Set Theoretic Symbols

$\in$	is in, is an element of
$\subset$	is a subset of, is contained in
$\cup$	union
$\cap$	intersection
$\varnothing$	the empty set

We shall also use the reverse inclusions, for instance $\supset$ stands for "contains". Negations will be written by 'striking out': $\notin$ is to be read "is not an element of". See section 1 below for the use of these symbols.

0.2 Numbers

We use the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ to denote the natural numbers, integers, rational numbers, real numbers and complex numbers respectively. For a real number x, we denote by $\lfloor x \rfloor$ the greatest integer which is less than or equal to x, and by $\lceil x \rceil$ the smallest integer which is greater than or equal to x. For example,

$$\lceil 5.25 \rceil = 6, \quad \lfloor 5.25 \rfloor = 5, \quad \lceil 8 \rceil = 8, \quad \lfloor 8 \rfloor = 8, \quad \lceil -7.3 \rceil = -7, \quad \lfloor -7.3 \rfloor = -8, \quad \lceil \pi \rceil = 4.$$

0.3 Logical Symbols

$\forall$	for all
$\exists$	there exists
$\Longrightarrow$	implies, is a sufficient condition for
$\Longleftrightarrow$	if and only if, iff

See Section 3 below for the use of these symbols.

1 Set Theory

> Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different.
>
> J. W. Goethe

This preliminary chapter sets up some notation and makes sure we have a common language in which to talk mathematically.

Every object that we encounter in mathematics belongs to a set. It is important therefore to know the language of sets and to know how sets can be manipulated.

1.1 Sets

We shall take a *set* to be 'a collection of objects'. The objects will be referred to as *elements.*

We specify a set in one of two ways.

1. By listing its elements. For instance $S = \{1, 2, 3, 4, 5\}$, or $A = \{-2, 0, 3, \pi\}$, or $X = \{\text{Alice}, \text{Bob}, \text{Carol}\}$.

2. By stating a property that determines membership, e.g.

$$\mathbb{Z}_+ = \{x \text{ integer} : x \geq 0\} \quad \text{or} \quad \mathbb{Z}_+ = \{x \text{ integer} \mid x \geq 0\}. \tag{1.1}$$

   This is to be read as "$\mathbb{Z}_+$ is the set of all integers x such that x is greater than or equal to 0". The colon ":" or the sign "|" here simply separate different parts of the definition from each other.

We use the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ to denote the natural numbers, integers, rational numbers, real numbers and complex numbers respectively:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$
$$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$$
$$\mathbb{Q} = \{a/b : a \in \mathbb{Z}, b \in \mathbb{N}\}$$

We will look at $\mathbb{R}$ more carefully soon. We can think of it intuitively as the set of all points on a straight line extending indefinitely in both directions.

In some books you will see $\mathbb{N}$ defined to include 0 too. It does not matter how you define it, provided that once you do define it you stick to your definition.

If x is an element of S we write $x \in S$. If it is not we write $x \notin S$. $x \in S$ may be written as $S \ni x$. With this notation, we can rewrite the definition (1.1) in a more standard way as

$$\mathbb{Z}_+ = \{x \in \mathbb{Z} : x \geq 0\}.$$

Consider the following example. Let U be the set defined by

$$U = \{(-1)^n : n \in \mathbb{N}\}.$$

The set U coincides with $\{-1, 1\}$, despite the fact that in the above definition, each of the numbers 1, -1 is listed infinitely many times.

We say that S is a *subset* of T and write $S \subset T$ if every element of S is an element of T. $S \subset T$ can equally well be written as $T \supset S$. The *empty set*, denoted $\varnothing$, is the set containing no elements. By convention, it is a subset of every set.

Example. Let us list all possible subsets of the set $\{1, 2, 3\}$. These are:

$$\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}.$$

Notice that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

To say that the sets A and B are equal is to say that they have the same elements. In other words, to say that $A = B$ is to say both that if $x \in A$ then also $x \in B$ and if $y \in B$ then also $y \in A$. We can write this as

$$A = B \text{ is the same as } \begin{cases} x \in A \Rightarrow x \in B \\ y \in B \Rightarrow y \in A. \end{cases}$$

Alternatively, using the notation $\subset$, this can be written as

$$A = B \text{ if and only if } (A \subset B \text{ and } B \subset A).$$

1.2 Finite and infinite sets

The difference between finite and infinite sets should be intuitively clear. For example, the sets

$$\{1, 2, 3, 4, 5\}, \quad \{-2, 0, 3, \pi\}, \quad \{\text{Alice}, \text{Bob}, \text{Carol}\},$$

are all finite, whereas the sets

$$\mathbb{N}, \quad \mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{R}$$

are all infinite. For any finite set X we can determine the number of elements of X, simply by counting its elements. The number of elements of a set X is called the *cardinality* of X and is denoted by $\#X$. For example,

$$\#\{-2, 0, 3, \pi\} = 4.$$

It should be clear that if $A \subset B$, then $(\#A) \leq (\#B)$. One can often see different notation for cardinality of A, e.g. $|A|$ or $\operatorname{card} A$.

Later on you will meet a more rigorous mathematical approach to the definition of finiteness and cardinality of sets. For now, it is sufficient to remember that one has to take extra care with infinite sets; they can behave in a way that at a first glance may seem counterintuitive. For example: if you take away an element of a finite set A, its cardinality decreases by one. If you take away an element of an infinite set, its cardinality remains the same; in a rigorous sense, the two sets

$$\mathbb{N} = \{1, 2, 3, 4, \dots\} \text{ and } \{2, 3, 4, \dots\}$$

have the same cardinality.

1.3 Operations with sets

We can form new sets from the given ones using the operations union, intersection and complement. The *union* of two sets A and B, denoted $A \cup B$, is the set of all elements which are either in A or in B (or both). Their *intersection*, $A \cap B$, is the set of those elements that are both in A and in B. Notation $A\backslash B$ stands for the *complement of B in A*, i.e. the set of all elements of A which are not in B.

Example Within the natural numbers $\mathbb{N}$ suppose $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 5\}$. Then

$$A \cup B = \{1, 2, 3, 4, 5\}\,, \quad A \cap B = \{1, 2\}\,, \quad A\backslash B = \{3, 4\}.$$

It is clear that for any sets A, B, one has $A \subset A \cup B$ and $B \subset A \cup B$ and also $A \cap B \subset A$ and $A \cap B \subset B$.

1.4 Intervals

The following subsets of $\mathbb{R}$ are called *intervals.*

(i) $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

(ii) $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.

(iii) $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$.

(iv) $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$.

(v) $(a, \infty) = \{x \in \mathbb{R} : a < x\}$.

(vi) $(-\infty, \infty) = \mathbb{R}$.

Note that the symbol ∞ is only used as part of the notation. It is not a real number!

$\boxed{\textbf{$\infty$ is not a real number! There is no such real number as ∞.}}$

(i), (iv) and (vi) are *closed* intervals. (ii), (v) and (vi) are *open* intervals. We shall not define precisely what it means to be 'open' or 'closed' here although you probably have an intuitive feel for these concepts.

Example. Which of the following statements are true?
(a) $3 \in (1,2)$, (b) $2 \in (-\infty, 3]$, (c) $1 \in [1,2]$, (d) $2 \in (0,2)$, (e) $[0,1] \subset (0,2)$.

We shall often apply the operations of union and intersection to intervals.

Example. One has:

$$\begin{aligned}
(-\infty, 1) \cap [-1, \infty) &= [-1, 1); \\
(-\infty, 1) \cup [-1, \infty) &= \mathbb{R}; \\
(-1, 0] \cup [0, 1] &= (-1, 1]; \\
(-1, 0] \cap [0, 1] &= \{0\}.
\end{aligned}$$

1.5 Multiple and infinite unions and intersections

The notions of union and intersection extend to the situation with more than just two sets. For example,

$$\begin{aligned}
A_1 \cup A_2 \cup A_3 &= \{x \mid x \in A_1 \text{ or } x \in A_2 \text{ or } x \in A_3\} \\
&= \{x \mid x \text{ belongs to at least one of the sets } A_1,\ A_2,\ A_3\} \\
&= \{x \mid x \in A_i \text{ for at least one of the indices } i = 1, 2, 3\}.
\end{aligned}$$

More generally, for n sets A_1, A_2, ..., A_n we have

$$A_1 \cup A_2 \cup \cdots \cup A_n = \{x \mid x \in A_i \text{ for at least one of the indices } i = 1, 2, \ldots, n\}. \tag{1.2}$$

There is a more convenient piece of notation for such a union:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n.$$

This notation works similarly to the $\sum$-notation for a sum of several terms. The index i here is a "dummy variable", i.e. it can be replaced by any other letter, e.g.

$$\bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{n} A_j = \bigcup_{\ell=1}^{n} A_\ell.$$

Example. Let $A_i = [0, i]$ for $i = 1, \ldots, 10$. Then

$$\bigcup_{i=1}^{10} A_i = [0, 10].$$

Let us rewrite our definition (1.2) as

$$\bigcup_{i=1}^{n} A_i = \{x \mid x \in A_i \text{ for some } i \in \{1, 2, \dots, n\}\}.$$

Let Λ denote the "index set" $\{1, 2, \dots, n\}$. This is just the set of labels for the collection of sets we are considering. Then the above can be conveniently written as

$$\bigcup_{i \in \Lambda} A_i = \{x \mid x \in A_i \text{ for some } i \in \Lambda\}.$$

This all makes sense for any non-empty index set Λ. In particular, Λ may be infinite. If $\Lambda = \mathbb{N}$, one often writes $\bigcup\limits_{i=1}^{\infty} A_i$ for $\bigcup\limits_{\lambda \in \Lambda} A_\lambda$.

Example. Let $\Lambda = \mathbb{N}$ and $A_j = [j, j+1]$ for $j \in \mathbb{N}$. Then

$$\bigcup_{j=1}^{\infty} A_j = [1, \infty).$$

A similar discussion can be made regarding intersections:

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \dots \cap A_n = \{x \mid x \in A_i \text{ for all } i \in \{1, 2, \dots, n\}\}$$

and more generally

$$\bigcap_{i \in \Lambda} A_i = \{x \mid x \in A_i \text{ for all } i \in \Lambda\}.$$

Example.

1. Consider the following infinite collection of open intervals:

$$(-1, 1), (-2, 2), (-3, 3), (-4, 4), \dots$$

We can form the union and intersection of these intervals:

$$\bigcup_{n=1}^{\infty} (-n, n) = \mathbb{R}, \qquad \bigcap_{n=1}^{\infty} (-n, n) = (-1, 1).$$

2. Consider the following infinite sequence of sets, each of which consists precisely of one point:

$$\{1\}, \{2\}, \{3\}, \dots$$

Then the union and intersection of these sets are:

$$\bigcup_{n=1}^{\infty} \{n\} = \mathbb{N}; \qquad \bigcap_{n=1}^{\infty} \{n\} = \varnothing.$$

3. Consider the following infinite collection of open intervals:

$$(-1,1),(-1/2,1/2),(-1/3,1/3),(-1/4,1/4),\ldots$$

   We can form the union and intersection of these intervals:

$$\bigcup_{n=1}^{\infty}(-1/n,1/n)=(-1,1), \qquad \bigcap_{n=1}^{\infty}(-1/n,1/n)=\{0\}.$$

4. Consider the intervals $(a-1,a+1)$ where $a\in[0,1]$. We have

$$\bigcup_{a\in[0,1]}[a-1,a+1]=[-1,2], \qquad \bigcap_{a\in[0,1]}[a-1,a+1]=[0,1].$$

We will discuss the above examples in more detail soon. You should do some exercises to get used to these concepts.

2 Functions

This section will be considered only briefly in lectures, with the foundational material for functions being considered more carefully in 4CCM111a (Calculus I).

2.1 Basics

The concept of function is very important in every branch of mathematics. Here is the formal definition:

Definition 2.1. *Let A and B be sets. Then a* function

$$f : A \to B$$

is a rule which assigns exactly one element of B to each element of A. The set A is called the domain *of the function, the set B is called the* codomain *or the* target set*.*

If x is an element of A then the element of B assigned to x by the function f is usually denoted $f(x)$. In this context, $x \in A$ is usually called the *argument* of the function f and $f(x) \in B$ is called its *value.* Depending on the situation, various synonyms to the term "function" are often used: *map, mapping, morphism.*

In order to set up a function, you have to specify three objects: (i) the set A; (ii) the set B; (iii) the rule f.

Some functions have standard names, others have names (usually a single letter) defined for a particular piece of work. You should be familiar with the following standard functions from $\mathbb{R}$ to $\mathbb{R}$: sin, cos, exp. Here are some other standard functions:

$$\begin{aligned}
\log\colon (0,\infty) &\to \mathbb{R}, & x &\mapsto \log(x);\\
\tan\colon \mathbb{R}\setminus\{\tfrac{\pi}{2}+\pi n \mid n\in\mathbb{Z}\} &\to \mathbb{R}, & x &\mapsto \tan(x);\\
\text{the ceiling function } \lceil\cdot\rceil\colon \mathbb{R} &\to \mathbb{Z}, & x &\mapsto \lceil x\rceil;\\
\text{the square root } \sqrt{\cdot}\colon [0,\infty) &\to [0,\infty), & x &\mapsto \sqrt{x};\\
\text{the square}\colon \mathbb{R} &\to \mathbb{R}, & x &\mapsto x^2;\\
\text{a linear function }: \mathbb{R} &\to \mathbb{R}, & x &\mapsto 2x+3.
\end{aligned}$$

Of course, in the last example, 2 and 3 can be replaced by any real numbers.

In setting up a function, there is sometimes a certain freedom in choosing the sets A and B. For example, the square root can be defined either as a function from $[0,\infty)$ to $[0,\infty)$ or as a function from $[0,\infty)$ to $\mathbb{R}$. We will discuss this later. For now, remember that you have to be very clear about your choices of the sets A, B; you can make any suitable choice, but you have to stick to this choice throughout your piece of work.

For functions from $\mathbb{R}$ to $\mathbb{R}$, plotting their graph is a very useful tool in analysing them. In this course we will mostly discuss functions $f : \mathbb{R} \to \mathbb{R}$, but in general, the domain and the target sets can be fairly arbitrary:

Example. Let A be the set of students in this class, let $B = \{1, 2, \ldots, 100\}$ and for a student $x \in A$ let $f(x)$ be the age of x in years. This sets up a function $f : A \to B$.

Example. Let P be the set of all polygons on the plane. We can define two functions, f and g, on the set P. For $p \in P$, let $f(p)$ be the number of edges of p and let $g(p)$ be the area of p. Thus, we have defined two functions $f : P \to \mathbb{N}$ and $g : P \to \mathbb{R}$.

2.2 Setting up a function

A function can be set up, or defined, in various ways. The most common way to define a function of real numbers is to write down an analytic expression for it. For example,

$$f(x) = 2x + 3, \qquad g(x) = \sqrt{x - 3}, \quad \text{and} \quad h(x) = \log(2x + 4)$$

define functions of a real variable x. In such simple cases, there is usually no need to specify the domain of a function explicitly: the domain is assumed to include all values of x for which the corresponding analytic expression makes sense, i.e. can be computed. For example, one immediately sees that the domain of f is $\mathbb{R}$, the domain of g is $[3, \infty)$ and the domain of h is $(-2, \infty)$.

In more complex cases, a function can be defined by a different analytic expression for different values of its argument. For example, the sign function is defined as

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

The *modulus*, or *absolute value* function is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

One can define more complex functions for a particular piece of work; for example,

$$f(x) = \begin{cases} 0 & \text{if } x \leq -1, \\ \sqrt{1 - x^2} & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 \leq x. \end{cases}$$

In mathematics, one often has to consider very complex and "weird" functions. A classic example is Thomae's function:

$$f(x) = \begin{cases} 1/n & \text{if } x \text{ is rational, } x = m/n,\ n \in \mathbb{N},\ m \text{ and } n \text{ have no common factors;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

One cannot even attempt to construct a graph of this function! However, this is a perfectly well defined function which can be mathematically analysed. In particular, f turns out to be continuous at all irrational points and discontinuous at all rational points.

Another example is the Weierstrass' function:

$$g(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \quad x \in \mathbb{R}, \tag{2.1}$$

where $0 < a < 1$, $b > 1$, $ab \geq 1$ are real parameters. This definition uses the notion of *series*, which we will only briefly mention at the end of this course. Weierstrass proved (with a later improvement by G. H. Hardy) that this function is continuous but nowhere differentiable.

2.3 Injective and surjective functions

There is some further terminology for functions which is useful.

Definition 2.2. *A function f is said to be* injective *or* one-to-one *if $f(x) = f(y) \Rightarrow x = y$.*

In words, a function is *injective* if distinct elements of the domain are always mapped to distinct elements of the codomain. (If two different elements of the domain are mapped to the same element of the codomain, then the function is not injective.)

Example. Consider the following functions:

$$\begin{array}{rccclcl}
\sin: & \mathbb{R} & \to & \mathbb{R}\,, & x & \mapsto & \sin x \\
\exp: & \mathbb{R} & \to & \mathbb{R}\,, & x & \mapsto & \exp x = e^x \\
\log: & (0,\infty) & \to & \mathbb{R}\,, & x & \mapsto & \log x \\
f: & \mathbb{R} & \to & \mathbb{R}\,, & x & \mapsto & x^2 \\
g: & (3,\infty) & \to & \mathbb{R}\,, & x & \mapsto & \frac{1}{x-3} \\
h: & \mathbb{R} & \to & \mathbb{R}\,, & x & \mapsto & x^3 - x
\end{array}$$

Which functions in the above examples are injective?

Definition 2.3. *Let $f : A \to B$ be a function. Then $f(A)$, the* image *(or* range*) of A under the function f, is the set*

$$\{y \in B \,|\, y = f(x) \textit{ for some } x \in A\} \subset B\,.$$

Example. What are the images of the functions in the above examples?

It is sometimes the case that the image of a function is equal to the entire codomain. In this case the function is said to be *surjective.*

Definition 2.4. *A function $f : A \to B$ is said to be* surjective *(or* onto*) if*

$$f(A) = B\,.$$

Example. Which functions in the above examples are surjective?

It is of course possible for a function to be both injective and surjective. This situation is sufficiently important to have a name of its own.

Definition 2.5. *A function which is both injective and surjective is said to be* bijective*.*

Example. Which functions in the above examples are bijective?

In principle, you can always change the definition of your function to make it surjective. For example, if you consider sin not as a function from $\mathbb{R}$ to $\mathbb{R}$, but as a function from $\mathbb{R}$ to $[-1, 1]$, then it becomes surjective. This is not always convenient though, because often the range of your function is unknown (or is complicated).

2.4 Composition and inverse

Definition 2.6. *Let A, B, C, be sets and $f : A \to B$ and $g : B \to C$ be functions. Then the* composition *$g \circ f$ is a function defined by*

$$(g \circ f)(x) = g(f(x)), \qquad x \in A.$$

Warning 1: in order to define the composition $g \circ f$, one has to make sure that the image of f is a subset of the domain of g. For example, consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$, $g : [0, \infty) \to \mathbb{R}$, $g(x) = \log x$. Then $g \circ f$ is NOT defined. Indeed, $g(f(x)) = \log(\sin x)$, and if $\sin x < 0$, this expression makes no sense!

Warning 2: in general $g \circ f \neq f \circ g$, even if both expressions are well defined. For example, if $f : \mathbb{R} \to \mathbb{R}$ is $f(x) = x + 1$ and $g : \mathbb{R} \to \mathbb{R}$ is $g(x) = x^2$, then $f \circ g(x) = x^2 + 1$ and $g \circ f(x) = (x + 1)^2 = x^2 + 2x + 1$.

An important feature of a bijective function is that it has a (unique) *inverse.*

Definition 2.7. *Let $f : A \to B$ be a bijective function. Then a function $g : B \to A$ is the inverse of the function f if and only if it satisfies*

$$(g \circ f)(x) = x, \quad \forall x \in A \quad \text{and} \quad (f \circ g)(y) = y, \quad \forall y \in B.$$

The function g with these properties is denoted f^{-1}.

Every bijective function has an inverse. On the other hand, if a function is not bijective then it does not have an inverse.

At a practical level, in order to find the inverse of a function f, you need to solve the equation $f(x) = y$ for x.

Example.

Find the inverse of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$.

Find the inverse of $\exp : \mathbb{R} \to (0, \infty)$.

3 Logic

We shall do a fair amount of theorem proving in this course, so it is important to be competent at logic.

3.1 Propositions

For our purposes it will be sufficient to define a *proposition* to be 'a statement which is either true or false'. The word *statement* is often used as a synonym to proposition.

Examples

A: 5 is a positive integer.

B: $3 > 5$.

C: For all real numbers x, we have $x^2 \geq 0$.

D: There exists a real number a such that $a^2 = 2$.

E: The number x is positive.

F: The absolute value of x is less than 1.

A proposition can contain a *variable*, such as in the examples E, F above. Here x is a variable, and the proposition can be true or false depending on the value of x. For example, E it is true for $x = 1$ and false for $x = -2$.

If A and B are propositions, one can form new propositions as follows:

- A AND B – is true if and only if both A and B are true;
- A OR B – is true if and only if either (A is true) or (B is true) or (both A and B are true);
- NOT A – is true if and only if A is false.

For example, for propositions E, F above we have:

- E AND F means $x \in (0, 1)$;
- E OR F means $x \in (-1, \infty)$;
- NOT E means $x \leq 0$.

3.2 Quantifiers

From propositions with variables one can form new propositions by using the quantifiers $\exists$ 'there exists' and $\forall$ 'for all'.

(i) The quantifier $\forall$ "for all". For example, from the proposition "$b^2 \geq \frac{1}{2}$" (which contains a variable b) we can form a new proposition "$\forall b \in \mathbb{N}$, one has $b^2 \geq \frac{1}{2}$" (which is true). Examples of other propositions formed in this way:

1. All students in this room are called Jane (false)
2. $\forall b \in \mathbb{R}$, one has $b^2 \geq \frac{1}{2}$ (false)
3. $\forall y \in \mathbb{R}$, one has $y^2 + 3 > 0$ (true)
4. $\forall t \in \mathbb{Z}$, one has $2t \in \mathbb{Z}$ (true)

Obviously, the notation for a variable in propositions of such type does not matter. For example, "$\forall k \in \mathbb{R}$, one has $k^2 + 3 > 0$" is the same proposition as in example 3 above.

Notation such as $\forall x, y \in \mathbb{R}$ is a shorthand for $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$; for example: $\forall x, y \in \mathbb{R}$ one has $x^2 + y^2 \geq 0$.

(ii) The quantifier $\exists$ "there exists". Examples:

1. There is a student in this room whose name is Jane (?)
2. $\exists x \in \mathbb{R}$ such that $|x| < 3$ (true)
3. $\exists n \in \mathbb{N}$ such that $n < 0$ (false)
4. $\exists n \in \mathbb{Z}$ such that $n < 0$ (true)

Remark: Instead of writing $\forall x \in (0, \infty)$, one often writes $\forall x > 0$. It is implied that all *real* positive x are considered. In the same way, one often writes $\exists x > 0$ instead of $\exists x \in (0, \infty)$. Of course, the same convention is applied to other intervals of the real line; e.g., one writes $\forall x \leq 5$ instead of $\forall x \in (-\infty, 5]$. Unfortunately, these conventions can lead to some ambiguity: for example, if it is understood that a is an integer variable, then $\forall a > 0$ means $\forall a \in \mathbb{N}$, and if a is understood to be a real number, then $\forall a > 0$ means $\forall a \in (0, \infty)$. Usually, the precise meaning is clear from the context and one should take extra care in these situations in mathematical writing to specify precisely the range of your variable.

Remember:

If a proposition contains a variable, then without a quantifier one cannot establish whether it is true or false!

For example, the proposition "N is divisible by 3" makes little sense on its own, since we don't know anything about N. On the other hand, the propositions "there exists an even integer N which is divisible by 3" (true) or "any even integer N is divisible by 3" (false) are precise mathematical statements, i.e. one can check whether they are true or false.

3.3 Negating propositions

It will be VERY IMPORTANT for us to be able to negate propositions, especially those containing variables and symbols $\exists$ and $\forall$. Negating simple propositions is straightforward, e.g. negation of $x > 0$ is $x \leq 0$; negation of $b = 2$ is $b \neq 2$. Clearly, if a proposition is true, then its negation is false, and vice versa.

When negating propositions containing operations AND, OR, one should replace AND by OR and vice versa:

1. Let $A =$ (the lectures are on Tuesdays AND Thursdays). Then (NOT A) $=$ (there are no lectures on Tuesdays OR there are no lectures on Thursdays)

2. Let $B =$ (You need to bring your passport OR driving licence). Then (NOT B) $=$ (You don't need to bring your passport AND you don't need to bring your driving licence).

When negating propositions containing symbols $\exists$ and $\forall$, one must replace $\exists$ by $\forall$ and vice versa:

1. "There exists a left-handed student in this room". Negation: "All students in this room are right-handed".

2. "$\forall x \in \mathbb{R}$, one has $x > 0$". Negation: "$\exists x \in \mathbb{R}$ such that $x \leq 0$"

3. "$\exists a \in \mathbb{R}$ such that $a^2 = 2$". Negation: "$\forall a \in \mathbb{R}$, one has $a^2 \neq 2$".

4. "In every country, there is a big city". Negation: "There is a country that has no big cities".

5. "$\forall k \in \mathbb{N}, \exists j \in \mathbb{N}$ such that $k < j$". Negation: "$\exists k \in \mathbb{N}$ such that $\forall j \in \mathbb{N}$ one has $k \geq j$".

6. "$\exists M \in \mathbb{R}$ such that $\forall k \in \mathbb{N}$ one has $k^2 \leq M$". Negation: "$\forall M \in \mathbb{R}$, $\exists k \in \mathbb{N}$ such that $k^2 > M$".

3.4 Implication

Let A and B be two propositions which contain the same variable. $A \Rightarrow B$ means 'if A is true then B is true'. In implications, you need to specify the range of the variable, so such statements will usually include the quantifier $\forall$.

Examples:

1. $\forall a \in \mathbb{R}$, one has: $a \geq 1 \Rightarrow a^2 \geq 1$;

2. $\forall x \in \mathbb{R}$: x is a solution of $x^2 = 1 \Rightarrow x$ is a solution of $x(x^2 - 1) = 0$;

3. $\forall a, b \in \mathbb{R}$, one has: $a \in (0,1)$ and $b \in (0,1) \Rightarrow (a+b)/2 \in (0,1)$.

$A \Rightarrow B$ can be equivalently rephrased in one of the following ways:

1. A implies B.
2. If A then B.
3. A is a sufficient condition for B.
4. B is a necessary condition for A.

Note that it is very often the case that $A \Rightarrow B$ but NOT $B \Rightarrow A$. Examples:

1. $k \in \mathbb{N} \Rightarrow k \in \mathbb{Z}$ but $k \in \mathbb{Z} \not\Rightarrow k \in \mathbb{N}$;
2. $\forall a \in \mathbb{R}$, one has $a \geq 1 \Rightarrow a^2 \geq 1$, but $a^2 \geq 1 \not\Rightarrow a \geq 1$.

It is a very common mistake to (explicitly or implicitly) confuse the statements $A \Rightarrow B$ and $B \Rightarrow A$

3.5 The usage of "$\Rightarrow$"

If A and B are propositions, then the expression A implies B (in symbols "$A \Rightarrow B$") in a mathematical text can mean at least two different things:

(a) strict logical meaning: "A implies B" is in itself a proposition, which may be true or false. One may refer to this proposition in the mathematical text. For example:

"it is not known whether the irrationality of $x \tan e^y$ implies the irrationality of $y \tan e^x$". In symbols:

"it is not known whether $x \tan e^y \in \mathbb{R} \setminus \mathbb{Q} \Rightarrow y \tan e^x \in \mathbb{R} \setminus \mathbb{Q}$ is true".

(b) much more often, the expression "A implies B" is used in mathematical arguments to indicate that this implication IS TRUE. For example, "we have proven that $x \in \mathbb{N}$; this implies that $x > 0$."

However, the symbol $\Rightarrow$ should be used in this sense very sparingly. It is considered a very bad style to use $\Rightarrow$ in a piece of continuous prose. For example, don't write something like "$x \in A$ and $A \subset B \Rightarrow x \in B$". Instead write "$x \in A$ and $A \subset B$, therefore $x \in B$".

3.6 Equivalence

Let A and B be two propositions which contain the same variable. Then $A \Leftrightarrow B$ means that B is true if and only if A is true. In equivalences, you need to specify the range of the variable, so such statements will usually include the quantifier $\forall$. Examples:

1. $\forall x \in \mathbb{R}$: $x > 0 \Leftrightarrow x + 1 > 1$
2. $\forall n \in \mathbb{Z}$: $n > \frac{1}{2} \Leftrightarrow n \geq 1$

3. $\forall k \in \mathbb{N}$: $k > 2 \Leftrightarrow k^2 > 4$
4. $\forall m \in \mathbb{Z}$: m is even $\Leftrightarrow m^2$ is even.

$A \Leftrightarrow B$ can be rephrased as 'A is a necessary and sufficient condition for B'.

3.7 Converse and contrapositive

Let A and B be propositions depending on a variable. For a statement $A \Rightarrow B$, the *contrapositive* is the statement (not B) $\Rightarrow$ (not A). A contrapositive is true if and only if the original statement is true. For example, the statement

"$\forall a \in \mathbb{R}$, one has $a \geq 1 \Rightarrow a^2 \geq 1$"

can be equivalently rewritten as

"$\forall a \in \mathbb{R}$, if $a^2 < 1$ then $a < 1$"

For a statement $A \Rightarrow B$, the *converse* is the statement $B \Rightarrow A$. A converse may or may not be true regardless of whether the original statement is true or not.

Example: "If you live in London, then you live in England" (true) Converse: "If you live in England, then you live in London" (false). Contrapositive: "If you don't live in England, then you don't live in London" (true)

4 Proofs

Reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess play: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.

G. H. Hardy

4.1 What is a proof?

A proof is a convincing demonstration (within the accepted standards of the field) that some mathematical statement is necessarily true. Proofs are obtained from deductive reasoning. In the deductive method one begins with a collection of statements (*axioms*) that are assumed to be true and then uses logical reasoning (*rules of inference*) to prove other statements (*theorems*). Once a theorem is proven, it can be used as the basis to prove further statements. A theorem may also be referred to as a *lemma*, especially if it is intended for use as a stepping stone in the proof of another theorem. An unproven proposition that is believed to be true is known as a *conjecture*.

By contrast, an inductive method, often used in natural sciences, is a type of reasoning which involves moving from a set of specific facts to a general conclusion. Inductive method is used in mathematics as a tool for informal discussion, but NOT for proofs. That is, a proof must demonstrate that a statement is true in all cases, without a single exception.

Proofs often follow the pattern

$$A_1 \Rightarrow A_2 \Rightarrow A_3 \Rightarrow \cdots \Rightarrow A_n,$$

where A_1 is a statement which is known to be true, A_n is the statement which needs to be proven and each of the steps $A_i \Rightarrow A_{i+1}$ is sufficiently clear and based on the rules of logical inference.

Proofs employ logic but usually include some amount of natural language which often admits some ambiguity. Ideally, mathematical proofs should be written in a style sufficiently clear so that they are accepted without dispute by anyone competent in the field. Since mathematicians are human, various minor individual deviations from this rule occur in practice.

4.2 The necessity of proofs in mathematics

Proof is a cornerstone of mathematics. It is imperative to establish a result by a general argument which applies in all possible cases. It is not enough to try out some particular cases, find that the result holds in each case tried, and conclude that the result will always hold. In a court of law all that is required is proof 'beyond reasonable doubt'. In mathematics we have to go beyond any possible doubt.

Here are two cautionary examples:

Example. If we evaluate the quadratic expression $P(n) = n^2 + n + 41$ for $n = 1, 2, 3, \ldots$ we find that $P(0) = 41$, which is prime, $P(1) = 43$, which is prime, $P(2) = 47$ which is prime, $P(3) = 9+3+41 = 53$ which is prime, etc. If we try any integer n between 0 and 39 inclusive, we will find that $P(n)$ is prime. We might easily conclude that the result would always be a prime number. However, $P(40) = 41^2$, and so the general result does not in fact hold.

Example. *Fermat numbers* are integers of the form $F_n = 2^{2^n} + 1$, where n are non-negative integers. These numbers were first studied by Pierre de Fermat (1601-1665), who conjectured that all Fermat numbers are prime. Indeed, the first five Fermat numbers $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, $F_4 = 65537$ are easily seen to be primes. However, in 1732 Leonhard Euler refuted the conjecture by computing the number F_5 and showing that it is composite.

Proving mathematical statements is part science and part art. There are some techniques but there is no method which would be sufficiently general to help you with every problem. Proofs require a great deal of insight and creativity and above all, hard work. As you get more experience with proofs, you will get better at it. Below are some examples of proofs.

4.3 Proving that two sets are equal

Let A and B be two sets. Earlier we saw that $A = B$ if and only if both $A \subset B$ and $B \subset A$. This is a common way of proving that the two sets are equal. Here are two examples:

Theorem 4.1. *Let* $A = \{n \in \mathbb{N} : n \textit{ is even}\}$, $B = \{n \in \mathbb{N} : n \textit{ is divisible by } 3\}$, $C = \{n \in \mathbb{N} : n \textit{ is divisible by } 6\}$. *Then* $C = A \cap B$.

Proof. 1. Let us show that $C \subset A \cap B$. If n is divisible by 6, then clearly n is even and n is divisible by 3. So $n \in A \cap B$, as required.

2. Let us show that $A \cap B \subset C$. Suppose that n is even and n is divisible by 3. Write $n = 3k$ with an integer k. Since n is even, k must also be even. Thus, $k = 2\ell$ with some integer ℓ. It follows that $n = 3k = 6\ell$, and so $n \in C$, as required. ∎

Theorem 4.2. *For any sets* A, B, *and* C, *we have*

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. (i) We must show that lhs $\subset$ rhs and rhs $\subset$ lhs (lhs = left hand side, rhs = right hand side).

1. Let us show that lhs $\subset$ rhs.

If lhs $= \varnothing$, we are done, because $\varnothing$ is a subset of any set. So now suppose that lhs $\neq \varnothing$ and let $x \in$ lhs $= A \cup (B \cap C)$. Then $x \in A$ or $x \in (B \cap C)$ (or both).

Case 1: Suppose $x \in A$. Then $x \in A \cup B$ and also $x \in A \cup C$ and therefore $x \in (A \cup B) \cap (A \cup C)$, that is, $x \in$ rhs.

Case 2: Suppose that $x \in (B \cap C)$. Then $x \in B$ and $x \in C$ and so $x \in A \cup B$ and also $x \in A \cup C$. Therefore $x \in (A \cup B) \cap (A \cup C)$, that is, $x \in$ rhs.

So in either case (1) or (2) (and at least one of these must be true), we find that $x \in$ rhs. Since $x \in$ lhs is arbitrary, we deduce that every element of the lhs is also an element of the rhs, that is lhs $\subset$ rhs.

2. Let us show that rhs $\subset$ lhs. If rhs $= \varnothing$, then there is no more to prove. So suppose that rhs $\neq \varnothing$, and let $x \in$ rhs. Then $x \in A \cup B$ and $x \in A \cup C$.

Case 1: suppose $x \in A$. Then $x \in A \cup (B \cap C)$ and so $x \in$ lhs.

Case 2: suppose $x \notin A$. Then since $x \in A \cup B$, we must have $x \in B$. Also $x \in A \cup C$ and so it follows that $x \in C$. Hence $x \in B \cap C$ and so $x \in A \cup (B \cap C)$ which means that $x \in$ lhs.

Thus, rhs $\subset$ lhs. Combining these two parts, we have lhs = rhs as required.

(ii) is left as an exercise. ∎

Note that the symbol ∎ above denotes the end of proof. This symbol is used in mathematical texts to render the proofs more visible in the text.

4.4 Proof by induction

This is a common method of proof; it is discussed in the Calculus course so we will not duplicate this discussion here.

4.5 Proof by contradiction

We will often prove mathematical statements by using the following trick. If we have to prove that A is true, we will first suppose that A is false and then, using a chain of logical implications, come to a conclusion which is definitely false. In this way we show that our initial assumption (A is false) was wrong, so A must be true. Proof by contradiction is essentially replacing a statement $A \Rightarrow B$ by its contrapositive NOT B $\Rightarrow$ NOT A. Proof by contradiction is used particularly often with uniqueness statements; see e.g. the proof of uniqueness of maximum in the next section.

4.6 Proof by constructing a counterexample

Let us discuss an example.

Proposition A: *For all natural numbers n, the expression $2n^2 - 20n + 49$ is positive.*

You need to find out whether Proposition A is true or false. First you try to prove it and fail. You suspect that the proposition might be false. You try substituting particular values of n and after trying $n = 1, 2, 3, 4$, (which give positive values to our expression) you discover that for $n = 5$ the expression is negative: $2 \times 5^2 - 20 \times 5 + 49 = -1$. Thus, the proposition is false. One says that the case $n = 5$ is a *counterexample* to Proposition A. We have *proven that Proposition A is false by constructing a counterexample.*

4.7 Further examples of proofs

Here we will prove several simple statements concerning unions and intersections of sets. This serves two purposes: firstly, it provides some examples of proofs and secondly, it introduces some important ideas which we will see again soon when we discuss convergence of sequences.

Theorem 4.3. *One has*

$$\bigcup_{n=1}^{\infty}(-n, n) = \mathbb{R}.$$

Proof. We need to prove that lhs $\subset$ rhs and rhs $\subset$ lhs. Since $(-n, n) \subset \mathbb{R}$ for all $n \in \mathbb{N}$, the first statement is obvious.

Let us prove the second statement: rhs $\subset$ lhs. Take any $x \in \mathbb{R}$; we need to prove that $x \in$ lhs. By the definition of union, for any $x \in \mathbb{R}$ we need to prove that there exists a positive integer n such that $x \in (-n, n)$. The easiest way to do it is to *explicitly specify* this integer n in terms of x. Consider two cases:

Case 1: $x \geq 0$. Take $n = \lceil x \rceil + 1$. Then $0 \leq x < n$ and so $x \in (-n, n)$. Thus, $x \in$ lhs.

Case 2: $x < 0$. Take $n = \lceil -x \rceil + 1$. Then $n > -x$, so $-n < x < 0$ and so $x \in (-n, n)$. Thus, $x \in$ lhs. ∎

Theorem 4.4. *One has*

$$\bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}.$$

Proof. As usual, we need to prove that lhs $\subset$ rhs and rhs $\subset$ lhs. First note that $0 \in (-1/n, 1/n)$ for all $n \in \mathbb{N}$, and therefore $0 \in$ lhs. Thus, rhs $\subset$ lhs.

Let us prove that lhs $\subset$ rhs. We need to prove that if $x \in$ lhs then $x = 0$. We will prove this by contradiction.

Suppose $x \in$ lhs and $x \neq 0$. Consider two cases:

Case 1: $x > 0$. Consider the number $1/x > 0$ and let us take $n = \lceil 1/x \rceil + 1$. Then n is a positive integer and $n > 1/x$, and so $x > 1/n$. It follows that $x \notin (-1/n, 1/n)$ (for this choice of n) and therefore $x \notin$ lhs.

Case 2: $x < 0$. Consider the number $-1/x > 0$ and let us take $n = \lceil -1/x \rceil + 1$. Then n is a positive integer and $n > -1/x$, and so $x < -1/n$. It follows that $x \notin (-1/n, 1/n)$ (for this choice of n) and therefore $x \notin$ lhs.

In either of the two cases, we have obtained a contradiction. Thus, our initial assumption $x \neq 0$ cannot be true. ■

4.8 Common mistakes

Here we refer to the excellent website "The most common errors in undergraduate mathematics":

`http://www.math.vanderbilt.edu/~schectex/commerrs/`

See the section ERRORS IN REASONING on this website.

Here we would like to particularly emphasise the common mistake of "working backwards". Below is an example of "proof" of the statement "$\forall n \in \mathbb{Z}$ one has $n \geq 1$" from this website:

- Let $n \in \mathbb{Z}$. Assume $n \geq 1$.
- Rearrange to $n - 1 \geq 0$.
- Since the l.h.s. is positive, we can multiply both sides by $n - 1$. We get $(n - 1)^2 \geq 0$.
- Since a square of any number is always non-negative, the inequality $(n - 1)^2 \geq 0$ is true. Thus, the assumption $n \geq 1$ must also be true.

What is wrong with this proof?

Let A be the statement "$\forall n \in \mathbb{N}$ one has $n \geq 1$" and let B be the statement "$\forall n \in \mathbb{N}$ one has $(n - 1)^2 \geq 0$". We have proven that $A \Rightarrow B$ is true and that B is true. Does it follow that A is true?

4.9 Some general advice

> If we knew what it was we were doing, it would not be called research, would it?
>
> — Albert Einstein

Although in some cases you will immediately be able to see how to construct a proof, generally when presented with a theorem and asked to provide a proof the first question is 'where do I start?' There is no fixed routine, recipe or algorithm. But there are strategies which it is advisable to adopt, which are summarised in the following list.

1. Make sure you have enough time and plenty of rough paper.
2. Write down the statement of the theorem you are required to prove.
3. Make sure you really understand the definitions of all the mathematical objects which appear in this statement. If you don't, go back to these definitions and revise.

4. Test the theorem by trying some special cases.
5. Try to find an example which contradicts the theorem. (This should be impossible, but trying to do this is often an effective way of seeing why a result has to be true.)
6. Write down mathematical facts which might be related to the theorem you are trying to prove.
7. Rewrite the statement of the theorem in several different ways.
8. Now try to prove the theorem and be persistent!

As well as these practical steps, remember that you need to really concentrate on the issue, get under its skin so to speak; you won't make any progress if half your mind is on the television or your supper. And believe in your ability to prove the result. Don't be feeble and give up.

Once you think you have the beginnings of an idea, start trying to set it down. You may have a few false starts. When you think you have finished, read through your proof critically, make sure you really have set down a logical chain of reasons from the assumptions at the beginning to the conclusion at the end. Try to simplify your proof or construct a different, simpler argument.

Do not become discouraged if constructing proofs seems difficult. It is difficult for nearly everyone. Don't be deceived by the form in which mathematicians generally display their finished products. Behind a polished proof there has often been a great deal of struggle and frustration and ruined paper. There is no other way to discover good mathematics, and for most of us there is no other way to learn it.

5 Real Numbers: Introduction

God created the integers; all
else is the work of man

Leopold Kronecker

There are two possible approaches to the rigorous study of real numbers. In some mathematical courses, one starts with the natural numbers $\mathbb{N}$, then *constructs* integers $\mathbb{Z}$ (by adding the negatives of natural numbers), then *constructs* rational numbers (as ratios of integers) and finally, by some limiting procedure, one *constructs* irrational numbers. This gives the whole set of real numbers by means of some *constructive procedure*. Another approach is to set the *axioms* of real numbers and then to derive the usual properties from these axioms. See, for example, Ivan Wilde's lecture notes in Analysis quoted on the course website.

In this introductory course, we will not follow consistently either of these rigorous approaches. Instead, we will briefly discuss basic properties of real numbers and then proceed to use them relying on our intuition and experience.

5.1 Arithmetic and order

We will use the usual rules of arithmetics to add, subtract, multiply and divide real numbers. We shall not question or analyse these rules but simply use them in an intuitively clear way. One thing worth reminding is that dividing by zero is not a valid operation:

$$\boxed{\frac{x}{0}, \frac{1}{0}, \frac{0}{0} \textbf{ are NOT DEFINED! It is NOT TRUE that } \frac{1}{0} = \infty\textbf{!}}$$

There is a relation $<$ ("less than") between elements of $\mathbb{R}$ such that for any $a, b \in \mathbb{R}$, exactly one of the following is true:

$$a < b, \quad b < a, \text{ or } a = b.$$

Of course, the notation $a > b$ means $b < a$. The relation $<$ satisfies the following properties (axioms of order):

- If $a < b$ and $b < c$ then $a < c$.
- If $a < b$, then $a + c < b + c$ for any $c \in \mathbb{R}$.
- If $a < b$ and $c > 0$, then $ac < bc$.

All the usual properties of inequalities can be derived from these ones. For convenience, we list some of these properties on the next page (which can be printed out separately and used as reference material).

Inequalities: some rules of manipulation

1. You can multiply an inequality by any positive number α:

$$x < y \Leftrightarrow \alpha x < \alpha y \qquad (\text{if } \alpha > 0),$$
$$x \leq y \Leftrightarrow \alpha x \leq \alpha y \qquad (\text{if } \alpha > 0).$$

2. You can multiply an inequality by any negative number α *and reverse the sign of the inequality* (i.e. replace $<$ by $>$ and vice versa):

$$x < y \Leftrightarrow \alpha x > \alpha y \qquad (\text{if } \alpha < 0),$$
$$x \leq y \Leftrightarrow \alpha x \geq \alpha y \qquad (\text{if } \alpha < 0).$$

Common mistake: forgetting to reverse the sign.
For example, one writes $-2x > 6 \Rightarrow x > -3$ (wrong!)

3. If x, y are both non-negative numbers, you can square the inequality $x < y$:

$$x < y \Leftrightarrow x^2 < y^2 \qquad (\text{if } x \geq 0,\ y \geq 0),$$
$$x \leq y \Leftrightarrow x^2 \leq y^2 \qquad (\text{if } x \geq 0,\ y \geq 0).$$

Common mistake: forgetting the requirements $x \geq 0$ and $y \geq 0$.
For example, one writes $-2 < x \Rightarrow 4 < x^2$ (wrong!)

4. If x, y are both non-negative numbers, you can take a square root of both sides of the inequality $x < y$:

$$x < y \Leftrightarrow \sqrt{x} < \sqrt{y} \qquad (\text{if } x \geq 0 \text{ and } y \geq 0),$$
$$x \leq y \Leftrightarrow \sqrt{x} \leq \sqrt{y} \qquad (\text{if } x \geq 0 \text{ and } y \geq 0).$$

5. More generally, if x, y are both positive numbers, you can raise the inequality $x < y$ to any positive power $p > 0$:

$$x < y \Leftrightarrow x^p < y^p \qquad (\text{if } x \geq 0,\ y \geq 0, \text{ and } p > 0),$$
$$x \leq y \Leftrightarrow x^p \leq y^p \qquad (\text{if } x \geq 0,\ y \geq 0, \text{ and } p > 0).$$

6. If x, y are both positive numbers, you can replace x and y in the inequality $x < y$ by their inverses $1/x$, $1/y$ *and reverse the sign of the inequality*:

$$x < y \Leftrightarrow \frac{1}{x} > \frac{1}{y} \qquad (\text{if } x > 0 \text{ and } y > 0),$$
$$x \leq y \Leftrightarrow \frac{1}{x} \geq \frac{1}{y} \qquad (\text{if } x > 0 \text{ and } y > 0).$$

7. If x, y are both positive numbers, you can take natural logarithm of both sides of the inequality:

$$x < y \Leftrightarrow \ln x < \ln y \qquad (\text{if } x > 0 \text{ and } y > 0),$$
$$x \leq y \Leftrightarrow \ln x \leq \ln y \qquad (\text{if } x > 0 \text{ and } y > 0).$$

5.2 Equations and inequalities

To *solve* an equation (resp. inequality) involving a variable means to list all possible values of this variable such that the equation (resp. inequality) becomes true.

Example.

1. $x^2 - x = 0$. Solution: $x \in \{0, 1\}$.
2. $x^2 < 4$. Solution: $x \in (-2, 2)$.

Often there are infinitely many solutions, so "listing" the values should be understood as "describing in an explicit way". Usually this is done by using the set theoretic notation discussed above.

Example.

1. $x^2 > 2$. Solution: $x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$.
2. $(x-1)^2 \leq 9$. Solution: $x \in [-2, 4]$.

Sometimes one is asked to find only the solutions (to a given inequality) in a given range, for example "Solve the inequality $(x-1)^2 \leq 9$ for $x > 0$". Solution: $x \in (0, 4]$.

5.3 Equations with parameters

Sometimes the equation or inequality that you need to solve contains an unknown parameter. Then this parameter will most likely appear in your answer.

Example. Let $a \in \mathbb{R}$. Solve the equation $(x-a)^2 = 4$ for $x \in \mathbb{R}$. Answer: $x \in \{a+2, a-2\}$.

Sometimes you have to consider different cases, depending on the value of your parameter.

Example. Let $a \in \mathbb{R}$. Solve the inequality $ax - a^2 \geq 0$ for $x \in \mathbb{R}$. Solution: Consider three cases:

Case 1: $a > 0$. Then $ax - a^2 \geq 0 \Leftrightarrow x - a \geq 0 \Leftrightarrow x \geq a \Leftrightarrow x \in [a, \infty)$.

Case 2: $a < 0$. Then $ax - a^2 \geq 0 \Leftrightarrow x - a \leq 0 \Leftrightarrow x \leq a \Leftrightarrow x \in (-\infty, a]$.

Case 3: $a = 0$. Then $ax - a^2 = 0 \geq 0$ for all $x \in \mathbb{R}$, so all $x \in \mathbb{R}$ are solutions to our inequality.

Answer: If $a > 0$, the solution is $x \in [a, \infty)$. If $a < 0$, the solution is $x \in (-\infty, a]$. If $a = 0$, the solution is $x \in \mathbb{R}$.

5.4 Roots

In this course we deal only with real numbers, and therefore we consider the expression $\sqrt{x}$ only for $x \geq 0$. If you've seen complex numbers, you may remember that i is sometimes written as $\sqrt{-1}$. However, from our point of view here the number i is "outside of our world", so we declare $\sqrt{-1}$ (and generally square roots of negative numbers) to be ill-defined.

Definition 5.1. *For $x \geq 0$, the square root of x, $a = \sqrt{x}$, is the (unique)* NON-NEGATIVE *solution to the equation $a^2 = x$.*

Existence and uniqueness of the solution to the above equation is not difficult to prove, but we will not discuss this in this course.

There is often some confusion about the definition of the square root. You may remember that the set of solutions to $x^2 = 4$, for example, is $x = 2$ and $x = -2$. But this does NOT mean that $\sqrt{4}$ has two values, 2 and -2. (Within the realm of complex numbers, the conventions are different, but this is another story!) So, in our course:

$$\boxed{\sqrt{x} \textbf{ is always NON-NEGATIVE! } \sqrt{4} \neq -2}$$

However, when you solve the equation $x^2 = 4$, you have indeed two solutions: $x = \sqrt{4} = 2$ and $x = -\sqrt{4} = -2$.

Similarly, if n is a positive integer and $x \geq 0$, then $a = \sqrt[n]{x} = x^{1/n}$ is defined as the unique positive solution to the equation $a^n = x$. The existence and uniqueness of the solutions to the above equation is not difficult to prove, but it is beyond the scope of this course.

5.5 Modulus

We will often work with modulus, or absolute value. It is very important that you know the definition and rules of manipulation with modulus.

Definition 5.2. *For $x \in \mathbb{R}$, the* modulus of x, *$|x|$, is defined as*

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

By definition, modulus is always non-negative: $|2| = 2 > 0$, $|-5| = -(-5) = 5 > 0$, $|0| = 0$.

Here are some properties of the modulus function:

1. $\forall x, y \in \mathbb{R}$: $|xy| = |x||y|$; in particular, $|ax| = a|x|$ if $a > 0$;
2. $\forall x, y \in \mathbb{R}$: $|x + y| \leq |x| + |y|$ (the triangle inequality);
3. $\forall x, y \in \mathbb{R}$: $|x - y| \leq |x| + |y|$ (another form of the triangle inequality).

These properties can be easily proven from the definition of modulus. Let us, for example, present the proof of the first of these properties in a theorem-proof style.

Theorem 5.3. *Let* $x, y \in \mathbb{R}$. *Then* $|xy| = |x||y|$.

Proof. We give a proof by exhaustion, i.e., prove the result separately for an exhaustive list of possible cases for x. The fact that the definition of modulus is given in terms of cases gives us this idea.

Case 1: suppose that $x = 0$. Then $|x| = 0$ and $|xy| = |0| = 0$ and so the statement to be proven becomes $0 \cdot |y| = 0$ which is obviously true.

Case 2: the case $y = 0$ is considered in the same way.

Case 3: let $x > 0$, $y > 0$. Then also $xy > 0$. We have $|x| = x$, $|y| = y$, $|xy| = xy$, so the statement to be proven becomes $xy = xy$, which is obviously true.

Case 4: let $x > 0$, $y < 0$. Then $xy < 0$. We have $|x| = x$, $|y| = -y$, $|xy| = -xy$, so the statement to be proven becomes $x \cdot (-y) = -xy$, which is obviously true.

Case 5: let $x < 0$, $y > 0$. This case is considered in the same way as Case 4.

Case 6: let $x < 0$, $y < 0$. Then $xy > 0$. We have $|x| = -x$, $|y| = -y$, $|xy| = xy$, so the statement to be proven becomes $(-x) \cdot (-y) = xy$, which is obviously true. ∎

At a practical level, we will need to solve simple equations and inequalities involving modulus. You should remember that:

1. if $a > 0$, then the equation $|x| = a$ has solutions $x = a$ and $x = -a$;
2. if $a > 0$, then the inequality $|x| < a$ has the solution $x \in (-a, a)$;
3. if $a > 0$, then the inequality $|x| > a$ has the solution $x \in (-\infty, -a) \cup (a, \infty)$.

What is the set of solutions of $|x| \leq a$? Of $|x| \geq a$?

Examples:

1. Solve $|x - 7| = 2$. Answer: $x \in \{5, 9\}$.
2. Solve $|x| \leq 3$. Answer: $x \in [-3, 3]$.
3. Solve $|x - 2| > 5$. Answer: $x \in (-\infty, -3) \cup (7, \infty)$.
4. Solve $|-3x| < 6$. Answer: $x \in (-2, 2)$.
5. Solve $|2x^2| \leq 8$. Answer: $x \in [-2, 2]$.

5.6 Irrational Numbers

Are all numbers rational? The following theorem was discovered in the Pythagorean school in the IVth or Vth century BC.

Theorem 5.4. *There is no rational number x that satisfies the equation $x^2 = 2$.*

Proof. We shall give the classic proof by contradiction. Suppose $(a/b)^2 = 2$ where a and b are integers which we may assume have no common factors. We have $a^2 = 2b^2$ and so a^2, and hence a itself, is divisible by 2. We may therefore write $a = 2c$ for some integer c. Squaring gives $a^2 = 4c^2$. But $a^2 = 2b^2$ and so $b^2 = 2c^2$. We now have b^2, and hence b itself, is divisible by 2. We have shown that both a and b are divisible by 2. This contradicts to them having no common factor. Hence our assumption that we can write $(a/b)^2 = 2$ must be false. ∎

There are many more irrational numbers! Indeed *most* real numbers are irrational, in a certain precise mathematical sense. We will not discuss this.

6 Boundedness; supremum and infimum

Up to now, we have been discussing some basic preliminary material which is common to all branches of mathematics. From now on, in this course we will be studying topics which provide the foundations of *Analysis*.

6.1 Boundedness

Definition 6.1 (bounded above/below, upper/lower bound)**.** *A subset S of $\mathbb{R}$ is said to be* bounded above *if there is a real number M such that $x \leq M$ for all $x \in S$. Alternatively: S is bounded above if $S \subset (-\infty, M]$ for some $M \in \mathbb{R}$. Such a number M is called an* upper bound *for S.*

A subset S of $\mathbb{R}$ is said to be bounded below *if there is a real number m such that $x \geq m$ for all $x \in S$. Alternatively: S is bounded below if $S \subset [m, \infty)$ for some $m \in \mathbb{R}$. Such a number m is called a* lower bound *for S.*

A subset S of $\mathbb{R}$ is said to be bounded *if it is both bounded above and bounded below. Alternatively: S is bounded if $S \subset [m, M]$ for some real numbers m, M.*

If S is not bounded, it is called unbounded.

As an exercise you should prove that a subset S of $\mathbb{R}$ is bounded if and only if there exists a real number H such that $|x| \leq H$ for all $x \in S$.

Example.

1. The set $\mathbb{N}$ is bounded from below but not bounded from above. The numbers $-1/2, 0, 1$ are lower bounds for $\mathbb{N}$.

2. The set $S = \{-n^2 \mid n \in \mathbb{N}\}$ is bounded from above but not bounded from below. Any number in the interval $[-1, \infty)$ can be taken for an upper bound of S.

3. The set $\mathbb{Z}$ is unbounded from below and from above.

4. The set $S = \{1, 1/2, 1/3, 1/4, \dots\}$ is bounded. All numbers from $[1, \infty)$ are upper bounds for S and all numbers from $(-\infty, 0]$ are lower bounds for S.

We shall often meet sets which are made up of the elements of a sequence of real numbers. In the last example, the sequence is $s_n = 1/n$, $n = 1, 2, 3, \dots$, and the set $S = \{s_n \mid n \in \mathbb{N}\}$.

Proving that a set is bounded above (or below) is often not difficult: you need to exhibit an upper bound for this set. Proving that a set is unbounded above (or below) is slightly more difficult. Let us consider this question for boundedness above. Let us write down the definition of boundedness above of a set S by using the quantifiers:

$$\exists M \in \mathbb{R} \text{ such that } \forall x \in S \text{ one has } x \leq M.$$

Using our rules of negating the propositions, we obtain the following definition of S being unbounded above:

$$\forall M \in \mathbb{R} \quad \exists x \in S \text{ such that } x > M.$$

Let us prove that $\mathbb{N}$ is unbounded above. Given $M \in \mathbb{R}$, we need to find a natural number $n \in \mathbb{N}$ such that $n > M$. Consider two cases:

Case 1: $M \leq 0$. Then we can take, for example, $n = 1$.

Case 2: $M > 0$. Let us take $n = \lceil M \rceil + 1$. Then, clearly, $n \in \mathbb{N}$ and $n > M$.

So in either case, we have found $n > M$ as required.

Example. Let us prove that the set $S = \{2^n \mid n \in \mathbb{Z}\}$ is unbounded above. As above, we need to prove that for any $M \in \mathbb{R}$ there exists $x \in S$ such that $x > M$. That is, given any $M \in \mathbb{R}$ we need to find $n \in \mathbb{Z}$ such that $2^n > M$. Consider two cases:

Case 1: $M \leq 0$. Then we can take, for example, $n = 0$: $M \leq 2^0 = 1$.

Case 2: $M > 0$. Take $n = \lceil \log_2(M) \rceil + 1$. Then, clearly, $n \in \mathbb{Z}$ and $2^n > M$, as required.

In the exercises, you are asked to apply the same reasoning to the proof of unboundedness of more complicated sets.

6.2 Maximum and minimum

Definition 6.2. *Let S be bounded above and suppose that there exists an upper bound M of S such that $M \in S$. Then M is called the* maximum *of S (or the* maximal element *of S): $M = \max S$.*

Let S be bounded below and suppose that there exists a lower bound m of S such that $m \in S$. Then m is called the minimum *of S (or the* minimal element *of S): $m = \min S$.*

Example.

1. For the set $S = [1, \infty)$, one has $\min S = 1$.
2. For the set $\mathbb{N}$, one has $\min \mathbb{N} = 1$.
3. For the set $[1, 2]$, one has $\max S = 2$, $\min S = 1$.
4. For the set $S = \{1/n \mid n \in \mathbb{N}\}$, one has $\max S = 1$. Does $\min S$ exist?

Let us prove that in the last example, $\min S$ does not exist. We use proof by contradiction. Assume that $\min S$ exists; denote it by m. Since $m \in S$, it must be of the form $m = 1/k$ for some $k \in \mathbb{N}$. Consider the number $1/(k+1)$. Clearly, $1/(k+1) \in S$ and

$$\frac{1}{k+1} < \frac{1}{k} = m. \tag{6.1}$$

On the other hand, by the definition of minimum, we must have $m \leq x$ for all $x \in S$; this contradicts (6.1). Thus, $\min S$ does not exist. ∎

Example. The set $S = (0, 1)$ is bounded. However, neither maximum nor minimum exist for this set.

Let us prove that $\max S$ does not exist. Suppose to the contrary that $M = \max S$ exists. Consider the number $M_1 = (M + 1)/2$. Since $M \in S$, we have

$$0 < M < 1 \quad \Rightarrow \quad 0 < M + 1 < 2 \quad \Rightarrow \quad 0 < M_1 < 1 \quad \Rightarrow \quad M_1 \in S.$$

Also

$$M < 1 \quad \Rightarrow \quad 2M < M + 1 \quad \Rightarrow \quad M < M_1.$$

Thus, we have found an element $M_1 \in S$ such that $M < M_1$, so M cannot be a maximum of S – contradiction! ∎

The proof that $\min S$ in the above example does not exist is left as an exercise.

Thus, we see that maximum and minimum may not exist even for a bounded set. However, one has

Theorem 6.3.

(i) Let S be bounded above. If $\max S$ exists, then it is unique.

(ii) Let S be bounded below. If $\min S$ exists, then it is unique.

Proof. We shall prove only part (i). Suppose that $M_1 \neq M_2$ are two different maxima of S. As M_1 is an upper bound for S and M_2 is an element of S, we get $M_1 \geq M_2$. Interchanging the roles of M_1 and M_2, we get $M_2 \geq M_1$. Thus, $M_1 = M_2$ — contradiction! ∎

6.3 Supremum and infimum

As we have just seen, maximum and minimum may not exist even for bounded sets. Below we discuss a proper substitute for maximum and minimum. Consider the set $(0, 1)$. Clearly, 1 is a candidate for a substitute for maximum. Similarly, 0 could be taken as a substitute for minimum.

Definition 6.4. *Let S be bounded below. Suppose that there exists the largest number m such that $S \subset [m, \infty)$. Then m is called the greatest lower bound for S, or the infimum of S, denoted* $\inf S$.

In our example, $S = (0, 1)$, we have $S \subset [0, \infty)$. In fact, 0 is the largest number m such that $S \subset [m, \infty)$. Let us prove this by contradiction. Suppose there exists $m > 0$ such that $S \subset [m, \infty)$. Clearly, we must have $m < 1$. Consider the number $m/2$; then $m/2 < m$ and $m/2 \in S$; this contradicts to $S \subset [m, \infty)$.

The above definition can be rephrased as follows: $\inf S$ is the maximum of the set of all lower bounds of S. Since maximum of any set is unique, the infimum of any set is uniquely defined.

The greatest lower bound is characterized by two properties:
(i) it is a lower bound;
(ii) it is greater than any other lower bound.

Example.

1. $\inf[1,2] = \min[1,2] = 1$.
2. For any $a < b$, one has $\inf(a,b) = \inf[a,b) = \inf(a,b] = \inf[a,b] = a$.
3. For $S = \{1/n \mid n \in \mathbb{N}\}$, one has $\inf S = 0$, and $\min S$ does not exist.

You should think of infimum as of the 'grown-ups' version of minimum.

Theorem 6.5. *Let S be bounded from below. If $\min S$ exists, then $\inf S$ also exists and coincides with $\min S$.*

Proof. Let $m = \min S$. Let us prove that $m = \inf S$. We need to prove that (i) m is a lower bound for S; (ii) if m_1 is another lower bound for S, then $m_1 < m$. The statement (i) follows from the definition of minumum. In order to prove (ii), argue by contradiction: suppose that there is a lower bound $m_1 > m$; this is impossible, as m is an element of S and so m should be greater than or equal to any lower bound. ∎

Thus, infimum is a satisfactory substitute for minimum. Of course, there is a similar substitute for maximum:

Definition 6.6. *Let S be bounded above. Suppose that there exists the smallest number M such that $S \subset (-\infty, M]$. Then M is called the least upper bound for S, or the supremum of S, denoted $\sup S$.*

Again, the least upper bound is characterized by two properties:
(i) it is an upper bound;
(ii) it is less than any other upper bound.
As an exercise, you should prove that if a maximum of a set exists, then its supremum also exists and coincides with the maximum.

Example.

1. For any $a < b$, one has $\sup(a,b) = \sup[a,b) = \sup(a,b] = \sup[a,b] = b$.
2. For $S = \{1 - 2^{-n} \mid n \in \mathbb{N}\}$, one has $\inf S = \min S = 1/2$, $\sup S = 1$, but $\max S$ does not exist.

The following property of $\inf S$ and $\sup S$ is simple to prove yet very important:

Theorem 6.7.

(i) Let $S \subset \mathbb{R}$ be a set bounded above and let $M = \sup S$. Then for any $\varepsilon > 0$ there exists $x \in S$ such that $M - \varepsilon < x$.

(ii) Let $S \subset \mathbb{R}$ be a set bounded below and let $m = \inf S$. Then for any $\varepsilon > 0$ there exists $x \in S$ such that $x < m + \varepsilon$.

Proof. (i) We have $M - \varepsilon < M$. Since M is the LEAST upper bound of S, the number $M - \varepsilon$ is NOT an upper bound of S. This means that there exists $x \in S$ such that $x \leq M - \varepsilon$ is FALSE. That is, there exists $x \in S$ such that $x > M - \varepsilon$, as required.

(ii) is left as an exercise. ∎

Example. Consider the set $S = (a, b)$; here $b = \sup S$. Let $\varepsilon > 0$ be given; let us find $x \in S$ such that $b - \varepsilon < x$. Consider two cases:

Case 1: $\varepsilon \geq b - a$. Then $b - \varepsilon \leq a$ and therefore we can take for x any element of the interval (a, b). For example, the midpoint $x = (b - a)/2$ will do.

Case 2: $\varepsilon < b - a$. Then $b - \varepsilon \in (a, b)$. We can take for x the midpoint of $(b - \varepsilon, b)$, i.e. $x = b - (\varepsilon/2)$. Clearly, $x \in (a, b)$ and $b - \varepsilon < x$.

Example. Let $S = \{1/n \mid n \in \mathbb{N}\}$; then $\inf S = 0$. Let $\varepsilon > 0$ be given; let us find $x \in S$ such that $x < \varepsilon$. That is, we need to find $n \in \mathbb{N}$ such that $1/n < \varepsilon$. We can equivalently rewrite this as $n > 1/\varepsilon$. Now it is clear that if we take $n = \lceil 1/\varepsilon \rceil + 1$, then we get $n > 1/\varepsilon$, and therefore $1/n < \varepsilon$, as required.

6.4 Completeness

In our examples, we have always been able to find supremum and infimum of a bounded set (although maximum and minimum may not exist). This is a consequence of a fundamental property of real numbers: completeness. This property cannot be deduced from any other axioms of real numbers and must be accepted as an independent axiom.

Definition 6.8 (Axiom of completeness)**.** *Every nonempty set of real numbers which is bounded above has a supremum. Every nonempty set of real numbers which is bounded from below has an infimum.*

It is this property that distinguishes real numbers from rational numbers. More precisely, rational numbers satisfy all the usual properties of arithmetics and order, but they do not satisfy the completeness axiom. Indeed, the set $\{x \in \mathbb{Q} \mid x^2 < 2\}$ does not have a supremum or infimum in the set of rational numbers; the supremum and infimum are $\sqrt{2}$ and $-\sqrt{2}$, which are irrational. You can think of $\mathbb{Q}$ as of a line with some elements (irrational numbers) missing, whereas $\mathbb{R}$ has no "missing elements".

7 Sequences: Convergence

Everything should be made as simple as possible, but no simpler.

Albert Einstein

The notion of convergence is conceptually the most important in the whole Numbers and Functions course. Once you *really* understand what convergence means for sequences the rest is easy!

7.1 Sequences

We shall deal with infinite sequences of real numbers: $s_1,\ s_2,\ s_3,\ \ldots$ One can specify a sequence either by giving a general formula, e.g.

$$s_n = \frac{1}{n^2} \quad n \in \mathbb{N},$$

or by writing out the first several terms of the sequence so that the pattern is clear, e.g.

$$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$$

Sometimes it is convenient to start labelling a sequence not from $n = 1$, but from a different integer n_0. For example, $s_n = 1/\log n$ is not defined for $n = 1$, so it is convenient to consider this sequence for $n = 2, 3, 4, \ldots$; so here $n_0 = 2$.

Notation: Strictly speaking, a proper piece of notation for a sequence is $\{s_n\}_{n=n_0}^{\infty}$ (sometimes written alternatively as $(s_n)_{n \geq n_0}$). But we will usually just write something like "let s_n be a sequence", assuming that n takes natural values unless explicitly stated otherwise.

7.2 Convergence: definition

First consider an example of the sequence $s_n = 1/n$. We know that s_n gets closer and closer to zero as n grows. Let us try to make this statement precise. We would like to say that s_n gets as small as we want for sufficiently large n. How small? Fix a margin of error 0.01. How large do we have to take n to make sure that $1/n \approx 0$ within the margin of error 0.01? More precisely, how large do we have to take n to make sure that $1/n < 0.01$? Obviously, it suffices to take $n > 100$. Next, let the margin of error be $\varepsilon = 0.001$. Then, taking $n > 1000$, we can guarantee that $1/n < \varepsilon$. In general, suppose we are given an arbitrary small number $\varepsilon > 0$. Then, taking $n > 1/\varepsilon$, we ensure that $1/n < \varepsilon$. Of course, the smaller ε we are given, the larger n we have to take to make sure that $1/n < \varepsilon$.

This motivates the following definition, which is the most important definition you will ever need to know in this course.

Definition 7.1 (convergent)**.** *The sequence s_n is said to* converge to the limit ℓ *if for all $\varepsilon > 0$ there exists a natural number n_0 such that for all $n \geq n_0$ we have*

$$|s_n - \ell| < \varepsilon\,.$$

Note that n_0 will depend on ε. The smaller ε is the larger n_0 will usually have to be. You can think of n_0 as of a function of ε, i.e. $n_0 = n_0(\varepsilon)$.

Also note that there are three quantifiers in this definition and that their order of appearance is crucial to the meaning. If you do not state the quantifiers or get them in the wrong order then your definition is wrong. In symbols the definition is:

$$\boxed{\forall\varepsilon > 0\ \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0 \text{ we have } |s_n - \ell| < \varepsilon.} \tag{7.1}$$

For brevity, when s_n converges to ℓ, we write

$$\lim_{n\to\infty} s_n = \ell \qquad \text{or} \qquad s_n \to \ell \quad \text{as } n \to \infty.$$

Also, for brevity, sometimes we even drop the "$n \to \infty$", and just say $s_n \to \ell$. Sometimes we just say that $|s_n - \ell| < \varepsilon$ *for n sufficiently large* in the definition without specifying the value n_0 beyond which the inequality is always true.

Let us consider some examples of sequences that converge to zero, i.e. $\ell = 0$.

Example.

1. Consider the sequences $s_n = \frac{1}{n}$, $s_n = -\frac{1}{n}$, $s_n = \frac{(-1)^n}{n}$. For all these sequences, we have

$$|s_n| = \frac{1}{n} < \varepsilon$$

   if $n \geq \frac{1}{\varepsilon}$. Thus, for all these sequences we can take $n_0(\varepsilon) = \lceil 1/\varepsilon \rceil + 1$.

   Note that the values of the first sequence are always greater than the limit 0, the values of the second one are always smaller than the limit, and the values of the third one are alternatively greater and smaller than the limit for even and odd n.

2. Consider the sequence $s_n = \frac{2+(-1)^n}{n}$. Here we have

$$|s_n| \leq \frac{3}{n} < \varepsilon$$

   as soon as $n > \frac{3}{\varepsilon}$. So we can take $n_0(\varepsilon) = \lceil 3/\varepsilon \rceil + 1$. Note that here the values of s_n alternatively get closer to the limit and further away from the limit:

$$s_n = 1, \frac{3}{2}, \frac{1}{3}, \frac{3}{4}, \frac{1}{5}, \frac{3}{6}, \ldots$$

3. Consider the sequence $s_n = \frac{1+(-1)^n}{n}$. Here we have

$$|s_n| \le \frac{2}{n} < \varepsilon$$

if $n > \frac{2}{\varepsilon}$. Thus, we can take $n_0(\varepsilon) = \lceil 2/\varepsilon \rceil + 1$. Note that for odd n the values of the sequence are equal to its limit 0.

These simple examples illustrate the multitude of possibilities that are covered by the definition of convergence. It doesn't matter whether the values of s_n are all on one side of the limit; it doesn't matter if s_n gets closer to the limit at every step; it doesn't matter whether s_n takes the values equal to the actual limit.

Example.

1. $\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$.
2. $\frac{1}{n^2} \to 0$ as $n \to \infty$.
3. $\frac{1}{2^n} \to 0$ as $n \to \infty$.
4. $s_n = 1, 0, 0, 0, 0, \dots$. That is, $s_1 = 1$ and $s_n = 0$ for all $n \ge 2$. Then $s_n \to 0$ as $n \to \infty$.

7.3 Convergence: discussion and further examples

In order to fully appreciate the definition of convergence, let us discuss the situation when convergence does NOT hold. The negation of the definition of convergence is:

$$\exists \varepsilon > 0 \text{ such that } \forall n_0 \in \mathbb{N} \quad \exists n \ge n_0 \text{ such that } |s_n - \ell| \ge \varepsilon.$$

Example.
1. The sequence $s_n = (-1)^n$ does NOT converge to 0. Indeed, take $\varepsilon = 1/2$. For a given $n_0 \in \mathbb{N}$, take any $n > n_0$. Then $|(-1)^n| = 1 > \varepsilon$.

2. Consider the sequence $s_n = \frac{1+(-1)^n}{2}$, i.e. $s_n = 0, 1, 0, 1, 0, 1, \dots$. This sequence does NOT converge to 0. Indeed, take $\varepsilon = 1/2$. For a given $n_0 \in \mathbb{N}$, take any *even* $n > n_0$. Then $|s_n| = 1 > \varepsilon$.

Let us look at some further examples of convergent sequences:

Example.

1. For any $k \in \mathbb{N}$ and $a \in \mathbb{R}$, $\frac{a}{n^k} \to 0$ as $n \to \infty$. The same holds true for any $k \in \mathbb{R}$, $k > 0$.

2. It is a trivial consequence of the definition (but nevertheless a useful fact) that $s_n \to 0$ if and only if $|s_n| \to 0$ as $n \to \infty$.

3. For any $A \in (-1, 1)$, one has $A^n \to 0$ as $n \to \infty$.

4. Let $s_n = 1/\log n$, $n \geq 2$. Then $s_n \to 0$ as $n \to \infty$.

5. $\frac{n}{n+1} \to 1$ as $n \to \infty$.

6. $\frac{n^2}{2n^2+1} \to \frac{1}{2}$ as $n \to \infty$.

Note that according to our definition of convergence, to find out whether a given sequence is convergent you need to know in advance the limit ℓ. Just so that we can be absolutely sure we are talking about *the* limit we had better prove the following theorem.

Theorem 7.2. *Every convergent sequence has one and only one limit.*

Proof. Let us use a proof by contradiction. Suppose s_n is a convergent sequence with two limits $\ell_1 \neq \ell_2$. Using, say, $\frac{1}{3}|\ell_1 - \ell_2|$ as a value for ε around one of these points, ℓ_1, we have that there exists n_1 such that $|s_n - \ell_1| < \frac{1}{3}|\ell_1 - \ell_2|$ for all $n \geq n_1$. In the same way, we obtain that there exists n_2 such that $|s_n - \ell_2| < \frac{1}{3}|\ell_1 - \ell_2|$ for all $n \geq n_2$.

Then for all $n \geq \max\{n_1, n_2\}$ using the triangle inequality we obtain

$$|\ell_1 - \ell_2| = |\ell_1 - s_n + s_n - \ell_2| \leq |\ell_1 - s_n| + |\ell_2 - s_n| < \tfrac{1}{3}|\ell_1 - \ell_2| + \tfrac{1}{3}|\ell_1 - \ell_2| = \tfrac{2}{3}|\ell_1 - \ell_2|,$$

which is impossible. ∎

7.4 Divergence

Happy families are all alike;
every unhappy family is
unhappy in its own way.

L. Tolstoy, *Anna Karenina*

Definition 7.3 (divergent)**.** *If the sequence s_n does not converge to any limit it is said to* diverge.

Note that this definition might not accord with what you were expecting. You might have been expecting the particular kind of divergence as in the following definition.

Definition 7.4 (divergence to infinity). *The sequence* s_n *is said to* diverge to $+\infty$, *for which we write* $s_n \to +\infty$*, if, for every positive real number* H*, there exists* n_0 *such that for all* $n \geq n_0$ *we have*

$$s_n > H\,.$$

The sequence s_n *is said to* diverge to $-\infty$, *for which we write* $s_n \to -\infty$*, if, for any negative real number* H*, there exists* n_0 *such that for all* $n \geq n_0$ *we have*

$$s_n < H\,.$$

Note that we use the same notation as above: $s_n \to +\infty$ looks like $s_n \to \ell$, but the definition is different. The reason is because we like to use the symbol ∞ as if it represented a real number. It does not! Again, the symbol ∞ is merely part of our shorthand notation.

The definition of divergence to $+\infty$ is similar in spirit to that of convergence: in each case you are given a challenge (ε or H) and are required to find a point (n_0) beyond which some condition is always satisfied.

Example.
1. $n^2 \to +\infty$ as $n \to \infty$.
2. $-\sqrt{n} \to -\infty$ as $n \to \infty$.

Let s_n be a sequence. Consider the following possibilities:

- s_n converges;
- s_n diverges to $+\infty$;
- s_n diverges to $-\infty$.

These possibilities are *mutually exclusive*, i.e. a sequence cannot, for example, diverge to $+\infty$ and diverge to $-\infty$ at the same time (in the exercises, you are asked to prove this).

However, this list of possibilities is not *exhaustive*, i.e. there are sequences that neither converge nor diverge to $+\infty$ or $-\infty$.

In some sense, all convergent sequences are alike, whereas there is a great multitude of different patterns of divergence.

Example. The following sequences neither converge nor diverge to $+\infty$ or $-\infty$:

$$s_n = (-1)^n; \quad t_n = (-1)^n n^2; \quad r_n = (1 + (-1)^n)n.$$

7.5 Boundedness

It is natural to make the following definition.

Definition 7.5 (bounded). *The sequence s_n is said to be* bounded *if its terms form a bounded set. i.e., if there exists a real number M such that $|s_n| \leq M$ for all $n \in \mathbb{N}$.*

Example. The sequences $(-1)^n$, $1/n$, $\sin(\pi n/2)$ are bounded. The sequences n^3, $(-1)^n n$ are unbounded.

Theorem 7.6. *Every convergent sequence is bounded.*

Proof. Let s_n be a convergent sequence with limit ℓ. We are looking to find a real number M such that $|s_n| \leq M$ for all $n \in \mathbb{N}$. We know that we may pick any $\varepsilon > 0$ and be guaranteed that there exists n_0 such that $|s_n - \ell| \leq \varepsilon$ for all $n \geq n_0$. Thus, picking $\varepsilon = 1$ say, there exists $n_0 \in \mathbb{N}$ such that $|s_n - \ell| \leq 1$ for all $n \geq n_0$. Can we use this to bound $|s_n|$ itself?

We use a familiar trick of adding and subtracting and the triangle inequality to get

$$|s_n| = |s_n - \ell + \ell| \leq |s_n - \ell| + |\ell| \,,$$

and hence deduce that $|s_n| \leq 1 + |\ell|$ for all $n \geq n_0$.

We must also consider what goes on before n_0. However, here there are only a finite number of terms and so we can immediately say that

$$|s_n| \leq \max\{|s_1|, |s_2|, \ldots, |s_{n_0-1}|\} \qquad \text{for } n < n_0 \,.$$

Putting these two inequalities together we have that

$$|s_n| \leq \max\{1 + |\ell|, |s_1|, |s_2|, \ldots, |s_{n_0-1}|\} \qquad \text{for all } n \,,$$

and so our desired value for M is $\max\{1 + |\ell|, |s_1|, |s_2|, \ldots, |s_{n_0-1}|\}$. ∎

Note that the proof should serve to remind you that finite sets of real numbers are easy to deal with, but infinite sets are hard. In particular remember that bounded infinite sets of real numbers do not necessarily have maxima and minima. They have suprema and infima, which are not necessarily actually members of the set.

8 The algebra of limits and the sandwich theorem

8.1 The algebra of limits

Here we discuss the technique which will allow us to compute limits of rather complicated expressions by reducing the question to much simpler limits.

Theorem 8.1 (Algebra of limits). *Let $s_n \to \ell$ and $t_n \to m$ as $n \to \infty$. Then*

1. *for any number $\alpha \in \mathbb{R}$, one has $\alpha s_n \to \alpha\ell$ as $n \to \infty$;*
2. *$s_n + t_n \to \ell + m$ as $n \to \infty$*
3. *$s_n t_n \to \ell m$ as $n \to \infty$*
4. *$s_n/t_n \to \ell/m$ as $n \to \infty$, provided $m \neq 0$ and $t_n \neq 0$ for all n.*

Proof. We shall only prove parts 2 and 3 here. The proof of parts 1 is a very simple exercise and the proof of part 4 is a slightly more difficult exercise.

Part 2: Let $\varepsilon > 0$ be given. We must show that there exists n_0 such that

$$|(s_n + t_n) - (\ell + m)| < \varepsilon \qquad \text{for all } n \geq n_0\,.$$

We know that given any $\varepsilon_1 > 0$ there exists n_1 such that $|s_n - \ell| < \varepsilon_1$ for all $n \geq n_1$, and that given any $\varepsilon_2 > 0$ there exists n_2 such that $|t_n - m| < \varepsilon_2$ for all $n \geq n_2$.

Using the triangle inequality we have

$$|(s_n + t_n) - (\ell + m)| = |(s_n - \ell) + (t_n - m)| \leq |s_n - \ell| + |t_n - m|\,.$$

Our goal is to ensure that this expression is less than ε for sufficiently large n. We shall do this by ensuring that each of the two terms on the right-hand side is less than $\frac{1}{2}\varepsilon$.

If we take $\varepsilon_1 = \frac{1}{2}\varepsilon$ (which is positive) then there exists n_1 such that $|s_n - \ell| < \frac{1}{2}\varepsilon$ for all $n \geq n_1$. Similarly, taking $\varepsilon_2 = \frac{1}{2}\varepsilon$ we get that there exists n_2 such that $|t_n - m| < \frac{1}{2}\varepsilon$ for all $n \geq n_2$. To ensure that both of these inequalities are satisfied we set $n_0 = \max\{n_1, n_2\}$ and see that $|(s_n + t_n) - (\ell + m)| < \varepsilon$ for all $n \geq n_0$ as required.

Part 3: Let $\varepsilon > 0$ be given. We must show that there exists n_0 such that

$$|s_n t_n - \ell m| < \varepsilon \qquad \text{for all } n \geq n_0\,.$$

We know that given any $\varepsilon_1 > 0$ there exists n_1 such that $|s_n - \ell| < \varepsilon_1$ for all $n \geq n_1$, and that given any $\varepsilon_2 > 0$ there exists n_2 such that $|t_n - m| < \varepsilon_2$ for all $n \geq n_2$. We also know that by Theorem 7.6, the sequences s_n and t_n are bounded: $|s_n| \leq L$ and $|t_n| \leq M$ for some $L > 0$, $M > 0$ and all $n \in \mathbb{N}$. We have:

$$\begin{aligned} |s_n t_n - \ell m| = |s_n t_n - s_n m + s_n m - \ell m| &\leq |s_n t_n - s_n m| + |s_n m - \ell m| \\ &\leq |s_n||t_n - m| + |m||s_n - \ell| \leq L|t_n - m| + |m||s_n - \ell|. \end{aligned} \tag{8.1}$$

Let us first assume that $m \neq 0$. Take $\varepsilon_1 = \frac{\varepsilon}{2|m|}$ and $\varepsilon_2 = \frac{\varepsilon}{2L}$ and set $n_0 = \max\{n_1(\varepsilon_1), n_2(\varepsilon_2)\}$. Then for any $n \geq n_0$ we have

$$|s_n t_n - \ell m| \leq L|t_n - m| + |m||s_n - \ell| < L\frac{\varepsilon}{2L} + |m|\frac{\varepsilon}{2|m|} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as required. Finally, if $m = 0$ then instead of (8.1) we have

$$|s_n t_n - \ell m| \leq L|t_n - m|.$$

Now one can take $\varepsilon_2 = \varepsilon/L$, $n_0 = n_2$ and we again obtain $|s_n t_n - \ell m| < \varepsilon$ for all $n \geq n_0$. ∎

Using the algebra of limits, we can tackle a great variety of examples. The general principle is to identify the fastest growing term of your expression and divide by this term. After this, consider each of the resulting terms separately and use the Algebra of Limits.

Example.

1. $\dfrac{1}{n^2+5} = \dfrac{1/n^2}{1+(5/n^2)} \to \dfrac{0}{1} = 0$ as $n \to \infty$

2. $\dfrac{n^2+n}{2n^2+3} = \dfrac{1+(1/n)}{2+(3/n^2)} \to 1/2$ as $n \to \infty$

3. $\dfrac{n+(-1)^n}{5n-3} + (1/2)^n = \dfrac{1+\frac{(-1)^n}{n}}{5-\frac{3}{n}} + (1/2)^n \to 1/5 + 0 = 1/5$ as $n \to \infty$.

4. $\dfrac{2^n-1}{2^n+\frac{5}{n}} = \dfrac{1-2^{-n}}{1+2^{-n}\frac{5}{n}} \to 1$ as $n \to \infty$.

8.2 The Sandwich Theorem

Theorem 8.2 (The Sandwich Theorem)**.** *Let $r_n \to \ell$ and $t_n \to \ell$ as $n \to \infty$ and suppose that $r_n \leq s_n \leq t_n$ for all $n \in \mathbb{N}$. Then $s_n \to \ell$ as $n \to \infty$.*

Proof. Let $\varepsilon > 0$ be given. We know there exist n_1 and n_2 such that $\ell - \varepsilon < r_n < \ell + \varepsilon$ for all $n \geq n_1$, and $\ell - \varepsilon < t_n < \ell + \varepsilon$ for all $n \geq n_2$. Hence, for $n \geq n_0 = \max\{n_1, n_2\}$, $\ell - \varepsilon < r_n \leq s_n \leq t_n < \ell + \varepsilon$, and in particular $\ell - \varepsilon < s_n < \ell + \varepsilon$ as required. ∎

Example.

1. Let $s_n = (-1)^n/(n+1)$. Take $r_n = -1/n$, $t_n = 1/n$. By the Sandwich Theorem, $s_n \to 0$ as $n \to \infty$.

2. Let $s_n = \frac{2n}{n^3+1}$. Take $r_n = 0$, $t_n = 2/n^2$. Then

$$0 \leq \frac{2n}{n^3+1} \leq \frac{2n}{n^3} = \frac{2}{n^2},$$

and therefore $s_n \to 0$ as $n \to \infty$.

3. Let $s_n = \frac{2+3\sin n}{n^2+3}$. We have

$$\frac{2+3\sin n}{n^2+3} \le \frac{2+3}{n^2+3} \le \frac{5}{n^2},$$

and

$$\frac{2+3\sin n}{n^2+3} \ge \frac{2-3}{n^2+3} \ge -\frac{1}{n^2}.$$

Thus,

$$-\frac{1}{n^2} \le \frac{2+3\sin n}{n^2+3} \le \frac{3}{n^2}$$

and therefore $s_n \to 0$ as $n \to \infty$.

Theorem 8.3. *If $s_n \to 0$ as $n \to \infty$ and t_n is a bounded sequence, then $s_n t_n \to 0$ as $n \to \infty$.*

Proof. We know that $|t_n| \le M$ for all $n \in \mathbb{N}$. Thus we have $-M|s_n| \le t_n s_n \le M|s_n|$. By the Algebra of Limits, we have $M|s_n| \to 0$ and $-M|s_n| \to 0$. By the Sandwich theorem it follows that $s_n t_n \to 0$ as $n \to \infty$. ∎

Example.

1. $\dfrac{(-1)^n n^2 + 2}{n^3 + 2n} = \dfrac{1}{n}\dfrac{(-1)^n + \frac{2}{n^2}}{1 + \frac{2}{n^2}} \to 0$ as $n \to \infty$.

2. $\dfrac{2^n + 3}{3^n + 2} = (2/3)^n \dfrac{1 + 3 \cdot 2^{-n}}{1 + 2 \cdot 3^{-n}} \to 0$ as $n \to \infty$.

8.3 Passing to the limit in inequalities

Theorem 8.4. *Let s_n, t_n be convergent sequences such that $s_n \le t_n$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} s_n \le \lim\limits_{n\to\infty} t_n$.*

Proof. Denote $\ell = \lim\limits_{n\to\infty} s_n$ and $m = \lim\limits_{n\to\infty} t_n$. We know that:

$\forall \varepsilon > 0\ \exists n_1 = n_1(\varepsilon) \in \mathbb{N}$ such that $\forall n \ge n_1$ one has $|s_n - \ell| < \varepsilon$;

$\forall \varepsilon > 0\ \exists n_2 = n_2(\varepsilon) \in \mathbb{N}$ such that $\forall n \ge n_2$ one has $|t_n - \ell| < \varepsilon$.

We need to prove that $\ell \le m$. Let us prove this by contradiction. Suppose $\ell > m$. Denote $a = \ell - m$. Take $\varepsilon = a/3$ in the definitions of convergence above. Then for all $n \ge \max\{n_1(a/3), n_2(a/3)\}$ we have

$$s_n \ge \ell - (a/3) \quad \text{and} \quad t_n \le m + (a/3),$$

or

$$s_n - t_n \ge \ell - (a/3) - (m + (a/3)) = \ell - m - (2a/3) = a - (2a/3) = a/3.$$

It follows that $s_n - t_n > 0$, which contradicts the hypothesis $s_n \le t_n$ for all n. This contradiction proves the required statement. ∎

Note that $s_n < t_n \Rightarrow s_n \le t_n$. Thus, we obtain an obvious corollary: if s_n and t_n are two convergent sequences and $s_n < t_n$, then $\lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n$.

It is important to note that in this situation the strict inequality $\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n$ may be FALSE! Consider the following example: $s_n = 0$, $t_n = 1/n$ for all $n \in \mathbb{N}$. Then $s_n < t_n$ for all n, yet the limits of s_n and t_n coincide (and equal to 0). To summarize:

One can pass to the limit in inequalities, but $<$ must be replaced by $\le$ and $>$ by $\ge$.

8.4 The algebra of limits: further results

Theorem 8.5. *Let $s_n \to \infty$ or $s_n \to -\infty$ as $n \to \infty$ and $s_n \neq 0$ for all n. Then $1/s_n \to 0$ as $n \to \infty$.*

The proof is an exercise.

Theorem 8.6. *Let $s_n \to \ell > 0$ and let $t_n \to \infty$ as $n \to \infty$. Then $s_n t_n \to \infty$ as $n \to \infty$.*

Proof. Let $H > 0$ be given; we need to find n_0 such that $s_n t_n > H$ for all $n \ge n_0$. We know that for all $H_1 > 0$ there exists n_1 such that $t_n > H_1$ for all $n \ge n_1$. We also know that there exists n_2 such that $s_n > \frac{\ell}{2}$ for all $n \ge n_2$ (this follows by taking $\varepsilon = \ell/2$ in the definition of the limit). Now take $H_1 = 2H/\ell$ and $n_0 = \max\{n_1, n_2\}$. For all $n \ge n_0$, we have $s_n > \ell/2$ and $t_n > 2H/\ell$, so $s_n t_n > H$, as required. ∎

Of course, there is a similar statement for $-\infty$: if $s_n \to \ell > 0$ and $t_n \to -\infty$, then $s_n t_n \to -\infty$ as $n \to \infty$.

Example.

1. $\dfrac{n^3+1}{n^2+5} = n\dfrac{1+\frac{1}{n^3}}{1+\frac{5}{n^2}} \to \infty$ as $n \to \infty$.

2. $\dfrac{10^n + n^{10}}{5^n - n^5} = 2^n \dfrac{1 + n^{10}10^{-n}}{1 - n^5 5^{-n}} \to \infty$ as $n \to \infty$.

9 Standard sequences and rates of convergence

9.1 Standard sequences; the o symbol

So far we have considered the following standard sequences that converge to zero:

$$
\begin{aligned}
a^n &\to 0 \text{ as } n \to \infty, \quad \forall a \in (-1,1) && \text{(exponentials)} \\
n^{-\gamma} &\to 0 \text{ as } n \to \infty, \quad \forall \gamma > 0 && \text{(powers)} \\
\frac{1}{\log n} &\to 0 \text{ as } n \to \infty && \text{(logarithms)}
\end{aligned}
$$

Alternatively, we can say that we have the following standard sequences that diverge to ∞:

$$
\begin{aligned}
A^n &\to \infty \text{ as } n \to \infty, \quad \forall A > 1 && \text{(exponentials)} \\
n^{\gamma} &\to \infty \text{ as } n \to \infty, \quad \forall \gamma > 0 && \text{(powers)} \\
\log n &\to \infty \text{ as } n \to \infty && \text{(logarithms)}
\end{aligned}
$$

There is a certain hierarchy in the set of sequences. For example, n^2 grows faster than n, n^3 grows faster than n^2, n^4 grows faster than n^3, etc. Similarly, in the row 2^n, 3^n, 4^n, each next sequence grows faster than the previous one. Let us introduce some notation and terminology for comparing sequences:

Definition 9.1. *Let s_n and t_n be two sequences, such that $t_n \neq 0$ for all n. Suppose that $\lim_{n\to\infty} \frac{s_n}{t_n} = 0$. Then we write $s_n = o(t_n)$, $n \to \infty$ (pronounced 's_n is little o of t_n as $n \to \infty$').*

Thus, we have $n = o(n^2)$ and $n^2 = o(n^3)$ as $n \to \infty$. In general, $n^p = o(n^q)$ as $n \to \infty$ for any $p < q$. Similarly, $A^n = o(B^n)$ as $n \to \infty$ for any $0 < A < B$.

In particular, $s_n = o(1)$ as $n \to \infty$ means $s_n \to 0$ as $n \to \infty$.

How do exponentials compare to powers?

9.2 Exponentials beat powers

The aim of this section is to prove

Theorem 9.2 (Exponentials beat powers)**.** *For any $a \in (0,1)$ and any $\gamma > 0$ one has*

$$n^\gamma a^n \to 0 \quad \text{as } n \to \infty.$$

This can be alternatively stated as follows:

Theorem 9.3. *For any $\gamma > 0$ and any $A > 1$, one has*

$$n^\gamma = o(A^n) \text{ as } n \to \infty.$$

Indeed, $n^\gamma A^{-n} = n^\gamma (A^{-1})^n \to 0$ since $0 < A^{-1} < 1$, and so Theorem 9.3 follows from Theorem 9.2.

Before attempting the proof of Theorem 9.2, let us prepare a useful inequality:

Lemma 9.4 (Bernoulli inequality). *For every $k \in \mathbb{N}$ and $x \geq -1$, one has*

$$(1+x)^k \geq 1 + kx.$$

Proof. The easiest proof is by induction. For $k = 1$ the inequality is clearly true (in this case it takes the form $1 + x = 1 + x$). Assume the inequality holds true for some k. Then

$$(1+x)^{k+1} = (1+x)^k(1+x) \geq (1+kx)(1+x) = 1 + (k+1)x + kx^2 \geq 1 + (k+1)x,$$

and so the inequality is true with k replaced by $k+1$. Thus, by induction the inequality is true for all $k \in \mathbb{N}$. ∎

Proof of Theorem 9.2.

1. Let $k = \lceil \gamma \rceil$. We have
$$0 \leq n^\gamma a^n \leq n^k a^n,$$
and so by the Sandwich Theorem it suffices to prove that $n^k a^n \to 0$ as $n \to \infty$.

2. Write
$$n^k a^n = (n^k a^{n/2}) a^{n/2}.$$
Since $0 < \sqrt{a} < 1$, we have $a^{n/2} = (\sqrt{a})^n \to 0$ as $n \to \infty$. By Theorem 8.3, it suffices to prove that the sequence $n^k a^{n/2}$ is bounded. Let us prove this.

3. We have
$$n^k a^{n/2} = (n a^{n/2k})^k.$$
Thus, it suffices to prove that the sequence $n a^{n/2k}$ is bounded.

4. Consider the number $a^{-1/2k} > 1$. Write this number as $a^{-1/2k} = 1 + x$ with $x > 0$. Then
$$n a^{n/2k} = \frac{n}{a^{-n/2k}} = \frac{n}{(1+x)^n}.$$
By Bernoulli's inequality, we have $(1+x)^n \geq 1 + nx$, and so
$$\frac{n}{(1+x)^n} \leq \frac{n}{1+nx} \leq \frac{n}{nx} = \frac{1}{x}.$$
This proves that $n a^{n/2k}$ is a bounded sequence, as required. ∎

Example. One has
$$n^3 (0.9)^n \to 0 \quad \text{as } n \to \infty.$$
Can you compute the first 10 terms of this sequence on your calculator? If you didn't know that the sequence converges to zero, what conclusion would you arrive to by looking at the values of these first 10 terms?

Using the ideas of the proof of Theorem 9.2, you can also prove

Theorem 9.5. *For any $A > 1$ and any $\gamma > 0$ one has*

$$A^n n^{-\gamma} \to \infty, \quad \text{as } n \to \infty.$$

The proof of this result is outlined in the exercises.

9.3 Factorials and logarithms

So, A^n grows faster than n^γ. Can anything grow faster than exponentials? In the exercises, you are asked to prove the following result.

Theorem 9.6 (Factorials beat exponentials)**.** *For any $A > 0$, one has*

$$A^n = o(n!) \quad \text{as } n \to \infty.$$

Of course, there are many sequences that grow even faster. For example, n^{n^n}, $n^{n^{n^n}}$, etc.

What about slowly growing sequences?

Theorem 9.7 (Powers beat logarithms)**.** *For any $\gamma > 0$, one has*

$$\log n = o(n^\gamma), \quad n \to \infty.$$

Proof. We will use without proof the following elementary inequality:

$$\log x \leq x, \quad \forall x > 0.$$

We have:

$$0 \leq \frac{\log n}{n^\gamma} = \frac{2}{\gamma} \cdot \frac{\log n^{\gamma/2}}{n^\gamma} \leq \frac{2}{\gamma} \cdot \frac{n^{\gamma/2}}{n^\gamma} = \frac{2}{\gamma} \cdot \frac{1}{n^{\gamma/2}}.$$

Using the Sandwich theorem, we conclude that $\log n/n^\gamma \to 0$, as $n \to \infty$, as required. ∎

It is easy to construct sequences that grow even more slowly than $\log n$: for example, $\log\log n$, $\log\log\log n$, etc.

9.4 The sequence $a^{1/n}$

Fix $a > 0$ and consider the sequence $s_n = a^{1/n}$, $n \in \mathbb{N}$.

Theorem 9.8. *For any $a > 0$, we have*

$$a^{1/n} \to 1 \text{ as } n \to \infty.$$

Proof. 1. Let $a > 1$. Then $a^{1/n} > 1$ for all n. Thus, denoting $b_n = a^{1/n} - 1$, we get $b_n > 0$ for all n. By the Bernoulli inequality (Lemma 9.4), we have

$$a = (1 + b_n)^n \geq 1 + nb_n.$$

It follows that $a - 1 \geq nb_n$ and so we have

$$0 < b_n \leq \frac{a-1}{n}.$$

By the Sandwich Theorem, we get that $b_n \to 0$ as $n \to \infty$. Thus, for any $a > 1$ we have $a^{1/n} = 1 + b_n \to 1$ as $n \to \infty$, as required.

2. Let $a < 1$. Set $\gamma = 1/a > 1$. Then by the first step, $\gamma^{1/n} \to 1$ as $n \to \infty$. Applying the Algebra of Limits, we get $a^{1/n} = (1/\gamma)^{1/n} = 1/\gamma^{1/n} \to 1$ as $n \to \infty$.

3. Finally, if $a = 1$, it is evident that $a^{1/n} = 1 \to 1$ as $n \to \infty$. ∎

9.5 Further examples

Armed with the Algebra of Limits and the above theorems, you can confidently compute a great variety of limits. The strategy is:

- Identify the fastest growing term in n in your expression;
- Divide by this term;
- Consider separately the limits of all the resulting terms;
- Apply the Algebra of Limits or related statements.

Example.

1. $\dfrac{(-1)^n}{\sqrt{n}} + \dfrac{n^2 + \sin n}{2^n + n^{10}} = \dfrac{(-1)^n}{\sqrt{n}} + \dfrac{n^2}{2^n}\dfrac{1 + (\sin n)/n^2}{1 + n^{10}2^{-n}} \to 0$, as $n \to \infty$.

2. $s_n = \dfrac{n2^{1/n} + 3}{n3^{1/n} + 2} = \dfrac{2^{1/n} + \frac{3}{n}}{3^{1/n} + \frac{2}{n}} \to \dfrac{1+0}{1+0} = 1$, as $n \to \infty$.

3. $s_n = \dfrac{3^{2n+\frac{1}{2n}} + 2^{2n}}{9^{n+2} + 4} = \dfrac{3^{\frac{1}{2n}} + (4/9)^n}{9^2 + 4 \cdot 9^{-n}} \to \dfrac{1+0}{81+0} = \dfrac{1}{81}$, as $n \to \infty$.

4.
$$s_n = \frac{(n^2 + n^3)(2^n + 3^n)}{n^2 3^n + n^3 2^n} = \frac{n^3 3^n (1 + \frac{1}{n})(1 + (\frac{2}{3})^n)}{n^2 3^n (1 + n(\frac{2}{3})^n)} = n\frac{(1 + \frac{1}{n})(1 + (\frac{2}{3})^n)}{1 + n(\frac{2}{3})^n} \to \infty.$$

10 The O symbol

Here is another definition useful in comparing sequences:

Definition 10.1. *Let s_n and t_n be sequences. Suppose that there exist $C > 0$ such that for all $n \in \mathbb{N}$, one has $|s_n| \le C|t_n|$. Then one writes*

$$s_n = O(t_n) \text{ as } n \to \infty$$

(pronounced 's_n is big O of t_n as $n \to \infty$.')

If $s_n = o(t_n)$, then $s_n = O(t_n)$, but the converse is false.

Example.

1. $n^{10} + (-1)^n n^5 = O(n^{10})$ as $n \to \infty$.
2. $\frac{n^2}{n+1} = O(n)$ as $n \to \infty$.
3. $\dfrac{n}{n+1} = \dfrac{1}{1+\frac{1}{n}} = 1 + O(\frac{1}{n}) = 1 - \frac{1}{n} + O(\frac{1}{n^2}) = 1 - \frac{1}{n} + \frac{1}{n^2} + O(\frac{1}{n^3})$ as $n \to \infty$.

There is a certain "calculus" of O, o symbols. For example, it's easy to prove the following statements:

$$s_n = O(n), \quad r_n = O(n) \quad \Rightarrow \quad s_n + r_n = O(n),$$

$$s_n = O(n), \quad r_n = o(n) \quad \Rightarrow \quad s_n r_n = o(n^2),$$

as $n \to \infty$.

11 Monotone Sequences

11.1 Theory

Definition 11.1 (increasing, decreasing, monotone). *A sequence s_n is said to be:*

- increasing, *if $s_{n+1} > s_n$ for all $n \in \mathbb{N}$;*
- decreasing, *if $s_{n+1} < s_n$ for all $n \in \mathbb{N}$;*
- non-decreasing, *if $s_{n+1} \geq s_n$ for all $n \in \mathbb{N}$;*
- non-increasing *if $s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$;*
- monotone *if it is either non-decreasing or non-increasing.*

Example 11.2.

1. $s_n = \frac{1}{n^2}$ is decreasing;
2. $s_n = \frac{n-2}{n}$ is increasing;
3. $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is increasing;
4. $s_n = \frac{n!}{n^n}$ is decreasing (because $\frac{s_{n+1}}{s_n} = \left(\frac{n}{n+1}\right)^n < 1$ for all n);
5. $s_n = (-1)^n$ is not monotone;
6. $s_n = n/2^n$ is non-increasing (because $\frac{s_{n+1}}{s_n} = \frac{n+1}{2n} \leq 1$ for all n). In particular, $s_n \leq s_1 = 1/2$ for all $n \in \mathbb{N}$.

Theorem 11.3.

1) *Every non-decreasing sequence which is bounded above is convergent. Moreover, the limit of the sequence is the supremum of its terms.*

2) *Every non-increasing sequence which is bounded below is convergent. Moreover, the limit of the sequence is the infimum of its terms.*

Proof. We only prove claim 1). For the proof you must understand the definition of supremum. Let our sequence be s_n. Since s_n is bounded above, we see that the set $\{s_n \mid n \in \mathbb{N}\}$ is bounded above and therefore, by the axiom of completeness, it has a supremum. Let ℓ denote this supremum. We have $s_n \leq \ell$ for all n. Let us prove that $s_n \to \ell$ as $n \to \infty$.

Let $\varepsilon > 0$ be given. Then $\ell - \varepsilon$ is not an upper bound for s_n, so there exists $n_0 \in \mathbb{N}$ such that $s_{n_0} > \ell - \varepsilon$. Since s_n is non-decreasing we have $s_n > \ell - \varepsilon$ for all $n \geq n_0$. Trivially $s_n < \ell + \varepsilon$ for all n (since it is less than or equal to ℓ), hence $\ell - \varepsilon < s_n < \ell + \varepsilon$ for all $n \geq n_0$ and thus s_n converges to ℓ.

The proof for non-increasing sequences is similar. Write it out in full yourself! ∎

11.2 Series

A series is just a special type of sequence.

Definition 11.4. *Let* $\{a_k\}_{k=1}^{\infty}$ *be a sequence of real numbers. We say that* the series $\sum_{k=1}^{\infty} a_k$ converges *if the limit*

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

exists. This limit is called the sum of the series $\sum_{k=1}^{\infty} a_k$. *So by definition,*

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k,$$

if the limit exists.

The sum $\sum_{k=1}^{n} a_k$ appearing in the above definition is called the partial sum of the series. Series are discussed in more detail in the Analysis course.

Remark. Sometimes it is more convenient to consider the series

$$\sum_{k=0}^{\infty} a_k,$$

i.e. the summation starts from 0 rather than 1. Of course, one can consider the series $\sum_{k=k_0}^{\infty} a_k$ for any choice of integer k_0, with some obvious modifications in all statements.

The following two facts about series are very simple yet important:

$$\boxed{\sum_{k=1}^{\infty} a_k \quad \text{convergent} \implies \lim_{k\to\infty} a_k = 0.}$$

Indeed, denoting by $s_n = \sum_{k=1}^{n} a_k$ the partial sum, we have $a_n = s_n - s_{n-1}$, and so if $s_n \to \ell$ as $n \to \infty$, then also $s_{n-1} \to \ell$ as $n \to \infty$, and therefore $s_n - s_{n-1} \to \ell - \ell = 0$.

$$\boxed{\lim_{k\to\infty} a_k = 0 \quad \not\Longrightarrow \quad \sum_{k=1}^{\infty} a_k \quad \text{convergent.}}$$

For example, the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges; we will discuss this very soon.

11.3 Geometric Series

Let $|a| < 1$. Consider the series

$$\sum_{k=0}^{\infty} a^k.$$

We have an explicit formula for the partial sum:

$$\sum_{k=0}^{n} a^k = 1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}.$$

This can be easily proven by induction (or by multiplying both sides by $1 - a$, expanding an comparing). Using this formula, we obtain:

$$\sum_{k=0}^{\infty} a^k = \lim_{n\to\infty} \sum_{k=0}^{n} a^k = \frac{1}{1 - a}.$$

It is clear that for $|a| \geq 1$, the series $\sum_{k=0}^{\infty} a^k$ diverges; indeed, in this case the k'th term of the series does not tend to zero.

11.4 Series with positive terms

An important application of Theorem 11.3 is the following statement:

Corollary 11.5. *Let $a_k \geq 0$ be a sequence of non-negative real numbers. Suppose that there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$*

$$\sum_{k=1}^{n} a_k \leq C.$$

Then the series $\sum_{k=1}^{\infty} a_k$ converges and the sum s of this series satisfies $s \leq C$.

Proof. Indeed, consider the sequence of partial sums

$$s_n = \sum_{k=1}^{n} a_k.$$

Since $a_k \geq 0$, this sequence is non-decreasing. By assumption, it is bounded, and so it converges. Passing to the limit as $n \to \infty$ in the inequality $s_n \leq C$, we obtain $s \leq C$. ∎

Example 11.6. Let

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}. \tag{11.1}$$

In contradistinction to geometric series, we do not have an explicit formula for s_n. However, we can still prove that s_n converges. Indeed, s_n is non-decreasing. Now we just have to prove that s_n is bounded above. For $n \geq 2$ one has

$$n! = n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1 \geq \underbrace{2 \cdot 2 \cdot \dots \cdot 2}_{n-1 \text{ times}} = 2^{n-1},$$

and so

$$s_n \leq 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}} \leq 1 + \frac{1-2^{-n}}{1-2^{-1}} \leq 3,$$

and so s_n is bounded above by 3. Thus, the sequence s_n converges. Equivalently, the series

$$\sum_{k=0}^{\infty} \frac{1}{k!}$$

converges. We will say more about the sum of this series soon.

Example 11.7. Suppose we have a sequence of digits $d_1, d_2, \dots$, where $d_k \in \{0, 1, 2, \dots, 9\}$. Consider the question of convergence of the series

$$\sum_{k=1}^{\infty} \frac{d_k}{10^k}.$$

According to the definition of convergence of a series, we need to study the sequence

$$x_n = \sum_{k=1}^{n} \frac{d_k}{10^k}, \quad n \in \mathbb{N}.$$

Of course, using the common notation for decimal representation of real numbers, we have

$$x_n = 0.d_1 d_2 \dots d_n.$$

Clearly, the sequence x_n is non-decreasing and

$$x_n \leq \sum_{k=1}^{n} \frac{9}{10^k} = \frac{9}{10} \frac{1-10^{-n-1}}{1-10^{-1}} \leq \frac{9}{10} \frac{1}{1-10^{-1}} = 1.$$

Thus, by Theorem 11.3 the sequence x_n converges. Let us denote its limit by x.

We have established that for any sequence of digits d_1, d_2, d_3, $\dots$, there exists a real number x which can be represented by the infinite decimal expansion $x = 0.d_1 d_2 d_3 \dots$, i.e. by

$$x = \lim_{N \to \infty} 0.d_1 d_2 \dots d_n = \sum_{k=1}^{\infty} \frac{d_k}{10^k}.$$

11.5 The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$

Here we consider the important series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

for various values of $p > 0$. We denote by s_n the partial sum: $s_n = \sum_{k=1}^{n} \frac{1}{k^p}$.

Case $0 < p < 1$. Let us prove that for $0 < p < 1$, the series diverges. For the partial sum of the series, we have, replacing all terms by the smallest one,

$$s_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} > \frac{n}{n^p} = n^{1-p} \to \infty,$$

as $n \to \infty$. It follows that the series diverges.

Case $p = 1$. This is called the *Harmonic Series*. Let us prove that the harmonic series diverges. Let us split the harmonic series into the groups of 2, 4, 8, ... terms:

$$\underbrace{\frac{1}{2} + \frac{1}{3}}_{2}; \quad \underbrace{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}}_{2^2}; \quad \underbrace{\frac{1}{8} + \cdots + \frac{1}{15}}_{2^3}; \ldots;$$
$$\underbrace{\frac{1}{2^k} + \cdots + \frac{1}{2^{k+1}-1}}_{2^k}; \ldots$$

Using the estimate

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1} > n\frac{1}{2n} = \frac{1}{2},$$

we see that the sum of each group above is at least 1/2. It follows that for the partial sum of the harmonic series we have:

$$s_{2^n} = \sum_{k=1}^{2^n} \frac{1}{k} > \frac{n-1}{2}.$$

So the sequence of partial sums is unbounded, hence the harmonic series diverges.

Case $p > 1$. Let us prove that the series converges for $p > 1$. Again, we split our series into groups of 2, 4, 8, ... terms:

$$\underbrace{\frac{1}{2^p} + \frac{1}{3^p}}_{2}; \quad \underbrace{\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}}_{2^2}; \quad \underbrace{\frac{1}{8^p} + \cdots + \frac{1}{15^p}}_{2^3}; \ldots;$$
$$\underbrace{\frac{1}{(2^k)^p} + \cdots + \frac{1}{(2^{k+1}-1)^p}}_{2^k}; \ldots$$

Using the simple inequality

$$\frac{1}{n^p} + \frac{1}{(n+1)^p} + \cdots + \frac{1}{(2n-1)^p} < n\frac{1}{n^p} = n^{1-p},$$

we see that the sums of these terms can be estimated above as follows:

$$\frac{1}{(2^k)^p} + \cdots + \frac{1}{(2^{k+1}-1)^p} < 2^k \frac{1}{(2^k)^p} = 2^{k(1-p)} = a^k,$$

where $a = 2^{1-p} < 1$. Thus, for the partial sums we have the estimate

$$s_n < a + a^2 + \cdots + a^{k-1} < \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \quad \text{if } n \le 2^k - 1.$$

So all partial sums are bounded above by a constant, and the series converges.

Remark. The sum of the series considered above is called *Riemann's zeta function:*

$$\zeta(p) = \sum_{k=1}^{\infty} \frac{1}{k^p}, \quad p > 1.$$

Although the definition of zeta function looks simple, some deepest fundamental unsolved mathematical questions are related to this function. The most famous of them is the Riemann hypothesis; the statement of this hypothesis is beyond the scope of this course and is related to the zeros of $\zeta(p)$ for complex values of p. A much more "elementary" question relates to the values of $\zeta(p)$ for natural values of p. The values $\zeta(2n)$ are known explicitly (they have the form $q_n \pi^{2n}$, where q_n is an explicit rational number). However, very little is known about the values $\zeta(2n+1)$. It is conjectured that these values are irrational, but this is only known for $\zeta(3)$.

11.6 The sequence $(1 + \frac{1}{n})^n$ and the number e

Theorem 11.8. *The sequence $t_n = (1+\frac{1}{n})^n$ is increasing and bounded above, and therefore is convergent.*

Proof. 1. Let us prove that the sequence t_n is increasing. It suffices to consider the ratio t_{n+1}/t_n and to prove that this ratio is greater than one for all n. We have:

$$\begin{aligned}\left(1+\frac{1}{n+1}\right)^{n+1} \Big/ \left(1+\frac{1}{n}\right)^n &= \left(1+\frac{1}{n+1}\right)\left(\frac{n+2}{n+1}\right)^n \frac{n^n}{(n+1)^n} \\ &= \left(1+\frac{1}{n+1}\right)\left(\frac{n^2+2n}{n^2+2n+1}\right)^n = \left(1+\frac{1}{n+1}\right)\left(1-\frac{1}{n^2+2n+1}\right)^n.\end{aligned}$$

Using Bernoulli's inequality (Lemma 9.4), we have

$$\left(1+\frac{1}{n+1}\right)\left(1-\frac{1}{n^2+2n+1}\right)^n \geq \left(1+\frac{1}{n+1}\right)\left(1-\frac{n}{n^2+2n+1}\right)$$
$$> \left(1+\frac{1}{n+1}\right)\left(1-\frac{n}{n^2+2n}\right) = \frac{n+2}{n+1}\left(1-\frac{1}{n+2}\right) = \frac{n+2}{n+1}\frac{n+1}{n+2} = 1.$$

Thus, we have proven that t_n is increasing.

2. Let us prove that t_n is bounded above. By the binomial expansion,

$$\begin{aligned} t_n = \left(1+\frac{1}{n}\right)^n &= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \dots + \left(\frac{1}{n}\right)^n \\ &= 2 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \\ &\quad + \dots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{n-1}{n}\right). \end{aligned} \tag{11.2}$$

Estimating all the expressions in brackets from above by 1, we get

$$t_n \leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} \leq 3.$$

Thus, t_n is bounded above. By Theorem 11.3, it follows that the sequence t_n converges. ∎

The limit of this sequence is denoted by e. This is the same number e as the one that serves as a base of natural logarithms, $e \approx 2.71828$:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

In fact, it is more convenient to calculate e according to a different formula:

Theorem 11.9. *One has*

$$\sum_{k=0}^{\infty}\frac{1}{k!} = e. \tag{11.3}$$

Proof. We use the notation s_n (see (11.1)) for the partial sum of the series in the l.h.s. of (11.3). We also denote $t_n = (1+\frac{1}{n})^n$. We have already seen in Example 11.6 that the series in the l.h.s. of (11.3) converges. We need to prove that

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n. \tag{11.4}$$

1. Recall formula (11.2). Consider the k'th term in the r.h.s.:

$$\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \leq \frac{1}{k!}.$$

It follows that $t_n \le s_n$ for all n. Passing to the limit in this inequality as $n \to \infty$, we get

$$\lim_{n\to\infty} t_n \le \lim_{n\to\infty} s_n. \tag{11.5}$$

2. Again, consider formula (11.2). Fix an integer k, $1 \le k \le n$, and let us keep the first k terms in the r.h.s. of (11.2) and erase the rest of the terms. Since we have erased positive numbers, we obtain an inequality

$$t_n > 2 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \dots + \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{k-1}{n}\right).$$

Now for fixed k let us pass to the limit $n \to \infty$ in this inequality. We get

$$\lim_{n\to\infty} t_n \ge 2 + \frac{1}{2!} + \dots + \frac{1}{k!} = s_k.$$

The inequality is true for all k, and so we can pass to the limit in this inequality. Bearing in mind that the l.h.s. is independent of k, we get

$$\lim_{n\to\infty} t_n \ge \lim_{k\to\infty} s_k = \lim_{n\to\infty} s_n. \tag{11.6}$$

Combining (11.5) and (11.6), we obtain (11.4). ∎

The last theorem gives a very efficient way of computing e. Indeed, taking the sum of the terms up to $1/10!$, we already obtain 7 correct digits of e after the decimal point.

Example.

1. Consider the sequence $t_n = \left(1 + \frac{1}{2n}\right)^{2n}$. Clearly, $t_n = s_{2n}$, where s_n is as in the above theorem. Thus, t_n is a *subsequence* of s_n (we will discuss subsequences in more detail shortly) and therefore it converges to the same limit: $t_n \to e$ as $n \to \infty$. In the same way, one can prove that
$$\lim_{n\to\infty}\left(1 + \frac{1}{kn}\right)^{kn} = e \tag{11.7}$$
for any natural number k. In fact, k doesn't have to be natural; one can prove that (11.7) holds true for any $k \in \mathbb{R}$.

2. Consider the sequence $r_n = \left(1 + \frac{1}{n}\right)^{2n}$. Clearly, $r_n = s_n^2$, where s_n is as in the above theorem. Thus, by the Algebra of limits, $r_n \to e^2$ as $n \to \infty$. In the same way, one can prove that
$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{kn} = e^k \tag{11.8}$$
for any natural number k. In fact, k doesn't have to be natural here; one can prove that (11.8) holds true for any $k \in \mathbb{R}$.

12 Subsequences and limit points

12.1 Subsequences

Definition 12.1 (subsequence)**.** *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers and let n_k be a strictly increasing sequence of natural numbers (i.e. $1 \le n_1 < n_2 < n_3 < \cdots$). Then $\{s_{n_k}\}_{k=1}^{\infty}$ is called a* subsequence *of $\{s_n\}_{n=1}^{\infty}$.*

Example. Let $s_n = \frac{1}{n}$ and $n_k = k^2$. Then $s_{n_k} = s_{k^2} = \frac{1}{k^2}$.

Theorem 12.2. *Let s_n be a convergent sequence with $s_n \to \ell$ as $n \to \infty$. Then every subsequence of s_n converges to ℓ.*

Proof. Let s_{n_k}, $k \in \mathbb{N}$, be a subsequence of the sequence s_n. By induction one easily establishes that $n_k \ge k$ for all k. Next, we know that for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have $|s_n - \ell| \le \varepsilon$. Now for given ε, take $k_0 = n_0$. Then for any $k \ge k_0$ we have $n_k \ge k \ge k_0 = n_0$ and therefore $|s_{n_k} - \ell| \le \varepsilon$. Thus, $s_{n_k} \to \ell$ as $k \to \infty$. ∎

Example.

1. Consider $s_n = (-1)^n + \frac{1}{n}$. This sequence does not converge. Consider the subsequence s_{n_k}, where $n_k = 2k$:
$$s_{n_k} = s_{2k} = (-1)^{2k} + \frac{1}{2k} = 1 + \frac{1}{2k}.$$
Thus, $s_{2k} \to 1$ as $k \to \infty$.

2. Consider $s_n = \frac{1}{10^n}$ and the subsequence s_{n_k}, where $n_k = 3k$. Then
$$s_{n_k} = s_{3k} = \frac{1}{1000^k} \to 0 \quad \text{as } n \to \infty.$$

Theorem 12.3 (The Bolzano-Weierstrass Theorem)**.** *Every bounded sequence has a convergent subsequence. More precisely, if s_n is a sequence of real numbers such that $a \le s_n \le b$ for all n, then there exists a subsequence of s_n which converges to a limit $\ell \in [a, b]$.*

Proof. If we bisect the closed interval $[a, b]$, at least one half must contain s_n for infinitely many n. Call such a half $[a_1, b_1]$. Denote
$$N_1 = \{n \in \mathbb{N} : s_n \in [a_1, b_1]\};$$
by our choice of the interval $[a_1, b_1]$, the set N_1 is infinite.

If we now bisect $[a_1, b_1]$, then, again, at least one half must contain s_n for infinitely many n. Call such a half $[a_2, b_2]$, and let N_2 be the infinite set $N_2 = \{n \in \mathbb{N} : s_n \in [a_2, b_2]\}$.

We can repeat this process indefinitely to obtain a sequence $[a_k, b_k]$ of closed intervals with the following properties.

(i) $a \le a_1 \le a_2 \le \cdots \le a_k \le b_k \le \cdots \le b_2 \le b_1 \le b$.

(ii) $b_k - a_k = \frac{1}{2^k}(b - a)$.

(iii) For every k, the set $N_k = \{n \in \mathbb{N} : s_n \in [a_k, b_k]\}$ is infinite.

From (i) we see that the sequence a_k is increasing and bounded above, therefore convergent, say to ℓ. Likewise, the sequence b_k is non-increasing and bounded below, therefore convergent, say to ℓ'. Taking limits in (ii) gives $\ell' - \ell = 0$, i.e., $\ell = \ell'$. Since $a \le a_k \le b$ for all k, passing to the limit we get $\ell \in [a, b]$.

Below we choose a sequence of natural numbers n_k such that $s_{n_k} \in [a_k, b_k]$ and $n_{k+1} > n_k$ for all k. Take

$$\begin{aligned} n_1 &= \min N_1, \\ n_2 &= \min N_2 \setminus \{1, 2, \ldots, n_1\}, \\ n_3 &= \min N_3 \setminus \{1, 2, \ldots, n_2\}, \\ n_4 &= \min N_4 \setminus \{1, 2, \ldots, n_3\}, \end{aligned}$$

and so on; in general, we set

$$n_k = \min N_k \setminus \{1, 2, \ldots, n_{k-1}\}.$$

Since each of the sets N_k is infinite, we get that at each step the set $N_k \setminus \{1, 2, \ldots, n_{k-1}\}$ is non-empty. By construction, we get $n_{k+1} > n_k$ for all k. Finally, condition $s_{n_k} \in [a_k, b_k]$ holds true by the definition of the sets N_k.

We now have $a_k \le s_{n_k} \le b_k$ for all k, and therefore by the Sandwich Theorem we must have s_{n_k} convergent to ℓ. ∎

Example. Let $s_n = \sin \frac{\pi n}{2}$. Then the subsequence s_{n_k}, $n_k = 4k$, converges to zero.

12.2 Limit points

Definition 12.4. *A real number a is called a* limit point *of a sequence s_n, $n \in \mathbb{N}$, if there exists a subsequence s_{n_k} such that $a = \lim_{k\to\infty} s_{n_k}$.*

Example.

1. Let $s_n = (-1)^n$. This sequence has two limit points: $a = -1$ and $a = 1$.
2. Let $s_n = \sin(\pi n/2)$. This sequence has three limit points: $a = -1, 0, 1$.
3. Let $s_n = n^{(-1)^n}$. This sequence has a limit point $a = 0$.
4. Let s_n be a convergent sequence: $s_n \to \ell$ as $n \to \infty$. Then s_n has a limit point ℓ.

It is clear that in the last example, there are no other limit points:

Theorem 12.5. *A convergent sequence has one and only one limit point; this point is the limit of the sequence.*

Proof. Suppose $s_n \to \ell$ as $n \to \infty$. Clearly, ℓ is a limit point of s_n: indeed, take any subsequence of s_n (for example, s_{2n}) and by Theorem 12.2 this subsequence converges to ℓ.

Next, assume that there is another limit point $\ell' \neq \ell$ of our sequence. This means that there exists a subsequence s_{n_k} such that $s_{n_k} \to \ell'$ as $k \to \infty$. But this contradicts the result of Theorem 12.2! ∎

In fact, for bounded sequences the converse of this theorem is true:

Theorem 12.6. *Let s_n, $n \in \mathbb{N}$, be a bounded sequence which has only one limit point ℓ. Then $s_n \to \ell$ as $n \to \infty$.*

Note that under the assumptions of this theorem, s_n has *at least one* limit point by the Bolzano-Weierstrass theorem.

The proof of Theorem 12.6 is a (challenging) exercise. Note that the assumption of boundedness in the hypothesis of Theorem 12.6 is important. Indeed, the sequence $s_n = n^{(-1)^n}$ has only one limit point 0 but it does not converge to 0.

12.3 Cauchy Sequences

As mentioned in the introductory section, the definition of convergence suffers from the "defect" that in order to know whether a sequence converges you need to know in advance what the limit is. Wouldn't it be nice if you could just look at the terms in the sequence and decide that there was a limit. You can!

There is a penalty to pay though. We need that all *pairs* of sequence elements are close together for n sufficiently large.

Definition 12.7 (Cauchy sequence)**.** *The sequence s_n is called a* Cauchy Sequence *if for all $\varepsilon > 0$ there exists a natural number n_0 such that for all m, $n \geq n_0$ we have*

$$|s_m - s_n| < \varepsilon\,.$$

First of all we show that if a sequence is convergent then it is a Cauchy sequence. (That is, being a Cauchy sequence is a *necessary* condition for convergence.)

Theorem 12.8. *Every convergent sequence is a Cauchy sequence.*

Proof. Let s_n be convergent with limit ℓ, and let $\varepsilon > 0$ be given. Then there exists n_0 such that $|s_n - \ell| < \frac{1}{2}\varepsilon$ for all $n \geq n_0$. Now

$$|s_m - s_n| = |s_m - \ell + \ell - s_n| \leq |s_m - \ell| + |\ell - s_n| = |s_m - \ell| + |s_n - \ell|.$$

So, if m, $n \geq n_0$, we have $|s_m - \ell| < \frac{1}{2}\varepsilon$, $|s_n - \ell| < \frac{1}{2}\varepsilon$ and hence $|s_m - s_n| < \varepsilon$ as required. ∎

Next we see what properties Cauchy sequences possess.

Theorem 12.9. *Every Cauchy sequence is bounded.*

Proof. (Similar to the proof of Theorem 7.6.) Let s_n be a Cauchy sequence. We take a concrete value of ε, for simplicity 1 say, then there exists a natural number n_0 such that

$$|s_m - s_n| \leq 1 \qquad \text{for all } m, n \geq n_0 \,.$$

Hence (fixing $m = n_0$) we have

$$|s_n - s_{n_0}| \leq 1 \qquad \text{for all } n \geq n_0 \,,$$

and so (writing $|s_n| = |s_n - s_{n_0} + s_{n_0}| \leq |s_n - s_{n_0}| + |s_{n_0}|$)

$$|s_n| \leq |s_{n_0}| + 1 \qquad \text{for all } n \geq n_0 \,.$$

This bounds s_n from n_0 onwards. Before this there are just a finite number of terms and so they have a maximum and a minimum. More precisely

$$|s_n| \leq \max\{|s_1|, |s_2|, \ldots, |s_{n_0-1}|\} \qquad \text{for all } n < n_0 \,.$$

Combining these two inequalities we have

$$|s_n| \leq \max\{|s_{n_0}| + 1, |s_1|, |s_2|, \ldots, |s_{n_0-1}|\} \qquad \text{for all } n \,,$$

showing that s_n is bounded. ∎

We now prove the converse of Theorem 12.8.

Theorem 12.10 (Cauchy's Convergence Criterion)**.** *Every Cauchy sequence is a convergent sequence.*

Proof. Let s_n be a Cauchy sequence. By Theorem 12.9, the sequence s_n is bounded and hence, by the The Bolzano-Weierstrass Theorem, there exists a convergent subsequence s_{n_k} of s_n. Denote by ℓ the limit of s_{n_k}. Let us prove that the whole sequence s_n also converges to this limit (NB: this is not automatic: cf. example $s_n = (-1)^n + \frac{1}{n}$ above). Given $\varepsilon > 0$, we need to find n_0 such that

$$|s_n - \ell| < \varepsilon, \qquad \forall n \geq n_0. \tag{12.1}$$

As $s_{n_k} \to \ell$, we can find n_1 such that

$$|s_{n_k} - \ell| < \frac{\varepsilon}{2}, \qquad \forall k \geq n_1.$$

As s_n is a Cauchy sequence, we can find n_2 such that

$$|s_m - s_n| < \frac{\varepsilon}{2} \quad \forall m, n \geq n_2.$$

Now take $n_0 = \max\{n_1, n_2\}$. Then, for any $n \geq n_0$, choose $k \geq n_0$ and write (noting that $n_k \geq k \geq n_2$):

$$|s_n - \ell| \leq |s_n - s_{n_k}| + |s_{n_k} - \ell| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves (12.1). ∎

This last theorem tells us that being a Cauchy sequence is a *sufficient* condition for convergence.

Thus, combining Theorems 12.8 and 12.10, we see that

a sequence of *real numbers* is a Cauchy sequence if and only if it is convergent.

This means that for real numbers, the two conditions are identical!

Warning It is not good enough for us to try to make Cauchy's condition more manageable by making it "for all $\varepsilon > 0$ there exists a natural number n_0 such that $|s_{n+1} - s_n| < \varepsilon$ for all $n \geq n_0$" for instance. The example $\sqrt{n}$ shows why not: in this case the difference between successive terms vanishes, but the sequence diverges.

13 Absolute and conditional convergence of series

13.1 Absolute convergence

The behaviour of series whose terms are positive is fairly straightforward to analyse as the sequence of partial sums is an increasing sequence. The series either increases to a limit or diverges to $+\infty$. Life is not always that easy! The terms in a series will be of a mixture of positive and negative values in general.

We can associate with any (real or complex) series an associated series, namely that made up of the absolute values of the terms. Maybe the behaviour of this new "nice" series will tell us something about the original.

Definition 13.1. *The series $\sum_{k=1}^{\infty} a_k$ is said to converge absolutely if the series $\sum_{k=1}^{\infty} |a_k|$ is convergent.*

Theorem 13.2. *Every absolutely convergent series is convergent. I.e., if $\sum_{k=1}^{\infty} |a_k|$ converges then $\sum_{k=1}^{\infty} a_k$ converges as well.*

Proof. Denote $\sigma_n = \sum_{k=1}^{n} a_k$ and $\tau_n = \sum_{k=1}^{n} |a_k|$. We know that τ_n converges and we need to prove that σ_n converges.

Let us use the Cauchy criterion and prove that σ_n is a Cauchy sequence. For any $n > m$, one has

$$|\sigma_n - \sigma_m| = \left| \sum_{k=m+1}^{n} a_k \right| \le \sum_{k=m+1}^{n} |a_k| = |\tau_n - \tau_m|.$$

As τ_n is a Cauchy sequence, we get that σ_n is also a Cauchy sequence. ∎

Example 13.3. The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}$ converges absolutely. This follows from Example 11.6 (see also Theorem 11.9).

It is useful to combine the notion of absolute convergence of the series with another idea: that of a comparison of two series.

Theorem 13.4 (The Comparison Test)**.** *Let $\sum_{k=1}^{\infty} b_k$ be a convergent series of nonnegative numbers and suppose that for some constant $M > 0$ we have $|a_k| \le M b_k$ for all k. Then the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.*

Proof. The proof is only a slight modification of the proof of Theorem 13.2.

Denote $\sigma_n = \sum_{k=1}^{n} |a_k|$ and $\tau_n = \sum_{k=1}^{n} b_k$. We know that τ_n converges and we need to prove that σ_n converges. Let us use the Cauchy criterion and prove that σ_n is a Cauchy sequence. For any $n > m$, one has

$$|\sigma_n - \sigma_m| = \sum_{k=m+1}^{n} |a_k| \le M \sum_{k=m+1}^{n} b_k = M|\tau_n - \tau_m|.$$

As τ_n is a Cauchy sequence, we get that σ_n is also a Cauchy sequence. ∎

Example 13.5.

1. The series $\sum_{k=1}^{\infty} \frac{\sin(k)}{2^k}$ converges absolutely; this can be seen by comparison with $\sum_{k=1}^{\infty} \frac{1}{2^k}$.
2. The series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$ converges absolutely (in fact, the sum of this series equals e^{-1}). The convergence can be seen by comparison with $\sum_{k=0}^{\infty} \frac{1}{k!}$.
3. We have mentioned the definition of the Weierstrass' function (2.1):
$$g(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x), \quad x \in \mathbb{R}.$$
By comparison with the geometric series $\sum_{k=0}^{\infty} a^k$, we see that the series is absolutely convergent, if $0 < a < 1$.

13.2 Conditional convergence

As we will see very soon, the converse of Theorem 13.2 is false:

$$\boxed{\sum_{k=1}^{\infty} a_k \quad \text{convergent} \quad \not\Longrightarrow \quad \sum_{k=1}^{\infty} |a_k| \quad \text{convergent.}}$$

Definition 13.6. *If the series $\sum_{k=1}^{\infty} a_k$ converges, but does not converge absolutely, it is said to converge* conditionally.

The following theorem is useful in the special case when the terms are strictly alternating in sign.

Theorem 13.7 (The Alternating Series Test)**.** *Let a_k be a non-increasing sequence of positive numbers such that $a_k \to 0$ as $k \to \infty$. Then the series*
$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots$$
converges. Moreover, its sum lies between $a_1 - a_2$ and a_1.

Proof. Write $s_n = \sum_{k=1}^{n} (-1)^{n+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1} a_n$.

Then $s_{2n} - s_{2n-2} = a_{2n-1} - a_{2n} \geq 0$, i.e., the sequence of even partial sums s_{2n} is increasing.

Now $s_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1$, so s_{2n} is also bounded above. Hence, by Theorem 11.3, it converges to a sum s, say, satisfying $s \leq a_1$.

We have $s_{2n+1} = s_{2n} + a_{2n+1} \to s$ as $n \to \infty$, i.e., the sequence of odd partial sums s_{2n+1} also converges to s. Thus s_n itself converges to s.

Since $s_2 = a_1 - a_2$ and s_{2n} is increasing it follows immediately that $s \geq a_1 - a_2$. ∎

Clearly the bounds stated in the theorem can be improved upon by taking any of the partial sums.

Example (The Alternating Harmonic Series)**.**

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\,.$$

This series is easily seen to be convergent by the Alternating Series Test (Theorem 13.7). It is not absolutely convergent, since when we take modulus of each term we get the Harmonic Series which is divergent. It is thus conditionally convergent.

Example. A more general example is given by the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}$$

for $0 < p < 1$. This series is conditionally convergent.

14 The limit of a function

Here we are going to relate the material of this course to that of Calculus I.

Throughout this section, $\Delta \subset \mathbb{R}$ is a non-empty interval, $a \in \Delta$ is a point and $f : \Delta \to \mathbb{R}$ is a function.

Definition 14.1. *We write* $\lim\limits_{x\to a} f(x) = A$*, if for any* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) > 0$ *such that for every* $x \in \Delta$ *if* $|x - a| < \delta$*, then* $|f(x) - A| < \varepsilon$*.*

Remark. The value of δ depends on ε; here the situation is similar to that of the definition of convergence of a sequence, where the value of N depends on ε.

You have seen many examples of limits of functions in Calculus 1 course. Here are some of them again as a reminder.

Example 14.2.

(1) $\lim\limits_{x\to 0} x^2 = 0$. Indeed, for a given $\varepsilon > 0$ let us take $\delta = \sqrt{\varepsilon}$. Then

$$|x| < \delta = \sqrt{\varepsilon} \quad \Rightarrow \quad |x^2| \leq \varepsilon.$$

(2) $\lim\limits_{x\to 4} \sqrt{x} = 2$. Indeed, for a given $\varepsilon > 0$ let us take $\delta = 2\varepsilon$. Then for $|x - 4| < \delta$ we have

$$|\sqrt{x} - 2| = \left|\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}+2}\right| = \frac{|x-4|}{\sqrt{x}+2} \leq \frac{|x-4|}{2} \leq \frac{\delta}{2} = \varepsilon,$$

as required.

Theorem 14.3. *Let* $f : \Delta \to \mathbb{R}$ *be a function and* $a \in \Delta$*. Then the following statements are equivalent:*

(i) *The limit* $\lim\limits_{x\to a} f(x)$ *exists and equals* A*.*

(ii) *For any sequence* $\{x_n\}_{n=1}^{\infty}$ *such that* $x_n \in \Delta$ *for all* $n \in \mathbb{N}$ *and* $x_n \to a$ *as* $n \to \infty$*, we have* $f(x_n) \to A$ *as* $n \to \infty$*.*

Proof. (i)⇒(ii): Assume that $A = \lim\limits_{x\to a} f(x)$, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $x_n \in \Delta$ for all n and $x_n \to a$ as $n \to \infty$. Let $\varepsilon > 0$ be given; we need to find N such that for all $n \geq N$ we have $|f(x_n) - A| < \varepsilon$.

According to Definition 14.1, there exists $\delta = \delta(\varepsilon)$ such that if $x \in \Delta$, $|x - a| < \delta$, then $|f(x) - A| < \varepsilon$. According to the definition of convergence of a sequence, there exists $n_0 = n_0(\delta)$ such that if $n \geq n_0$, then $|x_n - a| < \delta$. Thus, taking $N(\varepsilon) = n_0(\delta(\varepsilon))$, we get

$$n \geq N \quad \Rightarrow \quad |x_n - a| < \delta \quad \Rightarrow \quad |f(x_n) - A| < \varepsilon,$$

as required.

(ii)$\Rightarrow$(i): Assume to get a contradiction, that there exists $\varepsilon > 0$ such that for any $\delta > 0$ there exists $x \in \Delta$, $|x - a| < \delta$, but $|f(x) - A| \geq \varepsilon$.

Consider the sequence $\delta = \frac{1}{n}$, $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ there exists $x_n \in \Delta$ with the property $|x_n - a| < \frac{1}{n}$, but $|f(x_n) - A| \geq \varepsilon$. We obtain a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in \Delta$ for all n, $x_n \to a$ as $n \to \infty$, yet $f(x_n) \not\to A$ as $n \to \infty$. ∎

Using this theorem, many properties of limits which we have already established for sequences, can be easily transferred into the context of functions. For example:

Theorem 14.4 (The Algebra of Limits for Functions)**.** *Let $f, g : \Delta \to \mathbb{R}$ be functions and $a \in \Delta$. Assume that the limits*

$$A = \lim_{x \to a} f(x), \quad B = \lim_{x \to a} g(x)$$

exist. Then:

(i) $\lim_{x \to a}(f(x) \pm g(x)) = A \pm B$*;*

(ii) $\lim_{x \to a} f(x)g(x) = AB$*;*

(iii) *If $B \neq 0$ and $g(x) \neq 0$ for all $x \in \Delta$, then $\lim_{x \to a} f(x)/g(x) = A/B$.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $x_n \in \Delta$ for all n and $x_n \to a$ as $n \to \infty$. Then, by Theorem 14.3, $f(x_n) \to A$ and $g(x_n) \to B$ as $n \to \infty$. By Theorem 8.1 (the Algebra of Limits), we obtain

$$\begin{aligned} f(x_n) \pm g(x_n) &\to A \pm B \\ f(x_n)g(x_n) &\to AB \\ f(x_n)/g(x_n) &\to A/B \quad \text{(provided the denominators don't vanish)} \end{aligned}$$

as $n \to \infty$. Applying Theorem 14.3 again, we obtain the required statements. ∎

15 Finite, countable and uncountable sets

Definition 15.1. *A non-empty set A is said to be* finite *if for some $n \in \mathbb{N}$ there exists a bijection between A and the set $\{1, 2, \dots, n\}$.*

More generally, one makes the following definition.

Definition 15.2. *Two sets A and B are said to have the same* cardinality *if there exists a bijection $f : A \to B$.*

For finite sets, cardinality is the same as the number of elements.

For two sets A and B, let us write $A \sim B$ if A and B have the same cardinality. Then $\sim$ is an *equivalence relation*:

- For any set A, we have $A \sim A$ ($\sim$ is *reflexive*);
- For any sets A and B, we have $A \sim B$ if and only if $B \sim A$ ($\sim$ is *symmetric*);
- For any sets A, B, C, if $A \sim B$ and $B \sim C$, then $A \sim C$ ($\sim$ is *transitive*).

Definition 15.3. *A set A is said to be* countable *if it has the same cardinality as $\mathbb{N}$. An infinite set that is not countable is called* uncountable.

Example 15.4. $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ is countable. $\mathbb{Z}$ is countable.

It is not difficult to see that any subset of a countable set is either finite or countable.

Theorem 15.5. *A union of finitely many countable sets is countable.*

Proof. Using induction, one reduces the statement to the case of the union of two countable sets. Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$ be countable sets. The elements of $A \cup B$ can be listed as

$$a_1, b_1, a_2, b_2, a_3, b_3, \dots$$

This establishes a bijection $f : \mathbb{N} \to A \cup B$. ∎

If the intersection $A \cap B$ is non-empty, then the above proof needs to be corrected slightly, as the map f will not be a bijection (only a surjection). This problem is not difficult to fix.

Theorem 15.6. *A countable union of countable sets is countable.*

Proof. Let A_n be countable sets, where $n \in \mathbb{N}$. Let $A_n = \{a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \dots\}$. The elements of $A = \cup_{n=1}^{\infty} A_n$ can be listed as

$$a_1^{(1)}, a_2^{(1)}, a_1^{(2)}, a_3^{(1)}, a_2^{(2)}, a_1^{(3)}, a_4^{(1)}, \dots$$

(plot a diagram). This establishes a bijection $f : \mathbb{N} \to A$. ∎

Again, if the sets A_n are not disjoint, the above proof needs to be adjusted a little.

Example. The set $\mathbb{Q}$ is countable, as

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} \tfrac{1}{n}\mathbb{Z}.$$

Theorem 15.7. *For any non-empty set X, the set of all subsets of X (usually denoted by 2^X) cannot have the same cardinality as X.*

Proof. Assume $f : X \to 2^X$ is a bijection. Consider $S = \{x \in X \mid x \notin f(x)\} \subset X$. Let $s \in X$ be such that $f(s) = S$. Is $s \in S$? Both answers (yes and no) lead to a contradiction. ∎

Corollary 15.8. *The set $\mathbb{R}$ is uncountable.*

Proof. It suffices to show that the interval $(0, 1)$ is uncountable. Let us prove this by contradiction: assume that $(0, 1)$ is countable. It is easy to see that the interval $(0, 1)$ has the same cardinality as $2^{\mathbb{N}}$. Indeed, any real number from this interval can be written down uniquely in the form of a binary expansion $0.a_1a_2a_3\dots$ where $a_j \in \{0, 1\}$. Such a binary expansion uniquely defines a subset of $\mathbb{N}$. This establishes a bijection between $(0, 1)$ and $2^{\mathbb{N}}$. Thus, we obtain a bijection between $\mathbb{N}$ and $2^{\mathbb{N}}$, which contradicts the previous result. ∎