CCM115 Sequences and Series: Assignment 8

You should submit solutions to all of the problems on the list below that are marked with the symbol '†'. The deadline for submission of these solutions is

4pm, Monday 4th December.

(Submission is online, via the Keats pages for this course, and full instructions are available there). A selection of these problems will also be discussed in your tutorial this week.

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1. (*In this question we use the 'small o' notation from Definition 9.1 in the lecture notes - this notation will not occur in the examination, but is a bit of fun and not difficult!*) For each of the following statements, decide whether it is true or false and prove your claim (you are allowed to use any of the theorems from the lecture notes that you like, provided that you state clearly which results you are using).

   (a) $10^{-n} = o(1/n)$, as $n \to \infty$.

   (b) $n^2 + 5n + 3 = o(2^n)$, as $n \to \infty$.

   (c) $(-5)^n = o(4^n)$, as $n \to \infty$.

   (d)† $(n+1)^3 - n^3 = o(n^3)$, as $n \to \infty$.

   (e) $\dfrac{2^{-n} + 3^{-n}}{n - \frac{1}{2}} = o(2^{-n})$, as $n \to \infty$.

   (e) $\dfrac{n^2 - n}{n^2 + n} = o(1)$, as $n \to \infty$.

2. † Prove the following variant of the Sandwich Theorem. Let s_n, t_n be sequences such that $s_n \leq t_n$ for all n and $s_n \to +\infty$ as $n \to \infty$. Then $t_n \to +\infty$ as $n \to \infty$.

3. Prove, by induction (or otherwise), that for all $n \in \mathbb{N}$ and all $a \in \mathbb{R} \setminus \{1\}$, one has

$$1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}.$$

4. Let $a \in [0, 1]$. Using the theorem about bounded monotone sequences and the discussion of $\sum \frac{1}{k!}$ in the lecture notes, prove that the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{a^k}{k!}$$

   exists. Here we use the convention that $0! = 1$. (In fact, this limit equals e^a, but you don't need to prove this.)

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5. † Let $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$ for all $n \in \mathbb{N}$.

   (a) By induction prove that $s_n \leq 2 - \frac{1}{n}$ for all $n \geq 1$.

   (b) By using a general result about bounded monotone sequences (that you must state clearly), deduce that s_n is convergent.

6. Let the sequence a_n be defined by $a_1 = 1$ and $a_{n+1} = \frac{a_n^2+2}{2a_n}$ for $n \geq 1$. (Sequences defined in this way are said to be *iteratively, or recursively, defined.*) Prove that $a_n \to \sqrt{2}$. Proceed as follows:

   (a) By induction, prove that $a_n \geq \sqrt{2}$ for all $n > 1$. (Use the inequality $x^2 + y^2 \geq 2xy$).

   (b) Using the previous step prove, by induction, that $a_{n+1} \leq a_n$ for all $n > 1$.

   (c) Conclude that a_n converges.

   (d) Let $\ell = \lim_{n\to\infty} a_n$. Using the definition of a_n, prove that ℓ must satisfy $\ell = \frac{\ell^2+2}{2\ell}$.

   (e) Conclude that $a_n \to \sqrt{2}$ as $n \to \infty$.

   (f) Use your calculator to compute a_2, a_3, a_4. Compare with $\sqrt{2}$.

7. (a) † Identify two convergent subsequences of the sequence $s_n = \frac{1+(-1)^n}{2} + \frac{(-1)^n}{n}$.

   (b) Identify a convergent subsequence of the sequence $s_n = \cos(\pi(\sqrt{n} - n))$.

   (c) Identify a convergent subsequence of the sequence $s_n = (-1)^{\frac{n(n+1)}{2}}$.