FINITE GROUPS OF LIE TYPE

Conjugacy Classes and Complex Characters

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A Wiley–Interscience Publication

Wiley Classics Library Edition Published 1993

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British Library Cataloguing in Publication Data

A catalogue record for this book is available from the British Library

ISBN 0 471 94109 3

Printed and bound in Great Britain by Biddles Ltd, Guildford and King's Lynn

Preface

Since the appearance in 1972 of my book *Simple Groups of Lie Type*, the theory of finite simple groups has undergone substantial advances. In the first place the classification of the finite simple groups is now believed to have been completed, so that it is now known that any finite noncyclic simple group is either an alternating group, a simple group of Lie type or one of 26 sporadic simple groups. This shows clearly the importance of the groups of Lie type in the theory of finite groups. In the second place there has been a major advance in the understanding of the irreducible representations over an algebraically closed field of characteristic zero of the finite groups of Lie type. This advance was achieved by Deligne and Lusztig, who constructed families of irreducible characters using the theory of l-adic cohomology, and who thereby proved the conjectures on irreducible characters which had earlier been formulated by Macdonald. Further advances have subsequently been made by Lusztig, so that the degrees of the irreducible representations of the finite groups of Lie type are now completely known. It therefore seemed desirable to write an exposition of the Deligne–Lusztig theory which would help to make this representation theory accessible to a wider circle of mathematicians, and this is the main aim of the present volume.

The finite groups of Lie type are considered in this book as the groups of fixed points of reductive algebraic groups over an algebraically closed field of prime characteristic under the action of a Frobenius map. In this way results from the theory of reductive algebraic groups can be brought to bear in developing the structure and representation theory of the finite groups of Lie type. Some understanding of the theory of reductive algebraic groups is therefore necessary before one can discuss the finite groups in detail. There are now a number of good expositions of the theory of linear algebraic groups, and so we have been content to give in chapter 1 a summary without proofs of the basic results which will be used subsequently, together with details of references where proofs can be found.

Both connected reductive algebraic groups and finite groups of Lie type satisfy the axioms, introduced by Tits, for groups with a (B, N)-pair. We have developed a theory of algebraic groups with a split (B, N)-pair in chapter 2, which is applicable to both connected reductive algebraic groups and to their finite subgroups of fixed points under a Frobenius map. This is convenient since it enables us to derive results about the algebraic groups and the finite groups at the same time. Many of the results here can be found in standard references and are therefore quoted without proof.

Before one can understand the irreducible characters of a finite group one must have a reasonable understanding of the conjugacy classes of the group. The conjugacy classes of the connected reductive algebraic groups and of the finite groups of Lie type are discussed in chapters 3, 4 and 5. Chapter 3 deals with semisimple classes, chapter 4 with the concepts of geometric conjugacy and duality and chapter 5 with unipotent classes.

Our account of the representation theory begins in chapter 6 with a discussion of the Steinberg character. The properties of this character can be derived using the methods already available at this point. However in chapter 7 we must bring into play the techniques of l-adic cohomology theory. The l-adic cohomology was introduced by M. Artin and Grothendieck principally in order to attack the Weil conjectures about the number of points on an algebraic variety over a finite field. It has been outstandingly successful both for this and for other purposes. We shall make use of twelve basic properties of the l-adic cohomology which we take as axioms. The development of the theory of l-adic cohomology to the point where these twelve statements can be proved is a lengthy and elaborate body of mathematics which is far beyond the scope of this volume, and is in fact very different in character from the topics discussed in this book. We have therefore been content with giving a definition of the l-adic cohomology groups, which has been placed in an appendix so as not to interrupt the main development in the text. We have also indicated references in which the proofs of the basic properties can be found. The Deligne–Lusztig theory based on the l-adic cohomology modules has been developed in chapters 7 and 8.

The irreducible characters of a finite group with a split (B, N)-pair can be divided into a number of series, ranging from the principal series to the discrete series, by means of Harish-Chandra's concept of cuspidal characters. We describe these ideas in chapter 9 and in chapter 10 explain a method due to Howlett and Lehrer for decomposing into irreducible components a character induced from a cuspidal character of some parabolic subgroup.

The Deligne–Lusztig theory gives a different way of dividing the irreducible characters into families, ranging from the semisimple characters to the unipotent characters. The theory leads to an elegant description of the semisimple characters. The unipotent characters are, however, less well understood. Although their degrees and some of their character values have been determined by Lusztig, they have to be investigated to some extent in a case by case manner, and it is probable that a number of the proofs at present available may be capable of improvement and so have yet to reach a final form. We have therefore included in chapter 12 a discussion of the unipotent characters in which the various results are stated without proof. Some results on representations of Coxeter groups which are needed for this purpose are proved in chapter 11.

Finally in chapter 13 a variety of detailed information on simple groups has been collected together, including information on the unipotent classes in the algebraic group, the irreducible characters of the Weyl group, the generic degrees and the unipotent characters of the finite groups of Lie type. It is hoped that the collection of this information in one place will prove useful to subsequent investigators.

The aim throughout the book has been to proceed at a fairly relaxed pace of exposition. The results have not always been stated in the greatest possible generality, nor always proved in the shortest possible manner. The main aim has been rather to achieve as great a degree of clarity as possible regarding the main concepts of the theory and the techniques of proof which are used. It is hoped that this will stand the reader in good stead in coming to terms with the literature on the groups of Lie type.

I wish to thank a number of mathematicians for the help and inspiration they have given me in writing this book. Above all I am grateful to George Lusztig for many conversations about the representation theory of groups of Lie type. These conversations extended over a number of years while we were colleagues at Warwick University. I am also most grateful to Nick Spaltenstein for reading in detail large parts of the manuscript, for making many helpful comments and suggestions for improvement and for detecting a number of errors in an earlier version. The manuscript was greatly improved as a result of his comments. I am also grateful to Bob Howlett for helpful comments on the parts of the manuscript dealing with (B, N)-pairs and the Howlett–Lehrer theory, and to Ian Macdonald for a number of useful suggestions.

Work on the book was begun during the academic year 1980–81 which I spent on Sabbatical Leave from Warwick University. I am grateful to Warwick University for granting me leave, and to the Massachusetts Institute of Technology, the Australian National University at Canberra, the Universities of Sydney and New South Wales and the University of Notre Dame for the support and hospitality which they gave me during the course of that year. Particular thanks are due to members of the Mathematics Department at the University of Notre Dame for encouraging me to give a lecture course on the Deligne–Lusztig theory, and to Warren Wong and other members of the audience for their helpful comments.

Finally I would like to thank Mrs Susan Tall for the excellent work she has done once again in typing the manuscript.

University of Warwick, R. W. CARTER
April 1984

Contents

Chapter 1

INTRODUCTION TO ALGEBRAIC GROUPS

We shall be concerned in this book with the complex representation theory of the finite groups of Lie type. The representation theory of these groups is closely connected to certain aspects of the structure theory of the groups. In order to understand the relevant structure theory it is convenient, indeed essential, to regard the finite groups of Lie type as subgroups of algebraic groups over an algebraically closed field of prime characteristic. The structure of the finite groups is then seen to be closely related to the structure of the algebraic groups. We shall therefore begin by giving an account of the theory of algebraic groups over an algebraically closed field. We shall state the relevant results without proof. Proofs can be found in the standard texts on algebraic groups, for example the books of Borel [1], Humphreys [6], Springer [18] and Borel *et al.* [1].

1.1 ALGEBRAIC VARIETIES

An algebraic group is a mathematical structure which is at the same time a group and an algebraic variety. We therefore begin by describing the basic properties of algebraic varieties. Before discussing algebraic varieties in general we concentrate attention on affine varieties.

Let K be an algebraically closed field and K^n be the vector space of n-tuples over K. The polynomial ring $K[x_1, \ldots, x_n]$ then gives rise to a ring of functions from K^n into K. For any subset S of $K[x_1, \ldots, x_n]$ we denote by $\mathscr{V}(S)$ the set of $v \in K^n$ such that $f(v) = 0$ for all $f \in S$. Then $\mathscr{V}(S) = \mathscr{V}(I)$ where I is the ideal of $K[x_1, \ldots, x_n]$ generated by S. A subset of K^n of form $\mathscr{V}(I)$ for some ideal I is called an affine variety. Given an affine variety V in K^n we define its ideal $\mathscr{I}(V)$ to be the set of all $f \in K[x_1, \ldots, x_n]$ with $f(v) = 0$ for all $v \in V$. Thus every ideal I determines an affine variety $\mathscr{V}(I)$ and every affine variety V determines an ideal $\mathscr{I}(V)$.

The relation between the operations $\mathscr{V}$, $\mathscr{I}$ is as follows. It is easy to see that

$\mathscr{V}(\mathscr{I}(V)) = V$ for any affine variety V. It is not, however, true that $\mathscr{I}(\mathscr{V}(I)) = I$ for any ideal I. Rather one has $\mathscr{I}(\mathscr{V}(I)) = \sqrt{I}$, the radical of I. $\sqrt{I}$ is the set of all $f \in K[x_1, \ldots, x_n]$ such that $f^e \in I$ for some positive integer e. It follows that an ideal I has the form $\mathscr{I}(V)$ for some affine variety V if and only if $I = \sqrt{I}$. Such ideals are called radical ideals. The operations $\mathscr{V}, \mathscr{I}$ then give inverse bijections between the radical ideals of $K[x_1, \ldots, x_n]$ and the affine varieties in K^n.

The ring $K[V] = K[x_1, \ldots, x_n]/\mathscr{I}(V)$ is called the coordinate ring of V. It may be regarded as a ring of functions from V to K. $K[V]$ is a finitely generated K-algebra with no nilpotent elements. Conversely any finitely generated K-algebra R without nilpotent elements is isomorphic to $K[V]$ for some affine variety V in K^n for some n. Moreover the elements of V are in bijective correspondence with the maximal ideals of R. Two different affine varieties with the same coordinate ring may therefore be regarded as isomorphic and there is a bijective correspondence between isomorphism classes of affine varieties and finitely generated K-algebras with no nilpotent elements.

If V is an affine variety then the affine subvarieties of V form the closed sets in a topology called the Zariski topology. A topological space is said to be irreducible if it cannot be expressed as the union of two proper closed subsets. V is irreducible if and only if its coordinate ring $K[V]$ is an integral domain. Every affine variety V is the union of finitely many irreducible closed subsets and this decomposition is unique provided we assume that no irreducible component is contained in any other.

If V is an irreducible affine variety its coordinate ring $K[V]$, being an integral domain, can be embedded in its quotient field $K(V)$. $K(V)$ is called the function field of V. The dimension of V is then defined to be the transcendence degree of $K(V)$ over K. Every irreducible affine variety of dimension n has a maximal chain of irreducible subvarieties

$$V = V_n \supset V_{n-1} \supset \ldots \supset V_1 \supset V_0$$

and any other maximal chain of irreducible subvarieties of V has the same number of terms as this one.

Every affine variety has a sheaf of K-valued functions. This is defined as follows. Let X be a topological space and suppose that for each open subset U of X we have a K-algebra O_U of functions from U to K. The set $\{O_U\}$ is called a sheaf of K-valued functions on X if the following conditions are satisfied:

(i) If $U_1 \subseteq U_2$ are open subsets of X and $f \in O_{U_2}$ then $f|_{U_1} \in O_{U_1}$.

(ii) If $U = \bigcup_i U_i$ where U, U_i are open subsets of X and if $f: U \to K$ satisfies $f|_{U_i} \in O_{U_i}$ for each i, then $f \in O_U$.

An affine variety V may be given a sheaf of K-valued functions in the following way. For each open subset U of V we define O_U by

$O_U = \{f: U \to K$; For each $v \in U$ there exists an open subset $U(v)$ of U containing v and functions $g, h \in K[V]$ such that h does not vanish at any point of $U(v)$ and $f = g/h$ on $U(v)\}$.

The set $\{O_U\}$ is then a sheaf of K-valued functions on V. We have $O_V = K[V]$.

The general concept of an algebraic variety is broader than that of an affine variety. In order to explain it we shall first define an intermediate concept, that of a prevariety. A topological space is called Noetherian if it satisfies the maximal condition for open sets. A Noetherian space has only finitely many maximal irreducible subspaces. They are called the irreducible components of the space.

A prevariety over K is a Noetherian topological space X with a sheaf of K-valued functions which can be expressed as the union of a finite number of open subsets each isomorphic to some affine variety. The isomorphisms considered here are isomorphisms of topological spaces with a sheaf of K-valued functions. It is clear that every affine variety is a prevariety. There are, however, prevarieties which are not affine. These include the projective varieties which we shall now describe.

Let $P_n(K)$ be the set of one-dimensional subspaces of K^{n+1}. If v is a nonzero element of K^{n+1} we write $[v]$ for the one-dimensional subspace containing v. We shall consider $K[x_0, x_1, \ldots, x_n]$ as the polynomial ring giving rise to the ring of polynomial functions on K^{n+1}. Given $f \in K[x_0, x_1, \ldots, x_n]$ we may have $f(v) = 0$ but $f(\lambda v) \neq 0$ for $\lambda \in K$. However if f is homogeneous we have $f(\lambda v) = \lambda^m f(v)$ where m is the degree of f. Thus $f(v) = 0$ if and only if $f(\lambda v) = 0$. Thus for homogeneous polynomials the statement $f[v] = 0$ is meaningful.

A subset S of $K[x_0, x_1, \ldots, x_n]$ is called a homogeneous subset if $f \in S$ implies that each homogeneous component of f lies in S. For each homogeneous subset S we define a subset $\mathscr{V}_p(S)$ of $P_n(K)$ by

$$\mathscr{V}_p(S) = \{[v] \in P_n(K);\ f[v] = 0 \text{ for all homogeneous } f \in S\}.$$

Sets of the form $\mathscr{V}_p(S)$ are called projective varieties. For any projective variety V in $P_n(K)$ we may define an ideal $\mathscr{I}_p(V)$ to be the ideal of $K[x_0, x_1, \ldots, x_n]$ generated by all homogeneous polynomials of positive degree vanishing on V. $\mathscr{I}_p(V)$ is a homogeneous subset of $K[x_0, x_1, \ldots, x_n]$ and satisfies $\mathscr{V}_p\mathscr{I}_p(V) = V$. Thus every projective variety in $P_n(K)$ has the form $\mathscr{V}_p(I)$ for some homogeneous ideal I of $K[x_0, x_1, \ldots, x_n]$. However not every homogeneous ideal I has the form $\mathscr{I}_p(V)$ for some projective variety V in $P_n(K)$. I will have this form if and only if $I = \sqrt{I}$ and $I \neq K[x_0, x_1, \ldots, x_n]$. Thus the operators $\mathscr{V}_p$, $\mathscr{I}_p$ give inverse bijections between the homogeneous radical ideals other than $K[x_0, x_1, \ldots, x_n]$ and the projective varieties in $P_n(K)$.

The projective subvarieties of a projective variety V again form the closed sets in a topology on V—the Zariski topology. V is irreducible if it cannot be expressed as the union of two proper closed subsets, and this is so if and only if $\mathscr{I}_p(V)$ is a prime ideal of $K[x_0, x_1, \ldots, x_n]$. The ring $R = K[x_0, x_1, \ldots, x_n]/\mathscr{I}_p(V)$ is called the homogeneous coordinate ring of V. We have $R = \oplus_i R_i$ where R_i consists of images in R of homogeneous polynomials of degree i. Moreover $R_i R_j \subseteq R_{i+j}$ and so R has the structure of a graded ring.

We now define a sheaf of K-valued functions on a projective variety V. For each open subset U of V we define O_U by

$O_U = \{f: U \to K;$ For each $v \in U$ there exists an open subset $U(v)$ of U

containing v and functions $g, h \in R_i$ for some i such that h does not vanish at any point of $U(v)$ and $f = g/h$ on $U(v)\}$.

Thus f lies in O_U if and only if there is an open neighbourhood of each point in U on which f can be expressed as the quotient of two homogeneous polynomials of the same degree. If U_1, U_2 are open subsets of V with $U_1 \subseteq U_2$ we have a K-algebra homomorphism $\rho_{U_2, U_1}: O_{U_2} \to O_{U_1}$ obtained by restricting a K-valued function from U_2 to U_1. Then the K-algebras O_U and the homomorphisms ρ_{U_2, U_1} satisfy the axioms for a sheaf of K-valued functions on V. Moreover V satisfies the axioms of a prevariety. The finite covering of V by open subsets which are isomorphic to affine varieties is obtained as follows. Let

$$P_i = \{[v] \in P_n(K);\ v = (v_0, v_1, \ldots, v_n) \text{ with } v_i \neq 0\}.$$

P_i is an open subset of $P_n(K)$ and $P_n(K) = P_0 \cup P_1 \cup \ldots \cup P_n$. Thus $V \cap P_i$ is open in V and $V = (V \cap P_0) \cup (V \cap P_1) \cup \ldots \cup (V \cap P_n)$. Each $V \cap P_i$ which is non-empty is isomorphic to some affine variety.

Thus both affine varieties and projective varieties are examples of prevarieties. We now wish to give the definition of an algebraic variety in general. Before doing so we make some remarks on products of prevarieties. If X and X' are prevarieties their product $X \times X'$ can also be made into a prevariety in a natural way. We first observe that this is true of affine varieties. If V, V' are affine varieties then $V \times V'$ can be made into an affine variety. For if R, R' are the coordinate rings of V, V' then R, R' are finitely generated K-algebras with no nilpotent elements and the same will then be true of the tensor product $R \otimes_K R'$. This serves as the coordinate ring of the product variety $V \times V'$. Moreover if V and V' are irreducible so is $V \times V'$. Now consider more generally two prevarieties X, X'. Then the product $X \times X'$ can be given the structure of a prevariety in just one way such that the following condition is satisfied.

Given any isomorphisms $\phi: V \to U$, $\phi': V' \to U'$ where V, V' are affine varieties and U, U' are open subsets of X, X' respectively, the map $\phi \times \phi': V \times V' \to U \times U'$ is an isomorphism between the affine variety $V \times V'$ and the open subset $U \times U'$ of $X \times X'$. (The isomorphisms considered here are, as usual, isomorphisms of topological spaces with a sheaf of K-valued functions). Thus the product of two prevarieties can be regarded as a prevariety.

Having defined the product of two prevarieties we can now introduce the idea of an algebraic variety. An algebraic variety is a prevariety X such that the set $\Delta(X) = \{(x, x); x \in X\}$ is closed in the prevariety $X \times X$. It can be shown that if X is a prevariety in which every pair of points lies in a common open affine subset then X is a variety. This criterion makes it easy to see that both affine varieties and projective varieties are algebraic varieties in the sense defined above.

A morphism of algebraic varieties is a map $\phi: X \to X'$ such that ϕ is continuous and for each open subset U' of X' and each $f \in O_{U'}$ we have $f \circ \phi \in O_{\phi^{-1}(U')}$. An isomorphism of algebraic varieties is a bijective map ϕ such that ϕ and ϕ^{-1} are both morphisms.

If X is an algebraic variety and Y a closed subset of X then Y has a natural structure of an algebraic variety also. This follows from the way X is constructed as a union of open subsets which are affine varieties and from the fact that every closed subset of an affine variety is also an affine variety.

However it is also true that every open subset Y of an algebraic variety X has a natural structure of an algebraic variety. This is not so obvious as in the case of closed subsets. We first explain why every open subset of an affine variety has a natural structure of an algebraic variety (not necessarily affine). Let V be an affine variety and $f \in K[V]$ be a function in its coordinate ring. Let V_f be the subset of V given by

$$V_f = \{v \in V;\ f(v) \neq 0\}.$$

V_f is called a principal open subset of V. There is a natural bijection between V_f and the set

$$\{(v, \lambda) \in V \times K;\ \lambda f(v) = 1\}.$$

This latter set is clearly an affine variety and this enables us to give to V_f the structure of an affine variety. Thus each principal open subset of an affine variety may be regarded as an affine variety. The affine variety structure is in fact independent of the choice of f.

Now let U be any open subset of V. Then the complement $V - U$ is an affine variety, so there exist $f_1, \ldots, f_r \in K[V]$ with

$$V - U = \{v \in V;\ f_1(v) = 0, \ldots, f_r(v) = 0\}.$$

Hence

$$U = V_{f_1} \cup V_{f_2} \cup \ldots \cup V_{f_r}.$$

Thus we have an expression of U as a union of a finite number of open subsets each of which is an affine variety. This gives to U the structure of an algebraic variety. Thus every open subset of an affine variety may be given the structure of an algebraic variety.

Now let X be an algebraic variety and Y be an open subset of X. X is the union of finitely many open subsets U_i which are affine varieties. $Y \cap U_i$ is an open subset of U_i and so $Y \cap U_i$ is the union of finitely many principal open subsets U_{ij} of U_i. Each U_{ij} is an affine variety. Thus Y is the union of finitely many open subsets U_{ij} which are affine varieties, and this gives to Y the structure of an algebraic variety.

Thus both the closed subsets and the open subsets of an algebraic variety can themselves be regarded as algebraic varieties in a natural way. A subset of a topological space is called locally closed if it is the intersection of an open set and a closed set. A subset is locally closed if and only if it is open in its closure. Thus every locally closed subset of an algebraic variety X will inherit from X the structure of an algebraic variety.

If X is an irreducible variety covered by non-empty open subsets U_i which are affine varieties we shall have $U_i \cap U_j \neq \phi$ if $i \neq j$ since X is irreducible. It follows that $K(U_i) = K(U_j)$. Thus all the open affine subsets of X have the same function field. This is called the function field $K(X)$ of X. The dimension of an irreducible algebraic variety is defined as the transcendence degree of $K(X)$ over K. The dimension of an arbitrary algebraic variety (not necessarily irreducible) is the maximum of the dimensions of its irreducible components.

A general introduction to the theory of algebraic varieties can be found, for example, in the article of Serre [1].

1.2 ALGEBRAIC GROUPS

An algebraic group over K is a set G, which is an algebraic variety over K and also a group, such that the maps $\underset{(x,y)\to xy}{G \times G \to G}$ and $\underset{x \to x^{-1}}{G \to G}$ are morphisms of varieties. If the variety of G is affine then G is called an affine algebraic group.

Let G, G' be algebraic groups. A map $\alpha : G \to G'$ is called a homomorphism of algebraic groups if α is a morphism of varieties and also a homomorphism of groups. α is an isomorphism of algebraic groups if α is bijective and both α, α^{-1} are homomorphisms of algebraic groups.

If G is an algebraic group and H is a closed subgroup of G then H will also be an algebraic group. If G_1 and G_2 are algebraic groups then the direct product $G_1 \times G_2$ will also be an algebraic group.

We can obtain useful examples of algebraic groups as groups of nonsingular matrices over K. Let $GL_n(K)$ be the group of all nonsingular $n \times n$ matrices over K. Thus

$$GL_n(K) = \{(a_{ij}) \in K^{n^2}; \det(a_{ij}) \neq 0\}.$$

$GL_n(K)$ is called the general linear group of degree n over K. It may be regarded as a subset of K^{n^2}; however, as such it is not a closed subset. Being a principal open subset of K^{n^2} it can nevertheless be regarded as an affine algebraic variety as follows:

$$GL_n(K) = \{(a_{11}, \ldots, a_{nn}, b) \in K^{n^2+1}; b \det(a_{ij}) = 1\}.$$

This closed subset of K^{n^2+1} is an affine variety. Thus $GL_n(K)$ may be regarded as an affine algebraic variety as well as a group, and it satisfies the axioms of an affine algebraic group.

Any closed subgroup of $GL_n(K)$ will therefore also be an affine algebraic group. This gives a plentiful source of examples of affine algebraic groups. Closed subgroups of $GL_n(K)$ for various values of n are called linear algebraic groups. Thus every linear algebraic group is affine. In fact, however, the converse holds also. Every affine algebraic group is isomorphic to a closed subgroup of $GL_n(K)$ for some n. Thus the concepts of affine algebraic group and linear algebraic group coincide. We shall usually call such groups linear algebraic groups.

Let G be a linear algebraic group. As a topological space G will be expressible as the disjoint union of its connected components. As an affine variety G will be expressible as the union of finitely many irreducible components. In fact, however, these two decompositions coincide. The connected components of G as topological space coincide with the irreducible components of G as affine variety. There are thus only finitely many such components and they are disjoint. Let G^0 be the component containing the identity $1 \in G$. Then G^0 is a closed normal subgroup of G of finite index. The components of G are just the cosets G^0x of G with respect to G^0. G^0 is a connected linear algebraic group. It is called the connected component of G.

We now give two examples of linear algebraic groups which are of dimension 1. Consider the subgroup of $GL_2(K)$ of all matrices of form

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \qquad \lambda \in K.$$

This is clearly a closed subgroup of $GL_2(K)$. It is isomorphic as a group to the additive group of the field K, since

$$\begin{pmatrix} 1 & \lambda_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \lambda_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda_1 + \lambda_2 \\ 0 & 1 \end{pmatrix}.$$

An algebraic group isomorphic to this one is called the additive group and is denoted by $\mathbf{G}_a$. We have $\dim \mathbf{G}_a = 1$.

Now consider the group $GL_1(K)$. This consists of all matrices of form (λ) where $\lambda \neq 0$. It is isomorphic as a group to the multiplicative group K^* of nonzero elements of K. An algebraic group isomorphic to $GL_1(K)$ is called the multiplicative group and denoted by $\mathbf{G}_m$. We have $\dim \mathbf{G}_m = 1$.

We now mention some useful results about subgroups and homomorphisms between linear algebraic groups. If G is a linear algebraic group and H is any subgroup of G then its closure $\bar{H}$ is also a subgroup of G. If H_1 and H_2 are closed subgroups of G and H_2 lies in the normalizer of H_1 then H_1H_2 is also a closed subgroup of G. If G is generated by a set of closed connected subgroups H_i then G itself must be connected. If $\phi: G \to G'$ is a homomorphism of linear algebraic groups then the kernel of ϕ is a closed normal subgroup of G and the image of ϕ is a closed subgroup of G'. We also have

$$\dim(\ker \phi) + \dim(\operatorname{im} \phi) = \dim G.$$

A homomorphism from G into $GL_n(K)$ is called a rational representation of G. We have seen that every affine algebraic group G has a faithful rational representation.

1.3 THE TANGENT SPACE AND THE LIE ALGEBRA

In this section we shall show how one can associate to each linear algebraic group G of dimension n a Lie algebra $\mathfrak{L}(G)$, also of dimension n. Before doing so we describe some properties of the tangent space to an irreducible affine variety at a point on the variety. The Lie algebra $\mathfrak{L}(G)$ will then be obtained as the tangent space to G at the identity element.

Let V be an affine variety and suppose V is a closed subset of affine space K^n. Let $a \in K^n$. We say that a is a tangent vector to V at a point $v \in V$ if $d/dt\, f(v + ta) = 0$ at $t = 0$ for all $f \in \mathscr{I}(V)$. This means intuitively that the line joining v to $v + a$ is a tangent line to V at v. Let $\phi(t) = f(v + ta)$. We know that $\phi(0) = 0$ and a will be a tangent vector at v if and only if $\phi'(0) = 0$. Let $v = (v_1, \ldots, v_n)$ and $a = (a_1, \ldots, a_n)$. Then

$$\phi(t) = f(v_1 + ta_1, \ldots, v_n + ta_n).$$

We put $x_i = v_i + ta_i$. Then we have

$$\phi'(t) = a_1 \frac{\partial f}{\partial x_1} + \ldots + a_n \frac{\partial f}{\partial x_n}$$

and

$$\phi'(0) = \left[a_1 \frac{\partial f}{\partial x_1} + \ldots + a_n \frac{\partial f}{\partial x_n} \right](v).$$

Let $D_a : K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]$ be the map given by

$$D_a = a_1 \frac{\partial}{\partial x_1} + \ldots + a_n \frac{\partial}{\partial x_n}.$$

Then a is a tangent vector to V at v if and only if $D_a(\mathscr{I}(V)) \subseteq \mathscr{I}(\{v\})$. Now $K[x_1, \ldots, x_n]/\mathscr{I}(V) = K[V]$ and $K[x_1, \ldots, x_n]/\mathscr{I}(\{v\})$ is isomorphic to K. It is natural to consider K as a $K[x_1, \ldots, x_n]$-module here, with $f \in K[x_1, \ldots, x_n]$ acting on K as multiplication by $f(v)$. We write K_v for K regarded as a $K[x_1, \ldots, x_n]$-module in this way. A point-derivation of V at v is a K-linear map $D : K[V] \to K_v$ such that

$$D(fg) = Df . g(v) + f(v) . Dg$$

for all $f, g \in K[V]$. We see that any tangent vector a to V at v determines a point-derivation $D_a : K[V] \to K_v$. Furthermore these maps D_a are the only point-derivations of V at v. We therefore define the tangent space $T_v(V)$ to be the space $\mathrm{Der}(K[V], K_v)$ of point-derivations of V at v. This is a vector space over K.

Let $O_{V,v}$ be the localization of $K[V]$ at the ideal $\mathscr{I}(\{v\})$. $O_{V,v}$ is a local ring whose elements may be regarded as quotients g/h with $g, h \in K[V]$ and $h(v) \neq 0$. g_1/h_1, g_2/h_2 represent the same element of $O_{V,v}$ if and only if $g_1 h_2 = g_2 h_1$ on some open subset containing v.

Now any point-derivation $D : K[V] \to K$ can be extended uniquely to a point-derivation $D : O_{V,v} \to K$ by the formula

$$D(g/h) = \frac{Dg . h(v) - g(v) . Dh}{h(v)^2}.$$

Let m_v be the maximal ideal of the local ring $O_{V,v}$. Then D will vanish on m_v^2, for if $f = \sum_i m_i n_i$ with $m_i, n_i \in M_v$ then

$$Df = \sum (Dm_i . n_i(v) + m_i(v) . Dn_i) = 0$$

since $m_i(v) = n_i(v) = 0$. Also $D(1) = 0$ and D induces a linear map $m_v/m_v^2 \to K$. Conversely any K-linear map from m_v/m_v^2 into K can be extended uniquely to a point-derivation $D: O_{V,v} \to K$. Thus the tangent space $T_v(V)$ is canonically isomorphic to the dual space $\mathrm{Hom}(m_v/m_v^2, K)$ of m_v/m_v^2.

One can show that $\dim T_v(V) \geqslant \dim V$ for each point $v \in V$. v is defined to be a simple point of V if $\dim T_v(V) = \dim V$. The simple points form a non-empty open subset of V. V is called smooth, or nonsingular, if all its points are simple. Points on a variety which are not simple are called singularities.

Suppose now that we have two irreducible affine varieties V, V' and that $\phi: V \to V'$ is a morphism between them. ϕ determines a homomorphism ϕ^* between the coordinate rings $\phi^*: K[V'] \to K[V]$ given by

$$(\phi^* f)(x) = f(\phi(x))$$

for all $f \in K[V']$, $x \in V$. ϕ^* in turn gives rise to a map $(\mathrm{d}\phi)_v$ from $T_v(V)$ to $T_{v'}(V')$ where v is a point of V and $v' = \phi(v)$. $(\mathrm{d}\phi)_v$ is defined by

$$(\mathrm{d}\phi)_v D = D \circ \phi^*$$

where $D: K[V] \to K$ lies in $T_v(V) = \mathrm{Der}(K[V], K_v)$. $(\mathrm{d}\phi)_v$ is a linear map from the tangent space of V at v to the tangent space of V' at v'. It is called the differential of ϕ at v.

It is often useful to know when the differential $(\mathrm{d}\phi)_v$ is surjective. The following concepts are relevant in this connection. $\phi: V \to V'$ is called dominant if $\phi(V)$ is a dense subset of V'. If this is so we obtain a natural injective map of function fields $\phi^*: K(V') \to K(V)$. Thus $K(V)$ can be regarded as a field extension of $K(V')$. We say that a dominant morphism ϕ is separable if $K(V)$ is separably generated over $K(V')$, viz. $K(V)$ is a separable extension of a purely transcendental extension of $K(V')$. (This is always true, for example, for fields of characteristic 0.)

Suppose now that v, v' are simple points of V, V' respectively and that $\phi(v) = v'$. Suppose the differential $(\mathrm{d}\phi)_v$ is surjective. Then the morphism ϕ is dominant and separable. Now suppose conversely that $\phi: V \to V'$ is dominant and separable. Then there is a non-empty open subset U of V such that, for all $v \in U$, $\phi(v)$ is a simple point of V' and $(\mathrm{d}\phi)_v$ is surjective.

A useful property of dominant morphisms is that if $\phi: V \to V'$ is dominant then $\phi(V)$ contains some non-empty open subset of V'. There is in fact a non-empty open subset S of V' such that $\dim \phi^{-1}(s) = \dim V - \dim V'$ for all $s \in S$.

We now apply these facts about tangent spaces and differentials between them to the theory of linear algebraic groups. Let G be a connected linear algebraic group. Thus G is in particular an irreducible affine variety. For each $x \in G$ the right multiplication $g \to gx$ is a morphism of the variety G into itself. This map is invertible and its inverse $g \to gx^{-1}$ is also a morphism. Thus $g \to gx$ is an automorphism of G as an affine variety. Now every element of G can be obtained from every other element by a right multiplication of this kind. Since G contains a simple point and the image of a simple point under an automorphism is simple it follows that each point of G is simple, and so G is a smooth variety.

We consider the coordinate ring $K[G]$. If $f \in K[G]$ and $x \in G$ the map $f^x: G \to K$ defined by $f^x(t) = f(tx)$ lies in $K[G]$ since right multiplication by x is a morphism of G. Let $\alpha_x: K[G] \to K[G]$ be defined by $\alpha_x(f) = f^x$. Then α_x is a K-algebra automorphism of $K[G]$. Furthermore we have $\alpha_{xy} = \alpha_x \alpha_y$ and so we have a homomorphism from G into the group of K-algebra automorphisms of $K[G]$.

A linear map $D: K[G] \to K[G]$ is called a derivation if

$$D(f_1 f_2) = Df_1 . f_2 + f_1 . Df_2$$

for all $f_1, f_2 \in K[G]$. The set of derivations $\mathrm{Der}\, K[G]$ forms a Lie algebra under the Lie multiplication $[D_1\, D_2] = D_1 D_2 - D_2 D_1$. A derivation $D \in \mathrm{Der}\, K[G]$ is said to be invariant if $D(f^x) = (Df)^x$ for all $f \in K[G]$ and $x \in G$. This condition is equivalent to $D\alpha_x = \alpha_x D$ for all $x \in G$. The set $(\mathrm{Der}\, K[G])^G$ of invariant derivations forms a Lie subalgebra of $\mathrm{Der}\, K[G]$. If K has characteristic p and D is an invariant derivation, then D^p is an invariant derivation also. This gives the Lie algebra of invariant derivations the structure of a restricted Lie algebra.

Now we may define a map

$$(\mathrm{Der}\, K[G])^G \to \mathrm{Der}(K[G], K_1)$$

from invariant derivations of $K[G]$ to point derivations of G at 1 as follows. Let D be an invariant derivation of $K[G]$. Then the map $f \to Df(1)$ is a point-derivation of G at the identity. Moreover this map from invariant derivations to point-derivations at 1 is an isomorphism of vector spaces. Since $(\mathrm{Der}\, K[G])^G$ has a Lie algebra structure we may give a Lie algebra structure to $\mathrm{Der}\,(K[G], K_1)$ also using the above isomorphism. Thus the tangent space $T(G)_1$ to G at the identity has a Lie algebra structure. This Lie algebra is denoted by $\mathfrak{L}(G)$ and is called the Lie algebra of the linear algebraic group G. We have $\dim \mathfrak{L}(G) = \dim G$ since G is a smooth variety. One can also define the Lie algebra of a linear algebraic group which is not connected, but then $\mathfrak{L}(G) = \mathfrak{L}(G^0)$.

Suppose now that $\phi: G \to G'$ is a homomorphism of linear algebraic groups. The differential $d\phi = (d\phi)_1 : \mathfrak{L}(G) \to \mathfrak{L}(G')$ is then a homomorphism of Lie algebras. Moreover if we have homomorphisms $\phi_1: G_1 \to G_2$ and $\phi_2: G_2 \to G_3$ then their differentials satisfy $d(\phi_2 \circ \phi_1) = d\phi_2 \circ d\phi_1$.

For each element x of a linear algebraic group G we have the inner automorphism $i_x: G \to G$ defined by $i_x(g) = xgx^{-1}$. Also $i_x \circ i_{x^{-1}}$ is the identity map. We consider the differential $di_x: \mathfrak{L}(G) \to \mathfrak{L}(G)$ and write $\mathrm{Ad}x = di_x$. Then $\mathrm{Ad}x \circ \mathrm{Ad}x^{-1}$ is the identity and so $\mathrm{Ad}x$ is an automorphism of $\mathfrak{L}(G)$. Since $i_x \circ i_y = i_{xy}$ we have $\mathrm{Ad}x \circ \mathrm{Ad}y = \mathrm{Ad}xy$. Thus the map $x \to \mathrm{Ad}x$ gives a homomorphism from G into the group of automorphisms of $\mathfrak{L}(G)$. This representation of the linear algebraic group G on its Lie algebra $\mathfrak{L}(G)$ is called the adjoint representation of G.

If we take $G = GL_n(K)$ then the Lie algebra of G is $\mathfrak{L}(G) = \mathfrak{gl}_n(K)$, the Lie algebra of all $n \times n$ matrices over K under Lie multiplication $[AB] = AB - BA$. The adjoint representation of G can be considered as a homomorphism from G

into the group $GL(\mathfrak{L}(G))$ of all nonsingular maps of $\mathfrak{L}(G)$ into itself. Its differential will then be a homomorphism from $\mathfrak{L}(G)$ into the Lie algebra [End $\mathfrak{L}(G)$] of all linear maps of $\mathfrak{L}(G)$ into itself under Lie multiplication. We write $\mathrm{d}(\mathrm{Ad}) = \mathrm{ad}$ and for $x \in \mathfrak{L}(G)$ we have $\mathrm{ad}x \in \mathrm{End}\ \mathfrak{L}(G)$. In fact one has $\mathrm{ad}x \,.\, y = [xy]$. The map ad is called the adjoint representation of the Lie algebra $\mathfrak{L}(G)$.

The map $B\colon \mathfrak{L}(G) \times \mathfrak{L}(G) \to K$ given by $B(x, y) = \mathrm{trace}\,(\mathrm{ad}x\ \mathrm{ad}y)$ is called the Killing form on $\mathfrak{L}(G)$. This form is preserved by the adjoint G-action in the sense that

$$B(\mathrm{Ad}g\,.\,x, \mathrm{Ad}g\,.\,y) = B(x, y)$$

for all $g \in G$, x, $y \in \mathfrak{L}(G)$.

We shall discuss later some further connections between a linear algebraic group and its Lie algebra.

1.4 THE JORDAN DECOMPOSITION

The Jordan decomposition of an element of a linear algebraic group or of its Lie algebra is of key importance in the theory of algebraic groups. We discuss it first in the context of linear transformations of a finite-dimensional vector space into itself.

Let V be a finite-dimensional vector space over K and $T \in \mathrm{End}\ V$ be a linear transformation of V. T is said to be semisimple if T is diagonalizable, i.e. if V has a basis consisting of eigenvectors of T. T is called nilpotent if $T^e = 0$ for some positive integer e. This is equivalent to the condition that all eigenvalues of T are 0. The additive Jordan decomposition for V states that, given $T \in \mathrm{End}\ V$, there exists a semisimple element $T_s \in \mathrm{End}\ V$ and a nilpotent element $T_n \in \mathrm{End}\ V$ such that $T = T_s + T_n$ and $T_s T_n = T_n T_s$. Moreover T_s and T_n are uniquely determined by these conditions. T_s and T_n are called the semisimple and nilpotent parts of T respectively.

The multiplicative Jordan decomposition for V deals with nonsingular maps T, i.e. elements of $GL(V)$. An element $T \in GL(V)$ is called unipotent if all its eigenvalues are equal to 1. The multiplicative Jordan decomposition states that given $T \in GL(V)$ there exists a semisimple element $T_s \in GL(V)$ and a unipotent element $T_u \in GL(V)$ such that $T = T_s T_u = T_u T_s$. Moreover T_s and T_u are uniquely determined by these conditions. T_s and T_u are called the semisimple and unipotent parts of T respectively.

The semisimple part T_s is the same for the additive and multiplicative Jordan decompositions of $T \in GL(V)$. The nilpotent and unipotent parts are related by the formula $T_u = 1 + T_s^{-1} T_n$.

Now let G be a linear algebraic group. Then G is isomorphic to a closed subgroup of $GL_n(K)$. Consider the condition for an element $x \in G$ to be represented by a semisimple element of $GL_n(K)$ under this isomorphism. This condition turns out to be independent of the embedding of G in $GL_n(K)$. (There will in general be many such embeddings.) Thus we can give an unambiguous definition of the semisimple elements of G. Similarly the condition for x to be represented by a unipotent element of $GL_n(K)$ is independent of the embedding, and we can thus define what it means for an element $x \in G$ to be unipotent. The Jordan decomposition for G states that for each $x \in G$ there exists a semisimple element $x_s \in G$ and a unipotent element $x_u \in G$ such that $x = x_s x_u = x_u x_s$. Moreover x_s and x_u are uniquely determined by this condition. x_s and x_u are called the semisimple and unipotent parts of x. They satisfy the condition that if $\phi: G \to G'$ is a homomorphism of algebraic groups then, for $x \in G$, we have $\phi(x)_s = \phi(x_s)$ and $\phi(x)_u = \phi(x_u)$.

Let $\mathfrak{L}(G)$ be the Lie algebra of G. As before we express G as a closed subgroup of $GL_n(K)$. Thus we have an injective homomorphism $\phi: G \to GL_n(K)$. This gives rise to an injective homomorphism of Lie algebras $d\phi: \mathfrak{L}(G) \to \mathfrak{gl}_n(K)$. Let $X \in \mathfrak{L}(G)$. Then the condition for $d\phi(X)$ to be a semisimple $n \times n$ matrix is independent of the embedding of G in $GL_n(K)$. Thus we can define in this way what it means for X to be semisimple. Similarly the condition for $d\phi(X)$ to be a nilpotent matrix is independent of the embedding, and such elements X are called nilpotent elements of $\mathfrak{L}(G)$. The Jordan decomposition for $\mathfrak{L}(G)$ states that for each $X \in \mathfrak{L}(G)$ there exists a semisimple element $X_s \in \mathfrak{L}(G)$ and a nilpotent element $X_n \in \mathfrak{L}(G)$ such that $X = X_s + X_n$ and $[X_s X_n] = 0$. Moreover X_s and X_n are uniquely determined by this condition. X_s and X_n are called the semisimple and nilpotent parts of X. They satisfy the condition that if $\phi: G \to G'$ is a homomorphism of algebraic groups and $d\phi: \mathfrak{L}(G) \to \mathfrak{L}(G')$ the corresponding homomorphism of Lie algebras then for $X \in \mathfrak{L}(G)$ we have $(d\phi(X))_s = d\phi(X_s)$ and $(d\phi(X))_n = d\phi(X_n)$.

1.5 ACTIONS OF ALGEBRAIC GROUPS ON VARIETIES

Let G be an algebraic group and X an algebraic variety, both over K. We say that G acts on X if there is a morphism $\underset{(g,x) \to g.x}{G \times X \to X}$ such that $g_1(g_2 x) = (g_1 g_2)x$, $1x = x$ for all $g_1, g_2 \in G$, $x \in X$. Each element $g \in G$ thus gives rise to an isomorphism $x \to g.x$ of X. We then say that X is a G-space. If G acts transitively on X then X is called a homogeneous G-space.

If X is any G-space we may define an equivalence relation on X, x_1 and x_2 being equivalent if and only if $x_2 = g.x_1$ for some $g \in G$. The equivalence classes are called the orbits of G on X.

One of the key results on orbits is the following. Suppose G is an algebraic group acting on a variety X. They every orbit C of G on X is open in its closure $\bar{C}$. Thus the orbits are locally closed subsets of X.

If X is a G-space and $x \in X$ then the set $H = \{g \in G;\ g.x = x\}$ is a closed subgroup of G called the stabilizer of x. If we take two elements of X in the same orbit under G their stabilizers will be conjugate subgroups of G. In particular if X is a homogeneous space all the stabilizers of points of X will be conjugate in G. If $x \in X$ lies in the G-orbit C and has stabilizer H then we have

$$\dim \bar{C} = \dim G - \dim H.$$

We next discuss quotient varieties of an algebraic variety under the action of an affine algebraic group. Let G be an affine algebraic group which acts on an algebraic variety X. An algebraic variety Y is called a categorical quotient of X with respect to G if there exists a morphism $\phi: X \to Y$ satisfying the conditions:

(i) $\phi(g.x) = \phi(x)$ for all $x \in X$, $g \in G$.
(ii) For any morphism $\psi: X \to Z$ such that $\psi(g.x) = \psi(x)$ for all $x \in X, g \in G$ there is a unique morphism $\delta: Y \to Z$ such that $\psi = \delta \circ \phi$.

A categorical quotient of X with respect to G does not always exist, but if it does exist then it is uniquely determined up to isomorphism. However even if a categorical quotient exists it may not have the geometrical properties which one might desire in a quotient variety. For this reason we introduce a more stringent condition.

An algebraic variety Y is called a strict quotient (or geometric quotient) of X with respect to G if it satisfies the following conditions:

(i) There is a surjective morphism $\phi: X \to Y$ whose fibres are the orbits of G on X.
(ii) U is open in Y if and only if $\phi^{-1}(U)$ is open in X.
(iii) $f \in O_U$ if and only if $f \circ \phi \in O_{\phi^{-1}(U)}{}^G$

where $O_{\phi^{-1}(U)}{}^G = \{h \in O_{\phi^{-1}(U)};\ h^g = h \text{ for all } g \in G\}$.

A strict quotient of X with respect to G will not always exist, but if it does exist it is uniquely determined up to isomorphism, and is a categorical quotient. It will be denoted by X/G.

A surjective morphism $\phi: X \to Y$ whose fibres are the orbits of G on X is called an orbit map. It is useful to have a criterion which will ensure that an orbit map $\phi: X \to Y$ also satisfies conditions (ii), (iii) above, so that Y is isomorphic to the strict quotient X/G. Such a criterion is as follows, and is a special case of proposition 6.6 of Borel [1]. Suppose $\phi: X \to Y$ is an orbit map between irreducible varieties. Then ϕ satisfies (ii), (iii) provided Y is smooth and ϕ is separable.

We mention two important special cases of actions of an affine algebraic group G on an algebraic variety X for which a strict quotient X/G exists. The first is when X is an affine variety and G is finite. Then the strict quotient X/G exists and is itself affine (Fogarty [1], p. 187). The second is when we consider G/H where G is an affine algebraic group, H is a closed subgroup of G, and H acts on G by left multiplication. The elements of the strict quotient G/H are the cosets Hg for $g \in G$. G/H is an algebraic variety of dimension $\dim G - \dim H$ which will not in general be affine.

If H is a closed normal subgroup of G we can say more. G/H has a group structure as well as the structure of an algebraic variety. In fact G/H is in these circumstances a linear algebraic group. Thus, in particular, the variety G/H is affine. Also the natural map $\pi: G \to G/H$ is a homomorphism of algebraic groups.

If H is a closed subgroup of G which is not normal the variety G/H will not in general be affine. We shall in fact encounter important examples in which the quotient variety is projective.

In the case when G is an affine algebraic group acting on an affine variety X there is a useful process known as linearization. One can show that there is a finite-dimensional G-module V over K and a closed G-invariant subset X' of V such that there is an isomorphism between the affine varieties X, X' preserving the G-action (see Slodowy [2], p. 2). Thus the G-action on any affine variety can be extended to a linear G-action on a vector space containing the variety as a closed subset.

1.6 SOLVABLE ALGEBRAIC GROUPS

We now begin an outline of the structure theory of linear algebraic groups. Suppose G is a connected linear algebraic group. We start with the simplest case, when $\dim G = 1$. Then G must be isomorphic to either the additive group $\mathbf{G}_a$ or the multiplicative group $\mathbf{G}_m$. In particular every connected linear algebraic group of dimension 1 is commutative. We next suppose that G is a connected commutative group. Then the set of semisimple elements G_s forms a closed subgroup of G and so does the set of unipotent elements G_u. Also the map $\underset{(s,u)\to su}{G_s \times G_u \to G}$ is an isomorphism of algebraic groups. Thus G is a direct product of two groups, one consisting entirely of semisimple elements and the other entirely of unipotent elements. Furthermore the group G_s is isomorphic to the direct product of a finite number of factors isomorphic to $\mathbf{G}_m$. A group of the form $\mathbf{G}_m \times \ldots \times \mathbf{G}_m$ is called an algebraic torus. Thus the semisimple elements of G form a closed subgroup which is a torus. An algebraic group will be called unipotent if all its elements are unipotent. Thus G_u is a unipotent group. Examples of commutative unipotent groups are given by direct products $\mathbf{G}_a \times \ldots \times \mathbf{G}_a$, but not every connected commutative unipotent group is of this form.

We suppose next that G is a connected nilpotent group. Then G has a series of normal subgroups

$$G = G_1 \supset G_2 \supset \ldots \supset G_c \supset G_{c+1} = 1$$

where $G_{i+1} = [G_i, G]$ for each i. Each G_i is a closed connected normal subgroup of G. As in the commutative case the sets G_s and G_u are both closed connected normal subgroups of G and the product map $G_s \times G_u \to G$ is an isomorphism of algebraic groups. Moreover the subgroup G_s is commutative. It is therefore a torus, and lies in the centre of G. The unipotent group G_u need not be commutative. Thus every connected nilpotent group is a direct product of a torus with a unipotent group. Conversely it is true that every unipotent algebraic group, whether connected or not, is nilpotent.

We next suppose that G is a connected solvable group. This means that G has a series of normal subgroups

$$G = G^{(0)} \supset G^{(1)} \supset G^{(2)} \supset \ldots \supset G^{(d)} = 1$$

where $G^{(i+1)} = [G^{(i)}, G^{(i)}]$ for each i. Each $G^{(i)}$ is a closed connected normal subgroup of G. Every nilpotent group is solvable but not conversely. As in the nilpotent case the subset G_u is a closed normal subgroup of G. However G_s need not now be a subgroup of G. So G is no longer a direct product of a unipotent group with a torus. It is however a semidirect product. We consider the maximal tori in G, viz. closed subgroups which are tori and not properly contained in larger tori. All the maximal tori in G turn out to be conjugate in G. If T is one of them then $G = G_u T$ and $G_u \cap T = 1$. T will not in general be normal in G. In fact we have $N_G(T) = C_G(T)$, i.e. the normalizer of T coincides with the centralizer of T. Thus G is the semidirect product of the unipotent group G_u and the torus T. Moreover any semisimple element of G lies in some maximal torus of G. Thus G_s is the union of all the maximal tori of G. The product map $G_u \times T \to G$ is a bijective map which, although not an isomorphism of algebraic groups, is an isomorphism of varieties.

A useful tool in proving these results on connected solvable groups is the Lie–Kolchin theorem. Let G be a connected solvable group and suppose G is a closed subgroup of $GL_n(K)$. Then the Lie–Kolchin theorem asserts that G is conjugate in $GL_n(K)$ to a subgroup of the group $T_n(K)$ of upper-triangular matrices. In particular $T_n(K)$ is a maximal connected solvable subgroup of $GL_n(K)$ and any other maximal connected solvable subgroup is conjugate to $T_n(K)$.

1.7 BOREL SUBGROUPS

We now turn to a discussion of connected linear algebraic groups which are not necessarily solvable. If G is such a group a Borel subgroup of G is defined as a maximal closed connected solvable subgroup of G. Borel subgroups always exist since the set of closed connected solvable subgroups will have maximal elements. For example, if G is the group $GL_n(K)$ then $B = T_n(K)$ is a Borel subgroup of G.

In order to understand the main properties of Borel subgroups we need the concept of a complete variety. This is a concept which distinguishes projective varieties from affine ones. An algebraic variety X is complete if and only if, for any variety Y, the projection morphism $X \times Y \to Y$ is closed, i.e. takes closed subsets into closed subsets. X is complete if and only if all the irreducible components of X are complete. Every closed subset of a complete variety is complete. If X and Y are complete varieties so is $X \times Y$. If $\phi: X \to Y$ is a morphism and X is complete then $\phi(X)$ is closed in Y and is complete. Every projective variety is complete, although the converse is not true. In contrast to this, if X is a connected affine variety which is complete then X must consist of a single point.

Now if B is a Borel subgroup of G it can be shown that the quotient variety G/B is projective, hence complete. In fact any subgroup P of G containing B will have the property that G/P is complete. Conversely, if P is a closed subgroup of G such that G/P is complete, then P contains some Borel subgroup of G.

It is also true that any two Borel subgroups of G are conjugate. A key result in proving this, called Borel's fixed point theorem, states that a connected solvable group acting on a complete variety always has a fixed point. This is applied to the situation in which one Borel subgroup acts by multiplication on the quotient variety of the other. It follows from the conjugacy of Borel subgroups that any two maximal tori of G are conjugate. For any maximal torus lies in some Borel subgroup, whereas inside a given Borel subgroup any two maximal tori are conjugate, as we saw in section 1.6.

Next consider the union of all the Borel subgroups of G. This can be shown to be the whole of G. Thus every element of G lies in some Borel subgroup. It follows that every semisimple element of G lies in some maximal torus of G. For such an element lies in a Borel subgroup, but every semisimple element of a solvable group lies in a maximal torus, and a maximal torus of a Borel subgroup is a maximal torus of the whole group G.

Finally, each Borel subgroup can be shown to be its own normalizer. It follows that there is a bijective map between the quotient variety G/B and the set of all conjugates of B, which is the set of all Borel subgroups of G. In this way the set $\mathfrak{B}$ of all Borel subgroups of G can be given the structure of a projective variety. This variety will be of considerable importance in the topics to be discussed subsequently.

1.8 SIMPLE, SEMISIMPLE AND REDUCTIVE GROUPS

Let G be a connected linear algebraic group. Then the set of closed connected solvable normal subgroups of G has a unique maximal element. This is the product of all closed connected solvable normal subgroups of G and is called the radical $R(G)$. Similarly the set of closed connected unipotent normal subgroups of G has a unique maximal element, called the unipotent radical $R_u(G)$. Now every unipotent group is nilpotent. Thus $R_u(G)$ is nilpotent, and so lies in $R(G)$.

G is called semisimple if $G \neq 1$ and $R(G) = 1$, and reductive if $R_u(G) = 1$. Every semisimple group is reductive but the converse is not true. For example a torus is a reductive group which is not semisimple.

Suppose that G is a connected reductive group. Let $G' = [G, G]$ be its commutator subgroup and Z be the centre of G. Let Z^0 be the connected component of Z. Z^0 is called the connected centre of G. Then we have a factorization $G = G'Z^0$. G' is a connected semisimple group and Z^0 is a torus. This is almost, but not quite, a direct product decomposition. G and Z^0 are both normal subgroups of G but $G' \cap Z^0$, instead of being 1 as in a direct product, can be a nontrivial finite group. We say therefore that G is an almost direct product of the semisimple group G' and the torus Z^0. An example of such a decomposition is given by $G = GL_n(K)$. Then $G' = SL_n(K)$ is the set of all matrices of determinant 1, and Z^0 consists of all scalar multiples of the identity. Thus $G' \cap Z^0$ consists of all scalar matrices λI_n for which $\lambda^n = 1$. This is a finite group which may not be the identity.

This decomposition shows that an understanding of the structure of reductive groups can largely be reduced to that of semisimple groups. So let G be a connected semisimple group. We can again obtain an almost direct decomposition of G, this time into a product of simple groups. G is said to be simple if G has no proper closed connected normal subgroups. Any proper normal subgroup of a simple algebraic group must be finite and must lie in the centre of the group. An example of a simple algebraic group is $SL_n(K)$. A connected semisimple group G has a finite set of closed normal subgroups $G_1, \ldots, G_k$ such that:

(i) each G_i is simple,
(ii) $[G_i, G_j] = 1$ if $i \neq j$,
(iii) $G = G_1 G_2 \ldots G_k$,
(iv) $G_i \cap G_1 \ldots G_{i-1} G_{i+1} \ldots G_k$ is finite for each i.

The G_i are uniquely determined by these conditions. They are called the simple components of the semisimple group G. Thus each connected semisimple group is an almost direct product of simple groups. The problem of understanding the structure of semisimple groups can in this way be largely reduced to that of simple groups.

1.9 ROOTS, COROOTS AND THE WEYL GROUP

Let G be a connected group and T be a maximal torus of G. Let $N(T)$, $C(T)$ be the normalizer and centralizer of T in G respectively. Then $N(T)/C(T)$ is a finite group. In fact $C(T)$ is a connected group and is the connected component $N(T)^0$. The group $W(T) = N(T)/C(T)$ is called the Weyl group of T. It is uniquely determined up to isomorphism since any two maximal tori of G are conjugate. The abstract group W isomorphic to each $W(T)$ is called the Weyl group of G. If G is reductive we have $C(T) = T$ and so $W(T) = N(T)/T$. The Weyl group W is a finite group of a very special kind, whose structure can best be explained by introducing the roots of G.

Let $X = \text{Hom}\,(T, \mathbf{G}_m)$ be the set of algebraic group homomorphisms from T to $\mathbf{G}_m$. X can be made into a group under the operation $\chi_1 + \chi_2$ defined by

$$(\chi_1 + \chi_2)(t) = \chi_1(t)\chi_2(t) \qquad \chi_1, \chi_2 \in X,\ t \in T.$$

X is called the character group of T. Its structure can be described as follows. Suppose first that $\dim T = 1$. Then T is isomorphic to $\mathbf{G}_m$ and we are considering the group $\text{Hom}(\mathbf{G}_m, \mathbf{G}_m)$. It is easy to see that the only algebraic homomorphisms from $\mathbf{G}_m$ to itself are the maps $\lambda \to \lambda^n$ where $n \in \mathbb{Z}$. Thus $\text{Hom}(\mathbf{G}_m, \mathbf{G}_m) \cong \mathbb{Z}$. In general we shall have $T \cong \mathbf{G}_m \times \ldots \times \mathbf{G}_m$ (r factors). Then we have

$$X = \text{Hom}(T, \mathbf{G}_m) \cong \text{Hom}(\mathbf{G}_m \times \ldots \times \mathbf{G}_m, \mathbf{G}_m) \cong \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}.$$

Thus X is a free abelian group of rank r.

Let $Y = \text{Hom}(\mathbf{G}_m, T)$ be the set of algebraic homomorphisms of $\mathbf{G}_m$ into T. Y can be made into a group under the operation $\gamma_1 + \gamma_2$ given by

$$(\gamma_1 + \gamma_2)(\lambda) = \gamma_1(\lambda)\gamma_2(\lambda) \qquad \gamma_1, \gamma_2 \in Y,\ \lambda \in K^*.$$

We then have

$$Y \cong \mathrm{Hom}(\mathbf{G}_m, \mathbf{G}_m \times \ldots \times \mathbf{G}_m)\ \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}.$$

Thus Y is also a free abelian group of rank r. Y is called the group of cocharacters (or one-parameter subgroups) of T.

We now define a map from $X \times Y$ into $\mathbb{Z}$ taking (χ, γ) to an integer $\langle \chi, \gamma \rangle \in \mathbb{Z}$. This integer is defined as follows. Since $\chi \in X$ and $\gamma \in Y$, $\chi \circ \gamma$ lies in $\mathrm{Hom}(\mathbf{G}_m, \mathbf{G}_m)$. Hence $(\chi \circ \gamma)(\lambda) = \lambda^n$ for some $n \in \mathbb{Z}$ for all $\lambda \in \mathbf{G}_m$. We define $\langle \chi, \gamma \rangle = n$. The map

$$X \times Y \to \mathbb{Z}$$

$$(\chi, \gamma) \to \langle \chi, \gamma \rangle$$

is nondegenerate and gives rise to a duality between X and Y. It gives isomorphisms between X and $\mathrm{Hom}(Y, \mathbb{Z})$ and between Y and $\mathrm{Hom}(X, \mathbb{Z})$.

The Weyl group can be made to act on both X and Y as follows. If $w \in W$ and $\chi \in X$ we define ${}^{w}\chi \in X$ by

$${}^{w}\chi(t) = \chi(t^{w}) \qquad t \in T.$$

Then $\chi \to {}^{w}\chi$ is an automorphism of X and we have ${}^{w'}({}^{w}\chi) = {}^{(w'w)}\chi$. If $\gamma \in Y$ we define $\gamma^{w} \in Y$ by

$$\gamma^{w}(\lambda) = \gamma(\lambda)^{w} \qquad \lambda \in \mathbf{G}_m.$$

Then $\gamma \to \gamma^{w}$ is an automorphism of Y and we have $(\gamma^{w})^{w'} = \gamma^{(ww')}$. The W-actions on X and Y are related by the formula

$$\langle \chi, \gamma^{w} \rangle = \langle {}^{w}\chi, \gamma \rangle \qquad \chi \in X, \gamma \in Y, w \in W.$$

Let G be a connected reductive group and T a maximal torus of G. We consider the Borel subgroups of G containing T. There are only finitely many of these, and they are conjugate under the action of $N(T)$. In fact their number is equal to the order of the Weyl group W. Let B be a Borel subgroup of G containing T. Then B has a semidirect product decomposition $B = UT$ where $U = R_u(B)$. G has a unique Borel subgroup B^- containing T such that $B \cap B^- = T$. B, B^- are called opposite Borel subgroups. We have $B^- = U^-T$ where $U^- = R_u(B^-)$. U and U^- are connected groups normalized by T satisfying $U \cap U^- = 1$. They are maximal unipotent subgroups of G.

We consider the minimal proper subgroups of U and U^- which are normalized by T. These are all connected unipotent groups of dimension 1, so are isomorphic to the additive group $\mathbf{G}_a$. T acts on each of them by conjugation, giving a homomorphism $T \to \mathrm{Aut}\,\mathbf{G}_a$ from T to the group of algebraic automorphisms of $\mathbf{G}_a$. However the only algebraic automorphisms of $\mathbf{G}_a$ are the maps $\lambda \to \mu\lambda$ for some $\mu \in K$ with $\mu \neq 0$. Thus $\mathrm{Aut}\,\mathbf{G}_a$ is isomorphic to $\mathbf{G}_m$. Hence each of our 1-dimensional unipotent groups determines an element of $\mathrm{Hom}(T, \mathbf{G}_m) = X$. The elements of X arising in this way are called the roots. They are all nonzero elements of X. Distinct 1-dimensional unipotent subgroups give rise to distinct roots. The roots form a finite subset Φ of X, which is independent of the choice of Borel subgroup B containing T. For each root $\alpha \in \Phi$ the 1-dimensional unipotent subgroup giving rise to it is denoted by X_α. The X_α are called the root subgroups of G. The roots arising from root subgroups in U^- are the negatives of the roots arising from root subgroups in U. We also have $G = \langle T, X_\alpha \alpha \in \Phi \rangle$.

Let α, $-\alpha$ be a pair of opposite roots. Then we consider the subgroup $\langle X_\alpha, X_{-\alpha} \rangle$ of G generated by the root subgroups X_α, $X_{-\alpha}$. This subgroup is a 3-dimensional simple group isomorphic to either $SL_2(K)$ or to $PGL_2(K) = GL_2(K)/\{\pm I\}$. In fact there is a homomorphism $\phi: SL_2(K) \to \langle X_\alpha, X_{-\alpha} \rangle$ such that

$$\phi\left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\} = X_\alpha \qquad \phi\left\{\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\right\} = X_{-\alpha}.$$

$\phi\left\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right\}$ is then a 1-dimensional subgroup of T. Let $\alpha^\vee$ be the homomorphism $\mathbf{G}_m \to T$ given by

$$\alpha^\vee(\lambda) = \phi\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Then $\alpha^\vee$ is an element of $\mathrm{Hom}(\mathbf{G}_m, T) = Y$ which is uniquely determined by α. $\alpha^\vee$ is called the coroot corresponding to the root α. α and $\alpha^\vee$ are related by the condition $\langle \alpha, \alpha^\vee \rangle = 2$. The coroots form a finite subset of Y denoted by $\Phi^\vee$.

We now consider in more detail the actions of the Weyl group on X and Y. W acts faithfully on both these lattices. Each element of W permutes the set of roots Φ in X and the set of coroots $\Phi^\vee$ in Y. For each root α consider the element $\phi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \langle X_\alpha, X_{-\alpha} \rangle$. This element lies in $N(T)$ and gives rise to an element $w_\alpha \in W = N(T)/C(T)$. w_α acts on X by

$$w_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha \qquad \chi \in X$$

and on Y by

$$w_\alpha(\gamma) = \gamma - \langle \alpha, \gamma \rangle \alpha^\vee \qquad \gamma \in Y.$$

We have $w_\alpha = w_{-\alpha}$ and $w_\alpha^2 = 1$. Moreover the elements w_α for all $\alpha \in \Phi$ generate W.

We have now constructed for each connected reductive group a quadruple $(X, \Phi, Y, \Phi^\vee)$ called a root datum. This means that X and Y are free abelian groups of the same finite rank with a nondegenerate map $X \times Y \to \mathbb{Z}$ which puts them into duality. Φ and $\Phi^\vee$ are finite subsets of X and Y respectively, and there is a bijection $\alpha \to \alpha^\vee$ between them satisfying $\langle \alpha, \alpha^\vee \rangle = 2$. Finally for each $\alpha \in \Phi$ we have maps $w_\alpha: X \to X$ and $w_\alpha: Y \to Y$ defined as above and satisfying $w_\alpha(\Phi) = \Phi$, $w_\alpha(\Phi^\vee) = \Phi^\vee$.

The basic classification theorem for connected reductive groups asserts that, given any root datum, there is a unique connected reductive group G over K which gives rise to this root datum in the manner described above.

Let Φ^+ be the set of roots arising from root subgroups of U and Φ^- be those arising from subgroups of U^-. Roots in Φ^+, Φ^- are called positive and negative roots respectively. Let Δ be the set of positive roots which cannot be expressed as a sum of two positive roots. Δ is called the set of simple roots. The simple roots are linearly independent. Let $|\Delta| = l$ and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. Then each root in Φ^+ has the form $\sum_{i=1}^{l} n_i\alpha_i$ where $n_i \in \mathbb{Z}, n_i \geqslant 0$ and each root in Φ^- has the form $\sum_{i=1}^{l} n_i\alpha_i$ where $n_i \in \mathbb{Z}$ and $n_i \leqslant 0$. If $\alpha = \sum n_i\alpha_i \in \Phi$ then the integer $\sum n_i$ is called the height of α. We have $W(\Delta) = \Phi$, so that each root is the image of some simple root under an element of the Weyl group. Also there is a unique element $w_0 \in W$ such that $w_0(\Phi^+) = \Phi^-$.

The element w_{α_i} of W will also be denoted by s_i. The Weyl group W, which we previously observed is generated by the elements w_α for $\alpha \in \Phi$, can in fact be generated by the elements $s_1, \ldots, s_l$. In fact W is generated by these elements in a very special way. Let m_{ij} be the order of s_is_j for $i \neq j$. Then W is presented as an abstract group by the following system of generators and relations:

$$W = \langle s_1, \ldots, s_l; \ s_i^2 = 1 \ (s_is_j)^{m_{ij}} = 1 \text{ if } i \neq j \rangle.$$

A group given by such a presentation is called a Coxeter group. We shall discuss the properties of Coxeter groups in detail in chapter 2.

There is one conjugacy class of W which is of particular significance. The element $s_1s_2 \ldots s_l$ together with its conjugates in W are called Coxeter elements of W. Each element $s_{i_1}s_{i_2} \cdots s_{i_l}$, where $i_1, \ldots, i_l$ is a permutation of $1, \ldots, l$, is a Coxeter element. The order h of the Coxeter elements is called the Coxeter number. If G is simple this number can be described in a number of alternative ways. For example we have

$$h = |\Phi|/|\Delta|.$$

It is also true that $h - 1$ is the maximum height of any root in Φ.

There is a partial order relation on W which will be useful to us. A reduced expression for $w \in W$ is an expression

$$w = s_{i_1}s_{i_2} \cdots s_{i_k}$$

whose length k is as small as possible. We write $w' \leqslant w$ if there exist reduced expressions for $w, w' \in W$ such that the reduced expression for w' is obtained by omitting certain terms from the reduced expression for w. This gives a partial ordering on the Coxeter group W.

We illustrate the situation described in this section by considering the group $G = GL_n(K)$. We may take $T = D_n(K)$, the subgroup of diagonal matrices in $GL_n(K)$. We may take $B = T_n(K)$, the subgroup of upper-triangular matrices. Then $U = U_n(K)$, the subgroup of upper-unitriangular matrices. We also have for B^- the subgroup of lower-triangular matrices and for U^- the subgroup of lower-unitriangular matrices. The minimal proper subgroups of U normalized by T are the subgroups $X_{\alpha_{ij}} = \{I + \lambda E_{ij}; \lambda \in K\}$ where E_{ij} is the elementary matrix with 1 in the (i, j) position and zeros elsewhere, and $i < j$. The minimal proper subgroups of U^- normalized by T are the subgroups $X_{\alpha_{ij}}$ for $i > j$. The roots α_{ij} are the elements of $X = \mathrm{Hom}(T, \mathbf{G}_m)$ given by

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \xrightarrow{\alpha_{ij}} \lambda_i \lambda_j^{-1} \qquad (i \neq j).$$

For each root α_{ij} the corresponding coroot $\alpha_{ij}^\vee$ is the element of $Y = \mathrm{Hom}(\mathbf{G}_m, T)$ given by

$$\lambda \to \begin{array}{cl} \begin{bmatrix} 1 &&&&&&&&&& \\ & \ddots &&&&&&&&& \\ && 1 &&&&&&&& \\ &&& \lambda &&&&&&& \\ &&&& 1 &&&&&& \\ &&&&& \ddots &&&&& \\ &&&&&& 1 &&&& \\ &&&&&&& \lambda^{-1} &&& \\ &&&&&&&& 1 && \\ &&&&&&&&& \ddots & \\ &&&&&&&&&& 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \end{array} \end{array}$$

We observe that the root α_{ij} maps this matrix into λ^2, so that $\langle \alpha_{ij}, \alpha_{ij}^\vee \rangle = 2$.

The normalizer N of T is the subgroup of monomial matrices in $GL_n(K)$. Thus the Weyl group W, which is isomorphic to N/T, will in this example be isomorphic to the symmetric group S_n. For each root α_{ij} the corresponding element $w_{\alpha_{ij}} \in W$ is the permutation which transposes i, j and fixes the remaining symbols.

The set Δ of simple roots is given by $\Delta = \{\alpha_{12}, \alpha_{23}, \ldots, \alpha_{n-1.n}\}$. The corresponding set of elements $s_1, s_2, \ldots, s_{n-1}$ of W can be described as follows. s_i is the permutation which transposes i, $i + 1$ and fixes the remaining symbols. The order m_{ij} of $s_i s_j$ if $i \neq j$ is 2 if $|i - j| \neq 1$ and 3 if $|i - j| = 1$.

Thus we see that the group $GL_n(K)$ gives a most convenient illustration of the general theory. It is always helpful to gain an insight into the meaning of a theorem on reductive groups by seeing what the theorem asserts in the case of $GL_n(K)$.

1.10 SPLIT *BN*-PAIRS

We now describe some further structural properties of connected reductive groups. We first mention a commutation relation between root subgroups due to Chevalley. Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$. Let $[X_\alpha, X_\beta]$ be the subgroup generated by all commutators of elements in X_α with elements in X_β.

Then we have

$$[X_\alpha, X_\beta] \subseteq \prod_{\substack{i,j>0 \\ i,j\in\mathbf{Z} \\ i\alpha+j\beta\in\Phi}} X_{i\alpha+j\beta}.$$

The root subgroups $X_{i\alpha+j\beta}$ on the right-hand side can be taken in any order.

We next introduce the idea of a BN-pair or Tits system. As before G is a connected reductive group, T a maximal torus of G and B a Borel subgroup of G containing T. Let $N = N_G(T)$. Then the subgroups B and N generate G. In fact each double coset of B in G contains an element of N, so that $G = BNB$. We also have $B \cap N = T$, and consequently $N/(B \cap N)$ is isomorphic to the Weyl group W. We have seen that W is generated by elements $s_1, s_2, \ldots, s_l$, one for each simple root, and that these elements satisfy $s_i^2 = 1$. Let n_i be an element of N which maps to s_i under the natural homomorphism from N to W. Then one can show that $n_iBn_i \neq B$ and that, for each $n \in N$, we have

$$n_iBn \subseteq Bn_inB \cup BnB.$$

Thus, for all $b \in B$, the element n_ibn lies either in the double coset Bn_inB or in the double coset BnB. These properties form the basis of an axiom system, introduced by Tits, for groups with a BN-pair.

Let G be any group and B, N be subgroups of G. B, N form a BN-pair in G if the following axioms are satisfied:

(i) G is generated by B and N.
(ii) $B \cap N$ is normal in N.
(iii) $N/(B \cap N) = W$ is generated by a set of elements s_i with $s_i^2 = 1$.
(iv) Let $n_i \in N$ map to $s_i \in W$. Then $n_iBn_i \neq B$.
(v) $n_iBn \subseteq Bn_inB \cup BnB$.

Observe that axioms (iv), (v) are independent of which particular $n_i \in N$ is chosen mapping to s_i. The above discussion shows that every connected reductive group has a BN-pair. It follows from the axioms for a BN-pair that every double coset of B in G contains an element of N so has the form BnB, $n \in N$, and that $BnB = Bn'B$ if and only if $\pi(n) = \pi(n')$ where $\pi: N \to W$ is the natural homomorphism.

If G is a connected reductive group there is a most useful formula for the closure of such a double coset BnB. We have

$$\overline{BnB} = \bigcup_{\pi(n') \leqslant \pi(n)} Bn'B$$

where $\leqslant$ is the partial order relation on W introduced in section 1.9. Moreover BnB is an open subset of $\overline{BnB}$, so that in particular BnB is locally closed in G.

We shall discuss the consequences of the axioms for a BN-pair in detail in chapter 2.

There is a stronger version of the above axioms which is also relevant to connected reductive groups. Let G be any linear algebraic group. We shall say that G is an algebraic group with a split BN-pair if G has closed subgroups B, N satisfying the following axioms:

(i) B, N form a BN-pair in G.

(ii) $B = U(B \cap N)$ is the semidirect product of a closed normal unipotent group U and a closed commutative subgroup $B \cap N$, all of whose elements are semisimple.

(iii) $\bigcap_{n \in N} nBn^{-1} = B \cap N$.

Observe that a connected reductive group G satisfies these conditions. We have $B \cap N = T$ and $B = UT$ where $U = R_u(B)$. Also we have $\bigcap_{n \in N} nBn^{-1} = T$. One can in fact find an element $n \in N$ such that $nBn^{-1} = B^-$, and we know that $B \cap B^- = T$. Note also that in the definition of an algebraic group with split BN-pair we have not assumed that G is connected. The definition can be applied in particular to certain finite subgroups of $GL_n(K)$.

1.11 THE CLASSIFICATION OF SIMPLE ALGEBRAIC GROUPS

We have seen that each connected reductive group G can be specified uniquely by its root datum $(X, \Phi, Y, \Phi^\vee)$. Given such a root datum we can form the subgroup $\mathbb{Z}\Phi$ of X generated by the roots and the subgroup $\mathbb{Z}\Phi^\vee$ of Y generated by the coroots. $\mathbb{Z}\Phi$, $\mathbb{Z}\Phi^\vee$ are called the root lattice and coroot lattice respectively. We have rank $\mathbb{Z}\Phi$ = rank X if and only if rank $\mathbb{Z}\Phi^\vee$ = rank Y, and these conditions are equivalent to the property that the reductive group G is semisimple. Thus for a semisimple group G both $|X:\mathbb{Z}\Phi|$ and $|Y:\mathbb{Z}\Phi^\vee|$ will be finite. Recall now that X is canonically isomorphic to $\mathrm{Hom}(Y, \mathbb{Z})$. We have a restriction map

$$\mathrm{Hom}(Y, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z}) = \Omega$$

which is injective. Thus X can be identified with a subgroup of Ω, and X will have finite index in Ω since rank X = rank Ω. In fact we have $|\Omega:X| = |Y:\mathbb{Z}\Phi^\vee|$. It follows that

$$|X:\mathbb{Z}\Phi|\,|Y:\mathbb{Z}\Phi^\vee| = |\Omega/\mathbb{Z}\Phi|.$$

$\Omega/\mathbb{Z}\Phi$ is determined entirely by Φ and $\Phi^\vee$ and is independent of X and Y. $\Omega/\mathbb{Z}\Phi$ is called the fundamental group.

Let G be a connected semisimple group and $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ be a simple system of roots for G. Let $A_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$. The integers A_{ij} are called the Cartan integers and the matrix $A = (A_{ij})$ is called the Cartan matrix. One has $A_{ii} = 2$ and $A_{ij} \leqslant 0$ if $i \neq j$. The values which the Cartan integers can take are very restricted. If $i \neq j$ then A_{ij} must be one of $0, -1, -2, -3$. Moreover $A_{ij} = 0$ if and only if $A_{ji} = 0$. If $A_{ij} = -2$ or -3 then $A_{ji} = -1$. Thus the integer $n_{ij} = A_{ij}A_{ji}$ must take one of the values 0, 1, 2, 3. This integer n_{ij} is connected to the order m_{ij} of the element $s_i s_j$ of W. We have:

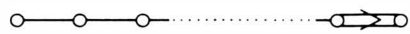
$$\begin{aligned}
n_{ij} &= 0 \quad \text{if and only if} \quad m_{ij} = 2\\
n_{ij} &= 1 \quad \text{if and only if} \quad m_{ij} = 3\\
n_{ij} &= 2 \quad \text{if and only if} \quad m_{ij} = 4\\
n_{ij} &= 3 \quad \text{if and only if} \quad m_{ij} = 6.
\end{aligned}$$

We now define the Dynkin diagram of G. This is a graph with l nodes, one for each of the simple roots α_i. The nodes corresponding to the roots α_i, α_j for $i \neq j$ are joined by n_{ij} bonds. If $n_{ij} = 2$ or 3 then one of the integers A_{ij}, A_{ji} will be -1 and the other will be -2 or -3. We place an arrow in the diagram pointing from α_i to α_j if $A_{ji} \neq -1$.

The connected semisimple group G is simple if and only if its Dynkin diagram is connected. Moreover if G is not simple the Dynkin diagram has connected components which are the Dynkin diagrams of the simple components of G. We assume subsequently that G is simple. Then the possible Dynkin diagrams are as follows:

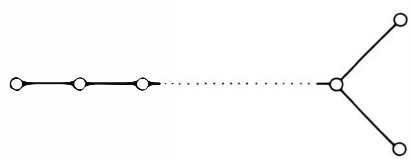
A_l $\quad l \geq 1$

B_l $\quad l \geq 2$

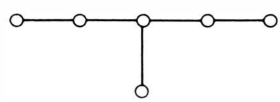


C_l $\quad l \geq 3$

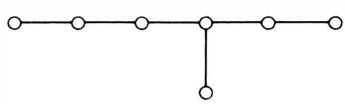
D_l $\quad l \geq 4$



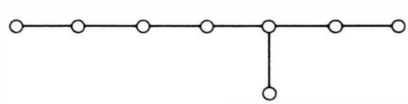
E_6



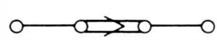
E_7



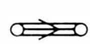
E_8



F_4



G_2



The Dynkin diagram is uniquely determined by G, being independent of the choice of the maximal torus T and Borel subgroup B containing T used to define it. Moreover the Dynkin diagram determines Φ and Φ^v up to isomorphism. For the diagram determines the Cartan integers and so also the transformations $s_1, \ldots, s_l$ on $\mathbb{Z}\Phi$ and $\mathbb{Z}\Phi^v$. However $s_1, \ldots, s_l$ generate W and we have $\Phi = W(\Delta)$, $\Phi^v = W(\Delta^v)$. Thus Φ and Φ^v are determined by the Dynkin diagram.

The way in which Φ and Φ^v occur as subsets of X and Y respectively is not, however, determined by the Dynkin diagram. Two simple groups are said to be isogenous if their Dynkin diagrams are the same. Given a root system Φ and its coroot system Φ^v we have seen the $\mathbb{Z}\Phi$ may be regarded as a subgroup of finite index in $\Omega = \mathrm{Hom}(\mathbb{Z}\Phi^v, \mathbb{Z})$. Ω is called the lattice of weights. Each simple group G with the given Dynkin diagram determines a subgroup X such that $\Omega \supseteq X \supseteq \mathbb{Z}\Phi$. Conversely each subgroup between Ω and $\mathbb{Z}\Phi$ arises in this way as the X-group of some such simple group G. Moreover the location of X between Ω and $\mathbb{Z}\Phi$ determines G up to isomorphism, although it is possible for distinct subgroups between Ω and $\mathbb{Z}\Phi$ to arise from isomorphic groups G.

We see therefore that there are only finitely many simple groups G in a given isogeny class, i.e. with a given Dynkin diagram. Among the groups in an isogeny class there are two extremes. G is called adjoint if $X = \mathbb{Z}\Phi$ and simply-connected if $X = \Omega$. The latter condition holds if and only if $Y = \mathbb{Z}\Phi^v$. If G_{ad}, G_{sc} are the adjoint and simply-connected groups isogenous to G there are surjective homomorphisms $G_{\mathrm{sc}} \to G$ and $G \to G_{\mathrm{ad}}$. The kernels of these homomorphisms are finite and lie in the centre. The kernel of the latter map is equal to the centre of G. The centre of the simply-connected group G_{sc} is isomorphic to the finite group $\mathrm{Hom}(\Omega/\mathbb{Z}\Phi, \mathbf{G}_m)$.

We now describe the possible groups for each individual type of Dynkin diagram. If G has type A_l then $\Omega/\mathbb{Z}\Phi$ is isomorphic to $\mathbb{Z}_{l+1}$, the cyclic group of order $l+1$. The simply-connected group of type A_l is the special linear group $SL_{l+1}(K)$ and the adjoint group is the projective general linear group $PGL_{l+1}(K)$. This is the general linear group factored by its centre. There may also be various other possibilities which are neither simply-connected nor adjoint.

If G has type C_l then $\Omega/\mathbb{Z}\Phi$ is isomorphic to $\mathbb{Z}_2$. There are therefore only two possibilities for G. If G is simply-connected then G is the symplectic group $Sp_{2l}(K)$ and if G is adjoint G is the projective conformal symplectic group $PCSp_{2l}(K)$. This is the conformal symplectic group $CSp_{2l}(K)$ factored by its centre. $CSp_{2l}(K)$ is the group of all symplectic similitudes. A symplectic similitude is a nonsingular map T such that $(Tx, Ty) = \lambda(x, y)$ for all x, y in the underlying vector space, where $(\ ,\ )$ is a nonsingular skew-symmetric form and λ is a constant independent of x, y.

If G has type B_l then $\Omega/\mathbb{Z}\Phi$ is isomorphic to $\mathbb{Z}_2$ and there are again two possibilities. If G is simply-connected then G is the spin group $\mathrm{Spin}_{2l+1}(K)$. If G is adjoint G is the special orthogonal group $SO_{2l+1}(K)$.

If G has type D_l then $\Omega/\mathbb{Z}\Phi$ has order 4 and is isomorphic to $\mathbb{Z}_4$ if l is odd and to $\mathbb{Z}_2 \times \mathbb{Z}_2$ if l is even. If l is odd there are three possibilities for G. If G is simply-connected G is the spin group $\mathrm{Spin}_{2l}(K)$. If G is neither simply-connected nor adjoint then G is $SO_{2l}(K)$. $SO_{2l}(K)$ is defined as the connected component $O_{2l}(K)^0$. It is equal to the set of orthogonal matrices of determinant 1 provided K does not have characteristic 2. Finally if G is adjoint then G is the projective group of the connected component of the conformal orthogonal group $CO_{2l}(K)$. Thus $G = P(CO_{2l}(K)^0)$. $CO_{2l}(K)$ is the group of all orthogonal similitudes acting on a $2l$-dimensional orthogonal space V over K with maximal Witt index l. The maximal isotropic subspaces of dimension l fall into two orbits under the action of $SO_{2l}(K)$. Let $CO_{2l}(K)^0$ be the subgroup of index 2 in $CO_{2l}(K)$ of elements which do not interchange these two orbits. Then $CO_{2l}(K)^0$ is the connected component of $CO_{2l}(K)$ containing the identity. If l is even there are, in addition, two further possibilities for X. However these give rise to isomorphic root data and hence isomorphic groups. Thus we obtain one additional possibility for G, called the half-spin group $HS_{2l}(K)$, l even.

Now consider the exceptional types. If G has type G_2, F_4 or E_8 then $\Omega = \mathbb{Z}\Phi$ and there is only one possibility for G, which is both adjoint and simply-connected. If G has type E_6 then $\Omega/\mathbb{Z}\Phi$ is isomorphic to $\mathbb{Z}_3$ and if G has type E_7 then $\Omega/\mathbb{Z}\Phi$ is isomorphic to $\mathbb{Z}_2$. In these two cases there are therefore two possibilities for G, the adjoint group and the simply-connected group. This concludes our description of the simple algebraic groups over the algebraically closed field K.

1.12 RELATIONS BETWEEN A TORUS AND ITS CHARACTER GROUP

Let T be a torus and $X = \mathrm{Hom}(T, \mathbf{G}_m)$ be the character group of T. For each closed subgroup S of T we define a subgroup $S^\perp$ of X by

$$S^\perp = \{\chi \in X;\ \chi(s) = 1 \quad \text{for all } s \in S\}.$$

If the base field K has characteristic p then the abelian group $X/S^\perp$ has no p-torsion, i.e. no non-identity elements of order a power of p. For if $\chi \in X$ satisfies $p^e\chi \in S^\perp$ then we have

$$(p^e\chi)(s) = 1 \quad \text{for all } s \in S$$

which implies $\chi(s)^{p^e} = 1$ and so $\chi(s) = 1$ for all $s \in S$. Thus $\chi \in S^\perp$. The group $X/S^\perp$ is naturally isomorphic to the character group $\mathrm{Hom}(S, \mathbf{G}_m)$ of S.

Now suppose that we have a subgroup A of X. We define a subgroup $A^\perp$ of T by

$$A^\perp = \{t \in T;\ \chi(t) = 1 \text{ for all } \chi \in A\}.$$

Then $A^\perp$ is a closed subgroup of T.

We may then consider $S^{\perp\perp} \subseteq T$ for each closed subgroup S of T and $A^{\perp\perp} \subseteq X$ for each subgroup A of X. We have $S^{\perp\perp} = S$ for each closed subgroup S of T. However it is not always true that $A^{\perp\perp} = A$. This is true if the field K has characteristic 0, but if K has characteristic p then $A \subseteq A^{\perp\perp}$ and $A^{\perp\perp}/A$ is the p-torsion subgroup of X/A. It follows that if K has characteristic 0 then the maps

$S \to S^{\perp}$ and $A \to A^{\perp}$ are inverse bijections between the closed subgroups of T and the subgroups of X, whereas if K has characteristic p then these maps are inverse bijections between the set of closed subgroups S of T and the set of subgroups A of X such that X/A has no p-torsion.

This information will be useful to us in the subsequent discussion.

1.13 THE CARTAN DECOMPOSITION

Let G be a connected reductive group and $\mathfrak{g} = \mathfrak{L}(G)$. The various structural properties we have described for G have analogues in the Lie algebra $\mathfrak{g}$.

Let T be a maximal torus of G and $\mathfrak{t} = \mathfrak{L}(T)$. For each root $\alpha \in \Phi$ let X_α be the corresponding root subgroup of G and $\mathfrak{x}_\alpha = \mathfrak{L}(X_\alpha)$. Then we have a direct decomposition of $\mathfrak{g}$ as vector space given by

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \Phi} \mathfrak{x}_\alpha.$$

This is called the Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{n} = \mathfrak{L}(U)$ and $\mathfrak{n}^- = \mathfrak{L}(U^-)$. Then we have $\mathfrak{n} = \sum_{\alpha \in \Phi^+} \mathfrak{x}_\alpha$ and $\mathfrak{n}^- = \sum_{\alpha \in \Phi^-} \mathfrak{x}_\alpha$. Hence we can write

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n} \oplus \mathfrak{n}^-.$$

Each of the spaces $\mathfrak{x}_\alpha$ is 1-dimensional and is invariant under the adjoint action of T on $\mathfrak{g}$. Moreover the 1-dimensional representation of T afforded by the module $\mathfrak{x}_\alpha$ is α.

When G is a simple group its Lie algebra $\mathfrak{g}$ will have certain nondegeneracy properties provided that the characteristic of K is either zero or a prime p which is not too small. For instance the centre of $\mathfrak{g}$ will be trivial provided K has characteristic 0 or $p > f$, where $f = \det(A_{ij})$ is the determinant of the Cartan matrix of $\mathfrak{g}$. If K has characteristic 0 or $p > f$ all the simple groups in the given isogeny class have isomorphic Lie algebras. If K has characteristic 0 or $p > h$, where h is the Coxeter number of G, we can say more. The Killing form on $\mathfrak{g}$ will then be nondegenerate. This follows from a formula for the discriminant of the Killing form given in Borel *et al.* [1], p. 180. We then have $G_{\text{ad}} = (\text{Aut } \mathfrak{g})^0$, i.e. the connected component of the group of automorphisms of $\mathfrak{g}$ is the adjoint group isogenous to G (cf. Ono [1]).

1.14 SOME RESULTS ON CENTRALIZERS

If x is an element of an algebraic group G then $C_G(x) = \{g \in G; gx = xg\}$ is the centralizer of x. This is a closed subgroup of G. Similarly, for any subset S of G, we define $C_G(S) = \{g \in G; gs = sg \text{ for all } s \in S\}$ to be the centralizer of S. In this section we review some results on centralizers which fall into three categories:

(a) results which assert that certain centralizers are connected,
(b) relations between centralizers in the group G and in its Lie algebra $L(G)$,
(c) properties of elements x for which the dimension of $C_G(x)$ is as small as possible.

We begin with results on connectedness. If G is a connected group and S a closed subgroup of G which is a torus then $C_G(S)$ is connected. Moreover if G is reductive and T is a maximal torus of G then $C_G(T) = T$.

We consider next the centralizer of a single semisimple element $s \in G$ when G is a semisimple group. The main result here is due to Steinberg [16]. It asserts that if G is a semisimple group which is connected and simply-connected then $C_G(s)$ is connected for each semisimple element $s \in G$. If G is not simply-connected then $C_G(s)$ need not be connected. However its connected component $C_G(s)^0$ is always a reductive group. We shall discuss the structure of such groups in more detail in chapter 3. If $C_G(s)$ is not connected we can still say that all the unipotent elements of $C_G(s)$ lie in $C_G(s)^0$ (Borel *et al.* [1], p. 204).

A similar result holds for $C_G(x)$ for an arbitrary element $x \in G$ if G is simple and if the characteristic of K is either 0 or a prime p which is not too small. A prime p is said to be bad for a simple group G if p divides the coefficient of some root α when expressed as a combination $\alpha = \sum_i n_i\alpha_i$ of simple roots. The bad primes for the individual types of simple group are as follows:

None when G has type A_l
$p = 2$ when G has type B_l, C_l, D_l
$p = 2$ or 3 when G has type G_2, F_4, E_6, E_7
$p = 2$, 3 or 5 when G has type E_8.

Primes which are not bad for G are called good. Now let x be any element of a simple group G and suppose the characteristic of K is either 0 or a good prime for G. Then every unipotent element of $C_G(x)$ lies in $C_G(x)^0$ (Borel *et al.* [1], p. 230).

We now turn to the relation between centralizers in an algebraic group G and its Lie algebra $\mathfrak{L}(G)$. We recall that G acts on $\mathfrak{L}(G)$ by the adjoint representation. We can define centralizers in the group or Lie algebra of elements in the group or algebra. Let $x \in G$ and $a \in \mathfrak{L}(G)$. Then we define

$$C_G(x) = \{g \in G;\ x^{-1}gx = g\}$$
$$C_{\mathfrak{L}(G)}(x) = \{b \in \mathfrak{L}(G);\ \mathrm{Ad}\, x.b = b\}$$
$$C_G(a) = \{g \in G;\ \mathrm{Ad}\, g.a = a\}$$
$$C_{\mathfrak{L}(G)}(a) = \{b \in \mathfrak{L}(G);\ \mathrm{ad}\, b.a = [ba] = 0\}.$$

The relation between these subgroups and subalgebras is as follows. One has

$$\mathfrak{L}(C_G(x)) \subseteq C_{\mathfrak{L}(G)}(x) \quad \text{for all } x \in G$$
$$\mathfrak{L}(C_G(a)) \subseteq C_{\mathfrak{L}(G)}(a) \quad \text{for all } a \in \mathfrak{L}(G)$$

One does not always have equality here. However equality does hold in certain important special cases. For example we have

$$\mathfrak{L}(C_G(s)) = C_{\mathfrak{L}(G)}(s) \quad \text{if } s \in G \text{ is semisimple}$$
$$\mathfrak{L}(C_G(a)) = C_{\mathfrak{L}(G)}(a) \quad \text{if } a \in \mathfrak{L}(G) \text{ is semisimple.}$$

We can obtain a generalization of these results for each closed subgroup H of G. We have

$$\mathfrak{L}(C_H(s)) = C_{\mathfrak{L}(H)}(s) \quad \text{if } s \in G \text{ is a semisimple element with } sHs^{-1} = H$$
$$\mathfrak{L}(C_H(a)) = C_{\mathfrak{L}(H)}(a) \quad \text{if } a \in \mathfrak{L}(G) \text{ is a semisimple element with } \operatorname{Ad} h.a - a \in \mathfrak{L}(H) \text{ for all } h \in H.$$

Furthermore we have

$$\mathfrak{L}(C_H(T)) = C_{\mathfrak{L}(H)}(T) \quad \text{if } T \subseteq G \text{ is a torus such that } T \subseteq N(H)$$

(Borel [1], p. 229).

Under certain circumstances one can relate the centralizers in G and $\mathfrak{L}(G)$ of elements which are not semisimple. Let G be a simple algebraic group. A prime p is said to be 'very good' for G if:

G is not of type A_l and p is a good prime for G,
G is of type A_l and p does not divide $l + 1$.

It is shown in Slodowy [2], p. 38 that if the characteristic of K is either 0 or a very good prime for G then we have

$$\mathfrak{L}(C_G(x)) = C_{\mathfrak{L}(G)}(x) \quad \text{for all } x \in G$$

$$\mathfrak{L}(C_G(a)) = C_{\mathfrak{L}(G)}(a) \quad \text{for all } a \in \mathfrak{L}(G)$$

We now consider elements of G for which the centralizer has minimum possible dimension. Suppose G is a connected reductive group. It was proved by Steinberg that, for all $g \in G$, $\dim C_G(g) \geqslant \operatorname{rank} G$, where the rank of G is defined to be the dimension of the maximal tori of G. Furthermore there exist elements g for which $\dim C_G(g) = \operatorname{rank} G$. Such elements are called regular. If g is a regular element then $C_G(g)^0$ is commutative. For example if g is a regular semisimple element then $C_G(g)^0$ is a maximal torus of G. Moreover a semisimple element is regular if and only if it lies in a unique maximal torus of G. On the other hand every connected reductive group G contains regular unipotent elements and any two are conjugate in G. Finally an arbitrary element $g \in G$ is regular if and only if g_u is a regular unipotent element of $C_G(g_s)^0$ where $g = g_s g_u$ is the Jordan decomposition of g.

1.15 THE UNIPOTENT AND NILPOTENT VARIETIES

We now concentrate on the unipotent elements of a simple group and the nilpotent elements of its Lie algebra and on the relation between them. Let G be a simple group and let $\mathfrak{U}$ be the set of all unipotent elements of G. Then $\mathfrak{U}$ is a closed subset of G. Thus $\mathfrak{U}$ is a subvariety of G which is in fact irreducible. The dimension of $\mathfrak{U}$ is the number of roots of G. Let $\mathfrak{N}$ be the set of all nilpotent elements in the Lie algebra $\mathfrak{L}(G)$. Then $\mathfrak{N}$ is a closed subset of $\mathfrak{L}(G)$. It is therefore a subvariety which again turns out to be irreducible. The dimension of $\mathfrak{N}$ is also equal to the number of roots of G.

The following connection between the unipotent variety $\mathfrak{U}$ and the nilpotent variety $\mathfrak{N}$ was proved by Springer (Borel *et al.* [1], p. 229). Suppose G is simple and simply-connected and that the characteristic of K is either 0 or a good prime for G. Then there is a bijective morphism of varieties

$$\phi:\mathfrak{U} \to \mathfrak{N}$$

which is a homeomorphism of topological spaces and which is consistent with the G-actions on $\mathfrak{U}$ and $\mathfrak{N}$. Thus

$$\phi(gug^{-1}) = \mathrm{Ad}\, g.\phi(u)$$

for all $u \in \mathfrak{U}$, $g \in G$. The number of G-orbits on $\mathfrak{U}$ and $\mathfrak{N}$ is known to be finite under these conditions by a theorem of Richardson (Borel *et al.* [1], p. 185). In fact such a map ϕ can be found which is an isomorphism of varieties.

If the characteristic of K is 0 or a very good prime for G we can obtain a bijective morphism $\phi:\mathfrak{U} \to \mathfrak{N}$ consistent with the G-actions even if G is not simply-connected. For let G_{sc} be the simply-connected covering of G and $\mathfrak{U}_{sc}$, $\mathfrak{N}_{sc}$ be the unipotent and nilpotent varieties for G_{sc}. Then there are natural maps $\mathfrak{U}_{sc} \to \mathfrak{U}$ and $\mathfrak{N}_{sc} \to \mathfrak{N}$ which are isomorphisms of varieties. Thus the Springer map $\phi:\mathfrak{U}_{sc} \to \mathfrak{N}_{sc}$ will give rise to a map from $\mathfrak{U}$ to $\mathfrak{N}$ with the above properties.

There is a further connection between unipotent and nilpotent elements which will be useful to us. This is valid for any linear algebraic group G and asserts that an element $a \in \mathfrak{L}(G)$ is nilpotent if and only if there exists a closed unipotent subgroup H of G such that $a \in \mathfrak{L}(H)$. This result is due to Borel [1], p. 355. It follows that every nilpotent element $a \in \mathfrak{L}(G)$ is conjugate under G to an element of $\mathfrak{n} = \mathfrak{L}(U)$.

1.16 ON THE EXISTENCE OF CERTAIN RATIONAL REPRESENTATIONS

A rational representation of a linear algebraic group G is a homomorphism of algebraic groups from G into $GL_n(K)$ for some n. Given a rational representation $\rho:G \to GL_n(K)$ we obtain a map $\mathrm{d}\rho:\mathfrak{L}(G) \to \mathfrak{gl}_n(K)$ which is a representation of $\mathfrak{L}(G)$. We define the trace-form of the representation $\mathrm{d}\rho$ to be the map $\mathfrak{L}(G) \times \mathfrak{L}(G) \to K$ given by

$$(a, b) \to \mathrm{trace}(\mathrm{d}\rho(a)\,\mathrm{d}\rho(b)).$$

The trace-form is said to be nondegenerate if trace $(\mathrm{d}\rho(a)\,\mathrm{d}\rho(b)) = 0$ for all $b \in \mathfrak{L}(G)$ implies that $a = 0$.

We consider the question of when a rational representation with nondegenerate trace-form exists. The following result in this direction will be useful. Suppose we are given a simple type, i.e. one of A_l, B_l, C_l, D_l, G_2, F_4, E_6, E_7, E_8, and an algebraically closed field K of characteristic either 0 or a good prime for this type. Then, if the type is not A_l, there exists a simple group G of the given type over K and a rational representation ρ of G such that the trace-form of $\mathrm{d}\rho$ is nondegenerate. If the type is B_l, C_l or D_l we may take for G a group $Sp(K)$ or

$SO(K)$ in the appropriate dimension and ρ to be the standard representation of G as a group of symplectic or orthogonal matrices. If the type is G_2, F_4, E_6, E_7, E_8 we may take for G the adjoint group and for ρ the adjoint representation of G on $\mathfrak{L}(G)$.

The fact that we do not always get this result for type A_l is of no serious inconvenience, since the required applications of the result can easily be proved for type A_l by other methods.

1.17 FROBENIUS MAPS

We now come to the key idea which relates the finite groups of Lie type to the corresponding algebraic groups. Let G be a linear algebraic group over an algebraically closed field of characteristic p. Then G is isomorphic to a closed subgroup of $GL_n(K)$ for some n. Let $q = p^e$, $e \geqslant 1$, be a power of p and F_q be the map of $GL_n(K)$ into itself given by

$$F_q:(a_{ij}) \to (a_{ij}{}^q).$$

F_q is a homomorphism of $GL_n(K)$ into itself.

A homomorphism $F:G \to G$ is called a *standard Frobenius map* if there exists an injective homomorphism $i:G \to GL_n(K)$ for some n, such that

$$i(F(g)) = F_q(i(g)) \text{ for some } q = p^e \text{ and all } g \in G.$$

A homomorphism $F:G \to G$ is called a *Frobenius map* if some power of F is a standard Frobenius map.

Here are some properties of Frobenius maps. Any Frobenius map $F:G \to G$ is bijective. F is a surjective homomorphism of algebraic groups but is not an isomorphism of algebraic groups. It is, however, an isomorphism of abstract groups.

If $F:G \to G$ is a Frobenius map and H is an F-stable closed subgroup of G then the restriction of F to H is a Frobenius map on H. If H is an F-stable closed normal subgroup of G then F induces a homomorphism of G/H into itself which is also a Frobenius map.

If $F: G \to G$ is a Frobenius map we define G^F by

$$G^F = \{g \in G;\ F(g) = g\}.$$

G^F is then a finite subgroup of G. Thus any Frobenius map is a surjective homomorphism of G to itself for which G^F is finite.

If G is connected and semisimple then any surjective homomorphism $F: G \to G$ for which G^F is finite is, conversely, a Frobenius map. However this is not always the case if G is a connected reductive group which is not semisimple. For further information about the properties of Frobenius maps the reader is referred to Steinberg [16].

The finite groups G^F arising from a Frobenius map $F: G \to G$ on a connected reductive group G are called the finite groups of Lie type. An example of such a group is as follows. Let $G = GL_n(K)$ and $F: G \to G$ be given by

$$(a_{ij}) \to ((a_{ij}{}^q)^t)^{-1}$$

where q is a power of p. Here we raise all matrix entries to the qth power and take the inverse transpose. Then $G^F = U_n(q^2)$ is the unitary group over the field of q elements.

We now state a fundamental theorem of Lang which applies in this situation. Lang's theorem asserts that if K is an algebraically closed field of characteristic p and if G is a closed connected subgroup of $GL_n(K)$ and if $F: G \to G$ is the map

$$(a_{ij}) \to (a_{ij}{}^q)$$

where $q = p^e$ and $e \geqslant 1$, then the morphism $L: G \to G$ given by $L(g) = g^{-1}F(g)$ is surjective.

Since Lang's theorem will be of key importance for us we shall indicate how it is proved. Since $L(1) = 1$ the differential $(\mathrm{d}L)_1$ maps $T(G)_1$ into itself. Since $L(g) = g^{-1}F(g)$ we obtain

$$(\mathrm{d}L)_1 a = -a + (\mathrm{d}F)_1 a$$

for $a \in T(G)_1$. However the differential of any function which is a qth power is zero, since $\mathrm{d}(f^q) = qf^{q-1}\,\mathrm{d}f = 0$ in K. It follows that $(\mathrm{d}F)_1 = 0$. Hence $(\mathrm{d}L)_1 a = -a$. Thus $(\mathrm{d}L)_1$ is surjective and consequently L is dominant and separable. In particular the image of L contains a non-empty open subset of G. Now let x be any element of G and consider the map $\phi: g \to g^{-1}xF(g)$. Since $\phi(1) = x$ the differential $(\mathrm{d}\phi)_1$ maps $T(G)_1$ into $T(G)_x$. Multiplication by x induces as its differential a bijection between $T(G)_1$ and $T(G)_x$ and we have

$$(\mathrm{d}\phi)_1 a = -a.x$$

since $(\mathrm{d}F)_1 = 0$ as before. In particular $(\mathrm{d}\phi)_1$ is surjective and so $\phi(G)$ contains a non-empty open subset of G. Since G is connected any two non-empty open subsets have a non-empty intersection. Thus we can choose an element which lies in the images of both L and ϕ. This element has form

$$g_1^{-1}F(g_1) = g_2^{-1}xF(g_2)$$

for certain elements $g_1, g_2 \in G$. Let $g = g_1 g_2^{-1}$. Then $x = g^{-1}F(g)$ and so the map L is surjective.

A generalization of Lang's theorem was proved by Steinberg [16], who showed that if G is a connected group over an algebraically closed field of characteristic p and if F is any surjective homomorphism $F: G \to G$ such that G^F is finite, then the map $L: G \to G$ given by $L(g) = g^{-1}F(g)$ is surjective. In particular L will be surjective whenever F is a Frobenius map. The proof can be reduced to the case in which G is semisimple. The proof then runs along similar lines to that of Lang's, but instead of having $(\mathrm{d}F)_1 = 0$ one has $(\mathrm{d}F)_1$ is nilpotent, and this is sufficient to carry through the argument. Steinberg's generalization of Lang's theorem will be crucial in deriving structural properties of the finite group G^F.

We give some examples of the way it is used. We show that G has a Borel subgroup which is F-stable, i.e. satisfies $F(B) = B$. Let B be any Borel subgroup of G. Then ${}^gB = gBg^{-1}$ will also be a Borel subgroup for any $g \in G$. Now we have

$$F({}^gB) = {}^gB \Leftrightarrow {}^{F(g)}F(B) = {}^gB \Leftrightarrow {}^{g^{-1}F(g)}F(B) = B.$$

However the Borel subgroups B and $F(B)$ are conjugate in G so there exists $x \in G$ with ${}^xF(B) = B$. This element x has the form $g^{-1}F(g)$ for some $g \in G$ by the Lang–Steinberg theorem. Thus ${}^gB$ is an F-stable Borel subgroup of G. One can prove in a similar way that G contains an F-stable maximal torus and that any F-stable Borel subgroup of G contains an F-stable maximal torus of G.

Another application of the Lang–Steinberg theorem is as follows. Suppose that H_1, H_2 are F-stable closed subgroups of G, that H_2 is a normal subgroup of H_1 and that H_2 is connected. Then F acts on the factor group H_1/H_2. We assert that

$$H_1{}^F/H_2{}^F \cong (H_1/H_2)^F.$$

For we see that the natural homomorphism $H_1 \to H_1/H_2$ maps $H_1{}^F$ into $(H_1/H_2)^F$ with kernel $H_2{}^F$. Thus we have an injective map $H_1{}^F/H_2{}^F \to (H_1/H_2)^F$. We show that this map is also surjective. Let $hH_2 \in (H_1/H_2)^F$. Then $F(hH_2) = hH_2$ and so $h^{-1}F(h) \in H_2$. Since H_2 is connected the Lang–Steinberg theorem shows there exists $x \in H_2$ with $h^{-1}F(h) = x^{-1}F(x)$. Thus $hx^{-1} \in H_1{}^F$ maps to $hH_2 \in (H_1/H_2)^F$. Hence the finite groups $H_1{}^F/H_2{}^F$ and $(H_1/H_2)^F$ are isomorphic.

1.18 THE FINITE GROUPS G^F

Let G be a connected reductive group over an algebraically closed field K of characteristic p and let $F: G \to G$ be a Frobenius map. We consider the corresponding finite group G^F and shall see that it has many structural properties in common with G. In particular G^F has a split BN-pair.

A Borel subgroup of G^F is defined to be a subgroup of the form B^F where B is an F-stable Borel subgroup of G. Any two Borel subgroups of G^F are conjugate in G^F. For let B and ${}^gB$ be F-stable Borel subgroups of G. Then ${}^{g^{-1}F(g)}B = B$ and, since $B = N(B)$, we have $g^{-1}F(g) \in B$. By the Lang–Steinberg theorem we have $g^{-1}F(g) = b^{-1}F(b)$ for some $b \in B$. Thus $gb^{-1} \in G^F$ and ${}^gB = {}^{gb^{-1}}B$. Thus the Borel subgroups B^F and $({}^gB)^F$ of G^F are conjugate by $gb^{-1} \in G^F$.

A maximal torus of G^F is defined to be a subgroup of the form T^F where T is an F-stable maximal torus of G. Although every maximal torus of G lies in a Borel subgroup of G it need not be true that every F-stable maximal torus of G lies in an F-stable Borel subgroup of G. Consequently it need not be true that every maximal torus of G^F lies in a Borel subgroup of G^F. An F-stable maximal torus of G is called maximally split if it lies in an F-stable Borel subgroup of G, and a maximal torus of G^F is called maximally split if it has the form T^F for some maximally split torus T in G. A proof similar to the above for Borel subgroups shows that any two maximal tori in B^F are conjugate in B^F. Consequently any two maximally split tori of G^F are conjugate in G^F. However it will not in general be true that any two maximal tori of G^F are conjugate in G^F. We discuss the situation in detail in chapter 3. We merely point out here that among the conjugacy classes of maximal tori in G^F there is one distinguished one, viz. the class of maximally split tori.

Let B be an F-stable Borel subgroup of G and T an F-stable maximal torus of G contained in B. Let $N = N_G(T)$. Then N is also F-stable and since $B \cap N = T$, we have $B^F \cap N^F = T^F$ also. Since F acts on N and T it also acts on $W = N/T$ by $F(nT) = F(n)T$. Let W^F be the subgroup of F-stable elements of W. Since T is connected we see, using a result in section 1.17, that

$$\frac{N^F}{T^F} \cong W^F.$$

Now since B is F-stable $U = R_u(B)$ will be F-stable also and the root subgroups X_α, $\alpha \in \Phi^+$, which are the minimal nontrivial subgroups of U normalized by T, will be permuted by F. There will therefore be a permutation ρ of the positive roots such that $F(X_\alpha) = X_{\rho(\alpha)}$. This permutation ρ satisfies $\rho(\Delta) = \Delta$. ρ therefore gives rise to a permutation of the simple roots, and so of the set $I = \{1, \ldots, l\}$.

Let J be an orbit of ρ on I. Let W_J be the subgroup of W given by

$$W_J = \langle s_i;\ i \in I \rangle$$

and let Φ_J be the subset of Φ defined by $\Phi_J = W_J(\Delta_J)$. Let $\Phi_J{}^+ = \Phi_J \cap \Phi^+$ and $\Phi_J{}^- = \Phi_J \cap \Phi^-$. Then there is a unique element s_J of W_J which satisfies $s_J(\Phi_J{}^+) = \Phi_J{}^-$. We have $s_J{}^2 = 1$. Moreover s_J lies in W^F and W^F is the subgroup of W generated by the elements s_J for all ρ-orbits J on I. Furthermore W^F is generated by the s_J as a Coxeter group.

Let n_J be an element of N^F which maps to $s_J \in W^F$. Then we have

$$n_J B^F n_J \neq B^F$$

and also

$$n_J B^F n \subseteq B^F n_J n B^F \cup B^F n B^F$$

for all $n \in N^F$. Since G^F is generated by B^F and N^F this shows that the subgroups B^F, N^F form a BN-pair in G^F. This is in fact a split BN-pair, for B^F has a decomposition $B^F = U^F T^F$ where U^F is the unique maximal normal unipotent subgroup of B^F and T^F is a maximally split torus of B^F, so is a commutative group all of whose elements are semisimple. Finally we have

$$\bigcap_{n \in N^F} n B^F n^{-1} = B^F \cap N^F = T^F.$$

Thus the finite group G^F satisfies the axioms for an algebraic group with a split BN-pair. We note here that for a finite algebraic group over a field of characteristic p an element is unipotent if and only if its order is a power of p and it is semisimple if and only if its order is prime to p.

Since both the connected reductive group G and its finite subgroup G^F satisfy the axioms for an algebraic group with a split BN-pair we shall be able to derive their properties simultaneously in an axiomatic manner in chapter 2.

We now define an action of F on the character and cocharacter groups of the maximally split torus T. We define $F: X \to X$ by

$$(F(\chi))t = \chi(F(t)) \qquad \chi \in X$$

and $F: Y \to Y$ by

$$(F(\gamma))\lambda = F(\gamma(\lambda)) \qquad \gamma \in Y\ \lambda \in K^*.$$

These actions of F on X and Y are related by

$$\langle \chi, F(\gamma) \rangle = \langle F(\chi), \gamma \rangle$$

for all $\chi \in X$, $\gamma \in Y$. The action of F on the roots in X is closely related to the permutation ρ of the roots defined above. In fact $F(\rho(\alpha))$ is a positive multiple of α for each $\alpha \in \Phi^+$ (see Steinberg [16], 11.2). In particular F transforms each simple root α into a positive multiple of $\rho^{-1}(\alpha)$.

Now some power of F is a standard Frobenius map. So there is a positive integer n such that F^n is an integral multiple of the identity on X. Let δ be the smallest such n. Then $F^\delta = kI$ on X and on Y for some $k \in \mathbb{Z}$ with $k > 1$. k is in fact a power of p. We define the positive real number q by $k = q^\delta$. Then $F^\delta = q^\delta I$. q is the absolute value of all the eigenvalues of F on X or Y.

Let $X_{\mathbb{R}} = X \otimes_{\mathbb{Z}} \mathbb{R}$. The action of F on X extends by linearity to an action on the vector space $X_{\mathbb{R}}$, and the determinant of F on $X_{\mathbb{R}}$ is a positive integral power of p. We define $F_0: X_{\mathbb{R}} \to X_{\mathbb{R}}$ by $F = qF_0$. Then $F_0^\delta = I$. Thus we have $F = qF_0$ where $q > 1$ and F_0 has finite order.

Let $X_{\mathbb{R}}{}^{F_0} = \{\chi \in X_{\mathbb{R}}; F_0(\chi) = \chi\}$. This is a subspace of $X_{\mathbb{R}}$. We define a linear map

$$\theta: X_{\mathbb{R}} \to X_{\mathbb{R}}{}^{F_0}$$

by

$$\theta(\chi) = \frac{1}{\delta}(\chi + F_0(\chi) + F_0{}^2(\chi) + \dots + F_0^{\delta-1}(\chi)).$$

This map gives a projection of $X_{\mathbb{R}}$ onto $X_{\mathbb{R}}{}^{F_0}$. The Weyl group W acts on $X_{\mathbb{R}}$ as a group of nonsingular linear transformations and the subgroup W^F maps the subspace $X_{\mathbb{R}}{}^{F_0}$ into itself.

We wish to introduce a root system for the group G^F. We begin with the root system Φ of G. We consider subsets of Φ of the form $w(\Phi_J{}^+)$ where J is a ρ-orbit on I and $w \in W^F$. Any two such subsets are either identical or have empty intersection and each element of Φ lies in at least one of them. Thus these subsets form a partition of Φ and so define an equivalence relation on Φ. This equivalence relation may also be defined in terms of the above projection map θ: $X_{\mathbb{R}} \to X_{\mathbb{R}}{}^{F_0}$. α, $\beta \in \Phi$ are equivalent if and only if $\theta(\alpha)$, $\theta(\beta)$ are positive multiples of one another. For each equivalence class A of roots we define α_A to be the vector of maximal length of the form $\theta(\alpha)$ for $\alpha \in A$. The set of vectors α_A is called the set of roots of G^F. It is a finite subset of $X_{\mathbb{R}}^{F\circ}$. The roots of G^F are permuted by the group W^F.

For each root we define a corresponding root subgroup $X_A{}^F$ of G^F, where

$$X_A = \prod_{\alpha \in A} X_\alpha.$$

(The order in which the factors X_α are taken is irrelevant.) In contrast to the root subgroups X_α of G the root subgroups $X_A{}^F$ of G^F need not be commutative. However each element $x \in U^F$ can be written uniquely in the form $x = \prod_{A>0} x_A$ with $x_A \in X_A{}^F$. (Note here that if one root in an equivalence class A lies in Φ^+ then all roots in A lie in Φ^+. Thus the statement $A > 0$ is meaningful.) The root subgroups $X_A{}^F$ also satisfy an analogue of Chevalley's commutator relation. Let A, B be two equivalence classes with $A \neq \pm B$. Then

$$[X_A{}^F, X_B{}^F] \subseteq \prod_C X_C{}^F$$

where the product is taken over all equivalence classes C such that $\alpha_C = i\alpha_A + j\alpha_B$ for $i, j > 0$.

We now discuss the choice of representatives $n_w \in N^F$ such that $\pi(n_w) = w \in W^F$ where $\pi: N^F \to W^F \cong N^F/T^F$ is the natural homomorphism. These coset representatives can be chosen in a favourable manner as follows. First recall that W^F is generated as a Coxeter group by the elements s_J as J runs over the ρ-orbits on I. For each such J it is possible to choose a corresponding representative $n_J \in N^F$ with $\pi(n_J) = s_J$ satisfying the additional condition

$$n_J \in \langle X_{A(J)}{}^F, X_{-A(J)}{}^F \rangle.$$

Here $A(J)$ is the equivalence class of roots containing J and $X_{A(J)}{}^F$ is the corresponding root subgroup of G^F. Suppose the elements n_J are chosen in this way. Let $w \in W^F$. Then w can be expressed in the form

$$w = s_{J_1} s_{J_2} \dots s_{J_k}$$

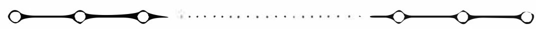
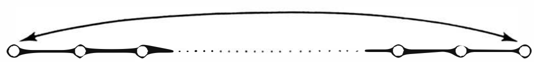
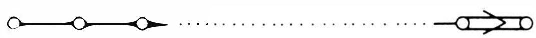
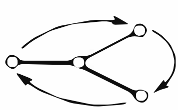
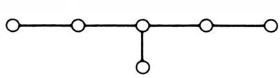
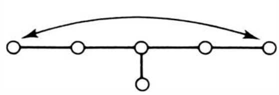
where $J_1, \dots, J_k$ are ρ-orbits on I and k is as small as possible. We define $n_w \in N^F$ by

$$n_w = n_{J_1} n_{J_2} \dots n_{J_k}.$$

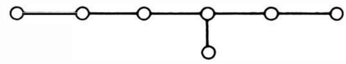
The element n_w is then independent of the reduced expression for w used to define it (Steinberg [15]). We shall find it useful to choose the representatives $n_w \in N^F$ in this way.

1.19 THE CLASSIFICATION OF THE FINITE GROUPS G^F

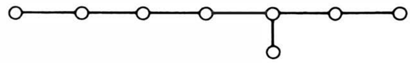
We now describe the possible finite groups G^F which arise from simple algebraic groups G. We have seen that the Frobenius map $F: G \to G$ determines a permutation ρ of the simple roots. This permutation gives rise to a symmetry of the Dynkin diagram of G if arrows are disregarded. Thus the number of bonds joining nodes i and j is the same as the number of bonds joining $\rho(i)$ and $\rho(j)$. The type of the simple group G together with the symmetry ρ of the Dynkin diagram determines the type of G^F. The possible types are shown in the accompanying list.

Type		*Dynkin diagram with ρ-action*
A_l	$l \geq 1$	
${}^2A_l$	$l \geq 2$	
B_l	$l \geq 2$	
${}^2B_2$		
C_l	$l \geq 3$	
D_l	$l \geq 4$	
${}^2D_l$	$l \geq 4$	
${}^3D_4$		
E_6		
${}^2E_6$		

E_7

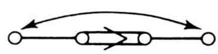


E_8



F_4

${}^2F_4$



G_2

${}^2G_2$

Let us now consider the possible groups G^F of a given type. Recall that there are finitely many isogenous simple groups G of a given type and that each of them is associated with some lattice X between the root lattice $\mathbb{Z}\Phi$ and the weight lattice Ω. $\mathbb{Z}\Phi$ and Ω are determined by the type of G and satisfy $\mathbb{Z}\Phi \subseteq \Omega$ and $|\Omega/\mathbb{Z}\Phi|$ is finite. In fact $|\Omega/\mathbb{Z}\Phi|$ is the determinant of the Cartan matrix (A_{ij}). Each symmetry ρ of the Dynkin diagram determines in a natural way a map of Ω into itself which leaves the sublattice $\mathbb{Z}\Phi$ invariant. In order for a group G^F to exist the lattice X for G must be invariant under this map $\rho:\Omega \to \Omega$. The lattice X determines the isogeny type of G.

We have seen that a simple algebraic group G is determined by its Dynkin diagram and its isogeny type. In order to specify a finite group G^F up to isomorphism we need a third invariant. This is the positive real number q which was defined in terms of the action of F on $X_{\mathbb{R}}$. We recall that $F = qF_0$ on $X_{\mathbb{R}}$ where $q > 1$ and F_0 has finite order. This determines q uniquely. It is the absolute value of all eigenvalues of F on $X_{\mathbb{R}}$. A finite group G^F of Lie type where G is simple is determined up to isomorphism by the following three invariants:

The Dynkin diagram with ρ-action.
The isogeny type.
The number q.

(There are circumstances under which groups G^F with different invariants can be isomorphic—we are merely asserting here that groups for which the invariants are the same must be isomorphic.)

We now describe the possible groups in the individual cases. G will always be a simple algebraic group over an algebraically closed field of characteristic p.

Type A_l q can take any value which is an integral power of p. If G is simply-connected then $G^F = (A_l)_{sc}(q)$ is the special linear group $SL_{l+1}(q)$ over the finite field $\mathbb{F}_q$. If G is adjoint then $G^F = (A_l)_{ad}(q)$ is the projective general linear group $PGL_{l+1}(q)$. There may also be various other possibilities corresponding to lattices X strictly between the root lattice and the weight lattice.

Type C_l q can take any value which is an integral power of p. If G is simply-connected then $G^F = (C_l)_{sc}(q)$ is the symplectic group $Sp_{2l}(q)$ and if G is adjoint then $G^F = (C_l)_{ad}(q)$ is the projective conformal symplectic group $PCSp_{2l}(q)$ of all symplectic similitudes modulo its centre.

Type B_l q can take any value which is an integral power of p. If G is simply-connected then $G^F = (B_l)_{sc}(q)$ is the spin group $\mathrm{Spin}_{2l+1}(q)$ and if G is adjoint then $G^F = (B_l)_{ad}(q)$ is the special orthogonal group $SO_{2l+1}(q)$ corresponding to a nondegenerate quadratic form over $\mathbb{F}_q$ of maximal Witt index l.

Type D_l q can take any value which is an integral power of p. If G is simply-connected then $G^F = (D_l)_{sc}(q)$ is the spin group $\mathrm{Spin}_{2l}(q)$ corresponding to a quadratic form over $\mathbb{F}_q$ of maximal Witt index l. If G is adjoint the group $G^F = (D_l)_{ad}(q)$ can be obtained as follows. Let $CO_{2l}(q)$ be the conformal orthogonal group corresponding to a quadratic form of maximal Witt index l. The isotropic subspaces of dimension l fall into two orbits under the action of $SO_{2l}(q)$. Let $CO_{2l}(q)^0$ be the subgroup of index 2 in $CO_{2l}(q)$ of elements which do not interchange these two families of maximal isotropic subspaces. Let $P(CO_{2l}(q)^0)$ be the factor group of $CO_{2l}(q)^0$ by its centre. Then G^F is isomorphic to $P(CO_{2l}(q)^0)$.

There is a further possibility, which is neither simply-connected nor adjoint, and this is the special orthogonal group $G^F = SO_{2l}(q)$ corresponding to a quadratic form of maximal index l.

If l is even there is one further possibility, and this is the half-spin group $G^F = HS_{2l}(q)$.

Type ${}^2A_l$ q can take any value which is an integral power of p. If G is simply-connected then $G^F = ({}^2A_l)_{sc}(q^2)$ is the special unitary group $SU_{l+1}(q^2)$. This is the subgroup of unitary matrices in $SL_{l+1}(q^2)$. If G is adjoint then $G^F = ({}^2A_l)_{ad}(q^2)$ is the projective unitary group $PU_{l+1}(q^2)$. There may also be various other possibilities corresponding to lattices X strictly between the lattice of roots and the lattice of weights.

Type ${}^2D_l$ q can take any value which is an integral power of p. If G is simply-connected then $G^F = ({}^2D_l)_{sc}(q^2)$ is the spin group $\mathrm{Spin}_{2l}^-(q)$ corresponding to a quadratic form of index $l-1$ relative to $\mathbb{F}_q$ but index l relative to $\mathbb{F}_{q^2}$. If G is adjoint the group $G^F = ({}^2D_l)_{\mathrm{ad}}(q^2)$ is obtained as follows. Let $CO_{2l}^-(q)$ be the conformal orthogonal group corresponding to a quadratic form of index $l-1$ relative to $\mathbb{F}_q$ but l relative to $\mathbb{F}_{q^2}$. Let $CO_{2l}^-(q)^0$ be the subgroup of index 2 in $CO_{2l}^-(q)$ of elements which do not interchange the two families of maximal isotropic subspaces of dimension l over $\mathbb{F}_{q^2}$. Let $P(CO_{2l}^-(q)^0)$ be the factor group of $CO_{2l}^-(q)^0$ with respect to its centre. Then G^F is isomorphic to $P(CO_{2l}^-(q)^0)$.

There is one further possibility, which is neither simply-connected nor adjoint. This is the group $G^F = SO_{2l}^-(q)$; the special orthogonal group corresponding to the quadratic form described above.

The identifications of the groups G^F with various classical groups over finite fields are shown in the accompanying list.

Identifications with classical groups

$(A_l)_{\mathrm{sc}}(q)$	$SL_{l+1}(q)$
$(A_l)_{\mathrm{ad}}(q)$	$PGL_{l+1}(q)$
$({}^2A_l)_{\mathrm{sc}}(q^2)$	$SU_{l+1}(q^2)$
$({}^2A_l)_{\mathrm{ad}}(q^2)$	$PU_{l+1}(q^2)$
$(B_l)_{\mathrm{sc}}(q)$	$\mathrm{Spin}_{2l+1}(q)$
$(B_l)_{\mathrm{ad}}(q)$	$SO_{2l+1}(q)$
$(C_l)_{\mathrm{sc}}(q)$	$Sp_{2l}(q)$
$(C_l)_{\mathrm{ad}}(q)$	$PCSp_{2l}(q)$
$(D_l)_{\mathrm{sc}}(q)$	$\mathrm{Spin}_{2l}(q)$
$(D_l)_{\mathrm{ad}}(q)$	$P(CO_{2l}(q)^0)$
$({}^2D_l)_{\mathrm{sc}}(q^2)$	$\mathrm{Spin}_{2l}^-(q)$
$({}^2D_l)_{\mathrm{ad}}(q^2)$	$P(CO_{2l}^-(q)^0)$

Type ${}^3D_4$ q can take any value which is an integral power of p. There are two possibilities for the isogeny type—G can be either adjoint or simply-connected. However the finite groups $({}^3D_4)_{\mathrm{sc}}(q^3)$ and $({}^3D_4)_{\mathrm{ad}}(q^3)$ turn out to be isomorphic. This common group G^F will be denoted simply by ${}^3D_4(q^3)$.

Type G_2 q can take any value which is an integral power of p and there is just one possibility $G^F = G_2(q)$ for each such q.

Type F_4 q can take any value which is an integral power of p and there is one possibility $G^F = F_4(q)$ for each such q.

Type E_6 q can take any value which is an integral power of p and there are two possibilities $(E_6)_{\mathrm{ad}}(q)$ and $(E_6)_{\mathrm{sc}}(q)$ for each such q.

Type ${}^2E_6$ q can take any value which is an integral power of p and there are two possibilities $({}^2E_6)_{\mathrm{ad}}(q^2)$ and $({}^2E_6)_{\mathrm{sc}}(q^2)$ for each such q.

Type E_7 q can take any value which is an integral power of p and there are two possibilities $(E_7)_{\mathrm{ad}}(q)$ and $(E_7)_{\mathrm{sc}}(q)$ for each q.

Type E_8 q can take any value which is an integral power of p and there is a single possibility $E_8(q)$ for each such q.

Type ${}^2B_2$ A group G^F can only exist in this case if $p = 2$ and $q^2 = 2^{2n+1}$ for some $n \geqslant 0$. G can be either simply-connected or adjoint but the groups $({}^2B_2)_{\mathrm{sc}}(q^2)$ and $({}^2B_2)_{\mathrm{ad}}(q^2)$ are isomorphic. This group will simply be denoted by ${}^2B_2(q^2)$. These are called the Suzuki groups. (This is the first example we have encountered in which the positive number q is not an integer.)

Type ${}^2G_2$ A group G^F can only exist in this case if $p = 3$ and $q^2 = 3^{2n+1}$ for some $n \geqslant 0$. There is one possibility ${}^2G_2(q^2)$ for each such q. These groups are called the Ree groups of type G_2.

Type ${}^2F_4$ A group G^F can only exist in this case if $p = 2$ and $q^2 = 2^{2n+1}$ for some $n \geqslant 0$. There is one possibility ${}^2F_4(q^2)$ for each such q. These groups are called the Ree groups of type F_4.

This completes the description of the possible groups G^F when G is simple. All the isogenous groups G^F of a given type with a given value of q have the same order. Certain of the groups described above which are apparently different may be isomorphic. For example $(C_l)_{\mathrm{ad}}(q)$ is isomorphic to $(C_l)_{\mathrm{sc}}(q)$ if q is a power of 2. We shall not need a complete list of such isomorphisms.

The groups G^F for which ρ acts trivially on the Dynkin diagram are called Chevalley groups, or split forms of G. The groups G^F for which the Dynkin diagram has only single bonds and ρ acts nontrivially are called twisted groups, or quasi-split forms of G. The remaining groups G^F are the Suzuki and Ree groups.

Further information about the finite groups G^F can be found in Carter [3] or in Steinberg [15].

Chapter 2

BN-PAIRS AND COXETER GROUPS

We have seen that if G is a connected reductive algebraic group and $F: G \to G$ is a Frobenius map then both G and G^F are groups with a BN-pair. In this chapter we shall discuss the properties of an arbitrary group with a BN-pair. These results are reasonably well known and so we shall give them without proof but with appropriate references. Each group with a BN-pair has a Weyl group $N/B \cap N$ which is a Coxeter group. In the case of the groups G and G^F the Weyl groups are W and W^F respectively. We shall therefore give an account of the basic properties of finite Coxeter groups. Most of these properties are well known and we shall not prove them here. However we have included proofs for certain properties which are somewhat less well known. The groups G and G^F both satisfy the axioms for an algebraic group with a split BN-pair. They also satisfy Chevalley's commutator relations. We shall derive some results on arbitrary algebraic groups with a split BN-pair satisfying the commutator relations. These results will be proved here, since the concept of an algebraic group with a split BN-pair is not a standard one. The results will be applied in subsequent chapters to the study of the groups G and G^F.

2.1 GROUPS WITH A BN-PAIR

Let G be a group with two subgroups B and N. These form a BN-pair if the following axioms are satisfied.

(i) $G = \langle B, N \rangle$.

(ii) $H = B \cap N$ is normal in N.

(iii) $N/H = W$ is generated by a set of elements s_i, $i \in I$, with $s_i^2 = 1$.

(iv) If $n_i \in N$ maps to $s_i \in W$ under the natural homomorphism $\pi: N \to W$ then $n_i B n_i \neq B$.

(v) For each $n \in N$ and each n_i we have $n_i B n \subseteq B n_i n B \cup BnB$.

We mention some consequences of these axioms. Proofs can be found in Bourbaki [2], chapter IV.

Proposition 2.1.1. *Let G be a group with a BN-pair. Then $G = BNB$.*

It follows from this proposition that every double coset of B in G will contain an element of N and so will be of the form BnB, $n \in N$.

Proposition 2.1.2. *Let $n, n' \in N$. Then $BnB = Bn'B$ if and only if $\pi(n) = \pi(n')$. Thus there is a bijective map between double cosets of B in G and elements of W under which BnB corresponds to $\pi(n)$.*

W is called the Weyl group of the BN-pair. If $w \in W$ we define $l(w)$ to be the minimal length of an expression of w as a product of the generators s_i, $i \in I$. In particular $l(w) = 0$ if and only if $w = 1$.

Proposition 2.1.3. *Let $n, n_i \in N$ and $w = \pi(n)$, $s_i = \pi(n_i)$. Then we have*

(i) $l(s_i w) = l(w) \pm 1$.
(ii) *If* $l(s_i w) = l(w) + 1$ *then* $n_i B n \subseteq B n_i n B$.
(iii) *If* $l(s_i w) = l(w) - 1$ *then* $n_i B n \not\subseteq B n_i n B$.

An expression of an element $w \in W$ as a product of generators s_i, $i \in I$, will be called reduced if it has length $l(w)$. It is clear that $l(s_i w) = l(w) - 1$ if and only if w has a reduced expression beginning with s_i.

Proposition 2.1.4. *Let J be a subset of the index set I. Let W_J be the subgroup of W generated by the elements s_i with $i \in J$ and let N_J be the subgroup of N satisfying $N_J/H = W_J$. Then BN_JB is a subgroup of G.*

We shall write $P_J = BN_JB$. Note that $P_I = G$ and $P_\varnothing = B$.

Proposition 2.1.5. *Let $n \in N$ and $w = \pi(n)$. Let $w = s_{i_1} \dots s_{i_k}$ be a reduced expression for w. Let $J = \{i_1, \dots, i_k\}$. Then the following subgroups of G are equal:*

(i) $\langle B, n \rangle$.
(ii) $\langle B, nBn^{-1} \rangle$.
(iii) P_J.

Note that the sequence $i_1, \dots, i_k$ may well contain repetitions, so that J is the set of elements of I which occur in this sequence.

Proposition 2.1.6. (i) *Any subgroup of G containing B is of the form P_J for some $J \subseteq I$.*

(ii) *If J, K are distinct subsets of I then P_J, P_K are distinct. In fact they are nonconjugate subgroups of G.*

(iii) *For all $J \subseteq I$ we have $N_G(P_J) = P_J$.*

Definition. A parabolic subgroup of G is a subgroup conjugate to P_J for some subset $J \subseteq I$.

Proposition 2.1.7. *If G has a BN-pair with Weyl group W then W is a Coxeter group with respect to the generators s_i, $i \in I$.*

2.2 THE GEOMETRICAL REPRESENTATION OF A FINITE COXETER GROUP

We have seen that the Weyl group of any group with a BN-pair is a Coxeter group. In general this group W need not be finite; indeed even the generating set s_i, $i \in I$, need not be finite. In the applications we have in mind, however, to the algebraic groups G and their finite subgroups G^F the Weyl group is always finite. We shall therefore concentrate attention on finite Coxeter groups and show how these may be regarded as groups generated by reflections in a Euclidean space.

Let W be a finite Coxeter group. Then W has a presentation $W = \langle s_1, s_2, \ldots, s_l; s_i^2 = 1\ (s_i s_j)^{m_{ij}} = 1\ i \neq j \rangle$ where m_{ij} is the order of $s_i s_j$. We shall describe an action of W on a vector space of dimension l.

Let V be a vector space over $\mathbb{R}$ of dimension l with basis $\alpha_1, \alpha_2, \ldots, \alpha_l$. We define a bilinear form on V, $(v, v') \to \langle v, v' \rangle$, by

$$\langle \alpha_i, \alpha_j \rangle = -\cos \frac{\pi}{m_{ij}}$$

and extended by linearity. This form is symmetric since $m_{ji} = m_{ij}$. In particular we have $\langle \alpha_i, \alpha_i \rangle = 1$ for each i. Let H_i be the subspace of V given by

$$H_i = \{ v \in V;\ \langle \alpha_i, v \rangle = 0 \}.$$

Then $\dim H_i = l - 1$ and we have

$$V = \mathbb{R}\alpha_i \oplus H_i.$$

We now define a linear map $\tau_i : V \to V$ by

$$\tau_i(v) = v - 2\langle \alpha_i, v \rangle \alpha_i.$$

Then $\tau_i(\alpha_i) = -\alpha_i$ and $\tau_i(v) = v$ whenever $v \in H_i$. Thus τ_i is the reflection in the hyperplane H_i. Thus $\tau_i^2 = 1$. It is also true that $(\tau_i \tau_j)^{m_{ij}} = 1$ if $i \neq j$. Thus there exists a homomorphism

$$\theta : W \to \langle \tau_1, \tau_2, \ldots, \tau_l \rangle$$

from W into the group generated by the τ_i given by $\theta(s_i) = \tau_i$. Since

$$\langle \tau v, \tau v' \rangle = \langle v, v' \rangle$$

for all $v, v' \in V$ this homomorphism θ gives a representation of W as a group of isometries of V.

We now state some properties of this representation. Proofs can be found in Bourbaki [2], chapter V.

Proposition 2.2.1. *The form $\langle v, v' \rangle$ on V is nonsingular and positive definite.*

We may therefore regard V as a Euclidean space.

We define

$$H_i^+ = \{ v \in V;\ \langle \alpha_i, v \rangle > 0 \}$$

$$H_i^- = \{ v \in V;\ \langle \alpha_i, v \rangle < 0 \}.$$

H_i^+ and H_i^- are the half-spaces separated by the hyperplane H_i. Let $C = H_1^+ \cap H_2^+ \cap \ldots \cap H_l^+$. C is a subset of V called the fundamental chamber.

Proposition 2.2.2. *Let $w \in W$. Then*
(i) *if $l(s_i w) = l(w) + 1$, $w(C) \subseteq H_i^+$*
(ii) *if $l(s_i w) = l(w) - 1$, $w(C) \subseteq H_i^-$.*

Proposition 2.2.3. *If $w \neq 1$ then $C \cap w(C)$ is empty.*

This follows from the fact that if we take a reduced expression $w = s_{i_1} s_{i_2} \ldots s_{i_k}$ then $l(s_{i_1} w) = l(w) - 1$ and so $C \subseteq H_{i_1}^+$ but $w(C) \subseteq H_{i_1}^-$.

Proposition 2.2.4. (i) *If $w(C) = C$ then $w = 1$.*
(ii) *W acts faithfully on V.*

This clearly follows from the previous proposition.

We shall now introduce the root system of W. A note of caution is needed here as this will not be quite the same concept as the root system of an algebraic group which was described in chapter 1.

Let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ and let $\Phi = W(\Delta)$. Then Φ is called the set of roots and Δ the set of simple roots. For each root $\alpha \in \Phi$ we define a corresponding hyperplane H_α given by

$$H_\alpha = \{\alpha \in V; \langle \alpha, v \rangle = 0\}.$$

The set of such hyperplanes is permuted by the group W. Consider the complement $V - \cup_\alpha H_\alpha$ of the union of the hyperplanes H_α. The connected components of this complement are called the chambers of V. The fundamental chamber C is a chamber in this sense. The image of any chamber under an element of W will also be a chamber.

We now state some basic properties of the roots. These properties do not depend upon whether the roots are defined as in this section or as in chapter 1. Proofs can be found, for example, in Bourbaki [2], chapter VI.

Proposition 2.2.5. *Each root $\alpha \in \Phi$ has the form $\alpha = \sum_{i=1}^{l} \lambda_i \alpha_i$ where either each $\lambda_i \geqslant 0$ or each $\lambda_i \leqslant 0$.*

We define Φ^+, Φ^- by

$$\Phi^+ = \{\alpha \in \Phi; \alpha = \textstyle\sum \lambda_i \alpha_i \quad \text{with each } \lambda_i \geqslant 0\}$$

$$\Phi^- = \{\alpha \in \Phi; \alpha = \textstyle\sum \lambda_i \alpha_i \quad \text{with each } \lambda_i \leqslant 0\}.$$

These are the positive and negative roots respectively.

Proposition 2.2.6. *The only positive root made negative by s_i is α_i.*

Proposition 2.2.7. *Let $w \in W$. Then $l(w)$ is the number of positive roots made negative by w.*

Proposition 2.2.8. *Let $w \in W$. Then*

(i) $l(s_i w) = l(w) + 1$ *if and only if* $w^{-1}(\alpha_i) \in \Phi^+$.
(ii) $l(s_i w) = l(w) - 1$ *if and only if* $w^{-1}(\alpha_i) \in \Phi^-$.
(iii) $l(w s_i) = l(w) + 1$ *if and only if* $w(\alpha_i) \in \Phi^+$.
(iv) $l(w s_i) = l(w) - 1$ *if and only if* $w(\alpha_i) \in \Phi^-$.

Proposition 2.2.9. *Let $w = s_{i_1} s_{i_2} \dots s_{i_k}$ be a reduced expression for w. Then the positive roots made negative by w are*

$$s_{i_k} s_{i_{k-1}} \dots s_{i_{j+1}}(\alpha_{i_j})$$

for $j = 1, 2, \dots, k$.

Proposition 2.2.10. *Let $w = s_{i_1} s_{i_2} \dots s_{i_k}$ be a reduced expression for w. Then if $l(s_j w) = l(w) - 1$ there exists a number h with $1 \leqslant h \leqslant k$ such that*

$$s_j s_{i_1} \dots s_{i_{h-1}} = s_{i_1} \dots s_{i_h}.$$

Consequently we have

$$s_j w = s_{i_1} \dots s_{i_{h-1}} s_{i_{h+1}} \dots s_{i_k}.$$

(This is called the exchange condition.)

Proposition 2.2.11. (i) *There is a unique element $w_0 \in W$ of maximal length.*

(ii) $w_0(\Phi^+) = \Phi^-$.
(iii) $l(w_0) = |\Phi^+|$.
(iv) $w_0^2 = 1$.

Proposition 2.2.12. *For each root $\alpha \in \Phi$ let w_α be the reflection in the hyperplane H_α.*

(i) *If $w \in W$, $v \in V$ and $w(v) = v$ then w is a product of reflections w_α each of which satisfies $w_\alpha(v) = v$.*

(ii) *The same holds for any set of vectors in V instead of a single vector v.*

Proposition 2.2.13. *Given any chamber C' there is a unique $w \in W$ with $w(C) = C'$. Thus the group W acts transitively and regularly on the set of chambers. In particular the number of chambers is $|W|$.*

Proposition 2.2.14. *The closure $\bar{C}$ of the fundamental chamber is a fundamental region for W on V. Thus each $v \in V$ can be transformed into a unique element of $\bar{C}$ by an element of W.*

We next state a result which may be regarded as a converse of the theorem that any finite Coxeter group has a representation as a group generated by reflections.

Proposition 2.2.15. *Let W be a finite group of transformations of a Euclidean* *space and suppose W is generated by reflections. Then W is a Coxeter group. Moreover any reflection in W is conjugate to one of the generating reflections.*

This is proved in Bourbaki [2], chapter V.

2.3 PARABOLIC SUBGROUPS OF A COXETER GROUP

Let $I = \{1, 2, \dots, l\}$ and J be a subset of I. Let $W_J = \langle s_i; i \in J \rangle$ be the subgroup of W generated by the s_i, $i \in J$. A parabolic subgroup of W is defined to be a subgroup conjugate to W_J for some $J \subseteq I$.

The proofs of the following statements about parabolic subgroups of W can be found in Bourbaki [2], chapter IV, or Carter [3], chapter 2.

Proposition 2.3.1. *Let J be a subset of I. Let $\Delta_J = \{\alpha_i; i \in J\}$ and $\Phi_J = W_J(\Delta_J)$. Let V_J be the subspace of V spanned by Δ_J. Then*

(i) $\Phi \cap V_J = \Phi_J$.

(ii) *W_J is generated by the s_i, $i \in J$, as a Coxeter group.*

(iii) *Φ_J is a root system for the Coxeter group W_J acting on the vector space V_J.*

Proposition 2.3.2. (i) *The subgroups W_J for distinct subsets $J \subseteq I$ are all distinct.*

(ii) $W_J \cap W_K = W_{J \cap K}$.

(iii) $\langle W_J, W_K \rangle = W_{J \cup K}$.

Thus the subgroups W_J form a lattice of 2^l subgroups of W.

We now consider the cosets of W_J in W. It is possible to find in each such coset a distinguished coset representative.

Proposition 2.3.3. (i) *Each coset wW_J has a unique element d_J of minimal length.*

(ii) *$l(d_J w_J) = l(d_J) + l(w_J)$ for all $w_J \in W_J$.*

(iii) *Let D_J be the set of all coset representatives of minimal length in their coset. Then $w \in D_J$ if and only if $w(\Delta_J) \subseteq \Phi^+$.*

The parabolic subgroups of W can be described geometrically in terms of the Coxeter complex of W. This is a family of subsets of V defined as follows. Let v, $v' \in V$. We write $v \sim v'$ if, for each hyperplane H_α, $\alpha \in \Phi$, the points v, v' are either both in H_α or both in $H_\alpha{}^+$ or both in $H_\alpha{}^-$. This is an equivalence relation on V. The equivalence classes are the simplices in the Coxeter complex. We have a partial ordering on such simplices in which two simplices are related in the ordering if and only if the first lies in the closure of the second. The maximal elements in this ordering are just the chambers. The elements of the Coxeter complex which lie in the closure $\bar{C}$ of the fundamental chamber are the sets C_J given by

$$\begin{aligned} C_J = \{v \in V; \langle \alpha_i, v \rangle &= 0 \quad \text{for all } i \in J \\ \langle \alpha_i, v \rangle &> 0 \quad \text{for all } i \notin J\} \end{aligned}$$

The Weyl group W acts on the Coxeter complex.

Proposition 2.3.4. *The following conditions on $w \in W$ are equivalent:*

(i) $w(C_J) = C_J$.

(ii) $w(v) = v$ for all $v \in C_J$.

(iii) $w \in W_J$.

Proposition 2.3.5. *The parabolic subgroups of W are the stabilizers in W of the elements of the Coxeter complex.*

2.4 POLYNOMIAL INVARIANTS OF A COXETER GROUP

Let W be a finite Coxeter group acting on a Euclidean space V in the manner described above. Let $\hat{V} = Hom(V, \mathbb{R})$ be the dual space of V. This consists of all linear functions on V with values in $\mathbb{R}$, and can be made into a W-module in a natural way. Let $\mathfrak{P}$ be the symmetric algebra of $\hat{V}$. Thus $\mathfrak{P}$ is the algebra of symmetric tensors on $\hat{V}$ and may be identified with the algebra of polynomial functions on V with values in $\mathbb{R}$. $\mathfrak{P}$ is a graded algebra $\mathfrak{P} = \oplus_i \mathfrak{P}_i$ where $\mathfrak{P}_i$ consists of the homogeneous polynomial functions on V of degree i. The W-action on $\hat{V} = \mathfrak{P}_1$ can be extended naturally to a W-action on $\mathfrak{P}$. $\mathfrak{P}$ is an integral domain, and we denote its field of fractions by $\mathfrak{F}$. $\mathfrak{F}$ is the field of rational functions on V. Let

$$\mathfrak{J} = \{f \in \mathfrak{P};\ w(f) = f \quad \text{for all } w \in W\}.$$

$\mathfrak{J}$ is an $\mathbb{R}$-subalgebra of $\mathfrak{P}$ called the algebra of polynomial invariants of W. The basic theorem on the structure of $\mathfrak{J}$ is as follows. The proof of this and subsequent results in this section can be found in Bourbaki [2], chapter V.

Proposition 2.4.1. (i) *$\mathfrak{J}$ is isomorphic to the polynomial ring $\mathbb{R}[x_1, \ldots, x_l]$ where* $l = \dim V$.

(ii) *$\mathfrak{J}$ can be generated as an $\mathbb{R}$-algebra by homogeneous polynomials $I_1, I_2, \ldots, I_l$.*

(iii) *The degrees $d_1, d_2, \ldots, d_l$ of $I_1, I_2, \ldots, I_l$ are uniquely determined, being independent of the choice of the homogeneous generators $I_1, \ldots, I_l$.*

(iv) *We have $d_1 + d_2 + \ldots + d_l = N + l$ where $N = |\Phi^+|$, and $d_1 d_2 \ldots d_l = |W|$.*

We now regard $\mathfrak{P}$ as an $\mathfrak{J}$-module, and consider its structure in this respect.

Proposition 2.4.2. *$\mathfrak{P}$ is a free $\mathfrak{J}$-module of rank equal to the order of W.*

The integral domain $\mathfrak{J}$ may also be embedded in its field of fractions $\mathfrak{F}'$. $\mathfrak{F}'$ is the field of W-invariant rational functions on V. It is a subfield of $\mathfrak{F}$. The W-action on $\mathfrak{P}$ can be extended naturally to an action on $\mathfrak{F}$, and the elements of W act trivially on the subfield $\mathfrak{F}'$.

Proposition 2.4.3. *$\mathfrak{F}$ is a Galois extension of $\mathfrak{F}'$ with degree $|\mathfrak{F}:\mathfrak{F}'| = |W|$. The Galois group of $\mathfrak{F}$ over $\mathfrak{F}'$ is W.*

We now consider how to choose convenient bases for $\mathfrak{P}$ as an $\mathfrak{I}$-module and for $\mathfrak{F}$ as a vector space over $\mathfrak{F}'$. Let $\mathfrak{P}^+ = \oplus_{i>0} \mathfrak{P}_i$ be the set of polynomial functions with constant term zero and $\mathfrak{I}^+ = \mathfrak{I} \cap \mathfrak{P}^+$. We consider the ideal $\mathfrak{P}\mathfrak{I}^+$ of $\mathfrak{P}$ generated by $\mathfrak{I}^+$. $\mathfrak{P}\mathfrak{I}^+$ is a graded subspace of $\mathfrak{P}$, in the sense that homogeneous components of elements of $\mathfrak{P}\mathfrak{I}^+$ lie in $\mathfrak{P}\mathfrak{I}^+$. Thus the quotient ring $\mathfrak{P}/\mathfrak{P}\mathfrak{I}^+$ is also a graded algebra. Moreover $\mathfrak{P}\mathfrak{I}^+$ is a W-submodule of $\mathfrak{P}$ and so $\mathfrak{P}/\mathfrak{P}\mathfrak{I}^+$ may be regarded as a W-module.

Proposition 2.4..4. (i) dim $\mathfrak{P}/\mathfrak{P}\mathfrak{I}^+ = |W|$.
(ii) *There is a graded subspace* $\mathfrak{M}$ *of* $\mathfrak{P}$ *which is a* W*-submodule and which satisfies* $\mathfrak{P} = \mathfrak{P}\mathfrak{I}^+ \oplus \mathfrak{M}$. *Thus* dim $\mathfrak{M} = |W|$.

We choose a basis for the vector space $\mathfrak{M}$. This turns out to be also a basis of $\mathfrak{P}$ as an $\mathfrak{I}$-module and a basis of $\mathfrak{F}$ as a vector space over $\mathfrak{F}'$.

Proposition 2.4.5. (i) *An* $\mathbb{R}$*-basis for* $\mathfrak{M}$ *is an* $\mathfrak{I}$*-basis for* $\mathfrak{P}$ *and also an* $\mathfrak{F}'$*-basis for* $\mathfrak{F}$.

(ii) *There are isomorphisms of* W*-modules*

$$\mathfrak{M} \otimes_{\mathbb{R}} \mathfrak{I} \to \mathfrak{P} \quad \textit{and} \quad \mathfrak{M} \otimes_{\mathbb{R}} \mathfrak{F}' \to \mathfrak{F}$$

$$\sum (m_i \otimes f_i) \to \sum m_i f_i \quad \sum (m_i \otimes f_i) \to \sum m_i f_i.$$

Now $\mathfrak{F}$, being a Galois extension of $\mathfrak{F}'$, will have a normal basis over $\mathfrak{F}'$, i.e. a basis whose elements are permuted by the Galois group W. We see in this way that $\mathfrak{F}$, regarded as an $\mathfrak{F}'W$-module, is isomorphic to $\mathfrak{F}'W$ and so gives the regular representation of W over $\mathfrak{F}'$. We obtain similar results for the W-actions on $\mathfrak{P}$ and $\mathfrak{M}$.

Proposition 2.4.6. (i) $\mathfrak{F}$ *is isomorphic to* $\mathfrak{F}'W$ *as an* $\mathfrak{F}'W$*-module and so gives the regular representation of* W *over* $\mathfrak{F}'$.

(ii) $\mathfrak{P}$ *is isomorphic to* $\mathfrak{I}W$ *as an* $\mathfrak{I}W$*-module and so gives the regular representation of* W *over* $\mathfrak{I}$.

(iii) $\mathfrak{M}$ *is isomorphic to* $\mathbb{R}W$ *as an* $\mathbb{R}W$*-module and so gives the regular representation of* W *over* $\mathbb{R}$.

This final result has important consequences for the representation theory of W when W is the Weyl group of a simple algebraic group G. In this case all the irreducible complex representations of W can be written over $\mathbb{R}$ (even over $\mathbb{Q}$) and so $\mathfrak{M}$ can be expressed as a direct sum of absolutely irreducible W-submodules. In this decomposition each summand can be taken inside one of the homogeneous components of $\mathfrak{M}$. Each absolutely irreducible representation of W occurs in this decomposition with multiplicity equal to its degree. Thus for each irreducible representation of W we obtain a set of integers giving the degrees of the graded components of $\mathfrak{M}$ in which the representation occurs. We shall see in chapter 11 how this information is useful in the representation theory of the Weyl group W of G.

2.5 ALGEBRAIC GROUPS WITH A SPLIT BN-PAIR

We recall that an algebraic group with a split BN-pair is a linear algebraic group G satisfying the following axioms:

(i) G has closed subgroups B and N which form a BN-pair.

(ii) $B = UH$, where $H = B \cap N$ is the semidirect product of a closed normal unipotent group U and a closed commutative subgroup H, all of whose elements are semisimple.

(iii) $\bigcap_{n \in N} nBn^{-1} = H$.

We do not assume that G is connected in this definition, so the definition can apply in particular to finite linear algebraic groups. A finite group G is said to have a split BN-pair of characteristic p if G has subgroups B, N which form a BN-pair; if $B = UH$ where U is a normal p-subgroup of B and H is an abelian subgroup of order prime to p; and if $\bigcap_{n \in N} nBn^{-1} = H$. Then any finite group with a split BN-pair of characteristic p can be regarded as a linear algebraic group over an algebraically closed field of characteristic p, and this will be an algebraic group with split BN-pair as defined above.

Thus the above axiom system for an algebraic group with split BN-pair will include both connected algebraic groups with a split BN-pair and finite groups with a split BN-pair of characteristic p. In particular, if G is a connected reductive group and $F: G \to G$ is a Frobenius map then both G and G^F satisfy the axioms for an algebraic group with split BN-pair.

We shall now derive some consequences of these axioms. In the case of finite groups these were proved by Richen [1]. We follow Richen's development quite closely, although changes are needed in some places because of the more general axiom system.

We first introduce some notation. For $x, y \in G$ we write $x^y = y^{-1}xy$ and ${}^yx = yxy^{-1}$. All the earlier results in this chapter apply to the present situation and we carry over the same notation. Let $n_0 \in N$ be such that $\pi(n_0) = w_0$. We define subgroups U^-, X_i, U_i, X_{-i}, U_w of G as follows:

$$U^- = U^{n_0}$$

$$X_i = U \cap (U^-)^{n_i} \qquad i \in \{1, 2, \dots, l\}$$

$$U_i = U \cap U^{n_i}$$

$$X_{-i} = X_i^{n_i}$$

$$U_w = U \cap U^{n_0 n_w} \qquad w \in W.$$

These subgroups are independent of the choice of elements n_0, n_i, $n_w \in N$ such that $\pi(n_0) = w_0$, $\pi(n_i) = s_i$, $\pi(n_w) = w$. It will sometimes be convenient to denote by $\dot{w}$ an element of N for which $\pi(\dot{w}) = w$. Note in particular that a coset $B\dot{w}$ or a double coset $B\dot{w}B$ is independent of the choice of this representative $\dot{w}$ but depends only upon the element $w \in W$.

In the following propositions we derive further properties of an algebraic group G with a split BN-pair.

Proposition 2.5.1. *U is a maximal unipotent subgroup of G.*

Proof. Suppose M is a unipotent subgroup of G properly containing U. If M lies in B then $M = U(H \cap M)$. But H contains only semisimple elements and M only unipotent elements, so $H \cap M = 1$. Thus $M = U$, a contradiction. Thus M does not lie in B. In fact $M \cap B = U$. Now every unipotent algebraic group is nilpotent, and in a nilpotent group every proper subgroup is properly contained in its normalizer. Let $P = N_G(U)$. Then P contains B and P contains $N_M(U)$, which is strictly larger than U. Thus P is strictly larger than B, since $M \cap B = U$. Hence P is a parabolic subgroup of G, so has the form P_J for some non-empty set J. Let $i \in J$. Then $n_i \in P_J$ and so $U^{n_i} = U$. Also $H^{n_i} = H$ and so $B^{n_i} = B$. This contradicts one of the axioms for a BN-pair.

Proposition 2.5.2. *G has no proper unipotent normal subgroup.*

Proof. Let M be a normal unipotent subgroup of G. Then UM is a subgroup of G. We have a surjective homomorphism $U \to UM/M$ so UM/M is unipotent. Since M and UM/M are both unipotent UM is unipotent also. Since U is a maximal unipotent subgroup of G we have $UM = U$ and so $M \subseteq U$. In particular $M \subseteq B$. It follows that $M \subseteq B^n$ for all $n \in N$ and so $M \subseteq \bigcap_{n \in N} B^n = H$. But all elements of H are semisimple and so $M = 1$.

Proposition 2.5.3. *If $l(s_i w) = l(w) + 1$ then $n_i B n_i \cap B\dot{w}B\dot{w}^{-1} \subseteq B$.*

Proof. Suppose this were false. Then, since $n_i B n_i \subseteq B \cup B n_i B$, we have

$$Bn_iB \cap B\dot{w}B\dot{w}^{-1} \neq \varnothing$$

Hence

$$Bn_iB\dot{w} \cap B\dot{w}B \neq \varnothing.$$

But we also have

$$Bn_iB\dot{w} \subseteq Bn_i\dot{w}B \cup B\dot{w}B.$$

Hence $Bn_iB\dot{w} \not\subseteq Bn_i\dot{w}B$. This contradicts the condition $l(s_i w) = l(w) + 1$ by 2.1.3.

Proposition 2.5.4. *If $l(s_i w) = l(w) + 1$ then $B \cap B^{n_i\dot{w}} \subseteq B \cap B^{\dot{w}}$.*

Proof. $\dot{w}(B \cap B^{n_i\dot{w}})\dot{w}^{-1} = \dot{w}B\dot{w}^{-1} \cap n_iBn_i \subseteq B$ by 2.5.3. Thus $B \cap B^{n_i\dot{w}} \subseteq B \cap B^{\dot{w}}$.

Proposition 2.5.5. (i) $B \cap U^- = 1$.
(ii) $UH \cap U^-H = H$.

Proof. Let $w \in W$ and define a sequence $w_1, w_2, \ldots, w_k$ of elements of W as follows. Let $w_1 = w$ and suppose w_i has already been defined. If $w_i = w_0$ we take $i = k$ and finish the sequence. If $w_i \neq w_0$ there exists $\alpha_j \in \Delta$ with $w_i(\alpha_j) \in \Phi^+$. Then let $w_{i+1} = w_i s_j$. We have

$$l(w_{i+1}) = l(w_i) + 1 \quad \text{by 2.2.8.}$$

Thus we shall eventually have $w_k = w_0$, the unique element of maximal length in W. We now apply 2.5.4 and obtain

$$B \cap B^{n_0} = B \cap B^{\dot{w}_k^{-1}} \subseteq B \cap B^{\dot{w}_{k-1}^{-1}} \subseteq \ldots \subseteq B \cap B^{\dot{w}_1^{-1}} = B \cap B^{\dot{w}^{-1}}.$$

Hence

$$H \subseteq B \cap B^{n_0} \subseteq \bigcap_{w \in W} (B \cap B^{\dot{w}^{-1}}) = \bigcap_{n \in N} B^n = H$$

and so $B \cap B^{n_0} = H$. This gives $UH \cap U^-H = H$.

Now consider $B \cap U^-$. We have $B \cap U^- \subseteq UH \cap U^-H = H$. However the elements of $B \cap U^-$ are all unipotent and those of H are all semisimple, so $B \cap U^- = 1$.

Proposition 2.5.6. (i) $B = (B \cap B^{n_i})(B \cap B^{n_0 n_i})$.
(ii) $B \cap B^{n_0 n_i} \neq H$.

Proof. (i) Suppose $w \in W$ satisfies $l(ws_i) = l(w) + 1$. Then $l(s_i w^{-1}) = l(w^{-1}) + 1$ and so

$$n_i B \dot{w}^{-1} \subseteq B n_i \dot{w}^{-1} B \qquad \text{by 2.1.3.}$$

Hence $B \subseteq n_i^{-1} B n_i \dot{w}^{-1} B \dot{w}$. Let $b \in B$. Then $b = b'b''$ with $b' \in B^{n_i}$ and $b'' \in B^{\dot{w}}$. Now $b' \in n_i B n_i \cap B \dot{w}^{-1} B \dot{w} \subseteq B$, by 2.5.3. Hence b', b'' both lie in B. Thus

$$B = (B \cap B^{n_i})(B \cap B^{\dot{w}}).$$

Applying this with $w = w_0 s_i$ gives

$$B = (B \cap B^{n_i})(B \cap B^{n_0 n_i})$$

as required.

(ii) If $B \cap B^{n_0 n_i} = H$ then it follows from (i) that $B = B \cap B^{n_i}$, so that $B^{n_i} \supseteq B$. Applying n_i we obtain $B \supseteq B^{n_i}$. Hence $B^{n_i} = B$, which contradicts one of the axioms for a BN-pair. ■

We define $B_i = B \cap B^{n_0 n_i}$ and $B_w = B \cap B^{n_0 \dot{w}}$ for each $w \in W$.

Proposition 2.5.7. (i) *If* $w(\alpha_i) \in \Phi^+$ *then* $B_i \subseteq B_{w_0 w}$.
(ii) *If* $w(\alpha_i) \in \Phi^-$ *then* $B_i \cap B_{w_0 w} = H$.

Proof. (i) Suppose $w(\alpha_i) \in \Phi^+$. Then $l(ws_i) = l(w) + 1$. Consider a reduced expression for $w_0 s_i w^{-1}$ given by

$$w_0 s_i w^{-1} = s_{i_1} s_{i_2} \ldots s_{i_k}.$$

Let $w_1, w_2, \ldots, w_k$ be defined by $w_j = s_{i_j} \ldots s_{i_k} w$. Then we have $l(w_j) = l(w_{j+1}) + 1$ for $j = 1, 2, \ldots, k-1$ and $l(w_k) = l(w) + 1$. By 2.5.4 we have

$$B_i = B \cap B^{n_0 n_i} = B \cap B^{\dot{w}_1} \subseteq B \cap B^{\dot{w}_2} \subseteq \ldots \subseteq B \cap B^{\dot{w}_k} \subseteq B \cap B^{\dot{w}}.$$

Thus $B_i \subseteq B_{w_0 w}$.

(ii) Now suppose $w(\alpha_i) \in \Phi^-$. Then

$$B_i \cap B_{w_0 w} = B \cap B^{n_0 n_i} \cap B^{\dot{w}} = (B^{n_i} \cap B^{n_0} \cap B^{\dot{w} n_i})^{n_i}.$$

But $B^{n_i} \cap B^{\dot{w} n_i} \subseteq B$ by 2.5.3 since $l(s_i . s_i w^{-1}) = l(s_i w^{-1}) + 1$. Hence $B_i \cap B_{w_0 w} \subseteq (B \cap B^{n_0})^{n_i} = H^{n_i} = H$. Since $H \subseteq B_i \cap B_{w_0 w}$ we obtain the required result.

Proposition 2.5.8. (i) *If* $w(\alpha_i) \in \Phi^+$ *then* $B_{ws_i} = B_i B_w{}^{n_i}$ *and* $B_i \cap B_w{}^{n_i} = H$.
(ii) *If* $w(\alpha_i) \in \Phi^-$ *then* $B_w = B_i B_{ws_i}{}^{n_i}$ *and* $B_i \cap B_{ws_i}{}^{n_i} = H$.

Proof. (i) Suppose $w(\alpha_i) \in \Phi^+$. Then we have $w_0 w s_i(\alpha_i) \in \Phi^+$ also. By 2.5.7 this implies $B_i \subseteq B_{ws_i}$. Now

$$B_{ws_i} = B_{ws_i} \cap B = B_{ws_i} \cap B_i B_{w_0 s_i} = B_i(B_{ws_i} \cap B_{w_0 s_i})$$

by 2.5.6. We also have

$$B_{ws_i} \cap B_{w_0 s_i} = B \cap B^{n_0 \dot{w} n_i} \cap B^{n_i} = (B^{n_i} \cap B^{n_0 \dot{w}} \cap B)^{n_i}.$$

Now $B^{n_i} \cap B^{n_0 \dot{w} n_i} \subseteq B$ by 2.5.3 since $w_0 w(\alpha_i) \in \Phi^-$. It follows that $B \cap B^{n_0 \dot{w}} \subseteq B^{n_i}$ and so

$$B_{ws_i} \cap B_{w_0 s_i} = (B \cap B^{n_0 \dot{w}})^{n_i} = B_w{}^{n_i}$$

and $B_{ws_i} = B_i B_w{}^{n_i}$. Also we have

$$B_i \cap B_w{}^{n_i} = B \cap B^{n_0 n_i} \cap B^{n_i} \cap B^{n_0 \dot{w} n_i} \subseteq (B^{n_0} \cap B)^{n_i}.$$

But $B^{n_0} \cap B = H$ by 2.5.5 and so $B_i \cap B_w{}^{n_i} = H$.

(ii) Suppose now that $w(\alpha_i) \in \Phi^-$. Then $ws_i(\alpha_i) \in \Phi^+$ and so, replacing w by ws_i in (i), we obtain

$$B_w = B_i B_{ws_i}{}^{n_i}, \quad B_i \cap B_{ws_i}{}^{n_i} = H.$$

Proposition 2.5.9. $B_w = U_w H$ *for all* $w \in W$.

Proof.

$$B \cap B^{\dot{w}} = UH \cap U^{\dot{w}} H \supseteq (U \cap U^{\dot{w}})H.$$

We show that equality holds here. Let $g \in UH \cap U^{\dot{w}}H$. Then $g = u^{\dot{w}} h = u'h'$ where $u, u' \in U$ and $h, h' \in H$. Hence $u^{\dot{w}} = u'h'h^{-1}$ is a unipotent element of B. However all the unipotent elements of B lie in U. Thus $u^{\dot{w}} \in U \cap U^{\dot{w}}$ and so $g \in (U \cap U^{\dot{w}})H$. Thus

$$B \cap B^{\dot{w}} = (U \cap U^{\dot{w}})H$$

for all $w \in W$. The result follows.

Proposition 2.5.10. (i) *If* $w(\alpha_i) \in \Phi^+$ *then* $U_{ws_i} = X_i U_w{}^{n_i}$ *and* $X_i \cap U_w{}^{n_i} = 1$.
(ii) *If* $w(\alpha_i) \in \Phi^-$ *then* $U_w = X_i U_{ws_i}{}^{n_i}$ *and* $X_i \cap U_{ws_i}{}^{n_i} = 1$.

Proof. (i) Let $w(\alpha_i) \in \Phi^+$. Then by 2.5.8 we have

$$B_{ws_i} = B_i B_w^{n_i}, \; B_i \cap B_w^{n_i} = H.$$

By 2.5.9 this implies that

$$U_{ws_i} H = X_i H U_w^{n_i} H = X_i U_w^{n_i} H.$$

Intersecting with U we obtain

$$U_{ws_i} = X_i U_w^{n_i}.$$

We also have

$$X_i \cap U_w^{n_i} \subseteq B_i \cap B_w^{n_i} \cap U \subseteq H \cap U = 1$$

again by 2.5.8.

(ii) Now let $w(\alpha_i) \in \Phi^-$. Then $ws_i(\alpha_i) \in \Phi^+$ and so by (i) we have

$$U_w = X_i U_{ws_i}^{n_i}, \; X_i \cap U_{ws_i}^{n_i} = 1.$$

Proposition 2.5.11. $U = X_i U_i = U_i X_i$ *and* $U_i \cap X_i = 1$.

Proof. We take $w = w_0$ in 2.5.10. Then $w(\alpha_i) \in \Phi^-$ and so

$$U_{w_0} = X_i U_{w_0 s_i}^{n_i}, \qquad X_i \cap U_{w_0 s_i}^{n_i} = 1.$$

Now $U_{w_0} = U$ and $U_{w_0 s_i}^{n_i} = (U \cap U^{n_i})^{n_i} = U^{n_i} \cap U = U_i$. Thus $U = X_i U_i$ and $X_i \cap U_i = 1$.

We note that $X_i \neq 1$ since $B_i = X_i H$ and $B_i \neq H$ by 2.5.6.

Proposition 2.5.12. *For each* $w \in W$ *we have* $U = U_{w_0 w} U_w$ *and* $U_{w_0 w} \cap U_w = 1$.

Proof. Let $w = s_{i_1} s_{i_2} \dots s_{i_k}$ be a reduced expression for w. We define elements $w_1, w_2, \dots, w_k \in W$ by $w_j = s_{i_j} s_{i_{j+1}} \dots s_{i_k}$. We have

$$w_{j+1}^{-1} = s_{i_k} s_{i_{k-1}} \dots s_{i_{j+1}}$$

and so

$$l(w_{j+1}^{-1} s_{i_j}) = l(w_{j+1}^{-1}) + 1.$$

Hence

$$l(w_0 w_{j+1}^{-1} s_{i_j}) = l(w_0 w_{j+1}^{-1}) - 1$$

and so $w_0 w_{j+1}^{-1}(\alpha_{i_j}) \in \Phi^-$ for $j = 1, 2, \dots, k-1$. We apply 2.5.10 repeatedly and obtain

$$\begin{aligned} U = U_{w_0} &= X_{i_k} (U_{w_0 w_k^{-1}})^{\dot{w}_k} \\ &= X_{i_k} X_{i_{k-1}}{}^{\dot{w}_k} (U_{w_0 w_{k-1}^{-1}})^{\dot{w}_{k-1}} \\ &= \dots\dots\dots\dots\dots \\ &= X_{i_k} X_{i_{k-1}}{}^{\dot{w}_k} \dots X_{i_1}{}^{\dot{w}_2} (U_{w_0 w_1^{-1}})^{\dot{w}_1}. \end{aligned}$$

However by applying 2.5.10 to the reduced expression $w = s_{i_1}s_{i_2} \dots s_{i_k}$ we obtain

$$U_w = X_{i_k}X_{i_{k-1}}{}^{\dot{w}_k}X_{i_{k-2}}{}^{\dot{w}_{k-1}} \dots X_{i_1}{}^{\dot{w}_2}.$$

We also have

$$(U_{w_0w_1^{-1}})^{\dot{w}_1} = (U_{w_0w^{-1}})^{\dot{w}} = (U \cap U^{\dot{w}^{-1}})^{\dot{w}} = U \cap U^{\dot{w}} = U_{w_0w}.$$

Thus $U = U_wU_{w_0w}$ as required.

We next wish to show that $U_w \cap U_{w_0w} = 1$. Since

$$U_w = X_{i_k}X_{i_{k-1}}{}^{\dot{w}_k}X_{i_{k-2}}^{\dot{w}_{k-1}} \dots X_{i_1}^{\dot{w}_2}$$

this would follow from the relations

$$U_{w_0w} \cap X_{i_1}{}^{\dot{w}_2} = 1$$
$$U_{w_0w}X_{i_1}^{\dot{w}_2} \cap X_{i_2}^{\dot{w}_3} = 1$$
$$U_{w_0w}X_{i_1}{}^{\dot{w}_2}X_{i_2}^{\dot{w}_3} \cap X_{i_3}^{\dot{w}_4} = 1$$
$$\vdots$$
$$U_{w_0w}X_{i_1}^{\dot{w}_2}X_{i_2}^{\dot{w}_3} \dots X_{i_{k-1}}^{\dot{w}_k} \cap X_{i_k} = 1.$$

We prove these relations in the following manner. The definition of the elements $w_1, w_2, \dots, w_k$ shows that $w_1^{-1}(\alpha_{i_1}) \in \Phi^-$, $w_2^{-1}(\alpha_{i_2}) \in \Phi^-, \dots, w_k^{-1}(\alpha_{i_k}) \in \Phi^-$. It follows that

$$w_0w_j^{-1}s_{i_j}(\alpha_{i_j}) \in \Phi^-$$

for $j = 1, 2, \dots, k$. We now apply 2.5.10, which asserts that if $w(\alpha_i) \in \Phi^-$ then $U_{ws_i}{}^{n_i} \cap X_i = 1$. Replacing w by $w_0w_j{}^{-1}s_{i_j}$ and α_i by α_{i_j} we obtain

$$U_{w_0w_j^{-1}}{}^{n_{i_j}} \cap X_{i_j} = 1.$$

Transforming by $\dot{w}_{j+1}$ we get

$$U_{w_0w_j^{-1}}{}^{\dot{w}_j} \cap X_{i_j}{}^{\dot{w}_{j+1}} = 1.$$

However

$$(U_{w_0w_j^{-1}})^{\dot{w}_j} = (U \cap U^{\dot{w}_j^{-1}})^{\dot{w}_j} = U_{w_0w_j}.$$

Hence

$$U_{w_0w_j} \cap X_{i_j}{}^{\dot{w}_{j+1}} = 1.$$

Now w_j is a final segment for the reduced expression for w, and so w_0w will be a final segment of a reduced expression for w_0w_j. In order to write down such an expression for w_0w_j in terms of w_0w we make the following observations. We have $l(w_0s_i) = l(w_0) - 1$ for each s_i. Hence $l(w_0s_iw_0) = l(w_0) - l(w_0s_i) = 1$ for each s_i. Thus $w_0s_iw_0 = s_{\bar{i}}$ for some $\bar{i} \in I$. Now we have

$$w = s_{i_1}s_{i_2} \dots s_{i_{j-1}}w_j$$

and so

$$\begin{aligned} w_0 w_j &= w_0 s_{i_{j-1}} \ldots s_{i_2} s_{i_1} w_0 w_0 w \\ &= s_{\bar{i}_{j-1}} \ldots s_{\bar{i}_2} s_{\bar{i}_1} w_0 w \end{aligned}$$

and $l(w_0 w_j) = l(w_0 w) + (j - 1)$. Since $w_0 w$ is a final segment of $w_0 w_j$ we can obtain a factorization of $U_{w_0 w_j}$ with $U_{w_0 w}$ as the initial factor. In fact we get

$$U_{w_0 w_j} = U_{w_0 w} X_{\bar{i}_0}{}^{n_0 \dot{w}} X_{\bar{i}_2}{}^{n_{\bar{i}_1} n_0 \dot{w}} \ldots X_{\bar{i}_{j-1}}{}^{n_{\bar{i}_{j-2}} \ldots n_{\bar{i}_1} n_0 \dot{w}}.$$

We next observe that $X_i{}^{n_i n_0} = X_{\bar{i}}$. For

$$\begin{aligned} X_i{}^{n_i n_0} &= (U \cap (U^-)^{n_i})^{n_i n_0} = U \cap (U^-)^{n_0 n_i n_0} \\ &= U \cap (U^-)^{n_{\bar{i}}} = X_{\bar{i}}. \end{aligned}$$

Thus we have

$$\begin{aligned} U_{w_0 w_j} &= U_{w_0 w} X_{i_1}{}^{n_{i_1} \dot{w}} X_{i_2}{}^{n_{i_2} n_{i_1} \dot{w}} \ldots X_{i_{j-1}}{}^{n_{i_{j-1}} \ldots n_{i_1} \dot{w}} \\ &= U_{w_0 w} X_{i_1}{}^{\dot{w}_2} X_{i_2}{}^{\dot{w}_3} \ldots X_{i_{j-1}}{}^{\dot{w}_j}. \end{aligned}$$

Finally we obtain the required relation

$$U_{w_0 w} X_{i_1}{}^{\dot{w}_2} X_{i_2}{}^{\dot{w}_3} \ldots X_{i_{j-1}}{}^{\dot{w}_j} \cap X_{i_j}{}^{\dot{w}_{j+1}} = 1$$

and the result is proved.

The mapping $i \to \bar{i}$ of I into itself introduced in the proof of 2.5.12 will be useful subsequently in a number of places. It is clearly involutary in the sense that $\bar{\bar{i}} = i$ for all i, and is called the *opposition involution*. The opposition involution determines a map $\alpha_i \to \alpha_{\bar{i}}$ of the set Δ of simple roots into itself. This map is given by $\alpha_{\bar{i}} = -w_0(\alpha_i)$.

Proposition 2.5.13. *$B\dot{w}B = B\dot{w}U_w$. Moreover each element of $B\dot{w}B$ is uniquely expressible in the form $b\dot{w}u$ with $b \in B$ and $u \in U_w$.*

Proof. We use 2.5.12. This gives

$$\begin{aligned} B\dot{w}B &= B\dot{w}HU = BH\dot{w}U = B\dot{w}U = B\dot{w}U_{w_0 w} U_w \\ &= B\dot{w}(U \cap U^{\dot{w}})U_w = B(U^{\dot{w}^{-1}} \cap U)\dot{w}U_w \\ &= B\dot{w}U_w. \end{aligned}$$

We now prove uniqueness of expression. Suppose $b\dot{w}u = b'\dot{w}u'$ where $b, b' \in B$ and $u, u' \in U_w$. Then

$$b^{-1}b' = \dot{w}uu'^{-1}\dot{w}^{-1} \in B \cap (U_w)^{\dot{w}^{-1}}.$$

However

$$B \cap (U_w)^{\dot{w}^{-1}} = B \cap U^{\dot{w}^{-1}} \cap U^{n_0} = 1$$

by 2.5.5. Hence $b' = b$ and $u' = u$.

Theorem 2.5.14. *Each element of G is uniquely expressible in the form* $uh\dot{w}u'$ *where* $u \in U$, $h \in H$, $w \in W$ *and* $u' \in U_w$.

Proof. We assume the representatives $\dot{w} \in N$ are chosen and kept fixed. Then each element of G lies in a double coset $B\dot{w}B$ for a unique $\dot{w} \in W$. Each element of this double coset can be expressed uniquely in the form $b\dot{w}u'$ where $b \in B$ and $u' \in U_w$. Finally each $b \in B$ is uniquely expressible in the form $b = uh$ with $u \in U$ and $h \in H$. ∎

The canonical form for elements of G given by this theorem is an extremely useful result. In particular it enables us to calculate the order of G when G is a finite group. We shall discuss the order formula later in the chapter.

We now introduce the root subgroups of an algebraic group with split BN-pair.

Proposition 2.5.15. *The set of subgroups* nX_in^{-1} *for* $n \in N$ *and* $i \in I$ *is in bijective correspondence with the set* Φ *of roots. If* $\pi(n) = w$ *the root corresponding to the subgroup* nX_in^{-1} *is* $w(\alpha_i)$.

Proof. We wish to show that the map $w(\alpha_i) \to \dot{w}X_i\dot{w}^{-1}$ is a bijective map from roots to root subgroups. It is sufficient to show that $w(\alpha_i) = \alpha_j$ if and only if $\dot{w}X_i\dot{w}^{-1} = X_j$. For this will imply

$$\begin{aligned} w'(\alpha_i) = w(\alpha_j) \quad &\text{if and only if } w^{-1}w'(\alpha_i) = \alpha_j \\ &\text{if and only if } \dot{w}^{-1}\dot{w}'X_i\dot{w}'^{-1}\dot{w} = X_j \\ &\text{if and only if } \dot{w}'X_i\dot{w}'^{-1} = \dot{w}X_j\dot{w}^{-1}. \end{aligned}$$

Suppose then that $w(\alpha_i) = \alpha_j$. Then $w_0s_jw(\alpha_i) \in \Phi^+$ and so $B_i \subseteq B_{w_0w}$ and $B_i \subseteq B_{s_jw}$ by 2.5.7. Hence

$$\dot{w}B_i\dot{w}^{-1} \subseteq B^{\dot{w}^{-1}} \cap B \cap B^{n_0n_j} \subseteq B_j.$$

The same argument applied to the equation $w^{-1}(\alpha_j) = \alpha_i$ shows that $\dot{w}^{-1}B_j\dot{w} \subseteq B_i$. Thus we have $\dot{w}B_i\dot{w}^{-1} = B_j$. Now $B_i = X_iH$ and $B_j = X_jH$ by 2.5.9. Hence $\dot{w}X_i\dot{w}^{-1}H = X_jH$ and it follows that $\dot{w}X_i\dot{w}^{-1} = X_j$.

Suppose conversely that $\dot{w}X_i\dot{w}^{-1} = X_j$. Then $\dot{w}B_i\dot{w}^{-1} = X_jH = B_j = B \cap B^{n_0n_j}$. Hence

$$B_i \subseteq B^{\dot{w}} \cap B^{n_0n_j\dot{w}}.$$

In particular $B_i \subseteq B \cap B^{\dot{w}} = B_{w_0w}$. We now apply 2.5.7. If $w(\alpha_i) \in \Phi^-$ then $B_i \cap B_{w_0w} = H$ and so $B_i = H$ which contradicts 2.5.6. Thus $w(\alpha_i) \in \Phi^+$. Also

$$B_i \subseteq B \cap B^{n_0n_j\dot{w}} = B_{s_jw}.$$

By 2.5.7 again we have $w_0s_jw(\alpha_i) \in \Phi^+$ and so $s_jw(\alpha_i) \in \Phi^-$. Since the only positive root made negative by s_j is α_j we have $w(\alpha_i) = \alpha_j$. ∎

If $w(\alpha_i) = \alpha$ we shall denote the root subgroup $\dot{w}X_i\dot{w}^{-1}$ by X_α.

Proposition 2.5.16. *Let $w = s_{i_1}s_{i_2}\dots s_{i_k}$ be a reduced expression for w and let $\beta_1, \beta_2, \dots, \beta_k \in \Phi$ be defined by*

$$\beta_1 = \alpha_{i_k}$$
$$\beta_2 = s_{i_k}(\alpha_{i_{k-1}})$$
$$\beta_3 = s_{i_k}s_{i_{k-1}}(\alpha_{i_{k-2}})$$
$$\vdots$$
$$\beta_k = s_{i_k}s_{i_{k-1}}\dots s_{i_2}(\alpha_{i_1}).$$

Thus $\beta_1, \beta_2, \dots, \beta_k$ are the positive roots made negative by w. Then we have

$$U_w = X_{\beta_1}X_{\beta_2}\dots X_{\beta_k}$$

with uniqueness. Thus each element $u \in U_w$ is uniquely expressible in the form $u = x_{\beta_1}x_{\beta_2}\dots x_{\beta_k}$ with $x_{\beta_i} \in X_{\beta_i}$.

Proof. We use induction on $l(w)$. The result is obvious if $l(w) = 0$ or 1. Assume the result for $l(w) = k - 1$. Let $w = s_{i_1}s_{i_2}\dots s_{i_k}$ have length k and write $w' = ws_{i_k}$. Then $l(w') = k - 1$ and

$$U_w = X_{i_k}U_{w'}{}^{n_{i_k}},\ X_{i_k} \cap U_{w'}{}^{n_{i_k}} = 1$$

by 2.5.10. By induction we have

$$U_{w'} = X_{\gamma_1}X_{\gamma_2}\dots X_{\gamma_{k-1}}$$

with uniqueness where

$$\gamma_1 = \alpha_{i_{k-1}},\ \gamma_2 = s_{i_{k-1}}(\alpha_{i_{k-2}}),\ \dots.$$

Thus

$$U_{w'}{}^{n_{i_k}} = X_{\beta_2}X_{\beta_3}\dots X_{\beta_k}$$

with uniqueness. It follows that

$$U_w = X_{\beta_1}X_{\beta_2}\dots X_{\beta_k}$$

with uniqueness.

Corollary 2.5.17. *$U = \prod_{\alpha\in\Phi^+} X_\alpha$ with uniqueness, if the positive roots are taken in a suitable order in the product.*

Results for U^- similar to those in 2.5.16 and 2.5.17 can be obtained by conjugating by n_0.

2.6 PARABOLIC SUBGROUPS AND THE LEVI DECOMPOSITION

As before G denotes an algebraic group with split BN-pair. We wish to find in G a subgroup which has a split BN-pair with root system $\{\alpha_i, -\alpha_i\}$. The first step is to prove the following result.

Proposition 2.6.1. *$X_iH \cup X_iHn_iX_i$ is a subgroup of G.*

Proof. Let $i \in I$ and let $w_0 s_i w_0 = s_j$. Then $j = \bar{i}$ is the image of i under the opposition involution. Let P_i, P_j be the parabolic subgroups P_J for $J = \{i\}$, $J = \{j\}$ respectively. Then

$$P_i = B \cup Bn_iB,\ P_j = B \cup Bn_jB.$$

We shall show that $X_iH \cup X_iHn_iX_i$ is a subgroup by proving that

$$X_iH \cup X_iHn_iX_i = P_i \cap P_j^{n_0}.$$

We have

$$\begin{aligned} P_j^{n_0} &= B^{n_0} \cup B^{n_0}n_iB^{n_0} = B^{n_0} \cup n_iB^{n_0n_i}B^{n_0} \\ &= U^-H \cup n_i(U^-)^{n_i}HU^-. \end{aligned}$$

Now $U^- = \prod_{\alpha\in\Phi^-} X_\alpha$ by 2.5.17, and so

$$(U^-)^{n_i} = \prod_{\alpha\in\Phi^-} X_\alpha^{n_i} = \prod_{\alpha\in\Phi^-} X_{s_i(\alpha)} = X_i \prod_{\substack{\alpha\in\Phi^- \\ \alpha\neq-\alpha_i}} X_\alpha.$$

Hence

$$\begin{aligned} P_j^{n_0} &= U^-H \cup n_iX_i \prod_{\substack{\alpha\in\Phi^- \\ \alpha\neq-\alpha_i}} X_\alpha HU^- \\ &= U^-H \cup n_iX_iHU^- \\ &= n_iU^-H \cup X_iHU^- \quad \text{since } n_i \in P_j^{n_0}. \end{aligned}$$

Thus we have

$$B \cap P_j^{n_0} = (B \cap n_iU^-H) \cup (B \cap X_iHU^-).$$

Now

$$B \cap n_iU^-H = B \cap n_in_0UHn_0 = (Bn_0 \cap n_in_0B)n_0$$

which is empty, since the double cosets Bn_0B and Bn_in_0B are disjoint. Also

$$B \cap X_iHU^- = X_iH(B \cap U^-) = X_iH$$

by 2.5.5. Hence $B \cap P_j^{n_0} = X_iH$. Now

$$P_i = B \cup Bn_iB = B \cup Bn_iX_i$$

by 2.5.13. Thus we have

$$\begin{aligned} P_i \cap P_j^{n_0} &= (B \cap P_j^{n_0}) \cup (Bn_iX_i \cap P_j^{n_0}) \\ &= X_iH \cup (B \cap P_j^{n_0})n_iX_i \end{aligned}$$

since $n_iX_i \subseteq P_j^{n_0}$

$$= X_iH \cup X_iHn_iX_i.$$

The result follows.

Corollary 2.6.2. $X_iH \cup X_iHn_iX_i = \langle X_i, X_{-i}, H\rangle$.

Proof. We have $X_iH \cup X_iHn_iX_i = \langle X_i, n_i, H\rangle$. Moreover $\langle X_i, n_i, H\rangle \supseteq \langle X_i, X_{-i}, H\rangle$ since $X_{-i} = X_i^{n_i}$. It is therefore sufficient to show that $n_i \in \langle X_i, X_{-i}, H\rangle$. This certainly holds if $X_{-i} \cap X_iHn_iX_i$ is non-empty. However if this set is empty then $X_{-i} \subseteq X_iH$. But $X_{-i} \subseteq U^-$, $X_iH \subseteq B$ and $B \cap U^- = 1$, Since $X_{-i} \neq 1$ we have a contradiction. ■

Since $n_i \in \langle X_i, X_{-i}, H\rangle = \langle X_i, X_{-i}\rangle H$ we may choose the representative n_i in $\langle X_i, X_{-i}\rangle$. We shall subsequently assume that n_i is chosen in this way.

We shall show that the subgroup $\langle X_i, X_{-i}, H\rangle$ of G has a split BN-pair. More generally we prove the following result.

For each subset $J \subseteq I$ define L_J by $L_J = \langle H, X_\alpha; \alpha \in \Phi_J\rangle$.

Proposition 2.6.3. *The group L_J has a split BN-pair corresponding to subgroups B_J, N_J where $B_J = U_{(w_0)_J}H$ and $N_J/H = W_J$.*

Here $(w_0)_J$ is the element of maximal length in W_J. (Note that for $J = \{i\}$ we have $L_J = \langle X_i, X_{-i}, H\rangle$.)

Proof. We must first show that B_J and N_J lie in L_J. The set of positive roots made negative by $(w_0)_J$ is $\Phi_J{}^+$, and so $U_{(w_0)_J} = \prod_{\alpha \in \Phi_J{}^+} X_\alpha$ by 2.5.16. This shows that B_J lies in L_J. Since each n_j lies in $\langle X_j, X_{-j}\rangle$ we see that N_J lies in L_J also. Thus $\langle B_J, N_J\rangle \subseteq L_J$.

Conversely we have $X_\alpha \subseteq B_J$ for all $\alpha \in \Phi_J{}^+$, while for $\alpha \in \Phi_J{}^-$ we have $X_\alpha = X_\beta{}^{(n_0)_J}$ for some $\beta \in \Phi_J{}^+$, and so $X_\alpha \subseteq \langle B_J, N_J\rangle$. Since $H \subseteq B_J$ we have $L_J \subseteq \langle B_J, N_J\rangle$. Thus $\langle B_J, N_J\rangle = L_J$.

Next consider $B_J \cap N_J$. We have $H \subseteq B_J \cap N_J \subseteq B \cap N = H$, and so $B_J \cap N_J = H$. Thus $B_J \cap N_J$ is normal in N_J. Moreover

$$N_J/(B_J \cap N_J) = N_J/H = W_J = \langle s_i; i \in J\rangle.$$

Now consider $n_jB_Jn_j$ where $j \in J$. We have $n_jX_jn_j^{-1} = X_{-j} \subseteq U^-$ whereas $B_J \subseteq B$. Since $B \cap U^- = 1$ and $X_j \neq 1$ we see that $n_jB_Jn_j \neq B_J$.

Finally we must show that for $j \in J$, $n \in N_J$ we have

$$n_jB_Jn \subseteq B_Jn_jnB_J \cup B_JnB_J.$$

We have

$$n_jB_Jn = n_jH\left(\prod_{\alpha \in \Phi_J{}^+} X_\alpha\right)n = n_jH\left(\prod_{\substack{\alpha \in \Phi_J{}^+ \\ \alpha \neq \alpha_j}} X_\alpha\right)X_jn$$

$$= H\left(\prod_{\substack{\alpha \in \Phi_J{}^+ \\ \alpha \neq \alpha_j}} X_{s_j(\alpha)}\right)n_jX_jn \subseteq B_Jn_jX_jn.$$

We consider the subset n_jX_jn. Let $\pi(n) = w \in W_J$. Suppose $w^{-1}(\alpha_j) \in \Phi^+$. Then

$$n_jX_jn = n_jnX_{w^{-1}(\alpha_j)} \subseteq n_jnB_J.$$

Now suppose $w^{-1}(\alpha_j) \in \Phi^-$. Then $(s_j w)^{-1}(\alpha_j) \in \Phi^+$ and so we have

$$\begin{aligned} n_j X_j n &= n_j X_j n_j^{-1} n_j n \subseteq (X_j H \cup X_j H n_j X_j) n_j n \\ &\subseteq B_J n_j n \cup B_J n_j X_j n_j n \\ &\subseteq B_J n_j n \cup B_J n X_j^{(n_j n)} \\ &\subseteq B_J n_j n \cup B_J n B_J. \end{aligned}$$

Hence $n_j B_J n \subseteq B_J n_j n B_J \cup B_J n B_J$ and we have shown that B_J, N_J form a BN-pair.

We next show that B_J, N_J form a split BN-pair. We have $U_{(w_0)_J}$ as a closed normal unipotent subgroup of B_J and H as a closed commutative subgroup of B_J all of whose elements are semisimple. Finally we must show that $\bigcap_{n \in N_J} nB_J n^{-1} = H$. We certainly have $H \subseteq \bigcap_{n \in N_J} nB_J n^{-1}$. However

$$B_J \cap (\dot{n}_0)_J B_J (n_0)_J^{-1} = U_{(w_0)_J} H \cap U_{(w_0)_J}{}^{(n_0)_J^{-1}} H \subseteq UH \cap U^- H = H.$$

Thus $\bigcap_{n \in N_J} nB_J n^{-1} = H$ as required. We have now shown that B_J, N_J form a split BN-pair in L_J. ■

In order to derive further results we need to use the commutator relations for root subgroups. The relations that we need can be expressed in the following form.

There exists a total ordering on the set Φ^+ of positive roots such that, for all $\alpha, \beta \in \Phi^+$ with $\alpha \neq \beta$,

$$[X_\alpha, X_\beta] \subseteq \prod_{\substack{\alpha < \gamma \\ \beta < \gamma \\ \gamma = i\alpha + j\beta \\ i > 0,\, j > 0}} X_\gamma$$

where the root subgroups on the right-hand side are taken in some suitable order.

By transforming by an element $w \in W$ we see that there exists a total ordering on the set $w(\Phi^+)$ such that, for all $\alpha, \beta \in w(\Phi^+)$ with $\alpha \neq \beta$,

$$[X_\alpha, X_\beta] \subseteq \prod_{\substack{\gamma = i\alpha + j\beta \\ i > 0,\, j > 0 \\ \alpha < \gamma,\, \beta < \gamma}} X_\gamma.$$

Since every pair of roots α, β with $\beta \neq \pm\alpha$ have the property that α, β lie in $w(\Phi^+)$ for some $w \in W$ these commutator relations apply to $[X_\alpha, X_\beta]$ for all pairs $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$.

These commutator relations do not appear to follow from the axioms of a split BN-pair in any very simple way. However the connected reductive groups G and their finite subgroups G^F, to which we shall apply the general results of this chapter, certainly satisfy the commutator relations in addition to the axioms for a split BN-pair. We shall therefore assume subsequently in this chapter that

G is an algebraic group with a split BN-pair which satisfies the commutator relations. The results will then be applicable to the connected reductive groups G and their finite subgroups G^F.

Proposition 2.6.4. *Let G be an algebraic group with split BN-pair which satisfies the commutator relations. Let $U_J = U \cap U^{(n_0)_J}$. Then U_J is a normal subgroup of P_J, $P_J = U_J L_J$ and $U_J \cap L_J = 1$.*

Proof. We shall first show that $P_J = U_J L_J$. We have $P_J = \bigcup_{n \in N_J} BnB$. Let $w \in W_J$. Then

$$B\dot{w}B = B\dot{w}U_w = HU\dot{w}U_w = HU_{w_0(w_0)_J}U_{(w_0)_J}\dot{w}U_w$$

by 2.5.12. Now $U_{w_0(w_0)_J} = U_J$ and $HU_{(w_0)_J}\dot{w}U_w \subseteq L_J$, thus $B\dot{w}B \subseteq U_J L_J$ for all $w \in W_J$. It follows that $P_J \subseteq U_J L_J$. However both U_J and L_J clearly lie in P_J and so $P_J = U_J L_J$.

We show next that $U_J \cap L_J = 1$. We have $L_J = \bigcup_{n \in N_J} B_J n B_J$ by 2.6.3. Thus $B \cap L_J = B_J$ and

$$U_J \cap L_J = U_J \cap B_J = U_J \cap U_{(w_0)_J}H = U_{w_0(w_0)_J} \cap U_{(w_0)_J} = 1$$

by 2.5.12.

It remains to show that U_J is normal in P_J. It will be sufficient to show that $L_J \subseteq N(U_J)$. Since $L_J = \langle H, X_\alpha, \alpha \in \Phi_J \rangle$ it will be sufficient to show that $X_\alpha \subseteq N(U_J)$ for all $\alpha \in \Phi_J$. Now we have

$$U_J = U_{w_0(w_0)_J} = \prod_{\substack{\beta \in \Phi^+ \\ w_0(w_0)_J(\beta) \in \Phi^-}} X_\beta = \prod_{\substack{\beta \in \Phi^+ \\ \beta \notin \Phi_J}} X_\beta$$

where the X_β are taken in some suitable order, by 2.5.16. Let $\alpha \in \Phi_J$, $\beta \in \Phi^+$, $\beta \notin \Phi_J$. Then, for all $i, j > 0$ for which $i\alpha + j\beta \in \Phi$, $i\alpha + j\beta$ will be a positive root not in Φ_J. For β will involve some simple root not in Δ_J with positive coefficient, and the same will then be true of $i\alpha + j\beta$. Thus we have

$$[X_\alpha, X_\beta] \subseteq \prod_{\substack{i > 0, j > 0 \\ \alpha, \beta < i\alpha + j\beta}} X_{i\alpha + j\beta} \subseteq U_J.$$

It follows from a repeated application of the commutator identity

$$[a, bc] = [a, c][a, b][[a, b], c]$$

that

$[X_\alpha, \prod_{\beta \notin \Phi_J}^{\beta \in \Phi^+} X_\beta] \subseteq U_J$. Hence $[X_\alpha, U_J] \subseteq U_J$ and so $X_\alpha \subseteq N(U_J)$. This completes the proof that U_J is normal in P_J.

Corollary 2.6.5. *U_J is the largest normal unipotent subgroup of P_J.*

Proof. Let M be the largest normal unipotent subgroup of P_J. Then M contains U_J. Thus $M = U_J(M \cap L_J)$. Now $M \cap L_J$ is a normal unipotent subgroup of L_J. Since L_J is an algebraic group with split BN-pair we have $M \cap L_J = 1$ by 2.5.2. Hence $M = U_J$. ∎

The decomposition $P_J = U_J L_J$ is called the Levi decomposition of P_J. P_J is called a standard parabolic subgroup of G and L_J a standard Levi subgroup. A Levi subgroup of P_J is by definition a P_J-conjugate of L_J.

Now we have seen that L_J is itself an algebraic group with split BN-pair and it will satisfy the commutator relations since G does. Thus for each subset $K \subseteq J$ L_J will have a standard parabolic subgroup and a standard Levi subgroup. We identify these in the next results.

Proposition 2.6.6. *The standard parabolic subgroup of L_J corresponding to the subset $K \subseteq J$ is $P_K \cap L_J$.*

Proof. The required subgroup is $B_J N_K B_J$. Now $B_J N_K B_J \subseteq BN_K B \cap L_J = P_K \cap L_J$. Conversely we have

$$P_K \cap L_J = BN_K B \cap L_J = BN_K B \cap B_J N_J B_J \subseteq B_J N_K B_J.$$

The result follows.

Proposition 2.6.7. *The maximal normal unipotent subgroup of $P_K \cap L_J$ is $U_K \cap L_J$. The standard Levi subgroup of $P_K \cap L_J$ is L_K.*

Proof. We have $L_K \subseteq P_K \cap L_J$. Since $P_K = U_K L_K$ we shall have $P_K \cap L_J = (U_K \cap L_J)L_K$ and $(U_K \cap L_J) \cap L_K = U_K \cap L_K = 1$. Moreover $U_K \cap L_J$ is a normal unipotent subgroup of $P_K \cap L_J$. The standard Levi subgroup of $P_K \cap L_J$ is $\langle H, X_\alpha, \alpha \in \Phi_K \rangle$, which is L_K. Hence the maximal normal unipotent subgroup of $P_K \cap L_J$ can be no larger than $U_K \cap L_J$. ∎

We discuss the Levi decomposition briefly in our two special cases of particular interest, viz. when our group is a connected reductive group G and when it is the finite group G^F of fixed points of G under a Frobenius map.

Suppose G is a connected reductive group. Then for any $J \subseteq I$ its subgroups P_J, U_J, L_J will also be connected, since they are generated by connected subgroups of the form X_α and H. In this case U_J is the unipotent radical of P_J and L_J is a connected reductive group.

Now consider the finite group G^F of fixed points of G under the Frobenius map F. F determines a permutation ρ of the set Δ of simple roots, as in section 1.18. ρ may also be regarded as a permutation of the index set I. The standard parabolic subgroups of G^F are the subgroups $P_J{}^F$ where J is a ρ-stable subset of I, and the standard Levi subgroups of G^F are the subgroups $L_J{}^F$ for ρ-stable J. The maximal normal unipotent subgroup of $P_J{}^F$ is $U_J{}^F$, and so $U_J{}^F$ is the largest normal p-subgroup of $P_J{}^F$. Hence the Levi decomposition of standard parabolic subgroups in G^F takes the form

$$P_J{}^F = U_J{}^F L_J{}^F \qquad J \ \rho\text{-stable}.$$

$U_J{}^F$ is the largest normal p-subgroup of $P_J{}^F$ and $L_J{}^F$ has no nontrivial normal p-subgroup.

2.7 DISTINGUISHED DOUBLE COSET REPRESENTATIVES

We shall now need further information about Coxeter groups. The results in this section are, perhaps, not so well known as those in section 2.3 about distinguished coset representatives, and we shall therefore give proofs for them. W will denote a finite Coxeter group.

Let J, K be subsets of I. We shall show that each double coset $W_J wW_K$ has a unique element of minimal length. However we first need two lemmas. We recall that the set D_J of distinguished coset representatives of W_J in W is given by $D_J = \{w \in W;\ w(\Delta_J) \subseteq \Phi^+\}$.

Definition. Let J, K be subsets of I. We define $D_{J,K} = D_J{}^{-1} \cap D_K$.

Lemma 2.7.1. *Suppose $xw(\alpha_k) \in \Phi^-$ where $x \in W_J$, $w \in D_{J,K}$ and $k \in K$. Then $w(\alpha_k) = \alpha_j$ for some $j \in J$.*

Proof. $w(\alpha_k) \in \Phi^+$ since $w \in D_K$. So $w(\alpha_k)$ is a positive root transformed by x into a negative root. Since $x \in W_J$ this implies that $w(\alpha_k) \in \Phi_J$. Thus we can write

$$w(\alpha_k) = \sum_{j \in J} \lambda_j \alpha_j \quad \text{with } \lambda_j \geqslant 0.$$

Hence

$$\alpha_k = \sum_{j \in J} \lambda_j w^{-1}(\alpha_j).$$

Since $w^{-1} \in D_J$ each $w^{-1}(\alpha_j)$ lies in Φ^+. Since a simple root cannot be expressed as a nontrivial positive combination of positive roots we have $\alpha_k = w^{-1}(\alpha_j)$ for some $j \in J$.

Lemma 2.7.2. *Let v, $w \in D_{J,K}$. Then $W_J \cap vW_K w^{-1}$ is empty unless $v = w$.*

Proof. Let $x \in W_J \cap vW_K w^{-1}$. We shall show that $v = w$ by induction on $l(x)$. If $l(x) = 0$ then $x = 1$ and so $w \in vW_K$. Since v, $w \in D_K$ we have $v = w$ by 2.3.3. So suppose $l(x) \geqslant 1$. Let $y = v^{-1}xw \in W_K$. If $y = 1$ then $v = xw$ and so $v = w$ and $x = 1$ since $x \in W_J$ and v, $w \in D_J{}^{-1}$. Hence $y \neq 1$. Thus we can write $y = y's_k$ where $k \in K$ and $l(y') = l(y) - 1$. We then have $y(\alpha_k) \in \Phi_K{}^-$. Since $v \in D_K$ this implies that $vy(\alpha_k) \in \Phi^-$. Thus $xw(\alpha_k) \in \Phi^-$. By 2.7.1 we have $w(\alpha_k) = \alpha_j$ for some $j \in J$. So $s_j = ws_k w^{-1}$.

Now $x(\alpha_j) \in \Phi^-$ so $x = x's_j$ with $l(x') = l(x) - 1$. We have

$$x' = xs_j = xws_k w^{-1} = vys_k w^{-1} = vy'w^{-1}.$$

Thus $x' \in W_J \cap vW_K w^{-1}$. By induction we may deduce that $v = w$. ∎

We can now identify the double coset representatives of minimal length.

Proposition 2.7.3. *Let J, K be subsets of I. Then*

(i) *Each double coset $W_J wW_K$ contains a unique element of $D_{J,K}$.*

(ii) *If $w \in D_{J,K}$ then w is the unique element of minimal length in its coset $W_J wW_K$.*

Proof. Let w be an element of minimal length in its double coset $W_J w W_K$. Then w has minimal length in $W_J w$ and in $w W_K$. Thus $w \in D_J^{-1} \cap D_K = D_{J,K}$. Thus part (i) of the proposition will imply part (ii).

Each double coset certainly contains an element of minimal length, hence an element of $D_{J,K}$. We must show the element is unique. So let $v, w \in D_{J,K}$ have the property that $v \in W_J w W_K$. Let $v = xwy$ with $x \in W_J$ and $y \in W_K$. Then $x = vy^{-1}w^{-1}$ and so $x \in W_J \cap vW_K w^{-1}$. By 2.7.2 we have $v = w$, as required. ■

We now come to a theorem of Kilmoyer which plays an important role in the theory of groups of Lie type.

Theorem 2.7.4. (*Kilmoyer*) *Let J, K be subsets of I and $w \in D_{J,K}$. Then*

$$W_J \cap {}^w W_K = W_L$$

where L is defined by $\Delta_L = \Delta_J \cap w(\Delta_K)$.

Proof. It is clear that $W_L \subseteq W_J \cap {}^w W_K$. So let $x \in W_J \cap {}^w W_K$. We show $x \in W_L$ by induction on $l(x)$. If $l(x) = 0$ then $x = 1$ and the result is clear. So suppose $l(x) > 0$. Let $y = w^{-1}xw \in W_K$. Then $y \neq 1$ so we can write $y = y's_k$ where $k \in K$ and $l(y') = l(y) - 1$. We then have $y(\alpha_k) \in \Phi_K^-$. Since $w \in D_K$ this implies that $wy(\alpha_k) \in \Phi^-$. Hence $xw(\alpha_k) \in \Phi^-$. By 2.7.1 we see that $w(\alpha_k) = \alpha_j$ for some $j \in J$. In fact $\alpha_j \in \Delta_J \cap w(\Delta_K) = \Delta_L$ and so $j \in L$.

Now $x(\alpha_j) \in \Phi^-$ and so $x = x's_j$ with $l(x') = l(x) - 1$. We have

$$x' = xs_j = xws_k w^{-1} = wys_k w^{-1} = wy'w^{-1}.$$

Thus $x' \in W_J \cap wW_K w^{-1}$. By induction we know that $x' \in W_L$. Thus $x = x's_j$ lies in W_L also. ■

Kilmoyer's theorem will be used many times in the subsequent work.

We now come to a uniqueness theorem for elements in a double coset $W_J w W_K$ with $w \in D_{J,K}$. This is not so straightforward as the corresponding result 2.3.3 for ordinary cosets. Although each element of this double coset can be written in the form awb with $a \in W_J$ and $b \in W_K$, such an expression is not in general unique. Nor will it be true in general that $l(awb) = l(a) + l(w) + l(b)$. There is, however, a result of this type due to Howlett.

Proposition 2.7.5. (*Howlett*) *Let $w \in D_{J,K}$. Then each element of the double coset $W_J w W_K$ is uniquely expressible in the form awb where $a \in W_J \cap D_L$ and $b \in W_K$. Here L is defined as before by $\Delta_L = \Delta_J \cap w(\Delta_K)$. Moreover this decomposition satisfies*

$$l(awb) = l(a) + l(w) + l(b).$$

Proof. Let $xwy \in W_J w W_K$ with $x \in W_J$ and $y \in W_K$. Since L is a subset of J we have a decomposition $W_J = (W_J \cap D_L)W_L$. Let $x = ax'$ where $a \in W_J \cap D_L$ and $x' \in W_L$. Then we have

$$xwy = ax'wy = awb$$

where $b = w^{-1}x'wy \in W_K$. We shall show that $l(awb) = l(a) + l(w) + l(b)$.

We have $l(aw) = l(a) + l(w)$ since $a \in W_J$ and $w \in D_J^{-1}$. In order to show $l(awb) = l(aw) + l(b)$ it is sufficient to prove that $aw \in D_K$. We must therefore show that $aw(\Delta_K) \subseteq \Phi^+$. Suppose if possible that $aw(\alpha_k) \in \Phi^-$ for some $k \in K$. By 2.7.1 we have $w(\alpha_k) = \alpha_j$ for some $j \in J$. In fact we have $j \in L$. Since $a \in D_L$ we have $aw(\alpha_k) = a(\alpha_j) \in \Phi^+$, a contradiction. Thus we have

$$l(awb) = l(a) + l(w) + l(b).$$

Finally we show the uniqueness of the decomposition. Suppose that $awb = a'wb'$ with $a, a' \in W_J \cap D_L$ and $b, b' \in W_K$. Then

$$a^{-1}a' = wbb'^{-1}w^{-1} \in W_J \cap {}^wW_K.$$

By Kilmoyer's theorem 2.7.4 we have $a^{-1}a' \in W_L$. Thus $aW_L = a'W_L$. Since $a, a' \in D_L$ this implies that $a = a'$. It follows that $b = b'$ and the decomposition is unique. ■

We now prove a theorem of Solomon based on Kilmoyer's result. Let J, K, L be subsets of I and $D_{J,K,L}$ be defined by

$$D_{J,K,L} = \{w \in D_{J,K};\ W_J \cap {}^wW_K = W_L\}.$$

Let $|D_{J,K,L}| = a_{JKL}$.

Proposition 2.7.6. (*Solomon*). *For each $J \subseteq I$ let ξ_J be the element of the group algebra $\mathbb{R}W$ given by $\xi_J = \sum_{w \in D_J} w$. Then*

$$\xi_K \xi_J = \sum_L a_{JKL} \xi_L.$$

Proof. Let $w \in W$. Then the coefficient of w in $\xi_K\xi_J$ is the number of pairs d_K, d_J with $d_K \in D_K, d_J \in D_J$ and $d_K d_J = w$. This relation is equivalent to $d_K{}^{-1}w = d_J$ and the number is therefore $|D_K{}^{-1}w \cap D_J|$. The coefficient of w in $\sum_L a_{JKL}\xi_L$ is $\sum^L_{w(L) \subseteq \Phi^+} a_{JKL}$. Let $J_w = \{i \in I;\ w(\alpha_i) \in \Phi^+\}$. Then we must show

$$|D_K{}^{-1}w \cap D_J| = \sum_{L \subseteq J_w} |D_{JKL}|.$$

We consider the intersection of $D_K{}^{-1}w \cap D_J$ with the double cosets $W_K x W_J$ with $x \in D_{K,J}$. Let $wx^{-1} = d_K w_K$ where $d_K \in D_K$, $w_K \in W_K$. Then d_K, w_K are uniquely determined and $d_K{}^{-1}w = w_K x$. We shall show that

$$D_K{}^{-1}w \cap D_J \cap W_K x W_J = \begin{cases} \{d_K{}^{-1}w\} & \text{if } d_K{}^{-1}w \in D_J \\ \varnothing & \text{if } d_K{}^{-1}w \notin D_J \end{cases}.$$

Let $y \in D_K{}^{-1}w \cap D_J \cap W_K x W_J$. By 2.7.5 there exist $a \in W_K$, $b \in W_J$ with $y = axb$ and $l(y) = l(a) + l(x) + l(b)$. Suppose $b \neq 1$. Then $l(bs_i) = l(b) - 1$ for some $i \in J$. But then $l(ys_i) = l(y) - 1$, which contradicts $y \in D_J$, $s_i \in W_J$. Thus we have $b = 1$ and $y = w_K x$ for some $w_K \in W_K$. Hence $w = d_K w_K x$ for some $d_K \in D_K$, $w_K \in W_K$. Thus $y = d_K{}^{-1}w$. Hence the set $D_K{}^{-1}w \cap D_J \cap W_K x W_J$ contains at most one element, viz. $d_K{}^{-1}w$. If it contains this element then certainly

$d_K{}^{-1}w \in D_J$. Conversely if $d_k{}^{-1}w \in D_J$ then $d_K{}^{-1}w$ lies in $D_K{}^{-1}w \cap D_J \cap W_KxW_J$ since $d_K{}^{-1}w = w_Kx$. Thus $D_K{}^{-1}w \cap D_J \cap W_KxW_J$ is as described above.

We have now seen that every double coset W_KxW_J, $x \in D_{K,J}$ meets $D_K{}^{-1}w \cap D_J$ in at most one element. We wish to identify which double cosets intersect $D_K{}^{-1}w \cap D_J$. Now $x \in D_{K,J}$ implies that $x^{-1} \in D_{J,K}$ and the element x^{-1} of $D_{J,K}$ lies in $D_{J,K,L}$ for a unique L. We shall show that W_KxW_J intersects $D_K{}^{-1}w \cap D_J$ if and only if $L \subseteq J_w$.

Suppose $d_K{}^{-1}w \in D_J$. Let $i \in L$. Then $i \in J$ and so $d_K{}^{-1}w(\alpha_i) \in \Phi^+$. We also have

$$d_K{}^{-1}ws_iw^{-1}d_K = w_Kxs_ix^{-1}w_K{}^{-1} \in W_K$$

since $xW_Lx^{-1} \subseteq W_K$. Thus $d_K{}^{-1}w(\alpha_i) \in \Phi_K$. We therefore have $d_K{}^{-1}w(\alpha_i) \in \Phi_K{}^+$. However, since $d_K \in D_K$, we have

$$d_K . d_K{}^{-1}w(\alpha_i) \in \Phi^+.$$

Thus $w(\alpha_i) \in \Phi^+$ for all $i \in L$. This shows that $L \subseteq J_w$.

Now suppose conversely that $L \subseteq J_w$. Let $i \in L$. Then $w(\alpha_i) \in \Phi^+$, and so

$$d_K . d_K{}^{-1}w(\alpha_i) \in \Phi^+.$$

However $d_K{}^{-1}w(\alpha_i) \in \Phi_K$ as above. Since $d_K \in D_K$ this implies that $d_K{}^{-1}w(\alpha_i) \in \Phi_K{}^+$.

Next suppose that $i \in J$ but $i \notin L$. Then $s_i \in W_J$ but $s_i \notin W_L$. Now $W_J \cap x^{-1}W_Kx = W_L$. Thus $s_i \notin x^{-1}W_Kx$ and so $xs_ix^{-1} \notin W_K$. Hence $x(\alpha_i) \notin \Phi_K$. Now $l(xs_i) = l(x) + 1$ since $x \in D_{K,J}$ and $s_i \in W_J$. Thus $x(\alpha_i) \in \Phi^+$. Since $x(\alpha_i) \notin \Phi_K$ this implies that $w_Kx(\alpha_i) \in \Phi^+$. Hence $d_K{}^{-1}w(\alpha_i) \in \Phi^+$.

We have now shown that $d_K{}^{-1}w(\alpha_i) \in \Phi^+$ for all $i \in J$. Hence $d_K{}^{-1}w \in D_J$.

Thus the double cosets W_KxW_J, $x \in D_{K,J}$, which intersect $D_K{}^{-1}w \cap D_J$ are just those with $x^{-1} \in D_{J,K,L}$ where $L \subseteq J_w$. It follows that

$$|D_K{}^{-1}w \cap D_J| = \sum_{L \subseteq J_w} |D_{J,K,L}|. \qquad \blacksquare$$

The following consequence of Solomon's result will be useful subsequently.

Proposition 2.7.7. *Let J, K, L be subsets of I. Then*

$$\sum_K (-1)^{|K|} a_{JKL} = (-1)^{|L|}$$

for all J, L with $J \supseteq L$.

Proof. We first observe that $\sum_J (-1)^{|J|}\xi_J = w_0$. For

$$\sum_J (-1)^{|J|}\xi_J = \sum_J (-1)^{|J|} \sum_{w(J) \subseteq \Phi^+} w = \sum_w \left(\sum_{J \subseteq J_w} (-1)^{|J|} \right) w$$

$$= \sum_w (1 - 1)^{|J_w|} w = w_0$$

since $|J_w| = 0$ if and only if $w = w_0$.

We show next that $w_0\xi_J = \sum_{K\subseteq J} (-1)^{|K|}\xi_K$. For

$$\sum_{K\subseteq J} (-1)^{|K|}\xi_K = \sum_{K\subseteq J} (-1)^{|K|} \sum_{w(K)\subseteq\Phi+} w = \sum_w \left(\sum_{K\subseteq J\cap J_w} (-1)^{|K|}\right) w$$
$$= \sum_{J\cap J_w=\varnothing} w = \sum_{w(J)\subseteq\Phi-} w = \sum_{w(J)\subseteq\Phi+} w_0 w = w_0\xi_J.$$

Substituting for w_0 from above we get

$$\left(\sum_K (-1)^{|K|}\xi_K\right)\xi_J = \sum_{L\subseteq J} (-1)^{|L|}\xi_L.$$

We now apply 2.7.6 and obtain

$$\sum_K (-1)^{|K|} \sum_L a_{JKL}\xi_L = \sum_{L\subseteq J} (-1)^{|L|}\xi_L.$$

This gives

$$\sum_L \left(\sum_K (-1)^{|K|} a_{JKL}\right)\xi_L = \sum_{L\subseteq J} (-1)^{|L|}\xi_L.$$

Now the ξ_L are linearly independent in $\mathbb{R}W$ and so, for all L with $L\subseteq J$, we obtain

$$\sum_K (-1)^{|K|} a_{JKL} = (-1)^{|L|}.$$

2.8 INTERSECTIONS OF PARABOLIC SUBGROUPS

In subsequent applications we shall need to know the structure of the intersection of two arbitrary parabolic subgroups of G. We follow the development due to Curtis [10], p. 671. We first show that, if $J, K\subseteq I$, the set $D_{J,K}$ of double coset representatives of W with respect to W_J and W_K gives rise to a set of double coset representatives of G with respect to P_J and P_K. For each $w\in D_{J,K}$ we have a representative $\dot{w}\in N$. We define $N_{J,K}$ by

$$N_{J,K} = \{\dot{w};\ w\in D_{J,K}\}$$

Proposition 2.8.1. (i) *$P_J n P_K \cap N = N_J n N_K$ for all $n\in N$.*

(ii) *$P_J n P_K = BN_J n N_K B$.*

(iii) *$N_{J,K}$ is a set of double coset representatives of G with respect to P_J, P_K.*

Proof. (i) We have

$$\begin{aligned} P_J n P_K \cap N &= BN_J BnBN_K B\cap N \\ &\subseteq BN_J nBN_K B\cap N, \text{ since } N_J Bn\subseteq BN_J nB \\ &\subseteq BN_J nN_K B\cap N, \text{ since } N_J nBN_K\subseteq BN_J nN_K B \\ &= N_J nN_K. \end{aligned}$$

Thus $P_J nP_K\cap N\subseteq N_J nN_K$. However the reverse inclusion is clear.

(ii) We have seen that $P_J n P_K \subseteq B N_J n N_K B$ and the reverse inclusion is clear.

(iii) Since $P_J n P_K = B N_J n N_K B$ any set of double coset representatives for N with respect to N_J, N_K will also be a set of double coset representatives for G with respect to P_J, P_K.

Proposition 2.8.2. *Any intersection $P_{J_1}{}^{g_1} \cap P_{J_2}{}^{g_2}$ of parabolic subgroups of G is conjugate to a subgroup of form $P_{J_1} \cap {}^n P_{J_2}$ where $n \in N_{J_1, J_2}$.*

Proof. $P_{J_1}{}^{g_1} \cap P_{J_2}{}^{g_2}$ is conjugate to a subgroup of the form $P_{J_1} \cap {}^g P_{J_2}$ where $g = g_1 g_2{}^{-1}$. Since $G = P_{J_1} N_{J_1, J_2} P_{J_2}$ we can write $g = p_{J_1} n p_{J_2}$ with $p_{J_1} \in P_{J_1}$, $p_{J_2} \in P_{J_2}$, $n \in N_{J_1, J_2}$. Then $P_{J_1} \cap {}^g P_{J_2}$ is conjugate to $P_{J_1} \cap {}^n P_{J_2}$. ■

We shall therefore investigate the structure of subgroups of the form $P_{J_1} \cap {}^n P_{J_2}$ where $n \in N_{J_1, J_2}$. Let $\pi(n) = w \in D_{J_1, J_2}$ and let $W_{J_1} \cap {}^w W_{J_2} = W_K$ as in Kilmoyer's theorem 2.7.4.

Proposition 2.8.3. $L_K \subseteq P_{J_1} \cap {}^n P_{J_2} \subseteq P_K$.

Proof. $L_K = \langle H, X_\alpha, \alpha \in \Phi_K \rangle \subseteq L_{J_1} \subseteq P_{J_1}$. Also we have

$$L_K \subseteq \langle H, X_\alpha, \alpha \in w(\Phi_{J_2}) \rangle \subseteq {}^n L_{J_2} \subseteq {}^n \mathrm{P}_{J_2}.$$

Thus $L_K \subseteq P_{J_1} \cap {}^n P_{J_2}$.

Now let $x \in P_{J_1} \cap {}^n P_{J_2}$. Then $x = b_1 n_1 b_1' = {}^n(b_2 n_2 b_2')$ where $b_1, b_1', b_2, b_2' \in B$ and $n_1 \in N_{J_1}$, $n_2 \in N_{J_2}$. Thus

$$b_1 n_1 b_1' n = n b_2 n_2 b_2'.$$

Now $b_1 n_1 b_1' n \in B n_1 B n B$. Let $w_1 = \pi(n_1) \in W_{J_1}$. $w = \pi(n) \in D_{J_1, J_2}$. Then $l(w_1 w) = l(w_1) + l(w)$. By 2.1.3 we have $n_1 B n \subseteq B n_1 n B$. Thus $b_1 n_1 b_1' n \in B n_1 n B$. A similar argument shows that $n b_2 n_2 b_2' \in B n n_2 B$. Thus $B n_1 n B = B n n_2 B$ and so $w_1 w = w w_2$ and $w_1 = w w_2 w^{-1}$. Hence $n_1 \in N_{J_1} \cap {}^n N_{J_2} = N_K$. It follows that $x = b_1 n_1 b_1' \in B N_K B = P_K$. Thus $P_{J_1} \cap {}^n P_{J_2} \subseteq P_K$.

Proposition 2.8.4. $P_K = U_{J_1}(P_{J_1} \cap {}^n P_{J_2})$.

Proof. We have $U_{J_1} \subseteq U_K \subseteq P_K$ and $P_{J_1} \cap {}^n P_{J_2} \subseteq P_K$. Also $P_{J_1} \cap {}^n P_{J_2} \subseteq N(U_{J_1})$. Thus $U_{J_1}(P_{J_1} \cap {}^n P_{J_2})$ is a subgroup of P_K.

Now $P_K = U_K L_K = U_K(P_{J_1} \cap {}^n P_{J_2})$ by 2.8.3. Also $U_K = \langle X_\alpha, \alpha \in \Phi^+, \alpha \notin \Phi_K \rangle$. Let α be a root satisfying $\alpha \in \Phi^+$ but $\alpha \notin \Phi_K$. If $\alpha \notin \Phi_{J_1}$ then $X_\alpha \subseteq U_{J_1}$. So suppose $\alpha \in \Phi_{J_1}$. Then $\alpha \notin \Phi_{w(J_2)}$ since $\Phi_{J_1} \cap \Phi_{w(J_2)} = \Phi_K$. Thus $w^{-1}(\alpha) \notin \Phi_{J_2}$. However $w^{-1}(\alpha) \in \Phi^+$ since $\alpha \in \Phi_{J_1}$ and $w \in D_{J_1, J_2}$. Hence $X_{w^{-1}(\alpha)} \subseteq U_{J_2}$ and so ${}^{n^{-1}}X_\alpha \subseteq U_{J_2}$ and $X_\alpha \subseteq {}^n U_{J_2}$. Thus $X_\alpha \subseteq P_{J_1} \cap {}^n U_{J_2} \subseteq P_{J_1} \cap {}^n P_{J_2}$. Thus for any $\alpha \in \Phi^+$ with $\alpha \notin \Phi_K$ we have either $X_\alpha \subseteq U_{J_1}$ or $X_\alpha \subseteq P_{J_1} \cap {}^n P_{J_2}$. Hence $X_\alpha \subseteq U_{J_1}(P_{J_1} \cap {}^n P_{J_2})$ for all such α. It follows that $U_K \subseteq U_{J_1}(P_{J_1} \cap {}^n P_{J_2})$. Thus

$$P_K = U_K(P_{J_1} \cap {}^n P_{J_2}) \subseteq U_{J_1}(P_{J_1} \cap {}^n P_{J_2})$$

and we therefore have equality.

Proposition 2.8.5. $U_K = U_{J_1}(P_{J_1} \cap {}^nU_{J_2})$.

Proof. As in the proof of 2.8.4 we have

$$U_K \subseteq \langle U_{J_1}, P_{J_1} \cap {}^n\dot{U}_{J_2} \rangle = U_{J_1}(P_{J_1} \cap {}^nU_{J_2}).$$

Conversely we have $U_{J_1} \subseteq U_K$. Consider $P_{J_1} \cap {}^nU_{J_2}$. This is a normal unipotent subgroup of $P_{J_1} \cap {}^nP_{J_2}$. However $P_{J_1} \cap {}^nP_{J_2}$ has a semi-direct product decomposition

$$P_{J_1} \cap {}^nP_{J_2} = (P_{J_1} \cap {}^nP_{J_2} \cap U_K)L_K$$

by 2.8.3, and L_K has no nontrivial normal unipotent subgroup. Thus $P_{J_1} \cap {}^nP_{J_2} \cap U_K$ is the largest normal unipotent subgroup of $P_{J_1} \cap {}^nP_{J_2}$. Hence $P_{J_1} \cap {}^nU_{J_2} \subseteq U_K$ and so $U_K = U_{J_1}(P_{J_1} \cap {}^nP_{J_2})$.

Proposition 2.8.6. $P_{J_1} \cap {}^nU_{J_2} = (U_{J_1} \cap {}^nU_{J_2})(L_{J_1} \cap {}^nU_{J_2})$.

Proof. The right-hand side is clearly contained in the left-hand side. So let $x \in P_{J_1} \cap {}^nU_{J_2}$. Then $x \in U_K$ by 2.8.5. Now

$$U_K = U_{w_0(w_0)_K} = \prod_{\substack{\alpha \in \Phi^+ \\ \alpha \notin \Phi_K}} X_\alpha \qquad \text{by 2.5.16}$$

where the factors X_α are taken in some suitable order in the product. Moreover the proof of 2.8.4 shows that $X_\alpha \subseteq U_{J_1}$ if $\alpha \notin \Phi_{J_1}$ and $X_\alpha \subseteq P_{J_1} \cap {}^nU_{J_2}$ if $\alpha \in \Phi_{J_1}$. We now use the commutator relations

$$[X_\alpha, X_\beta] \subseteq \prod_{\substack{i > 0,\, j > 0 \\ \alpha,\, \beta < i\alpha + j\beta}} X_{i\alpha + j\beta}.$$

Note that if $\alpha \notin \Phi_{J_1}$ and $\beta \in \Phi_{J_1}$ then $i\alpha + j\beta \notin \Phi_{J_1}$.

Let $x = \prod x_\alpha$ with $x_\alpha \in X_\alpha$. We use the commutator relations to bring the factors x_α with $\alpha \in \Phi_{J_1}$ to the right. All commutators introduced in making this change in the order are elements of an X_α with $\alpha \notin \Phi_{J_1}$. Hence we can write $x = yz$ where $y \in U_{J_1}$ and $z \in P_{J_1} \cap {}^nU_{J_2}$. In fact each X_α for $\alpha \in \Phi_{J_1}$ lies in $L_{J_1} \cap {}^nU_{J_2}$, so $z \in L_{J_1} \cap {}^nU_{J_2}$. Thus $x = yz$ with $x \in P_{J_1} \cap {}^nU_{J_2}$, $z \in L_{J_1} \cap {}^nU_{J_2}$. Hence $y \in {}^nU_{J_2}$ also. Thus

$$P_{J_1} \cap {}^nU_{J_2} = (U_{J_1} \cap {}^nU_{J_2})(L_{J_1} \cap {}^nU_{J_2}). \qquad \blacksquare$$

We now come to the main result of this section.

Theorem 2.8.7. *Let $n \in N_{J_1, J_2}$ and $N_{J_1} \cap {}^nN_{J_2} = N_K$. Then*

(i) $P_{J_1} \cap {}^nP_{J_2} = (U_{J_1} \cap {}^nU_{J_2})(U_{J_1} \cap {}^nL_{J_2})(L_{J_1} \cap {}^nU_{J_2})L_K$.

(ii) *This factorization is with uniqueness.*

(iii) *The product of the first three factors is the largest normal unipotent subgroup of $P_{J_1} \cap {}^nP_{J_2}$.*

Proof. By 2.8.3 we have

$$P_{J_1} \cap {}^nP_{J_2} = (U_K \cap P_{J_1} \cap {}^nP_{J_2})L_K$$

with uniqueness. The first factor is the largest normal unipotent subgroup of $P_{J_1} \cap {}^nP_{J_2}$.

By 2.8.5 we then have

$$U_K \cap P_{J_1} \cap {}^nP_{J_2} = (U_{J_1} \cap {}^nP_{J_2})(P_{J_1} \cap {}^nU_{J_2})$$

and by 2.8.6 we obtain

$$\begin{aligned} U_K \cap P_{J_1} \cap {}^nP_{J_2} &= (U_{J_1} \cap {}^nP_{J_2})(U_{J_1} \cap {}^nU_{J_2})(L_{J_1} \cap {}^nU_{J_2}) \\ &= (U_{J_1} \cap {}^nP_{J_2})(L_{J_1} \cap {}^nU_{J_2}). \end{aligned}$$

This is a product with uniqueness also, since $U_{J_1} \cap L_{J_1} = 1$.

By 2.8.6 again with J_1, J_2 interchanged we have

$$\begin{aligned} U_{J_1} \cap {}^nP_{J_2} &= {}^n({}^{n^{-1}}U_{J_1} \cap P_{J_2}) = {}^n(({}^{n^{-1}}U_{J_1} \cap U_{J_2})({}^{n^{-1}}U_{J_1} \cap L_{J_2})) \\ &= (U_{J_1} \cap {}^nU_{J_2})(U_{J_1} \cap {}^nL_{J_2}). \end{aligned}$$

We have uniqueness again here since $U_{J_2} \cap L_{J_2} = 1$. Thus we have

$$P_{J_1} \cap {}^nP_{J_2} = (U_{J_1} \cap {}^nU_{J_2})(U_{J_1} \cap {}^nL_{J_2})(L_{J_1} \cap {}^nU_{J_2})L_K$$

with uniqueness. Moreover the product of the first three factors is $U_K \cap P_{J_1} \cap {}^nP_{J_2}$, the largest normal unipotent subgroup of $P_{J_1} \cap {}^nP_{J_2}$.

Corollary 2.8.8. *Let $J_1, J_2 \subseteq I$, $n \in N_{J_1,J_2}$, $w = \pi(n)$, and $\Delta_K = \Delta_{J_1} \cap w(\Delta_{J_2})$. Then the following statements are equivalent*:

(i) $P_K = P_{J_1}$.

(ii) $P_{J_1} \cap {}^nU_{J_2} \subseteq U_{J_1}$.

(iii) $L_{J_1} \subseteq {}^nL_{J_2}$.

(iv) $L_{J_1} \cap {}^nU_{J_2} = 1$.

Proof. If $P_K = P_{J_1}$ then $K = J_1$ and $L_{J_1} = L_K \subseteq {}^nL_{J_2}$. Thus (i) implies (iii). If $L_{J_1} \subseteq {}^nL_{J_2}$ then $L_{J_1} \cap {}^nU_{J_2} = 1$ and so $P_{J_1} \cap {}^nU_{J_2} \subseteq U_{J_1}$ by 2.8.6. Thus (iii) implies (ii). If $P_{J_1} \cap {}^nU_{J_2} \subseteq U_{J_1}$ then $U_K = U_{J_1}$ by 2.8.5 and so $K = J_1$ and $P_K = P_{J_1}$. Thus (ii) implies (i). Finally 2.8.6 shows that (ii) and (iv) are equivalent.

Proposition 2.8.9. *$L_{J_1} \cap {}^nP_{J_2} = L_{J_1} \cap P_K$ is a standard parabolic subgroup of L_{J_1}. The Levi decomposition of this parabolic subgroup is*

$$L_{J_1} \cap {}^nP_{J_2} = (L_{J_1} \cap {}^nU_{J_2})L_K.$$

Proof. We know by 2.6.6 and 2.6.7 that $L_{J_1} \cap P_K$ is a standard parabolic subgroup of L_{J_1} and that its Levi decomposition is

$$L_{J_1} \cap P_K = (L_{J_1} \cap U_K)L_K.$$

Now we have

$$L_{J_1} \cap {}^nP_{J_2} \subseteq P_{J_1} \cap {}^nP_{J_2} \subseteq P_K \quad \text{by 2.8.3 and so} \quad L_{J_1} \cap {}^nP_{J_2} \subseteq L_{J_1} \cap P_K.$$

Conversely we have

$$\begin{aligned} L_{J_1} \cap P_K &= (L_{J_1} \cap U_K)L_K = (L_{J_1} \cap U_{J_1}(P_{J_1} \cap {}^nP_{J_2}))L_K \\ &= (L_{J_1} \cap U_{J_1}(L_{J_1} \cap {}^nU_{J_2})L_K)L_K \\ &= (L_{J_1} \cap {}^nU_{J_2})L_K \subseteq L_{J_1} \cap {}^nP_{J_2} \end{aligned}$$

using 2.8.5 and 2.8.7. The same argument shows that $L_{J_1} \cap U_K \subseteq L_{J_1} \cap {}^nU_{J_2}$. Conversely $L_{J_1} \cap {}^nU_{J_2}$ is a normal unipotent subgroup of $L_{J_1} \cap {}^nP_{J_2}$ so lies in $L_{J_1} \cap U_K$. ■

We now obtain a rather different decomposition of $P_{J_1} \cap {}^nP_{J_2}$ when $n \in N_{J_1, J_2}$ which will be useful subsequently. We first need a preliminary result.

Proposition 2.8.10. *If* $w_1, w_2 \in W$ *satisfy* $l(w_1w_2) = l(w_1) + l(w_2)$ *then* $B_{w_0w_1}B_{w_0w_2^{-1}} = B$.

Proof. We have $w_0 = w_0w_1w_2.w_2^{-1}.w_1^{-1}$ with $l(w_0) = l(w_0w_1w_2) + l(w_2^{-1}) + l(w_1^{-1})$. Thus w_2^{-1} is a final segment of some reduced expression for w_0w_1. Applying 2.5.4 repeatedly to this reduced expression we obtain $B_{w_1} \subseteq B_{w_0w_2^{-1}}$. However $B_{w_0w_1}B_{w_1} = B$ by 2.5.12. Thus $B_{w_0w_1}B_{w_0w_2^{-1}} = B$ also.

Proposition 2.8.11 (Howlett) $P_{J_1} \cap {}^nP_{J_2} = (B \cap {}^nB)N_K(B \cap {}^nB)$.

Proof. Let $n_1 \in N_{J_1}$, $n_2 \in N_{J_2}$, $w_1 = \pi(n_1)$, $w_2 = \pi(n_2)$, $w = \pi(n)$. Then we have

$$Bn_1B \cap nBn_2Bn^{-1} = n(n^{-1}Bn_1B \cap Bn_2Bn^{-1}).$$

Now $l(w^{-1}w_1) = l(w^{-1}) + l(w_1)$ and $l(w_2w^{-1}) = l(w_2) + l(w^{-1})$ since $w \in D_{J_1, J_2}$. Thus by 2.8.10 we have

$$B = (B \cap nBn^{-1})(B \cap n_1Bn_1^{-1})$$

$$B = (B \cap n_2^{-1}Bn_2)(B \cap n^{-1}Bn).$$

It follows that

$$\begin{aligned} &Bn_1B \cap nBn_2Bn^{-1} \\ &\quad = n(n^{-1}(B \cap nBn^{-1})(B \cap n_1Bn_1{}^{-1})n_1B \cap Bn_2(B \cap n_2{}^{-1}Bn_2)(B \cap n^{-1}Bn)n^{-1}) \\ &\quad = n((B \cap n^{-1}Bn)n^{-1}n_1B \cap Bn_2n^{-1}(B \cap nBn^{-1})). \end{aligned}$$

Comparing the double cosets in which the terms in the intersection lie, we see that this set is empty unless $w^{-1}w_1 = w_2w^{-1}$. Thus

$$w_1 \in W_{J_1} \cap {}^wW_{J_2} = W_K.$$

It follows that, if the set is non-empty, then

$$Bn_1B \cap nBn_2Bn^{-1} = n((B \cap n^{-1}Bn)n^{-1}n_1(B \cap nBn^{-1}))$$
$$= (B \cap nBn^{-1})n_1(B \cap nBn^{-1}).$$

Thus

$$P_{J_1} \cap {}^nP_{J_2} = (B \cap nBn^{-1})N_K(B \cap nBn^{-1}).$$

2.9 APPLICATIONS TO THE FINITE GROUPS G^F

The results we have established in this chapter are valid for any algebraic group with a split BN-pair which satisfies the commutator relations. They are in particular valid for a finite group of the form G^F where G is a connected reductive group and $F: G \to G$ is a Frobenius map. For G^F, being a closed subgroup of G, may itself be regarded as an algebraic group. We describe some of the consequences of the above axioms in the groups G^F.

We first recall some results about the groups G^F mentioned in section 1.18. Let T be an F-stable maximal torus of G contained in an F-stable Borel subgroup B. Then we can define F-actions also on the character group X of T and on $V = X_{\mathbf{R}}$. Moreover $F = qF_0$ on V where $q > 1$ is a real number and F_0 has finite order.

Now $G = G'Z^0$ is the product of the semisimple group G' and the central torus Z^0, and we get a corresponding decomposition $T = SZ^0$ where $S = T \cap G'$ is a maximal torus of G' and $S \cap Z^0$ is finite. This in turn gives rise to subgroups $S^\perp$, $(Z^0)^\perp$ of X as in section 1.12 which satisfy

$$S^\perp \cap (Z^0)^\perp = 0, \; |X : S^\perp + (Z^0)^\perp| \text{ is finite.}$$

The vector space $V = X_{\mathbf{R}}$ then has a direct decomposition $V = V_1 \oplus V_2$ where $V_1 = (Z^0)^\perp{}_{\mathbf{R}}$ is the subspace spanned by the roots and $V_2 = (S^\perp)_{\mathbf{R}}$ is a complementary subspace. We have

$$\dim V = \dim T = \operatorname{rank} G = r$$
$$\dim V_1 = \dim S = \operatorname{rank} G' = l.$$

l is called the semisimple rank of G. The subspaces V_1, V_2 are both invariant under the F-action on V.

Now we saw in section 1.18 how to define a permutation ρ of the set of simple roots of G, or of the index set $I = \{1, \ldots, l\}$ labelling these simple roots. The eigenvalues of the map $F_0: V_1 \to V_1$ are related to the ρ-orbits on I. For each ρ-orbit J on I the subspace $(V_1)_J$ spanned by the roots in J is F_0-invariant. Moreover the eigenvalues of F_0 on $(V_1)_J$ form a complete set of $|J|$th roots of unity. Hence the characteristic polynomial of F_0 on V_1 is

$$\prod_{i=1}^{l} (t - \eta_i) = \prod_J (t^{|J|} - 1)$$

where $\eta_1, \ldots, \eta_l$ are the eigenvalues of F_0 on V_1 and the product on the right-hand side runs over all ρ-orbits J on I. Moreover we have $(T^F)^\perp = (F - 1)X$ and so

$$|T^F| = |X/(F-1)X| = |\det_V(F-1)| = |\det_V(qF_0 - 1)|$$
$$= |\det_V(q1 - F_0^{-1})| = \det_V(q1 - F_0^{-1}) = \det_V(q1 - F_0)$$

since F_0 has finite order and $q > 1$. More details regarding these results will be given in chapter 3. Thus $|T^F|$ is obtained from the characteristic polynomial $\chi_V(t)$ of F_0 by replacing t by q. In particular we have $|S^F| = \chi_{V_1}(q)$ and $|(Z^0)^F| = \chi_{V_2}(q)$. Thus

$$|S^F| = \prod_{\substack{J \\ \rho\text{-orbit}}} (q^{|J|} - 1).$$

These results are proved in Carter [3] or Steinberg [15].

We also recall from section 1.18 that the map F_0 on V determines an equivalence relation on the set of roots of G. Moreover each equivalence class A of roots gives rise to an F-stable subgroup $X_A = \prod_{\alpha \in A} X_\alpha$ and the finite groups $(X_A)^F$ are the root subgroups of the group G^F with respect to the split BN-pair B^F, N^F. The orders of the root subgroups are given by

$$|X_A{}^F| = q^{|A|}.$$

The equivalence classes A can be described as being positive or negative without ambiguity and we have

$$U^F = \prod_{A > 0} X_A{}^F$$

with uniqueness. Hence

$$|U^F| = q^N \qquad \text{where } N = |\Phi^+|.$$

For $w \in W^F$ we also have

$$U_w{}^F = \prod_{\substack{A > 0 \\ w(A) < 0}} X_A{}^F$$

with uniqueness. Note here that W^F permutes the equivalence classes, and so the statement $w(A) < 0$ is meaningful. Thus $U_w{}^F$ has order given by

$$|U_w{}^F| = q^{l(w)}.$$

(Here $l(w)$ is the length of w as an element of W, not of W^F.) Since each element of G^F is uniquely expressible in the form $uh\dot{w}u'$ where $u \in U^F$, $h \in T^F$, $w \in W^F$, $u' \in U_w{}^F$ we see that the order of G^F is given by

$$|G^F| = |(Z^0)^F| . q^N \prod_{\substack{J \\ \rho\text{-orbit}}} (q^{|J|} - 1) \sum_{w \in W^F} q^{l(w)}.$$

Now the expression $\sum_{w \in W^F} q^{l(w)}$ has a nice factorization. This is connected with the algebra of polynomial invariants of W described in section 2.4. Let $\mathfrak{P}$ be the algebra of polynomial functions on V_1. We may define an action of F_0 on $\mathfrak{P}$ by

$$(F_0 f)v = f(F_0 v) \qquad f \in \mathfrak{P},\ v \in V.$$

This map $F_0 : \mathfrak{P} \to \mathfrak{P}$ has the property that it transforms the subalgebra $\mathfrak{J}$ of W-invariants of $\mathfrak{P}$ into itself. We recall from section 2.4 that $\mathfrak{J}$ is isomorphic to a polynomial ring in l variables and that generators $I_1, I_2, \ldots, I_l$ of $\mathfrak{J}$ may be chosen which are homogeneous polynomials. Let $d_1, d_2, \ldots, d_l$ be the degrees of $I_1, I_2, \ldots, I_l$ respectively. By extending the base field of V_1 from $\mathbb{R}$ to $\mathbb{C}$ we can choose the generators $I_1, \ldots, I_l$ of $\mathfrak{J}$ to be eigenvectors of F_0. Thus we shall have

$$F_0(I_i) = \varepsilon_i I_i$$

where ε_i is a root of unity. The required factorization of $\sum_{w \in W^F} t^{l(w)}$ is then given by

$$\sum_{w \in W^F} t^{l(w)} = \frac{\prod_{i=1}^{l} (t^{d_i} - \varepsilon_i)}{\prod_{i=1}^{l} (t - \eta_i)}.$$

It follows from this that the order of G^F is given by

$$|G^F| = |(Z^0)^F| q^N \prod_{i=1}^{l} (q^{d_i} - \varepsilon_i).$$

These results can be found in Steinberg [15] or Carter [3].

The orders of the individual groups G^F when the algebraic group G is simple are given in the following table:

$$|A_l(q)| = q^{l(l+1)/2}(q^2 - 1)(q^3 - 1)\ldots(q^{l+1} - 1)$$

$$|{}^2A_l(q^2)| = q^{l(l+1)/2}(q^2 - 1)(q^3 + 1)\ldots(q^{l+1} - (-1)^{l+1})$$

$$|B_l(q)| = q^{l^2}(q^2 - 1)(q^4 - 1)\ldots(q^{2l} - 1)$$

$$|C_l(q)| = q^{l^2}(q^2 - 1)(q^4 - 1)\ldots(q^{2l} - 1)$$

$$|D_l(q)| = q^{l(l-1)}(q^2 - 1)(q^4 - 1)\ldots(q^{2l-2} - 1)(q^l - 1)$$

$$|{}^2D_l(q^2)| = q^{l(l-1)}(q^2 - 1)(q^4 - 1)\ldots(q^{2l-2} - 1)(q^l + 1)$$

$$|{}^3D_4(q^3)| = q^{12}(q^2 - 1)(q^4 - \varepsilon)(q^4 - \varepsilon^2)(q^6 - 1) \qquad \varepsilon = e^{2\pi i/3}$$

$$|G_2(q)| = q^6(q^2 - 1)(q^6 - 1)$$

$$|F_4(q)| = q^{24}(q^2 - 1)(q^6 - 1)(q^8 - 1)(q^{12} - 1)$$

$$|E_6(q)| = q^{36}(q^2 - 1)(q^5 - 1)(q^6 - 1)(q^8 - 1)(q^9 - 1)(q^{12} - 1)$$

$$|{}^2E_6(q^2)| = q^{36}(q^2 - 1)(q^5 + 1)(q^6 - 1)(q^8 - 1)(q^9 + 1)(q^{12} - 1)$$

$$|E_7(q)| = q^{63}(q^2 - 1)(q^6 - 1)(q^8 - 1)(q^{10} - 1)(q^{12} - 1)(q^{14} - 1)(q^{18} - 1)$$

$$|E_8(q)| = q^{120}(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)$$
$$(q^{24}-1)(q^{30}-1)$$

$$|{}^2B_2(q^2)| = q^4(q^2-1)(q^4+1) \qquad q^2 = 2^{2m+1}$$

$$|{}^2G_2(q^2)| = q^6(q^2-1)(q^6+1) \qquad q^2 = 3^{2m+1}$$

$$|{}^2F_4(q^2)| = q^{24}(q^2-1)(q^6+1)(q^8-1)(q^{12}+1) \qquad q^2 = 2^{2m+1}.$$

Now consider the special case when G is a semisimple group of adjoint type. Thus G is a direct product of simple groups all of adjoint type. In this case the action of a maximal torus T on the unipotent group U is particularly favourable. We have $U = \prod_{\alpha \in \Phi^+} X_\alpha$ and we put $U^* = \prod_{\alpha \notin \Delta}^{\alpha \in \Phi^+} X_\alpha$. The commutator relations show that U^* is a normal subgroup of U and U/U^* is abelian. T normalizes both U and U^* so acts on the quotient U/U^*. U/U^* is isomorphic to the direct product $X_{\alpha_1} \times \ldots \times X_{\alpha_l}$ of the simple root subgroups. T is isomorphic to the direct product $T_{\alpha_1} \times \ldots \times T_{\alpha_l}$ of subgroups T_{α_i} each isomorphic to K^*.

The action of T on U/U^* can be described as follows. T_{α_i} acts trivially on X_{α_j} if $i \neq j$. If $i = j$ T_{α_i} acts on X_{α_i} by multiplication by an element of K^*. Thus there exist isomorphisms

$$K^* \to T_{\alpha_i} \qquad K \to X_{\alpha_i}$$
$$\mu \to t_{\alpha_i}(\mu) \qquad \lambda \to x_{\alpha_i}(\lambda)$$

such that

$$t_{\alpha_i}(\mu)x_{\alpha_i}(\lambda)t_{\alpha_i}(\mu)^{-1} = x_{\alpha_i}(\lambda\mu).$$

One has a similar phenomenon in the finite group G^F. Again we suppose that G is semisimple of adjoint type. Let T be an F-stable maximal torus contained in an F-stable Borel subgroup $B = UT$ of G. Then U and U^* are F-stable and both U^F, $(U^*)^F$ are normalized by T^F. Thus T^F acts on the factor group $U^F/(U^*)^F$. For each ρ-orbit J on I let X_J be the direct product of the root subgroups X_{α_i} for $i \in J$. Since $F(X_{\alpha_i}) = X_{\rho(\alpha_i)}$ we can define a natural F-action on this direct product X_J. Then $U^F/(U^*)^F$ is isomorphic to the direct product $\prod_{\rho\text{-orbit}}^{J} (X_J)^F$ taken over all ρ-orbits J of I.

Similarly let T_J be the direct product of the subgroups T_{α_i} for $i \in J$. This is an F-stable subgroup of T and T^F is the direct product of the subgroups $T_J{}^F$ for all ρ-orbits J on I.

The action of T^F on $U^F/(U^*)^F$ can be described as follows. The subgroups $X_J{}^F$ are isomorphic to the additive group of the field $F_{q^{|J|}}$ and the subgroups $T_J{}^F$ are isomorphic to the multiplicative group of $F_{q^{|J|}}$. $T_J{}^F$ acts trivially on $X_{J'}{}^F$ if $J \neq J'$. If $J = J'$ $T_J{}^F$ acts on $X_J{}^F$ by multiplication by a nonzero element of $F_{q^{|J|}}$. Thus there exist isomorphisms

$$F_{q^{|J|}}{}^* \to T_J{}^F \qquad F_{q^{|J|}} \to X_J{}^F$$
$$\mu \to t_J(\mu) \qquad \lambda \to x_J(\lambda)$$

such that

$$t_J(\mu)x_J(\lambda)t_J(\mu)^{-1} = x_J(\lambda\mu).$$

This favourable action of T^F on $U^F/(U^*)^F$ will be useful to us later. It is valid only when G is of adjoint type.

It is also useful to know that if $x_J \in X_J{}^F$ and $x_J = \prod_{i\in J} x_{\alpha_i}$ with $x_{\alpha_i} \in X_{\alpha_i}$ then $x_J \neq 1$ implies $x_{\alpha_i} \neq 1$ for all $i \in J$. This follows from the structure of the root subgroups of G^F (see Steinberg [15] or Carter [3]).

2.10 DIFFERENT CONCEPTS OF ROOT SYSTEM

The reader will have noticed that we have introduced the concept of a root system in two different contexts. In section 1.9 we defined the root system of a connected reductive algebraic group G. This was a finite subset of the character group X of a maximal torus T of G. It is well defined up to isomorphism since any two maximal tori of G are conjugate. Also in section 2.2 we defined the root system of a finite Coxeter group W. This is a finite set of vectors in a Euclidean space on which W acts. It is necessary to be clear about the relation between these two concepts of root system and why it is necessary to have them both.

If G is connected reductive and T is a maximal torus of G then the Weyl group $W = N_G(T)/T$ is a Coxeter group and $V = X_{\mathbf{R}}$ is a real vector space on which W has a natural action. Moreover $X_{\mathbf{R}}$ can be made into a Euclidean space on which W acts as a group of isometries, as in section 2.2. The roots of G then lie in $X_{\mathbf{R}}$. They need not all have the same length, so do not in general form a root system in the sense of section 2.2 since all roots of a Coxeter group W were defined as unit vectors. However the unit vectors in the directions of the roots of G form the roots of the Coxeter group W in the subspace of $X_{\mathbf{R}}$ which they span. Thus we obtain the roots of W in the sense of section 2.2 from the roots of G by ignoring root lengths of G and taking only the subspace of $X_{\mathbf{R}}$ spanned by the roots.

However not every root system of a Coxeter group W^F of a finite group G^F can be derived from the root system of some connected reductive group by ignoring root lengths. The group $G^F = {}^2F_4(q^2)$ provides a counter-example. The Coxeter group W^F of G^F is isomorphic to the dihedral group of order 16, and there are 16 roots which are unit vectors inclined at an angle of $\pi/8$ to their neighbours. This root system and Coxeter group does not arise from any connected reductive group G. It is in fact the only indecomposable root system of a group G^F which does not arise in this way.

We therefore need both concepts of root system. However most arguments involving roots and Coxeter groups do not depend on the lengths of the roots and so are independent of which concept of root system we use. The fact that we have two slightly different concepts should not therefore lead to any serious problems in the subsequent development.

The subsets of Euclidean space which can arise as root systems of connected reductive groups can be described in the following axiomatic manner. Every such root system Φ in $V = X_{\mathbf{R}}$ satisfies:

(a) If α, $\beta \in \Phi$ then $w_\alpha(\beta) \in \Phi$.

(b) If α, $\beta \in \Phi$ then $\dfrac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$.

(c) If α, $\lambda\alpha \in \Phi$ then $\lambda = \pm 1$.

Conversely if Φ is any subset of a Euclidean space V satisfying axioms (a), (b), (c) then Φ is similar, up to scale, to the root system of some connected reductive group G.

Chapter 3

MAXIMAL TORI AND SEMISIMPLE CLASSES

In the present chapter we discuss certain properties of the conjugacy classes of semisimple elements in a connected reductive group G over an algebraically closed field K and in a finite subgroup of the form G^F where F is a Frobenius map on G. Since every semisimple element lies in a maximal torus, results on semisimple conjugacy classes are closely related to results on maximal tori in G and in G^F. We shall therefore prove results on these maximal tori also. We concentrate on results which will be needed in subsequent applications to representation theory. Further properties of semisimple conjugacy classes can be found in Borel *et al.* [1], part E, and in Steinberg [14].

Throughout this chapter K will be the algebraic closure of the field $\mathbb{F}_p$ with p elements, although a few of the results will clearly be valid for any algebraically closed field.

3.1 SOME RESULTS ON TORI

Let T be a torus over K and $X = \mathrm{Hom}(T, \mathbf{G}_m)$, $Y = \mathrm{Hom}(\mathbf{G}_m, T)$ be its character and cocharacter groups. Let K^* be the multiplicative group of nonzero elements of K.

Proposition 3.1.1. (i) *$Y \otimes K^*$ is isomorphic as abelian group to* $\mathrm{Hom}(X, K^*)$. (ii) *$X \otimes K^*$ is isomorphic to* $\mathrm{Hom}(Y, K^*)$.

Proof. We prove (i) and then (ii) will be entirely similar with X and Y interchanged. We first define a balanced map

$$\theta: Y \times K^* \to \mathrm{Hom}(X, K^*)$$
$$(\gamma, \lambda) \to (\chi \to \lambda^{\langle \chi, \gamma \rangle}).$$

We assert that every element of $\mathrm{Hom}(X, K^*)$ has the form $\sum \theta(\gamma_i, \lambda_i)$. For suppose $\chi_1, \ldots, \chi_r$ is a basis for X and $\gamma_1, \ldots, \gamma_r$ is the dual basis for Y with

$\langle \chi_i, \gamma_j \rangle = \delta_{ij}$. Take any homomorphism of X into K^* and let it map χ_i to λ_i. Then this homomorphism is $\sum_{i=1}^{l} \theta(\gamma_i, \lambda_i)$.

Now let $\psi: Y \times K^* \to A$ be any balanced map of $Y \times K^*$ into an abelian group A. We must show there is a homomorphism $\phi: \mathrm{Hom}(X, K^*) \to A$ such that $\psi = \phi \circ \theta$. We define ϕ as follows. Take a homomorphism of X into K^* which maps χ_i to λ_i. The image of this under ϕ is defined to be $\psi(\gamma_1, \lambda_1) + \ldots + \psi(\gamma_r, \lambda_r)$. ϕ is certainly a homomorphism of abelian groups. Also we have

$$\begin{aligned}\phi \circ \theta(\gamma, \lambda) &= \phi(\theta(\gamma, \lambda))\\ &= \psi(\gamma_1, \lambda^{\langle \chi_1, \gamma \rangle}) + \ldots + \psi(\gamma_r, \lambda^{\langle \chi_r, \gamma \rangle})\\ &= \psi(\langle \chi_1, \gamma \rangle \gamma_1, \lambda) + \ldots + \psi(\langle \chi_r, \gamma \rangle \gamma_r, \lambda)\\ &= \psi(\gamma, \lambda)\end{aligned}$$

since $\gamma = \langle \chi_1, \gamma \rangle \gamma_1 + \ldots + \langle \chi_r, \gamma \rangle \gamma_r$. Thus $\phi \circ \theta = \psi$. It follows that $\mathrm{Hom}(X, K^*)$ is isomorphic to $Y \otimes K^*$.

Proposition 3.1.2. (i) $\mathrm{Hom}(X, K^*)$ *is isomorphic as abelian group to* T.
(ii) $Y \otimes K^*$ *is isomorphic to* T.

Proof. It will be sufficient to prove (i). We define a map

$$\theta: T \to \mathrm{Hom}(X, K^*)$$
$$t \to (\chi \to \chi(t))$$

which sends t to evaluation at t. This is clearly a homomorphism of abelian groups. Now T is isomorphic to $\mathbf{G}_m \times \ldots \times \mathbf{G}_m$ and, in terms of such an isomorphism, the elements of X are the maps $(\lambda_1, \ldots, \lambda_r) \to \lambda_1^{n_1} \ldots \lambda_r^{n_r}$ for $n_1, \ldots, n_r \in \mathbb{Z}$. Let χ_i map $(\lambda_1, \ldots, \lambda_r)$ to λ_i. Suppose t lies in the kernel of θ. Then $\chi_i(t) = 1$ for each i. Hence $\lambda_i = 1$ for each i, and so $t = 1$. Thus θ is injective.

Now take any homomorphism from X to K^* and suppose it maps χ_i to λ_i. Then it is given by evaluation at the element $(\lambda_1, \ldots, \lambda_r)$ of $\mathbf{G}_m \times \ldots \times \mathbf{G}_m$. Thus θ is surjective. Hence θ is an isomorphism.

We therefore also obtain an isomorphism $Y \otimes K^* \to T$. We observe that the image of $\gamma \otimes \lambda$ under this isomorphism is $\gamma(\lambda) \in T$. ■

We next prove a result about the structure of the group K^* when K is the algebraic closure of $\mathbb{F}_p$. Let Ω be the group of all complex roots of unity and $\Omega_{p'}$ the subgroup of Ω given by

$$\Omega_{p'} = \{z \in \Omega;\ z^n = 1 \text{ for some } n \text{ not divisible by } p\}.$$

Let $\mathbb{Q}_{p'}$ be the additive group of rational numbers of the form r/s where $r, s \in \mathbb{Z}$ and s is not divisible by p.

Proposition 3.1.3. *The following three groups are isomorphic:*

(i) K^* (ii) $\Omega_{p'}$ (iii) $\mathbb{Q}_{p'}/\mathbb{Z}$.

Proof. There is a homomorphism $\mathbb{Q} \to \Omega$ given by $r/s \to e^{2\pi i r/s}$. This homomorphism maps $\mathbb{Q}_{p'}$ onto $\Omega_{p'}$. Its kernel is $\mathbb{Z}$. Thus $\mathbb{Q}_{p'}/\mathbb{Z}$ is isomorphic to $\Omega_{p'}$.

We now wish to construct a similar homomorphism from $\mathbb{Q}_{p'}$ to K^*. Each element of K^*, being algebraic over $\mathbb{F}_p$, lies in some finite extension $\mathbb{F}_{p^e}$ of $\mathbb{F}_p$ and so satisfies $x^{p^e-1} - 1 = 0$. Thus each element of K^* is a root of unity of order prime to p. On the other hand, given any positive integer s prime to p we consider the elements $\lambda \in K^*$ satisfying $\lambda^s = 1$. The polynomial $x^s - 1$ factorizes into linear factors in $K[x]$ and its roots are distinct since $x^s - 1$ has no root in common with its derivative. The s elements of K satisfying $\lambda^s = 1$ are all algebraic over the prime subfield $\mathbb{F}_p$ of K and so lie in some finite extension $\mathbb{F}_{p^e}$ of $\mathbb{F}_p$. However the multiplicative group of $\mathbb{F}_{p^e}$ is cyclic. Hence the sth roots of unity in K will form a cyclic group Γ_s of order s, and $K^* = \bigcup_{p \nmid s} \Gamma_s$. Let $n_i = (i!)_{p'}$ for $i = 1, 2, \ldots$. Then we have

$$\Gamma_{n_1} \subseteq \Gamma_{n_2} \subseteq \Gamma_{n_3} \subseteq \ldots$$

and $\bigcup_i \Gamma_{n_i} = K^*$. Moreover we can find generators γ_{n_i} of Γ_{n_i} such that $(\gamma_{n_i})^{n_i/n_{i-1}} = \gamma_{n_{i-1}}$ for each i. Given any element $r/s \in \mathbb{Q}_{p'}$ with $r, s \in \mathbb{Z}$ and $(p, s) = 1$ we know that s divides n_i for some i. Thus $r/s = m/n_i$ for $m \in \mathbb{Z}$. The map $r/s \to \gamma_{n_i}{}^m$ is then a homomorphism from $\mathbb{Q}_{p'}$ into K^*. This map is surjective and its kernel is $\mathbb{Z}$. Thus it induces an isomorphism between $\mathbb{Q}_{p'}/\mathbb{Z}$ and K^*.

We shall choose an isomorphism between $\mathbb{Q}_{p'}/\mathbb{Z}$ and K^* and keep it fixed throughout the subsequent development.

3.2 TORI WITH *F*-ACTION

Let G be a connected reductive group and $F: G \to G$ a Frobenius map. Let T be an F-stable maximal torus of G. We have seen in section 1.18 how to define actions of F on X and Y. We have

$$(F(\chi))t = \chi(F(t)) \qquad \chi \in X,\ t \in T$$

$$(F(\gamma))\lambda = F(\gamma(\lambda)) \qquad \gamma \in Y,\ \lambda \in \mathbf{G}_m.$$

Proposition 3.2.1. *Let $\chi_1, \ldots, \chi_r \in X$, $\gamma_1, \ldots, \gamma_r \in Y$ be dual bases of X, Y. Then the matrices representing F with respect to these bases of X, Y are transposes of one another.*

Proof. Let $\lambda \in \mathbf{G}_m$. Then we have

$$\lambda^{\langle \chi, F(\gamma) \rangle} = \chi((F(\gamma))(\lambda)) = \chi(F(\gamma(\lambda))) = (F(\chi))(\gamma(\lambda)) = \lambda^{\langle F(\chi), \gamma \rangle}$$

for all $\chi \in X$, $\gamma \in Y$. Hence

$$\langle \chi, F(\gamma) \rangle = \langle F(\chi), \gamma \rangle.$$

Let $F(\chi_i) = \sum_{j=1}^{r} a_{ij}\chi_j$, $F(\gamma_i) = \sum_{j=1}^{r} b_{ij}\gamma_j$. Then we have

$$a_{ij} = \langle F(\chi_i), \gamma_j \rangle = \langle \chi_i, F(\gamma_j) \rangle = b_{ji}.$$

Thus (a_{ij}) and (b_{ij}) are transpose matrices.

Proposition 3.2.2. *T^F is isomorphic to $Y/(F-1)Y$.*

Proof. We first extend the F-action on Y to an F-action on $Y \otimes \mathbb{Q}$ given by

$$F(\gamma \otimes \lambda) = F(\gamma) \otimes \lambda \qquad \gamma \in Y,\ \lambda \in \mathbb{Q}.$$

Now $Y \otimes \mathbb{Q}_{p'}$ is a subgroup of $Y \otimes \mathbb{Q}$ which is transformed into itself by F. There is a natural homomorphism

$$Y \otimes \mathbb{Q}_{p'} \to Y \otimes \mathbb{Q}_{p'}/\mathbb{Z}$$

with kernel $Y \otimes \mathbb{Z} = Y$. Thus F induces an action on $Y \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ which satisfies

$$F(\gamma \otimes \bar{\lambda}) = F(\gamma) \otimes \bar{\lambda} \qquad \gamma \in Y,\ \bar{\lambda} \in \mathbb{Q}_{p'}/\mathbb{Z}.$$

Now $\mathbb{Q}_{p'}/\mathbb{Z}$ is isomorphic to K^* by 3.1.3 and $Y \otimes K^*$ is isomorphic to T by 3.1.2. This latter isomorphism maps $\gamma \otimes \lambda$ to $\gamma(\lambda) \in T$. Since

$$F(\gamma(\lambda)) = (F(\gamma))(\lambda)$$

the F-action on T carries over to $Y \otimes K^*$ by mapping $\gamma \otimes \lambda$ to $F(\gamma) \otimes \lambda$. Thus the F-action on $Y \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ defined above is consistent with the F-action on T under the isomorphism between T and $Y \otimes \mathbb{Q}_{p'}/\mathbb{Z}$.

We now consider the following diagram:

$$\begin{tikzcd}
 & & & 1 \arrow[d] & \\
 & & & T^F \arrow[d] & \\
0 \arrow[r] & Y \arrow[r] \arrow[d, "F-1"] & Y \otimes \mathbb{Q}_{p'} \arrow[r] \arrow[d, "F-1"] & Y \otimes \mathbb{Q}_{p'}/Z \arrow[r] \arrow[d, "F-1"] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & Y \otimes \mathbb{Q}_{p'} \arrow[r] & Y \otimes \mathbb{Q}_{p'}/Z \arrow[r] & 0 \\
 & Y/(F-1)Y \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

This diagram has commutative squares and exact rows and columns. We may therefore apply the snake lemma (Rotman [1], p. 119) to obtain a homomorphism $T^F \to Y/(F-1)Y$. Moreover this will be an isomorphism provided we can show that the middle map

$$F - 1 : Y \otimes \mathbb{Q}_{p'} \to Y \otimes \mathbb{Q}_{p'}$$

is bijective.

Now $F = qF_0$ on $Y \otimes \mathbb{Q}$ where $q > 1$ and F_0 has finite order. Thus F has no eigenvalue 1 on $Y \otimes \mathbb{Q}$. It follows that the map $F - 1 : Y \otimes \mathbb{Q}_{p'} \to Y \otimes \mathbb{Q}_{p'}$ is injective.

We next observe that the determinant of $F - 1$ on $Y \otimes \mathbb{Q}$ is an integer prime to p. This determinant can be computed on Y and $\det_Y(F - 1)$ is prime to p if and only if $\det_{Y/pY}(F - 1)$ is nonzero in $\mathbb{Z}/p\mathbb{Z}$. However some power of F is multiplication by a power of p, thus F is nilpotent on Y/pY. Thus $F - 1$ is invertible on Y/pY and so $\det_{Y/pY}(F - 1) \neq 0$.

Thus the determinant of $F - 1$ on $Y \otimes \mathbb{Q}$ is an integer prime to p and this implies that the map $F - 1 : Y \otimes \mathbb{Q}_{p'} \to Y \otimes \mathbb{Q}_{p'}$ is surjective.

We therefore obtain an isomorphism $T^F \to Y/(F - 1)Y$. We note that this isomorphism is not canonical, as we have made use of the noncanonical isomorphism between $\mathbb{Q}_{p'}/\mathbb{Z}$ and K^* in defining it. ■

We now consider $\hat{T}^F = \mathrm{Hom}(T^F, \mathbb{C}^*)$. This is the group of complex characters of T^F. Since T^F is a finite group of order prime to p we have $\hat{T}^F = \mathrm{Hom}(T^F, \Omega_{p'})$. Using the isomorphism between $\Omega_{p'}$ and $\mathbb{Q}_{p'}/\mathbb{Z}$ we have

$$\hat{T}^F \cong \mathrm{Hom}(T^F, \mathbb{Q}_{p'}/\mathbb{Z}).$$

Proposition 3.2.3. *$\hat{T}^F$ is isomorphic to $X/(F - 1)X$.*

Proof. By considering the elementary divisors of the map $F - 1 : X \to X$ we see that

$$|X/(F - 1)X| = |\det_{X \otimes \mathbb{Q}}(F - 1)|.$$

As in the proof of 3.2.2 we know that $\det_{X \otimes \mathbb{Q}}(F - 1)$ is prime to p. Thus $(F - 1)X$ is a subgroup of X of index prime to p. Consider its annihilator $S = ((F - 1)X)^\perp$ in T. This is the set of all $s \in T$ satisfying

$$F(\chi)\chi^{-1}(s) = 1 \qquad \text{for all } \chi \in X$$

which is equivalent to $\chi(F(s)s^{-1}) = 1$ for all $\chi \in X$. However the only element of T annihilated by all $\chi \in X$ is the identity. Thus $F(s)s^{-1} = 1$ and so $s \in T^F$. Hence $S = T^F$.

Now we know from section 1.12 that $S^\perp = (F - 1)X$ since $|X : (F - 1)X|$ is prime to p, and also that $X/(F - 1)X$ is isomorphic to $\mathrm{Hom}(T^F, K^*)$. Using the isomorphism of 3.1.3 we obtain

$$X/(F - 1)X \cong \hat{T}^F.$$

Proposition 3.2.4. *$\hat{T}^F$ is isomorphic to the kernel of $F - 1$ on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$.*

Proof. If we replace X by Y in the diagram used to prove 3.2.2 we must replace $Y/(F - 1)Y$ by $X/(F - 1)X$, which is isomorphic to $\hat{T}^F$ by 3.2.3. The same argument as in 3.2.2 will then show that $\hat{T}^F$ is isomorphic to the kernel of $F - 1$ on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$.

3.3 G^F-CLASSES OF MAXIMAL TORI

We recall from chapter 1 that any two maximal tori in the connected reductive group G are conjugate in G and that there exist F-stable maximal tori of G. In fact there exist F-stable maximal tori of G which lie in F-stable Borel subgroups of G. Such F-stable maximal tori are called maximally split. Any two maximally split F-stable maximal tori of G are conjugate by an element of G^F. However not every F-stable maximal torus will in general be maximally split. We consider in this section how the set of F-stable maximal tori of G falls into conjugacy classes under the action of G^F.

We begin with a maximally split F-stable maximal torus T_0. Any maximal torus of G will then have the form ${}^gT_0$ for some $g \in G$.

Proposition 3.3.1. *${}^gT_0$ is F-stable if and only if $g^{-1}F(g) \in N_0 = N(T_0)$.*

Proof. We have

$$F({}^gT_0) = {}^{F(g)}F(T_0) = {}^{F(g)}T_0.$$

Thus $F({}^gT_0) = {}^gT_0$ if and only if ${}^gT_0 = {}^{F(g)}T_0$, which is equivalent to $g^{-1}F(g) \in N_0$. ■

Now $N_0/T_0 \cong W$ and we have the natural map $\pi: N_0 \twoheadrightarrow W$. Thus if ${}^gT_0$ is F-stable we obtain an element $\pi(g^{-1}F(g)) \in W$. We consider to what extent this element depends upon the choice of the conjugating element g.

Proposition 3.3.2. *Suppose ${}^gT_0 = {}^{g'}T_0$ is F-stable. Let $\pi(g^{-1}F(g)) = w$ and $\pi(g'^{-1}F(g')) = w'$. Then there exists $x \in W$ such that $w' = x^{-1}wF(x)$.*

Proof. Since ${}^gT_0 = {}^{g'}T_0$ we have $g^{-1}g' \in N_0$. Let $g^{-1}g' = n$. Then $g' = gn$ and so $g'^{-1}F(g') = n^{-1}g^{-1}F(g)F(n)$. We now apply π and obtain $w' = x^{-1}wF(x)$ where $x = \pi(n)$. ■

We say that w, $w' \in W$ are F-conjugate if there exists $x \in W$ such that $w' = x^{-1}wF(x)$. F-conjugacy is clearly an equivalence relation on W. The equivalence classes under this relation are called F-conjugacy classes.

We see from 3.3.1 and 3.3.2 that each F-stable maximal torus of G determines an F-conjugacy class in W.

Proposition 3.3.3 *The map ${}^gT_0 \to \pi(g^{-1}F(g))$ determines a bijection between the G^F-classes of F-stable maximal tori of G and the F-conjugacy classes of W.*

Proof. Each F-stable maximal torus determines an F-conjugacy class of W. Suppose ${}^gT_0$, ${}^{g'}T_0$ are F-stable and determine the same F-conjugacy class of W. Then there exists $x \in W$ with

$$\pi(g'^{-1}F(g')) = x^{-1}\pi(g^{-1}F(g))F(x).$$

Let $n \in N_0$ have $\pi(n) = x$. Then $g'^{-1}F(g')$ and $n^{-1}g^{-1}F(g)F(n)$ lie in the same coset of T_0 in N_0, and so

$$n^{-1}g^{-1}F(g)F(n)F(g')^{-1}g' \in T_0.$$

It follows that

$$g'n^{-1}g^{-1}F(g)F(n)F(g')^{-1} \in {}^{g'}T_0.$$

By the Lang–Steinberg theorem applied to ${}^{g'}T_0$ there exists $t \in T_0$ such that

$$g'n^{-1}g^{-1}F(g)F(n)F(g')^{-1} = ({}^{g'}t)^{-1}F({}^{g'}t).$$

This gives

$$g'n^{-1}g^{-1}F(g)F(n)F(g')^{-1} = g't^{-1}g'^{-1}F(g')F(t)F(g'^{-1})$$

which is equivalent to

$$g'tn^{-1}g^{-1} = F(g')F(t)F(n)^{-1}F(g)^{-1}.$$

Hence $g'tn^{-1}g^{-1} \in G^F$. Moreover we have ${}^{g'tn^{-1}g^{-1}}({}^{g}T_0) = {}^{g'}T_0$ and so ${}^{g}T_0$ and ${}^{g'}T_0$ are G^F-conjugate.

Conversely it is clear that whenever ${}^{g}T_0$, ${}^{g'}T_0$ are G^F-conjugate they will give rise to the same F-conjugacy class in W.

Finally we must show that each F-conjugacy class of W arises from some F-stable maximal torus. This is proved by a further application of the Lang–Steinberg theorem. Let $w \in W$ and $n \in N_0$ satisfy $\pi(n) = w$. By the Lang–Steinberg theorem there exists $g \in G$ such that $g^{-1}F(g) = n$. Then ${}^{g}T_0$ is F-stable by 3.3.1 and gives rise to the F-conjugacy class of W containing w. ∎

If T is an F-stable maximal torus of G for which the corresponding F-conjugacy class of W contains w, we say that T is obtained from the maximally split torus T_0 by twisting with w.

Let T be such an F-stable maximal torus and let $T = {}^{g}T_0$ where $\pi(g^{-1}F(g)) = w$. Let X, Y be the character and cocharacter groups of T and X_0, Y_0 those of T_0. We wish to define maps $X_0 \to X$ and $Y_0 \to Y$ derived from the conjugation map $t_0 \to {}^{g}t_0$ from T_0 to T. In the special case when $g \in N_0$ these maps will take X_0 and Y_0 into themselves and should therefore agree with the operation of the element $\pi(g) \in W$. We therefore define conjugation maps as follows:

$$X_0 \to X \qquad Y_0 \to Y$$

$$\chi_0 \to {}^{g}\chi_0 \qquad \gamma_0 \to {}^{g}\gamma_0$$

where

$${}^{g}\chi_0({}^{g}t_0) = \chi_0(t_0), \qquad ({}^{g}\gamma_0)(\lambda) = {}^{g}(\gamma_0(\lambda)).$$

Note that when $g = \dot{x} \in N$ for $x \in W$ we obtain

$${}^{\dot{x}}\chi_0 = {}^{x}\chi_0 \quad \text{and} \quad {}^{\dot{x}}\gamma_0 = \gamma_0{}^{x^{-1}}.$$

Thus in this case the conjugation operations reduce to the actions of W on X_0 and Y_0 defined earlier in section 1.9.

Proposition 3.3.4. (i) *The map $F: X \to X$ corresponds under the conjugation map from X to X_0 to the map $F \circ w^{-1}: X_0 \to X_0$.*

(ii) *The map $F: Y \to Y$ corresponds under the conjugation map from Y to Y_0 to the map $w^{-1} \circ F: Y_0 \to Y_0$.*

Proof. (i) We recall the definitions of the actions of F and w on X. These were given by

$$(F(\chi))(t) = \chi(F(t)) \qquad \chi \in X,\ t \in T$$
$$({}^{w}\chi)(t) = \chi(t^{\dot w}) \qquad w \in W.$$

Let $\chi_0 \in X_0$, so that ${}^{g}\chi_0 \in X$. Then we have

$$\begin{aligned}(F({}^{g}\chi_0))({}^{g}t_0) &= ({}^{g}\chi_0)(F({}^{g}t_0)) = ({}^{g}\chi_0)({}^{F(g)}F(t_0)) \\ &= ({}^{g}\chi_0)({}^{g\dot w}F(t_0)) = ({}^{g}\chi_0)({}^{g}(F(t_0)^{\dot w^{-1}})) = \chi_0(F(t_0)^{\dot w^{-1}}) \\ &= {}^{w^{-1}}\chi_0(F(t_0)) = (F({}^{w^{-1}}\chi_0))(t_0) = {}^{g}(F({}^{w^{-1}}\chi_0))({}^{g}t_0).\end{aligned}$$

Thus F maps ${}^{g}\chi_0$ to ${}^{g}(F({}^{w^{-1}}\chi_0))$. Therefore F acts on X in the same way that $F \circ w^{-1}$ acts on X_0.

(ii) We recall the definitions of the actions of F and w on Y. These were given by

$$(F(\gamma))(\lambda) = F(\gamma(\lambda)) \qquad \gamma \in Y,\ \lambda \in \mathbf{G}_m$$
$$(\gamma^{w})(\lambda) = (\gamma(\lambda))^{\dot w} \qquad w \in W.$$

Let $\gamma_0 \in Y_0$ so that ${}^{g}\gamma_0 \in Y$. Then we have

$$\begin{aligned}(F({}^{g}\gamma_0))(\lambda) &= F({}^{g}\gamma_0(\lambda)) = F({}^{g}(\gamma_0(\lambda))) = {}^{F(g)}F(\gamma_0(\lambda)) \\ &= {}^{g\dot w}F(\gamma_0(\lambda)) = {}^{g\dot w}((F(\gamma_0))(\lambda)) = {}^{g}((F(\gamma_0)^{w^{-1}})(\lambda)) \\ &= ({}^{g}(F(\gamma_0)^{w^{-1}}))(\lambda).\end{aligned}$$

Hence F maps ${}^{g}\gamma_0$ to ${}^{g}((F(\gamma_0))^{w^{-1}})$. Thus F acts on Y in the same way that $w^{-1} \circ F$ acts on Y_0.

Proposition 3.3.5. *Let T be an F-stable maximal torus of G obtained from the maximally split torus T_0 by twisting with w. Then the order of T^F is given by*

$$|T^F| = |\det_{Y_0 \otimes \mathbb{R}}(w^{-1} \circ F - 1)|.$$

Moreover if we write $F = qF_0$ where $q > 1$ and F_0 has finite order then

$$|T^F| = \chi(q)$$

where $\chi(x)$ is the characteristic polynomial of ${F_0}^{-1} \circ w$ on $Y_0 \otimes \mathbb{R}$.

Proof. By 3.2.2 we know that T^F is isomorphic to $Y/(F - 1)Y$. We apply the conjugation map $Y \to Y_0$. By 3.3.4 we have

$$Y/(F - 1)Y \cong Y_0/(w^{-1} \circ F - 1)Y_0.$$

$(w^{-1} \circ F - 1)Y_0$ has finite index in Y_0 which is equal to

$$|\det_{Y_0 \otimes \mathbf{R}}(w^{-1} \circ F - 1)|.$$

We now write $F = qF_0$ where $q > 1$ and F_0 has finite order. Then

$$(w^{-1} \circ F) - 1 = (qw^{-1} \circ F_0) - 1 = (q1 - F_0^{-1}w)w^{-1}F_0.$$

Since $|\det w| = 1$ and $|\det F_0| = 1$ we have

$$|\det_{Y_0 \otimes \mathbf{R}}(w^{-1} \circ F - 1)| = |\det_{Y_0 \otimes \mathbf{R}}(q1 - F_0^{-1} \circ w)|.$$

We show finally that $\det_{Y_0 \otimes \mathbf{R}}(q1 - F_0{}^{-1} \circ w) > 0$ so that the modulus sign can be omitted. $q1 - F_0{}^{-1} \circ w$ is a real transformation and so its eigenvalues will either be real or occur in complex conjugate pairs. Let λ be a real eigenvalue corresponding to an eigenvector v. Then

$$(q1 - F_0^{-1} \circ w)v = \lambda v$$

and so

$$(q - \lambda)v = (F_0^{-1} \circ w)v.$$

We compare the lengths of these vectors. $(F_0^{-1} \circ w)v$ has the same length as v and so $|q - \lambda| = 1$. Since $q > 1$ this implies that $\lambda > 0$. Thus all real eigenvalues are positive. It follows that the product of the eigenvalues is positive, so that $\det_{Y_0 \otimes \mathbf{R}}(q1 - F_0^{-1} \circ w) > 0$.

Finally we have

$$\chi(x) = \det_{Y_0 \otimes \mathbf{R}}(x1 - F_0^{-1} \circ w)$$

and so $|T^F| = \chi(q)$. ■

Let $w \in W$. The set $C_{W,F}(w)$ defined by $C_{W,F}(w) = \{x \in W; x^{-1}wF(x) = w\}$ is a subgroup of W called the F-centralizer of w. It has the property that its index $|W : C_{W,F}(w)|$ is the number of elements in the F-conjugacy class containing w.

Proposition 3.3.6. *Let T be an F-stable maximal torus of G obtained from the maximally split torus T_0 by twisting with w. Let N be the normalizer of T. Then N^F/T^F is isomorphic to $C_{W,F}(w)$.*

Proof. Let $T = {}^gT_0$. Then we have

$$F({}^gt_0) = {}^{F(g)}F(t_0) = {}^{g\dot{w}}F(t_0) = {}^g((F(t_0))^{w^{-1}}).$$

Thus F acts on T as $w^{-1} \circ F$ acts on T_0. Now N^F/T^F is isomorphic to $(N/T)^F$ since T is connected, as in section 1.17. The conjugation map transforms T to T_0, N to N_0 and N/T to N_0/T_0. It also transforms $(N/T)^F$ to $(N_0/T_0)^{w^{-1} \circ F}$. Thus $(N/T)^F$ is isomorphic to the subgroup of W fixed by $w^{-1} \circ F$. Let $x \in W$. Then x is fixed by $w^{-1} \circ F$ if and only if $wF(x)w^{-1} = x$, which is equivalent to $x^{-1}wF(x) = w$, i.e. $x \in C_{W,F}(w)$. Thus $(N/T)^F \cong C_{W,F}(w)$. The required result follows.

Note that in the particular case when T is maximally split this result reduces to $N^F/T^F \cong W^F$. ■

We next discuss the relation between the maximal tori of a connected reductive group and the maximal tori of a connected semisimple group. We recall that if G is a connected reductive group then $G = G'Z^0$ and $G' \cap Z^0$ is finite. G' is a connected semisimple group and Z^0, the connected centre of G, is a torus. Moreover every maximal torus T of G contains Z^0.

If T is a maximal torus of G then $S = T \cap G'$ is a maximal torus of G'. For S^0 is certainly a maximal torus of G', but in a connected reductive group a maximal torus is its own centralizer. Since S lies in the centralizer of S^0 in G' we have $S = S^0$. Conversely if S is any maximal torus of G' then $T = Z^0 S$ is a maximal torus of G. We thus obtain a bijective map between maximal tori of G and G'. Furthermore we have

$$N_G(T)/T \cong N_{G'}(S)/S$$

and so the Weyl groups of G, G' are isomorphic.

Given a Frobenius map $F: G \to G$ we see that T is F-stable if and only if S is F-stable and T is maximally split in G if and only if S is maximally split in G'.

Proposition 3.3.7. *Let S, T be F-stable maximal tori of G', G respectively with $T = SZ^0$ and $S = T \cap G'$. Then $|T^F| = |S^F|.\,|(Z^0)^F|$.*

Proof. We consider the cocharacter groups $Y(T)$, $Y(S)$, $Y(Z^0)$. Each homomorphism from $\mathbf{G}_m$ into S can be regarded as a homomorphism from $\mathbf{G}_m$ into T, and so $Y(S)$ can be regarded as a subgroup of $Y(T)$. Similarly $Y(Z^0)$ can be regarded as a subgroup of $Y(T)$. Let $\gamma \in Y(S) \cap Y(Z^0)$. Then $\gamma(\mathbf{G}_m) \subseteq S \cap Z^0$ and so $\gamma(\mathbf{G}_m)$ is finite. However $\gamma(\mathbf{G}_m)$ is connected and so $\gamma(\mathbf{G}_m) = 1$ and $\gamma = 0$. Thus $Y(S) \cap Y(Z^0) = 0$. $Y(S) \oplus Y(Z^0)$ is thus a subgroup of $Y(T)$. These groups have the same rank and so $Y(S) \oplus Y(Z^0)$ has finite index in $Y(T)$. It follows that

$$Y(T) \otimes \mathbb{R} = (Y(S) \otimes \mathbb{R}) \oplus (Y(Z^0) \otimes \mathrm{R}).$$

Now T^F is isomorphic to $Y(T)/(F-1)Y(T)$ by 3.2.2 and we have similar results for S^F and $(Z^0)^F$. Thus

$$|T^F| = |\det_{Y(T)\otimes\mathbb{R}}(F-1)|.$$

Now $F - 1$ acts on $Y(T) \otimes \mathbb{R}$ and leaves the subspaces $Y(S) \otimes \mathbb{R}$ and $Y(Z^0) \otimes \mathbb{R}$ invariant. Thus

$$\det_{Y(T)\otimes\mathbb{R}}(F-1) = \det_{Y(S)\otimes\mathbb{R}}(F-1).\det_{Y(Z^0)\otimes\mathbb{R}}(F-1).$$

It follows that $|T^F| = |S^F|.|(Z^0)^F|$.

Proposition 3.3.8. $|T^F| = |(Z^0)^F|.\chi_{G'}(q)$ *where $\chi_{G'}(x)$ is the characteristic polynomial of $F_0^{-1} \circ w$ on $Y_0(S) \otimes \mathbb{R}$, and T is obtained from a maximally split torus T_0 by twisting with w.*

Proof. This follows from 3.3.7 and 3.3.5.

3.4 A THEOREM OF STEINBERG

We shall now prove a theorem of Steinberg ([16], 14.16) which determines the number of F-stable maximal tori in a connected reductive group G.

Theorem 3.4.1. *The number of F-stable maximal tori in the connected reductive group G is q^{2N} where $N = |\Phi^+|$. This number can also be written $|G^F|_p{}^2$, where $|G^F|_p$ is the highest power of p dividing $|G^F|$.*

Proof. The discussion in section 3.3 shows that we may assume G is semisimple. Let T be a maximally split F-stable maximal torus of G. The other maximal tori then have form ${}^gT$. Now ${}^gT$ is F-stable if and only if gN is F-stable where $N = N(T)$. Let G/N denote the set of cosets gN. Thus the number we require is $|(G/N)^F|$.

Now there is a natural map $G/T \to G/N$ given by $gT \to gN$. W acts on G/T by $gT \xrightarrow{w} gT\dot{w}$ and the fibres of the above map from gT to gN are the W-orbits on G/T. Thus the number we require is $((G/T)/W)^F$ where $(G/T)/W$ denotes the set of W-orbits on G/T.

Let $A = G/T$ and A_0 be the subset of A given by

$$A_0 = \{a \in A;\ F(a) = a^w \text{ for some } w \in W\}.$$

Then A_0 is stable under W and the F-stable W-orbits on A are precisely the W-orbits on A_0. Thus the required number is $|A_0/W|$.

We next observe that $|A_0/W|$ is the average over W of the number of $a \in A$ with $F(a) = a^w$. For let $W_a = \{w \in W;\ a^w = a\}$. Then each $a \in A_0$ lies in a W-orbit with $|W: W_a|$ elements. Thus we have

$$|A_0/W| = \sum_{a \in A_0} \frac{|W_a|}{|W|} = \frac{1}{|W|} \sum_{a \in A_0} \sum_{\substack{w \in W \\ a^w = a}} 1$$

$$= \frac{1}{|W|} \sum_{a \in A_0} \sum_{\substack{w \in W \\ a^w = F(a)}} 1 \qquad \text{since each } W\text{-orbit in } A_0 \text{ is } F\text{-stable}$$

$$= \frac{1}{|W|} \sum_{w \in W} \sum_{\substack{a \in A_0 \\ a^w = F(a)}} 1 = \frac{1}{|W|} \sum_{w \in W} \Bigg(\sum_{\substack{a \in A \\ a^w = F(a)}} 1 \Bigg)$$

which is the required average. Thus the number we need is

$$\frac{1}{|W|} \sum_{w \in W} \left| \left(\frac{G}{T}\right)^{w^{-1}F} \right|.$$

By the Lang–Steinberg theorem there exists $g \in G$ with $\dot{w} = g^{-1}F(g)$. Consider the F-stable torus ${}^gT$ obtained from T by twisting with w. Then the conjugation map takes elements of G/T fixed by $w^{-1} \circ F$ to elements of $G/{}^gT$

fixed by F. Thus the required number is

$$\frac{1}{|W|}\sum_{w\in W}\left|\left(\frac{G}{{}^gT}\right)^F\right| = \frac{1}{|W|}\sum_{w\in W}\frac{|G^F|}{|({}^gT)^F|}$$

since ${}^gT$ is connected. Thus our required number is

$$\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det_{Y\otimes\mathbb{R}}(q1 - F_0{}^{-1}\circ w)}$$

by 3.3.8. Since $F = qF_0$ and $\dim Y = l$ this may be written

$$\frac{1}{q^l|W|}\sum_{w\in W}\frac{1}{\det_{Y\otimes\mathbb{R}}(1 - F^{-1}\circ w)}.$$

Before going further we need the following lemma.

Lemma 3.4.2. *Let G be a connected semisimple group of rank l, $F:G\to G$ be a Frobenius map, T be a maximally split torus of G, Y be the cocharacter group of T, and $V = Y\otimes\mathbb{R}$. Let $d_1,\dots,d_l$ and $\varepsilon_1,\dots,\varepsilon_l$ be defined as in section* 2.9. *Then*

$$\frac{1}{|W|}\sum_{w}\frac{1}{\det_V(1 - F^{-1}\circ w)} = \prod_{i=1}^{l}\frac{1}{(1 - \varepsilon_i{}^{-1}q^{-d_i})}.$$

Proof. Let $\mathfrak{P}$ be the algebra of polynomial functions on V. $\mathfrak{P}$ is a graded algebra with homogeneous components $\mathfrak{P}_n$. The subalgebra $\mathfrak{I}$ of W-invariants of $\mathfrak{P}$ is isomorphic to a polynomial ring in l variables by 2.4.1. $\mathfrak{I}$ can be generated by homogeneous polynomials $I_1,\dots,I_l$ of degrees $d_1,\dots,d_l$. If we extend the base field from $\mathbb{R}$ to $\mathbb{C}$ we can choose $I_1,\dots,I_l$ to be eigenvalues of F_0, where $F = qF_0$ with $q > 1$ and F_0 of finite order. Thus we have $F_0(I_i) = \varepsilon_i I_i$ for $i = 1,\dots,l$ where ε_i are roots of unity. It follows that $F(I_i) = \varepsilon_i q^{d_i} I_i$.

We now consider the expression

$$\frac{1}{\det_V(1 - tF^{-1}w)}$$

where t is an indeterminate. Let $\lambda_1,\dots,\lambda_l$ be the eigenvalues of $F^{-1}w$ on $V\otimes\mathbb{C}$. Then the above expression can be expanded in a power series in t given by

$$\prod_{i=1}^{l}\frac{1}{(1-\lambda_i t)} = \prod_{i=1}^{l}(1 + \lambda_i t + \lambda_i{}^2t^2 + \dots)$$

$$= \sum_{n\geq 0}\Bigg(\sum_{\substack{k_1,k_2,\dots,k_l\geq 0\\ k_1+\dots+k_l=n}}\lambda_1{}^{k_1}\lambda_2{}^{k_2}\dots\lambda_l{}^{k_l}\Bigg)t^n.$$

Now $\lambda_1,\dots,\lambda_l$ are also the eigenvalues of $F^{-1}w$ on the dual space $\hat{V}\otimes\mathbb{C}$. Thus the eigenvalues of $F^{-1}w$ on $\mathfrak{P}_n$ will be the numbers $\lambda_1{}^{k_1}\lambda_2{}^{k_2}\dots\lambda_l{}^{k_l}$ for all sets of nonnegative integers $k_1,\dots,k_l$ with $\sum k_i = n$. Thus the coefficient of t^n in the power series for $1/(\det_V(1 - tF^{-1}w))$ is the trace of $F^{-1}w$ on $\mathfrak{P}_n$.

Thus we have

$$\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det_V(1-tF^{-1}w)}=\frac{1}{|W|}\sum_{w\in W}\sum_{n\geqslant 0}(\operatorname{trace}_{\mathfrak{P}_n}F^{-1}w)t^n$$

$$=\sum_{n\geqslant 0}\left(\operatorname{trace}_{\mathfrak{P}_n}F^{-1}\left(\frac{1}{|W|}\sum_{w\in W}w\right)\right)t^n.$$

Now $\frac{1}{|W|}\sum_{w\in W}w$ maps $\mathfrak{P}_n$ into $\mathfrak{I}_n=\mathfrak{I}\cap\mathfrak{P}_n$ and acts trivially on $\mathfrak{I}_n$. Moreover F^{-1} transforms $\mathfrak{I}_n$ into itself. Thus we have

$$\operatorname{trace}_{\mathfrak{P}_n}F^{-1}\left(\frac{1}{|W|}\sum_{w\in W}w\right)=\operatorname{trace}_{\mathfrak{I}_n}F^{-1}.$$

Now $F^{-1}(I_i)=\varepsilon_i^{-1}q^{-d_i}I_i$ and the elements $I_1^{p_1}I_2^{p_2}\dots I_l^{p_l}$ with $p_1d_1+p_2d_2+\dots+p_ld_l=n$ form a basis for $\mathfrak{I}_n$. Thus

$$\operatorname{trace}_{\mathfrak{I}_n}F^{-1}=\sum_{\substack{p_1,\dots,p_l\geqslant 0\\ p_1d_1+\dots+p_ld_l=n}}(\varepsilon_1^{-1}q^{-d_1})^{p_1}\dots(\varepsilon_l^{-1}q^{-d_l})^{p_l}.$$

However this is the coefficient of t^n in the power series expansion of the function

$$\prod_{i=1}^{l}\frac{1}{(1-\varepsilon_i^{-1}q^{-d_i}t^{d_i})}.$$

Thus the functions $\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det_V(1-tF^{-1}w)}$ and $\prod_{i=1}^{l}\frac{1}{(1-\varepsilon_i^{-1}q^{-d_i}t^{d_i})}$ have the same power series expansion. However both functions have no singularities in the region $|t|<q$. Since $q>1$ we put $t=1$ and obtain

$$\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det_V(1-F^{-1}w)}=\prod_{i=1}^{l}\frac{1}{(1-\varepsilon_i^{-1}q^{-d_i})}.$$

This completes the proof of the lemma.

We can now complete the proof of 3.4.1. The number of F-stable maximal tori of the connected semisimple group G is now known to be

$$\frac{|G^F|}{q^l}\prod_{i=1}^{l}\frac{1}{(1-\varepsilon_i^{-1}q^{-d_i})}.$$

We now recall from section 2.9 that

$$|G^F|=q^N\prod_{i=1}^{l}(q^{d_i}-\varepsilon_i).$$

Since $\prod_{i=1}^{l}(q^{d_i}-\varepsilon_i)$ is real and the ε_i are roots of unity we have

$$\prod_{i=1}^{l}(q^{d_i}-\varepsilon_i)=\prod_{i=1}^{l}(q^{d_i}-\varepsilon_i^{-1}).$$

It follows that the number of F-stable maximal tori is $q^{N+d_1+\dots+d_l-l}$. Moreover we have $d_1 + \dots + d_l = N + l$ by 2.4.1. Thus the number of F-stable maximal tori of G is q^{2N}.

3.5 CENTRALIZERS OF SEMISIMPLE ELEMENTS

Let G be a connected reductive group and s a semisimple element of G. We consider some properties of the centralizer $C_G(s)$.

Proposition 3.5.1. *If s is semisimple then s lies in $C_G(s)^0$.*

Proof. Every semisimple element s lies in a maximal torus T of G. Thus $T \subseteq C_G(s)$. Since T is connected $T \subseteq C_G(s)^0$. Hence $s \in C_G(s)^0$.

Proposition 3.5.2. *Let s be a semisimple element of G and T a maximal torus of G. Then $s \in T$ if and only if $T \subseteq C_G(s)^0$.*

Proof. We have seen that if $s \in T$ then $T \subseteq C_G(s)^0$. Suppose conversely that $T \subseteq C_G(s)^0$. Since $s \in C_G(s)^0$ s lies in some maximal torus of $C_G(s)^0$. However any two maximal tori of $C_G(s)^0$ are conjugate and so s lies in all maximal tori of $C_G(s)^0$. Hence $s \in T$.

Theorem 3.5.3. *Let s be semisimple and T be a maximal torus of G containing s. Then*

(i) $C_G(s)^0 = \langle T, X_\alpha, \alpha(s) = 1\rangle$.
(ii) $C_G(s) = \langle T, X_\alpha, \alpha(s) = 1; \dot{w}, s^w = s\rangle$.

where X_α are the root subgroups with respect to the torus T.

Proof. We first observe that T, the root subgroups X_α with $\alpha(s) = 1$, and the $\dot{w} \in N$ with $s^w = s$ all lie in $C_G(s)$.

Conversely let $g \in C_G(s)$. Then g has the form

$$g = ut\dot{w}u', \; u \in U, \; t \in T, \; w \in W, \; u' \in U_w$$

by 2.5.14. Thus

$$sg = sus^{-1}.st.\dot{w}.u'$$

$$gs = u.t\dot{w}s\dot{w}^{-1}.\dot{w}.s^{-1}u's$$

and $sus^{-1} \in U$, $st \in T$, $t\dot{w}s\dot{w}^{-1} \in T$, $s^{-1}u's \in U_w$. By the uniqueness of expression in canonical form we have

$$sus^{-1} = u, \; \dot{w}s\dot{w}^{-1} = s, \; s^{-1}u's = u'.$$

Now $\dot{w}s\dot{w}^{-1} = s$ implies that $s^w = s$. We consider the implications of $sus^{-1} = u$. We have $U = \prod_{\alpha\in\Phi^+} X_\alpha$ and each root subgroup X_α is isomorphic to the additive group $\mathbf{G}_a$. There is an isomorphism $\lambda \to x_\alpha(\lambda)$ from $\mathbf{G}_a$ into X_α such that T acts on X_α by

$$t x_\alpha(\lambda) t^{-1} = x_\alpha(\alpha(t)\lambda) \qquad t \in T.$$

Let $u = \prod_{\alpha \in \Phi^+} x_\alpha(\lambda_\alpha)$. Then we have $sus^{-1} = \prod_{\alpha \in \Phi^+} x_\alpha(\alpha(s)\lambda_\alpha)$. Since $sus^{-1} = u$ we have $\alpha(s) = 1$ whenever $\lambda_\alpha \neq 0$. Thus u lies in the subgroup generated by the X_α for which $\alpha(s) = 1$. Similarly, since $s^{-1}u's = u'$, u' lies in the subgroup generated by the X_α with $\alpha(s) = 1$. Thus $g = ut\dot{w}u'$ lies in the subgroup $\langle T; X_\alpha, \alpha(s) = 1; \dot{w}, s^{w} = s \rangle$.

Now the subgroup $\langle T; X_\alpha, \alpha(s) = 1 \rangle$ is connected, being generated by connected groups T, X_α. This subgroup is normal in $C_G(s)$ since it is normalized by all $\dot{w}$ with $s^w = s$. For if $\alpha(s) = 1$ and $\beta = w(\alpha)$ then $\beta(s) = 1$ also. This normal subgroup has finite index in $C_G(s)$, the index being at most $|W|$. This subgroup is therefore the connected component $C_G(s)^0$.

Theorem 3.5.4. *$C_G(s)^0$ is reductive. Its root system is $\Phi_1 = \{\alpha \in \Phi; \alpha(s) = 1\}$ and its Weyl group is $W_1 = \langle w_\alpha; \alpha \in \Phi_1 \rangle$.*

Proof. Let V be the unipotent radical of $C_G(s)^0$. Then VT is a connected solvable subgroup of G. Thus VT lies in a Borel subgroup B of G. We have $B = UT$ where U is the unipotent radical of B. Thus $V \subseteq U$. Now $U = \prod_{\alpha \in \Phi^+} X_\alpha$ where Φ^+ is the positive system of roots defined by the Borel subgroup B. Any subgroup of U normalized by T must have the form $\prod_{\alpha \in \Psi} X_\alpha$ for some subset Ψ of Φ^+. Since V is normalized by T we have $V = \prod_{\alpha \in \Psi} X_\alpha$ for some such Ψ. Since $V \subseteq C_G(s)^0$ each X_α for $\alpha \in \Psi$ is centralized by s. Thus

$$\Psi \subseteq \{\alpha \in \Phi^+; \alpha(s) = 1\}.$$

Suppose $\alpha \in \Phi^+$ satisfies $\alpha(s) = 1$. Then X_α and $X_{-\alpha}$ both lie in $C_G(s)^0$ so $n_\alpha \in C_G(s)^0$ where $n_\alpha \in \langle X_\alpha, X_{-\alpha} \rangle \cap N$ satisfies $\pi(n_\alpha) = w_\alpha$. Also ${}^{n_\alpha}X_\alpha = X_{-\alpha}$. Thus if $\alpha \in \Psi$ we shall have $X_\alpha \subseteq V$ and so $X_{-\alpha} \subseteq V$ since V is normal in $C_G(s)^0$. This is a contradiction, so Ψ is empty and $V = 1$. Thus $C_G(s)^0$ is reductive.

A Borel subgroup of $C_G(s)^0$ containing T will lie in a Borel subgroup of G so will have the form $(\prod_{\alpha \in \Omega} X_\alpha)T$ for some subset Ω of Φ. However the subgroups X_α which lie in $C_G(s)^0$ are just those for which $\alpha(s) = 1$. Thus the root system of $C_G(s)^0$ is $\Phi_1 = \{\alpha \in \Phi; \alpha(s) = 1\}$ and the Weyl group of $C_G(s)^0$ is $W_1 = \langle w_\alpha; \alpha \in \Phi_1 \rangle$.

Corollary 3.5.5. *Let $s \in G$ be semisimple. Let T be a maximal torus of G containing s and $B = UT$ be a Borel subgroup of G. Then $U \cap C_G(s)^0$ is a maximal unipotent subgroup of $C_G(s)^0$.*

We now come to a result of basic importance due to Steinberg.

Theorem 3.5.6. *Let G be a connected reductive group whose derived group G' is simply-connected. Let s be a semisimple element of G. Then $C_G(s)$ is connected.*

Proof. We first reduce to the case when G is semisimple. We have $G = G'Z^0$ and $s = s'z$ where $s' \in G'$ and $z \in Z^0$. Both s' and z are semisimple. We have

$$C_G(s) = C_G(s') = C_{G'}(s')Z^0.$$

If the result is true for semisimple groups $C_{G'}(s')$ will be connected. Since Z^0 is connected $C_G(s)$ will be connected also.

We therefore assume that G is semisimple. By 3.5.3 and 3.5.4 it will be sufficient to show that each element $w \in W$ such that $s^w = s$ is a product of reflections $w_\alpha \in W$ with $s^{w_\alpha} = s$. Since $s^{w_\alpha} = s$ if and only if $\alpha(s) = 1$ and since $\dot{w}_\alpha \in \langle X_\alpha, X_{-\alpha}\rangle$ this will show that all the generators of $C_G(s)$ in 3.5.3 lie in $C_G(s)^0$.

In order to prove that each $w \in W$ satisfying $s^w = s$ is a product of reflections each satisfying $s^{w_\alpha} = s$ it is convenient to translate the context from the W-action on an algebraic torus T to the W-action on a topological torus $\hat{T}$. This is done as follows. Let $X = \mathrm{Hom}(T, \mathbf{G}_m)$ be the character group of T and $\hat{T} = \mathrm{Hom}(X, S^1)$ be the set of topological group homomorphisms from X to the circle group S^1. $\hat{T}$ is a topological torus of the same dimension as the algebraic torus T. Let $\hat{Y} = \mathrm{Hom}(S^1, \hat{T})$ be the set of topological group homomorphisms from S^1 into $\hat{T}$. $\hat{Y}$ is the cocharacter group of $\hat{T}$. Since $\mathrm{Hom}(S^1, S^1) \cong \mathbb{Z}$ (again homomorphisms of topological groups) and $X \cong \mathrm{Hom}(\hat{T}, S^1)$ we have a map

$$X \times \hat{Y} \to \mathbb{Z}$$

which gives rise to an isomorphism $\hat{Y} \cong \mathrm{Hom}(X, \mathbb{Z})$. Since $Y = \mathrm{Hom}(\mathbf{G}_m, T)$ is also naturally isomorphic to $\mathrm{Hom}(X, \mathbb{Z})$ we obtain a canonical isomorphism $Y \to \hat{Y}$ between the cocharacter groups of T and $\hat{T}$.

We are given an element $s \in T$ which determines a cyclic subgroup $\langle s\rangle$ of T. Consider the annihilator $\langle s\rangle^\perp$ of this subgroup in X. The torsion subgroup C of $X/\langle s\rangle^\perp$ will be cyclic, being isomorphic to a factor group of $\langle s\rangle$. Now consider the annihilator of $\langle s\rangle^\perp$ in $\hat{T}$. This will be the product of a subtorus $\hat{S}$ of $\hat{T}$ with a finite cyclic group $\hat{C}$ isomorphic to C. Now a topological torus contains an element which lies in no proper closed subgroup. Taking the product of such an element in $\hat{S}$ with a generator of $\hat{C}$ we obtain an element $\hat{s} \in \hat{S} \times \hat{C}$ which lies in no proper closed subgroup of $\hat{S} \times \hat{C}$. Now let $w \in W$. We have $s^w = s$ if and only if w acts trivially on $X/\langle s\rangle^\perp$ and this holds if and only if w acts trivially on $\hat{S} \times \hat{C}$. This holds in turn if and only if $\hat{s}^w = \hat{s}$ since the set of elements of $\hat{T}$ fixed by w will be a closed subgroup. Thus we have

$$\{w \in W;\ s^w = s\} = \{w \in W;\ \hat{s}^w = \hat{s}\}.$$

We shall call this subgroup W_1.

We have now transformed our problem into the context of the Weyl group acting on a topological torus. We consider the universal covering space V of $\hat{T}$. V consists of the set of homotopy classes of paths in $\hat{T}$ beginning from 1 (Hilton and Wylie [1]). We have a covering map $p: V \to \hat{T}$ which takes each path to its end point. The kernel of p consists of the homotopy classes of closed paths in $\hat{T}$.

Since there is just one homomorphism $S^1 \to \hat{T}$ corresponding to each homotopy class of closed paths in $\hat{T}$ the kernel of p may be identified with the cocharacter group $\hat{Y} = \mathrm{Hom}(S^1, \hat{T})$.

Now W acts on Y and is generated by reflections with respect to the coroots Φ^v in Y. We identify Φ^v with the corresponding set of elements of $\hat{Y}$ under the canonical isomorphism $Y \to \hat{Y}$. Since G is semisimple and simply-connected the cocharacter group is generated by the coroots and so $\hat{Y} = \mathbb{Z}\Phi^v$. V is the Euclidean space $\hat{Y} \otimes \mathbb{R}$.

Now let $\hat{s} \in \hat{T}$ be as above. We choose $v \in V$ such that $p(v) = \hat{s}$ and the length $|v|$ is as small as possible. Let $w \in W_1$. Then $\hat{s}^w = \hat{s}$ and so

$$w(v) - v \in \ker p = \hat{Y} = \mathbb{Z}\Phi^v.$$

Thus there exist coroots $\alpha_i{}^v$ such that

$$w(v) - v = \sum_{i=1}^{k} \alpha_i{}^v.$$

We choose the coroots on the right-hand side so that the number of terms k is as small as possible. This clearly implies that $\langle \alpha_i{}^v, \alpha_j{}^v \rangle \geqslant 0$ for all i, j, for $\langle \alpha_i{}^v, \alpha_j{}^v \rangle < 0$ would imply that $\alpha_i{}^v + \alpha_j{}^v$ is a coroot and k could be decreased. We show in fact that all the coroots $\alpha_i{}^v$ in the sum are distinct, and they are mutually orthogonal. For

$$\begin{aligned} k|v|^2 &\leqslant \sum_{i=1}^{k} |v + \alpha_i{}^v|^2 \qquad \text{since } |v| \leqslant |v + \alpha_i{}^v| \\ &\leqslant \sum_{i=1}^{k} |v + \alpha_i{}^v|^2 + 2\sum_{i<j} \langle \alpha_i{}^v, \alpha_j{}^v \rangle \\ &= (k-1)|v|^2 + \Big|v + \sum_{i=1}^{k} \alpha_i{}^v\Big|^2 \\ &= (k-1)|v|^2 + |w(v)|^2 \\ &= k|v|^2. \end{aligned}$$

It follows that $|v + \alpha_i{}^v| = |v|$ for all i and that $\langle \alpha_i{}^v, \alpha_j{}^v \rangle = 0$ for all $i \neq j$. The former equation implies that $2\langle v, \alpha_i{}^v \rangle = -\langle \alpha_i{}^v, \alpha_i{}^v \rangle$, i.e. that $w_{\alpha_i}(v) = v + \alpha_i{}^v$. The latter equation implies that all the $\alpha_i{}^v$ are distinct and that $w_{\alpha_i}(\alpha_j{}^v) = \alpha_j{}^v$ if $i \neq j$. Thus we have

$$w_{\alpha_1} \dots w_{\alpha_k}(v) = v + \alpha_1{}^v + \dots + \alpha_k{}^v = w(v).$$

Hence $(w_{\alpha_1} \dots w_{\alpha_k})^{-1} w$ fixes v. As mentioned in 2.2.12 this implies that $(w_{\alpha_1} \dots w_{\alpha_k})^{-1} w$ is a product of reflections all fixing v. Thus

$$w = w_{\alpha_1} \dots w_{\alpha_k} w_{\beta_1} \dots w_{\beta_k}$$

where $w_{\beta_i}(v) = v$. Each reflection w_{α_i} and w_{β_i} fixes $\hat{s}$ and so the theorem is proved.

3.6 NONDEGENERATE MAXIMAL TORI

In section 3.3 we considered G^F-classes of F-stable maximal tori of G. We now consider conjugacy classes of maximal tori in G^F. We show that these are also in bijective correspondence with the F-conjugacy classes in W provided q is not too small. The reason for this condition on q can be seen by taking $G = GL_n(K)$, $G^F = GL_n(q)$, T to be the diagonal subgroup of G, and $q = 2$. Then $T^F = 1$. Thus T^F is, by definition, a maximal torus of G^F but is properly contained in other maximal tori of G^F. A similar situation will occur for split groups of any type when $q = 2$. We must therefore exercise caution in dealing with maximal tori of G^F when q is small.

Proposition 3.6.1. *Let G be connected reductive with Frobenius map F and let T be an F-stable maximal torus of G. Then the following conditions are equivalent:*

(i) *T is the only maximal torus of G containing T^F.*

(ii) $T = C_G(T^F)^0$.

(iii) *No root of G with respect to T satisfies $\alpha(t) = 1$ for all $t \in T^F$.*

Proof. Suppose T is the only maximal torus of G containing T^F. Then $T \subseteq C_G(T^F)^0$. If $T \neq C_G(T^F)^0$ then T is not normal in $C_G(T^F)^0$, for T is not normal in any connected subgroup of G larger than itself. Hence $C_G(T^F)^0$ would contain a conjugate of T which would also contain T^F. This cannot happen and so $T = C_G(T^F)^0$.

Conversely if $T = C_G(T^F)^0$ then T is the only maximal torus of G containing T^F. For any maximal torus of G containing T^F lies in $C_G(T^F)^0$. Thus conditions (i), (ii) are equivalent.

Let $t \in T$. Then by 3.5.3 we have

$$C_G(t)^0 = \langle T, X_\alpha, \alpha(t) = 1\rangle.$$

Exactly the same argument shows that for any subset S of T

$$C_G(S)^0 = \langle T, X_\alpha, \alpha(t) = 1 \text{ for all } t \in S\rangle.$$

We apply this with $S = T^F$. Thus

$$C_G(T^F)^0 = \langle T, X_\alpha, \alpha(t) = 1 \text{ for all } t \in T^F\rangle.$$

Thus $C_G(T^F)^0 = T$ if and only if no root α satisfies $\alpha(t) = 1$ for all $t \in T^F$. Hence conditions (ii), (iii) are equivalent. ∎

The maximal torus T^F of G^F will be called nondegenerate if the conditions of 3.6.1 are satisfied.

Proposition 3.6.2. *Let T_1, T_2 be F-stable maximal tori of G and suppose ${T_1}^F$, ${T_2}^F$ are nondegenerate. Then T_1, T_2 are G^F-conjugate maximal tori of G if and only if ${T_1}^F$, ${T_2}^F$ are conjugate subgroups of G^F.*

Proof. If T_1, T_2 are G^F-conjugate it is clear that ${T_1}^F$, ${T_2}^F$ will be conjugate subgroups of G^F. So suppose conversely that $T_2{}^F = (T_1{}^F)^g$ where $g \in G^F$. Then $T_2{}^F \subseteq T_1{}^g \cap T_2$. Since $T_2{}^F$ is nondegenerate we must have $T_1{}^g = T_2$. Thus T_1, T_2 are conjugate by an element of G^F.

Proposition 3.6.3. *If all the maximal tori of G^F are nondegenerate there is a bijective map between conjugacy classes of maximal tori in G^F and F-conjugacy classes in W.*

Proof. This follows from 3.3.3 and 3.6.2.

Proposition 3.6.4. *Let T be an F-stable maximal torus of G and $N = N_G(T)$. Suppose T^F is nondegenerate. Then $N^F = N_{G^F}(T^F)$.*

Proof. It is clear that T^F is normal in N^F and so $N^F \subseteq N_{G^F}(T^F)$. Suppose conversely that $g \in N_{G^F}(T^F)$. Then g also lies in the normalizer of $C_G(T^F)^0$. But $C_G(T^F)^0 = T$. Thus $g \in N \cap G^F = N^F$. ■

We note that the result of 3.6.4 fails if T is a split torus and $q = 2$, provided G is not itself a torus.

Corollary 3.6.5. *Let T be an F-stable maximal torus of G such that T^F is nondegenerate. Then $N_{G^F}(T^F)/T^F$ is isomorphic to W^F.*

Proof. This follows from 3.6.4 and the fact that N^F/T^F is isomorphic to W^F.

Proposition 3.6.6. *All the maximal tori of G^F are nondegenerate provided q is sufficiently large.*

Proof. Suppose G^F has a degenerate maximal torus T^F. Then there is a root $\alpha \in \Phi$ such that $\alpha(t) = 1$ for all $t \in T^F$. Let $S = T \cap G'$. Then S is a maximal torus of G' which is degenerate, since $\alpha(s) = 1$ for all $s \in S^F$.

Let X be the character group of S. We know from 3.2.3 that

$$(S^F)^\perp = \{\chi \in X;\ \chi(s) = 1 \quad \text{for all } s \in S^F\} = (F - 1)X.$$

Thus $\alpha \in (F - 1)X$.

Now $F = qF_0$ where $q > 1$ and F_0 has finite order. Moreover $X \otimes \mathbb{R}$ can be regarded as a Euclidean space on which the finite group $\langle W, F_0 \rangle$ acts as a group of isometries. We may choose the metric on $X \otimes \mathbb{R}$ such that $|\chi| \geqslant 1$ for all $\chi \in X$ with $\chi \neq 0$.

Let $\alpha = (F - 1)\chi = (qF_0 - 1)\chi$ with $\chi \in X$. Then we have

$$\begin{aligned} |\alpha| &= |(qF_0 - 1)\chi| \geqslant |qF_0\chi| - |\chi| \\ &= q|F_0\chi| - |\chi| = q|\chi| - |\chi| = (q - 1)|\chi| \geqslant q - 1 \end{aligned}$$

since $\chi \neq 0$.

Thus if there is a maximal torus T^F and a root α which satisfies $\alpha(t) = 1$ for all

$t \in T^F$ then $|\alpha| \geqslant q - 1$. Thus if we choose q so that $q > |\alpha| + 1$ for all roots $\alpha \in \Phi$ all the maximal tori of G^F will be nondegenerate. ■

We now consider when it can happen that an F-stable maximal torus T of G satisfies $T^F = 1$.

Proposition 3.6.7. *If $T^F = 1$ and $q \geqslant 2$ then we must have $q = 2$, $F_0 = 1$, and T is a split torus.*

Proof. We may assume without loss of generality that G is semisimple. Suppose T is obtained from a maximally split torus T_0 by twisting with w. Then by 3.3.5 we have

$$|T^F| = \det_{Y_0 \otimes \mathbf{R}} (q1 - F_0{}^{-1} \circ w)$$

where Y_0 is the cocharacter group of T_0. Now $F_0{}^{-1}w$ has finite order, so its eigenvalues $\lambda_1, \ldots, \lambda_l$ on $Y_0 \otimes \mathbb{C}$ are roots of unity. Thus the eigenvalues of $q1 - F_0{}^{-1}w$ are $q - \lambda_1, \ldots, q - \lambda_l$. Thus

$$|T^F| = (q - \lambda_1)(q - \lambda_2) \ldots (q - \lambda_l).$$

Now $q > 1$ and so $|q - \lambda_i| \geqslant q - 1$ with equality only if $\lambda_i = 1$. Thus

$$|T^F| = \prod_{i=1}^{l} |q - \lambda_i| \geqslant (q - 1)^l$$

with equality only if each $\lambda_i = 1$.

Now suppose $q > 2$. Then $(q - 1)^l > 1$ and so $|T^F| > 1$. Thus with the given assumptions $q \geqslant 2$ and $|T^F| = 1$ we must have $q = 2$. Moreover, since each $\lambda_i = 1$, we have $F_0{}^{-1}w = 1$ and so $w = F_0$. But F_0 transforms each simple root into a positive multiple of a simple root. Since w permutes the roots w must therefore permute the simple roots. But this implies $w = 1$. Thus T is maximally split. Since $F_0 = 1$ we have $F = q1$ on Y_0 and so T is actually split. ■

Note. If $q = \sqrt{2}$ or $\sqrt{3}$, as can happen in the smallest Suzuki and Ree groups, one can have $T^F = 1$ without T being maximally split.

We also include in this section the following proposition, which will be useful subsequently.

Proposition 3.6.8. *Let G be a connected reductive group and $F: G \to G$ be a Frobenius map. Then $Z(G^F) = Z(G)^F$.*

Proof. It is clear that $Z(G)^F \subseteq Z(G^F)$ and so we must show that $Z(G^F) \subseteq Z(G)$.

We shall first show that $|Z(G^F)|$ is prime to p. Suppose this is false. Then $Z(G^F)$ contains an element $u \neq 1$ of order a power of p. We may choose $u \in U^F$ since U^F is a Sylow p-subgroup of G^F. However U^F is conjugate to $(U^-)^F$ in G^F so $u \in (U^-)^F$ also. Thus $u \in U^F \cap (U^-)^F = 1$, and we have a contradiction.

We now know that $|Z(G^F)|$ is prime to p. Thus each element $s \in Z(G^F)$ is semisimple. Hence $C_G(s)^0$ is a connected reductive subgroup of G which is F-stable. Consider the group $(C_G(s)^0)^F$. We have $C_G(s)^F = G^F$ and so $(C_G(s)^0)^F$ is normal in G^F. Moreover

$$|G^F:(C_G(s)^0)^F| = |C_G(s)^F:(C_G(s)^0)^F|$$

which divides $|C_G(s):C_G(s)^0|$. But we know from section 1.14 that $|C_G(s):C_G(s)^0|$ is prime to p. Thus $|G^F:(C_G(s)^0)^F|$ is prime to p and so $U^F \subseteq (C_G(s)^0)^F$. Hence $|U^F|$ is the same for G as for $C_G(s)^0$. But $|U^F| = q^N$ where N is the number of positive roots. Thus G and $C_G(s)^0$ have the same number of roots, hence the same root system in view of 3.5.4. Hence $C_G(s)^0 = G$ by 3.5.3. Thus $s \in Z(G)$.

3.7 CLASSES OF SEMISIMPLE ELEMENTS

We now consider the conjugacy classes of semisimple elements in G and in G^F where G is a connected reductive group.

Proposition 3.7.1. *There is a bijection between semisimple conjugacy classes in the connected reductive group G and orbits T/W of the Weyl group on a maximal torus.*

Proof. Every semisimple element of G lies in a maximal torus, so has a conjugate which lies in the fixed torus T. Now let t_1, t_2 be elements of T which are conjugate in G. Let $t_2 = gt_1g^{-1}$ where $g = ut\dot{w}u'$ with $u \in U$, $t \in T$, $w \in W$, $u' \in U_w$. Then

$$ut\dot{w}u't_1 = t_2ut\dot{w}u'.$$

This can be written

$$u.t\dot{w}t_1\dot{w}^{-1}.\dot{w}.t_1{}^{-1}u't_1 = t_2ut_2{}^{-1}.t_2t.\dot{w}.u'.$$

By the uniqueness of expression of elements of G in canonical form we have in particular $t_2 = \dot{w}t_1\dot{w}^{-1}$. Thus t_1, t_2 are in the same orbit of W on T. It follows that we have a natural bijection between semisimple classes in G and orbits in T/W.

Corollary 3.7.2. *Let T be an F-stable maximal torus of G. Then there is a bijection between F-stable semisimple conjugacy classes of G and F-stable orbits in T/W.*

Proof. This follows from 3.7.1 since the F-stable semisimple classes are those which intersect T in an F-stable W-orbit.

Proposition 3.7.3. *Suppose G is a connected reductive group whose derived group G' is simply-connected. Then there is a bijection between semisimple conjugacy classes of G^F and F-stable orbits in T/W.*

Proof. We know from Steinberg's theorem that the centralizer of any semisimple element in G' is connected, since G' is simply-connected. We show that the same is true in G. Let $s \in G$ be semisimple. Then $s = s'z$ where $s' \in G'$ and $z \in Z^0$. Thus

$$C_G(s) = C_G(s'z) = C_G(s') = C_{G'}(s')Z^0.$$

Now s' is a semisimple element of G' so $C_{G'}(s')$ is connected. Thus $C_G(s)$ will be connected also.

We show now that every F-stable semisimple class of G contains an F-stable element. For let x lie in such a class. Then $F(x) = y^{-1}xy$ for some $y \in G$. By the Lang–Steinberg theorem $y = g^{-1}F(g)$ for some $g \in G$. Thus we have

$$F(x) = F(g)^{-1}gxg^{-1}F(g).$$

It follows that gxg^{-1} is F-stable.

We next show that any two F-stable elements of this class are conjugate in G^F. For let x, x' be F-stable with $x' = gxg^{-1}$. Then we have $x' = F(g)xF(g^{-1}) = gxg^{-1}$ and so $g^{-1}F(g) \in C_G(x)$. Since G' is simply-connected we know that $C_G(x)$ is connected. Thus we may apply the Lang–Steinberg theorem to $C_G(x)$ to see that there exists $y \in C_G(x)$ with $g^{-1}F(g) = y^{-1}F(y)$. Thus $gy^{-1} \in G^F$ and

$$x' = gy^{-1}x(gy^{-1})^{-1}.$$

Thus x, x' are conjugate in G^F.

We now have a bijection between the F-stable semisimple conjugacy classes of G and the semisimple conjugacy classes of G^F. The required result now follows from 3.7.2.

Proposition 3.7.4. *Let T be an F-stable maximal torus of the connected reductive group G. For each $w \in W$ let $T^{w^{-1}\circ F}$ be the set of $t \in T$ with $F(t)^{w^{-1}} = t$. Then*

$$\left|\left(\frac{T}{W}\right)^F\right| = \frac{1}{|W|}\sum_{w\in W} |T^{w^{-1}\circ F}|.$$

Proof. Let $A = \{t \in T; F(t) = t^w \text{ for some } w \in W\}$. Thus $A = \bigcup_w T^{w^{-1}\circ F}$. W acts on A and $|(T/W)^F|$ is the number of W-orbits on A. For each $t \in A$ the number of W-conjugates of t is $|W: W_t|$ where $W_t = \{w \in W; t^w = t\}$. Thus the number of W-orbits on A is

$$\sum_{t\in A}\frac{|W_t|}{|W|} = \frac{1}{|W|}\sum_{t\in A}\sum_{\substack{w\in W\\ t^w = t}} 1 = \frac{1}{|W|}\sum_{t\in A}\sum_{\substack{w\in W\\ t^w = {}_tF(t)}} 1$$

$$= \frac{1}{|W|}\sum_{w\in W}\sum_{\substack{t\in A\\ t^w = F(t)}} = \frac{1}{|W|}\sum_{w\in W} |T^{w^{-1}\circ F}|.$$

Now let T_0 be an F-stable maximal torus of G which is maximally split and T be an F-stable maximal torus obtained from T_0 by twisting with $w \in W$. Then F

acts on T as $w^{-1}\circ F$ acts on T_0. Thus

$$|T^F| = |T_0{}^{w^{-1}\circ F}|$$

and by 3.3.8 we have

$$|T^F| = |(Z^0)|\,\det_{Y(S_0)\otimes}\mathbb{R}(q1 - F_0{}^{-1}\circ w)$$

where $S_0 = T_0 \cap G'$

$$= q^l|(Z^0)^F|\,\det_{Y(S_0)\otimes\mathbb{R}}(1 - F^{-1}\circ w).$$

We now need a lemma.

Lemma 3.7.5. *Let* $V = Y(S_0)\otimes\mathbb{R}$. *Then*

$$\frac{1}{|W|}\sum_{w\in W}\det_V(1-\theta w) = 1$$

for any linear map $\theta: V\to V$.

Proof. We consider the exterior powers $\Lambda^i V$ of V for $i = 1,\ldots,l$. Let $e_1,\ldots,e_l$ be a basis for V. Then $e_1\wedge\ldots\wedge e_l$ is a basis for the 1-dimensional space $\Lambda^l V$ and

$$(1-\theta w)(e_1\wedge\ldots\wedge e_l) = \det(1-\theta w)(e_1\wedge\ldots\wedge e_l).$$

Hence

$$\det(1-\theta w)e_1\wedge\ldots\wedge e_l = (1-\theta w)e_1\wedge\ldots\wedge(1-\theta w)e_l$$

$$= \sum_{S\subseteq\{1,\ldots,l\}}(-1)^{|S|}v_1\wedge\ldots\wedge v_l$$

where

$$v_i = \begin{cases}\theta w e_i & \text{if } i\in S\\ e_i & \text{if } i\notin S\end{cases}$$

$$= \sum_{S\subseteq\{1,\ldots,l\}}\pm\left(\left(\bigwedge_{i\notin S}e_i\right)\wedge\theta w\left(\bigwedge_{i\in S}e_i\right)\right).$$

Thus

$$\frac{1}{|W|}\sum_{w\in W}\det_V(1-\theta w)(e_1\wedge\ldots\wedge e_l)$$

$$= \sum_{S\subseteq\{1,\ldots,l\}}\pm\left(\left(\bigwedge_{i\notin S}e_i\right)\wedge\theta\frac{1}{|W|}\sum_{w\in W}\left(\bigwedge_{i\in S}e_i\right)\right).$$

We now use a result about the action of a Coxeter group W on the exterior powers $\Lambda^i V$ of its natural representation. It was proved by Steinberg ([16], p. 90) that if $i>0$ the only element of $\Lambda^i V$ fixed by all $w\in W$ is zero. However $\frac{1}{|W|}\sum_{w\in W}w(\wedge_{i\in S}e_i)$ is fixed by all $w\in W$. Hence this element is zero. The only nonzero summand of the above expression therefore comes from the term with $S = \varnothing$. Hence we have

$$\frac{1}{|W|} \sum_{w \in W} \det_V(1 - \theta w)(e_1 \wedge \ldots \wedge e_l) = e_1 \wedge \ldots \wedge e_l.$$

Thus $\frac{1}{|W|} \sum_{w \in W} \det_V(1 - \theta w) = 1$ as required.

We can now prove the following theorem.

Theorem 3.7.6. (i) *The number of F-stable semisimple classes of the connected reductive group G is $|(Z^0)^F|q^l$ where l is the semisimple rank of G.*

(ii) *If the derived group G' is simply-connected then the number of semisimple conjugacy classes of G^F is $|(Z^0)^F|q^l$.*

Proof. By 3.7.2 the number of F-stable semisimple classes of G is $|(T_0/W)^F|$. By 3.7.4 this is equal to

$$\frac{1}{|W|} \sum_{w \in W} |T_0{}^{w^{-1} \circ F}|$$

and we have seen that

$$|T_0{}^{w^{-1} F}| = q^l |(Z^0)^F| \det_{Y(S_0) \otimes \mathbb{R}}(1 - F^{-1} \circ w).$$

Thus the required number is

$$\left(\frac{1}{|W|} \sum_{w \in W} \det_{Y(S_0) \otimes \mathbb{R}}(1 - F^{-1} \circ w) \right) q^l |(Z^0)^F|.$$

We now apply 3.7.5 with $\theta = F^{-1}$. This shows that the required number is $q^l|(Z^0)^F|$.

If G' is simply-connected 3.7.3 shows that the number of semisimple classes in G^F is also equal to $q^l|(Z^0)^F|$.

3.8 THE BRAUER COMPLEX

We assume in this section that G is simple and simply-connected. The group G^F then has q^l semisimple conjugacy classes, by 3.7.6. We show that the semisimple classes of G^F are in a bijective correspondence with the simplices of maximum dimension in a certain simplicial complex called the Brauer complex. This complex was originally studied by Humphreys ([7], p. 65) in connection with the modular representation theory of the group G^F.

Let T be a maximally split F-stable maximal torus of G and Y its cocharacter group. We recall from 3.1.2 and 3.1.3 that there is an isomorphism between T and $Y \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ and F-actions on both groups which are compatible with this isomorphism. Now we have a natural homomorphism $Y \otimes \mathbb{Q}_{p'} \to Y \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ with kernel Y. Thus T is isomorphic to $Y \otimes \mathbb{Q}_{p'}/Y$. We also have W-actions on T and on $Y \otimes \mathbb{Q}_{p'}/Y$ which are compatible with this isomorphism. Thus we have a bijection between the set of orbits T/W and the set of orbits $(Y \otimes \mathbb{Q}_{p'}/Y)W$. The latter is in natural bijection with $Y \otimes \mathbb{Q}_{p'}/\langle Y, W\rangle$. We also have F-actions on both T/W and $Y \otimes \mathbb{Q}_{p'}/\langle Y, W\rangle$ which are compatible with the above bijection between them.

We therefore consider the action of the group $\langle Y, W\rangle$ on the vector space $Y \otimes \mathbb{R}$. The elements of Y act by translations $v \to v + y$, $y \in Y$, and the elements of W by orthogonal transformations. Since G is simply-connected we know from section 1.11 that Y is generated by the coroots of G. Under these circumstances $\langle Y, W\rangle$ is a Coxeter group called the affine Weyl group. We recall from Bourbaki [2], p. 173, the following facts about the affine Weyl group. Since G is simple G has a unique root α_0 of maximal height. Then the affine Weyl group $\langle Y, W\rangle$ has a fundamental region in $Y \otimes \mathbb{R}$ given by

$$\bar{A} = \{v \in Y \otimes \mathbb{R};\ \langle \alpha_i, v\rangle \geqslant 0 \text{ for all } \alpha_i \in \Delta,\ \langle \alpha_0, v\rangle \leqslant 1\}.$$

Thus each element of $Y \otimes \mathbb{R}$ is equivalent under $\langle Y, W\rangle$ to just one element of $\bar{A}$. Note here that since $\langle \chi, y\rangle \in \mathbb{Z}$ for all $\chi \in X$, $y \in Y$ the above expressions $\langle \alpha_i, v\rangle$ and $\langle \alpha_0, v\rangle$ are uniquely defined elements of $\mathbb{R}$.

Let A be the interior of $\bar{A}$. Then $\bar{A}$ is the closure of A. The subsets of $Y \otimes \mathbb{R}$ of the form $\omega(A)$ for $\omega \in \langle Y, W\rangle$ are called alcoves. Each alcove has the form $\omega(A)$ for a unique $\omega \in \langle Y, W\rangle$ and $\langle Y, W\rangle$ is generated as a Coxeter group by the reflections in the $l + 1$ walls of A.

Let $\bar{A}_{p'} = \bar{A} \cap (Y \otimes \mathbb{Q}_{p'})$. We now define an F-action on $\bar{A}_{p'}$.

Proposition 3.8.1. (i) *We may define an F-action on $\bar{A}_{p'}$ by taking the image of a under F, $a \in \bar{A}_{p'}$, to be the unique element of $\bar{A}$ equivalent to $F(a)$ under $\langle Y, W\rangle$.*

(ii) *There are exactly q^l elements of $\bar{A}_{p'}$ which are stable under this F-action.*

Proof. Given $a \in \bar{A}_{p'}$ we have $a \in Y \otimes \mathbb{Q}_{p'}$. Hence $F(a) \in Y \otimes \mathbb{Q}_{p'}$. There is a unique element of $\bar{A}$ equivalent to $F(a)$ under $\langle Y, W\rangle$. We must show that this element lies in $Y \otimes \mathbb{Q}_{p'}$ also. However this is clear because elements of the affine Weyl group $\langle Y, W\rangle$ transform Y into itself and therefore $Y \otimes \mathbb{Q}_{p'}$ into itself also.

Now the elements of $\bar{A}_{p'}$ which are fixed under this action of F are in bijective correspondence with the orbits in $(Y \otimes \mathbb{Q}_{p'})/\langle Y, W\rangle$ which are F-stable. These in turn are in bijective correspondence with the F-stable orbits in T/W. However

$$\left|\left(\frac{T}{W}\right)^F\right| = q^l$$

by 3.7.6. Thus $\bar{A}_{p'}$ has q^l F-stable elements. ∎

We now consider the action of the group $\langle F^{-1}(Y), W\rangle$ on $Y \otimes \mathbb{R}$ where $F^{-1}(Y) = \{y \in Y \otimes \mathbb{R}; F(y) \in Y\}$. $F^{-1}(Y)$ acts on $Y \otimes \mathbb{R}$ by translations $v \to v + y$ for $y \in F^{-1}(Y)$ and this group of translations is normalized by W. For we have $F(y^w) = F(y)^{F(w)}$ for $y \in Y \otimes \mathbb{R}$, $w \in W$ and so if $F(y) \in Y$ it follows that $F(y^w) \in Y$ also. Thus $y^w \in F^{-1}(Y)$. Each element of $\langle F^{-1}(Y), W\rangle$ is uniquely expressible in the form yw where $y \in F^{-1}(Y)$ and $w \in W$.

Consider the map $\langle F^{-1}(Y), W\rangle \xrightarrow{F} \langle Y, W\rangle$ given by $yw \to F(y)F(w)$. This map is an isomorphism of groups. We compare the actions of these two groups on $Y \otimes \mathbb{R}$. For all $\omega \in \langle F^{-1}(Y), W\rangle$ and all $y \in Y \otimes \mathbb{R}$ it is readily seen that

$$F(\omega)(F(y)) = F(\omega(y)).$$

Thus the isomorphism of groups $\langle F^{-1}(Y), W\rangle \xrightarrow{F} \langle Y, W\rangle$ together with the bijection $Y \otimes \mathbb{R} \xrightarrow{F} Y \otimes \mathbb{R}$ gives an isomorphism of permutation groups between $\langle F^{-1}(Y), W\rangle$ acting on $Y \otimes \mathbb{R}$ and $\langle Y, W\rangle$ acting on $Y \otimes \mathbb{R}$. In particular the set $\bar{A}_1$ given by

$$\bar{A}_1 = \{y \in Y \times \mathbb{R};\ F(y) \in \bar{A}\}$$

is a fundamental region for the action of $\langle F^{-1}(Y), W\rangle$ on $Y \otimes \mathbb{R}$. The volumes of the regions $\bar{A}$, $\bar{A}_1$ are related by

$$\operatorname{vol} \bar{A}_1 = \frac{1}{q^l} \operatorname{vol} \bar{A}$$

since $F = qF_0$ where $q > 1$ and F_0 has finite order.

Now $\langle Y, W\rangle$ is a subgroup of $\langle F^{-1}(Y), W\rangle$. Thus every affine reflecting hyperplane for the affine Weyl group $\langle Y, W\rangle$ will be a reflecting hyperplane for $\langle F^{-1}(Y), W\rangle$, but not conversely. In particular the walls of $\bar{A}$ are all reflecting hyperplanes for $\langle F^{-1}(Y), W\rangle$. Thus $\bar{A}$ is the union of certain transforms of $\bar{A}_1$ under the action of $\langle F^{-1}(Y), W\rangle$. Since each transform has the same volume the number of such transforms will be q^l.

Now $\bar{A}_1$ is a closed simplex whose open faces form a simplicial complex. We consider all transforms of open faces of $\bar{A}_1$ under elements of $\langle F^{-1}(Y), W\rangle$ which lie in $\bar{A}$. These form a simplicial complex called the Brauer complex. The elements of the Brauer complex are thus simplices for the form $\omega(B)$ where $\omega \in \langle F^{-1}(Y), W\rangle$, B is an open face of $\bar{A}_1$, and $\omega(B) \subseteq \bar{A}$. Thus $\bar{A}$ is the disjoint union of the faces of the Brauer complex. We note in particular that the Brauer complex has just q^l simplices of maximum dimension l. These are the subsets of $\bar{A}$ of the form $\omega(A_1)$ where A_1 is the interior of $\bar{A}_1$.

We shall now consider the positions of the q^l F-stable elements of $\bar{A}_{p'}$, under the F-action defined in 3.8.1, in connection with the decomposition of $\bar{A}$ into the disjoint union of the faces of the Brauer complex.

Theorem 3.8.2. (Deriziotis) *Let G be simple and simply connected and $F: G \to G$ be a Frobenius map.*

(i) *Let B_1 be any face of the Brauer complex of maximum dimension l. Then its closure $\bar{B}_1$ contains a unique F-stable point under the F-action defined in* 3.8.1. *(This action can be defined on $\bar{A}$, not just on $\bar{A}_{p'}$). Moreover this F-stable point lies in $\bar{A}_{p'}$.*

(ii) *If B_1' is also a face of dimension l and $\overline{B_1}$, $\overline{B_1'}$ contain the same F-stable point then $B_1 = B_1'$.*

We shall therefore obtain a bijection between the q^l F-stable points in $\bar{A}_{p'}$ and the q^l simplices of maximum dimension in the Brauer complex.

Proof. $F(B_1)$ is an alcove for the affine Weyl group $\langle Y, W\rangle$. Thus there is a unique element $\omega \in \langle Y, W\rangle$ such that $F(B_1) = \omega(A)$. We therefore have $F(\bar{B}_1) = \omega(\bar{A})$ and so $F^{-1}\omega(\bar{A}) = \bar{B}_1$.

Consider the map $F^{-1}\omega:\bar{A} \to \bar{A}$. We have $F = qF_0$ where $q > 1$ and F_0 has finite order. $Y \otimes \mathbb{R}$ may be regarded as a metric space on which ω and F_0 act as isometries. With respect to this metric we have

$$d(F^{-1}\omega x, F^{-1}\omega y) = \frac{1}{q}\, d(x, y).$$

Since $q > 1$ we may apply the contraction mapping theorem to show that $F^{-1}\omega$ has a unique fixed point $y \in \bar{A}$.

This fixed point y certainly lies in $\bar{B}_1$ since $F^{-1}\omega(\bar{A}) = \bar{B}_1$. It satisfies

$$y \equiv F(y) \bmod \langle Y, W\rangle$$

and so is an F-stable point under the F-action defined in 3.8.1.

Conversely, suppose $y \in \bar{B}_1$ satisfies

$$y \equiv F(y) \bmod \langle Y, W\rangle.$$

Then $F(y)$, $\omega(y)$ both lie in $\omega(\bar{A})$ and we have

$$\omega(y) \equiv F(y) \bmod \langle Y, W\rangle.$$

It follows that $\omega(y) = F(y)$ and so y is fixed under $F^{-1}\omega$.

Thus we have shown that $\bar{B}_1$ contains a unique F-stable point y of $\bar{A}$. Since $\bar{A}$ is the union of q^l closed simplices of the form $\bar{B}_1$ we see that there are at most q^l F-stable points y in $\bar{A}$. However we know by 3.8.1 that the subset $\bar{A}_{p'}$ contains exactly q^l F-stable points. Thus all the F-stable points in $\bar{A}$ must lie in $\bar{A}_{p'}$. Moreover if B_1, B_1' are simplices such that $\bar{B}_1$, $\bar{B}_1'$ contain the same F-stable point then we must have $B_1 = B_1'$.

Corollary 3.8.3. *There is a bijection between the semisimple classes of G^F and the simplices of maximum dimension in the Brauer complex.*

Proof. Under the given assumptions on G the semisimple classes of G^F are in bijective correspondence with $(T/W)^F$. Moreover $(T/W)^F$ is in bijective correspondence with $((Y \otimes \mathbb{Q}_{p'})/\langle Y, W\rangle)^F$. The elements in this latter set are in bijective correspondence with F-stable elements of the fundamental region $\bar{A}$. Each of these lies in the closure of just one simplex of maximum dimension in the Brauer complex. ■

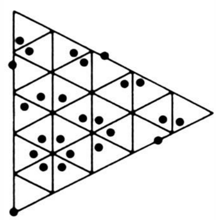
Note One can investigate further the properties of the semisimple elements of G^F in a given class with reference to the position of the corresponding F-stable point in its simplex in the Brauer complex. For example the semisimple elements in the class are regular if and only if the F-stable point lies in the interior of its simplex.

One could also use the arguments of this section to give an alternative proof of the fact that G^F has q^l semisimple classes.

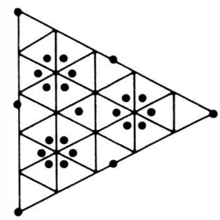
For further details see Deriziotis [1].

We show in the figures the Brauer complexes of the groups $A_2(5)$, ${}^2A_2(5^2)$, $B_2(5)$, ${}^2B_2(2^5)$. The F-stable point in the closure of each alcove is shown in the figures.

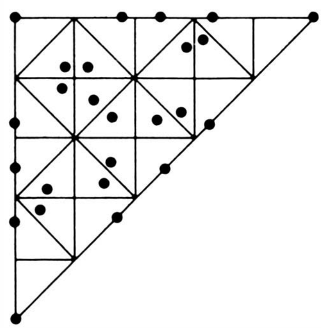
$A_2(5)$



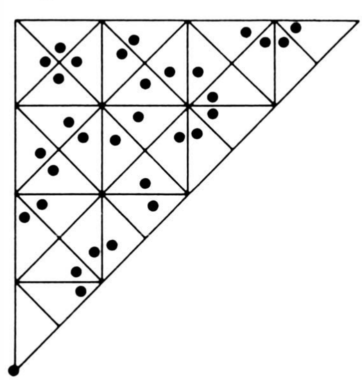
${}^2A_2(5^2)$



$B_2(5)$



${}^2B_2(2^5)$



Chapter 4

GEOMETRIC CONJUGACY AND DUALITY

4.1 GEOMETRIC CONJUGACY

Let G be a connected reductive group with Frobenius map F. In this chapter we shall consider complex characters of a maximal torus T^F of G^F, viz. elements of $\mathrm{Hom}(T^F, \mathbb{C}^*) = \hat{T}^F$. In particular we wish to compare $\hat{T}^F$ with $({}^g\hat{T})^F$ when $T, {}^gT$ are both F-stable. Since T^F and $({}^gT)^F$ will not in general be conjugate in G^F we cannot compare $\hat{T}^F$ and $({}^g\hat{T})^F$ by conjugation within G^F. There are, however, two methods which do enable such a comparison to be made.

The first is to recall from 3.2.2 that T^F is isomorphic to $Y/(F-1)Y$. Since

$$\hat{T}^F = \mathrm{Hom}(T^F, \Omega_{p'}) \cong \mathrm{Hom}(T^F, \mathbb{Q}_{p'}/\mathbb{Z})$$

each element of $\hat{T}^F$ determines an element of $\mathrm{Hom}(Y, \mathbb{Q}_{p'}/\mathbb{Z})$ with $(F-1)Y$ in the kernel. Similarly each element of $({}^g\hat{T})^F$ determines an element of $\mathrm{Hom}(Y({}^gT), \mathbb{Q}_{p'}/\mathbb{Z})$. These characters of $Y(T)$ and $Y({}^gT)$ may then be compared by conjugation within G.

The second approach is to observe that each character $\theta \in \hat{T}^F$ determines a character in $\hat{T}^{F^n}$ for any positive integer n. For there is a homomorphism

$$N: T^{F^n} \to T^F$$

called the norm map, given by

$$t \to t.F(t).F^2(t) \ldots F^{n-1}(t).$$

Thus each $\theta \in \hat{T}^F$ determines $\theta \circ N \in \hat{T}^{F^n}$. However we shall see that T^{F^n} and $({}^gT)^{F^n}$ are conjugate in G^{F^n} for some n. It will then be possible to compare characters of T^{F^n} and $({}^gT)^{F^n}$ by conjugation within G^{F^n}.

Proposition 4.1.1. *Let T be an F-stable maximal torus of G. Then T is F^n-stable for any $n > 0$, and T is maximally split with respect to F^n for some n.*

Proof. We must show that T lies in an F^n-stable Borel subgroup for some n.

Let B be a Borel subgroup containing T. Then $F(B)$ is also a Borel subgroup containing T. Thus F permutes the Borel subgroups of G containing T. However there are only finitely many Borel subgroups of G containing T, by section 1.9. Thus some power of F will fix them all. So for some n there will be an F^n-stable Borel subgroup containing T.

Proposition 4.1.2. *Let T be an F-stable maximal torus of G and $N : T^{F^n} \to T^F$ be the norm map. Let $\theta \in \hat{T}^F$. Then θ and $\theta \circ N$ give rise to the same element of* $\mathrm{Hom}(Y, \mathbb{Q}_{p'}/\mathbb{Z})$.

Proof. We know from 3.2.2 that T^F is isomorphic to $Y/(F-1)Y$ and similarly T^{F^n} is isomorphic to $Y/(F^n-1)Y$. The diagram which shows this is

$$\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & T^{F^n} & & \\
 & & & & & & \downarrow & & \\
0 & \to & Y & \to & Y \otimes \mathbb{Q}_{p'} & \to & Y \otimes \mathbb{Q}_{p'}/\mathbb{Z} & \to & 0 \\
 & & \downarrow{\scriptstyle F^n-1} & & \downarrow{\scriptstyle F^n-1} & & \downarrow{\scriptstyle F^n-1} & & \\
0 & \to & Y & \to & Y \otimes \mathbb{Q}_{p'} & \to & Y \otimes \mathbb{Q}_{p'}/\mathbb{Z} & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & Y/(F^n-1)Y & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

The map $F^n - 1$ in this diagram may be factorized as $F^n - 1 = (F-1)\circ(F^{n-1} + F^{n-2} + \dots + F + 1)$ and this factorization gives rise to homomorphisms

$$T^{F^n} \xrightarrow{N} T^F \quad \text{and} \quad Y/(F^n-1)Y \xrightarrow{\phi} Y/(F-1)Y$$

where N is the norm map and ϕ is the natural homomorphism. Moreover these homomorphisms have the property that the diagram

$$\begin{array}{ccc}
Y/(F^n-1)Y & \to & T^{F^n} \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle N} \\
Y/(F-1)Y & \to & T^F
\end{array}$$

is commutative, where the horizontal arrows represent the isomorphisms given by the snake lemma. It follows that if $\theta \in \hat{T}^F$ then $\theta \circ N \in \hat{T}^{F^n}$ and $\theta, \theta \circ N$ give rise to the same character of Y, i.e. to the same element of $\mathrm{Hom}(Y, \mathbb{Q}_{p'}/\mathbb{Z})$. ■

We can now prove the equivalence of our two methods for comparing characters of T^F and T'^F for distinct F-stable maximal tori of G.

Proposition 4.1.3. *Let T, T' be F-stable maximal tori of G. Let $\theta \in \hat{T}^F$ and $\theta' \in \hat{T}'^F$. Then the following two conditions are equivalent:*

(i) *There is an element $g \in G$ which transforms T to T' and θ, regarded as a character of $Y(T)$, to θ', regarded as a character of $Y(T')$.*

(ii) *For some $n > 0$ there is an element $g \in G^{F^n}$ which transforms T to T' and $\theta \circ N$, a character of T^{F^n}, to $\theta' \circ N$, a character of T'^{F^n}.*

Proof. Suppose (ii) is satisfied. The given $g \in G^{F^n}$ then satisfies (i). For by 4.1.2 θ and $\theta \circ N$ are the same, as characters of $Y(T)$, and θ' and $\theta' \circ N$ are the same, as characters of $Y(T')$.

Conversely, suppose (i) is satisfied. By 4.1.1 T and T' are F^n-stable for any $n > 0$ and there exists an n for which T and T' are maximally split with respect to F^n. We can choose this n to satisfy also the condition that F^n acts trivially on the Weyl group W. Now T and T', being maximally split with respect to F^n, are conjugate by an element of G^{F^n}. It is therefore sufficient to prove (ii) in the case when $T = T'$. The given element g then lies in $N = N_G(T)$. Since F^n acts trivially on W we have

$$\frac{N}{T} = W = W^{F^n} \cong \frac{N^{F^n}}{T^{F^n}}.$$

Thus each element of N lies in the same coset with respect to T as some element of N^{F^n}. Let $Tg = Tg'$ where $g' \in N^{F^n}$. Then g' transforms θ to θ', as characters of $Y(T)$. Thus g' transforms $\theta \circ N$ to $\theta' \circ N$, as characters of $Y(T)$, by 4.1.2. Thus g' is an element of G^{F^n} which transforms the character $\theta \circ N$ of T^{F^n} to the character $\theta' \circ N$. ∎

Let T, T' be F-stable maximal tori of G and $\theta \in \hat{T}^F$, $\theta' \in \hat{T}'^F$. We say that the pairs (T, θ) (T', θ') are geometrically conjugate if the conditions of 4.1.3 are satisfied. Geometric conjugacy is thus an equivalence relation on such pairs (T, θ). This equivalence relation was first introduced in Deligne and Lusztig [1].

We now wish to establish a bijection between the set of geometric conjugacy classes of pairs (T, θ) and another set given in a more direct way. We recall from 3.1.1 that $\mathrm{Hom}(Y, \mathbb{Q}_{p'}/\mathbb{Z})$ is isomorphic to $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$. Each pair (T, θ) where T is F-stable and $\theta \in T^F$ gives rise to an element of $\mathrm{Hom}(Y, \mathbb{Q}_{p'}/\mathbb{Z})$ and hence to an element of $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$.

Proposition 4.1.4. *Let T be an F-stable maximal torus of G and X the character group of T. Then there is a bijective correspondence between geometric conjugacy classes of pairs (T', θ') where T' is F-stable and $\theta' \in \hat{T}'^F$ and F-stable W-orbits on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$.*

Proof. We begin with a pair (T', θ') with $\theta' \in \hat{T}'^F$. T' is conjugate to T, so let $T' = {}^gT$. We define a conjugation map $X \to X' = X(T')$ by $\chi \to {}^g\chi$ where ${}^g\chi({}^gt) = \chi(t)$ for $\chi \in X$, $t \in T$. This conjugation map extends to an isomorphism $X \otimes \mathbb{Q}_{p'}/\mathbb{Z} \to X' \otimes \mathbb{Q}_{p'}/\mathbb{Z}$. θ' gives rise to an element of $X' \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ which corresponds under the conjugation map to an element of $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$. We consider to what extent this element of $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ is uniquely determined, as it appears to depend upon the choice of conjugating element g. Suppose $T' = {}^gT = {}^{g'}T$. Then $g^{-1}g' \in N$ and so $g' = g\dot{w}$ for some $w \in W$. Suppose θ' gives the element $\sum_i \psi_i \otimes \lambda_i \in X' \otimes \mathbb{Q}_{p'}/\mathbb{Z}$. This maps to $\sum_i {}^{g^{-1}}\psi_i \otimes \lambda_i$ or to

$\sum_i {}^{g'^{-1}}\psi_i \otimes \lambda_i$ in $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$. However we have

$$\sum_i {}^{g'^{-1}}\psi_i \otimes \lambda_i = \sum_i {}^{\dot{w}^{-1}g^{-1}}\psi_i \otimes \lambda_i = {}^{w^{-1}}\left(\sum {}^{g^{-1}}\psi_i \otimes \lambda_i\right).$$

Thus the W-orbit on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ is uniquely determined by θ'.

Now F acts on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ and on W, and so on the set of W-orbits on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$. We have

$$F\left(\sum_i \psi_i \otimes \lambda_i\right) = \sum_i (F(\psi_i) \otimes \lambda_i).$$

We shall show that the W-orbit on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ determined by (T', θ') is F-stable.

Now θ' is an F-stable character of Y'. For $(F - 1)Y'$ lies in the kernel of θ' and so for all $\gamma \in Y'$ we have $\theta'((F - 1)\gamma) = 1$. Thus

$$F(\theta')\theta'^{-1}(\gamma) = 1 \quad \text{for all } \gamma \in Y'.$$

Thus $F(\theta') = \theta'$ and θ' is F-stable. θ' therefore gives rise to an F-stable element of $X' \otimes \mathbb{Q}_{p'}/\mathbb{Z}$. As F acts on $X' \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ in the same way that $F \circ w^{-1}$ acts on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ for some $w \in W$, this element in turn gives rise to an F-stable W-orbit on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ under the conjugation maps. Thus (T', θ') determines an F-stable W-orbit on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$.

If (T'', θ'') is a pair geometrically conjugate to (T', θ') then, by property 4.1.3 (i), the W-orbits on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ determined by (T', θ') and (T'', θ'') will be the same. Thus each geometric conjugacy class gives rise to an F-stable W-orbit on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$.

Conversely, suppose we are given an F-stable W-orbit on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$. Suppose it contains the element $\sum_i \psi_i \otimes \lambda_i$. Then

$$F\left(\sum_i \psi_i \otimes \lambda_i\right) = {}^{w^{-1}}\left(\sum_i \psi_i \otimes \lambda_i\right)$$

for some $w \in W$. By the Lang–Steinberg theorem there exists $g \in G$ with $g^{-1}F(g) = \dot{w}$. Let $T' = {}^gT$ and $X' = X(T')$. Consider the element $\sum_i {}^g\psi_i \otimes \lambda_i \in X' \otimes \mathbb{Q}_{p'}/\mathbb{Z}$. Then we have

$$\begin{aligned} F\left(\sum_i {}^g\psi_i \otimes \lambda_i\right) &= \sum_i {}^{F(g)}F(\psi_i) \otimes \lambda_i = \sum_i {}^{g\dot{w}}F(\psi_i) \otimes \lambda_i \\ &= {}^{g\dot{w}}F\left(\sum_i \psi_i \otimes \lambda_i\right) = \sum_i {}^g\psi_i \otimes \lambda_i. \end{aligned}$$

Thus $\sum_i {}^g\psi_i \otimes \lambda_i$ is an F-stable element of $X' \otimes \mathbb{Q}_{p'}/\mathbb{Z}$. It corresponds to an F-stable element $\theta' \in \mathrm{Hom}(Y', \mathbb{Q}_{p'}/\mathbb{Z})$. We then have

$$\theta'((F - 1)\gamma) = \theta'(F(\gamma)\gamma^{-1}) = F(\theta')\theta'^{-1}(\gamma) = 1$$

for all $\gamma \in Y'$. Thus $(F - 1)Y'$ lies in the kernel of θ' and θ' can be regarded as a character of $T'^F \cong Y'/(F - 1)Y'$. Thus (T', θ') is a pair giving rise to the given F-

stable W-orbit on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$. It is easy to see that (T', θ') is determined up to geometric conjugacy, using property 4.1.3(i).

We have therefore established a bijection between the set of geometric conjugacy classes and the set

$$((X \otimes \mathbb{Q}_{p'}/\mathbb{Z})/W)^F. \qquad \blacksquare$$

We note here a similarity to the result proved in 3.7.2. We there established a bijection between the set of F-stable semisimple classes of G and the set

$$((Y \otimes \mathbb{Q}_{p'}/\mathbb{Z})/W)^F.$$

The next section on duality will show how to exploit the similarity between these two results.

4.2 DUALITY OF CONNECTED REDUCTIVE GROUPS

Let G be a connected reductive group and T a maximal torus of G. Let X, Y be the character and cocharacter groups of T. X contains the set Φ of roots and Y contains the set of $\Phi^\vee$ of coroots.

We recall from section 1.9 that a quadruple $(X, \Phi, Y, \Phi^\vee)$ is called a root datum if the following conditions are satisfied:

(a) X and Y are free abelian groups of the same finite rank with a nondegenerate map $X \times Y \to \mathbb{Z}$ denoted by $(\chi, \gamma) \to \langle \chi, \gamma \rangle$ which puts them into duality.

(b) Φ and $\Phi^\vee$ are finite subsets of X, Y respectively and there is a bijection $\alpha \dashrightarrow \alpha^\vee$ from Φ to $\Phi^\vee$ satisfying $\langle \alpha, \alpha^\vee \rangle = 2$.

(c) For each $\alpha \in \Phi$ the maps $w_\alpha : X \to X$ and $w_{\alpha^\vee} : Y \to Y$ defined by

$$w_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha \qquad \chi \in X$$

$$w_{\alpha^\vee}(\gamma) = \gamma - \langle \alpha, \gamma \rangle \alpha^\vee \qquad \gamma \in Y$$

satisfy $w_\alpha(\Phi) = \Phi$ and $w_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$.

Thus the quadruple $(X, \Phi, Y, \Phi^\vee)$ determined as above by the maximal torus T in the connected reductive group G is a root datum.

We now define the concept of isomorphic root data. Let $(X, \Phi, Y, \Phi^\vee)$ and $(X', \Phi', Y', \Phi'^\vee)$ be two root data. These are said to be isomorphic if there exist maps $\delta : X \to X'$ and $\varepsilon : Y \to Y'$ such that:

(a) δ, ε are isomorphisms of abelian groups.
(b) $\langle \delta(\chi), \varepsilon(\gamma) \rangle = \langle \chi, \gamma \rangle$ for all $\chi \in X$, $\gamma \in Y$.
(c) $\delta(\Phi) = \Phi'$ and $\varepsilon(\Phi^\vee) = \Phi'^\vee$.
(d) $\varepsilon(\alpha^\vee) = \delta(\alpha)^\vee$ for all $\alpha \in \Phi$.

If we take the root data derived from two different maximal tori of the same connected reductive group G it is clear that they will be isomorphic. Thus the group G determines a root datum uniquely up to isomorphism. As mentioned in section 1.9 the connected reductive groups over a fixed algebraically closed field K can be classified up to isomorphism by their root data. Two connected reductive groups are isomorphic if and only if their root data are isomorphic. This gives a very useful way of describing the isomorphism classes of connected reductive groups.

Proposition 4.2.1. *If $(X, \Phi, Y, \Phi^\vee)$ is a root datum so is $(Y, \Phi^\vee, X, \Phi)$.*

Proof. This is clear from the definition of a root datum. ■

$(Y, \Phi^\vee, X, \Phi)$ is called the root datum dual to $(X, \Phi, Y, \Phi^\vee)$.

Two connected reductive groups G, G^* are said to be dual if their root data are dual. Thus each connected reductive group G has a dual group G^* which is unique up to isomorphism.

Proposition 4.2.2. *Let $(X, \Phi, Y, \Phi^\vee)$ and $(X', \Phi', Y', \Phi'^\vee)$ be two root data. Suppose there is an isomorphism $\delta: X \to Y'$ such that $\delta(\Phi) = \Phi'^\vee$ and*

$$\langle \chi, \alpha^\vee \rangle = \langle \alpha', \delta(\chi) \rangle \quad \textit{for all } \chi \in X, \alpha \in \Phi$$

where α' is defined by $\delta(\alpha) = \alpha'^\vee$. Then the root data are dual.

Proof. We must show that the root data $(X, \Phi, Y, \Phi^\vee)$ and $(Y', \Phi'^\vee, X', \Phi')$ are isomorphic. Thus we must prove the existence of isomorphisms

$$\delta: X \to Y' \qquad \varepsilon: Y \to X'$$

such that $\delta(\Phi) = \Phi'^\vee$, $\varepsilon(\Phi^\vee) = \Phi'$,

$$\langle \varepsilon(\gamma), \delta(\chi) \rangle = \langle \chi, \gamma \rangle \qquad \text{for all } \chi \in X, \gamma \in Y$$

and $\varepsilon(\alpha^\vee) = \delta(\alpha)^\vee$ for all $\alpha \in \Phi$.

Now the map δ is already given to us. We wish to define ε. Let $\gamma \in Y$. Then the map $\chi \to \langle \chi, \gamma \rangle$ lies in $\mathrm{Hom}(X, \mathbb{Z})$. So the map $\delta(\chi) \to \langle \chi, \gamma \rangle$ lies in $\mathrm{Hom}(Y', \mathbb{Z})$. Since $\mathrm{Hom}(Y', \mathbb{Z})$ is isomorphic to X' there is a unique element $\varepsilon(\gamma) \in X'$ such that

$$\langle \varepsilon(\gamma), \delta(\chi) \rangle = \langle \chi, \gamma \rangle \qquad \text{for all } \chi \in X.$$

Thus we have defined a map $\varepsilon: Y \to X'$ which is clearly an isomorphism of abelian groups.

Let $\alpha \in \Phi$. Taking $\gamma = \alpha^\vee$ we have

$$\langle \varepsilon(\alpha^\vee), \delta(\chi) \rangle = \langle \chi, \alpha^\vee \rangle \qquad \text{for all } \chi \in X.$$

However we are given that

$$\langle \alpha', \delta(\chi) \rangle = \langle \chi, \alpha^\vee \rangle \qquad \text{for all } \chi \in X$$

where $\delta(\alpha) = \alpha'^\vee$. It follows that $\varepsilon(\alpha^\vee) = \alpha'$. Thus $\varepsilon(\Phi^\vee) = \Phi'$. Finally we have

$$\varepsilon(\alpha^\vee) = \alpha' = \delta(\alpha)^\vee \qquad \text{for all } \alpha \in \Phi$$

and so the root data are dual to each other.

Proposition 4.2.3. *Let G, G^* be two connected reductive groups which are dual. Let $(X, \Phi, Y, \Phi^\vee)$ and $(X^*, \Phi^*, Y^*, \Phi^{*\vee})$ be their root data and let $\delta: X \to Y^*$ be an isomorphism of the type given in 4.2.2. Let W, W^* be the Weyl groups of G, G^*. Then there is an isomorphism $\delta: W \to W^*$ which maps w_α to $w_{\delta(\alpha)}$ for each $\alpha \in \Phi$. This isomorphism satisfies*

$$\delta({}^w\chi) = \delta(\chi)^{\delta(w^{-1})} \qquad \text{for all } \chi \in X,\ w \in W.$$

Proof. The isomorphism $\delta: X \to Y^*$ transforms the map $w_\alpha: X \to X$ into the map $w_{\delta(\alpha)}: Y^* \to Y^*$. For we have

$$w_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha \qquad \chi \in X$$

$$w_{\delta(\alpha)}(\delta(\chi)) = \delta(\chi) - \langle \alpha', \delta(\chi) \rangle \delta(\alpha)$$

where $\delta(\alpha) = \alpha'^\vee$, and $\langle \chi, \alpha^\vee \rangle = \langle \alpha', \delta(\chi) \rangle$. Hence δ induces an isomorphism between the group of transformations of X generated by the w_α for all $\alpha \in \Phi$ and the group of transformations of Y^* generated by the $w_{\delta(\alpha)}$ for all $\delta(\alpha) \in \Phi^{*\vee}$. The former group is isomorphic to W and the latter group to W^*. Thus we have an isomorphism $\delta: W \to W^*$.

Each $w \in W$ can be written as $w = w_{\alpha_1} \dots w_{\alpha_k}$ for suitable $\alpha_i \in \Phi$. We have seen that

$$\delta(w_{\alpha_i}(\chi)) = w_{\delta(\alpha_i)}(\delta(\chi)) \qquad \text{for all } \chi \in X.$$

Now recall that we defined W-actions on X and Y in section 1.9 giving elements ${}^w\chi \in X$ and $\gamma^w \in Y$ for $\chi \in X$ and $\gamma \in Y$. In this notation we have

$$\delta({}^{w_{\alpha_i}}\chi) = \delta(\chi)^{w_{\delta(\alpha_i)}}.$$

It follows that

$$\delta({}^w\chi) = \delta({}^{w_{\alpha_1} \dots w_{\alpha_k}}\chi) = \delta(\chi)^{w_{\delta(\alpha_k)} \dots w_{\delta(\alpha_1)}}$$
$$= \delta(\chi)^{\delta(w_{\alpha_k} \dots w_{\alpha_1})} = \delta(\chi)^{\delta(w^{-1})}.$$

4.3 DUALITY OF REDUCTIVE GROUPS OVER FINITE FIELDS

We now consider the duality of connected reductive groups with Frobenius map. Let G be a connected reductive group with Frobenius map $F: G \to G$. Let T be an F-stable maximal torus of G. We recall that F acts on $X = X(T)$ and $Y = Y(T)$ by

$$(F(\chi))(t) = \chi(F(t)) \qquad \chi \in X,\ t \in T$$

$$(F(\gamma))(\lambda) = F(\gamma(\lambda)) \qquad \gamma \in Y,\ \lambda \in \mathbf{G}_m.$$

These F-actions are related by

$$\langle \chi, F(\gamma) \rangle = \langle F(\chi), \gamma \rangle$$

by section 1.18. We wish to describe what it means for two such pairs (G, F) and (G^*, F^*) to be in duality.

Proposition 4.3.1. *Let T be an F-stable maximal torus of G and X, Y be its character and cocharacter groups. Let T^* be an F^*-stable maximal torus of G^* and X^*, Y^* be its character and cocharacter groups. Then the following two conditions are equivalent:*

(a) *There exists an isomorphism $\delta: X \to Y^*$ such that*
   (i) $\delta(\Phi) = (\Phi^*)^\vee$.
   (ii) $\langle \chi, \alpha^\vee \rangle = \langle \alpha^*, \delta(\chi) \rangle$ *for all* $\chi \in X$, $\alpha \in \Phi$ *where* $\delta(\alpha) = (\alpha^*)^\vee$.
   (iii) $\delta(F(\chi)) = F^*(\delta(\chi))$ *for all* $\chi \in X$.

(b) *There exists an isomorphism $\varepsilon: Y \to X^*$ such that*
   (i) $\varepsilon(\Phi^\vee) = \Phi^*$.
   (ii) $\langle \alpha, \gamma \rangle = \langle \varepsilon(\gamma), \varepsilon(\alpha^\vee)^\vee \rangle$ *for all* $\gamma \in Y$, $\alpha \in \Phi$.
   (iii) $\varepsilon(F(\gamma)) = F^*(\varepsilon(\gamma))$ *for all* $\gamma \in Y$.

Proof. Suppose $\delta: X \to Y^*$ satisfies the conditions of (a). We then define $\varepsilon: Y \to X^*$ as in the proof of 4.2.2. In order to show that ε satisfies the conditions of (b) we can apply 4.2.2 and need only show in addition that

$$\varepsilon(F(\gamma)) = F^*(\varepsilon(\gamma)) \qquad \gamma \in Y.$$

Now $\varepsilon(F(\gamma))$ and $F^*(\varepsilon(\gamma))$ lie in X^*. Let $\chi \in X$. Then we have

$$\langle \varepsilon(F(\gamma)), \delta(\chi) \rangle = \langle \chi, F(\gamma) \rangle = \langle F(\chi), \gamma \rangle$$

and so

$$\langle F^*(\varepsilon(\gamma)), \delta(\chi) \rangle = \langle \varepsilon(\gamma), F^*(\delta(\chi)) \rangle = \langle \varepsilon(\gamma), \delta(F(\chi)) \rangle = \langle F(\chi), \gamma \rangle.$$

Thus we have

$$\langle \varepsilon(F(\gamma)), \delta(\chi) \rangle = \langle F^*(\varepsilon(\gamma)), \delta(\chi) \rangle$$

for all $\delta(\chi) \in Y^*$. Since the map $X^* \times Y^* \to \mathbb{Z}$ is nondegenerate it follows that $\varepsilon(F(\gamma)) = F^*(\varepsilon(\gamma))$.

Thus the conditions of (a) imply those of (b). Since the conditions in (a), (b) are of dual form there will be an entirely analogous argument showing that the conditions of (b) imply those of (a). ■

We say that the pairs (G, F) and (G^*, F^*) are in duality if the conditions of 4.3.1 hold for tori T, T^* which are maximally split in G, G^* respectively. The conditions are independent of which maximally split tori are chosen since any two maximally split tori of G are conjugate by an element of G^F.

We shall show subsequently, however, that if (G, F) and (G^*, F^*) are in duality then for any F-stable maximal torus T of G, not necessarily maximally split, there is an F^*-stable maximal torus T^* of G^* for which the conditions of 4.3.1 hold.

Proposition 4.3.2. *Suppose the pairs (G, F) and (G^*, F^*) are in duality and that W, W^* are their Weyl groups. Let $\delta: W \to W^*$ be an isomorphism of the type given* *in* 4.2.3. *Then we have*

$$\delta(F(w)) = F^{*-1}(\delta(w))$$

for all $w \in W$. *Thus the Frobenius maps* F, F^* *operate in inverse ways on the Weyl group.*

Proof. Let T, T^* be maximally split tori of G, G^* and X, Y, X^*, Y^* be their character and cocharacter groups. We have an isomorphism $\delta: X \to Y^*$ of the type given in 4.2.2 which induces a map $\delta: W \to W^*$ as in 4.2.3. This isomorphism satisfies

$$\delta({}^{w}\chi) = \delta(\chi)^{\delta(w^{-1})} \qquad \text{by 4.2.3}$$

$$\delta(F(\chi)) = F^*(\delta(\chi)) \qquad \text{by 4.3.1.}$$

We also have an action of F on W which satisfies

$$F(t^w) = F(t)^{F(w)} \qquad \text{for all } t \in T,\ w \in W.$$

Consequently we have

$$^{w}(F(\chi)) = F({}^{F(w)}\chi) \qquad \text{for all } \chi \in X,\ w \in W$$

$$F(\gamma^w) = F(\gamma)^{F(w)} \qquad \text{for all } \gamma \in Y,\ w \in W.$$

Now consider the map $\delta: X \to Y^*$. Under this map χ is transformed to $\delta(\chi)$, $F(\chi)$ maps to $F^*(\delta(\chi))$, ${}^{w}F(\chi)$ maps to $(F^*(\delta(\chi)))^{\delta(w^{-1})}$, ${}^{F(w)}\chi$ maps to $\delta(\chi)^{\delta(F(w)^{-1})}$ and $F({}^{F(w)}\chi)$ maps to $F^*(\delta(\chi)^{\delta(F(w)^{-1})})$.

Since ${}^{w}(F(\chi)) = F({}^{F(w)}\chi)$ it follows that

$$(F^*(\delta(\chi)))^{\delta(w^{-1})} = F^*(\delta(\chi)^{\delta(F(w)^{-1})})$$

for all $\delta(\chi) \in Y^*$. However we also have

$$F^*(\delta(\chi)^{\delta(F(w)^{-1})}) = (F^*(\delta(\chi))^{F^*(\delta(F(w)^{-1}))}.$$

We therefore deduce that

$$\delta(w^{-1}) = F^*(\delta(F(w)^{-1})).$$

Replacing w by w^{-1} we obtain

$$\delta(w) = F^*(\delta(F(w))).$$

Thus $\delta(F(w)) = F^{*-1}(\delta(w))$ as required. ■

We now consider maximal tori in G, G^* which need not be maximally split. These have the form ${}^{g}T$, ${}^{g^*}T^*$ where T, T^* are maximally split and $g^{-1}F(g) \in N$, $g^{*-1}F^*(g^*) \in N^*$. Let $\pi(g^{-1}F(g)) = w \in W$, $\pi(g^{*-1}F^*(g^*)) = w^* \in W^*$. We assume that the pairs (G, F) and (G^*, F^*) are in duality. Thus we have a duality map $\delta: X \to Y^*$ satisfying the conditions of 4.3.1. This gives rise to a map $\bar{\delta}: X({}^{g}T) \to Y^*({}^{g^*}T^*)$ defined by

$$\bar{\delta}({}^{g}\chi) = {}^{g^*}\delta(\chi) \qquad \chi \in X$$

where the conjugation maps $\chi \to {}^g\chi$, $\delta(\chi) \to {}^{g^*}\delta(\chi)$ are defined as in section 3.3. We wish to know under what circumstances the map $\bar{\delta}$ itself satisfies the conditions of 4.3.1. This will be so if and only if

$$\bar{\delta}(F({}^g\chi)) = F^*(\bar{\delta}({}^g\chi)) \qquad \text{for all } \chi \in X.$$

We now need the following lemma.

Lemma 4.3.3. *$\bar{\delta}(F({}^g\chi)) = F^*(\bar{\delta}({}^g\chi))$ for all $\chi \in X$ if and only if $w^* = F^*(\delta(w)^{-1})$.*

Proof. We have $F({}^g\chi) = {}^g(F({}^{w^{-1}}\chi))$ by 3.3.4. Thus

$$\begin{aligned}\bar{\delta}(F({}^g\chi)) &= \bar{\delta}({}^g(F({}^{w^{-1}}\chi))) = {}^{g^*}(\delta(F({}^{w^{-1}}\chi)))\\ &= {}^{g^*}(F^*(\delta({}^{w^{-1}}\chi))) = {}^{g^*}(F^*(\delta(\chi)^{\delta(w)})) = {}^{g^*}(F^*(\delta(\chi))^{F^*(\delta(w))}).\end{aligned}$$

On the other hand we have

$$\begin{aligned}F^*(\bar{\delta}({}^g\chi)) &= F^*({}^{g^*}\delta(\chi)) = {}^{F^*(g^*)}F^*(\delta(\chi)) = {}^{g^*w^*}F^*(\delta(\chi))\\ &= {}^{g^*}(F^*(\delta(\chi))^{(w^*)^{-1}}).\end{aligned}$$

Comparing these two expressions we see that $\bar{\delta}(F({}^g\chi)) = F^*(\bar{\delta}({}^g\chi))$ for all $\chi \in X$ if and only if $F^*(\delta(w)) = (w^*)^{-1}$. Thus

$$w^* = (F^*(\delta(w)))^{-1} = F^*(\delta(w)^{-1}).$$

Proposition 4.3.4. *Let (G, F) and (G^*, F^*) be in duality with respect to maximally split tori T, T^*. Let X, Y, X^*, Y^* be the character and cocharacter groups of T, T^*. Let $\delta: X \to Y^*$ be a duality map as in 4.2.2 inducing an isomorphism $\delta: W \to W^*$ as in 4.2.3. Then we have:*

(i) *Two elements $w, w' \in W$ are F-conjugate if and only if $\delta(w)^{-1}$, $\delta(w')^{-1}$ are F^*-conjugate in W^*.*

(ii) *The map $w \to \delta(w)^{-1}$ is a bijection between W and W^* which induces a bijection between the F-conjugacy classes of W and the F^*-conjugacy classes of W^*.*

(iii) *The map $w \to \delta(w)^{-1}$ gives rise to a bijection between G^F-classes of F-stable maximal tori of G and $(G^*)^{F^*}$-classes of F^*-stable maximal tori of G^*.*

(iv) *Suppose ${}^gT$ and ${}^{g^*}T^*$ are maximal tori of G, G^* which are F-stable and F^*-stable respectively and which are in corresponding classes under the bijection in (iii). Then these two tori satisfy the conditions of 4.3.1 and are therefore in duality.*

Proof. (i) Suppose w, w' are F-conjugate. Then $w' = x^{-1}wF(x)$ for some $x \in W$. Applying δ we obtain

$$\begin{aligned}\delta(w') &= \delta(x^{-1})\delta(w)\delta(F(x))\\ &= \delta(x)^{-1}\delta(w)F^{*-1}(\delta(x))\end{aligned}$$

by 4.3.2. Thus $\delta(w')^{-1} = F^{*-1}(\delta(x)^{-1})\delta(w)^{-1}\delta(x)$. Hence

$$F^*(\delta(w')^{-1}) = \delta(x)^{-1}F^*(\delta(w)^{-1})F^*(\delta(x)).$$

Thus $F^*(\delta(w)^{-1})$ and $F^*(\delta(w')^{-1})$ are F^*-conjugate. However

$$F^*(y) = y^{-1}.y.F^*(y) \qquad \text{for any } y \in W$$

and so $F^*(y)$ is F^*-conjugate to y. Thus $\delta(w)^{-1}$ and $\delta(w')^{-1}$ are F^*-conjugate. The converse clearly holds in a similar manner.

(ii) The map $w \to \delta(w)^{-1}$ is clearly a bijective map from W to W^*. It induces a bijection between the F-conjugacy classes of W and the F^*-conjugacy classes of W^* because of (i).

(iii) This follows from (ii) and 3.3.3.

(iv) Let ${}^gT$ be obtained from T by twisting with the element $w \in W$. Then ${}^{g^*}T^*$ is obtained from T^* by twisting with an element of W^* which is F^*-conjugate to $\delta(w)^{-1}$. We may choose any such F^*-conjugate of $\delta(w)^{-1}$ and there will be an appropriate transforming element g^*. We choose the F^*-conjugate $F^*(\delta(w)^{-1})$. Then 4.3.3 shows that the map $\bar{\delta}: X({}^gT) \to Y({}^{g^*}T^*)$ satisfies the conditions of 4.3.1. Thus the tori ${}^gT$ and ${}^{g^*}T^*$ are in duality. ∎

This proposition shows that if (G, F) and (G^*, F^*) are in duality and if we take an F-stable maximal torus of G obtained from a maximally split torus by twisting with $w \in W$ and in F^*-stable maximal torus of G^* obtained from a maximally split torus by twisting with $w^* \in W^*$ then these tori will be in duality in the sense of 4.3.1 provided w^* is F^*-conjugate to $\delta(w)^{-1}$. Thus every F-stable maximal torus of G is in duality with some F^*-stable maximal torus of G^* and conversely.

4.4 PROPERTIES OF FINITE REDUCTIVE GROUPS IN DUALITY

We suppose that the pairs (G, F) and (G^*, F^*) are in duality and consider the relation between the finite groups G^F and $(G^*)^{F^*}$.

Proposition 4.4.1. *Let T be an F-stable maximal torus of G and T^* an F^*-stable maximal torus of G^* such that T, T^* are in duality in the sense of* 4.3.1. *Then the duality map $\delta: X \to Y^*$ gives rise to an isomorphism between $(T^*)^{F^*}$ and the character group $\hat{T}^F$.*

Proof. By 3.2.3 we have $\hat{T}^F \cong X/(F-1)X$. We observe that $\delta((F-1)X) = (F^*-1)Y^*$. This follows from the fact that $\delta(F(\chi)) = F^*(\delta(\chi))$ for all $\chi \in X$. Hence

$$\hat{T}^F \cong X/(F-1)X \cong Y^*/(F^*-1)Y^* \cong (T^*)^{F^*}$$

by 3.2.2. Thus we have an isomorphism which maps characters of a torus to elements of a dual torus.

Corollary 4.4.2. *If T, T^* are in duality then $|T^F| = |(T^*)^{F^*}|$.*

Proposition 4.4.3. *If (G, F) and (G^*, F^*) are in duality then $q = q^*$.*

Proof. We recall the definition of q. We take a maximally split torus T of G

with character group X. Then the Frobenius map F acts on X in such a way that some power of F acts as a positive integral multiple of the identity. We defined q to satisfy

$$F^n = q^n 1 \text{ on } X \qquad q > 1.$$

It also satisfies $F^n = q^n 1$ on Y since by 3.2.1 the F-actions on X, Y with respect to dual bases are given by matrices which are transposes of one another. We also have $(F^*)^{n^*} = (q^*)^{n^*} 1$ on X^* and Y^* for some positive integer n^*. Since (G, F) and (G^*, F^*) are in duality we have an isomorphism $\delta: X \to Y^*$ satisfying

$$\delta(F(\chi)) = F^*(\delta(\chi)) \qquad \text{for all } \chi \in X.$$

It follows that

$$\delta(F^{nn^*}(\chi)) = F^{*nn^*}(\delta(\chi)).$$

Hence $\delta(q^{nn^*}\chi) = (q^*)^{nn^*}\delta(\chi)$. Thus

$$q^{nn^*}\delta(\chi) = (q^*)^{nn^*}\delta(\chi) \qquad \text{for all } \delta(\chi) \in Y^*.$$

It follows that $q^{nn^*} = (q^*)^{nn^*}$ and so $q = q^*$, since we have $q > 1$ and $q^* > 1$.

Proposition 4.4.4. *If (G, F) and (G^*, F^*) are in duality then $|G^F| = |(G^*)^{F^*}|$. Thus finite groups of the form G^F in duality have the same order.*

Proof. We recall from section 2.9 that the order of G^F is given by

$$|G^F| = |(Z^0)^F| q^N \prod_{\substack{J \\ \rho\text{-orbit}}} (q^{|J|} - 1) \sum_{w \in W^F} q^{l(w)}$$

where $N = |\Phi^+|$ and J runs over the ρ-orbits on I.

Now if T is a maximally split torus of G then $|T^F|$ is given by

$$|T^F| = |(Z^0)^F| \prod_{\substack{J \\ \rho\text{-orbit}}} (q^{|J|} - 1).$$

Thus we have

$$|G^F| = q^{|\Phi^+|} |T^F| \sum_{w \in W^F} q^{l(w)}.$$

Similarly

$$|G^{*F^*}| = q^{*|\Phi^{*+}|} |T^{*F^*}| \sum_{w^* \in W^{*F^*}} q^{*l(w^*)}.$$

Since (G, F) and (G^*, F^*) are in duality we have $q = q^*$ by 4.4.3. Also $|\Phi| = |\Phi^*|$ and so $|\Phi^+| = |\Phi^{*+}|$. The maximally split tori T, T^* are in duality and so $|T^F| = |T^{*F^*}|$ by 4.4.2. We also have an isomorphism $\delta: W \to W^*$ by 4.2.3 which maps s_α to $s_{\delta(\alpha)}$ for each simple root $\alpha \in \Delta$. Thus δ maps the simple reflections of W into a set of simple reflections of W^*. Hence $l(w) = l(\delta(w))$ for all $w \in W$.

Finally we observe that δ maps W^F to $(W^*)^{F^*}$. For by 4.3.2 we have $\delta(F(w)) = F^{*-1}(\delta(w))$. Thus $F(w) = w$ if and only if $F^*(\delta(w)) = \delta(w)$. Hence we may compare terms in the above order formulae and deduce that $|G^F| = |G^{*F^*}|$.

Proposition 4.4.5. *If (G, F) and (G^*, F^*) are in duality then $|G'^F| = |(G^*)'^{F^*}|$ and $|(Z^0)^F| = |(Z^{*0})^{F^*}|$.*

Proof. We have

$$|G'^F| = q^{|\Phi^+|} \prod_{\substack{J \\ \rho\text{-orbit}}} (q^{|J|} - 1) \sum_{w \in W^F} q^{l(w)}.$$

Comparing this with the analogous formula for $|(G^*)'^{F^*}|$ we see that it is sufficient to show that the sizes $|J|$ of the ρ-orbits on I are the same as those of the ρ^*-orbits on I^*. We know from section 1.18 that

$$F(\alpha_i) \text{ is a positive multiple of } \rho^{-1}(\alpha_i) \quad i \in I.$$

Now $\delta: X \to Y^*$ transforms a set of simple roots in X to a set of simple coroots in Y. Also we have

$$\delta(F(\alpha_i)) = F^*(\delta(\alpha_i)).$$

Thus $F^*(\delta(\alpha_i))$ is a positive multiple of $\delta(\rho^{-1}(\alpha_i))$. However $F^*(\delta(\alpha_i))$ is a positive multiple of the coroot $(\rho^*)^{-1}(\delta(\alpha_i))$. It follows that

$$(\rho^*)^{-1}(\delta(\alpha_i)) = \delta(\rho^{-1}(\alpha_i)) \qquad i \in I.$$

Thus each ρ-orbit on I is transformed by δ to a ρ^*-orbit on I^*. It follows that

$$|G'^F| = |(G^*)'^{F^*}|.$$

Since $|G^F| = |G'^F| . |(Z^0)^F|$ it follows from 4.4.4 that

$$|(Z^0)^F| = |(Z^{*0})^{F^*}|$$

Theorem 4.4.6. *Suppose the pairs (G, F) and (G^*, F^*) are in duality. Then:*

(i) *There is a bijective correspondence between geometric conjugacy classes of pairs (T, θ) in G, where T is an F-stable maximal torus and $\theta \in \hat{T}^F$, and F^*-stable semisimple conjugacy classes in G^*.*

(ii) *The number of geometric conjugacy classes in G is $|(Z^0)^F|q^l$.*

Proof. Let T, T^* be maximally split tori in G, G^* respectively and X, Y, X^*, Y^* their character and cocharacter groups. By 4.1.4 there is a bijective correspondence between geometric conjugacy classes in G and the set

$$((X \otimes \mathbb{Q}_{p'}/\mathbb{Z})/W)^F$$

of F-stable W-orbits on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$. By 3.7.2 there is a bijective correspondence between F^*-stable semisimple classes of G^* and the set

$$((Y^* \otimes \mathbb{Q}_{p'}/\mathbb{Z})/W^*)^{F^*}$$

of F^*-stable W^*-orbits on $Y^* \otimes \mathbb{Q}_{p'}/\mathbb{Z}$.

Since (G, F) and (G^*, F^*) are in duality there is an isomorphism $\delta: X \to Y^*$ of the type given in 4.3.1. This extends to an isomorphism between $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ and $Y^* \otimes \mathbb{Q}_{p'}/\mathbb{Z}$. We also have an isomorphism $\delta: W \to W^*$ by 4.2.3 which satisfies

$$\delta({}^w\chi) = \delta(\chi)^{\delta(w^{-1})} \qquad \text{for all } \chi \in X,\ w \in W.$$

Thus we obtain a bijection between the set of W-orbits on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ and the set of W^*-orbits on $Y^* \otimes \mathbb{Q}_{p'}/\mathbb{Z}$. Finally we consider the actions of F and F^*. Since $\delta(F(\chi)) = F^*(\delta(\chi))$ we see that the F-stable W-orbits on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ correspond under the above bijection to the F^*-stable W^*-orbits on $Y^* \otimes \mathbb{Q}_{p'}/\mathbb{Z}$. This gives a bijection between geometric conjugacy classes in G and F^*-stable semisimple classes in G^*. By 3.7.6 the number of F^*-stable semisimple classes of G^* is $|(Z^*)^{0F^*}|q^{*l^*}$. However $q^* = q$, $l^* = l$ and $|(Z^*)^{0F^*}| = |(Z^0)^F|$ and so the number of geometric conjugacy classes in G is $|(Z^0)^F|q^l$. ∎

The description of the possible pairs (G, F) when G is simple, given in section 1.19, shows that each such pair (G, F) has a dual pair (G^*, F^*). The same is true more generally if G is semisimple or reductive.

It is sometimes convenient to say that G^F, G^{*F^*} are dual groups when the pairs (G, F), (G^*, F^*) are in duality.

In the accompanying table we give a list of dual pairs G^F when G is simple. (Not all the possible groups G^F are included here, just those which can be described conveniently.) Each of the Suzuki and Ree groups G^F is self-dual in the sense defined above.

Pairs of dual groups G^F

$(A_l)_{sc}(q)$	$(A_l)_{ad}(q)$
$({}^2A_l)_{sc}(q^2)$	$({}^2A_l)_{ad}(q^2)$
$(B_l)_{sc}(q)$	$(C_l)_{ad}(q)$
$(C_l)_{sc}(q)$	$(B_l)_{ad}(q)$
$(D_l)_{sc}(q)$	$(D_l)_{ad}(q)$
$({}^2D_l)_{sc}(q^2)$	$({}^2D_l)_{ad}(q^2)$
$SO_{2l}(q)$	$SO_{2l}(q)$
$SO_{2l}^-(q)$	$SO_{2l}^-(q)$
$HS_{2l}(q)$ (l even)	$HS_{2l}(q)$ (l even)
$G_2(q)$	$G_2(q)$
$F_4(q)$	$F_4(q)$
$(E_6)_{sc}(q)$	$(E_6)_{ad}(q)$
$({}^2E_6)_{sc}(q^2)$	$({}^2E_6)_{ad}(q^2)$
$(E_7)_{sc}(q)$	$(E_7)_{ad}(q)$
$E_8(q)$	$E_8(q)$
${}^3D_4(q^3)$	${}^3D_4(q^3)$

4.5 THE DUAL OF A GROUP WITH CONNECTED CENTRE

We shall in this section determine the condition on a connected reductive group for its dual group to have connected centre.

Lemma 4.5.1. *Let G be a connected reductive group, T be a maximal torus of G, X the character group of T and Φ the set of roots. Then the centre $Z(G)$ is connected if and only if $X/\mathbb{Z}\Phi$ has no p'-torsion (i.e. no non-identity elements of finite order prime to p).*

Proof. We have $G = \langle T, X_\alpha, \alpha \in \Phi\rangle$ and we know $Z(G)$ lies in T. Moreover

$$tx_\alpha(\lambda)t^{-1} = x_\alpha(\alpha(t)\lambda) \qquad t \in T.$$

It follows that $Z(G) = \{t \in T; \alpha(t) = 1 \text{ for all } \alpha \in \Phi\}$. This means $Z = Z(G) = \mathbb{Z}\Phi^\perp$ in the notation of section 1.12. Hence

$$Z^\perp = \mathbb{Z}\Phi^{\perp\perp}$$

where $\mathbb{Z}\Phi^{\perp\perp}/\mathbb{Z}\Phi$ is the p-torsion subgroup of $X/\mathbb{Z}\Phi$. It follows that

$$(Z^0)^\perp = \overline{\mathbb{Z}\Phi}$$

where $\overline{\mathbb{Z}\Phi}/\mathbb{Z}\Phi$ is the torsion subgroup of $X/\mathbb{Z}\Phi$. Hence $Z^\perp = (Z^0)^\perp$ if and only if $X/\mathbb{Z}\Phi$ has no p'-torsion, and this condition is equivalent to $Z = Z^0$. ■

Now the dual condition to that in 4.5.1 is that $Y/\mathbb{Z}\Phi^\vee$ has no p'-torsion. Our next lemmas will therefore concentrate on properties of the Y-subgroup.

Lemma 4.5.2. *Let G be connected reductive, $T(G)$ a maximal torus of G, and $T(G') = T(G) \cap G'$ a maximal torus of G'. Then $Y(T(G'))$ is a subgroup of $Y(T(G))$ and $Y(T(G))/Y(T(G'))$ is torsion free.*

Proof. Any homomorphism $\mathbf{G}_m \to T(G')$ is a homomorphism $\mathbf{G}_m \to T(G)$ and so $Y(T(G'))$ is a subgroup of $Y(T(G))$. In order to show that the quotient is torsion free we must show that if $\gamma: \mathbf{G}_m \to T(G)$ satisifes $\gamma^n(\mathbf{G}_m) \subseteq T(G')$ for $n > 0$ then $\gamma(\mathbf{G}_m) \subseteq T(G')$. But this is clear since γ and γ^n have the same image if $n > 0$.

Lemma 4.5.3. *Let G be connected and semisimple and let G_{sc} be the simply-connected covering of G. Let $T(G_{sc})$ be a maximal torus of G_{sc} and $T(G)$ be its image under the natural homomorphism. Then $Y(T(G_{sc}))$ may be identified with a subgroup of $Y(T(G))$ of finite index. It is the subgroup generated by the coroots $\Phi^\vee$.*

Proof. Let F be the kernel of the natural homomorphism $G_{sc} \to G$. Let $\theta: T(G_{sc}) \to T(G)$ be the natural map between the tori. Then for each $\gamma \in Y(T(G_{sc}))$ we have $\theta \circ \gamma \in Y(T(G))$. If $\theta \circ \gamma = 1$ then $\gamma(K^*) \subseteq F$ so $\gamma(K^*)$ is finite. This implies that $\gamma(K^*) = 1$, so $\gamma = 1$. Thus the map $\gamma \to \theta \circ \gamma$ is injective and $Y(T(G_{sc}))$ can be identified with a subgroup of $Y(T(G))$. This subgroup has finite index since both groups have the same rank. Finally $Y(T(G_{sc}))$ is generated by the coroots of G_{sc} and these are identified with the coroots of G under the above map.

Lemma 4.5.4. *Let G be connected reductive. Then $Y(T(G'))/Y(T(G'_{sc}))$ is the torsion subgroup of $Y(T(G))/Y(T(G'_{sc}))$.*

Proof. We have $Y(T(G)) \supseteq Y(T(G')) \supseteq Y(T(G'_{sc}))$ by 4.5.2 and 4.5.3. Moreover $Y(T(G))/Y(T(G'))$ is torsion free and $Y(T(G'))/Y(T(G'_{sc}))$ is finite. The result follows.

Lemma 4.5.5. *Let G be connected and semisimple. Then we have injective maps*

$$\alpha: X(T(G)) \to X(T(G_{sc})),\ \beta: Y(T(G_{sc})) \to Y(T(G))$$

which give a commutative diagram

$$\begin{tikzcd} & X(T(G)) \times Y(T(G)) \arrow[dr] & \\ X(T(G)) \times Y(T(G_{sc})) \arrow[ur, "id \times \beta"] \arrow[dr, "\alpha \times id"'] & & \mathbb{Z} \\ & X(T(G_{sc})) \times Y(T(G_{sc})) \arrow[ur] & \end{tikzcd}$$

Proof. We have defined β in 4.5.3. Now each $\chi \in X(T(G))$ gives $\chi \circ \theta \in X(T(G_{sc}))$ and we define $\alpha(\chi) = \chi \circ \theta$. If $\alpha(\chi) = 1$ then χ induces the unit character on T/F where F is finite. This can only happen if $\chi = 1$, thus α is injective.

Next consider the commutativity of the above diagram. Let $\chi \in X(T(G))$, $\gamma \in Y(T(G_{sc}))$. The element of $\mathrm{Hom}(\mathbf{G}_m, \mathbf{G}_m) \cong \mathbb{Z}$ given by the two paths in the diagram are $\chi \circ (\theta \circ \gamma)$ and $(\chi \circ \theta) \circ \gamma$, and so these are equal.

Lemma 4.5.6. *Let G be connected and semisimple. Then the finite abelian groups $X(T(G_{sc}))/X(T(G))$ and $Y(T(G))/Y(T(G_{sc}))$ are isomorphic.*

Proof. Let $\chi_1, \ldots, \chi_l$ and $\gamma_1, \ldots, \gamma_l$ be dual bases of $X(T(G))$ and $Y(T(G))$ and $\bar{\chi}_1, \ldots, \bar{\chi}_l$ and $\bar{\gamma}_1, \ldots, \bar{\gamma}_l$ be dual bases of $X(T(G_{sc}))$ and $Y(T(G_{sc}))$. Then we have

$$\chi_i = \sum_j m_{ij}\bar{\chi}_j \qquad m_{ij} \in \mathbb{Z}$$

$$\bar{\gamma}_i = \sum_j n_{ij}\gamma_j \qquad n_{ij} \in \mathbb{Z}.$$

Consider the scalar product $\langle \chi_i, \bar{\gamma}_j \rangle$ in two ways, as in 4.5.5. We can evaluate this scalar product in G or in G_{sc}, but the result is the same by 4.5.5. We have

$$\langle \chi_i, \bar{\gamma}_j \rangle = n_{ji} \text{ in } G$$

$$\langle \chi_i, \bar{\gamma}_j \rangle = m_{ij} \text{ in } G_{sc}.$$

Hence $n_{ji} = m_{ij}$. (m_{ij}) is the relation matrix of $X(T(G_{sc}))/X(T(G))$ and (n_{ij}) is the relation matrix of $Y(T(G))/Y(T(G_{sc}))$. These relation matrices are transposes of one another, so one of the groups is isomorphic to the character group of the other. Thus the two groups are isomorphic.

Lemma 4.5.7. *Let G be connected and semisimple. Consider $X(T(G))$ as a subgroup of $X(T(G_{sc}))$ and let $\tilde{X}(T(G))/X(T(G))$ be the p-torsion subgroup of*

$X(T(G_{sc}))/X(T(G))$. Then $X(T(G_{sc}))/\tilde{X}(T(G))$ is isomorphic to the kernel F of $G_{sc} \to G$.

Proof. Consider the surjective homomorphism $T(G_{sc}) \to T(G)$ with kernel F. We have

$$F = \{t \in T(G_{sc});\ \chi(t) = 1 \text{ for all } \chi \in X(T(G))\}$$

or, in other words, $F = X(T(G))^{\perp}$. Hence

$$F^{\perp} = X(T(G))^{\perp\perp} = \tilde{X}(T(G)).$$

It follows from this that

$$F \cong X(T(G_{sc}))/F^{\perp} = X(T(G_{sc}))/\tilde{X}(T(G)).$$ ■

We are now able to prove our main result of this section.

Theorem 4.5.8. *The following two conditions on a connected reductive group G are dual conditions in the sense that one holds in G if and only if the other holds in the dual group of G:*

(i) *$Z(G)$ is connected.*
(ii) *The natural map $G'_{sc} \to G'$ is bijective.*

Proof. By 4.5.1 $Z(G)$ is connected if and only if $X(T(G))/\mathbb{Z}\Phi$ has no p'-torsion. The dual of this condition asserts that $Y(T(G))/\mathbb{Z}\Phi^{\vee}$ has no p'-torsion.

Let $\tilde{Y}(T(G'_{sc}))/Y(T(G'_{sc}))$ be the p-torsion subgroup of $Y(T(G))/Y(T(G'_{sc}))$. By 4.5.3 we have $Y(T(G'_{sc})) = \mathbb{Z}\Phi^{\vee}$ and by 4.5.4 $Y(T(G'))/Y(T(G'_{sc}))$ is the torsion subgroup of $Y(T(G))/Y(T(G'_{sc}))$. Thus $Y(T(G))/\mathbb{Z}\Phi^{\vee}$ has no p'-torsion if and only if $Y(T(G'))/\tilde{Y}(T(G'_{sc})) = 1$. This condition is stated entirely in terms of the semisimple group G'. We may therefore assume subsequently that G is semisimple and we shall investigate the condition $Y(T(G))/\tilde{Y}(T(G_{sc})) = 1$.

By 4.5.6 we know that $X(T(G_{sc}))/X(T(G))$ and $Y(T(G))/Y(T(G_{sc}))$ are isomorphic. Factoring out by the p-torsion subgroup we see that

$$X(T(G_{sc}))/\tilde{X}(T(G)) \cong Y(T(G))/\tilde{Y}(T(G_{sc})).$$

Thus $Y(T(G))/\tilde{Y}(T(G_{sc})) = 1$ if and only if $X(T(G_{sc}))/\tilde{X}(T(G)) = 1$. Finally we know by 4.5.7 that $X(T(G_{sc}))/\tilde{X}(T(G))$ is isomorphic to F. $F = 1$ holds if and only if the natural homomorphism $G_{sc} \to G$ is bijective. The result follows. ■

Note. The map $G'_{sc} \to G'$ need not be an isomorphism of algebraic groups even if it is bijective.

Theorem 4.5.9. *Let G be a connected reductive group in which $Z(G)$ is connected. Let G^* be the dual group of G and s^* be a semisimple element of G^*. Then $C_{G^*}(s^*)$ is connected.*

Proof. Let G be a connected reductive group in which the natural homomorphism $G'_{sc} \to G'$ is bijective. We must show that centralizers of semisimple elements in G are connected. As in the proof of 3.7.3 this will be true provided the same is true in G'.

Consider the bijective homomorphism $G'_{sc} \xrightarrow{\theta} G'$. Under this homomorphism semisimple elements map to semisimple elements and centralizers to centralizers. Let $s \in G'$ be semisimple. There is a unique $s' \in G'_{sc}$ with $\theta(s') = s$ and s' is semisimple. $C_{G'_{sc}}(s')$ maps to $C_{G'}(s)$ under θ. By Steinberg's theorem 3.5.6 $C_{G'_{sc}}(s')$ is connected. Thus its image $C_{G'}(s)$ is connected also. ■

The concept of duality of connected reductive groups with Frobenius map first appeared in Deligne and Lusztig [1]. We shall see that this idea is very useful in relating certain irreducible characters of one group G^F to certain conjugacy classes in the dual group G^{*F^*}.

Chapter 5

UNIPOTENT CLASSES

We now turn our attention to the unipotent classes in a connected reductive group G. We show first that the natural homomorphism $G \to G/Z$ induces a bijection between the unipotent classes of G and the unipotent classes of G/Z. Thus we may assume without loss of generality that G is semisimple with trivial centre. There is then a bijective homomorphism from G to a semisimple group of adjoint type which preserves the unipotent classes. It is therefore sufficient to consider the case when G is semisimple of adjoint type. G is then a direct product of simple groups of adjoint type. We therefore need only consider simple groups of adjoint type or, alternatively, one simple group of each type (not necessarily the adjoint group) if this is more convenient for us.

We recall from section 1.15 that the set $\mathfrak{U}$ of unipotent elements of G forms a closed irreducible subset of G and that the set $\mathfrak{N}$ of nilpotent elements of the Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ forms a closed irreducible subset of $\mathfrak{g}$. Moreover if G is simply connected and the characteristic of K is either zero or a good prime for G there is an isomorphism of varieties $\phi : \mathfrak{U} \to \mathfrak{N}$ which is consistent with the G-actions on $\mathfrak{U}$ and $\mathfrak{N}$. Thus we have a bijection between the unipotent conjugacy classes in G and the nilpotent orbits in $\mathfrak{g}$ under the adjoint G-action under this assumption on the characteristic.

We shall in this chapter describe the unipotent classes in G and the nilpotent orbits in $\mathfrak{g}$ when the characteristic of K is either 0 or a prime p sufficiently large. To be precise we assume that the characteristic is either 0 or $p > 3(h - 1)$ where h is the Coxeter number of G. The key idea of the classification is to use a theorem of Jacobson and Morozov, which relates orbits of nilpotent elements in $\mathfrak{g}$ to G-orbits of 3-dimensional subalgebras of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(K)$. The restriction on the characteristic is imposed in order to be able to make use of the Jacobson–Morozov theorem in the characteristic p situation.

The G-orbits of subalgebras of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2)$ were first determined by Dynkin [2] in the characteristic 0 case. These were related to the nilpotent orbits in $\mathfrak{g}$ by Kostant [1] using the Jacobson–Morozov theorem, again in the characteristic 0 case. An exposition of this theory in characteristic 0 can be found in Bourbaki [3], chapter VIII, §11. An exposition of the theory valid also in characteristic p when p is sufficiently large was given by Springer and

Steinberg in Borel *et al.* [1], part E. A particularly simple description of the unipotent classes when the characteristic is 0 or p sufficiently large was obtained by Bala and Carter [1]. This makes use of a theorem of Richardson to relate the unipotent classes to the parabolic subgroups of G. The proof given by Bala and Carter has since been shortened by a result of Jantzen (cf. 5.7.6). In this chapter we shall follow Jantzen's modification of the Bala–Carter argument. We also give a formula for the dimension of the centralizer of a unipotent element, which was proved by Steinberg [20] using the Bala–Carter theorem.

Finally we discuss the situation when the characteristic is $p \leqslant 3(h-1)$, but without giving proofs. It was shown by Pommerening [1], [2] that the Bala–Carter classification remains valid whenever p is a good prime for G. Finally, if p is a bad prime for G, the classification may be different. There are in fact two classification problems in this case, since the unipotent classes need not be in bijective correspondence with the nilpotent orbits when p is a bad prime for G. We conclude by describing the unipotent classes and the nilpotent orbits in this case.

One general result from algebraic groups which will be useful in this chapter is a theorem of Rosenlicht which asserts that in a unipotent group each conjugacy class is closed. A proof of this fact can be found in Steinberg [18], p. 35.

We begin in section 5.1 by giving some properties of regular unipotent elements, both in the algebraic group G and in the finite group G^F. In section 5.2 we prove Richardson's dense orbit theorem about the action of a parabolic subgroup of G on its unipotent radical, and also the analogous result for the Lie algebra $\mathfrak{g}$. In section 5.3 we prove the Jacobson–Morozov theorem that every nonzero nilpotent element of $\mathfrak{g}$ lies in a subalgebra isomorphic to $\mathfrak{sl}_2(K)$. In section 5.4 we discuss the representation theory of $\mathfrak{sl}_2(K)$ and in section 5.5 show there is a bijection between nonzero nilpotent orbits in $\mathfrak{g}$ and G-orbits of subalgebras isomorphic to $\mathfrak{sl}_2(K)$. In section 5.6 we show how to associate with each nilpotent orbit a Dynkin diagram with weights 0, 1 or 2 attached to each node. In sections 5.7 and 5.8 we introduce the concept of distinguished nilpotent elements and distinguished parabolic subgroups and in section 5.9 we prove the main classification theorem. In section 5.10 we derive some results on dimension and in section 5.11 we discuss the situation over small characteristic.

5.1 REGULAR UNIPOTENT ELEMENTS

Proposition 5.1.1. *Let G be a connected reductive group and Z be the centre of G. Then the natural homomorphism $G \to G/Z$ restricts to a bijective morphism from the unipotent variety of G to that of G/Z. It also induces a bijection between the unipotent classes of G and the unipotent classes of G/Z.*

Proof. Let $g \in G$ and $\bar{g} = Zg \in G/Z$. If g is unipotent so is $\bar{g}$. Thus we have a map from unipotent elements of G to unipotent elements of G/Z.

We show this map is injective. Suppose u_1, u_2 are unipotent elements of G such that $\bar{u}_1 = \bar{u}_2$. Then $u_2 \in Zu_1$ and $u_2 = zu_1$ for some $z \in Z$. Now z is semisimple and commutes with u_1. By the uniqueness of the Jordan decomposition we have $z = 1$ and $u_1 = u_2$.

We next show that the map is surjective. Let g be an element of G for which $\bar{g}$ is unipotent. Let $g = su = us$ be the Jordan decomposition of g. Then $\bar{g} = \bar{s}\bar{u} = \bar{u}\bar{s}$ is the Jordan decomposition of $\bar{g}$. Thus $\bar{s} = 1$ and $\bar{g} = \bar{u}$.

Thus we have a bijective morphism from the variety of unipotent elements of G to the variety of unipotent elements of G/Z. Moreover if u_1, u_2 are conjugate unipotent elements of G then $\bar{u}_1, \bar{u}_2$ are clearly conjugate unipotent elements of G/Z. Thus we obtain a bijection between the unipotent classes in G and the unipotent classes in G/Z. ■

It is therefore sufficient to know the unipotent classes for connected reductive groups G with $Z = 1$. Such groups G are semisimple. Moreover if G is any semisimple group with $Z = 1$ there is a bijective homomorphism from G to the corresponding semisimple group of adjoint type. This in turn is a direct product of simple groups of adjoint type. A knowledge of the unipotent classes for simple groups will thus give a knowledge of the unipotent classes for connected reductive groups.

Proposition 5.1.2. *Let G be a connected reductive group. Then regular unipotent elements exist in G and any two are conjugate. They form a dense open subset of* $\mathfrak{U}$.

Proof. We recall from section 1.15 that the dimension of the unipotent variety $\mathfrak{U}$ is given by

$$\dim \mathfrak{U} = \dim G - \operatorname{rank} G.$$

Since G has only finitely many unipotent classes (Borel *et al.* [1], p. 185, Lusztig [6]) there will be at least one class C with $\dim \bar{C} = \dim \mathfrak{U}$.† For $x \in C$ we have

$$\dim C_G(x) = \dim G - \dim \bar{C} = \operatorname{rank} G$$

and so x is regular unipotent by section 1.14.

Now $\mathfrak{U}$ is irreducible and so $\bar{C} = \mathfrak{U}$. Also C is open in $\bar{C}$ by section 1.5 and so $\mathfrak{U} - C$ is a proper closed subset of $\mathfrak{U}$. Thus any unipotent class $C' \neq C$ satisfies $\dim \bar{C}' < \dim \bar{C}$ and so cannot be regular. Thus C is the only regular unipotent class.

Proposition 5.1.3. *Let G be a connected reductive group and* $u \in G$ *be unipotent. Then the following conditions on u are equivalent:*

(i) *u is regular.*
(ii) *u lies in a unique Borel subgroup of G.*
(iii) *u is conjugate to an element of the form* $\prod_{\alpha \in \Phi^+} x_\alpha(\lambda_\alpha)$ *with* $\lambda_{\alpha_i} \neq 0$ *for all* $\alpha_i \in \Delta$.

† It is convenient to use the fact that G has only finitely many unipotent classes. However the proof of this fact is highly nontrivial, particularly if the characteristic is a bad prime for G. A proof of the existence and conjugacy of regular unipotent elements which does not use the finiteness of the number of unipotent classes can be found in Steinberg [14].

Proof. (ii) $\Rightarrow$ (iii) Let $B = UT$ be a Borel subgroup and $U = \prod_{\alpha\in\Phi^+} X_\alpha$. We may assume that u lies in U. Thus $u = \prod_{\alpha\in\Phi^+} x_\alpha(\lambda_\alpha)$ and u lies in no Borel subgroup other than B. Suppose if possible that $\lambda_{\alpha_i} = 0$ for some $\alpha_i \in \Delta$. Then u lies in $\prod_{\alpha \neq \alpha_i}^{\alpha\in\Phi^+} X_\alpha = U_i$. Now U_i is normalized by $\langle X_{\alpha_i}, X_{-\alpha_i}\rangle$ by 2.6.4. Thus for all $x \in X_{-\alpha_i}$ we have $u^x \in U_i$. Hence $u \in {}^xB$. But $B = N(B)$ and $B \cap X_{-\alpha_i} = 1$. Thus the Borel subgroups ${}^xB$ for $x \in X_{-\alpha_i}$ are all distinct. Thus u lies in infinitely many Borel subgroups and we have a contradiction.

(iii) $\Rightarrow$ (i) The set of unipotent elements of the type defined in (iii) is dense in $\mathfrak{U}$. However the set of regular unipotent elements forms an open dense subset of $\mathfrak{U}$, by 5.1.2. Thus these two sets must intersect. There is therefore a regular unipotent element of the form

$$u' = \prod_{\alpha\in\Phi^+} x_\alpha(\mu_\alpha) \qquad \mu_\alpha \neq 0 \text{ for } \alpha \in \Delta.$$

Let $u = \prod_{\alpha\in\Phi^+} x_\alpha(\lambda_\alpha)$ have the property that $\lambda_\alpha \neq 0$ for all $\alpha \in \Delta$. We must show that u is regular. It will be sufficient to show that u is conjugate to u'. Now there exists $t \in T$ such that

$$tut^{-1} = \prod_{\alpha\in\Phi^+} x_\alpha(\nu_\alpha) \qquad \text{with } \nu_\alpha = \mu_\alpha \text{ for all } \alpha \in \Delta.$$

We have seen in section 2.9 that this is true for a semisimple group G_{ad} of adjoint type, and it follows by considering the surjective homomorphism $G \to G_{ad}$ that the same holds in the given connected reductive group G. Thus

$$tut^{-1}u'^{-1} \in U^* = \prod_{\alpha\in\Phi^+-\Delta} X_\alpha.$$

We recall from section 2.9 that U^* is a normal subgroup of U and that U/U^* is abelian.

Let A be the conjugacy class of u' in U. Since U/U^* is abelian we have $A \subseteq U^*u'$. By Rosenlicht's theorem A is a closed subset of U^*u'. We consider the dimension of A. We have

$$\dim A = \dim U - \dim C_U(u') \geqslant \dim U - \dim C_{G'}(u') = \dim U - l$$

since u' is regular. But

$$\dim U^*u' = \dim U^* = \dim U - l.$$

It follows that $\dim A = \dim U^*u'$. But U^* is irreducible, so U^*u' is irreducible. Hence $A = U^*u'$.

Now we have seen that $tut^{-1} \in U^*u'$ thus tut^{-1} is conjugate to u'. Thus u is conjugate to u'.

(i) $\Rightarrow$ (ii) We have seen that there is a regular unipotent element of the form $u = \prod_{\alpha\in\Phi^+} x_\alpha(\lambda_\alpha)$ with $\lambda_\alpha \neq 0$ for all $\alpha \in \Delta$. We have $u \in B$ and must show that u lies in no Borel subgroup other than B.

Suppose $u \in B^g$. Then $g = b\dot{w}u'$ for $b \in B$, $w \in W$, $u' \in U_w$ by 2.5.14. Hence

$B^g = B^{\dot{w}u'}$ and so $u'uu'^{-1} \in B^w$. Thus

$$u'uu'^{-1} \in U \cap B^w = U \cap U^w = \prod_{\substack{\alpha > 0 \\ w(\alpha) > 0}} X_\alpha.$$

However we have

$$u'uu'^{-1} = \prod_{\alpha > 0} x_\alpha(\mu_\alpha) \qquad \text{with } \mu_\alpha = \lambda_\alpha \text{ for all } \alpha \in \Delta.$$

Thus $\mu_\alpha \neq 0$ for all $\alpha \in \Delta$. Since $U = \prod_{\alpha > 0} X_\alpha$ with uniqueness we see that each $\alpha \in \Delta$ satisfies $w(\alpha) > 0$. Thus w transforms each positive root to a positive root, and so $w = 1$. Thus $g \in B$ and $B^g = B$.

Proposition 5.1.4. *Let G be a connected reductive group and u be a regular unipotent element of G. Suppose n is a positive integer not divisible by the characteristic of K. Then u^n is also regular unipotent.*

Proof. u is conjugate to an element of the form $\prod_{\alpha > 0} x_\alpha(\lambda_\alpha)$ with $\lambda_\alpha \neq 0$ for all $\alpha \in \Delta$, by 5.1.3. Also we have

$$\left(\prod_{\alpha > 0} x_\alpha(\lambda_\alpha)\right)^n = \prod_{\alpha > 0} x_\alpha(\mu_\alpha)$$

where $\mu_\alpha = n\lambda_\alpha$ if $\alpha \in \Delta$, since U/U^* is abelian. Now $n\lambda_\alpha \neq 0$ if $\alpha \in \Delta$, since $n \neq 0$, $\lambda_\alpha \neq 0$. Thus u^n is regular.

Proposition 5.1.5. *Let G be a connected reductive group and u be a regular unipotent element of G. Then every semisimple element in $C_G(u)$ lies in Z.*

Proof. Let $s \in C_G(u)$ be semisimple. Then $u \in C_G(s)$ and so $u \in C_G(s)^0$ by section 1.14. Let T be a maximal torus of G containing s. Then

$$C_G^0(s) = \langle T, X_\alpha, \alpha(s) = 1 \rangle \qquad \text{by 3.5.3.}$$

Let $U_1 = \prod_{\substack{\alpha > 0 \\ \alpha(s) = 1}} X_\alpha$. Then U_1 is a maximal unipotent subgroup of $C_G{}^0(s)$. Thus u is conjugate in $C_G^0(s)$ to an element $u_1 \in U_1$. Let

$$u_1 = \prod_{\substack{\alpha > 0 \\ \alpha(s) = 1}} x_\alpha(\lambda_\alpha) = \prod_{\alpha > 0} x_\alpha(\lambda_\alpha)$$

where $\lambda_\alpha = 0$ if $\alpha(s) \neq 1$. Since u_1 is regular unipotent it follows from 5.1.3 that $\lambda_\alpha \neq 0$ for all $\alpha \in \Delta$, and so $\alpha(s) = 1$ for all $\alpha \in \Delta$. Hence $\alpha(s) = 1$ for all roots α. Thus $C_G(s)^0 = G$ and so $s \in Z(G)$.

Proposition 5.1.6. *Let G be a connected reductive group and suppose the centre Z of G is connected. Suppose the characteristic of K is either 0 or a good prime for G. Let u be a regular unipotent element of G. Then $C_G(u)$ is connected.*

Proof. Let $g \in C_G(u)$. Let $g = g_s g_u = g_u g_s$ be the Jordan decomposition of g. Then g_s, g_u lie in $C_G(u)$. By 5.1.5 we have $g_s \in Z$ and hence $g_s \in C_G(u)^0$ since Z is connected. Also, since the characteristic is not a bad prime for G, every unipotent element of $C_G(u)$ lies in $C_G(u)^0$, by section 1.14. Thus $g_u \in C_G(u)^0$ and hence $g \in C_G(u)^0$. Thus $C_G(u)$ is connected. ∎

We now consider regular unipotent elements in the finite group G^F.

Proposition 5.1.7. *Let G be a connected reductive group and $F:G \to G$ be a Frobenius map. Then:*

(a) *G^F contains regular unipotent elements.*

(b) *If Z is connected and the characteristic of K is a good prime for G then any two regular unipotent elements of G^F are conjugate in G^F.*

Proof. (a) By 5.1.2 the regular unipotent elements of G form a single conjugacy class C. We have $F(C) = C$ since F transforms regular unipotent elements to regular unipotent elements. Let $u \in C$. Then $F(u) \in C$ so $F(u) = x^{-1}ux$ for some $x \in G$. By the Lang–Steinberg theorem we have $x = g^{-1}F(g)$ for some $g \in G$. It follows that gug^{-1} is a regular unipotent element of G^F.

(b) Suppose that Z is connected and that p is a good prime for G. Let u, u' be regular unipotent elements of G^F. Then $u' = gug^{-1}$ for some $g \in G$ by 5.1.2. Applying F we have $u' = F(g)uF(g^{-1})$. Hence $g^{-1}F(g) \in C_G(u)$. By 5.1.6 $C_G(u)$ is connected. Thus, by the Lang–Steinberg theorem, there exists $x \in C_G(u)$ such that $g^{-1}F(g) = x^{-1}F(x)$. Hence $gx^{-1} \in G^F$. Also $u' = gx^{-1}.u.(gx^{-1})^{-1}$, so u, u' are conjugate in G^F.

Proposition 5.1.8. *Let G be a connected reductive group and suppose the centre Z of G is connected. Let $B = UT$ be an F-stable Borel subgroup of G containing the F-stable maximal torus T. Suppose u, u' are regular unipotent elements of U^F. Then there exists $t \in T^F$ such that $tut^{-1} \in U^{*F}u'$. Moreover if $Z = 1$ the element t is unique.*

Proof. We first reduce to the case when $Z = 1$. Suppose we already have the result for G/Z. Then, since Zu, Zu' are F-stable regular unipotent elements of G/Z, there exists an element $Zt \in (T/Z)^F$ such that

$$ZtZu(Zt)^{-1} \in (ZU^*/Z)^F.Zu'.$$

Thus $tut^{-1}u'^{-1} \in ZU^* \cap U = U^*$. Since Z is connected the element t can be chosen in T^F by the Lang–Steinberg theorem. Thus $tut^{-1}u'^{-1} \in U^{*F}$ as required.

Thus we now assume that $Z = 1$. Let U_{reg} be the set of regular unipotent elements which lie in U. The proof of 5.1.3 shows that any two elements of U_{reg} are conjugate by an element of B. Moreover U_{reg} is a union of cosets of U^*. Let U_{reg}/U^* be this set of cosets. Then B acts by conjugation on U_{reg}/U^* and U acts trivially since U/U^* is abelian. Since $B = UT$ it follows that T acts transitively on U_{reg}/U^*.

We show next that T acts simply transitively on U_{reg}/U^*. Let $t \in T$ and

$u \in U_{\text{reg}}$ satisfy $tU^*ut^{-1} = U^*u$. Then u and tut^{-1} have the form

$$u = \prod_{\alpha > 0} x_\alpha(\lambda_\alpha) \qquad \lambda_\alpha \neq 0 \text{ if } \alpha \in \Delta$$

$$tut^{-1} = \prod_{\alpha > 0} x_\alpha(\alpha(t)\lambda_\alpha).$$

Thus $\alpha(t)\lambda_\alpha = \lambda_\alpha$ for all $\alpha \in \Delta$. Since $\lambda_\alpha \neq 0$ for $\alpha \in \Delta$ we have $\alpha(t) = 1$ for all $\alpha \in \Delta$. Thus $\alpha(t) = 1$ for all $\alpha \in \Phi$ and so $t \in C_T(u)$. However $C_T(u) = 1$ by 5.1.5 since $Z = 1$. Thus $t = 1$ and so T acts simply transitively on U_{reg}/U^*.

Given $u, u' \in U_{\text{reg}}{}^F$ there exists a unique $t \in T$ such that $tU^*ut^{-1} = U^*u'$. By applying F we have $F(t)U^*uF(t)^{-1} = U^*u'$ and so $F(t) = t$. Thus $t \in T^F$. Hence T^F acts transitively on $U_{\text{reg}}{}^F/U^*$. Thus there is an element $t \in T^F$ satisfying

$$tut^{-1} \in U^{*F}u'$$

and t is unique if $Z = 1$.

Proposition 5.1.9. *Let G be a connected reductive group and F be a Frobenius map. Then the number of regular unipotent elements of G^F is $|G^F|/|(Z^0)^F|q^l$ where Z^0 is the connected centre of G and l is the semisimple rank of G.*

Proof. By 5.1.3 each regular unipotent element of G lies in a unique Borel subgroup. If the unipotent element is F-stable the Borel subgroup will be also. Thus each regular unipotent element of G^F will lie in just one Borel subgroup B^F of G^F. Now the number of Borel subgroups of G^F is $|G^F:B^F|$ since B^F is self-normalizing in G^F. It is therefore sufficient to know the number of regular unipotent elements in a fixed Borel subgroup B^F. They will all lie in U^F where $U = R_u(B)$.

In order to determine the number of regular unipotent elements in U^F we consider the map $G \to G/Z$. By 5.1.1 this induces a bijection between unipotent elements of G and G/Z, hence between F-stable regular unipotent elements in G and G/Z. Moreover ZU^F/Z is the subgroup of G/Z corresponding to U^F in G.

Let T be an F-stable maximal torus contained in B. By 5.1.8 the torus $(T/Z)^F$ acts simply transitively on the cosets of regular unipotent elements in ZU^F/Z with respect to the subgroup ZU^*/Z. Thus the number of such cosets is $|(T/Z)^F|$. Each such coset contains $|ZU^{*F}/Z| = |U^{*F}|$ F-stable elements. Thus the total number of regular unipotent elements in ZU^F/Z is $|U^{*F}|.|(T/Z)^F|$, and this is also the number of regular unipotent elements in U^F. Thus the total number of regular unipotent elements in G^F is

$$|G^F:B^F|.|U^{*F}|.|(T/Z)^F|.$$

In order to evaluate $|(T/Z)^F|$ we first consider the group G/Z^0. Since Z^0 is connected we have

$$|(T/Z^0)^F| = |T^F|/|Z^{0F}|$$

and T/Z^0 is a maximally split torus of G/Z^0. Now G/Z is obtained from G/Z^0 by factoring out the finite centre Z/Z^0. T/Z and T/Z^0 are maximally split tori in G/Z and G/Z^0 respectively. However the results of chapter 3 show that the order of the group of F-stable elements in a maximally split torus is unaltered by factoring out a finite centre. (This order depends only on the characteristic polynomial of F_0 in its action on a vector space over $\mathbb{Q}$.) Thus we have

$$|(T/Z)^F| = |(T/Z^0)^F| = |T^F|/|(Z^0)^F|.$$

Hence the number of regular unipotent elements in G^F is

$$\frac{|G^F:B^F|.|U^{*F}||T^F|}{|Z^{0F}|} = \frac{|G^F|}{|U^F:U^{*F}||Z^{0F}|}.$$

Also U^F/U^{*F} has a direct product decomposition

$$U^F/U^{*F} \cong \prod_J X_J{}^F$$

by section 2.9, where J runs over the ρ-orbits on I and $X_J{}^F$ is isomorphic to the additive group of $F_{q^{|J|}}$. Thus

$$|U^F/U^{*F}| = \prod_J q^{|J|} = q^l$$

where l is the semisimple rank of G. Thus the number of regular unipotent elements in G^F is

$$\frac{|G^F|}{|Z^{0F}|q^l}.$$

5.2 RICHARDSON'S DENSE ORBIT THEOREMS

In order to discuss further the equivalent problems (when the characteristic is not a bad prime for G) of determining the unipotent conjugacy classes in G or the nilpotent orbits in $\mathfrak{g}$ we shall need a theorem of Richardson regarding the action of a parabolic subgroup on its unipotent radical. We shall prove two versions of this theorem, one for the group G and the other for the Lie algebra $\mathfrak{g}$. The proofs given here are due to Steinberg [18].

Theorem 5.2.1. *Let G be a connected reductive group and P_J be a parabolic subgroup of G. Let U_J be the unipotent radical of P_J. Let C be the unique unipotent conjugacy class in G such that $C \cap U_J$ is an open dense subset of U_J. Then $C \cap U_J$ is a single P_J-orbit under the P_J-action by conjugation on U_J.*

Note. The existence of a unique unipotent class C such that $C \cap U_J$ is open and dense in U_J follows from the fact that the number of unipotent conjugacy classes in G is finite. Let these classes be $C_1, \ldots, C_k$. Then $U_J \subset C_1 \cup \ldots \cup C_k$ so $U_J = (U_J \cap C_1) \cup \ldots \cup (U_J \cap C_k)$. Since U_J is irreducible at least one of the $U_J \cap C_i$ is dense in U_J. We shall then have $U_J \cap \bar{C}_i = U_J$. However C_i is open in $\bar{C}_i$ and so $U_J \cap C_i$ is open in $U_J \cap \bar{C}_i = U_J$. Thus $U_J \cap C_i$ is an open dense subset of U_J.

Furthermore, if $j \neq i$, $U_J \cap C_j \subseteq U_J - (U_J \cap C_i)$ and so $U_J \cap C_j$ cannot be dense in U_J. Thus there is a unique unipotent class C such that $C \cap U_J$ is open and dense in U_J.

If $P_J = B$, a Borel subgroup, then C is the class of regular unipotent elements.

Proof. We recall that $D_J = \{w \in W;\ w(\Phi_J{}^+) \subseteq \Phi^+\}$ is the set of distinguished coset representatives of W_J in W. We consider the subgroup $U_J \cap {}^wU_J$ of U where $w \in D_J$. We show first that

$$\dim(U_J \cap {}^wU_J) \leqslant \dim U_J - l(w).$$

Now we have

$$U_J = \prod_{\alpha \in \Phi^+ - \Phi_J} X_\alpha, \qquad U_J{}^w = \prod_{w(\alpha) \in \Phi^+ - \Phi_J} X_\alpha.$$

Thus

$$U_J \cap U_J{}^w = \prod_{\alpha \in S} X_\alpha$$

where $S = \{\alpha \in \Phi;\ \alpha \in \Phi^+,\ w(\alpha) \in \Phi^+,\ \alpha \notin \Phi_J,\ w(\alpha) \notin \Phi_J\}$. Since $w \in D_J$ all the positive roots made negative by w do not lie in Φ_J. Thus

$$\{\alpha \in \Phi;\ \alpha \in \Phi^+,\ w(\alpha) \in \Phi^-\} = \{\alpha \in \Phi;\ \alpha \in \Phi^+,\ w(\alpha) \in \Phi^-,\ \alpha \notin \Phi_J\}$$

has cardinality $l(w)$, and

$$\{\alpha \in \Phi;\ \alpha \in \Phi^+,\ w(\alpha) \in \Phi^+,\ \alpha \notin \Phi_J\}$$

has cardinality $\dim U_J - l(w)$. It follows that

$$|S| \leqslant \dim U_J - l(w)$$

and so

$$\dim(U_J \cap {}^wU_J) = \dim(U_J \cap U_J{}^w) \leqslant \dim U_J - l(w).$$

We next recall that $U_{w^{-1}} = \prod_{w^{-1}(\alpha) \in \Phi^-}^{\alpha \in \Phi^+} X_\alpha$ and define a map

$$U_{w^{-1}} \times (U_J \cap {}^wU_J) \xrightarrow{f} U_J$$

given by $(u, y) \xrightarrow{f} uyu^{-1} = {}^uy$. Note that $uyu^{-1} \in U_J$ since U_J is normal in U. The map f defined in this way is a morphism of varieties. We show that there is a dense open subset $U_{J,w}{}^*$ of U_J for which $f^{-1}(z)$ is finite for all $z \in U_{J,w}{}^*$.

Suppose first that the morphism f is dominant. Then by section 1.3 U_J contains a dense open subset $U_{J,w}{}^*$ such that, for all $z \in U_{J,w}{}^*$

$$\begin{aligned} \dim f^{-1}(z) &= \dim(U_{w^{-1}} \times (U_J \cap {}^wU_J)) - \dim U_J \\ &\leqslant \dim U_{w^{-1}} + (\dim U_J - l(w)) - \dim U_J \\ &= 0 \end{aligned}$$

Thus $\dim f^{-1}(z) = 0$ and so $f^{-1}(z)$ is finite.

So suppose that f is not dominant. Then we define $U_{J,w}^* = U_J - \overline{\mathrm{Im} f}$. This is a non-empty open subset of U_J, so is dense in U_J since U_J is irreducible. $f^{-1}(z)$ is empty for all $z \in U_{J,w}^*$.

Thus in either case we have a dense open subset $U_{J,w}^*$ of U_J such that $f^{-1}(z)$ is finite for all $z \in U_{J,w}^*$.

Let $U_J^* = \bigcap_{w \in D_J} U_{J,w}^*$. U_J^* is thus an open subset of U_J. It is non-empty, since the intersection of two non-empty open subsets of U_J is non-empty. Since any non-empty open subset of U_J is dense, U_J^* is a dense open subset of U_J.

We shall now show that each element of U_J^* lies in only finitely many conjugates of U_J. Let $z \in U_J^*$ and suppose $z \in {}^gU_J$. We have

$$G = \bigcup_w B\dot{w}U_w = \bigcup_w U_w\dot{w}^{-1}B = \bigcup_w U_{w^{-1}}\dot{w}B.$$

Moreover $w = d_Jw_J$ for $d_J \in D_J$, $w_J \in W_J$ and so

$$U_{w^{-1}} = U_{w_J^{-1}d_J^{-1}} = U_{d_J^{-1}}U_{w_J^{-1}}{}^{d_J^{-1}}$$

by 2.5.10. Thus

$$U_{w^{-1}}\dot{w}B = U_{d_J^{-1}}U_{w_J^{-1}}{}^{d_J^{-1}}\dot{d}_J\dot{w}_JB = U_{d_J^{-1}}\dot{d}_JU_{w_J^{-1}}\dot{w}_JB \subseteq U_{d_J^{-1}}\dot{d}_JP_J.$$

Thus each element $g \in G$ is expressible in the form $g = u\dot{w}p$ where $p \in P_J$, $w \in D_J$, $u \in U_{w^{-1}}$. Now we have

$$z \in {}^gU_J = {}^{u\dot{w}p}U_J = {}^{u\dot{w}}U_J.$$

Thus $z = {}^uy$ where $y \in {}^{\dot{w}}U_J$ and $u \in U_{w^{-1}}$. Note here that $y \in U_J \cap {}^wU_J$ since $z \in U_J$ and U_J is normal in U.

Now $z \in U_{J,w}^*$ and so $f^{-1}(z)$ is finite. Thus there are only finitely many pairs $(u, y) \in U_{w^{-1}} \times (U_J \cap {}^wU_J)$ with $uyu^{-1} = z$. So for each $w \in D_J$ there are only finitely many $u \in U_{w^{-1}}$ such that $z \in {}^{uw}U_J$. However there are only finitely many $w \in D_J$, and we have seen that each conjugate of U_J has the form ${}^{uw}U_J$ with $u \in U_{w^{-1}}$, $w \in D_J$. Thus z lies in only finitely many conjugates of U_J.

We next consider the subset ${}^GU_J = \bigcup_{g \in G} {}^gU_J$ of G. We shall show that ${}^GU_J$ is a closed irreducible subset of G of dimension $\dim G - \dim L_J$, where L_J is a Levi subgroup of P_J. In order to prove this we consider the subset S of $G/P_J \times G$ given by

$$S = \{(gP_J, x) \in G/P_J \times G;\ g^{-1}xg \in U_J\}.$$

Note that S is unambiguously defined since U_J is normal in P_J. Moreover S is a closed subset of $G/P_J \times G$. We have a map

$$\begin{aligned} G \times U_J &\to S \\ (g, y) &\to (gP_J, gyg^{-1}) \end{aligned}$$

which is a surjective morphism of varieties. Since G and U_J are both irreducible $G \times U_J$ will be irreducible and so S is irreducible also. Consider the map $S \to G/P_J$ given by $(gP_J, x) \to gP_J$. The fibres of this map are all conjugates of

U_J, so all have dimension dim U_J. It follows that

$$\dim S = \dim G/P_J + \dim U_J = \dim G - \dim L_J.$$

We now consider the map $S \to G$ given by $(gP_J, x) \to x$. The image of the map is ${}^GU_J$. Now G/P_J is a complete variety and so the projection map $G/P_J \times X \to X$ maps closed subsets into closed subsets for any algebraic variety X. In particular, considering the map $G/P_J \times G \to G$, since S is closed in $G/P_J \times G$ its image ${}^GU_J$ is closed in G. Since S is irreducible ${}^GU_J$ must be irreducible also.

Now if $x \in U_J{}^*$ x lies in only finitely many conjugates of U_J so the fibre of the map $S \to {}^GU_J$ corresponding to x is finite. The same applies if $x \in {}^g(U_J{}^*)$ for any $g \in G$. However $U_J{}^*$ is dense in U_J and so $\bigcup_{g \in G} {}^g(U_J{}^*)$ is dense in $\bigcup_{g \in G} ({}^gU_J) = {}^GU_J$. Thus there is a dense subset of ${}^GU_J$ on which the fibres of the morphism $S \to {}^GU_J$ are finite. However there is also a dense open subset of ${}^GU_J$ on which the fibres have dimension $\dim S - \dim {}^GU_J$. This dense open subset must intersect the above dense subset, and it follows that $\dim S = \dim {}^GU_J$. Hence

$$\dim {}^GU_J = \dim G - \dim L_J.$$

Now G has finitely many unipotent classes. Thus ${}^GU_J$ is a finite union of conjugacy classes of G. Let ${}^GU_J = \bigcup_i C_i$ where each C_i is a unipotent class in G. Since ${}^GU_J$ is closed we have ${}^GU_J = \bigcup_i \bar{C}_i$. Since ${}^GU_J$ is irreducible we have ${}^GU_J = \bar{C}_i$ for some i. However C_i is open in $\bar{C}_i$ and so C_i is an open dense subset of ${}^GU_J$. So no C_j with $j \neq i$ can satisfy $\bar{C}_j = {}^GU_J$. We write $C_i = C$. Thus C is the unique unipotent class of G such that $\bar{C} = {}^GU_J$.

C is also the unique unipotent class of G such that $C \cap U_J$ is open and dense in U_J. For $C \cap U_J$ is non-empty, as $C \cap U_J = \varnothing$ would imply $C \cap {}^GU_J = \varnothing$, and since C is open in $\bar{C}$, $C \cap U_J$ is open in U_J. Thus $C \cap U_J$ is a non-empty open subset of U_J, so is a dense open subset of U_J since U_J is irreducible.

Let $x \in C \cap U_J$ and let A be the conjugacy class of P_J containing x. Then we have

$$\begin{aligned}
\dim \bar{A} &= \dim P_J - \dim C_{P_J}(x) \\
&\geqslant \dim P_J - \dim C_G(x) \\
&= \dim P_J - \dim G + \dim \bar{C} \\
&= \dim P_J - \dim G + \dim {}^GU_J \\
&= \dim U_J.
\end{aligned}$$

But $\bar{A} \subseteq U_J$ and so $\dim \bar{A} = \dim U_J$. Since U_J is irreducible we have $\bar{A} = U_J$. Now A is open in $\bar{A}$ and so A is a dense open subset of U_J.

Now let $x, y \in C \cap U_J$. The conjugacy classes of P_J containing these two elements are both dense open subsets of U_J, so intersect one another. Thus x, y are conjugate in P_J. Hence $C \cap U_J$ is a single P_J-orbit. ■

The unipotent class C is called the Richardson class corresponding to P_J.

Corollary 5.2.2. *Suppose $x \in C \cap U_J$. Then $C_G(x)^0 \subseteq P_J$.*

Proof. We see from the proof of 5.2.1 that $\dim C_G(x) = \dim C_{P_J}(x)$. Thus $C_{P_J}(x)^0 \subseteq C_G(x)^0$ are connected groups of the same dimension, so must be equal. Thus $C_G(x)^0 \subseteq P_J$.

We now state and prove the analogous result for the adjoint action of G on its Lie algebra $\mathfrak{g}$. The proof is very similar to that of 5.2.1 but we include it because it provides a good introduction to the ideas on Lie algebras which we shall use subsequently in this chapter.

Theorem 5.2.3. *Let G be a connected reductive group and P_J be a parabolic subgroup of G. Let U_J be the unipotent radical of P_J and $\mathfrak{u}_J = \mathfrak{L}(U_J)$. Suppose the characteristic is not a bad prime for G. Let C be the unique nilpotent orbit in $\mathfrak{g} = \mathfrak{L}(G)$ under the adjoint G-action such that $C \cap \mathfrak{u}_J$ is an open dense subset of $\mathfrak{u}_J$. Then $C \cap \mathfrak{u}_J$ is a single P_J-orbit under the adjoint P_J-action on $\mathfrak{u}_J$.*

Proof. We define a map

$$\begin{array}{ccc} U_{w^{-1}} \times (\mathfrak{u}_J \cap \operatorname{Ad} \dot{w}.\mathfrak{u}_J) & \xrightarrow{f} & \mathfrak{u}_J \\ (u \quad , \quad y) & \to & \operatorname{Ad} u.y \end{array}$$

where $w \in D_J$. Note that $\operatorname{Ad} u.y \in \mathfrak{u}_J$ since $\operatorname{Ad} U.\mathfrak{u}_J \subseteq \mathfrak{u}_J$. f is a morphism of varieties. We show there is a dense open subset $\mathfrak{u}_{J,w}{}^*$ of $\mathfrak{u}_J$ for which $f^{-1}(z)$ is finite for all $z \in \mathfrak{u}_{J,w}{}^*$.

If f is dominant $\mathfrak{u}_J$ contains a dense open subset $\mathfrak{u}_{J,w}{}^*$ such that, for all $z \in \mathfrak{u}_{J,w}{}^*$,

$$\begin{aligned} \dim f^{-1}(z) &= \dim(U_{w^{-1}} \times (\mathfrak{u}_J \cap \operatorname{Ad} \dot{w}.\mathfrak{u}_J)) - \dim \mathfrak{u}_J \\ &\leqslant \dim U_{w^{-1}} + (\dim U_J - l(w)) - \dim U_J \leqslant 0 \end{aligned}$$

as in 5.2.1. Thus $f^{-1}(z)$ is finite.

If f is not dominant we define $\mathfrak{u}_{J,w}{}^* = \mathfrak{u}_J - \overline{\operatorname{Im} f}$. This is a dense open subset of $\mathfrak{u}_J$ and $f^{-1}(z)$ is empty for all $z \in \mathfrak{u}_{J,w}{}^*$.

Thus in either case we have a non-empty open subset $\mathfrak{u}_{J,w}{}^*$ of $\mathfrak{u}_J$ for which $f^{-1}(z)$ is finite for all $z \in \mathfrak{u}_{J,w}{}^*$. Let $\mathfrak{u}_J{}^* = \bigcap_{w \in D_J} \mathfrak{u}_{J,w}{}^*$. We show that every element of $\mathfrak{u}_J{}^*$ lies in only finitely many G-conjugates of $\mathfrak{u}_J$.

Let $z \in \mathfrak{u}_J{}^*$ and suppose $z \in \operatorname{Ad} g.\mathfrak{u}_J$. Then $g = u\dot{w}p$ with $p \in P_J$, $w \in D_J$, $u \in U_{w^{-1}}$ as in 5.2.1. Thus $z \in \operatorname{Ad}(u\dot{w}).\mathfrak{u}_J$. Thus $z = \operatorname{Ad} u.y$ where $y \in \mathfrak{u}_J \cap \operatorname{Ad} \dot{w}.\mathfrak{u}_J$, $u \in U_{w^{-1}}$. Since $z \in \mathfrak{u}_J{}^*$ there are only finitely many pairs (u, y) with $\operatorname{Ad} u.y = z$. So, given $w \in D_J$, there are only finitely many $u \in U_{w^{-1}}$ with $z \in \operatorname{Ad}(u\dot{w}).\mathfrak{u}_J$. Since D_J is finite it follows that z lies in only finitely many G-conjugates of $\mathfrak{u}_J$.

We now consider the subset $\operatorname{Ad} G.\mathfrak{u}_J$ of $\mathfrak{g}$ and show that it is a closed irreducible subset of $\mathfrak{g}$ of dimension $\dim G - \dim L_J$. Let S be the subset of $G/P_J \times \mathfrak{g}$ given by

$$S = \{(gP_J, x) \in G/P_J \times \mathfrak{g};\ x \in \operatorname{Ad} g.\mathfrak{u}_J\}.$$

S is a closed subset of $G/P_J \times \mathfrak{g}$. The map $G \times \mathfrak{u}_J \to S$ given by $(g, y) \to (gP_J, \operatorname{Ad} g.y)$ is a surjective morphism. Since G, $\mathfrak{u}_J$ are irreducible $G \times \mathfrak{u}_J$ is irreducible and hence S is irreducible. Consider the projection $S \to G/P_J$. The fibres of this projection are all G-conjugates of $\mathfrak{u}_J$ so all have dimension $\dim \mathfrak{u}_J$. Thus

$$\dim S = \dim G/P_J + \dim U_J = \dim G - \dim L_J.$$

Now consider the projection $S \to \mathfrak{g}$ given by $(gP_J, x) \to x$. The image of this map is $\operatorname{Ad} G.\mathfrak{u}_J$. Since S is closed in $G/P_J \times \mathfrak{g}$ and G/P_J is complete $\operatorname{Ad} G.\mathfrak{u}_J$ is closed in $\mathfrak{g}$. Since S is irreducible $\operatorname{Ad} G.\mathfrak{u}_J$ must be irreducible also. We wish to determine its dimension.

If $x \in \mathfrak{u}_J^*$ the fibre above x is finite and the same applies if $x \in \operatorname{Ad} g.\mathfrak{u}_J^*$ for $g \in G$. However $\mathfrak{u}_J^*$ is dense in $\mathfrak{u}_J$ so $\bigcup_g \operatorname{Ad} g.\mathfrak{u}_J^*$ is dense in $\operatorname{Ad} G.\mathfrak{u}_J$. However there is a dense open subset of $\operatorname{Ad} G.\mathfrak{u}_J$ on which the fibres have dimension $\dim S - \dim \operatorname{Ad} G.\mathfrak{u}_J$. Hence

$$\dim \operatorname{Ad} G.\mathfrak{u}_J = \dim S = \dim G - \dim L_J.$$

Now there are only finitely many nilpotent G-orbits on $\mathfrak{g}$, by section 1.15. Thus $\operatorname{Ad} G.\mathfrak{u}_J$ is a finite union of G-orbits. Let $\operatorname{Ad} G.\mathfrak{u}_J = \bigcup_i C_i$ where C_i are G-orbits. Then $\operatorname{Ad} G.\mathfrak{u}_J = \bigcup_i \bar{C}_i$ and there exists $C = C_i$ with $\operatorname{Ad} G.\mathfrak{u}_J = \bar{C}$ since $\operatorname{Ad} G.\mathfrak{u}_J$ is irreducible. Since C is open in $\bar{C}$ we see that C is open and dense in $\operatorname{Ad} G.\mathfrak{u}_J$.

Now consider $C \cap \mathfrak{u}_J$. This is non-empty since $C \cap \mathfrak{u}_J = \varnothing$ would imply $C \cap \operatorname{Ad} G.\mathfrak{u}_J = \varnothing$. Let $x \in C \cap \mathfrak{u}_J$. Then we have

$$\begin{aligned}
\dim \overline{\operatorname{Ad} P_J.x} &= \dim P_J - \dim C_{P_J}(x) \\
&\geqslant \dim P_J - \dim C_G(x) \\
&= \dim P_J - \dim G + \dim \bar{C} \\
&= \dim P_J - \dim G + \dim G - \dim L_J \\
&= \dim U_J = \dim \mathfrak{u}_J.
\end{aligned}$$

But $\overline{\operatorname{Ad} P_J.x} \subseteq \mathfrak{u}_J$ and so we have $\overline{\operatorname{Ad} P_J.x} = \mathfrak{u}_J$ for all $x \in C \cap \mathfrak{u}_J$. Since $\operatorname{Ad} P_J.x$ is open in $\overline{\operatorname{Ad} P_J.x}$ we see that $\operatorname{Ad} P_J.x$ is an open dense subset of $\mathfrak{u}_J$. If y is also in $C \cap \mathfrak{u}_J$ then $\operatorname{Ad} P_J.x$ and $\operatorname{Ad} P_J.y$ will intersect, so that x, y are in the same P_J-orbit. Thus $C \cap \mathfrak{u}_J$ is a single P_J-orbit.

Finally $C \cap \mathfrak{u}_J = \operatorname{Ad} P_J.x$ is an open dense subset of $\mathfrak{u}_J$ and so C is the nilpotent orbit defined in the theorem. ■

The nilpotent orbit C is called the Richardson orbit in $\mathfrak{g}$ corresponding to P_J.

Corollary 5.2.4. *Suppose $x \in C \cap \mathfrak{u}_J$. Then $C_G(x)^0 \subseteq P_J$.*

Proof. This follows from the fact that $\dim C_G(x) = \dim C_{P_J}(x)$.

Corollary 5.2.5. *Suppose G is simple and the characteristic is either* 0 *or a very good prime for G. Let $x \in C \cap \mathfrak{u}_J$. Then $[\mathfrak{p}_J\, x] = \mathfrak{u}_J$ where $\mathfrak{p}_J = \mathfrak{L}(P_J)$.*

Proof. We certainly have $[\mathfrak{p}_J\, x] \subseteq \mathfrak{u}_J$. Thus $[\mathfrak{p}_J\, x]$ is a subspace of $\mathfrak{u}_J$. It will therefore be sufficient to show $\dim [\mathfrak{p}_J\, x] = \dim \mathfrak{u}_J$. We have

$$\dim \overline{\mathrm{Ad}\, P_J . x} = \dim P_J - \dim C_{P_J}(x)$$

$$\dim \overline{\mathrm{ad}\, \mathfrak{p}_J . x} = \dim \mathfrak{p}_J - \dim C_{\mathfrak{p}_J}(x).$$

Also $L(C_{P_J}(x)) = C_{\mathfrak{p}_J}(x)$ by section 1.14. Thus

$$\dim \mathfrak{u}_J = \dim \overline{\mathrm{Ad}\, P_J . x} = \dim \overline{[\mathfrak{p}_J\, x]} = \dim [\mathfrak{p}_J\, x].$$

The result follows.

5.3 THE JACOBSON–MOROZOV THEOREM

In discussing the unipotent conjugacy classes of a simple algebraic group G and the nilpotent orbits of its Lie algebra $\mathfrak{g}$ we shall concentrate on the nilpotent orbits in $\mathfrak{g}$. This is because the fact that $\mathfrak{g}$ is a vector space makes it easier to work in this latter situation. The Jacobson–Morozov theorem shows that under suitable conditions a nonzero nilpotent element e in $\mathfrak{g}$ lies in a 3-dimensional subalgebra of $\mathfrak{g}$ with basis $\langle e, h, f\rangle$ where

$$[he] = 2e \quad [hf] = -2f \qquad [ef] = h.$$

Such a subalgebra is isomorphic to the Lie algebra $\mathfrak{sl}_2$ of 2×2 matrices of trace 0. For the elements

$$e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

form a basis for $\mathfrak{sl}_2$ and satisfy the above relations.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an arbitrary field k and let M be a finite-dimensional $k\mathfrak{g}$-module. Let ρ be the representation of $\mathfrak{g}$ afforded by M. The map $\mathfrak{g} \times \mathfrak{g} \to k$ given by

$$(x, y) \to \mathrm{trace}_M(\rho(x)\rho(y))$$

is called the trace-form of M. We write

$$\langle x, y\rangle = \mathrm{trace}_M(\rho(x)\rho(y)).$$

The trace-form is said to be non-degenerate if $\langle x, y\rangle = 0$ for all $y \in M$ implies $x = 0$.

Proposition 5.3.1. *Let x be a finite-dimensional Lie algebra over a field k and let M be a finite-dimensional $k\mathfrak{g}$-module. Let e be an element of $\mathfrak{g}$ satisfying* $(\mathrm{ad}\, e)^m = 0$ *where $m \leqslant p - 2$ if k has characteristic p. (There is no restriction on m if k has characteristic* 0.) *Suppose that the trace-form on M is nondegenerate and* *that $\rho(e)$ is a nilpotent endomorphism of M. Then there exist elements $h, f \in \mathfrak{g}$ such that* $[he] = 2e$, $[hf] = -2f$, $[ef] = h$.

Proof. We have a homomorphism $\rho : \mathfrak{g} \to [\text{End } M]$ where $[\text{End } M]$ is the algebra of endomorphisms of M under Lie multiplication. The image $\rho(\mathfrak{g})$ is a Lie subalgebra of $[\text{End } M]$ isomorphic to $\mathfrak{g}$, since the trace-form on M is nondegenerate. Let $\langle \alpha, \beta \rangle = \text{trace}_M(\alpha\beta)$ for $\alpha, \beta \in \text{End } M$. Let

$$\rho(\mathfrak{g})^{\perp} = \{\alpha \in \text{End } M;\ \langle \alpha, \rho(x) \rangle = 0 \quad \text{for all } x \in \mathfrak{g}\}.$$

Then we have

$$\dim(\rho(\mathfrak{g})^{\perp}) = \dim(\text{End } M) - \dim \rho(\mathfrak{g})$$

since $\langle \alpha, \beta \rangle = 0$ for all $\beta \in \text{End } M$ implies $\alpha = 0$. Also we have $\rho(\mathfrak{g}) \cap \rho(\mathfrak{g})^{\perp} = 0$ since the trace-form of $\mathfrak{g}$ on M is nondegenerate. It follows that End M has a direct decomposition

$$\text{End } M = \rho(\mathfrak{g}) \oplus \rho(\mathfrak{g})^{\perp}.$$

We shall show that $[\rho(\mathfrak{g}), \rho(\mathfrak{g})^{\perp}] \subseteq \rho(\mathfrak{g})^{\perp}$. Let $x, y \in \mathfrak{g}$ and $a \in \rho(\mathfrak{g})^{\perp}$. Consider the element $\langle [\rho(x), a], \rho(y) \rangle$ of k. We have

$$\begin{aligned}
\langle [\rho(x), a], \rho(y) \rangle &= \text{trace}(\rho(x)a - a\rho(x))\rho(y) \\
&= \text{trace } \rho(x)a\rho(y) - \text{trace } a\rho(x)\rho(y) \\
&= \text{trace } a\rho(y)\rho(x) - \text{trace } a\rho(x)\rho(y) \\
&= \text{trace } a(\rho(y)\rho(x) - \rho(x)\rho(y)) \\
&= \text{trace}(a\rho[yx]) = 0.
\end{aligned}$$

Hence $[\rho(\mathfrak{g}), \rho(\mathfrak{g})^{\perp}] \subseteq \rho(\mathfrak{g})^{\perp}$ as required.

Now $\rho(e)$ is nilpotent, so there is a basis of M with respect to which $\rho(e)$ is a diagonal sum of Jordan blocks. Let $e_0 = \rho(e)$. Then we can find $h_0, f_0 \in \text{End } M$ such that

$$[h_0 e_0] = 2e_0 \quad [h_0 f_0] = -2f_0 \quad [e_0 f_0] = h_0.$$

We choose for each Jordan block of e_0 of size k

$$\begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}_k$$

a diagonal matrix with (i, i)-coefficient $2i - k - 1$

$$\begin{pmatrix} 1-k & & & \\ & 3-k & & \\ & & \ddots & \\ & & & k-1 \end{pmatrix}$$

and a superdiagonal matrix with $(j, j+1)$-coefficient $j(k-j)$

$$\begin{pmatrix} 0 & 1.(k-1) & & & \\ & 0 & 2(k-2) & & \\ & & 0 & \ddots & \\ & & & \ddots & (k-1).1 \\ & & & & 0 \end{pmatrix}$$

h_0 is the diagonal sum of the above diagonal matrices and f_0 is the diagonal sum of the above superdiagonal matrices. e_0, h_0, f_0 then satisfy the required relations.

We wish, however, to find elements h, f in $\mathfrak{g}$ rather than in End M. Now there exist unique elements $h, z \in \mathfrak{g}$ such that

$$\rho(h) \equiv h_0 \bmod \rho(\mathfrak{g})^{\perp}$$

$$\rho(z) \equiv f_0 \bmod \rho(\mathfrak{g})^{\perp}$$

since End $M = \rho(\mathfrak{g}) \oplus \rho(\mathfrak{g})^{\perp}$ and $\rho(\mathfrak{g}) \cong \mathfrak{g}$. We consider the relations between e, h, z. We have

$$\rho[he] = [\rho(h), \rho(e)] = [h_0 + a, e_0] \text{ with } a \in \rho(\mathfrak{g})^{\perp}.$$

Thus

$$\rho[he] \equiv [h_0, e_0] \bmod \rho(\mathfrak{g})^{\perp}$$

since $[\rho(\mathfrak{g}), \rho(\mathfrak{g})^{\perp}] \subseteq \rho(\mathfrak{g})^{\perp}$. Hence

$$\rho[he] \equiv 2e_0 = \rho(2e) \bmod \rho(\mathfrak{g})^{\perp}$$

and it follows that $\rho[he] = \rho(2e)$, and so $[he] = 2e$.

We also have

$$\rho[ez] = [\rho(e), \rho(z)] = [e_0, f_0 + a] \text{ with } a \in \rho(\mathfrak{g})^{\perp}.$$

Thus

$$\rho[ez] \equiv [e_0, f_0] \bmod \rho(\mathfrak{g})^{\perp}$$

$$\rho[e, z] \equiv h_0 \equiv \rho(h) \bmod \rho(\mathfrak{g})^{\perp}.$$

It follows that $\rho[ez] = \rho(h)$ and so $[ez] = h$. In particular we see that $h \in [e\mathfrak{g}]$.

We now wish to choose $f \in \mathfrak{g}$ to satisfy the relations $[hf] = -2f$ and $[ef] = h$. Consider the map

$$\text{ad } e : \mathfrak{g} \to \mathfrak{g}$$

$$x \to [ex]$$

and let $\mathfrak{k} = \ker(\text{ad } e)$. If $a \in \mathfrak{k}$ then $[ha] \in \mathfrak{k}$ also. For

$$[e[ha]] = [[ae]h] + [[eh]a] = 0$$

since $[he] = 2e$. Thus ad h maps $\mathfrak{k}$ into itself. We consider the map

$$\text{ad } h + 2 : \mathfrak{k} \to \mathfrak{k}$$

and wish to show this map is nonsingular. Suppose this has been done. Let u be defined by $u = (\operatorname{ad} h + 2)z = [hz] + 2z$. We have

$$\begin{aligned}[eu] &= [e[hz]] + 2[ez] = [[ze]h] + [[eh]z] + 2h \\ &= -2[ez] + 2h = 0.\end{aligned}$$

Thus $u \in \mathfrak{k}$. Since $\operatorname{ad} h + 2 : \mathfrak{k} \to \mathfrak{k}$ is assumed nonsingular there exists $v \in \mathfrak{k}$ with $(\operatorname{ad} h + 2)v = u$. Let $f = z - v$. Then

$$\begin{aligned}[hf] &= [h, z - v] = [hz] - [hv] = [hz] - u + 2v \\ &= u - 2z - u + 2v = -2f \\ [ef] &= [e, z - v] = [ez] - [ev] = h.\end{aligned}$$

Thus f satisfies $[hf] = -2f$ and $[ef] = h$ and e, h, f satisfy all the required relations.

It remains to show that the map

$$\operatorname{ad} h + 2 : \mathfrak{k} \to \mathfrak{k}$$

is nonsingular. Let $\mathfrak{k}_i = \mathfrak{k} \cap (\operatorname{ad} e)^i \mathfrak{g}$. Then

$$\mathfrak{k} = \mathfrak{k}_0 \supseteq \mathfrak{k}_1 \supseteq \ldots \supseteq \mathfrak{k}_m = 0.$$

We shall show that $(i + 1)(\operatorname{ad} h - i)\mathfrak{k}_i \subseteq \mathfrak{k}_{i+1}$. Let $E = \operatorname{ad} e$, $Z = \operatorname{ad} z$, $H = \operatorname{ad} h$. Then we have $HE - EH = 2E$, $EZ - ZE = H$. We show that $E^{i+1}Z - ZE^{i+1} = (i + 1)(H - i)E^i$. This is true if $i = 0$. Assume inductively that $E^iZ - ZE^i = i(H - i + 1)E^{i-1}$. Then

$$\begin{aligned}E^{i+1}Z - ZE^{i+1} &= E(ZE^i + i(H - i + 1)E^{i-1}) - ZE^{i+1} \\ &= (ZE + H)E^i + i(HE - 2E)E^{i-1} - i(i - 1)E^i - ZE^{i+1} \\ &= ((i + 1)H - i(i + 1))E^i \\ &= (i + 1)(H - i)E^i.\end{aligned}$$

Now let $b \in \mathfrak{k}_i$. Thus $b = (\operatorname{ad} e)^i c$ for some $c \in \mathfrak{g}$ and $(\operatorname{ad} e)^{i+1}c = 0$. We have

$$\begin{aligned}(\operatorname{ad} e)^{i+1}[zc] &= (\operatorname{ad} e)^{i+1} \operatorname{ad} z\, c \\ &= ((\operatorname{ad} e)^{i+1} \operatorname{ad} z - \operatorname{ad} z(\operatorname{ad} e)^{i+1})c \\ &= (i + 1)(\operatorname{ad} h - i)(\operatorname{ad} e)^i c \qquad \text{as above} \\ &= (i + 1)(\operatorname{ad} h - i)b.\end{aligned}$$

Thus we see that $(i + 1)(\operatorname{ad} h - i)\mathfrak{k}_i \subseteq \mathfrak{k}_{i+1}$.

Now $i \leqslant m - 1$ and so $i + 1 \leqslant m \leqslant p - 2$ if k has characteristic p. Thus $i + 1 \neq 0$ in k. Hence $(\operatorname{ad} h - i)\mathfrak{k}_i \subseteq \mathfrak{k}_{i+1}$, and so

$$((\operatorname{ad} h + 2) - (i + 2))\mathfrak{k}_i \subseteq \mathfrak{k}_{i+1}.$$

Thus $\operatorname{ad} h + 2$ acts on $\mathfrak{k}_i/\mathfrak{k}_{i+1}$ as multiplication by $i + 2$. But $i + 2 \neq 0$ in k. It follows that $\operatorname{ad} h + 2$ is invertible on $\mathfrak{k}$, and the result is proved. ∎

We shall apply this result to the case when $\mathfrak{g}$ is the Lie algebra of a simple algebraic group G over an algebraically closed field K of characteristic 0 or a good prime for G. We recall from section 1.16 that for each simple type other than possibly A_l there exists a simple algebraic group G of this type and a rational representation ρ of G such that the trace-form of the representation $d\rho$ of $\mathfrak{g} = \mathfrak{L}(G)$ is nondegenerate.

Theorem 5.3.2 (Jacobson–Morozov) *Let G be a simple algebraic group over an algebraically closed field K of characteristic 0 or a good prime for G. Let $\mathfrak{g} = \mathfrak{L}(G)$ and e be a nilpotent element of $\mathfrak{g}$ with $(\operatorname{ad} e)^m = 0$ where $m \leqslant p-2$ if K has characteristic p. Then there exist elements $h, f \in \mathfrak{g}$ with $[he] = 2e$, $[hf] = -2f$, $[ef] = h$.*

Proof. If G is of type A_l then e is in the same G-orbit as an element in Jordan canonical form and h, f can be written down directly as in the proof of 5.3.1.

If G is not of type A_l we may apply section 1.16. There is then a simple group $\tilde{G}$ isogenous to G and rational representation ρ of $\tilde{G}$ such that the trace-form of $d\rho$ is non-degenerate. We have $\mathfrak{g} = \mathfrak{L}(G) \cong \mathfrak{L}(\tilde{G})$ so that $d\rho$ is a representation of $\mathfrak{g}$. Since e is a nilpotent element of $\mathfrak{g}$, $d\rho(e)$ is a nilpotent matrix. We may therefore apply 5.3.1 to show the existence of elements $h, f \in \mathfrak{g}$ satisfying

$$[he] = 2e \quad [hf] = -2f \quad [ef] = h.$$

5.4 REPRESENTATIONS OF $\mathfrak{sl}_2(k)$

The Jacobson–Morozov theorem shows that under suitable conditions every nonzero nilpotent element of the Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ lies in a 3-dimensional subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(k)$. In order to study the nilpotent elements of $\mathfrak{g}$ we can therefore make use of the representation theory of the Lie algebra $\mathfrak{sl}_2(k)$. Since $\mathfrak{sl}_2(k)$ has a basis e, h, f satisfying

$$[he] = 2e \quad [hf] = -2f \quad [ef] = h$$

the universal enveloping algebra $\mathfrak{U}$ of $\mathfrak{sl}_2(k)$ is the associative k-algebra with identity generated by elements E, H, F subject to the relations

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = H.$$

The representation theory of the Lie algebra $\mathfrak{sl}_2(k)$ is the same as that of the associative k-algebra $\mathfrak{U}$. We shall need some results on this representation theory due to Jacobson [6]. We first observe that $EH = (H - 21)E$ and $E^n H = (H - 2n1)E^n$ for $n \geqslant 1$.

Lemma 5.4.1. *Let $D:\mathfrak{U} \to \mathfrak{U}$ be defined by $D(u) = Fu - uF$. Then, for all $r \geqslant 1$,*

$$D^r(E^m) = \sum_{k=0}^{[\frac{1}{2}r]} (-1)^{r-k}\binom{m}{k}\binom{m-k}{r-2k} r!F^k \sum_{j=1}^{r-2k} (H - m1 + j1)E^{m-r+k}.$$

Proof. We first consider the case $r = 1$ and show that

$$D(E^m) = -m(H-(m-1)1)E^{m-1}.$$

This is true when $m = 1$ since $FE - EF = -H$. Assuming inductively that

$$D(E^{m-1}) = -(m-1)(H-(m-2)1)E^{m-2}$$

we have

$$\begin{aligned} D(E^m) &= D(E^{m-1})E + E^{m-1}D(E) \\ &= -(m-1)(H-(m-2)1)E^{m-1} - E^{m-1}H \\ &= -(m-1)(H-(m-2)1)E^{m-1} - (H-(2m-2)1)E^{m-1} \\ &= -m(H-(m-1)1)E^{m-1}. \end{aligned}$$

We now prove the required result by induction on r. We assume inductively that

$$\begin{aligned} D^{r-1}(E^m) = \sum_{k=0}^{[\frac{1}{2}(r-1)]} (-1)^{r-1-k}\binom{m}{k}\binom{m-k}{r-1-2k}(r-1)!F^k \\ \times \prod_{j=1}^{r-1-2k} (H - m1 + j1).E^{m-r+1+k}. \end{aligned}$$

Now we have, for $k \leqslant \frac{1}{2}r - 2$

$$\begin{aligned} D\prod_{j=1}^{r-1-2k}(H-(m-j)1) &= F\prod_{j=1}^{r-1-2k}(H-(m-j)1) - \prod_{j=1}^{r-1-2k}(H-(m-j)1)F \\ &= F\prod_{j=1}^{r-1-2k}(H-(m-j)1) \\ &\quad - F\prod_{j=1}^{r-1-2k}(H-21-(m-j)1) \\ &= F\left(\prod_{j=1}^{r-3-2k}(H-(m-j)1)\right).(r-1-2k) \\ &\quad \times (2H - 2m1 + (r-2-2k)1) \\ &= (r-1-2k)F.\left(\prod_{j=1}^{r-3-2k}(H-(m-j)1).\right) \\ &\quad \times (2H-(2m-r+2+2k)1). \end{aligned}$$

Using this formula for $2k \leqslant r - 4$, and checking separately terms for which $r - 3 \leqslant 2k \leqslant r - 1$ we have:

$$D^r(E^m) = \sum_{k=0}^{[\frac{1}{2}(r-1)]} (-1)^{r-1-k}\binom{m}{k}\binom{m-k}{r-1-2k}(r-1)!F^k$$
$$\times D\left(\prod_{j=1}^{r-1-2k} (H-(m-j)1).E^{m-r+1+k}\right)$$
$$= \sum_{k=0}^{[\frac{1}{2}(r-1)]} (-1)^{r-1-k}\binom{m}{k}\binom{m-k}{r-1-2k}(r-1)!F^{k+1}(r-1-2k)$$
$$\times\left(\prod_{j=1}^{r-3-2k} (H-(m-j)1)\right).(2H-(2m-r+2+2k)1)E^{m-r+1+k}$$
$$- \sum_{k=0}^{[\frac{1}{2}(r-1)]} (-1)^{r-1-k}\binom{m}{k}\binom{m-k}{r-1-2k}(r-1)!F^k$$
$$\times\left(\prod_{j=1}^{r-1-2k} (H-(m-j)1)\right).(m-r+1+k)$$
$$\times (H-(m-r+k)1)E^{m-r+k}$$
$$= \sum_{k=1}^{[\frac{1}{2}(r-1)]+1} (-1)^{r-k}\binom{m}{k-1}\binom{m-k+1}{r+1-2k}(r-1)!(r+1-2k)F^k$$
$$\times\left(\prod_{j=1}^{r-1-2k} (H-(m-j)1)\right).(2H-(2m-r+2k)1)E^{m-r+k}$$
$$+ \sum_{k=0}^{[\frac{1}{2}(r-1)]} (-1)^{r-k}\binom{m}{k}\binom{m-k}{r-1-2k}(r-1)!(m-r+1+k)F^k$$
$$\times\left(\prod_{j=1}^{r-1-2k} (H-(m-j)1)\right).(H-(m-r+k)1)E^{m-r+k}$$
$$= \sum_{k=0}^{[\frac{1}{2}r]} (-1)^{r-k}(r-1)!F^k \prod_{j=1}^{r-1-2k} (H-(m-j)1).X.E^{m-r+k}$$

where

$$X = \binom{m}{k-1}\binom{m-k+1}{r+1-2k}(r+1-2k)(2H-(2m-r+2k)1)$$
$$+\binom{m}{k}\binom{m-k}{r-1-2k}(m-r+1+k)(H-(m-r+k)1)$$
$$= r\binom{m}{k}\binom{m-k}{r-2k}(H-(m-r+2k)1).$$

(Observe that if r is odd the value $k = [\frac{1}{2}(r-1)] + 1$ gives no contribution to the sum since the factor $r + 1 - 2k$ is then zero. If r is even the second sum can be taken over the range $0 \leqslant k \leqslant [\frac{1}{2}r]$ since $k = \frac{1}{2}r$ gives no contribution, and the first sum can be taken to include $k = 0$.)

Thus we have

$$D^r(E^m) = \sum_{k=0}^{[\frac{1}{2}r]} (-1)^{r-k} r! \binom{m}{k}\binom{m-k}{r-2k} F^k . \prod_{j=1}^{r-2k} (H - (m-j)1) . E^{m-r+k}.$$

Lemma 5.4.2. *Suppose ρ is a representation of $\mathfrak{U}$ such that $\rho(E^m) = 0$ for some positive integer m satisfying $m \leqslant p - 1$ if k has characteristic p. (There is no restriction on m if k has characteristic 0.) Then*

$$\rho\left(\left(\prod_{j=1}^{2r-1} (H - (m-j)1)\right) E^{m-r}\right) = 0$$

for $r = 1, 2, \ldots, m$.

Proof. We first prove the result for $r = 1$. We have

$$D(E^m) = -m(H - (m-1)1)E^{m-1}$$

and $\rho(E^m) = 0$ implies $\rho(D(E^m)) = 0$. Since $m \neq 0$ in k this gives

$$\rho((H - (m-1)1)E^{m-1}) = 0.$$

We now assume the result inductively for $r = 1, 2, \ldots, s-1$ and prove it for $r = s$. By 5.4.1 we have

$$D^s(E^m) = \sum_{k=0}^{[\frac{1}{2}s]} (-1)^{s-k} \binom{m}{k}\binom{m-k}{s-2k} s! F^k . \prod_{j=1}^{s-2k} (H - (m-j)1) . E^{m-s+k}.$$

$\rho(E^m) = 0$ implies $\rho(D^s(E^m)) = 0$. Thus

$$\sum_{k=0}^{[\frac{1}{2}r]} (-1)^{s-k} \binom{m}{k}\binom{m-k}{s-2k} s! \rho\left(F^k . \prod_{j=1}^{s-2k} (H - (m-j)1) . E^{m-s+k}\right) = 0.$$

We multiply on the left by $\rho(\prod_{j=s+1}^{2s-1} (H - (m-j)1))$. Then

$$\sum_{k=0}^{[\frac{1}{2}s]} (-1)^{s-k} \binom{m}{k}\binom{m-k}{s-2k} s! \rho\left(\prod_{j=s+1}^{2s-1} (H - (m-j)1)F^k \right.$$
$$\left. \times \prod_{j=1}^{s-2k} (H - (m-j)1) . E^{m-s+k}\right) = 0.$$

This gives

$$\sum_{k=0}^{[\frac{1}{2}s]} (-1)^{s-k} \binom{m}{k}\binom{m-k}{s-2k} s! \rho\left(F^k \prod_{j=s+1}^{2s-1} (H - (m-j+2k)1) \right.$$
$$\left. \times \prod_{j=1}^{s-2k} (H - (m-j)1) . E^{m-s+k}\right) = 0$$

which is equivalent to

$$\sum_{k=0}^{[\frac{1}{2}s]} (-1)^{s-k}\binom{m}{k}\binom{m-k}{s-2k} s!\rho\left(F^k.\prod_{j=1}^{2s-2k-1}(H-(m-j)1).E^{m-s+k}\right)=0.$$

By induction we have

$$\rho\left(\prod_{j=1}^{2s-2k-1}(H-(m-j)1)E^{m-s+k}\right)=0$$

if $s-k=1,2,\ldots,s-1$, viz. $k=1,2,\ldots,s-1$. Thus all the terms in the above sum vanish except possibly the term with $k=0$. Thus this term must vanish also and we have

$$(-1)^s\binom{m}{s}s!\rho\left(\left(\prod_{j=1}^{2s-1}(H-(m-j)1)\right).E^{m-s}\right)=0.$$

Now $\binom{m}{s}\neq 0$ in k and, since $s\leqslant m$, $s!\neq 0$ in k. Thus

$$\rho\left(\left(\prod_{j=1}^{2s-1}(H-(m-j)1)\right).E^{m-s}\right)=0$$

and the result follows.

Corollary 5.4.3. *Suppose ρ is a representation of $\mathfrak{U}$ such that $\rho(E^m)=0$ for some positive integer m satisfying $m\leqslant p-1$ if k has characteristic p. Then*

$$\rho\left(\prod_{j=1}^{2m-1}(H-(m-j)1)\right)=0.$$

We now define a representation ρ_j of $\mathfrak{sl}_2(k)$. This is afforded by a module V_j of dimension j with basis $x_1, x_2, \ldots, x_j$ such that

$$ex_i = x_{i+1} \quad i=1,2,\ldots,j-1, ex_j=0$$
$$hx_i=(2i-j-1)x_i$$
$$fx_{i+1}=i(j-i)x_i \quad i=1,2,\ldots,j-1 \quad fx_1=0.$$

It can readily be checked that this is in fact a representation of $\mathfrak{sl}_2(k)$. The matrices representing e, h, f respectively are

$$\begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & & \\ & & & 1 & 0 \end{bmatrix}\begin{bmatrix} 1-j & & & \\ & 3-j & & \\ & & \ddots & \\ & & & j-1 \end{bmatrix}\begin{bmatrix} 0 & 1(j-1) & & \\ & 0 & 2(j-2) & \\ & & \ddots & \ddots \\ & & & (j-1).1 \\ & & & 0 \end{bmatrix}$$

Proposition 5.4.4. *The representation ρ_j of $\mathfrak{sl}_2(k)$ is irreducible provided k has characteristic 0 or k has characteristic p and $j \leqslant p$.*

Proof. Let $V = V_j$ and U be a nonzero submodule of V. Let $v \in U$ *with* $v \neq 0$. Let $v = \sum_{i=1}^{j} \lambda_i x_i$. There exists an integer m such that $v = \lambda_m x_m + \ldots + \lambda_j x_j$ with $\lambda_m \neq 0$. Consider the action of the enveloping algebra $\mathfrak{U}$ on V. We have $E^{j-m}v = \lambda_m x_j$. $v \in U$ implies that $E^{j-m}v \in U$ and since $\lambda_m \neq 0$ we have $x_j \in U$. We now consider elements of the form $F^i x_j$. Since $i(j-i) \neq 0$ in k for $i = 1, 2, \ldots, j-1$ we have $F^i x_j = \lambda x_{j-i}$ where $\lambda \neq 0$. Hence $x_{j-i} \in U$ for $i = 1, 2, \ldots, j-1$ and so $U = V$. ∎

We note that if k has characteristic p and $j > p$ the representation ρ_j will not be irreducible. In fact it will not even be completely reducible.

We observe that, under the hypotheses of 5.4.4, all the eigenvalues of h on V_j are distinct provided $p \neq 2$. We shall assume that $p \neq 2$ for the remainder of this section.

Let V be any finite-dimensional $\mathfrak{sl}_2(k)$-module. A vector $x \in V$ is called a maximal vector if $x \neq 0$, $hx = \lambda x$ for some $\lambda \in k$, and $ex = 0$. $x \in V$ is called a minimal vector if $x \neq 0$, $hx = \lambda x$ for some $\lambda \in k$, and $fx = 0$.

Proposition 5.4.5. *Let V be a finite-dimensional $\mathfrak{sl}_2(k)$-module affording a representation ρ. Suppose m is a positive integer such that $\rho(E^{m-1}) = 0$ and $\rho(F^{m-1}) = 0$. Suppose also that $m \leqslant p$ if k has characteristic p and that $p \neq 2$. Let x be a minimal vector in V. Then the submodule generated by x affords a representation ρ_j for some $j \leqslant m-1$.*

The same holds for any maximal vector in V.

Proof. Let x be a minimal vector. Then $x \neq 0$, $hx = \lambda x$, $fx = 0$. Let $x_i = E^{i-1}x$ for $i \geqslant 1$. There exists $j \geqslant 1$ such that $x_1, \ldots, x_j$ are nonzero and $x_{j+1} = 0$. We have $j \leqslant m-1$ since $E^{m-1}x = 0$. Now $Hx = \lambda x$ and so $Hx_i = (\lambda + 2i - 2)x_i$. But $\lambda, \lambda + 2, \ldots, \lambda + 2j - 2$ are distinct in k since char $k \neq 2$ and $j < p$ if char $k = p$. Thus the eigenvectors $x_1, x_2, \ldots, x_j$ of H are linearly independent.

Now $Fx_1 = 0$ and by induction we see that $Fx_{i+1} = -i(\lambda + i - 1)x_i$.

Let U be the subspace of V spanned by $x_1, \ldots, x_j$. Then U has dimension j and is a submodule of V, being invariant under E, H and F. It is the submodule generated by x. In order to obtain explicitly the effect of E, H, F on the basis vectors of U we must determine λ. We have

$$\mathrm{trace}_U H = \mathrm{trace}_U(EF - FE) = 0.$$

Thus $\sum_{i=1}^{j} (\lambda + 2i - 2) = 0$. Hence $j(\lambda - 2) + j(j+1) = 0$. Since $j \neq 0$ in k we have $\lambda = 1 - j$. It follows that

$$Ex_i = x_{i+1} \quad i = 1, 2, \ldots, j-1, \ Ex_j = 0$$

$$Hx_i = (2i - j - 1)x_i$$

$$Fx_{i+1} = i(j-i) \quad i = 1, 2, \ldots, j-1, \ Fx_1 = 0.$$

Thus U affords the representation ρ_j.

A similar argument applies if x is a maximal vector, interchanging the rôles of E and F.

Corollary 5.4.6. *Let V be a finite-dimensional irreducible $\mathfrak{sl}_2(k)$-module affording the representation ρ. Suppose m is a positive integer such that $\rho(E^{m-1}) = 0$ and $\rho(F^{m-1}) = 0$. Suppose that $m \leqslant p$ and $p \neq 2$ if k has characteristic p. Then $\rho = \rho_j$ for some $j \leqslant m - 1$.*

Proof. We apply 5.4.3 with m replaced by $m - 1$. This gives

$$\rho\left(\prod_{j=1}^{2m-3} (H - (m - 1 - j))\right) = 0.$$

Thus the minimum polynomial of $\rho(H)$ divides

$$\prod_{j=1}^{2m-3} (t - (m - 1 - j)) = (t - (m-2))(t - (m-3)) \ldots (t + (m-2)).$$

Now all the eigenvalues of $\rho(H)$ are zeros of the minimum polynomial. So all the eigenvalues of $\rho(H)$ lie in the prime subfield of k. Thus we can find an eigenvector x of H in V. We now define $x_i = F^{i-1}x$. Since $F^{m-1}x = 0$ there exists a positive integer k with $x_k \neq 0$, $x_{k+1} = 0$. x_k is then a minimal vector in V. We then see by 5.4.5 that $\rho = \rho_j$ for some $j \leqslant m - 1$.

Proposition 5.4.7. *Let V be a finite-dimensional $\mathfrak{sl}_2(k)$-module affording the representation ρ. Suppose there is a positive integer $m \geqslant 2$ such that $\rho(E^{m-1}) = 0$ and $\rho(F^{m-1}) = 0$. Suppose that $p \neq 2$ and $m \leqslant p$ if k has characteristic p. Then V is completely reducible.*

Proof. (Spaltenstein) Consider the composition factors of V. By 5.4.6 each of these affords one of the representations ρ_j for some $j \leqslant m - 1$. We prove the result by induction on the length $l(V)$ of a composition series of V. The result is trivial if $l(V) = 1$. Suppose we can prove it when $l(V) = 2$. Consider a module V with $l(V) = r > 2$. Then V has submodules V_1, V_2 with $V \supset V_1 \supset V_2 \supset 0$. By induction V_1 is completely reducible. Thus there exists a submodule V_3 with $V_1 = V_2 \oplus V_3$. V/V_3 is also completely reducible by induction. So there exists a submodule V_4 such that $V/V_3 = V_1/V_3 \oplus V_4/V_3$. But then $V = V_2 \oplus V_4$. V_2, V_4 are completely reducible by induction so V is also.

It is therefore sufficient to prove the result when $l(V) = 2$. Suppose this is so and that V is not completely reducible. Then V has a unique proper submodule

V_1. Let $\dim V = n$. Let the representation afforded by V_1 be ρ_j. Then the representation afforded by V/V_1 must be ρ_{n-j}.

We consider the kernel $\mathrm{Ker}_V E$. We have $\dim \mathrm{Ker}_{V_1} E = 1$ and $\dim \mathrm{Ker}_{V/V_1} E = 1$ and so $\dim \mathrm{Ker}_V E \leqslant 2$. It follows that $\rho(E)$ has at most two Jordan blocks. We now distinguish between two cases.

Case 1 Suppose $\rho(E)$ has just one Jordan block. Then $\rho(E^{n-1}) \neq 0$ and so $n < m$. Let $U_i = \mathrm{Ker}_V E^i$. Then we have

$$0 \subset U_1 \subset U_2 \subset \ldots \subset U_{n-1} \subset U_n = V$$

and $\dim U_i = i$ if $i \leqslant n$.

Now $\dim V_1 = j$ and $V_1 \subseteq U_j$ and so $V_1 = U_j$. Each subspace U_i is invariant under the action of H. Consider the action of H on U_i/U_{i-1}. Let the eigenvalue of H on U_i/U_{i-1} be λ_i. Since $HE - EH = 2E$ we shall have $\lambda_i = \lambda_{i+1} + 2$ for $1 \leqslant i < n$. We observe that these eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all distinct in k. This is clear if k has characteristic 0. If k has characteristic p and $\lambda_i = \lambda_{i'}$ in k then p divides $2(i - i')$. Since $p \neq 2$ and $n < p$ this implies that $i = i'$. Thus H has n distinct eigenvalues on V. Let L_{λ_i} be the eigenspace for H with eigenvalue λ_i. Then $\dim L_{\lambda_i} = 1$ and we have

$$V = L_{\lambda_1} \oplus L_{\lambda_2} \oplus \ldots \oplus L_{\lambda_n}.$$

Since $HE - EH = 2E$ we see that $E(L_{\lambda_i}) = L_{\lambda_{i-1}}$ for $1 < i \leqslant n$ and $E(L_{\lambda_1}) = 0$.

Now $U_j = V_1$ gives rise to the representation ρ_j hence H acts as multiplication by $1 - j$ on L_{λ_j}. Thus $\lambda_j = 1 - j$. Also $V/U_j = V/V_1$ gives rise to the representation ρ_{n-j} hence H acts as multiplication by $n - j - 1$ on $L_{\lambda_{j+1}}$. Thus $\lambda_{j+1} = n - j - 1$. Since $\lambda_j = \lambda_{j+1} + 2$ we have

$$1 - j = n - j + 1$$

and so $n = 0$ in k, which is impossible.

Case 2 Suppose $\rho(E)$ has two Jordan blocks. Then $\dim \mathrm{Ker}_V E = 2$. Consider the maximal vectors in V. These all lie in $\mathrm{Ker}_V E$ and are all in V_1 by 5.4.5. However $\dim(V_1 \cap \mathrm{Ker}_V E) = 1$. Thus there is only one maximal vector in V, up to scalar multiplication. Now $\mathrm{Ker}_V E$ is invariant under H and so H can only have one eigenvalue on $\mathrm{Ker}_V E$ rather than two distinct ones. This eigenvalue is $j - 1$. Thus V/V_1 also has a maximal vector with eigenvalue $j - 1$. V/V_1 therefore affords the representation ρ_j also and we have $n = 2j$.

Now $\rho\{(H + (j-1)I)(H + (j-3)I) \ldots (H - (j-1)I)\} = 0$ on V_1 and V/V_1 so

$$\rho\{(H + (j-1)I)^2(H + (j-3)I)^2 \ldots (H - (j-1)I)^2\} = 0$$

on V. Let $M_i = \mathrm{Ker}_V (H - (2i - j - 1)I)^2$. Then we have

$$V = M_1 \oplus M_2 \oplus \ldots \oplus M_j$$

and $\dim M_i = 2$ for $1 \leqslant i \leqslant j$. Moreover $\dim(M_i \cap V_1) = 1$ for $1 \leqslant i \leqslant j$.

Let $x_1, \ldots, x_j$ be a standard basis of V_1, so that $Ex_i = x_{i+1}$ and $Hx_i = (2i - j - 1)x_i$. Let y_1 lie in M_1 but not in V_1. Let $y_i = E^{i-1}y_1$ for $2 \leqslant i \leqslant j$.

Since $HE - EH = 2E$ we have $y_i \in M_i$. Moreover $y_i \notin V_1$ for $i \leqslant j$ since V/V_1 has no maximal vector with eigenvalue other than $j - 1$. Thus x_i, y_i form a basis for M_i and $x_1, \ldots, x_j, y_1, \ldots, y_j$ form a basis for V.

Now each subspace M_i is invariant under H. We consider Hy_1. This lies in M_1 so is a linear combination of x_1 and y_1. y_1 gives rise to an eigenvector of H in V/V_1 with eigenvalue $1 - j$. Thus we have

$$Hy_1 = \alpha x_1 + (1 - j)y_1 \qquad \text{for some } \alpha \in k.$$

We show by induction that

$$Hy_i = \alpha x_i + (2i - j - 1)y_i.$$

This follows since

$$\begin{aligned} Hy_i = HEy_{i-1} &= EHy_{i-1} + 2Ey_{i-1} \\ &= E(\alpha x_{i-1} + (2i - j - 3)y_{i-1}) + 2Ey_{i-1} \\ &= \alpha x_i + (2i - j - 1)Ey_{i-1} \\ &= \alpha x_i + (2i - j - 1)y_i. \end{aligned}$$

In particular we have

$$Hy_j = \alpha x_j + (j - 1)y_j.$$

This implies that $\alpha \neq 0$. For if $\alpha = 0$ y_j would be a maximal vector and so would lie in an irreducible submodule of V by 5.4.5.

Now $Fy_1 = 0$ since $-1 - j$ is not an eigenvalue of H. We show by induction that

$$Fy_{i+1} = -i\alpha x_i + i(j - i)y_i.$$

This follows since

$$\begin{aligned} Fy_{i+1} &= FEy_i = EFy_i - Hy_i \\ &= E(-(i-1)\alpha x_{i-1} + (i-1)(j - i + 1)y_{i-1}) - \alpha x_i - (2i - j - 1)y_i \\ &= -(i-1)\alpha x_i + (i-1)(j - i + 1)y_i - \alpha x_i - (2i - j - 1)y_i \\ &= -i\alpha x_i + i(j - i)y_i. \end{aligned}$$

Thus we have

$$\begin{aligned} Hy_j = (EF - FE)y_j &= EFy_j \\ &= E(-(j-1)\alpha x_{j-1} + (j-1)y_{j-1}) \\ &= -(j-1)\alpha x_j + (j-1)y_j. \end{aligned}$$

However we have seen that

$$Hy_j = \alpha x_j + (j - 1)y_j.$$

Thus $j\alpha = 0$ in k. Since $j \neq 0$ and $\alpha \neq 0$ in k we have a contradiction. ■

We summarize the results we have obtained about the representations of $\mathfrak{sl}_2(k)$ in the following theorem.

Theorem 5.4.8. *Let V be a finite-dimensional $\mathfrak{sl}_2(k)$-module affording the representation ρ. Suppose there is a positive integer $m \geqslant 2$ such that $\rho(E^{m-1}) = 0$, $\rho(F^{m-1}) = 0$. Suppose that $p \neq 2$ and $m \leqslant p$ if k has characteristic p. (There is no restriction on m if k has characteristic 0.) Then V is a direct sum of irreducible submodules each of which gives one of the representations ρ_j for some $j \leqslant m - 1$.*

Proof. This follows from 5.4.6 and 5.4.7.

5.5 NILPOTENT ORBITS AND ORBITS OF $\mathfrak{sl}_2$'S

We now return to the situation when G is a simple group over the algebraically closed field K and $\mathfrak{g}$ is the Lie algebra of G. Our aim in the present section is to show that if K has characteristic 0 or characteristic p sufficiently large then there is a natural bijection between G-orbits of nonzero nilpotent elements in $\mathfrak{g}$ and G-orbits of subalgebras of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(K)$.

We begin with a preliminary result giving a sufficient condition for an algebraic group to be reductive.

Proposition 5.5.1. *Let G be an affine algebraic group over K and $R = R_u(G)$ be its unipotent radical. Let $\mathfrak{g} = \mathfrak{L}(G)$ and $\mathfrak{r} = \mathfrak{L}(R)$ be the Lie algebras of G, R. Let V be a finite-dimensional rational G-module affording a representation ρ of G and a representation $\mathrm{d}\rho$ of $\mathfrak{g}$. Let*

$$B(x, y) = \operatorname{trace}(\mathrm{d}\rho(x)\,\mathrm{d}\rho(y)) \qquad x, y \in \mathfrak{g}$$

Then we have:

(i) $B(\mathfrak{g}, \mathfrak{r}) = 0$.
(ii) *If B is nondegenerate on $\mathfrak{g}$ then G is reductive.*

Proof. It is clear that (ii) follows from (i). For then we must have $\mathfrak{r} = 0$. Thus $R = 1$ and G is reductive.

Let $V = V_0 \supset V_1 \supset \ldots \supset V_{k-1} \supset V_k = 0$ be a composition series of V as a $\mathfrak{g}$-module. R acts on V by unipotent endomorphisms, so $\mathfrak{r}$ acts on V by nilpotent endomorphisms. So $\mathfrak{r}$ acts on each $\bar{V}_i = V_i/V_{i+1}$ by nilpotent endomorphisms. By a basic result on representations of Lie algebras there exists $\bar{v} \in \bar{V}_i$ with $\bar{v} \neq 0$ and $\mathfrak{r}\bar{v} = 0$ (Humphreys [3], p. 12). Let $\bar{W}_i = \{\bar{v} \in \bar{V}_i;\, \mathfrak{r}\bar{v} = 0\}$. Then $\bar{W}_i$ is a nonzero $\mathfrak{g}$-submodule of $\bar{V}_i$. For let $x \in \mathfrak{g}$, $y \in \mathfrak{r}$, $\bar{v} \in \bar{W}_i$. Then

$$y(x\bar{v}) = x(y\bar{v}) + [yx]\bar{v} = 0$$

and so $x\bar{v} \in \bar{W}_i$. Since $\bar{V}_i$ is irreducible we have $\bar{W}_i = \bar{V}_i$. Thus $\mathfrak{r}\bar{V}_i = 0$ and $\mathfrak{r}V_i \subseteq V_{i+1}$.

Again let $x \in \mathfrak{g}$, $y \in \mathfrak{r}$. Then $d\rho(y).V_i \subseteq V_{i+1}$. Hence

$$d\rho(x)\, d\rho(y) V_i \subseteq d\rho(x) V_{i+1} \subseteq V_{i+1}.$$

It follows that $\operatorname{trace}(d\rho(x)\, d\rho(y)) = 0$. Hence $B(x, y) = 0$ for all $x \in \mathfrak{g}$, $y \in \mathfrak{r}$.

Proposition 5.5.2. *Let h be the Coxeter number of the simple group G and e be a nilpotent element of $\mathfrak{g} = \mathfrak{L}(G)$. Then $(\operatorname{ad} e)^{2h-1} = 0$.*

Proof. We make use of the Cartan decomposition of $\mathfrak{g}$ described in section 1.13. We have $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}^-$ where $\mathfrak{n} = \mathfrak{L}(U)$, $\mathfrak{t} = \mathfrak{L}(T)$, $\mathfrak{n}^- = \mathfrak{L}(U^-)$. We recall from section 1.15 that each nilpotent element of $\mathfrak{g}$ lies in the same G-orbit as an element of $\mathfrak{n}$. We may therefore assume that $e \in \mathfrak{n}$. Now we have $\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{x}_\alpha$ where $\mathfrak{x}_\alpha = \mathfrak{L}(X_\alpha)$. Let $\mathfrak{x}_\alpha = K e_\alpha$. Then $e = \sum_{\alpha \in \Phi^+} \lambda_\alpha e_\alpha$. For each weight $\beta \in X(T)$ we have a weight space $\mathfrak{g}_\beta$ defined by

$$\mathfrak{g}_\beta = \{x \in \mathfrak{g};\ \operatorname{Ad} t.x = \beta(t)x \text{ for all } t \in T\}.$$

We have $\mathfrak{g} = \bigoplus_{\beta \in X(T)} \mathfrak{g}_\beta$ and only finitely many $\mathfrak{g}_\beta$ are nonzero. Moreover $[\mathfrak{g}_\beta \mathfrak{g}_\gamma] \subseteq \mathfrak{g}_{\beta+\gamma}$. The weights β for which $\mathfrak{g}_\beta \neq 0$ are the roots Φ together with 0.

Now the height of the highest root of G is $h - 1$ by section 1.9. Moreover the smallest height of any root is $-(h - 1)$. Since the addition of $\alpha \in \Phi^+$ increases the height of each root by at least one we see that

$$\operatorname{ad} e_{\alpha_1} . \operatorname{ad} e_{\alpha_2} \ldots \operatorname{ad} e_{\alpha_{2h-1}} = 0$$

for any set $e_{\alpha_1}, \ldots, e_{\alpha_{2h-1}}$ of positive roots. Hence

$$(\operatorname{ad} e)^{2h-1} = \left(\operatorname{ad}\left(\sum_{\alpha \in \Phi^+} \lambda_\alpha e_\alpha\right)\right)^{2h-1} = 0. \qquad \blacksquare$$

We now assume that the characteristic of K is either 0 or $p > 2h$. By 5.3.2, given any nonzero nilpotent element $e \in \mathfrak{g}$, we can find elements $h, f \in \mathfrak{g}$ such that $[he] = 2e$, $[hf] = -2f$, $[ef] = h$. Thus the subalgebra $\langle e, h, f\rangle$ is isomorphic to $\mathfrak{sl}_2(K)$. We may then regard $\mathfrak{g}$ as an $\langle e, h, f\rangle$-module. By 5.4.8 $\mathfrak{g}$ is a direct sum of irreducible $\langle e, h, f\rangle$-submodules each affording one of the standard representations ρ_j of $\mathfrak{sl}_2(K)$.

Lemma 5.5.3. *Let $j \leqslant p$ and $\mathfrak{V}_j$ be the vector space of polynomials in x, y over K which are homogeneous of degree $j - 1$. Thus $\mathfrak{V}_j$ has basis*

$$v_1 = x^{j-1},\ v_2 = (j-1)x^{j-2}y,\ v_3 = (j-1)(j-2)x^{j-3}y^2, \ldots, v_j = (j-1)!\,y^{j-1}.$$

Then $\mathfrak{V}_j$ is a left $\mathfrak{sl}_2(K)$-module in which the elements of $\mathfrak{sl}_2(K)$ act as derivations satisfying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . x = ax + cy \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} . y = bx + dy.$$

The elements $E = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $H = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ *of the enveloping algebra of* $\mathfrak{sl}_2(K)$ *act on the above basis by*

$$Ev_i = v_{i+1} \qquad i = 1, 2, \ldots, j-1, \; Ev_j = 0$$

$$Hv_i = (2i - j - 1)v_i$$

$$Fv_{i+1} = i(j-i)v_i \qquad i = 1, 2, \ldots, j-1, \; Fv_1 = 0.$$

Thus $\mathfrak{V}_j$ *is a module giving the standard representation* ρ_j *of* $\mathfrak{sl}_2(K)$ *with respect to the above basis.*

Proof. Straightforward.

Lemma 5.5.4. (i) *The space* $\mathfrak{V}_j$ *is an* $SL_2(K)$*-module under the action derived from*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . x = ax + cy \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} . y = bx + dy.$$

(ii) *The elements* E^j, F^j *of the enveloping algebra of* $\mathfrak{sl}_2(K)$ *act as zero on* $\mathfrak{V}_j$.

(iii) *If* $j \leqslant p$ *the element* $\sum_{r=0}^{j-1} \dfrac{\lambda^r E^r}{r!}$ *acts on* $\mathfrak{V}_j$ *in the same way as* $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \in SL_2(K)$ *and the element* $\sum_{r=0}^{j-1} \dfrac{\lambda^r F^r}{r!}$ *acts on* $\mathfrak{V}_j$ *in the same way as* $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in SL_2(K)$.

Proof. Straightforward.

Proposition 5.5.5. *Suppose* $p > 2h$. *Then* (i) $\mathfrak{g}$ *may be regarded as an* $SL_2(K)$*-module with* $SL_2(K)$*-action on each standard summand* ρ_j *given as in* 5.5.4.

(ii) *The elements* E^{2h-1}, F^{2h-1} *of the enveloping algebra of* $\mathfrak{sl}_2(K)$ *act as zero on* $\mathfrak{g}$.

(iii) *The element* $\sum_{r=0}^{2h-2} \dfrac{\lambda^r E^r}{r!}$ *acts on* $\mathfrak{g}$ *in the same way as* $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \in SL_2(K)$ *and the element* $\sum_{r=0}^{2h-2} \dfrac{\lambda^r F^r}{r!}$ *acts on* $\mathfrak{g}$ *in the same way as* $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in SL_2(K)$.

(iv) *If* $p > 3(h-1)$ *the elements* $\sum_{r=0}^{2h-2} \dfrac{\lambda^r E^r}{r!}$ *and* $\sum_{r=0}^{2h-2} \dfrac{\lambda^r F^r}{r!}$ *act as automorphisms of* $\mathfrak{g}$.

Proof. (i) follows directly from 5.5.4. (ii) follows from 5.5.2. (iii) follows from 5.5.4. We therefore have only to consider (iv). Here we shall adapt the proof which shows that, over a field of characteristic 0, the exponential of a nilpotent derivation is an automorphism. Let δ be equal to λE. Then, for all $a, b \in \mathfrak{g}$, we have

$$\delta[ab] = [\delta a, b] + [a, \delta b].$$

Hence

$$\delta^r[ab] = \sum_{i=0}^{r} \binom{r}{i} [\delta^i a, \delta^{r-i} b] \qquad \text{for all } r \geqslant 0.$$

Thus

$$\begin{aligned}\frac{\delta^r}{r!}[ab] &= \sum_{i=0}^{r} \left[\frac{\delta^i}{i!} a, \frac{\delta^{r-i}}{(r-i)!} b\right] \qquad \text{for all } r < p \\ &= \sum_{\substack{i,j \\ i+j=r}} \left[\frac{\delta^i}{i!} a, \frac{\delta^j}{j!} b\right].\end{aligned}$$

Now a is a linear combination of terms of height $\geqslant -h+1$. Thus $(\delta^i/i!)\, a$ is a linear combination of terms of height $\geqslant -h+1+i$. Similarly $(\delta^j/j!)\, b$ is a linear combination of terms of height $\geqslant -h+1+j$. Thus $[(\delta^i/i!)\, a, (\delta^j/j!)\, b]$ is a linear combination of terms of height $\geqslant -2h+2+i+j$. Hence

$$\left[\frac{\delta^i}{i!} a, \frac{\delta^j}{j!} b\right] = 0 \quad \text{if} \quad -2h+2+i+j \geqslant h, \text{ i.e. if } i+j \geqslant 3h-2.$$

Thus we have

$$\left[\sum_{i=0}^{2h-2} \frac{\delta^i}{i!} a, \sum_{i=0}^{2h-2} \frac{\delta^i}{i!} b\right] = \sum_{i=0}^{3h-3} \frac{\delta^i}{i!} [ab]$$

provided $3(h-1) < p$, since $\delta^i/i! = 0$ on $\mathfrak{g}$ for $i > 2(h-1)$. Hence

$$\left[\sum_{i=0}^{2h-2} \frac{\delta^i}{i!} a, \sum_{i=0}^{2h-2} \frac{\delta^i}{i!} b\right] = \sum_{i=0}^{2h-2} \frac{\delta^i}{i!} [ab]$$

provided $3(h-1) < p$, and so we have an automorphism of $\mathfrak{g}$. The same argument applies if δ is λF instead of λE. ■

We subsequently assume that the characteristic of K is either 0 or $p > 3(h-1)$. This assumption has the following consequences. In the first place each nonzero nilpotent element e of $\mathfrak{g}$ lies in a subalgebra isomorphic to $\mathfrak{sl}_2(K)$. This follows from the Jacobson–Morozov theorem 5.3.2, making use of 5.5.2. The condition on p needed for this is $p > 2h$. However $3(h-1) \geqslant 2h$ except when the Coxeter number h is 2, and any prime $p > 3(h-1)$ satisfies $p > 2h$ even when $h = 2$. Secondly the result of 5.4.8 is valid for the module $V = \mathfrak{g}$, again using 5.5.2, and so $\mathfrak{g}$ is a direct sum of standard $\mathfrak{sl}_2(K)$-modules. These can be regarded as $SL_2(K)$-modules as in 5.5.5 on which E^{2h-1} and F^{2h-1} act as zero. The elements

$$\sum_{r=0}^{2h-2} \frac{\lambda^r E^r}{r!}, \qquad \sum_{r=0}^{2h-2} \frac{\lambda^r F^r}{r!}$$

act as automorphisms of $\mathfrak{g}$, and these are the actions induced by $\begin{pmatrix} 1 & 1 \\ \lambda & 0 \end{pmatrix}, \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ respectively. Finally all the elements of $SL_2(K)$ act as automorphisms of $\mathfrak{g}$ in the action described in 5.5.5, since $SL_2(K)$ is generated by its elements of form $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$.

Proposition 5.5.6. *Suppose the characteristic of K is 0 or $p > 3(h-1)$. Then the element $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ of $SL_2(K)$ acts on $\mathfrak{g}$ by the automorphism $\gamma(\lambda)$ defined as follows. Decompose $\mathfrak{g}$ into a direct sum of standard modules ρ_j for $\mathfrak{sl}_2(K)$ and let $x_1, \ldots, x_j$ be a standard basis of such a module. Then*

$$\gamma(\lambda).x_k = \lambda^{2k-j-1}x_k.$$

Proof. We see from 5.5.3 and 5.5.4 that $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ acts on the standard basis by

$$x_1 \to \lambda^{1-j}x_1,\ x_2 \to \lambda^{3-j}x_2, \ldots, x_k \to \lambda^{2k-j-1}x_k. \qquad \blacksquare$$

Let $\mathfrak{g}(i) = \{x \in \mathfrak{g};\ \gamma(\lambda)x = \lambda^i x \text{ for all } \lambda \in K^*\}$

Proposition 5.5.7 (i) $\mathfrak{g} = \bigoplus_{i \in Z} \mathfrak{g}(i)$.
(ii) $[\mathfrak{g}(i), \mathfrak{g}(j)] \subseteq \mathfrak{g}(i+j)$.
(iii) $e \in \mathfrak{g}(2)$, $f \in \mathfrak{g}(-2)$, $h \in \mathfrak{g}(0)$.
(iv) $[hx] = ix$ *for all* $x \in \mathfrak{g}(i)$.

Proof. (i) holds because each standard basis vector lies in some $\mathfrak{g}(i)$.
(ii) Let $a \in \mathfrak{g}(i)$, $b \in \mathfrak{g}(j)$. Then

$$\gamma(\lambda)[ab] = [\gamma(\lambda)a, \gamma(\lambda)b] = [\lambda^i a, \lambda^j b] = \lambda^{i+j}[ab].$$

Hence $[ab] \in \mathfrak{g}(i+j)$.
(iii) If we restrict the $\langle e, h, f\rangle$-module $\mathfrak{g}$ to $\langle e, h, f\rangle$ we obtain the adjoint representation of $\langle e, h, f\rangle$. If we take the basis f, h, $-2e$ of this module we obtain the standard module ρ_j for $\langle e, h, f\rangle$ with $j = 3$ and with its standard basis. Hence $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ acts on this basis by $f \to \lambda^{-2}f$, $h \to h$, $-2e \to \lambda^2(-2e)$. It follows that $e \in \mathfrak{g}(2)$, $f \in \mathfrak{g}(-2)$, $h \in \mathfrak{g}(0)$.
(iv) This is clear from the way h acts on the standard basis. $\blacksquare$

Let $\gamma: K^* \to \operatorname{Aut}\mathfrak{g}$ be defined by $\lambda \to \gamma(\lambda)$ where $\gamma(\lambda)$ is as in 5.5.6. γ is a rational homomorphism from K^* into $\operatorname{Aut}\mathfrak{g}$. Its image is connected, so lies in $(\operatorname{Aut}\mathfrak{g})^0$. Now $(\operatorname{Aut}\mathfrak{g})^0 = G_{\mathrm{ad}}$, the adjoint group of the same type as G, by section 1.13.

We shall assume subsequently in this chapter that G is a simple group of adjoint type. We have remarked earlier that there is no loss of generality in assuming this. We then have $\gamma: K^* \to G$. Let $S = \gamma(K^*)$. Since γ is nontrivial S must be a 1-dimensional subtorus of G. Let $\mathfrak{s} = \mathfrak{L}(S)$. Taking differentials we have a map $\mathrm{d}\gamma: K \to \mathfrak{s}$ satisfying

$$\mathrm{ad}(\mathrm{d}\gamma(\lambda)).x = i\lambda x \qquad \text{for all } x \in \mathfrak{g}(i).$$

Thus $\mathrm{ad}(\mathrm{d}\gamma(\lambda)) = \mathrm{ad}(\lambda h)$. Since the centre of $\mathfrak{g}$ is 0 by section 1.13 it follows that $\mathfrak{s} = Kh$.

Proposition 5.5.8. *Suppose the characteristic of* K *is* 0 *or* $p > 3(h-1)$. *Then*:

(i) $\mathfrak{g}(i) = \{x \in \mathfrak{g}; [hx] = ix\}$ *for* $-2 \leqslant i \leqslant 2$.

(ii) $C_{\mathfrak{g}}(h) = \mathfrak{g}(0)$.

Proof. (ii) clearly follows from (i). We know already that $[hx] = ix$ when $x \in \mathfrak{g}(i)$. All such i lie in the range

$$-2(h-1) \leqslant i \leqslant 2(h-1)$$

since the dimension of each standard component of $\mathfrak{g}$ is at most $2h - 1$. If $-2 \leqslant i \leqslant 2$ and $p > 2h$ then i is incongruent mod p to any other integer between $-2(h-1)$ and $2(h-1)$. Thus $\mathfrak{g}(i) = \{x \in \mathfrak{g}; [hx] = ix\}$ if $-2 \leqslant i \leqslant 2$ and K has characteristic 0 or $0 > 2h$. We show that the condition $p > 3(h-1)$ is sufficient. The Coxeter number h always satisfies $h \geqslant 2$. If $h \geqslant 3$ then $3(h-1) \geqslant 2h$ so $p > 3(h-1)$ implies $p > 2h$. If $h = 2$, $p > 3(h-1)$ again implies $p > 2h$, so the result follows.

Proposition 5.5.9. *Suppose the characteristic of* K *is* 0 *or* $p > 3(h-1)$. *Let* $M = C_G(e)^0$, R *be the unipotent radical of* M *and* $C = C_M(S)$. *Then we have*:

(i) $M = RC$ *and* $R \cap C = 1$.

(ii) C *is a connected reductive group.*

Proof. We first observe that C is connected. For recall from section 1.14 that the centralizer of a torus in a connected group is connected. It follows that the centralizer of a torus acting as a group of automorphisms on a connected group is connected. Since M is connected $C = C_M(S)$ must also be connected.

We show next that S lies in $N_G(M)$. Let $s \in S$ and $g \in C_G(e)$. Then we have

$$\mathrm{Ad}(s^{-1}gs).e = \mathrm{Ad}\, s^{-1}g.\mathrm{Ad}\, s.e.$$

Now $e \in \mathfrak{g}(2)$ and so $\mathrm{Ad}\, s.e = \lambda e$ for some $\lambda \in K$. Thus

$$\mathrm{Ad}(s^{-1}gs).e = \mathrm{Ad}\, s^{-1}g.(\lambda e) = \mathrm{Ad}\, s^{-1}.(\lambda e) = e.$$

Hence $s^{-1}gs \in C_G(e)$. Thus S normalizes $C_G(e)$ and therefore also normalizes $M = C_G(e)^0$.

Now the given assumptions on the characteristic certainly imply that $C_{\mathfrak{g}}(e) = \mathfrak{L}(C_G(e))$ by section 1.14. Thus we have

$$\mathfrak{m} = \mathfrak{L}(M) = \mathfrak{L}(C_G(e)) = C\mathfrak{g}(e).$$

We now regard $\mathfrak{g}$ as a module for $\langle e, h, f\rangle$. By 5.4.8 $\mathfrak{g}$ is a direct sum of irreducible $\langle e, h, f\rangle$-modules each affording one of the standard representations ρ_j of $\langle e, h, f\rangle \cong \mathfrak{sl}_2(K)$. Let $\mathfrak{g} = \bigoplus V_r$ be such a decomposition. Each irreducible module V_r has a standard basis, and just one standard basis vector in each V_r lies in $C_{\mathfrak{g}}(e) = \mathfrak{m}$, viz. the one on which the weight of S is maximal.

Let $\mathfrak{c} = \mathfrak{L}(C) = \mathfrak{L}(C_M(S)) = C_{\mathfrak{m}}(S)$ by section 1.14. Now there is at most one basis vector in each V_r which lies in $C_{\mathfrak{g}}(S) = C_{\mathfrak{g}}(\mathfrak{s})$, viz the one on which ad h has eigenvalue 0. Hence $\mathfrak{c} = \mathfrak{m} \cap C_{\mathfrak{g}}(S)$ must be the direct sum of all the components V_r with $\dim V_r = 1$. For these are the only ones containing a basis vector annihilated by both ad e and ad h. Thus we have

$$\mathfrak{m} = \mathfrak{r}_1 \oplus \mathfrak{c}$$

where $\mathfrak{r}_1$ is the subspace of $\mathfrak{g}$ spanned by the standard basis vectors which lie in $\mathfrak{m}$ and in a component V_r with $\dim V_r > 1$. We note also that $\mathfrak{r}_1 = \mathfrak{m} \cap [e\mathfrak{g}]$.

Now the assumptions on the characteristic certainly imply that the Killing form $B(x, y) = \operatorname{trace}(\operatorname{ad} x \operatorname{ad} y)$ on $\mathfrak{g}$ is nondegenerate, by section 1.13. Let $\mathfrak{m}^{\perp}$ be the subspace of $\mathfrak{g}$ given by

$$\mathfrak{m}^{\perp} = \{x \in \mathfrak{g};\ B(x, y) = 0 \text{ for all } y \in \mathfrak{m}\}.$$

We claim that $\mathfrak{m}^{\perp} = [e\mathfrak{g}]$. For ad $e : \mathfrak{g} \to [e\mathfrak{g}]$ is a surjective map with kernel $\mathfrak{m}$ and so

$$\dim[e\mathfrak{g}] = \dim \mathfrak{g} - \dim \mathfrak{m} = \dim \mathfrak{m}^{\perp}.$$

Also if $x \in \mathfrak{g}$, $y \in \mathfrak{m}$ we have

$$B([ex], y) = -B([xe], y) = -B(x, [ey]) = 0$$

so that $[e\mathfrak{g}] \subseteq \mathfrak{m}^{\perp}$. It follows that $[e\mathfrak{g}] = \mathfrak{m}^{\perp}$. Hence we have

$$\mathfrak{r}_1 = \mathfrak{m} \cap [e\mathfrak{g}] = \mathfrak{m} \cap \mathfrak{m}^{\perp}.$$

We can now show that C is reductive. For let $x \in \mathfrak{c} \cap \mathfrak{c}^{\perp}$. Then $x \in \mathfrak{r}_1{}^{\perp}$ since for $y \in \mathfrak{r}_1$ we have $B(x, y) = 0$ as $x \in \mathfrak{m}$ and $y \in \mathfrak{m}^{\perp}$. Hence $x \in \mathfrak{c}^{\perp} \cap \mathfrak{r}_1{}^{\perp} = \mathfrak{m}^{\perp}$. Also $x \in \mathfrak{m} \cap \mathfrak{m}^{\perp} = \mathfrak{r}_1$. Thus $x \in \mathfrak{c} \cap \mathfrak{r}_1 = 0$. It follows that $\mathfrak{c} \cap \mathfrak{c}^{\perp} = 0$ and so the form $B(x, y)$ is nondegenerate on $\mathfrak{c}$. It follows from 5.5.1 that C is reductive.

Let $\mathfrak{r} = \mathfrak{L}(R)$. Again by 5.5.1 we have $B(\mathfrak{r}, \mathfrak{m}) = 0$. Thus $\mathfrak{r} \subseteq \mathfrak{m} \cap \mathfrak{m}^{\perp} = \mathfrak{r}_1$. We shall show that in fact $\mathfrak{r} = \mathfrak{r}_1$. To see this consider the adjoint action of S on $\mathfrak{g}$. This gives rise to certain weights which lie in $X(S) \cong \mathbb{Z}$. Since $S \subseteq N_G(M)$, $\mathfrak{m}$ will be invariant under Ad S. Moreover the weights of S on $\mathfrak{m}$ are all nonnegative. $\mathfrak{r}$ is also invariant under Ad S, and so S acts on $\mathfrak{m}/\mathfrak{r}$, which is reductive. Now M/R is reductive and MS/M is torus and so MS/R must also be reductive. Now the Lie algebra of MS is $\mathfrak{m} + \mathfrak{s}$, since $\mathfrak{m} + \mathfrak{s} \subseteq \mathfrak{L}(MS)$ and $\dim(\mathfrak{m} + \mathfrak{s}) = \dim MS$. Thus the Lie algebra of MS/R is $(\mathfrak{m} + \mathfrak{s})/\mathfrak{r}$ and this differs from $\mathfrak{m}/\mathfrak{r}$ at most by the addition of a 1-dimensional central subalgebra. However the subtorus $RS/R \cong S$ of MS/R acts on $(\mathfrak{m} + \mathfrak{s})/\mathfrak{r}$ in such a way that for each positive weight in $X(S)$ arising there is a corresponding negative weight. Thus the same applies to the action of S on the subalgebra $\mathfrak{m}/\mathfrak{r}$. But we have seen that all weights of S on $\mathfrak{m}/\mathfrak{r}$ are nonnegative. Thus all weights of S on $\mathfrak{m}/\mathfrak{r}$ must be 0. Now $\mathfrak{m}$ is completely reducible as an S-module and so there exists an S-module $\mathfrak{c}'$ with $\mathfrak{m} = \mathfrak{r} \oplus \mathfrak{c}'$. Since the weights of S on $\mathfrak{m}/\mathfrak{r}$ are all 0 we have $\mathfrak{c}' \subseteq C_{\mathfrak{m}}(S) = \mathfrak{c}$. Thus $\mathfrak{m} = \mathfrak{r} + \mathfrak{c}$. Since $\mathfrak{m} = \mathfrak{r}_1 \oplus \mathfrak{c}$ and $\mathfrak{r} \subseteq \mathfrak{r}_1$ this implies that $\mathfrak{r} = \mathfrak{r}_1$ and $\mathfrak{m} = \mathfrak{r} \oplus \mathfrak{c}$.

We may now complete the proof. $R \cap C$ is a normal unipotent subgroup of C and C is connected reductive, so $R \cap C = 1$. Also we have

$$\dim RC = \dim R + \dim C = \dim \mathfrak{r} + \dim \mathfrak{c} = \dim \mathfrak{m} = \dim M.$$

It follows that $RC = M$ since M is connected.

Proposition 5.5.10. Suppose the characteristic of K is 0 or $p > 3(h-1)$. Let $\langle e, h, f\rangle$ and $\langle e, h_1, f_1\rangle$ be 3-dimensional subalgebras of $\mathfrak{g}$ satisfying

$$[he] = 2e \qquad [hf] = -2f \qquad [ef] = h$$

$$[h_1e] = 2e \qquad [h_1f_1] = -2f_1 \qquad [ef_1] = h_1.$$

Then there is an element $g \in C_G(e)^0$ such that $\operatorname{Ad} g.h = h_1$ and $\operatorname{Ad} g.f = f_1$.

Proof. Let $Q = \{g \in G;\ \operatorname{Ad} g.Ke = Ke\}$. Let S be a 1-dimensional torus defined as before with respect to $\langle e, h, f\rangle$ and S_1 be a 1-dimensional torus defined with respect to $\langle e, h_1, f_1\rangle$. Let $M = C_G(e)^0$. Then $S \subseteq Q^0$, $M \subseteq Q^0$ and $S \nsubseteq M$. Also $\dim Q^0 \leqslant \dim M + 1$ since $\operatorname{Aut} \mathbf{G}_a \cong \mathbf{G}_m$ is 1-dimensional. Thus we have $Q^0 = SM$.

Now $M = CR$ by 5.5.9 where $C = C_M(S)$ and $R = R_u(M)$. Let T be a maximal torus of C. Then T is a maximal torus of M and ST is a maximal torus of Q^0. Similarly there is a maximal torus T_1 of M such that S_1T_1 is a maximal torus of Q^0. Since $Q^0 = SM$, ST and S_1T_1 are conjugate by an element of M. Let ${}^m(ST) = S_1T_1$. By replacing the elements h, f in the statement of the proposition by the elements $\operatorname{Ad} m.h$, $\operatorname{Ad} m.f$ we see that it is sufficient to consider the case where $m = 1$. Then $ST = S_1T_1$ and so, taking Lie algebras, we have

$$Kh + \mathfrak{t} = Kh_1 + \mathfrak{t}_1 \qquad \text{where } \mathfrak{t} = \mathfrak{L}(T), \mathfrak{t}_1 = \mathfrak{L}(T_1).$$

Thus $h_1 = \lambda h + x$ for some $x \in \mathfrak{t}$ and some $\lambda \in K$. Hence

$$[h_1e] = [\lambda h + x, e] = \lambda[he]$$

since $[xe] = 0$ as $x \in \mathfrak{m} = C_{\mathfrak{g}}(e)$. Thus $2e = \lambda(2e)$ and so $\lambda = 1$. Hence $h_1 = h + x$. We also have

$$[e, f_1 - f] = [ef_1] - [ef] = h_1 - h = x.$$

Thus $x \in [e\mathfrak{g}]$. Let $\mathfrak{c} = \mathfrak{L}(C)$. Then we have

$$x \in \mathfrak{c} \cap [e\mathfrak{g}] = \mathfrak{c} \cap \mathfrak{m}^{\perp} \qquad \text{as in 5.5.9}$$

$$= \mathfrak{c} \cap \mathfrak{m} \cap \mathfrak{m}^{\perp} = \mathfrak{c} \cap \mathfrak{r} = 0.$$

Thus $x = 0$ and $h_1 = h$.

We also have $[e, f_1 - f] = 0$ and so $f_1 - f \in \mathfrak{m}$. Also

$$[h_1, f_1 - f] = -2(f_1 - f).$$

Thus $f_1 - f$ lies in the intersection of $\mathfrak{m}$ and the space $\mathfrak{g}(-2)$ with respect to $\langle e, h_1, f_1\rangle$. But $\mathfrak{m} \cap \mathfrak{g}(-2) = 0$. Hence $f_1 - f = 0$ and $f_1 = f$. ■

We now come to the main result in this section.

Theorem 5.5.11. *Suppose the characteristic of K is 0 or $p > 3(h-1)$. Then there is a bijection between G-orbits of nonzero nilpotent elements of $\mathfrak{g}$ and G-orbits of subalgebras isomorphic to $\mathfrak{sl}_2(K)$ in $\mathfrak{g}$.*

Proof. By 5.3.2 each nonzero nilpotent element $e \in \mathfrak{g}$ lies in a 3-dimensional subalgebra $\langle e, h, f\rangle$ of $\mathfrak{g}$ where $[he] = 2e$, $[hf] = -2f$, $[ef] = h$. This subalgebra is isomorphic to $\mathfrak{sl}_2(K)$. Suppose we take a second such subalgebra $\langle e, h_1, f_1\rangle$ containing e. By 5.5.10 these two subalgebras lie in the same G-orbit. Thus we have a well-defined map from G-orbits of nonzero nilpotent elements to G-orbits of subalgebras isomorphic to $\mathfrak{sl}_2(K)$. Now each subalgebra $\langle e, h, f\rangle$ isomorphic to $\mathfrak{sl}_2(K)$ contains a nonzero nilpotent element e of $\mathfrak{g}$. For $\langle e, h, f\rangle$ is the Lie algebra of a subgroup of G isomorphic to $SL_2(K)$ or $PGL_2(K)$ as in 5.5.5, and e is certainly nilpotent as an element of $\mathfrak{L}(SL_2(K))$ so must also be nilpotent as an element of $\mathfrak{g} = \mathfrak{L}(G)$, by section 1.4. Thus the map from G-orbits of nonzero nilpotent elements to G-orbits of subalgebras isomorphic to $\mathfrak{sl}_2(K)$ is surjective.

To show it is bijective one must verify that two nonzero nilpotent elements in a subalgebra isomorphic to $\mathfrak{sl}_2(K)$ lie in the same G-orbit. However any nonzero nilpotent element of $\mathfrak{sl}_2(K)$ is conjugate under $GL_2(K)$ to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Moreover $GL_2(K) = SL_2(K).C_{GL_2}(K)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Thus any nonzero nilpotent element of $\mathfrak{sl}_2(K)$ is conjugate under $SL_2(K)$ to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Thus any two nonzero nilpotent elements in a subalgebra isomorphic to $\mathfrak{sl}_2(K)$ are conjugate under $SL_2(K)$. They are therefore in the same G-orbit since we know as in 5.5.6 that G contains a subgroup isomorphic to $SL_2(K)$ or $PGL_2(K)$ whose Lie algebra is the given subalgebra.

5.6 THE WEIGHTED DYNKIN DIAGRAM

We shall show in this section how to attach to each nilpotent orbit of $\mathfrak{g}$ an object called a weighted Dynkin diagram.

We assume as in section 5.5 that G is simple of adjoint type, that K has characteristic 0 or $p > 3(h-1)$, that e is a nonzero nilpotent element of $\mathfrak{g}$, and that $\langle e, h, f\rangle$ is a 3-dimensional subalgebra containing e satisfying the usual relations. Let S be the 1-dimensional subtorus of G constructed in section 5.5

with $\mathfrak{L}(S) = Kh$. Let $L = C_G(S)$ and $\mathfrak{l} = \mathfrak{L}(L) = C_{\mathfrak{g}}(S)$ by section 1.14. Then we also have

$$C_{\mathfrak{g}}(\mathfrak{s}) = C_{\mathfrak{g}}(h) = \mathfrak{g}(0) = \mathfrak{l}$$

by 5.5.8.

Proposition 5.6.1. *The map* ad $e:\mathfrak{g}(0) \to \mathfrak{g}(2)$ *is surjective.*

Proof. Since $e \in \mathfrak{g}(2)$ by 5.5.7 ad e maps $\mathfrak{g}(0)$ into $\mathfrak{g}(2)$. We consider $\mathfrak{g}$ as an $\langle e, h, f\rangle$-module. $\mathfrak{g}$ is a direct sum of irreducible $\langle e, h, f\rangle$-submodules each affording a standard representation ρ_j of $\mathfrak{sl}_2(K)$. Let $\mathfrak{g} = \bigoplus V_r$ be such a decomposition. $\mathfrak{g}(2)$ is spanned by the standard basis vectors in the submodules V_r which lie in the weight space for S corresponding to the weight 2. Each such vector is the image under ad e of a basis vector in the 0-weight space of S. Thus the map ad $e:\mathfrak{g}(0) \to \mathfrak{g}(2)$ is surjective.

Proposition 5.6.2. *The morphism $L \to \mathfrak{g}(2)$ given by $x \to$ Ad $x.e$ is dominant and separable. In particular the orbit of L on $\mathfrak{g}(2)$ containing e is a dense open subset of $\mathfrak{g}(2)$.*

Proof. Since $L = C_G(S)$ preserves the weight spaces for S we see that $\mathfrak{g}(2)$ is an L-module under the adjoint action. Thus $x \to$ Ad $x.e$ is a morphism from L into $\mathfrak{g}(2)$. The differential of this morphism at the identity is the map from $\mathfrak{l} = \mathfrak{g}(0)$ into $\mathfrak{g}(2)$ given by $x \to [xe]$. By 5.6.1 this is surjective. The given morphism must therefore be dominant and separable by section 1.3. Since the image is open in its closure the orbit of L on $\mathfrak{g}(2)$ containing e must therefore be a non-empty open subset of $\mathfrak{g}(2)$.

Proposition 5.6.3. *Suppose $\langle e, h, f\rangle$ and $\langle e_1, h_1, f_1\rangle$ are subalgebras of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(K)$ satisfying the usual relations, with $h_1 = h$. Then e, e_1 are in the same G-orbit on $\mathfrak{g}$.*

Proof. By 5.5.8 $\mathfrak{g}(2)$ is determined by h. Both e and e_1 lie in $\mathfrak{g}(2)$. By 5.6.2 the L-orbit of e is a dense open subset of $\mathfrak{g}(2)$ and so is the L_1-orbit of e_1 where $L_1 = C_G(S_1)$ is the corresponding group for $\langle e_1, h_1, f_1\rangle$. These orbits must therefore intersect and so e and e_1 lie in the same G-orbit.

Proposition 5.6.4. *Let $\langle e, h, f\rangle$ and $\langle e_1, h_1, f_1\rangle$ be two subalgebras of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(K)$ satisfying the usual relations. Then the following conditions are equivalent:*

(i) *e, e_1 lie in the same G-orbit.*
(ii) *$\langle e, h, f\rangle$, $\langle e_1, h_1, f_1\rangle$ lie in the same G-orbit.*
(iii) *h, h_1 lie in the same G-orbit.*

Proof. We have already shown in 5.5.11 that conditions (i), (ii) are equivalent. We show that (ii) implies (iii) and that (iii) implies (i). Suppose condition (ii) holds. We may assume without loss of generality that $\langle e, h, f\rangle =$

$\langle e_1, h_1, f_1\rangle$. Now any two nonzero nilpotent elements e, e_1 of $\mathfrak{sl}_2(K)$ are conjugate under $SL_2(K)$, so lie in the same G-orbit. We may therefore assume also that $e = e_1$. But then h, h_1 are conjugate under $C_G(e)^0$ by 5.5.10. Thus h, h_1 lie in the same G-orbit.

Now suppose condition (iii) holds. We may assume without loss of generality that $h = h_1$. Then e, e_1 are in the same G-orbit by 5.6.3. ■

Let T be a maximal torus of G containing the 1-dimensional torus S. Let $\mathfrak{t} = \mathfrak{L}(T)$. We decompose $\mathfrak{g}$ into weight spaces under the action of T giving a Cartan decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha\in\Phi} Ke_\alpha.$$

Now each root space Ke_α is invariant under Ad S. Thus $e_\alpha \in \mathfrak{g}(i)$ for a unique $i \in \mathbb{Z}$. Let $\eta:\Phi \to \mathbb{Z}$ be the function defined by $\eta(\alpha) = i$. Then we have $\eta(-\alpha) = -\eta(\alpha)$ and $\eta(\alpha_1 + \alpha_2) = \eta(\alpha_1) + \eta(\alpha_2)$ whenever α_1, α_2, $\alpha_1 + \alpha_2$ are all in Φ.

Lemma 5.6.5. *Given any function $\eta:\Phi \to \mathbb{Z}$ satisfying $\eta(-\alpha) = -\eta(\alpha)$ for all $\alpha \in \Phi$ and $\eta(\alpha_1 + \alpha_2) = \eta(\alpha_1) + \eta(\alpha_2)$ whenever $\alpha_1, \alpha_2, \alpha_1 + \alpha_2 \in \Phi$ there exists a system Δ of simple roots for which $\eta(\alpha) \geqslant 0$ for all $\alpha \in \Delta$.*

Proof. η can be chosen arbitrarily on any system of simple roots in Φ and is then uniquely determined by the given conditions. Thus there is a unique element of $\mathrm{Hom}(X, \mathbb{Z})$ extending η, where $X = X(T) = \mathbb{Z}\Phi$. Let η map to $\gamma \in Y$ under the isomorphism $\mathrm{Hom}(X, \mathbb{Z}) \cong Y$. Then $\eta(\alpha) = \langle\alpha, \gamma\rangle$ for all $\alpha \in \Phi$. We choose a simple system Δ of roots for which γ lies in the closure of the fundamental chamber. Then $\eta(\alpha) = \langle\alpha, \gamma\rangle \geqslant 0$ for all $\alpha \in \Delta$.

If the function $\eta:\Phi \to \mathbb{Z}$ is defined as above by $\eta(\alpha) = i$ where $e_\alpha \in \mathfrak{g}(i)$ then the corresponding cocharacter $\gamma \in Y$ satisfying $\eta(\alpha) = \langle\alpha, \gamma\rangle$ is the map $\gamma:K^* \to T$ defined in 5.5.6.

Proposition 5.6.6 (Dynkin) *Let Δ be a simple system of roots for which $\langle\alpha, \gamma\rangle \geqslant 0$ for all $\alpha \in \Delta$. Then $\langle\alpha, \gamma\rangle \in \{0, 1, 2\}$ for all $\alpha \in \Delta$.*

Proof. Let $\langle e, h, f\rangle$ be the 3-dimensional subalgebra giving rise to the cocharacter γ. Let $\alpha \in \Delta$ and suppose $\langle\alpha, \gamma\rangle = j$. Then $e_\alpha \in \mathfrak{g}(j)$. Now $f \in \mathfrak{g}(-2)$ and so f is a linear combination $f = \sum_{\langle\beta,\gamma\rangle=-2} \lambda_\beta e_\beta$. Each β occurring in this sum is a negative root, hence $[fe_\alpha]$ is a linear combination of elements of $\mathfrak{t}$ and elements e_β with $\beta < 0$. In particular we have $[fe_\alpha] \in \oplus_{i\leqslant 0}\, \mathfrak{g}(i)$.

Suppose $[fe_\alpha] \neq 0$. Then $[fe_\alpha] \in \mathfrak{g}(j-2)$ and $[fe_\alpha] \in \oplus_{i\leqslant 0}\, \mathfrak{g}(i)$. Hence $j - 2 \leqslant 0$ and $j \leqslant 2$. However we know that $j \geqslant 0$ so $j \in \{0, 1, 2\}$.

Now suppose $[fe_\alpha] = 0$. Then $e_\alpha \in C_{\mathfrak{g}}(f)$. By applying the argument of 5.5.9 to f instead of to e we see that $C_{\mathfrak{g}}(f) \subseteq \oplus_{i\leqslant 0}\, \mathfrak{g}(i)$. Hence $e_\alpha \in \mathfrak{g}(j)$ and $e_\alpha \in \oplus_{i\leqslant 0}\, \mathfrak{g}(i)$. It follows that $j \leqslant 0$. Since we also have $j \geqslant 0$ we deduce that $j = 0$.

Thus $\langle\alpha, \gamma\rangle \in \{0, 1, 2\}$ for all $\alpha \in \Delta$. ■

We now define a weighted Dynkin diagram to be a Dynkin diagram with a number from the set $\{0, 1, 2\}$ attached to each node. We have described a procedure for constructing a weighted Dynkin diagram associated to any nonzero nilpotent element $e \in \mathfrak{g}$. We must verify however that the weighted Dynkin diagram is uniquely determined by e.

Proposition 5.6.7. *The weighted Dynkin diagram is uniquely determined by the nilpotent element $e \in \mathfrak{g}$.*

Proof. We recall the procedure for determining the diagram. The nonzero nilpotent element e can be embedded in a 3-dimensional subalgebra $\langle e, h, f\rangle$ of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(K)$ satisfying the standard relations. Any two such subalgebras are conjugate under the action of $C_G(e)^0$ by 5.5.10. The subalgebra $\langle e, h, f\rangle$ determines uniquely the map $\gamma: K^* \to G$ of 5.5.6 so also determines the 1-dimensional torus $S = \gamma(K^*)$. We then choose a maximal torus of G containing S. Any two such maximal tori will be conjugate in $C_G(S)$. Let Φ be the root system of G with respect to T. We choose a simple system Δ in Φ with respect to which γ is dominant. If Δ' is another such simple system there exists $w \in W$ such that $w(\Delta) = \Delta'$ and $w(\gamma) = \gamma$. The numbers $\langle \alpha_i, \gamma\rangle$ for $\alpha_i \in \Delta$ are then determined.

The process of determining the weighted Dynkin diagram from the nilpotent element e is thus unique up to conjugacy. The diagram is therefore uniquely determined by e. ■

It is natural also to define the weighted Dynkin diagram of the nilpotent element $e = 0$. This has the number 0 at each node of the diagram.

The weighted Dynkin diagram of the nilpotent element e will be denoted by $\Delta(e)$.

Proposition 5.6.8. *Let e, e_1 be nilpotent elements of $\mathfrak{g}$. Then $\Delta(e) = \Delta(e_1)$ if and only if e, e_1 lie in the same G-orbit.*

Proof. If e, e_1 lie in the same G-orbit we certainly have $\Delta(e) = \Delta(e_1)$. Suppose conversely that $\Delta(e) = \Delta(e_1)$. If all the weights of this diagram are 0 then $e = e_1 = 0$. So we may suppose that e, e_1 are nonzero. Then there are 3-dimensional subalgebras $\langle e, h, f\rangle$ and $\langle e_1, h_1, f_1\rangle$ satisfying the standard relations. These subalgebras determine maps $\gamma: K^* \to G$, $\gamma_1: K^* \to G$ as in 5.5.6. Let $S = \gamma(K^*)$, $S_1 = \gamma_1(K^*)$. Let T, T_1 be maximal tori of G containing S, S_1 respectively. We may assume without loss of generality that $T = T_1$. Thus γ, γ_1 lie in $Y(T)$. By conjugating by an element of W we may assume that γ, γ_1 are both dominant with respect to the same simple system Δ of roots. However $\gamma \in Y$ is uniquely determined by the numbers $\langle \alpha_i, \gamma\rangle$ for $\alpha_i \in \Delta$. Since $\Delta(e) = \Delta(e_1)$ we therefore have $\gamma = \gamma_1$. Hence $S = S_1$ and $Kh = \mathfrak{L}(S) = \mathfrak{L}(S_1) = Kh_1$. Moreover $h = d\gamma(1) = d\gamma_1(1) = h_1$. By 5.6.4 we deduce that e, e_1 lie in the same G-orbit.

5.7 DISTINGUISHED NILPOTENT ELEMENTS

We assume as before that K has characteristic 0 or $p > 3(h-1)$. Let e be a nonzero nilpotent element of $\mathfrak{g}$, $\langle e, h, f\rangle$ a 3-dimensional subalgebra containing e satisfying the standard relations and $\gamma: K^* \to G$ be the homomorphism arising from it as in 5.5.6. Let $S = \gamma(K^*)$ and T be a maximal torus of G containing S. Let Δ be a simple system of roots of G with respect to T for which $\gamma \in Y(T)$ is dominant.

We now introduce a parabolic subgroup of G which will play an important rôle in what follows. Let P be the subgroup of G given by

$$P = \langle T, X_\alpha; \langle \alpha, \gamma\rangle \geqslant 0\rangle.$$

Then P is a parabolic subgroup of G which has a Levi decomposition $P = U_P L$ where

$$U_P = \langle X_\alpha; \langle \alpha, \gamma\rangle > 0\rangle$$

$$L = \langle T, X_\alpha; \langle \alpha, \gamma\rangle = 0\rangle.$$

Let $\mathfrak{p} = \mathfrak{L}(P)$, $\mathfrak{u}_\mathfrak{p} = \mathfrak{L}(U_P)$, $\mathfrak{l} = \mathfrak{L}(L)$ be the Lie algebras of these subgroups. Then we have

$$\mathfrak{p} = \mathfrak{t} \oplus \sum_{\langle \alpha, \gamma\rangle \geqslant 0} K e_\alpha$$

$$\mathfrak{u}_\mathfrak{p} = \sum_{\langle \alpha, \gamma\rangle > 0} K e_\alpha$$

$$\mathfrak{l} = \mathfrak{t} \oplus \sum_{\langle \alpha, \gamma\rangle = 0} K e_\alpha.$$

The relation between these subalgebras and the grading $\mathfrak{g} = \oplus_i \mathfrak{g}(i)$ is as follows. We recall that $e_\alpha \in \mathfrak{g}(i)$ if and only if $\langle \alpha, \gamma\rangle = i$. Thus we have

$$\mathfrak{p} = \bigoplus_{i \geqslant 0} \mathfrak{g}(i), \quad \mathfrak{l} = \mathfrak{g}(0), \quad \mathfrak{u}_\mathfrak{p} = \bigoplus_{i > 0} \mathfrak{g}(i).$$

Proposition 5.7.1. (i) *P is uniquely determined by e.*
(ii) *$C_G(e) \subseteq P$.*

Proof. Since $P = N_G(P)$ it is clear that (i) implies (ii). Let $M = C_G(e)^0$. Then $\mathfrak{m} = \mathfrak{L}(M) = C_\mathfrak{g}(e)$. Let us decompose $\mathfrak{g}$ as a direct sum of standard irreducible $\langle e, h, f\rangle$-modules. Then $\dim \mathfrak{m}$ is the number of such standard irreducible components. Since each such component contains either a 1-dimensional subspace in $\mathfrak{g}(0)$ or a 1-dimensional subspace in $\mathfrak{g}(1)$, but not both, we have $\dim \mathfrak{m} = \dim \mathfrak{g}(0) + \dim \mathfrak{g}(1)$.

Now the P-orbit containing e lies in $\oplus_{i \geqslant 2} \mathfrak{g}(i)$ and so we have

$$\dim C_P(e) \geqslant \dim P - \dim\left(\bigoplus_{i \geqslant 2} \mathfrak{g}(i)\right) = \dim \mathfrak{g}(0) + \dim \mathfrak{g}(1) = \dim M.$$

Since $C_p(e)^0 \subseteq M$ we must have $C_p(e)^0 = M$ and so $M \subseteq P$.

We now consider the choices which appear in the definition of P. Given the element e we first construct the algebra $\langle e, h, f\rangle$. This is unique up to conjugacy by an element of M by 5.5.10 so its choice does not affect the definition of P, since $M \subseteq P$. The algebra $\langle e, h, f\rangle$ determines the 1-dimensional torus $S = \gamma(K^*)$. We then choose a maximal torus T containing S. This is unique up to conjugacy by an element of $C_G(S)^0$ and its choice does not affect the definition of P since $C_G(S)^0 \subseteq P$. Thus P is uniquely determined by e.

Proposition 5.7.2. *The map* $\operatorname{ad} e : \mathfrak{p} \to \bigoplus_{i\geqslant 2} \mathfrak{g}(i)$ *is surjective.*

Proof. Since $e \in \mathfrak{g}(2)$ and $\mathfrak{p} = \bigoplus_{i\geqslant 0} \mathfrak{g}(i)$ it is clear that $\operatorname{ad} e$ maps $\mathfrak{p}$ into $\bigoplus_{i\geqslant 2} \mathfrak{g}(i)$. Let us choose a standard basis of $\mathfrak{g}$ as an $\langle e, h, f\rangle$-module. Then each basis vector lies in some $\mathfrak{g}(i)$. If $i \geqslant 2$ each standard basis vector in $\mathfrak{g}(i)$ is the image under $\operatorname{ad} e$ of a standard basis vector in $\mathfrak{g}(i-2)$, i.e. of an element of $\mathfrak{p}$. Thus the map $\operatorname{ad} e : \mathfrak{p} \to \bigoplus_{i\geqslant 2}\mathfrak{g}(i)$ is surjective.

Proposition 5.7.3. *The morphism* $P \to \bigoplus_{i\geqslant 2} \mathfrak{g}(i)$ *given by* $x \to \operatorname{Ad} x.e$ *is dominant and separable. In particular the orbit of P on* $\bigoplus_{i\geqslant 2} \mathfrak{g}(i)$ *containing e is a dense open subset of* $\bigoplus_{i\geqslant 2} \mathfrak{g}(i)$.

Proof. It is clear that $\bigoplus_{i\geqslant 2} \mathfrak{g}(i)$ is a P-module under the adjoint action. Also $e \in \mathfrak{g}(2) \subseteq \bigoplus_{i\geqslant 2} \mathfrak{g}(i)$. Thus $x \to \operatorname{Ad} x.e$ is a morphism from P to $\bigoplus_{i\geqslant 2} \mathfrak{g}(i)$. Its differential is the map $-\operatorname{ad} e : \mathfrak{p} \to \bigoplus_{i\geqslant 2} \mathfrak{g}(i)$. This is surjective by 5.7.2. Thus the given morphism is dominant and separable by section 1.3. Since the image is open in its closure the P-orbit of e is a non-empty open subset of $\bigoplus_{i\geqslant 2} \mathfrak{g}(i)$. ■

A nilpotent element $e \in \mathfrak{g}$ is said to be distinguished if whenever $[es] = 0$ for $s \in \mathfrak{g}$ semisimple we have $s = 0$. Thus the distinguished nilpotent elements are those which commute with no nonzero semisimple element of $\mathfrak{g}$.

Proposition 5.7.4. *A nilpotent element* $e \in \mathfrak{g}$ *is distinguished if and only if* $\operatorname{ad} e : \mathfrak{g}(0) \to \mathfrak{g}(2)$ *is bijective.*

Proof. By 5.6.1 this map is surjective. We therefore consider its kernel. This is $C_{\mathfrak{g}(0)}(e) = C_{\mathfrak{l}}(e)$. Now we have

$$C_{\mathfrak{g}}(e) \subseteq \bigoplus_{i\geqslant 0} \mathfrak{g}(i) = \mathfrak{p} \qquad \text{as in the proof of 5.5.9.}$$

Thus $C_{\mathfrak{g}}(e) = C_{\mathfrak{p}}(e)$. Also from the proof of 5.5.9 we have $C_{\mathfrak{p}}(e) = \mathfrak{c} \oplus \mathfrak{r}$ where $\mathfrak{c} = C_{\mathfrak{l}}(e)$ and $\mathfrak{r} = C_{\mathfrak{u}_{\mathfrak{p}}}(e)$. Thus we have

$$C_{\mathfrak{g}}(e) = C_{\mathfrak{p}}(e) = C_{\mathfrak{l}}(e) \oplus C_{\mathfrak{u}_{\mathfrak{p}}}(e).$$

$$\text{Now } \mathfrak{c} = \mathfrak{L}(C) \text{ where } C = C_L(e) = L \cap C_G(e)^0 = C_G(S) \cap C_G(e)^0.$$

By 5.5.9 we know that C is reductive. On the other hand $\mathfrak{r}$ is a Lie algebra in which all elements are nilpotent. Thus $C_{\mathfrak{g}}(e)$ will contain a nonzero semisimple element if and only if $C_{\mathfrak{l}}(e)$ contains one. $C_{\mathfrak{l}}(e)$, being the Lie algebra of a reductive group, will certainly contain a nonzero semisimple element if it is nonzero. Thus $C_{\mathfrak{g}}(e)$ contains a nonzero semisimple element if and only if $C_{\mathfrak{l}}(e) \neq 0$. Thus e is distinguished if and only if $C_{\mathfrak{l}}(e) = 0$.

Corollary 5.7.5. *e is distinguished if and only if* $\dim \mathfrak{g}(0) = \dim \mathfrak{g}(2)$.

A nilpotent element e of $\mathfrak{g}$ is said to be even if all the numbers in the weighted Dynkin diagram $\Delta(e)$ are even (viz. 0 or 2).

If e is even we have $\langle \alpha_i, \gamma \rangle \in \{0, 2\}$ for all $\alpha_i \in \Delta$ and so $\langle \alpha, \gamma \rangle$ is even for all $\alpha \in \Phi$. It follows that $\mathfrak{g}(i) = 0$ for all odd i. We also observe that e is even if and only if $\mathfrak{g}(1) = 0$. For if e is not even it will contain a 1 in its weighted Dynkin diagram.

Proposition 5.7.6. *Every distinguished nilpotent element is even.*

Proof. (Jantzen) This is a consequence of Richardson's theorem of section 5.2. Let e be distinguished. Then we have a corresponding decomposition $\mathfrak{g} = \oplus_i \mathfrak{g}(i)$ with $\dim \mathfrak{g}(0) = \dim \mathfrak{g}(2)$, by 5.7.5. We must show that $\mathfrak{g}(1) = 0$. We have $\mathfrak{p} = \oplus_{i \geqslant 0} \mathfrak{g}(i)$ and $\mathfrak{u}_{\mathfrak{p}} = \oplus_{i>0} \mathfrak{g}(i)$. By 5.2.5 there exists an element $x \in \mathfrak{u}_{\mathfrak{p}}$ with $[\mathfrak{p}x] = \mathfrak{u}_{\mathfrak{p}}$. Let $x = \sum_{i>0} x_i$ with $x_i \in \mathfrak{g}(i)$. Then

$$\left[\bigoplus_{i \geqslant 0} \mathfrak{g}(i), \sum_{i>0} x_i\right] = \bigoplus_{i>0} \mathfrak{g}(i).$$

We intersect both sides of this equation with $\mathfrak{g}(1) \oplus \mathfrak{g}(2)$. Then

$$[\mathfrak{g}(0), x_1 + x_2] + [\mathfrak{g}(1), x_1] = \mathfrak{g}(1) \oplus \mathfrak{g}(2).$$

Suppose if possible that $\mathfrak{g}(1) \neq 0$. Since $x_1 \in \mathfrak{g}(1)$ we then have

$$\dim[\mathfrak{g}(1), x_1] < \dim \mathfrak{g}(1).$$

It follows that

$$\dim[\mathfrak{g}(0), x_1 + x_2] + \dim[\mathfrak{g}(1), x_1] < \dim \mathfrak{g}(0) + \dim \mathfrak{g}(1) = \dim \mathfrak{g}(2) + \dim \mathfrak{g}(1)$$

and we have a contradiction. Hence $\mathfrak{g}(1) = 0$ and e is even.

5.8 DISTINGUISHED PARABOLIC SUBGROUPS

Let P_J be a standard parabolic subgroup of G with Levi decomposition $P_J = U_J L_J$. We describe a decomposition of $\mathfrak{g} = \mathfrak{L}(G)$ into graded components $\mathfrak{g} = \oplus_i \mathfrak{g}_J(i)$ corresponding to P_J. We first define a function $\eta_J : \Phi \to 2\mathbb{Z}$ by

$$\eta_J(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \Delta_J \\ 2 & \text{if } \alpha \in \Delta - \Delta_J \end{cases}$$

and extending to arbitrary roots by linearity. We then define $\mathfrak{g}_J(i)$ by

$$\mathfrak{g}_J(i) = \begin{cases} \sum_{\eta_J(\alpha)=i} Ke_\alpha & \text{if } i \neq 0 \\ \mathfrak{t} + \sum_{\eta_J(\alpha)=0} Ke_\alpha & \text{if } i = 0. \end{cases}$$

Then we have $\mathfrak{g} = \bigoplus_i \mathfrak{g}_J(i)$.

Proposition 5.8.1. $\dim \mathfrak{g}_J(0) = \dim L_J, \dim \mathfrak{g}_J(2) = \dim U_J/U_J'$.

Proof. We have $\mathfrak{g}_J(0) = \mathfrak{l}_J = \mathfrak{L}(L_J)$ and so $\dim \mathfrak{g}_J(0) = \dim L_J$. We also have $U_J = \prod_{\eta_J(\alpha)\geqslant 2} X_\alpha$. We shall show that $U_J' = \prod_{\eta_J(\alpha)>2} X_\alpha$. This will follow from Chevalley's commutator formula. The commutator formula implies immediately that $U_J' \subseteq \prod_{\eta_J(\alpha)>2} X_\alpha$. Let $\alpha \in \Phi$ satisfy $\eta_J(\alpha) > 2$. We show that there exist $\beta, \gamma \in \Phi$ with $\alpha = \beta + \gamma$, $\eta_J(\beta) = 2$, $\eta_J(\gamma) \geqslant 2$. We first choose an element $w \in W_J$ such that

$$(w(\alpha), \alpha_i) \leqslant 0 \qquad \text{for all } i \in J.$$

Such an element w can certainly be found in W_J. Since $\alpha \notin \Phi_J$ we have $w(\alpha) > 0$ and $\eta_J(w(\alpha)) = \eta_J(\alpha)$. Let $w(\alpha) = \sum_{\alpha_i \in \Delta} n_i\alpha_i$. Then we have

$$0 < (w(\alpha), w(\alpha)) = \sum_{\alpha_i\in\Delta} n_i(w(\alpha), \alpha_i).$$

Since $(w(\alpha), \alpha_i) \leqslant 0$ for all $i \in J$ there exists $i \in I - J$ with $(w(\alpha), \alpha_i) > 0$. Thus $w(\alpha) - \alpha_i \in \Phi$. Hence $\alpha - w^{-1}(\alpha_i) \in \Phi$. Let $\beta = w^{-1}(\alpha_i)$, $\gamma = \alpha - w^{-1}(\alpha_i)$. Then $\alpha = \beta + \gamma$ with $\eta_J(\beta) = 2$ and $\eta_J(\gamma) \geqslant 2$. Chevalley's commutator formula then shows that

$$[x_\beta(\lambda), x_\gamma(\mu)] = \prod_{\substack{i,j>0 \\ i,j\in\mathbb{Z} \\ i\beta+j\gamma\in\Phi}} x_{i\beta+j\gamma}(C_{ij\beta\gamma}\lambda^i\mu^j)$$

where each $C_{ij\beta\gamma}$ satisfies $|C_{ij\beta\gamma}| = 1, 2$ or 3. We assume the characteristic of K is not 2 or 3. (This certainly follows from the general assumptions about the characteristic in this chapter.) Thus

$$[x_\beta(\lambda), x_\gamma(C_{11\beta\gamma}{}^{-1})] = x_\alpha(\lambda) \prod_{\eta_J(\delta)>\eta_J(\alpha)} x_\delta(\mu_\delta).$$

By induction we may assume that $X_\delta \subseteq U_J'$ for all $\delta \in \Phi$ with $\eta_J(\delta) > \eta_J(\alpha)$. It follows that $x_\alpha(\lambda) \in U_J'$ for all $\lambda \in K$. Thus $X_\alpha \subseteq U_J'$ for all α with $\eta_J(\delta) > 2$. Hence $U_J' = \prod_{\eta_J(\alpha)>2} X_\alpha$ and $\dim U_J/U_J' = \dim \mathfrak{g}_J(2)$.

Proposition 5.8.2. *For any parabolic subgroup P_J of G we have* $\dim L_J \geqslant \dim U_J/U_J'$.

Proof. This is a consequence of Richardson's theorem 5.2.1. P_J has a dense orbit on U_J, so also has a dense orbit on U_J/U_J'. Now U_J acts trivially by conjugation on U_J/U_J', so P_J/U_J acts on U_J/U_J' with a dense orbit. It follows that $\dim P_J/U_J \geqslant \dim U_J/U_J'$. ∎

A parabolic subgroup P of G is called distinguished if $\dim P/U_P = \dim U_P/U_P'$.

Corollary 5.8.3. *The standard parabolic subgroup P_J is distinguished if and only if* $\dim \mathfrak{g}_J(0) = \dim \mathfrak{g}_J(2)$.

Proof. This follows from 5.8.1.

Following these remarks on parabolic subgroups we return to the general theme of the chapter, assuming as usual that the characteristic is either 0 or $p > 3(h-1)$.

Proposition 5.8.4. *Let e be a distinguished nilpotent element of $\mathfrak{g}$ and P_J be the parabolic subgroup associated with e in section 5.7. Then:*

(i) *P_J is a distinguished parabolic subgroup of G.*
(ii) *e lies in the dense orbit of P_J on $\mathfrak{u}_J$.*

Proof. Since e is distinguished, e is even by 5.7.5. P_J is given by

$$P_J = \langle T, X_\alpha; \langle \alpha, \gamma \rangle \geqslant 0 \rangle.$$

Since e is even $\langle \alpha_i, \gamma \rangle = 0$ or 2 for all $\alpha_i \in \Delta$. $\langle \alpha_i, \gamma \rangle = 0$ if and only if $i \in J$. Thus the function η_J on the simple roots is given by $\eta_J(\alpha_i) = \langle \alpha_i, \gamma \rangle$. By linearity we have $\eta_J(\alpha) = \langle \alpha, \gamma \rangle$ for all $\alpha \in \Phi$. Thus $\mathfrak{g}_J(i) = \mathfrak{g}(i)$ for all $i \in \mathbb{Z}$. Since e is distinguished we have $\dim \mathfrak{g}(0) = \dim \mathfrak{g}(2)$ by 5.7.5. Thus $\dim \mathfrak{g}_J(0) = \dim \mathfrak{g}_J(2)$ and so P_J is distinguished by 5.8.3.

Now the P_J-orbit on $\oplus_{i \geqslant 2} \mathfrak{g}(i)$ containing e is a dense open subset of $\oplus_{i \geqslant 2} \mathfrak{g}(i)$ by 5.7.3. Since $\mathfrak{g}(1) = 0$ we have $\mathfrak{u}_J = \oplus_{i \geqslant 2} \mathfrak{g}(i)$. Thus e lies in the dense open orbit of P_J on $\mathfrak{u}_J$. ∎

We now begin conversely with a distinguished parabolic subgroup P_J and consider the nilpotent elements in the dense open orbit of P_J on $\mathfrak{u}_J$. Let $P_J = U_J L_J$ be the Levi decomposition, let T be a maximal torus of L_J and Δ be a simple system of roots with respect to T such that the corresponding Borel subgroup lies in P_J.

Proposition 5.8.5. *Let $e = \sum_{\eta_J(\alpha) > 0} \lambda_\alpha e_\alpha$ lie in the dense orbit of P_J on $\mathfrak{u}_J$. Then the U_J-orbit containing e consists precisely of elements of the form $\sum_{\eta_J(\alpha) \geqslant 2} \mu_\alpha e_\alpha$ for which $\mu_\alpha = \lambda_\alpha$ for all α with $\eta_J(\alpha) = 2$. In particular the element $\sum_{\eta_J(\alpha) = 2} \lambda_\alpha e_\alpha$ lies in the same U_J-orbit as e.*

Proof. Let C be the U_J-orbit containing e and let

$$D = \left\{ \sum_{\eta_J(\alpha) \geqslant 2} \mu_\alpha e_\alpha;\ \mu_\alpha = \lambda_\alpha \text{ when } \eta_J(\alpha) = 2 \right\}.$$

D is a closed irreducible subset of $\mathfrak{u}_J$ of dimension

$$\dim D = \dim \mathfrak{u}_J - \dim \mathfrak{g}_J(2).$$

It is evident that, for $x \in U_J$, Ad $x.e$ has the same coefficients as e for root vectors e_α with $\eta_J(\alpha) = 2$. Thus C is contained in D. C is a closed subset of D by the theorem of Rosenlicht mentioned at the beginning of this chapter. We calculate the dimension of C. We have

$$\dim C = \dim U_J - \dim C_{U_J}(e)$$

and so $\dim C_{P_J}(e) \geqslant \dim U_J - \dim C$. However we also have

$$\dim C_{P_J}(e) = \dim P_J - \dim \mathfrak{u}_J$$

since e lies in the dense orbit of P_J on $\mathfrak{u}_J$. It follows that

$$\begin{aligned} \dim C &\geqslant \dim U_J - \dim C_{P_J}(e) \\ &= 2 \dim U_J - \dim P_J \\ &= \dim U_J - \dim \mathfrak{g}_J(0). \end{aligned}$$

But P_J is distinguished and so $\dim \mathfrak{g}_J(0) = \dim \mathfrak{g}_J(2)$ by 5.8.3. Thus we have

$$\dim C \geqslant \dim U_J - \dim \mathfrak{g}_J(2) = \dim D.$$

Since C is a subset of D this implies that $\dim C = \dim D$. Since D is irreducible we have $C = D$.

Corollary 5.8.6. *If P_J is a distinguished parabolic subgroup and e lies in the dense orbit of P_J on $\mathfrak{u}_J$ then:*

(i) $\dim C_{P_J}(e) = \dim C_{U_J}(e)$.
(ii) *$C_{P_J}(e)^0$ lies in U_J.*

Proof. This follows from the proof of 5.8.5.

Proposition 5.8.7. *Let P_J be a distinguished parabolic subgroup and let e lie in the dense orbit of P_J on $\mathfrak{u}_J$. Suppose $e \in \mathfrak{g}_J(2)$. (We can choose e to satisfy this condition, by 5.8.5.) Then:*

(i) *e is a distinguished nilpotent element.*
(ii) *The L_J-orbit of e is dense and open in $\mathfrak{g}_J(2)$.*

Proof. We use Richardson's theorem 5.2.4. This shows that $C_G(e)^0 = C_{P_J}(e)^0$, and this is equal to $C_{U_J}(e)^0$ by 5.8.6. On the Lie algebra level we have

$$C_{\mathfrak{g}}(e) = C_{\mathfrak{p}_J}(e) = C_{\mathfrak{u}_J}(e).$$

Thus $C_{\mathfrak{g}}(e)$ contains no nonzero semisimple element and so e is distinguished.

We now consider the map ad e: $\mathfrak{g}_J(0) \to \mathfrak{g}_J(2)$ and recall that $\mathfrak{g}_J(0) = \mathfrak{l}_J = \mathfrak{L}(L_J)$. Since $C_{\mathfrak{l}_J}(e) = 0$ this map has kernel 0. Since P_J is distinguished we have dim $\mathfrak{g}_J(0) = \dim \mathfrak{g}_J(2)$ by 5.8.3 and so our map must be bijective. It is the differential of the map

$$L_J \to \mathfrak{g}_J(2)$$
$$x \to \mathrm{Ad}\, x.e.$$

This map is therefore dominant, by section 1.3. Let its image be C. Then C is open in $\bar{C} = \mathfrak{g}_J(2)$. Thus the L_J-orbit of e is a dense open subset of $\mathfrak{g}_J(2)$.

Proposition 5.8.8. *Let P_J be a distinguished parabolic subgroup of G and e lie in the dense orbit of P_J on $\mathfrak{u}_J$. Suppose $e \in \mathfrak{g}_J(2)$. Then the parabolic subgroup associated to e in section 5.7 is equal to P_J.*

Proof. Let C be the L_J-orbit containing e. C is a dense open subset of $\mathfrak{g}_J(2)$ by 5.8.7. Thus $C \cap Ke$ is a non-empty open subset of Ke. It must therefore contain all but a finite number of elements of Ke. Let

$$N_{L_J}(e) = \{x \in L_J;\ \mathrm{Ad}\, x.e \in Ke\}.$$

Then $N_{L_J}(e)$ is infinite and so $\dim N_{L_J}(e) \geqslant 1$. However $\dim C_{L_J}(e) = 0$ by 5.8.6. Also $\dim N_{L_J}(e)/C_{L_J}(e) \leqslant 1$. Thus we have $\dim N_{L_J}(e) = 1$. Let $S = N_{L_J}(e)^0$. S is a 1-dimensional connected subgroup of L_J. It is a torus since it has a nontrivial 1-dimensional representation $x \to \rho(x)$ where $\mathrm{Ad}\, x.e = \rho(x)e$. Let $\mathfrak{s} = \mathfrak{L}(S)$. We show there is a 3-dimensional subalgebra $\langle e, h, f\rangle$ satisfying the standard relations with $h \in \mathfrak{s}$.

Now e certainly lies in some 3-dimensional subalgebra $\langle e, h', f'\rangle$. h' lies in the Lie algebra of some 1-dimensional torus S' in $N_G(e)^0$. Now e is distinguished and so $C_G(e)^0$ contains only unipotent elements. Thus we have

$$\dim N_G(e)^0 = \dim C_G(e)^0 + 1$$

and S, S' are maximal tori of $N_G(e)^0$. Thus there exists $z \in C_G(e)^0$ such that ${}^zS' = S$. We define $h = \mathrm{Ad}\, z.h'$ and $f = \mathrm{Ad}\, z.f'$ and obtain a subalgebra $\langle e, h, f\rangle$ with $h \in \mathfrak{s}$.

Let T be the maximal torus L_ϕ of G. We do not know that the 1-dimensional torus S lies in T. So let T' be a maximal torus of L_J containing S. T and T' are conjugate in L_J. Let $T = {}^mT'$ where $m \in L_J$. Then T contains ${}^mS$ and Ad $m.e$ lies in the dense orbit of P_J on $\mathfrak{u}_J$ and also in $\mathfrak{g}_J(2)$.

We shall prove the proposition with e replaced by Ad $m.e$, so that we may assume that $T = T'$. In the general case we note that replacing e by Ad $m.e$ involves only conjugation by an element of P_J, so that the parabolic subgroup associated to e is also equal to P_J.

We thus assume that S lies in T. We decompose $\mathfrak{g}$ as an $\langle e, h, f\rangle$-module and obtain a homomorphism $\gamma: K^* \to G$ as in 5.5.6. We then consider the parabolic subgroup

$$P = \langle T, X_\alpha;\ \langle \alpha, \gamma\rangle \geqslant 0\rangle \qquad \text{as in section 5.7.}$$

We wish to show that $P = P_J$. Now $e \in \mathfrak{g}_J(2)$ must have the form

$$e = \sum_{\eta_J(\alpha)=2} \lambda_\alpha e_\alpha.$$

Roots α with $\eta_J(\alpha) = 2$ involve a single simple root not in Δ_J which must occur with multiplicity 1. There is a unique element $h \in \mathfrak{t} = \mathfrak{L}(T)$ such that

$$d\alpha_i(h) = \begin{cases} 0 & \text{if } \alpha_i \in \Delta_J \\ 2 & \text{if } \alpha_i \in \Delta - \Delta_J \end{cases}$$

and this element will satisfy $[he] = 2e$. Since e is distinguished there is a unique element $h \in \mathfrak{t}$ satisfying $[he] = 2e$. This must therefore be the element h considered above. This element h satisfies

$$[he_{\alpha_i}] = 0 \quad \text{if } \alpha_i \in \Delta_J$$

$$[he_{\alpha_i}] = 2e_{\alpha_i} \quad \text{if } \alpha_i \in \Delta - \Delta_J.$$

By 5.5.8 we have

$$\langle \alpha_i, \gamma \rangle = 0 \quad \text{if } \alpha_i \in \Delta_J$$

$$\langle \alpha_i, \gamma \rangle = 2 \quad \text{if } \alpha_i \in \Delta - \Delta_J.$$

It follows that

$$P = \langle T, X_\alpha; \langle \alpha, \gamma \rangle \geqslant 0 \rangle = P_J.$$

Corollary 5.8.9. *Let P_J be a distinguished parabolic subgroup of G and e lie in the dense orbit of P_J on $\mathfrak{u}_J$. Then the parabolic subgroup associated to e by section 5.7 is equal to P_J.*

5.9 THE BALA–CARTER THEOREM

In this section we shall prove a result which describes the nilpotent orbits in $\mathfrak{g}$ when the characteristic is 0 or $p > 3(h-1)$.

We first need some preliminary results.

Proposition 5.9.1. *Let L_J be a Levi subgroup of G and $\mathfrak{l}_J = \mathfrak{L}(L_J)$. Let $\mathfrak{z}_J$ be the centre of $\mathfrak{l}_J$. Then $C_{\mathfrak{g}}(\mathfrak{z}_J) = \mathfrak{l}_J$.*

Proof. We have $\mathfrak{l}_J = \mathfrak{t} + \sum_{\alpha \in \Phi_J} K e_\alpha$. $\mathfrak{t}$ has a basis h_α, $\alpha \in \Delta$, where $[h_\alpha e_\beta] = A_{\alpha\beta} e_\beta$. The numbers $A_{\alpha\beta}$ are the Cartan integers defined by $w_\alpha(\beta) = \beta - A_{\alpha\beta}\alpha$. Now the restrictions on the characteristic certainly imply that all the $A_{\alpha\beta}$ which are nonzero in $\mathbb{Z}$ are nonzero in K. Thus we see that $e_\beta \notin \mathfrak{z}_J$ and that $\mathfrak{z}_J \subseteq \mathfrak{t}$. The restrictions on the characteristic also imply that the Killing form is nondegenerate, by section 1.13. Let $\sum_{\alpha \in \Delta} \lambda_\alpha h_\alpha$ lie in $\mathfrak{z}_J$. Then we have

$$\left[\sum_{\alpha \in \Delta} \lambda_\alpha h_\alpha, e_\beta\right] = 0 \qquad \text{for all } \beta \in \Phi_J.$$

Hence

$$\sum_{\alpha\in\Delta} \lambda_\alpha A_{\alpha\beta} e_\beta = 0 \qquad \text{for all } \beta \in \Phi_J$$

and so

$$\sum_{\alpha\in\Delta} \lambda_\alpha A_{\alpha\beta} = 0 \qquad \text{for all } \beta \in \Phi_J.$$

This gives

$$\left(\sum_{\alpha\in\Delta} \lambda_\alpha h_\alpha, h_\beta\right) = 0 \qquad \text{for all } \beta \in \Phi_J.$$

Let $\mathfrak{t}_J$ be the subspace of $\mathfrak{t}$ spanned by the elements h_β with $\beta \in \Delta_J$. Let $\mathfrak{t}_J{}^\perp$ be the orthogonal space of $\mathfrak{t}_J$ in $\mathfrak{t}$ with respect to the Killing form. Then we have shown that $\mathfrak{z}_J = \mathfrak{t}_J{}^\perp$.

We now consider $C_\mathfrak{g}(\mathfrak{z}_J) = C_\mathfrak{g}(\mathfrak{t}_J{}^\perp)$. This clearly contains $\mathfrak{l}_J$. On the other hand it cannot contain any nonzero element of $\sum_{\beta\in\Phi-\Phi_J} Ke_\beta$. For suppose

$$\left[h, \sum_{\beta\notin\Phi_J} \lambda_\beta e_\beta\right] = 0 \qquad \text{for all } h \in \mathfrak{t}_J{}^\perp.$$

Then $\sum_{\beta\notin\Phi_J} \lambda_\beta \beta(h) e_\beta = 0$ for all $h \in \mathfrak{t}_J{}^\perp$, and so for all $\beta \notin \Phi_J$ with $\lambda_\beta \neq 0$ we must have $\beta(h) = 0$ for all $h \in \mathfrak{t}_J{}^\perp$. But this implies $h_\beta \in \mathfrak{t}_J$ and so $\beta \in \Phi_J$, a contradiction. Thus we have $\sum_{\beta\notin\Phi_J} \lambda_\beta e_\beta = 0$. It follows that $C_\mathfrak{g}(\mathfrak{z}_J) = \mathfrak{l}_J$.

Proposition 5.9.2. *Let $\mathfrak{s}$ be any subset of $\mathfrak{t}$. Then $C_\mathfrak{g}(\mathfrak{s})$ is a Levi subalgebra $\mathfrak{l}_J$ of $\mathfrak{g}$ for some simple system Δ of roots and some J with $\Delta_J \subseteq \Delta$.*

Proof. We have $\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha\in\Phi} Ke_\alpha$. $C_\mathfrak{g}(\mathfrak{s})$ certainly contains $\mathfrak{t}$. So suppose $\sum_{\alpha\in\Phi} \lambda_\alpha e_\alpha \in C_\mathfrak{g}(\mathfrak{s})$. Then we have

$$0 = \left[x, \sum_{\alpha\in\Phi} \lambda_\alpha e_\alpha\right] = \sum_{\alpha\in\Phi} \lambda_\alpha \alpha(x) e_\alpha = \sum_{\alpha\in\Phi} \lambda_\alpha (h_\alpha, x) e_\alpha$$

for all $x \in \mathfrak{s}$. Hence $(h_\alpha, x) = 0$ whenever $\lambda_\alpha \neq 0$ for all $x \in \mathfrak{s}$. Thus we have

$$C_\mathfrak{g}(\mathfrak{s}) = \mathfrak{t} + \sum_{\alpha\in\Psi} Ke_\alpha$$

where $\Psi = \{\alpha \in \Phi; (h_\alpha, x) = 0 \text{ for all } x \in \mathfrak{s}\}$. In particular Ψ is a set of roots which is closed under the formation of rational linear combinations. However any set of roots closed under rational linear combinations has the form Φ_J for some simple system Δ and some J with $\Delta_J \subseteq \Delta$ (Slodowy [2], p. 23). Then

$$C_\mathfrak{g}(\mathfrak{s}) = \mathfrak{t} \oplus \sum_{\alpha\in\Phi_J} Ke_\alpha = \mathfrak{l}_J. \qquad \blacksquare$$

Given a nilpotent element $e \in \mathfrak{g}$ we recall from 5.5.9 that $C_G(e)^0$ factorizes as $C_G(e)^0 = RC$ where R is a normal unipotent subgroup and C is connected reductive.

Let $\bar{S}$ be a maximal torus of C and $\bar{T}$ be a maximal torus of G containing $\bar{S}$. Let $\bar{\mathfrak{s}} = \mathfrak{L}(\bar{S})$ and $\mathfrak{l} = C_{\mathfrak{g}}(\bar{\mathfrak{s}})$. Since $\bar{\mathfrak{s}} \subseteq \bar{\mathfrak{t}} = \mathfrak{L}(\bar{T})$ we know from 5.9.2 that $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$. Moreover $[e, \bar{\mathfrak{s}}] = 0$ and so $e \in \mathfrak{l}$.

Proposition 5.9.3. *$\mathfrak{l}$ is minimal among the Levi subalgebras of $\mathfrak{g}$ containing e. Further, any other minimal Levi subalgebra of $\mathfrak{g}$ containing e is conjugate to $\mathfrak{l}$ by an element of $C_G(e)^0$.*

Proof. Let $\mathfrak{l}^*$ be any Levi subalgebra of $\mathfrak{g}$ containing e. Then $\mathfrak{l}^* = \mathfrak{L}(L^*)$ for some Levi subgroup L^* of G. Let $C^* = Z^0(L^*)$ be the connected centre of L^*. C^* is a torus and $\mathfrak{c}^* = \mathfrak{L}(C^*)$ is the centre of $\mathfrak{l}^*$. We have $\mathfrak{c}^* \subseteq C_{\mathfrak{g}}(e)$ and $C^* \subseteq C_G(e)$. Now $\bar{S}$ is a maximal torus of $C_G(e)^0$, thus there exists $z \in C_G(e)^0$ such that ${}^zC^* \subseteq \bar{S}$. Thus $\operatorname{Ad} z . \mathfrak{c}^* \subseteq \bar{\mathfrak{s}}$. It follows that

$$\mathfrak{l} = C_{\mathfrak{g}}(\bar{\mathfrak{s}}) \subseteq C_{\mathfrak{g}}(\operatorname{Ad} z . \mathfrak{c}^*) = \operatorname{Ad} z . C_{\mathfrak{g}}(\mathfrak{c}^*) = \operatorname{Ad} z . \mathfrak{l}^*$$

by 5.9.1. Thus $\mathfrak{l} \subseteq \operatorname{Ad} z . \mathfrak{l}^*$. In particular we have $\dim \mathfrak{l} \leqslant \dim \mathfrak{l}^*$ and so $\mathfrak{l}$ must be a minimal Levi subalgebra containing e.

Now suppose that $\mathfrak{l}^*$ is also minimal. Then $\operatorname{Ad} z . \mathfrak{l}^*$ must also be minimal and we have $\mathfrak{l} = \operatorname{Ad} z . \mathfrak{l}^*$. Thus $\mathfrak{l}$, $\mathfrak{l}^*$ are conjugate by an element of $C_G(e)^0$.

Proposition 5.9.4. *Let $\mathfrak{l}$ be a minimal Levi subalgebra of $\mathfrak{g}$ containing e and let $\mathfrak{k}$ be the semisimple part of $\mathfrak{l}$. Then e lies in $\mathfrak{k}$ and is a distinguished nilpotent element of $\mathfrak{k}$.*

Proof. Let $\mathfrak{c}$ be the centre of $\mathfrak{l}$. Then $\mathfrak{l} = \mathfrak{c} \oplus \mathfrak{k}$ and all nilpotent elements of $\mathfrak{l}$ lie in $\mathfrak{k}$. Suppose if possible that there is a nonzero semisimple element $s \in \mathfrak{k}$ with $[es] = 0$. Let $\mathfrak{s} = \mathfrak{c} \cup \{s\}$. The set $\mathfrak{s}$ lies in any Cartan subalgebra of $\mathfrak{l}$ containing s. By 5.9.2 $C_{\mathfrak{l}}(\mathfrak{s})$ is a Levi subalgebra of $\mathfrak{l}$. It must therefore be a Levi subalgebra of $\mathfrak{g}$ contained in $\mathfrak{l}$. However $C_{\mathfrak{l}}(\mathfrak{s})$ contains e and is strictly contained in $\mathfrak{l}$. This contradicts the minimality of $\mathfrak{l}$. Thus e is a distinguished nilpotent element of $\mathfrak{k}$. ■

We are now in a position to state and prove the main theorems on the classification of nilpotent G-orbits on $\mathfrak{g}$.

Theorem 5.9.5. (Bala, Carter) *Let G be a simple algebraic group of adjoint type over K and suppose the characteristic of K is either 0 or $p > 3(h - 1)$ where h is the Coxeter number of G. Let $\mathfrak{g} = \mathfrak{L}(G)$. Then:*

(i) *There is a bijective map between G-orbits of distinguished nilpotent elements of $\mathfrak{g}$ and conjugacy classes of distinguished parabolic subgroups of G. The G-orbit corresponding to a given parabolic subgroup P contains the dense orbit of P acting on the Lie algebra of its unipotent radical.*

(ii) *There is a bijective map between G-orbits of nilpotent elements of $\mathfrak{g}$ and G-classes of pairs $(L, P_{L'})$ where L is a Levi subgroup of G and $P_{L'}$ is a distinguished parabolic subgroup of the semisimple part L' of L. The G-orbit corresponding to a* *given pair $(L, P_{L'})$ contains the dense orbit of $P_{L'}$ acting on the Lie algebra of its unipotent radical.*

Proof. (i) We are here comparing classes of distinguished parabolic subgroups and orbits of distinguished nilpotent elements. Let P be a distinguished parabolic subgroup, U_P its unipotent radical and $\mathfrak{u}_p = \mathfrak{L}(U_p)$. Then P has a unique dense orbit on $\mathfrak{u}_p$ by 5.2.3. This determines a G-orbit on $\mathfrak{g}$. Thus we have a map ϕ from conjugacy classes of distinguished parabolic subgroups of G to nilpotent G-orbits on $\mathfrak{g}$. We must show that the image of ϕ consists of the distinguished nilpotent orbits and that ϕ is injective.

By 5.8.7 the dense orbit of P on $\mathfrak{u}_p$ consists of distinguished nilpotent elements. By 5.8.4 every distinguished nilpotent element lies in the dense orbit of P on $\mathfrak{u}_p$ for some distinguished parabolic subgroup P. Thus the image of ϕ consists of the distinguished nilpotent orbits. However by 5.8.9 any two distinguished parabolic subgroups giving rise to the same nilpotent orbit must be conjugate. Thus ϕ is injective.

(ii) We now compare the class of pairs $(L, P_{L'})$ with the set of all nilpotent orbits of $\mathfrak{g}$. We again have a well determined map ϕ from the class of pairs $(L, P_{L'})$ to the set of nilpotent orbits. The nilpotent orbit corresponding to $(L, P_{L'})$ is that containing the dense orbit of $P_{L'}$ on $u_{P_{L'}}$. We must show that ϕ is bijective.

Given any nilpotent element $e \in \mathfrak{g}$ let $\mathfrak{l}$ be a minimal Levi subalgebra of $\mathfrak{g}$ containing e. Let L be the Levi subgroup of G such that $\mathfrak{l} = \mathfrak{L}(L)$. Let L' be the semisimple part of L and $\mathfrak{l}' = \mathfrak{L}(L')$. Then, by 5.9.4, e lies in $\mathfrak{l}'$ and e is distinguished in $\mathfrak{l}'$. By (i) there exists a distinguished parabolic subgroup $P_{L'}$ of L' such that e lies in the dense orbit of $P_{L'}$ on $\mathfrak{u}_{P_{L'}}$. Thus the map ϕ is surjective.

By 5.9.3 any two minimal Levi subalgebras of $\mathfrak{g}$ containing e are conjugate by an element of $C_G(e)^0$. Thus L' is determined by e up to conjugacy by an element of $C_G(e)^0$. Moreover by (i) the distinguished parabolic subgroup $P_{L'}$ is determined up to conjugacy in L'. Thus the pair $(L, P_{L'})$ is determined up to conjugacy in G. Hence the map ϕ is bijective. ■

Since we have a bijection between the unipotent conjugacy classes of G and the nilpotent orbits of $\mathfrak{g}$ we can derive a corresponding description of the unipotent classes of G. A unipotent element $u \in G$ will be called distinguished if $C_G(u)^0$ contains no non-identity semisimple element.

Theorem 5.9.6. *Let G be a simple algebraic group of adjoint type over K where the characteristic of K is either 0 or $p > 3(h - 1)$. Then:*

(i) *There is a bijective map between conjugacy classes of distinguished unipotent elements of G and conjugacy classes of distinguished parabolic subgroups of G. The unipotent class corresponding to a given parabolic subgroup P contains the dense orbit of P on its unipotent radical U_P.*

(ii) *There is a bijective map between conjugacy classes of unipotent elements of G and G-classes of pairs $(L, P_{L'})$ where L is a Levi subgroup of G and $P_{L'}$ is a* *distinguished parabolic subgroup of the semisimple part L' of L. The unipotent class corresponding to the pair $(L, P_{L'})$ contains the dense orbit of $P_{L'}$ on its unipotent radical.*

Proof. This result will follow from 5.9.5 by making use of Springer's map $\phi:\mathfrak{U} \to \mathfrak{N}$ from the unipotent variety to the nilpotent variety. The conditions on the characteristic certainly imply that this is either 0 or a very good prime for G. Thus by section 1.15 there is a bijective morphism of varieties $\phi:\mathfrak{U} \to \mathfrak{N}$ which preserves the G-actions on $\mathfrak{U}$ and $\mathfrak{N}$.

Now $\phi(U) = \mathfrak{L}(U)$. For let u be a regular unipotent element in U and $\phi(u) = e$. Then e is a regular nilpotent element in $\mathfrak{N}$. u lies in a unique Borel subgroup B and there is a unique Borel subgroup B' such that $\mathfrak{L}(B')$ contains e. Moreover we have

$$u \in C_G(u) = C_G(e) \subseteq B'$$

and so $B = B'$. Now U is the closure of the B-orbit of u. So $\phi(U)$ is the closure of the B-orbit of e, which is $\mathfrak{L}(U)$.

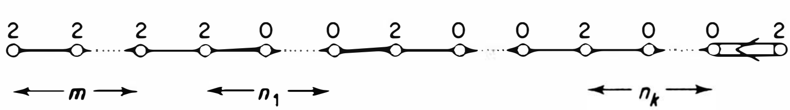
We see in a similar way that $\phi(U_J) = \mathfrak{L}(U_J)$ for each J. Moreover ϕ induces a bijection between unipotent conjugacy classes of G and nilpotent orbits of $\mathfrak{g}$ which transforms distinguished unipotent elements to distinguished nilpotent elements and vice-versa. Thus the required result follows from 5.9.5 by the application of ϕ. ■

In connection with the classification of unipotent and nilpotent elements it is useful to know explicitly which parabolic subgroups are distinguished. This was determined by Bala and Carter in [2], [3] and the results are listed in the following table.

Distinguished parabolic subgroups

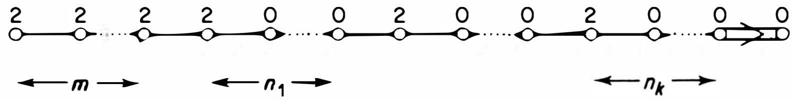
Type A_l The only distinguished parabolic subgroups are the Borel subgroups. These have diagram

Type C_l The diagrams of distinguished parabolic subgroups are those of the form



where $m + n_1 + \ldots + n_k + 1 = l$, $n_1 = 2$, and $n_{i+1} = n_i$ or $n_i + 1$ for each i. ($k = 0$, $m = l - 1$ is a special case.)

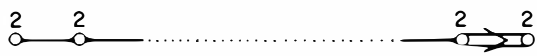
Type B_l The diagrams of distinguished parabolic subgroups are those of the form



where $m + n_1 + \ldots + n_k + 1 = l$, $n_1 = 2$, $n_{i+1} = n_i$ or $n_i + 1$ for $i = 1, 2, \ldots, k - 2$, and

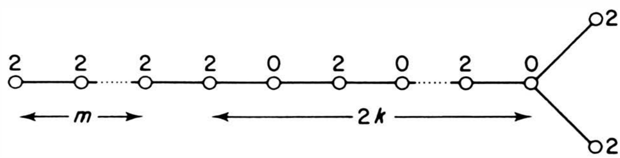
$$n_k = \begin{cases} \frac{1}{2}n_{k-1} & \text{if } n_{k-1} \text{ is even} \\ \frac{1}{2}(n_{k-1} - 1) & \text{if } n_{k-1} \text{ is odd.} \end{cases}$$

In addition the diagram

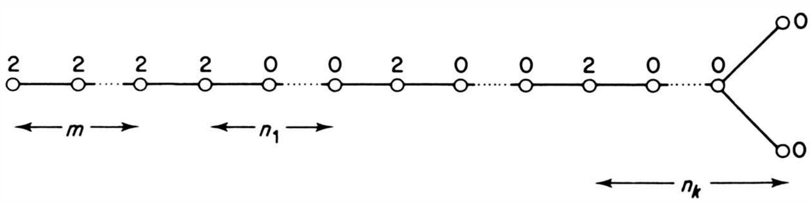


is distinguished.

Type D_l The diagrams of distinguished parabolic subgroups are those of the form



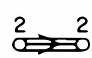
with $m + 2k + 2 = l$, and those of form



with $m + n_1 + \ldots + n_k = l$, $n_1 = 2$, $n_{i+1} = n_i$ or $n_i + 1$ for $i = 1, 2, \ldots, k - 2$, and

$$n_k = \begin{cases} \frac{1}{2}n_{k-1} & \text{if } n_{k-1} \text{ is even} \\ \frac{1}{2}(n_{k-1} + 1) & \text{if } n_{k-1} \text{ is odd.} \end{cases}$$

Type G_2 There are two possibilities:

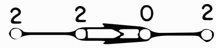
G_2 

$G_2(a_1)$ 

Type F_4 There are four possibilities

F_4

$F_4(a_1)$

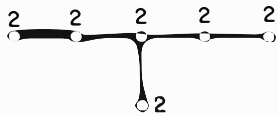


$F_4(a_2)$

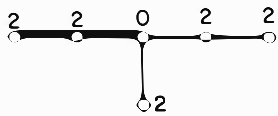
$F_4(a_3)$

Type E_6 There are three possibilities:

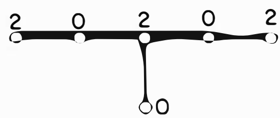
E_6



$E_6(a_1)$

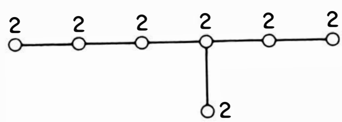


$E_6(a_3)$

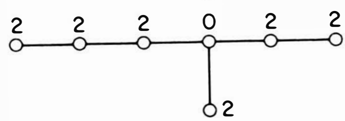


Type E_7 There are six possibilities:

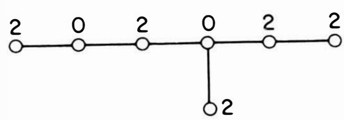
E_7



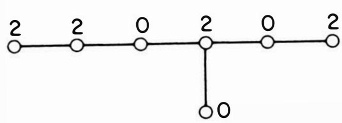
$E_7(a_1)$



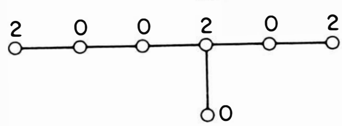
$E_7(a_2)$



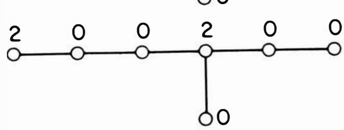
$E_7(a_3)$



$E_7(a_4)$

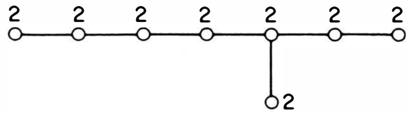


$E_7(a_5)$

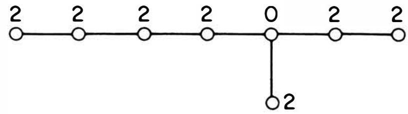


Type E_8 There are 11 possibilities

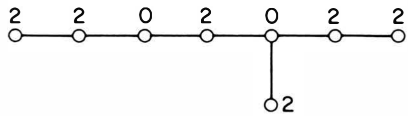
E_8



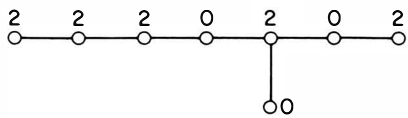
$E_8(a_1)$



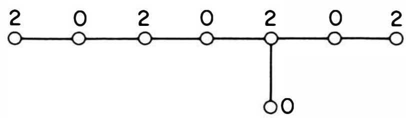
$E_8(a_2)$



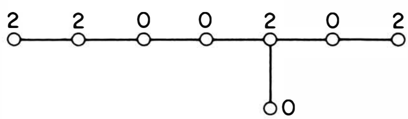
$E_8(a_3)$



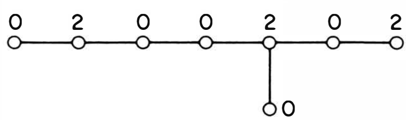
$E_8(a_4)$



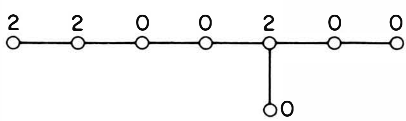
$E_8(b_4)$



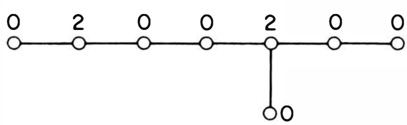
$E_8(a_5)$



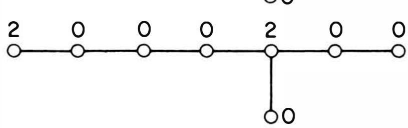
$E_8(b_5)$



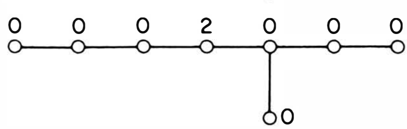
$E_8(a_6)$



$E_8(b_6)$



$E_8(a_7)$



5.10 SOME RESULTS ON DIMENSION

Theorem 5.10.1. (Steinberg) *Let G be a simple algebraic group over a field of characteristic 0 or $p > 3(h-1)$. Let u be a unipotent element of G. Let $\mathfrak{B}$ be the variety of all Borel subgroups of G and $\mathfrak{B}_u$ be the subvariety of Borel subgroups containing u. Then*

$$\dim C_G(u) = \operatorname{rank} G + 2\dim \mathfrak{B}_u.$$

Proof. Let C be the conjugacy class of u in G. C is locally closed in G so inherits from G the structure of an algebraic variety. Consider the product variety $\mathfrak{B} \times \mathfrak{B} \times C$. Let S be the closed subset of $\mathfrak{B} \times \mathfrak{B} \times C$ defined by

$$S = \{(B_1, B_2, y) \in \mathfrak{B} \times \mathfrak{B} \times C;\ y \in B_1 \cap B_2\}.$$

Let B be a fixed Borel subgroup of G. For each $w \in W$ define S_w by

$$S_w = \{(B_1, B_2, y) \in S;\ B_1 = {}^gB,\ B_2 = {}^{g\dot{w}}B \text{ for some } g \in G\}.$$

Then S_w is a locally closed subset of S and S is the disjoint union of the S_w for all $w \in W$. Thus we have

$$\dim S \geqslant \dim S_w \qquad \text{for all } w \in W$$

$$\dim S = \dim S_w \qquad \text{for some } w \in W.$$

Consider the projection $S \to C$ given by $(B_1, B_2, y) \to y$. This map is surjective and the fibre of $x \in C$ is $\mathfrak{B}_x \times \mathfrak{B}_x$. It follows that

$$\dim S = \dim C + \dim(\mathfrak{B}_u \times \mathfrak{B}_u) = \dim C + 2\dim \mathfrak{B}_u.$$

We now consider the subvariety S_w. Since $({}^{g_1}B, {}^{g_1\dot{w}}B) = ({}^{g_2}B, {}^{g_2\dot{w}}B)$ if and only if $g_1(B \cap {}^{\dot{w}}B) = g_2(B \cap {}^{\dot{w}}B)$ we have a bijective map between

$$\{(B_1, B_2) \in \mathfrak{B} \times \mathfrak{B};\ B_1 = {}^gB,\ B_2 = {}^{g\dot{w}}B \text{ for some } g \in G\}$$

and $G/B \cap {}^{\dot{w}}B$. We therefore have an injective morphism $S_w \to G/B \cap {}^{\dot{w}}B \times C$ given by

$$({}^gB, {}^{g\dot{w}}B, y) \to (g(B \cap {}^{\dot{w}}B), y).$$

We consider the projection $S_w \to G/B \cap {}^{\dot{w}}B$ to the first factor. The fibre above $g(B \cap {}^{\dot{w}}B)$ is the set of elements $({}^gB, {}^{g\dot{w}}B, y)$ for which the first two components are fixed and $y \in {}^gB \cap {}^{g\dot{w}}B$. Thus this fibre is isomorphic to

$$C \cap {}^g(B \cap {}^{\dot{w}}B) = {}^g(C \cap B \cap {}^{\dot{w}}B) = {}^g(C \cap U \cap {}^{\dot{w}}U).$$

All the fibres therefore have dimension equal to $\dim(C \cap U \cap {}^{\dot{w}}U)$. Thus we have

$$\begin{aligned}\dim S_w &= \dim(G/B \cap {}^{\dot{w}}B) + \dim(C \cap U \cap {}^{\dot{w}}U)\\ &= \dim G - \dim(B \cap {}^{\dot{w}}B) + \dim(C \cap U \cap {}^{\dot{w}}U)\\ &= \dim G - \dim T - \dim(U \cap {}^{\dot{w}}U) + \dim(C \cap U \cap {}^{\dot{w}}U).\end{aligned}$$

Now $\dim S \geqslant \dim S_w$ with equality for some $w \in W$. Hence

$$\dim C + 2\dim \mathfrak{B}_u \geqslant \dim G - \dim T - \dim(U \cap {}^{\dot{w}}U) + \dim(C \cap U \cap {}^{\dot{w}}U)$$

with equality for some $w \in W$. This gives

$$\begin{aligned}\dim C_G(u) &= \dim G - \dim C \\ &\leqslant \operatorname{rank} G + 2\dim \mathfrak{B}_u + \dim(U \cap {}^{\dot{w}}U) - \dim(C \cap U \cap {}^{\dot{w}}U)\end{aligned}$$

with equality for some $w \in W$.

However it follows from 5.9.6 that $C \cap U \cap {}^{\dot{w}}U$ is dense in $U \cap {}^{\dot{w}}U$ for some $w \in W$. For there exists a Levi subgroup L_J and a parabolic subgroup $P_K \cap L_J$ of L_J with $K \subseteq J$ such that $C \cap (U_K \cap L_J)$ is dense in $U_K \cap L_J$. But we have

$$U_K \cap L_J = \prod_{\substack{\alpha > 0 \\ \alpha \in \Phi_J - \Phi_K}} X_\alpha = \prod_{\substack{\alpha > 0 \\ (w_0)_J(w_0)_K(\alpha) < 0}} X_\alpha = U \cap {}^{\dot{w}}U$$

where $w = (w_0)_K(w_0)_J w_0$. For this element $w \in W$ we know that $C \cap U \cap {}^{\dot{w}}U$ is dense in $U \cap {}^{\dot{w}}U$ and so that

$$\dim(U \cap {}^{\dot{w}}U) = \dim(C \cap U \cap {}^{\dot{w}}U).$$

Thus the minimum value of

$$\operatorname{rank} G + 2\dim \mathfrak{B}_u + \dim(U \cap {}^{\dot{w}}U) - \dim(C \cap U \cap {}^{\dot{w}}U)$$

for all $w \in W$ is $\operatorname{rank} G + 2\dim \mathfrak{B}_u$. We therefore have

$$\dim C_G(u) = \operatorname{rank} G + 2\dim \mathfrak{B}_u$$

as required.

Theorem 5.10.2. (Spaltenstein) *Let G be a simple algebraic group, u be a unipotent element of G, C the conjugacy class of u in G, $\mathfrak{B}_u$ the variety of Borel subgroups containing u, and U a maximal unipotent subgroup of G. Then the following statements are equivalent*:

(i) $\dim C_G(u) = \operatorname{rank} G + 2\dim \mathfrak{B}_u$.
(ii) $\dim(C \cap U) = \frac{1}{2}\dim C$.
(iii) $\dim(C \cap U) = \dim U - \dim \mathfrak{B}_u$.

All these statements are true if the characteristic is 0 *or* $p > 3(h-1)$.

Proof. C is open in $\bar{C}$ and so C is locally closed. Thus $C \cap U$ will also be locally closed. Thus $C \cap U$ inherits from G the structure of an algebraic variety. We consider the following morphisms π_1, π_2:

$$\begin{aligned} \pi_1 &: G \to C & \pi_2 &: G \to \mathfrak{B} \\ & g \to g^{-1}ug & & g \to {}^{g}B. \end{aligned}$$

Now we have $\pi_1^{-1}(C \cap U) = \pi_2^{-1}(\mathfrak{B}_u)$. For

$$g \in \pi_1^{-1}(C \cap U) \Leftrightarrow g^{-1}ug \in U \Leftrightarrow u \in {}^gU$$
$$\Leftrightarrow u \in {}^gB \Leftrightarrow g \in \pi_2^{-1}(\mathfrak{B}_u).$$

Let $Y = \pi_1^{-1}(C \cap U) = \pi_2^{-1}(\mathfrak{B}_u)$. Then Y is a closed subset of G. Now all the fibres of π_1 have dimension equal to $\dim C_G(u)$ and all the fibres of π_2 have dimension equal to $\dim B$. Thus we have

$$\dim Y = \dim(C \cap U) + \dim C_G(u)$$
$$\dim Y = \dim \mathfrak{B}_u + \dim B.$$

It follows that

$$\dim(C \cap U) + \dim C_G(u) = \dim \mathfrak{B}_u + \dim B.$$

This equation can be written

$$\dim(C \cap U) + (\dim C_G(u) - \dim T - 2 \dim \mathfrak{B}_u) = \dim U - \dim \mathfrak{B}_u$$

which shows that the statements (i), (iii) are equivalent. It can also be written

$$\tfrac{1}{2} \dim C - \dim(C \cap U) = \tfrac{1}{2}(\dim C_G(u) - \dim T - 2 \dim \mathfrak{B}_u)$$

which shows that statements (i), (ii) are equivalent.

Finally it follows from 10.5.1 that all the statements are true when the characteristic is 0 or $p > 3(h - 1)$. ∎

Note. The statements of 5.10.1 and 5.10.2 are known to be true for all p (see Spaltenstein [3], p. 54).

5.11 UNIPOTENT CLASSES AND NILPOTENT ORBITS IN SMALL CHARACTERISTIC

We have determined the unipotent classes of G and the nilpotent orbits of G on $\mathfrak{g} = \mathfrak{L}(G)$ when G is a simple group of adjoint type over an algebraically closed field K of characteristic 0 or $p > 3(h - 1)$. We now describe without proof what happens when $p \leqslant 3(h - 1)$. The above classification of unipotent classes and nilpotent orbits is in fact valid for a considerably wider range of values of p than that for which we have proved it. Pommerening has shown ([1], [2]) that this classification is valid whenever p is a good prime for G, but his proof is more complicated than the one given in this chapter.

We therefore consider the case when p is a bad prime for G, i.e. $p = 2$ for types B_l, C_l, D_l; $p = 2$ or 3 for types G_2, F_4, E_6, E_7; $p = 2$, 3 or 5 for type E_8. The unipotent classes are no longer necessarily in bijective correspondence with the nilpotent orbits so we must consider the two problems separately.

We begin with the classical groups B_l, C_l, D_l in characteristic 2. The unipotent classes in these groups were determined by G. E. Wall [1] and the nilpotent orbits by Hesselink [3]. We use the terminology of Hesselink to describe both the unipotent classes and the nilpotent orbits.

First let $G = Sp_{2l}(K) = (C_l)_{sc}(K)$ and $\mathfrak{g} = \mathfrak{L}(G)$. For each nilpotent $2l \times 2l$ matrix T we define two invariants $p(T)$ and $\chi(T)$. $p(T)$ is a partition of $2l$ and $\chi(T)$ is a function from the parts of this partition into the non-negative integers. Unipotent elements $1 + T$, $1 + T'$ of G will be conjugate in G if and only if $p(T) = p(T')$ and $\chi(T) = \chi(T')$. Nilpotent elements T, T' of $\mathfrak{g}$ will lie in the same G-orbit if and only if $p(T) = p(T')$ and $\chi(T) = \chi(T')$.

$p(T)$ and $\chi(T)$ are defined in terms of a basis with respect to which T has Jordan canonical form. Such a basis has form

$$v_1, Tv_1, T^2v_1, \ldots, T^{m_1-1}v_1, v_2, Tv_2, \ldots, T^{m_2-1}v_2, \ldots, T^{m_r-1}v_r$$

where $T^{m_1}v_1 = 0$, $T^{m_2}v_2 = 0, \ldots, T^{m_r}v_r = 0$ and $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_r$

$$p(T) = \{m_1, m_2, \ldots, m_r\}, \text{ a partition of } 2l$$

$$(\chi(T))(m) = \min\{n \geqslant 0;\ T^m v = 0 \Rightarrow \langle T^{n+1}v, T^n v\rangle = 0\}$$

where $\langle\ ,\ \rangle$ is the symplectic form.

To describe the unipotent classes and nilpotent orbits we must therefore specify which pairs $p(T)$, $\chi(T)$ can arise from the group or the algebra. In the group $G = Sp_{2l}(K)$ the situation is as follows. A partition of $2l$ has the form $p(T)$ for some unipotent element $1 + T \in G$ if and only if each odd part occurs with even multiplicity. A function χ on the parts of such a partition has the form $\chi(T)$ for some such T if and only if

$$\chi(m) = \begin{cases} \frac{1}{2}(m-1) & \text{if } m \text{ is odd} \\ \frac{1}{2}m & \text{if } m \text{ is even and the multiplicity of } m \text{ is odd} \\ \frac{1}{2}m - 1 \text{ or } \frac{1}{2}m & \text{if } m \text{ is even and the multiplicity of } m \text{ is even.} \end{cases}$$

In the Lie algebra $\mathfrak{g} = \mathfrak{sp}_{2l}(K)$ the situation is as follows. A partition of $2l$ has the form $p(T)$ for some nilpotent element $T \in \mathfrak{g}$ if and only if each odd part has even multiplicity. A function χ on the parts of such a partition has the form $\chi(T)$ for some such T if and only if

$$\begin{cases} 0 \leqslant \chi(m) \leqslant \frac{1}{2}m & \text{for all } m \\ \chi(m') \leqslant \chi(m) \leqslant \chi(m') + m - m' & \text{for all } m > m' \\ \chi(m) = \frac{1}{2}m \quad \text{if } m \text{ is even and the multiplicity of } m \text{ is odd.} \end{cases}$$

This describes the unipotent classes and nilpotent orbits when $G = Sp_{2l}(K)$.

Now suppose $G = SO_{2l+1}(K)$ and $\mathfrak{g} = \mathfrak{L}(G)$ where K is algebraically closed of characteristic 2. G is the adjoint group of type B_l. In fact $G = O_{2l+1}(K)$ since the orthogonal group is connected in this case. We again define invariants $p(T)$ and $\chi(T)$ for each nilpotent $(2l + 1) \times (2l + 1)$ matrix T which will have the property that unipotent elements $1 + T$, $1 + T'$ of G will be conjugate in G if and only if $p(T) = p(T')$ and $\chi(T) = \chi(T')$, and nilpotent elements T, T' of $\mathfrak{g}$ will lie in the same G-orbit if and only if $p(T) = p(T')$ and $\chi(T) = \chi(T')$. $p(T)$ is, as before, the partition of $2l + 1$ given by the elementary divisors of T. $\chi(T)$ is defined this time by

$$(\chi(T))(m) = \min\{n \geqslant 0;\ T^m v = 0 \Rightarrow Q(T^n v) = 0\}$$

where Q is the underlying quadratic form.

We will again describe which pairs $p(T)$, $\chi(T)$ can arise from the group or the Lie algebra. In the group $G = SO_{2l+1}(K)$ the situation is as follows. A partition of $2l + 1$ has the form $p(T)$ for some unipotent element $1 + T \in G$ if and only if each odd part occurs with even multiplicity. A function χ on the parts of such a partition has the form $\chi(T)$ for some such T if and only if

$$\chi(m) = \begin{cases} \frac{1}{2}(m+1) & \text{if } m \text{ is odd} \\ \frac{1}{2}(m+2) & \text{if } m \text{ is even and the multiplicity of } m \text{ is odd} \\ \frac{1}{2}m \text{ or } \frac{1}{2}(m+2) & \text{if } m \text{ is even and the multiplicity of } m \text{ is even.} \end{cases}$$

In the Lie algebra $\mathfrak{g} = \mathfrak{so}_{2l+1}(K)$ the situation is as follows. A partition of $2l + 1$ has the form $p(T)$ for some nilpotent element $T \in \mathfrak{g}$ if and only if the set of parts with odd multiplicity is one of $\varnothing$, $\{1\}$, $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\} \ldots \{k, k+1\} \ldots$. A function χ on the parts of such a partition has the form $\chi(T)$ for some such T if and only if

$$\begin{cases} \frac{1}{2}m \leqslant \chi(m) \leqslant m & \text{for all } m \\ \chi(m') \leqslant \chi(m) \leqslant \chi(m') + m - m' & \text{for all } m > m' \\ \chi(m) = m \quad \text{if } m \text{ has odd multiplicity.} & \end{cases}$$

This describes the unipotent classes and nilpotent orbits when $G = SO_{2l+1}(K)$.

Now suppose $G = SO_{2l}(K)$ and $\mathfrak{g} = \mathfrak{L}(G)$. Then G is one of the simple groups of type D_l. The orthogonal group $O_{2l}(K)$ is not connected, and we have $|O_{2l}(K):SO_{2l}(K)| = 2$. It will be convenient first to describe the unipotent classes in $O_{2l}(K)$ and the $O_{2l}(K)$-orbits of nilpotent elements in $\mathfrak{g}$, before descending from $O_{2l}(K)$ to $SO_{2l}(K)$. We define invariants $p(T)$ and $\chi(T)$ for each nilpotent $2l \times 2l$ matrix T exactly as for the group $O_{2l+1}(K)$. Two unipotent elements $1 + T, 1 + T'$ of $O_{2l}(K)$ will be conjugate in $O_{2l}(K)$ if and only if $p(T) = p(T')$ and $\chi(T) = \chi(T')$. Two nilpotent elements T, T' of $\mathfrak{g}$ will be in the same $O_{2l}(K)$-orbit if and only if $p(T) = p(T')$ and $\chi(T) = \chi(T')$. The condition which describes which pairs $p(T)$, $\chi(T)$ can arise from the group $O_{2l}(K)$ or the algebra $\mathfrak{g}$ are exactly as in the case of $O_{2l+1}(K)$.

Having described the unipotent classes of $O_{2l}(K)$ and the $O_{2l}(K)$-orbits of nilpotent elements of $\mathfrak{g}$ we must now describe what modifications are necessary in descending from $O_{2l}(K)$ to $SO_{2l}(K)$. Not every unipotent class of $O_{2l}(K)$ lies in $SO_{2l}(K)$, and if a unipotent class of $O_{2l}(K)$ does lie in $SO_{2l}(K)$ it may either constitute a unipotent class of $SO_{2l}(K)$ or split into two unipotent classes of $SO_{2l}(K)$. Again, an $O_{2l}(K)$-orbit of nilpotent elements of $\mathfrak{g}$ may either be an $SO_{2l}(K)$-orbit or may split into two $SO_{2l}(K)$-orbits. The required conditions on the invariants $p(T)$, $\chi(T)$ are as follows.

The unipotent element $1 + T$ of $O_{2l}(K)$ lies in $SO_{2l}(K)$ if and only if $p(T)$ has an even number of parts.

If $p(T)$ has an even number of parts the unipotent class of $O_{2l}(K)$ containing $1 + T$ splits into two $SO_{2l}(K)$-classes if and only if $\chi(T)(m) \leqslant \frac{1}{2}m$ for all parts m of $p(T)$.

The $O_{2l}(K)$-orbit of $\mathfrak{g}$ containing T splits into two $SO_{2l}(K)$-orbits if and only if $\chi(T)(m) \leqslant \frac{1}{2}m$ for all parts m of $p(T)$. This completes the description of the unipotent classes and nilpotent orbits for the simple classical groups in characteristic 2.

We now consider the unipotent classes of G when G is a simple group of exceptional type over an algebraically closed field of characteristic p, where p is a bad prime for G.

If G has type G_2 and $p = 2$ or 3 the unipotent classes were determined by Stuhler [1]. If $p = 2$ the unipotent classes are parametrized in the same way as for good primes. If $p = 3$ there is one additional unipotent class.

If G has type F_4 we must again consider the primes 2 or 3. If $p = 3$ the unipotent classes were determined by Shoji [1] and are parametrized in the same way as for good primes. If $p = 2$ it was shown by Shinoda [1] that there are four additional unipotent classes.

If G has type E_6 and $p = 2$ or 3 it was shown by Mizuno [1] that the unipotent classes are parametrized in the same way as for good primes.

If G has type E_7 and $p = 3$ it was shown by Mizuno [2] that the unipotent classes are parametrized in the same way as for good primes, whereas if $p = 2$ the same paper shows that there is one additional unipotent class.

If G has type E_8 we must consider the primes 2, 3 and 5. The unipotent classes in these cases were also determined by Mizuno in [2]. If $p = 5$ they are parametrized in the same way as for good primes. If $p = 3$ there is one additional class and if $p = 2$ there are four additional classes.

Finally we consider the nilpotent orbits of G acting on $\mathfrak{g}$ when G is a simple group of exceptional type and p is a bad prime.

If G has type G_2 and $p = 2$ or 3 the nilpotent orbits were determined by Stuhler [1]. If $p = 2$ they are parametrized in the same way as for good primes, whereas if $p = 3$ there is one additional orbit.

If G has type F_4 and $p = 2$ or 3 the nilpotent orbits were determined by Spaltenstein [6], [7]. If $p = 3$ they are parametrized in the same way as for good primes, whereas if $p = 2$ there are six additional orbits.

If G has type E_6 and $p = 2$ or 3 it was shown by Spaltenstein [5] that the nilpotent orbits are parametrized in the same way as for good primes.

If G has type E_7 and $p = 2$ or 3 the nilpotent orbits have been determined by Spaltenstein [7]. If $p = 3$ they are parametrized in the same way as for good primes, whereas if $p = 2$ there are two additional orbits.

If G has type E_8 and $p = 2$, 3 or 5 the nilpotent orbits have been determined by Spaltenstein [7]. If $p = 5$ they are parametrized in the same way as for good primes. If $p = 3$ there is one additional orbit, and if $p = 2$ there are six additional orbits.

We see, therefore, that the situation for bad primes can differ substantially from that for characteristic 0.

Chapter 6

THE STEINBERG CHARACTER

Let G be a connected reductive group and $F:G \to G$ be a Frobenius map. We now turn our attention to the representation theory of the finite group G^F over an algebraically closed field of characteristic 0. We shall for convenience consider representations of G^F over the field of complex numbers. We begin by recalling some basic definitions and properties of representations.

6.1 ELEMENTARY PROPERTIES OF REPRESENTATIONS

Let G be any finite group. A representation of G is a homomorphism $\rho:G \to GL_n(\mathbb{C})$ for some n. If ρ is a representation of G so is the map ρ' given by

$$\rho'(g) = T^{-1}\rho(g)T \qquad T \in GL_n(\mathbb{C}).$$

Representations ρ and ρ' related in this way are called equivalent. If ρ is a representation of G the function $\chi:G \to \mathbb{C}$ given by

$$\chi(g) = \text{trace } \rho(g)$$

is called the character of ρ. Equivalent representations have the same character.

Every representation of G can be obtained from a $\mathbb{C}G$-module, i.e. a finite-dimensional vector space V for which there is a map $(v, g) \to vg$ from $V \times G$ into V satisfying:

(i) $v \to vg$ is a linear map for all $g \in G$.
(ii) $v1 = v \qquad$ for all $v \in V$.
(iii) $(vg_1)g_2 = v(g_1g_2) \qquad$ for all $g_1, g_2 \in G$, $v \in V$.

We shall also frequently consider left $\mathbb{C}G$-modules, i.e. vector spaces on which G acts on the left rather than on the right.

If $e_1 \dots e_n$ is a basis for V and

$$e_ig = \sum_{j=1}^{n} \rho_{ij}(g)e_j$$

then the map ρ given by $g \to (\rho_{ij}(g))$ is a representation of G. We obtain in this way a bijection between isomorphism classes of $\mathbb{C}G$-modules and classes of equivalent representations. We say that the representation ρ is afforded by the $\mathbb{C}G$-module V.

A representation ρ is called reducible† if ρ is equivalent to a diagonal sum

$$g \to \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$$

of two representations ρ_1, ρ_2 of smaller degree. A representation which is not reducible is called irreducible. Two irreducible representations are equivalent if and only if they have the same character. The characters of the irreducible representations are called irreducible characters. G has only finitely many irreducible characters $\chi^1, \chi^2, \ldots, \chi^k$ and their number k is equal to the number of conjugacy classes in G.

The irreducible characters of G are class functions, i.e. they are constant on conjugacy classes of G. If $C_1, C_2, \ldots, C_k$ are the conjugacy classes of G then the matrix

$$(\chi^i(g_j)) \qquad g_j \in C_j$$

is called the character table of G. When studying the complex representation theory of a given finite group G one aims to find out as much as possible about the character table of G. One of the irreducible characters, which we shall take as χ^1, has the property that $\chi^1(g) = 1$ for all $g \in G$. χ^1 is called the principal character and we sometimes write $\chi^1 = 1$.

If ρ is any representation of G then its character χ will have the form

$$\chi = n_1\chi^1 + \ldots + n_k\chi^k$$

where $n_1, n_2, \ldots, n_k$ are non-negative integers. More generally, a function $\chi: G \to \mathbb{C}$ of the form $\chi = \sum_{i=1}^k n_i\chi^i$ where each $n_i \in \mathbb{Z}$ is called a generalized character of G. Note that in a generalized character some of the n_i may be negative integers.

We next recall that the scalar product of two class functions ϕ_1, ϕ_2 is defined by

$$(\phi_1, \phi_2) = \frac{1}{|G|} \sum_{g \in G} \phi_1(g)\overline{\phi_2(g)}.$$

The connection between this scalar product and the $\mathbb{C}G$-modules is as follows. If V_1, V_2 are $\mathbb{C}G$-modules affording characters ϕ_1, ϕ_2 respectively then we have

$$(\phi_1, \phi_2) = \dim \operatorname{Hom}_G(V_1, V_2).$$

Thus the scalar product is the dimension of the space of module homomorphisms from V_1 into V_2.

The irreducible characters of G satisfy the relations

$$(\chi^i, \chi^i) = 1 \qquad (\chi^i, \chi^j) = 0 \qquad \text{if } i \neq j$$

† A reducible representation is the same as a decomposable representation over the field $\mathbb{C}$.

called the orthogonality relations. It follows from these relations that the multiplicity n_i of an irreducible character χ^i in a generalized character χ is given by

$$n_i = (\chi, \chi^i).$$

We also have orthogonality relations for the coefficient functions of the irreducible representations of G. If ρ^i is an irreducible representation of G with character χ^i then we have

$$(\rho^i_{ab}, \rho^j_{cd}) = 0 \qquad \text{unless } i = j,\ a = d,\ b = c$$

and

$$(\rho^i_{ab}, \rho^i_{ba}) = \frac{1}{d_i}$$

where $d_i = \chi^i(1)$ is the degree of the representation ρ^i.

If ρ^i is an irreducible representation of G with character χ^i then $\chi^i(g)$ is an algebraic integer for all $g \in G$. Moreover ρ^i is equivalent to a representation all of whose matrix coefficients for each $g \in G$ are algebraic integers.

If V is a $\mathbb{C}G$-module and $V^* = \operatorname{Hom}(V, \mathbb{C})$ is the dual space of V then V^* can be made into a $\mathbb{C}G$-module by the rule

$$(fg)v = f(vg^{-1}) \qquad \text{for all } f \in V^*,\ v \in V,\ g \in G.$$

We then have

$$(fg_1)g_2 = f(g_1g_2) \qquad \text{for all } g_1, g_2 \in G.$$

If χ is the character of G afforded by the module V then the character afforded by the dual space V^* is the complex conjugate of χ, i.e. the map $g \to \overline{\chi(g)}$. We have $\overline{\chi(g)} = \chi(g^{-1})$ for all $g \in G$.

We next recall the basic facts relating representations of a group G with representations of a subgroup H of G. If ρ is a representation of G then the restriction ρ_H of ρ to H is a representation of H. If σ is a representation of H then we can construct a representation of G as follows. Let G be the disjoint union of the cosets $Hx_1, Hx_2, \ldots, Hx_m$. Then for each x_i and each $g \in G$ there is a unique x_j such that $x_igx_j^{-1} \in H$. Let $\rho(g)$ be the $m \times m$ block matrix whose (i, j) block is $\sigma(x_igx_j^{-1})$ if $x_igx_j^{-1} \in H$ and zero otherwise. Then ρ is a representation of G called the induced representation of σ. Let ψ be the character of σ. Then the character ψ^G of the induced representation ρ is given by

$$\psi^G(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ g^x \in H}} \psi(g^x).$$

Induction of characters is transitive. Thus if H_1 is a subgroup with $H \subseteq H_1 \subseteq G$ then $(\psi^{H_1})^G = \psi^G$.

A $\mathbb{C}G$-module giving the induced representation can be obtained as follows. If V is a $\mathbb{C}H$-module affording the representation σ of H then $V \otimes_{\mathbb{C}H} \mathbb{C}G$ is a $\mathbb{C}G$-module affording the induced representation ρ of G.

There is a close connection between the operations of restriction and induction. Let ϕ be a character of G and ψ be a character of a subgroup H of G. Then we may form the restriction ϕ_H of ϕ to H and the induced character ψ^G of ψ to G. These characters are related by the formula

$$(\psi^G, \phi)_G = (\psi, \phi_H)_H$$

which is called Frobenius reciprocity.

There is also a useful formula for the scalar product of induced characters from two subgroups. Let H, K be subgroups of the finite group G and let ψ be a character of K. Let ψ^G be the induced character and $(\psi^G)_H$ denote its restriction to H. Let R be a set of double coset representatives of G with respect to H and K. Then we have

$$(\psi^G)_H = \sum_{r \in R} (({}^r\psi)_{H \cap {}^rK})^H$$

where ${}^r\psi$ is the character of ${}^rK$ defined by ${}^r\psi({}^rk) = \psi(k)$. This is called Mackey's subgroup formula. Using this subgroup formula one can prove the following result which is also due to Mackey. Let ϕ, ψ be characters of the subgroups H, K of G. Then

$$(\phi^G, \psi^G) = \sum_{r \in R} (\phi, {}^r\psi)_{H \cap {}^rK}.$$

These results of Mackey will be useful to us in the subsequent development.

All the elementary properties of representations which we have mentioned are discussed in more detail in the book of Curtis and Reiner [1].

6.2 IRREDUCIBILITY OF THE STEINBERG CHARACTER

We now suppose that G is a finite group with a BN-pair. We recall from chapter 2 that such a group G has parabolic subgroups P_J for all subsets J of the index set I. We define the Steinberg character of G to be the generalized character given by

$$\mathrm{St} = \sum_{J \subseteq I} (-1)^{|J|} 1_{P_J}{}^G.$$

Thus the Steinberg character is the alternating sum of the induced characters from the principal characters of the parabolic subgroups P_J. We shall show that St is in fact an irreducible character of G.

In order to derive the properties of the Steinberg character we first consider the analogous expression in the Weyl group $W = N/B \cap N$. Let ε be the generalized character of W defined by

$$\varepsilon = \sum_{J \subseteq I} (-1)^{|J|} 1_{W_J}{}^W.$$

The following result shows that ε can be described in a very simple way.

Proposition 6.2.1. (i) $\varepsilon(w) = (-1)^{l(w)}$ *for all* $w \in W$.

(ii) $\varepsilon(w) = \det w$, *the determinant of* w *in the natural representation of* W *described in section* 2.2.

Proof. W is generated by involutions $s_1, s_2, \ldots, s_l$ and $\det s_i = -1$ for each i. Each element $w \in W$ can be expressed as a product of elements s_i with $l(w)$ factors. Thus $\det w = (-1)^{l(w)}$.

We recall from section 2.3 that the group W acts on the Coxeter complex, and that the simplex C_J has the property that $w(C_J) = C_J$ if and only if $w \in W_J$. Moreover if $w \in W_J$ then $w(v) = v$ for all $v \in C_J$. Now we have

$$1_{W_J}{}^W(w) = \frac{1}{|W_J|} \sum_{\substack{x \in W \\ w^x \in W_J}} 1$$

summed over all $x \in W$ for which $w^x \in W_J$. Now $w^x \in W_J$ if and only if w fixes $x(C_J)$. Suppose $n_J(w)$ is the number of distinct simplices of the form $x(C_J)$ fixed by w. Then we have

$$1_{W_J}{}^W(w) = \frac{1}{|W_J|} |W_J| n_J(w) = n_J(w).$$

Let $U = \{v \in V;\ w(v) = v\}$. U is a subspace of V and the simplices of the Coxeter complex fixed by w are just those which lie in U. They form a simplicial complex κ in U. Let n_i be the number of simplices in κ of dimension i. Then we have

$$\sum_i (-1)^i n_i = (-1)^{\dim U}.$$

This result is in fact true for any simplicial complex obtained by dividing up a vector space by means of a finite number of hyperplanes through the origin. It is proved by induction on the number of hyperplanes. It is clearly true if there is just one hyperplane. Suppose it is true for a system of n hyperplanes and that an additional hyperplane H is then added. Each simplex of dimension i which is cut in two by H has corresponding to it a simplex of dimension $i - 1$ separating the two parts. Thus the sum $\sum_i (-1)^i n_i$ is unchanged by the addition of H.

We apply this result to the simplicial complex κ in U. We have dim $C_J = l - |J|$ and so

$$n_i = \sum_{\substack{J \\ |J| = l - i}} n_J(w).$$

Thus

$$\sum_i (-1)^i n_i = (-1)^l \sum_J (-1)^{|J|} n_J(w) = (-1)^{\dim U}.$$

Now w, being an orthogonal transformation, has eigenvalues which are 1, -1 or pairs of complex conjugate roots of unity. Thus

$$\det w = (-1)^{l - \dim U}.$$

It follows that

$$\sum_J (-1)^{|J|} n_J(w) = \det w$$

and so we have

$$\sum_J (-1)^{|J|} 1_{W_J}{}^W(w) = \det w.$$

The representation ε of W given by $\varepsilon(w) = \det w$ is called the sign representation of W.

Proposition 6.2.2. (St, St) = 1.

Proof. We have

$$(\mathrm{St}, \mathrm{St}) = \sum_J \sum_K (-1)^{|J|}(-1)^{|K|}(1_{P_J}{}^G, 1_{P_K}{}^G).$$

We apply Mackey's formula, using from 2.8.1 the fact the $N_{J,K}$ is a set of double coset representatives of G with respect to P_J and P_K. Thus

$$\begin{aligned}(\mathrm{St}, \mathrm{St}) &= \sum_J \sum_K (-1)^{|J|}(-1)^{|K|} \sum_{n \in N_{J,K}} (1, {}^n 1)_{P_J \cap {}^n P_K} \\ &= \sum_J \sum_K (-1)^{|J|}(-1)^{|K|}|N_{J,K}| = \sum_J \sum_K (-1)^{|J|}(-1)^{|K|}|D_{J,K}|\end{aligned}$$

where $D_{J,K}$ is the set of distinguished double coset representatives for W with respect to W_J and W_K.

We now consider the corresponding scalar product in W. The sign representation ε of W, having degree 1, is certainly irreducible. Thus we have

$$\begin{aligned}1 = (\varepsilon, \varepsilon) &= \sum_J \sum_K (-1)^{|J|}(-1)^{|K|}(1_{W_J}{}^W, 1_{W_K}{}^W) \\ &= \sum_J \sum_K (-1)^{|J|}(-1)^{|K|} \sum_{w \in D_{J,K}} (1, {}^w 1) = \sum_J \sum_J (-1)^{|J|}(-1)^{|K|}|D_{J,K}|.\end{aligned}$$

We therefore deduce that (St, St) = 1. ∎

Since St is a generalized character of G we have $\mathrm{St} = n_1\chi^1 + \ldots + n_k\chi^k$ with $n_i \in \mathbb{Z}$. Hence $(\mathrm{St}, \mathrm{St}) = \sum n_i{}^2$. Since (St, St) = 1 it follows that $\mathrm{St} = \pm\chi^i$ for some i. Thus $\pm$St is irreducible. In order to determine the sign we prove the following proposition.

Proposition 6.2.3. $(1_B{}^G, \mathrm{St}) = 1.$

Proof. We have $\mathrm{St} = \sum_J (-1)^{|J|} 1_{P_J}{}^G$. Thus

$$\begin{aligned}
(1_B{}^G, \mathrm{St}) &= \left(1_B{}^G, \sum_J (-1)^{|J|} 1_{P_J}{}^G\right) \\
&= \sum_J (-1)^{|J|} (1_B{}^G, 1_{P_J}{}^G) \\
&= \sum_J (-1)^{|J|} |N_{\phi, J}| \qquad \text{by Mackey's formula} \\
&= \sum_J (-1)^{|J|} |D_{\phi, J}| \\
&= \sum_J (-1)^{|J|} (1_1{}^W, 1_{W_J}{}^W) \qquad \text{again by Mackey's formula} \\
&= \left(1_1{}^W, \sum_J (-1)^{|J|} 1_{W_J}{}^W\right) \\
&= (1_1{}^W, \varepsilon) \\
&= (1, 1)_1 \qquad \text{by Frobenius reciprocity} \\
&= 1.
\end{aligned}$$

Corollary 6.2.4. *The Steinberg character is an irreducible character of* G.

Proof. This follows from $(\mathrm{St}, \mathrm{St}) = 1$ and $(1_B{}^G, \mathrm{St}) = 1$. ■

The Steinberg character was first introduced by Steinberg in his papers [2]. It was defined for arbitrary finite groups with a BN-pair by Curtis [5].

6.3 RESTRICTION TO A PARABOLIC SUBGROUP

We now assume that G is a finite group with a split BN-pair satisfying the commutator relations. Then each parabolic subgroup P_J of G has a Levi decomposition $P_J = U_J L_J$ as in section 2.6. The Levi subgroup L_J itself has a BN-pair. Thus we may consider the Steinberg characters St_G, St_{L_J} of G and of L_J. Our main result in this section is that St_G restricted to P_J is equal to St_{L_J} induced to P_J. We first need some preliminary results.

Lemma 6.3.1. *For each* $w \in W$ *let* K_w *be the set of* $\alpha_i \in \Delta$ *such that* $w(\alpha_i) \in \Phi^+$. *Then the only element of* W *satisfying the conditions*

(i) $w^{-1} \in D_J$
(ii) $K_w \subseteq w^{-1}(\Delta_J)$

is the element $(w_0)_J w_0$. *(*w_0 *is the element of maximal length in* W *and* $(w_0)_J$ *the element of maximal length in* W_J.*)*

Proof. It is easily seen that the element $(w_0)_J w_0$ satisfies the given conditions. Suppose now that w satisfies (i), (ii). Condition (ii) shows that, for each $\alpha_i \in \Delta$, if $w(\alpha_i) \in \Phi^+$ then $w(\alpha_i) \in \Delta_J$. Thus we have either $w(\alpha_i) \in \Delta_J$ or $w(\alpha_i) \in \Phi^-$. If $w(\alpha_i) \in \Delta_J$ then $(w_0)_J w(\alpha_i) \in \Phi^-$. Suppose that $w(\alpha_i) \in \Phi^-$ and if possible that $(w_0)_J w(\alpha_i) \in \Phi^+$. Then $-w(\alpha_i) \in \Phi_J{}^+$ and so $-\alpha_i \in w^{-1}(\Phi_J{}^+)$. However all roots in $w^{-1}(\Phi_J{}^+)$ are positive by condition (i) and so we have a contradiction. Thus if $w(\alpha_i) \in \Phi^-$ then $(w_0)_J w(\alpha_i) \in \Phi^-$ also.

We have now shown that, for each simple root $\alpha_i \in \Delta$, $(w_0)_J w(\alpha_i) \in \Phi^-$. Thus $(w_0)_J w$ makes each simple root negative and so $(w_0)_J w = w_0$ and $w = (w_0)_J w_0$.

Lemma 6.3.2. $(B \cap {}^{(w_0)_J w_0}B) N_K (B \cap {}^{(w_0)_J w_0}B) = L_J \cap P_K$ *for all* $K \subseteq J$.

Proof. We have

$$B \cap {}^{(w_0)_J w_0}B = B \cap B^{w_0 (w_0)_J} = (U \cap U^{w_0 (w_0)_J})H = U_{(w_0)_J} H.$$

Now $U_{(w_0)_J} H = B_J$ by 2.6.3, where (B_J, N_J) is a BN-pair for L_J. Thus

$$(B \cap {}^{(w_0)_J w_0}B) N_K (B \cap {}^{(w_0)_J w_0}B) = B_J N_K B_J.$$

We now have

$$L_J \cap P_K = B_J N_J B_J \cap B N_K B = B_J N_K B_J$$

which gives the required result.

Proposition 6.3.3. *Let* $(\mathrm{St}_G)_{P_J}$ *denote the restriction of* St_G *to* P_J. *Then we have*

$$(\mathrm{St}_G)_{P_J} = \mathrm{St}_{L_J}{}^{P_J}.$$

Proof. (Howlett) We have

$$(\mathrm{St}_G)_{P_J} = \left(\sum_K (-1)^{|K|} 1_{P_K}{}^G\right)_{P_J} = \sum_K (-1)^{|K|} (1_{P_K}{}^G)_{P_J}$$
$$= \sum_K (-1)^{|K|} \sum_{n \in N_{J,K}} (1_{P_J \cap {}^n P_K})^{P_J}$$

by Mackey's subgroup formula, since $N_{J,K}$ is a system of double coset representatives for G with respect to P_J, P_K. By 2.8.11 we have

$$P_J \cap {}^n P_K = (B \cap {}^n B) N_{J \cap w(K)} (B \cap {}^n B)$$

where $w = \pi(n)$. Thus

$$(\mathrm{St}_G)_{P_J} = \sum_K (-1)^{|K|} \sum_{n \in N_{J,K}} (1_{(B \cap {}^n B) N_{J \cap w(K)} (B \cap {}^n B)})^{P_J}$$
$$= \sum_K (-1)^{|K|} \sum_{w \in D_{J,K}} (1_{(B \cap {}^w B) N_{J \cap w(K)} (B \cap {}^w B)})^{P_J}$$
$$= \sum_K (-1)^{|K|} \sum_{\substack{w \in D_J{}^{-1} \\ K \subseteq K_w}} (1_{(B \cap {}^w B) N_{J \cap w(K)} (B \cap {}^w B)})^{P_J}$$

where K_w is as in 6.3.1, since $D_{J,K} = D_J^{-1} \cap D_K$

$$= \sum_{w \in D_J^{-1}} \sum_{K \subseteq K_w} (-1)^{|K|} (1_{(B \cap {}^wB)N_{J \cap w(K)}(B \cap {}^wB)})^{P_J}.$$

We now decompose the set K into the union of two disjoint sets K_1, K_2 where

$$K_1 = K \cap w^{-1}(J),\ K_2 = K \cap (\Phi - w^{-1}(J)).$$

(We are here ignoring the distinction between the set K and the set Δ_K of simple roots corresponding to K, but the meaning remains quite clear.) Then $J \cap w(K) = J \cap w(K_1)$. Thus

$$(\mathrm{St}_G)_{P_J} = \sum_{w \in D_J^{-1}} \sum_{K_1 \subseteq K_w \cap w^{-1}(J)} (-1)^{|K_1|} \sum_{K_2 \subseteq K_w \cap (\Phi - w^{-1}(J))} (-1)^{|K_2|} \times (1_{(B \cap {}^wB)N_{J \cap w(K_1)}(B \cap {}^wB)})^{P_J}.$$

Since $\sum_{K_2 \subseteq K_w \cap (\Phi - w^{-1}(J))} (-1)^{|K_2|} = 0$ unless $K_w \cap (\Phi - w^{-1}(J))$ is empty we need only sum over those w for which $K_w \subseteq w^{-1}(J)$. Thus we have

$$(\mathrm{St}_G)_{P_J} = \sum_{\substack{w \in D_J^{-1} \\ K_w \subseteq w^{-1}(J)}} \sum_{K_1 \subseteq K_w} (-1)^{|K_1|} (1_{(B \cap {}^wB)N_{J \cap w(K_1)}(B \cap {}^wB)})^{P_J}.$$

But by 6.3.1 the only element $w \in W$ satisfying $w \in D_J^{-1}$ and $K_w \subseteq w^{-1}(J)$ is $w = (w_0)_J w_0$. For this element w we have $K_w = J' = -w_0(J)$. It follows that

$$(\mathrm{St}_G)_{P_J} = \sum_{K_1 \subseteq J'} (-1)^{|K_1|} (1_{(B \cap {}^{(w_0)_J w_0}B)N_{J \cap (w_0)_J w_0(K_1)}(B \cap {}^{(w_0)_J w_0}B)})^{P_J}.$$

However $(w_0)_J w_0(K_1) \subseteq J$. Let $K = (w_0)_J w_0(K_1)$. Then

$$\begin{aligned}
(\mathrm{St}_G)_{P_J} &= \sum_{K \subseteq J} (-1)^{|K|} (1_{(B \cap {}^{(w_0)_J w_0}B)N_K(B \cap {}^{(w_0)_J w_0}B)})^{P_J} \\
&= \sum_{K \subseteq J} (-1)^{|K|} 1_{L_J \cap P_K}{}^{P_J} \qquad \text{by 6.3.2} \\
&= \sum_{K \subseteq J} (-1)^{|K|} (1_{L_J \cap P_K}{}^{L_J})^{P_J} \\
&= \left(\sum_{K \subseteq J} (-1)^{|K|} 1_{L_J \cap P_K}{}^{L_J} \right)^{P_J} \\
&= (\mathrm{St}_{L_J})^{P_J}
\end{aligned}$$

since the standard parabolic subgroups of L_J are the subgroups $L_J \cap P_K$ for $K \subseteq J$, by 2.6.6.

6.4 THE VALUES OF THE STEINBERG CHARACTER

We now use the result of section 6.3 to determine the values of the Steinberg character at each element of G. We again assume that G is a finite group with split BN-pair which satisfies the commutator relations.

Proposition 6.4.1. *Suppose $g \in G$ does not lie in any proper parabolic subgroup of G. Then $St(g) = (-1)^{|I|}$.*

Proof. $\mathrm{St}(g) = \sum_{J \subseteq I} (-1)^{|J|} 1_{P_J}{}^G(g)$.
If $J \neq I$ then no conjugate of g lies in P_J, thus $1_{P_J}{}^G(g) = 0$. Thus

$$\mathrm{St}(g) = (-1)^{|I|} 1_G{}^G(g) = (-1)^{|I|}.$$

Proposition 6.4.2. *Suppose $g \in P_J$ is not conjugate in G to any element of P_K for $K \subset J$. Then*

(i) *$St(g) = 0$ if g is not conjugate in P_J to an element of L_J.*
(ii) *$St(g) = (-1)^{|J|}|C_{U_J}(g)|$ if $g \in L_J$.*

Proof. (i) Since $g \in P_J$ we have by 6.3.3

$$\mathrm{St}_G(g) = St_{L_J}{}^{P_J}(g) = \frac{1}{|L_J|} \sum_{\substack{x \in P_J \\ g^x \in L_J}} St_{L_J}(g^x).$$

If g is not conjugate in P_J to an element of L_J the sum is empty and so $St_G(g) = 0$.

(ii) Suppose $g \in L_J$. We consider elements $x \in P_J$ for which $g^x \in L_J$. We have $P_J = L_J U_J$ and so $x = yz$ where $y \in L_J$, $z \in U_J$. Thus $g^x = (g^y)^z \in L_J$ and $g^y \in L_J$. Hence

$$[g^y, z] = (g^y)^{-1}(g^y)^z \in U_J \cap L_J = 1$$

and so $z \in C_{U_J}(g^y)$. Conversely if $z \in C_{U_J}(g^y)$ then $g^x \in L_J$. Thus we have

$$\begin{aligned}
\mathrm{St}_G(g) &= \frac{1}{|L_J|} \sum_{y \in L_J} \sum_{z \in C_{U_J}(g^y)} St_{L_J}(g^y) \\
&= \frac{1}{|L_J|} \sum_{y \in L_J} |C_{U_J}(g^y)| St_{L_J}(g) \\
&= |C_{U_J}(g)| St_{L_J}(g) \\
&= (-1)^{|J|} |C_{U_J}(g)| \qquad \text{by 6.4.1}
\end{aligned}$$

since g lies in no proper parabolic subgroup of L_J.

Corollary 6.4.3. *$St(1) = |U|$.*

Proof. Take $g = 1$ and $J = \varnothing$ in 6.4.2.

Proposition 6.4.4. *Suppose $g \in L_J$ and g is not conjugate in G to any element of P_K for $K \subset J$. Then*

(i) *$|C_{U_J}(g)| = |C_G(g)|_p$.*
(ii) *$St_G(g) = (-1)^{|J|}|C_G(g)|_p$.*

Proof. We use the fact, well known in the representation theory of finite groups, that if χ^i is any irreducible character of G then the expression

$$\frac{\chi^i(g)}{\chi^i(1)}|G:C_G(g)|$$

is an algebraic integer. See, for example, Curtis and Reiner [1], p. 235. We apply this result to the Steinberg character. Thus

$$\frac{\mathrm{St}(g)}{\mathrm{St}(1)}\frac{|G|}{|C_G(g)|}=\frac{(-1)^{|J|}|C_{U_J}(g)||G|}{|U||C_G(g)|}=(-1)^{|J|}\frac{|G:U|}{|C_G(g):C_{U_J}(g)|}$$

is an algebraic integer. Since it is also a rational number it must be a rational integer. Thus $|C_G(g):C_{U_J}(g)|$ divides $|G:U|$. Now $|G:U|=|G|_{p'}$ and so $|C_G(g):C_{U_J}(g)|$ is prime to p. Since $|C_{U_J}(g)|$ is a power of p we have

$$|C_{U_J}(g)|=|C_G(g)|_p.$$

This proves (i), and (ii) follows from (i) and 6.4.2.

Proposition 6.4.5. *Suppose $g\in G$ does not lie in any proper parabolic subgroup of G. Then*

(i) $|C_G(g)|$ *is prime to p.*
(ii) *The order of g is prime to p.*

Proof. (i) follows from 6.4.4 by taking $J=I$ since $U_I=1$. (ii) follows immediately from (i).

Proposition 6.4.6. *Suppose $g\in P_J$ and g is not conjugate in G to any element of P_K for $K\subset J$.* Then *g is conjugate in P_J to an element of L_J if and only if the order of g is prime to p.*

Proof. If g is conjugate to an element of L_J then this conjugate lies in no proper parabolic subgroup of L_J. Thus the order of this element is prime to p by 6.4.5 and so g has order prime to p.

Conversely suppose the order of g is prime to p. Then $\langle U_J,g\rangle=U_J\langle g\rangle$ and $U_J\cap\langle g\rangle=1$. U_J is a normal subgroup of $\langle U_J,g\rangle$ of order a power of p and index prime to p. Thus any two complements of U_J in $\langle U_J,g\rangle$ are conjugate, by the Schur–Zassenhaus theorem (Huppert [1], p. 126). Now $\langle U_J,g\rangle\cap L_J$ and $\langle g\rangle$ are both complements of U_J in $\langle U_J,g\rangle$. Thus $\langle g\rangle$ is conjugate to $\langle U_J,g\rangle\cap L_J$ and so g is conjugate in P_J to an element of L_J. ■

We now combine the results of the last few propositions to give the following theorem, which gives all the values of the Steinberg character.

Theorem 6.4.7. *Let $g\in G$. Then*

(i) *$St(g)=0$ if the order of g is divisible by p.*

(ii) *If the order of g is prime to p then*

$$St(g) = \pm |C_G(g)|_p.$$

(iii) *If the order of g is prime to p then $St(g) = (-1)^{|J|}|C_G(g)|_p$ where g is conjugate to an element of L_J but not conjugate to an element of L_K for any $K \subset J$.*

Proof. Let $g \in G$. Then there exists J such that g is conjugate to an element of P_J but not to an element of P_K for any $K \subset J$. Suppose the order of g is divisible by p. Then g is not conjugate to an element of L_J by 6.4.6, and so $\mathrm{St}(g) = 0$ by 6.4.2.

Now suppose the order of g is prime to p. Then g is conjugate to an element of L_J by 6.4.6, and so $\mathrm{St}(g) = (-1)^{|J|}|C_G(g)|_p$ by 6.4.4.

6.5 THE SIGNS OF THE STEINBERG CHARACTER

We shall now consider the group G^F where G is a connected reductive group and $F: G \to G$ is a Frobenius map. In this case there is a more convenient form for the signs of the values of the Steinberg character, which we shall derive in the present section.

We adapt the general definition of the Steinberg character valid for any finite group with a BN-pair to the particular situation we have in the group G^F. As in section 1.18 the Frobenius map $F: G \to G$ gives rise to a permutation ρ of the set of simple roots, and so of the index set I labelling the simple roots. For each ρ-orbit J of I we have a corresponding element $s_J \in W^F$, and the elements s_J for all ρ-orbits of I generate W^F as a Coxeter group. Let $I' = I/\rho$ be the set of all ρ-orbits on I. We then have a bijection between subsets J' of I' and ρ-stable subsets J of I given by $J' = J/\rho$. Each such subset J' of I' gives rise to a parabolic subgroup $P_{J'}$ of G^F given by $P_{J'} = P_J{}^F$. Thus the Steinberg character of G^F is given by

$$\mathrm{St} = \sum_{J' \subseteq I'} (-1)^{|J'|} 1_{P_{J'}}{}^{G^F} = \sum_{\substack{J \subseteq I \\ \rho(J) = J}} (-1)^{|J/\rho|} 1_{P_J{}^F}{}^{G^F}.$$

6.4.7 shows that $\mathrm{St}(g) = 0$ for all $g \in G^F$ which are not semisimple and that, if $s \in G^F$ is semisimple, then

$$\mathrm{St}(s) = (-1)^{|J'|}|C_{G^F}(s)|_p$$

where some conjugate of s lies in $L_{J'}$ but no conjugate of s lies in $L_{K'}$ for any proper subset K' of J'. Such a subset J' clearly exists, but may not be unique. It is however unique up to conjugacy under the Weyl group W^F, as we show in the next proposition.

Proposition 6.5.1. *Let s be a semisimple element of G^F and consider the set of $J' \subseteq I'$ such that some conjugate of s lies in $L_{J'}$. Then any two minimal elements of this set are equivalent under the Weyl group W^F of G^F.*

Proof. Suppose s is conjugate to an element of $L_{J_1'}$ and of $L_{J_2'}$. Thus s lies in ${}^{g_1}L_{J_1'} \cap {}^{g_2}L_{J_2'}$ for some $g_1, g_2 \in G^F$. Thus some conjugate of s lies in $L_{J_1'} \cap {}^{g}L_{J_2'}$ for some $g \in G^F$. Now we have

$$G^F = P_{J_1'} N_{J_1', J_2'} P_{J_2'} = L_{J_1'} U_{J_1'} N_{J_1', J_2'} U_{J_2'} L_{J_2'}.$$

Let $g = l_1 u_1 n u_2 l_2$ where $l_1 \in L_{J_1'}$, $u_1 \in U_{J_1'}$, $n \in N_{J_1', J_2'}$, $u_2 \in U_{J_2'}$, $l_2 \in L_{J_2'}$. Then $L_{J_1'} \cap {}^{g}L_{J_2'} = L_{J_1'} \cap {}^{l_1 u_1 n u_2 l_2}L_{J_2'}$, and this subgroup is conjugate to ${}^{u_1^{-1}}L_{J_1'} \cap {}^{n u_2}L_{J_2'}$, which lies in $P_{J_1'} \cap {}^{n}P_{J_2'}$. We now recall from 2.8.7 that

$$P_{J_1'} \cap {}^{n}P_{J_2'} = U_1 L_{K'} \qquad U_1 \cap L_{K'} = 1$$

where U_1 is the maximal normal unipotent subgroup of $P_{J_1'} \cap {}^{n}P_{J_2'}$ and K' is given by $\Delta_{K'} = \Delta_{J_1'} \cap w(\Delta_{J_2'})$ where $w = \pi(n)$.

We show next that any semisimple element of $P_{J_1'} \cap {}^{n}P_{J_2'}$ is conjugate to an element of $L_{K'}$. Let x be such a semisimple element. Consider the subgroup $\langle U_1, x\rangle$. This contains U_1 as a normal subgroup of order a power of p, and the subgroups $\langle x\rangle$ and $\langle U_1, x\rangle \cap L_{K'}$ are complements of U_1 of order prime to p. By the Schur–Zassenhaus theorem x is conjugate to an element of $L_{K'}$.

We have therefore shown that if s is conjugate to an element of $L_{J_1'}$ and of $L_{J_2'}$ then s is conjugate to an element of $L_{K'}$ where $K' = J_1' \cap w(J_2')$ for some $w \in D_{J_1', J_2'}$. Suppose J_1' and J_2' are minimal with respect to the above property. Then $K' = J_1'$ and $w^{-1}(K') = J_2'$, and so $J_1' = w(J_2')$. ∎

Now if a semisimple element $s \in G^F$ lies in $L_{J'} = L_J^F$ it will lie in some F-stable maximal torus of L_J. We consider which F-stable maximal tori of G lie in L_J. Recall that each F-stable maximal torus of G is obtained from a maximally split torus by twisting by an element $w \in W$, which is determined up to F-conjugacy.

Proposition 6.5.2. *Let L_J be an F-stable standard Levi subgroup of G and let $w \in W$. Then L_J contains an F-stable maximal torus of type w if and only if w is F-conjugate to an element of W_J.*

Proof. Let $T = L_\phi$ be maximally split and suppose ${}^{g}T$ is F-stable and lies in L_J. Then T and ${}^{g}T$ are maximal tori of L_J so are conjugate in L_J. We may therefore assume that $g \in L_J$. Since ${}^{g}T$ is F-stable we have $g^{-1}F(g) \in N$. Hence $g^{-1}F(g) \in N \cap L_J = N_J$ and $\pi(g^{-1}F(g)) \in W_J$. Thus ${}^{g}T$ is obtained from T by twisting by an element of W_J.

Now suppose conversely that $w \in W_J$. Let $n \in N_J$ satisfy $\pi(n) = w$. Then there exists $g \in L_J$ with $g^{-1}F(g) = n$. ${}^{g}T$ is then an F-stable maximal torus of L_J of type w.

Corollary 6.5.3. *Some G^F-conjugate of L_J contains an F-stable maximal torus of type w if and only if w is F-conjugate to an element of W_J.*

Proof. This follows from 6.5.2 and the fact that G^F-classes of F-stable maximal tori of G correspond to F-conjugacy classes in W.

Proposition 6.5.4. *Let s be a semisimple element of G^F. Then s lies in a G^F-conjugate of $L_{J'} = L_J{}^F$ if and only if $C(s)^0$ contains a maximal torus of type w for some $w \in W_J$.*

Proof. Since s is F-stable s lies in $L_{J'}$ if and only if s lies in L_J. s lies in a G^F-conjugate of $L_{J'}$ if and only if s lies in a G^F-conjugate of L_J. s lies in a G^F-conjugate of L_J if and only if s lies in an F-stable maximal torus of a G^F-conjugate of L_J. By 6.5.3 this is so if and only if s lies in an F-stable maximal torus of type w for some w which is F-conjugate to an element of W_J. By 3.5.2 this is equivalent to the condition that the connected centralizer $C(s)^0$ contains a maximal torus of type w for some w which is F-conjugate to an element of W_J. If this is so we can choose w to lie in W_J. ■

We must therefore consider the maximal tori of the group $C(s)^0$ and determine the minimal F-stable subsets J of I for which $C(s)^0$ contains a maximal torus of type w for some $w \in W_J$.

We shall need an invariant called the relative rank, or F_q-rank, of G. (We prefer to use the term relative rank since in our general set-up q need not be an integer and so the finite field F_q is not defined.) Let T be a maximally split torus of G, X be its character group, and $V = X \otimes \mathbb{R}$. F acts on V by $F = qF_0$ where $q > 1$ and F_0 has finite order.

Definition. The relative rank of G is $\dim V^{F_0}$. This is the number of eigenvalues of F_0 on V equal to 1, or the number of eigenvalues of F on V equal to q. This definition implies that the relative rank of G is equal to the relative rank of a maximally split torus of G. We show that this is greater than the relative rank of any F-stable maximal torus which is not maximally split. We recall from 3.3.4 that if ${}^gT$ is an F-stable torus obtained from the maximally split torus T by twisting with w, then F acts on the character group of ${}^gT$ as Fw^{-1} acts on X. Thus the relative rank of ${}^gT$ is $\dim V^{F_0w^{-1}}$.

Proposition 6.5.5. *Let $w \in W$. Suppose that w is F-conjugate to an element of W_J where J is F-stable but is not F-conjugate to an element of W_K for any F-stable $K \subset J$. Then*

$$\dim V^{F_0w^{-1}} = \dim V^{F_0} - |J'|.$$

Proof. We may assume without loss of generality that $w \in W_J$. Consider the decomposition

$$V = V_J \oplus V_J{}^{\perp}.$$

F_0 and w both act on V_J and, since they are isometries, on $V_J{}^{\perp}$ also. However w acts as the identity on $V_J{}^{\perp}$ since $w \in W_J$. Thus

$$\dim V^{F_0w^{-1}} = \dim V_J{}^{F_0w^{-1}} + \dim (V_J{}^{\perp})^{F_0}.$$

Now we shall show in the following lemma that if F_0w^{-1} fixes a nonzero vector in V_J then w is F-conjugate to an element of W_K where K is a proper F-stable subset of J. Since this does not happen we have $V_J^{F_0 w^{-1}} = 0$. Thus we have

$$\dim V^{F_0 w^{-1}} = \dim (V_J^{\perp})^{F_0} = \dim V^{F_0} - \dim V_J^{F_0} = \dim V^{F_0} - |J'|$$

since the number of eigenvalues of F_0 on V_J equal to 1 is the number of orbits of F_0 on J.

Lemma 6.5.6. *Suppose $w \in W_J$ and $F_0 w^{-1}$ fixes a nonzero vector of V_J. Then w is F-conjugate to an element of W_K for some proper F-stable subset K of J.*

Proof. Let $F_0 w^{-1}(v) = v$ where $v \neq 0$. Let C be the fundamental chamber of W_J in V_J. Then there exists $x \in W_J$ with $x(v) \in \bar{C}$. Let $v' = x(v)$. Then we have

$$xF_0 w^{-1} x^{-1}(v') = v'$$

and so

$$F_0^{-1} x F_0 w^{-1} x^{-1}(v') = F_0^{-1}(v').$$

Now $F_0^{-1} x F_0 = F^{-1} x F$, and the definitions of the actions of W and F on V imply that $F^{-1} x F$ acts on V in the same way as $F(x)$. Thus we have

$$F(x) w^{-1} x^{-1}(v') = F_0^{-1}(v').$$

Now F_0 transforms $\bar{C}$ into itself. Hence v' and $F_0^{-1}(v')$ both lie in $\bar{C}$. Since $\bar{C}$ is a fundamental region for W_J this implies that

$$F(x) w^{-1} x^{-1}(v') = v' \quad \text{and} \quad F_0^{-1}(v') = v'.$$

Let $F(x) w^{-1} x^{-1} = w'^{-1}$. Then $w'(v') = v'$ and $w' = xwF(x^{-1})$. Thus w' is F-conjugate to w and w' fixes a nonzero vector v' in $\bar{C}$. Since w' fixes v', w' is a product of reflections each fixing v' by 2.2.12. Thus $w' \in W_K$ where $K = \{i \in J; (\alpha_i, v') = 0\}$. Since v' is F_0-stable K must be F_0-stable also. Since $v' \neq 0$ K is a proper subset of J.

Corollary 6.5.7. *The relative rank of any F-stable maximal torus of G is less than or equal to the relative rank of G. Equality holds if and only if the torus is maximally split.*

Proof. This follows from 6.5.5 since $|J'| = 0$ if and only if w is F-conjugate to the identity.

Proposition 6.5.8. *Let s be a semisimple element of G^F. Suppose some conjugate of s lies in $L_{J'}$ but that no conjugate of s lies in $L_{K'}$ for any proper subset K' of J'. Then the relative rank of $C(s)^0$ is given by*

$$\text{rel. rank } C(s)^0 = \textit{rel. rank } G - |J'|.$$

Proof. By 6.5.4 we know that $C(s)^0$ contains an F-stable maximal torus T of type w for some $w \in W_J$ but no F-stable maximal torus of type w' where w' is F-conjugate to an element of W_K for a proper F-stable subset K of J. By 6.5.5 the relative rank of T is given by

$$\text{rel. rank } T = \text{rel. rank } G - |J'|.$$

Now consider any other F-stable maximal torus T' of $C(s)^0$. Suppose T' has type w'. Then

$$\text{rel. rank } T' = \text{rel. rank } G - |L'|$$

where w' is F-conjugate to an element of W_L but not to an element of W_M for any proper F-stable subset M of L. However 6.5.1 and 6.5.4 together show that $|L'| \geqslant |J'|$. Hence

$$\text{rel. rank } T' \leqslant \text{rel. rank } T.$$

It follows that T must be a maximally split torus of $C(s)^0$. Hence we have

$$\text{rel. rank } C(s)^0 = \text{rel. rank } T = \text{rel. rank } G - |J'|.$$

Definition. We define $\varepsilon_G = (-1)^{\text{rel. rank } G}$.

We may now obtain our desired formula for the values of the Steinberg character of G^F.

Theorem 6.5.9. *Let $s \in G^F$. Then*

$$St(s) = \begin{cases} \varepsilon_G \varepsilon_{C(s)^0} |C(s)^F|_p & \text{if } s \text{ is semisimple} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows from 6.4.7 and 6.5.8. ■

We recall from section 1.14 that every unipotent element of $C(s)$ lies in $C(s)^0$ when s is semisimple. Thus $|C(s)^F : C(s)^{0F}|$ is prime to p and $|C(s)^F|_p = |C(s)^{0F}|_p$. Hence the value of St($s$) when s is semisimple can also be written

$$\mathrm{St}(s) = \varepsilon_G \varepsilon_{C(s)^0} |(C(s)^0)^F|_p.$$

6.6 THE NUMBER OF UNIPOTENT ELEMENTS IN G^F

We now prove a theorem of Steinberg which determines the number of unipotent elements of G^F and whose proof makes essential use of the properties of the Steinberg character.

Theorem 6.6.1. *Let G be a connected reductive group and $F: G \to G$ be a Frobenius map. Then the number of unipotent elements of G^F is $|G^F|_p{}^2$.*

Proof. We use induction on the dimension of G, the result being trivially true if G is a torus. Since (St, St) $= 1$ we have, using 6.5.9,

$$\frac{1}{|G^F|} \sum_{\substack{s \in G^F \\ \text{semisimple}}} |C(s)^F|_p{}^2 = 1$$

and so

$$|G^F| = \sum_{\substack{s \in G^F \\ \text{semisimple}}} |C(s)^F|_p{}^2.$$

We may obtain a second equation for $|G^F|$ by using the Jordan decomposition. Each element $g \in G^F$ is uniquely expressible in the form $g = su = us$ where s, $u \in G^F$ and s is semisimple, u is unipotent. Let Q_s be the number of unipotent elements in $C(s)^F$. Then we have

$$|G^F| = \sum_{\substack{s \in G^F \\ \text{semisimple}}} Q_s.$$

Now all the unipotent elements of $C(s)$ lie in $C(s)^0$ which is a connected reductive group. Moreover $C(s)^0$ has smaller dimension than G unless s lies in the centre $Z(G)$. Thus we may assume inductively that $Q_s = |(C(s)^0)^F|_p{}^2$ when $s \notin Z(G)$. Since we know also that $|(C(s)^0)^F|_p = |C(s)^F|_p$ we have $Q_s = |C(s)^F|_p{}^2$ when $s \notin Z(G)$. It follows that

$$\sum_{s \in Z(G)^F} |C(s)^F|_p{}^2 = \sum_{s \in Z(G)^F} Q_s.$$

Thus we have

$$|Z(G)^F||G^F|_p{}^2 = |Z(G)^F|Q_1$$

and so $Q_1 = |G^F|_p{}^2$. Q_1 is the number of unipotent elements of G^F and so the theorem is proved.

Chapter 7

THE GENERALIZED CHARACTERS OF DELIGNE–LUSZTIG

In this chapter we shall define certain generalized characters $R_{T,\theta}$ of the group G^F which were introduced by Deligne and Lusztig in [1]. One obtains such a generalized character for each F-stable maximal torus T of G and each character $\theta \in \hat{T}^F$. These generalized characters turn out to be of basic importance for the understanding of the irreducible complex characters of G^F.

In order to define the $R_{T,\theta}$ we must first have the concept of the l-adic cohomology groups of an algebraic variety over an algebraically closed field of characteristic p, where l is a prime different from p. We shall consider l-adic cohomology groups with compact support, and shall give a list of properties which such cohomology groups satisfy. We shall then take this list of properties as axioms on which are based the subsequent development of the representation theory. The definition of the l-adic cohomology groups can be found in the appendix. The proof that these l-adic cohomology groups with compact support satisfy our given list of axioms is complicated and uses techniques of algebraic geometry which are far beyond the scope of the present volume. For further information on this subject we refer the reader to Deligne [1], Grothendieck *et al.* [1] and Milne [1].

7.1 *l*-ADIC COHOMOLOGY WITH COMPACT SUPPORT

Let l be a prime. Then $\mathbb{Z}/l^i\mathbb{Z}$ is a finite ring for each i and we have a sequence of ring homomorphisms

$$\ldots \to \mathbb{Z}/l^3\mathbb{Z} \to \mathbb{Z}/l^2\mathbb{Z} \to \mathbb{Z}/l\mathbb{Z} \to \mathbb{Z}/\mathbb{Z}$$

under which the element $l^i\mathbb{Z} + a$ of $\mathbb{Z}/l^i\mathbb{Z}$ maps to the element $l^{i-1}\mathbb{Z} + a$ of $\mathbb{Z}/l^{i-1}\mathbb{Z}$. The inverse limit $\mathbb{Z}_l = \varprojlim \mathbb{Z}/l^i\mathbb{Z}$ is the ring of l-adic integers. $\mathbb{Z}_l$ is an integral domain. Its field of fractions $\mathbb{Q}_l$ is the field of l-adic numbers. $\mathbb{Q}_l$ is a field of characteristic 0. Let $\bar{\mathbb{Q}}_l$ be an algebraic closure of $\mathbb{Q}_l$. $\bar{\mathbb{Q}}_l$, being an algebraically closed field of characteristic 0, will contain as a subfield the field of algebraic numbers.

We shall be considering complex characters of various finite groups, and all such character values lie in the ring of algebraic integers. Thus it does not matter whether we consider characters with values in $\mathbb{C}$ or with values in $\bar{\mathbb{Q}}_l$, since in either case the character values lie in the common subring of algebraic integers. We must take care, however, with the operation of complex conjugation, which appears in a number of character formulae. This operation is not defined in any natural way in $\bar{\mathbb{Q}}_l$. It is, however, defined for the elements of $\bar{\mathbb{Q}}_l$ which are roots of unity, where it coincides with the operation of inversion, and is also defined for all elements of $\bar{\mathbb{Q}}_l$ which are sums of roots of unity. These are the only elements of $\bar{\mathbb{Q}}_l$ for which we shall need it. We shall therefore consider $\mathbb{C}$-valued characters or $\bar{\mathbb{Q}}_l$-valued characters, depending on which is most convenient in a given context.

Let X be an algebraic variety over the field $K = \bar{F}_p$. Let l be a prime distinct from p. Then one can associate with X a finite-dimensional vector space $H_c^i(X, \bar{\mathbb{Q}}_l)$ over $\bar{\mathbb{Q}}_l$, the ith l-adic cohomology group of X with compact support. The way in which this group is defined is described in the appendix.

The cohomology groups $H_c^i(X, \bar{\mathbb{Q}}_l)$ satisfy the following conditions.

Property 7.1.1. *$H_c^i(X, \bar{\mathbb{Q}}_l) = 0$ for all i except possibly when $0 \leqslant i \leqslant 2\dim X$.*

Property 7.1.2. *If X is n-dimensional affine space K^n then*

$$\dim H_c^i(X, \bar{\mathbb{Q}}_l) = \begin{cases} 1 & \text{if } i = 2n \\ 0 & \text{if } i \neq 2n \end{cases}.$$

Property 7.1.3. *Each automorphism of X induces a nonsingular linear map of $H_c^i(X, \bar{\mathbb{Q}}_l)$ into itself in such a way as to make $H_c^i(X, \bar{\mathbb{Q}}_l)$ a module for the group of automorphisms of X.*

Property 7.1.4. *Let g be an automorphism of X of finite order and define*

$$\mathscr{L}(g, X) = \sum_i (-1)^i \operatorname{trace}(g, H_c^i(X, \bar{\mathbb{Q}}_l)).$$

Then $\mathscr{L}(g, X) \in \mathbb{Z}$ and is independent of the choice of l. $\mathscr{L}(g, X)$ is called the Lefschetz number of g on X.

Property 7.1.5. *Suppose $f: X \to Y$ is a morphism of algebraic varieties such that, for each $y \in Y$, $f^{-1}(y)$ is isomorphic to the affine space K^n for fixed n. Let g, g' be automorphisms of finite order of X, Y respectively such that $fg = g'f$. Then $\mathscr{L}(g, X) = \mathscr{L}(g', Y)$. This holds in particular if $f: X \to Y$ is a bijective morphism.*

Property 7.1.6. *Suppose $X = \bigcup_{j=1}^n X_j$ is a disjoint union of locally closed* *subsets X_j and that g is an automorphism of X of finite order which leaves each X_j invariant. Then*

$$\mathscr{L}(g, X) = \sum_{j=1}^{n} \mathscr{L}(g, X_j).$$

Suppose in addition that $\bigcup_{j=1}^{k} X_j$ is closed in X for all k and that G is a finite group of automorphisms of X leaving each X_j invariant. Let θ be an irreducible character of G and $H_c^i(X_j, \bar{\mathbb{Q}}_l)_\theta$ be the subspace of $H_c^i(X_j, \bar{\mathbb{Q}}_l)$ on which G has character θ. Then if $H_c^i(X_j, \bar{\mathbb{Q}}_l)_\theta = 0$ for all i, j we have $H_c^i(X, \bar{\mathbb{Q}}_l)_\theta = 0$ for all i.

Property 7.1.7. *Suppose $X = \bigcup_{j=1}^{n} X_j$ is a disjoint union of closed subsets X_j and that G is a finite group of automorphisms of X permuting the X_j transitively. Let H be the subgroup of G which is the stabilizer of X_1. Then the generalized character $g \to \mathscr{L}(g, X)$ of G is the induced character of the generalized character $h \to \mathscr{L}(h, X_1)$ of H.*

Property 7.1.8. *Suppose that G is a finite group of automorphisms of X such that a strict quotient X/G exists. (This will be the case, for example, if X is affine.) Then*

$$H_c^i(X/G, \bar{\mathbb{Q}}_l) \cong H_c^i(X, \bar{\mathbb{Q}}_l)^G.$$

Moreover, if g is an automorphism of X of finite order which commutes with all the elements of G and if g' is an automorphism of X/G such that $fg = g'f$ where $f: X \to X/G$ is the natural map, then

$$\mathscr{L}(g', X/G) = \frac{1}{|G|} \sum_{x \in G} \mathscr{L}(gx, X).$$

Property 7.1.9. *Let X_1, X_2 be algebraic varieties. Then*

$$H_c^i(X_1 \times X_2, \bar{\mathbb{Q}}_l) \cong \bigoplus_{\substack{j,k \\ j+k=i}} (H_c^j(X_1, \bar{\mathbb{Q}}_l) \otimes H_c^k(X_2, \bar{\mathbb{Q}}_l)).$$

Moreover, if g_1, g_2 are automorphisms of finite order of X_1, X_2 respectively, then $g_1 \times g_2$ is an automorphism of $X_1 \times X_2$ of finite order and

$$\mathscr{L}(g_1 \times g_2, X_1 \times X_2) = \mathscr{L}(g_1, X_1)\mathscr{L}(g_2, X_2).$$

Property 7.1.10. *Let g be an automorphism of X of finite order and let $g = su = us$ where s has order prime to p and u has order a power of p. Let $X^s = \{x \in X; sx = x\}$. Then*

$$\mathscr{L}(g, X) = \mathscr{L}(u, X^s).$$

In particular

$$\mathscr{L}(g, X) = 0 \text{ if } X^s \text{ is empty}.$$

Property 7.1.11. *If X is finite and g is an automorphism of X then*

$$\mathscr{L}(g, X) = |X^g|.$$

Property 7.1.12. *Let G be a connected algebraic group acting on X as a group of automorphisms. Then each element of G acts trivially on $H_c^i(X, \bar{\mathbb{Q}}_l)$.*

We now indicate where proofs of these properties can be found. Some of the statements involve properties of the cohomology groups and others merely properties of the Lefschetz number. The proofs of most of the statements about Lefschetz numbers can be found in Lusztig [10]. The facts about Lefschetz numbers stated in 7.1.4, 7.1.5, 7.1.6, 7.1.7, 7.1.8, 7.1.9 and 7.1.11 are proved in this paper of Lusztig. The proofs make use of the fact that a definition can be given for the Lefschetz number $\mathscr{L}(g, X)$ which makes no mention of the l-adic cohomology groups of X. This definition, which is described in section (h) of our appendix, follows from Grothendieck's trace formula. This trace formula of Grothendieck, which is described in section (g) of the appendix, relates the number of elements of X fixed by a Frobenius map F to the trace of F on the l-adic cohomology groups of X. A proof of Grothendieck's trace formula can be found in Deligne [1] (SGA $4\frac{1}{2}$), p. 76–109.

The statements about Lefschetz numbers in 7.1.8 and 7.1.9, as well as being provable directly, are also immediate consequences of the corresponding statements about l-adic cohomology groups, once these have been established.

The property 7.1.10 of Lefschetz numbers cannot be proved in the same way as the other properties. This property was proved for the first time in Deligne and Lusztig [1], and uses statements from M. Artin *et al.* [1] (SGA 4) and Grothendieck *et al.* [1] (SGA 5).

We now turn to the statements involving l-adic cohomology groups. Property 7.1.3 follows immediately from the definition of the groups $H_c^i(X, \bar{\mathbb{Q}}_l)$. The definition of these groups can be found in the appendix. They are obtained from groups $H_c^i(X, \mathbb{Z}_l)$ by extending the base ring from $\mathbb{Z}_l$ to $\bar{\mathbb{Q}}_l$. The groups $H_c^i(X, \mathbb{Z}_l)$ are themselves obtained from groups $H_c^i(X, \mathbb{Z}/l^j\mathbb{Z})$ by means of an inverse limit. The groups $H_c^i(X, \mathbb{Z}/l^j\mathbb{Z})$ are all finite.

The basic property 7.1.1 is proved in SGA 4 XVII 5.2.8.1 and 5.3.8. Property 7.1.2 about the l-adic cohomology of affine space follows from a specialization theorem and a comparison theorem. The specialization theorem, proved in SGA 4 XVI 2.1 and 2.5, relates l-adic cohomology groups for varieties over characteristic p and characteristic 0 and implies that

$$H_c^i(K^n, \mathbb{Z}/k\mathbb{Z}) \cong H_c^i(\mathbb{C}^n, \mathbb{Z}/k\mathbb{Z})$$

for all k not divisible by p. The comparison theorem, proved in SGA 4 XI, 4.4, relates l-adic cohomology groups and classical cohomology groups for varieties over $\mathbb{C}$ and implies that

$$H_c^i(\mathbb{C}^n, \mathbb{Z}/k\mathbb{Z}) \cong H_c^i((\mathbb{C}^n)_{\mathrm{an}}, \mathbb{Z}/k\mathbb{Z})$$

where $(\mathbb{C}^n)_{\mathrm{an}}$ is n-dimensional affine space regarded as an analytic manifold with its classical topology. These two results together imply that

$$H_c^i(K^n, \mathbb{Z}/l^j\mathbb{Z}) \cong H_c^i((\mathbb{C}^n)_{\mathrm{an}}, \mathbb{Z}/l^j\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/l^j\mathbb{Z} & \text{if } i = 2n \\ 0 & \text{otherwise} \end{cases}$$

and so a classical knowledge of the right-hand groups gives a knowledge of the groups $H_c^i(K^n, \mathbb{Z}/l^j\mathbb{Z})$ and so of $H_c^i(K^n, \bar{\mathbb{Q}}_l)$.

The proof of property 7.1.8 is sketched in Srinivasan [9], p. 53, and is based on a result in SGA 4 XVII p 426, 6.2.5 on the existence of a transfer map. The proof of property 7.1.9 can be found in SGA 4 XVII p. 368, 5.4.3. Finally, the proofs of the properties 7.1.6 and 7.1.12 are to be found in Deligne and Lusztig [1]. Property 7.1.6 is proved on p. 137 and uses the existence of a spectral sequence corresponding to a filtration in a variety. Property 7.1.12 is proved on p. 136 in Proposition 6.4, and makes use of a base change theorem proved in Deligne [1] (SGA $4\frac{1}{2}$), p. 49.

We shall take the properties 7.1.1–7.1.12 as the basic axioms on l-adic cohomology on which the development of the representation theory is based.

7.2 THE GENERALIZED CHARACTERS $R_{T,\theta}$

Let T be an F-stable maximal torus of the connected reductive group G. Let us choose a Borel subgroup B containing T. (B may not be F-stable.) Then $B = UT$ where $U = R_u(B)$. We recall that Lang's map L is defined by $L(g) = g^{-1}F(g)$. Thus $G^F = L^{-1}(1)$. Let $\tilde{X} = L^{-1}(U)$. Then $\tilde{X}$ is an algebraic subset of G, so is an affine algebraic variety.

We may define an action of G^F on $\tilde{X}$ by left multiplication. For if $x \in \tilde{X}$, $g \in G^F$ then $gx \in \tilde{X}$ since

$$L(gx) = x^{-1}g^{-1}F(g)F(x) = x^{-1}F(x) \in U.$$

We may also define an action of T^F on $\tilde{X}$ by right multiplication. For if $x \in \tilde{X}$, $t \in T^F$ then $xt \in \tilde{X}$ since

$$L(xt) = t^{-1}x^{-1}F(x)F(t) = t^{-1}x^{-1}F(x)t \in t^{-1}Ut = U.$$

Moreover left multiplication by $g \in G^F$ commutes with right multiplication by $t \in T^F$. By applying 7.1.3 to the direct product $G^F \times T^F$ we see that $H_c^i(\tilde{X}, \bar{\mathbb{Q}}_l)$ is a left G^F-module and a right T^F-module such that

$$(gv)t = g(vt) \qquad g \in G^F, t \in T^F, v \in H_c^i(\tilde{X}, \bar{\mathbb{Q}}_l).$$

Lemma 7.2.1. *Let A be a finite abelian group and V be a finite-dimensional right A-module over an algebraically closed field of characteristic 0. Let θ be an irreducible character of A and e be the element of the group algebra of A given by*

$$e = \frac{1}{|A|}\sum_{a \in A} \theta(a^{-1})a.$$

Then $e^2 = e$ and $Ve = V_\theta = \{v \in V; va = \theta(a)v \text{ for all } a \in A\}$.

Proof.

$$e^2 = \frac{1}{|A|^2}\sum_{a \in A}\sum_{b \in A} \theta(a^{-1})\theta(b^{-1})ab = \frac{1}{|A|}\sum_{c \in A} \theta(c^{-1})c = e.$$

Thus e is idempotent. Let $v \in V_\theta$. Then we have

$$ve = \frac{1}{|A|} \sum_{a \in A} \theta(a^{-1}) va = \frac{1}{|A|} \sum_{a \in A} v = v.$$

Thus $V_\theta \subseteq Ve$. Conversely let $v \in Ve$. Then $v = ue$ for some $u \in V$. Thus

$$\begin{aligned} va = uea &= \frac{1}{|A|} u \sum_{b \in A} \theta(b^{-1}) ba \\ &= \frac{1}{|A|} u \sum_{c \in A} \theta(a)\theta(c^{-1}) c = \theta(a) ue = \theta(a) v. \end{aligned}$$

Hence $Ve \subseteq V_\theta$ and so we have equality.

Lemma 7.2.2. *Let Γ, A be finite groups and suppose A is abelian. Let V be a (Γ, A)-bimodule over an algebraically closed field of characteristic* 0 *with Γ acting on the left and A on the right. Let θ be an irreducible character of A. Let*

$$e = \frac{1}{|A|} \sum_{a \in A} \theta(a^{-1}) a.$$

Then $\operatorname{trace}_V (g, e) = \operatorname{trace}_{V_\theta} g$ *for* $g \in \Gamma$.

Proof. We use 7.2.1. We have $e^2 = e$ and so $V = Ve \oplus V(1 - e)$. Consider the action of (g, e) on Ve. We have

$$ve \underset{(g,e)}{\rightarrow} g(ve)e = g(ve) = (gv)e.$$

Thus (g, e) transforms Ve into itself and acts on Ve in the same way as g.

Next consider the action of (g, e) on $V(1 - e)$. We have

$$v(1 - e) \underset{(g,e)}{\rightarrow} gv(1 - e)e = 0.$$

Thus (g, e) annihilates $V(1 - e)$. It follows that

$$\operatorname{trace}_V(g, e) = \operatorname{trace}_{Ve}(g, e) = \operatorname{trace}_{Ve} g = \operatorname{trace}_{V_\theta} g$$

by 7.2.1. ∎

We shall now define the Deligne–Lusztig generalized characters $R_{T,\theta}$ where $\theta \in \hat{T}^F$ is an irreducible character of T^F. We recall that $\hat{T}^F = \operatorname{Hom}(T^F, \mathbb{C}^*)$, so that θ is an irreducible complex character of T^F. However the values of θ will all be algebraic integers. Since $\bar{\mathbb{Q}}_l$ contains the field of algebraic numbers the values of θ may be regarded as elements of $\bar{\mathbb{Q}}_l$. Thus

$$\hat{T}^F = \operatorname{Hom}(T^F, \mathbb{C}^*) = \operatorname{Hom}(T^F, \bar{\mathbb{Q}}_l^*).$$

Now $H_c^i(\tilde{X}, \bar{\mathbb{Q}}_l)_\theta$ is a right T^F-module. Let $H_c^i(\tilde{X}, \bar{\mathbb{Q}}_l)_\theta$ be the T^F-submodule of elements on which T^F acts by the character θ. This will be a left G^F-module. For all $g \in G^F$ we define $R_{T,\theta}\colon G^F \to \bar{\mathbb{Q}}_l$ by

$$R_{T,\theta}(g) = \sum_{i \geqslant 0} (-1)^i \operatorname{trace}(g, H_c^i(\tilde{X}, \bar{\mathbb{Q}}_l)_\theta).$$

Note that the values $R_{T,\theta}(g)$ are again algebraic integers so that it is irrelevant whether we map into $\bar{\mathbb{Q}}_l$ or into $\mathbb{C}$. $R_{T,\theta}$ is a generalized character of G^F.

We recall that in defining the generalized character $R_{T,\theta}$ we chose a Borel subgroup B of G containing T. The generalized character thus appears to depend not only upon the F-stable maximal torus T and the character θ of T^F but also upon the choice of Borel subgroup B containing T. It should therefore, strictly speaking, be denoted by $R_{T,\theta,B}$. However we shall show in section 7.3 that $R_{T,\theta,B}$ is in fact independent of the choice of B. This is proved by means of a nontrivial argument. We shall, however, denote the generalized character by $R_{T,\theta}$ from the beginning and prove various properties of this character in 7.2.3–7.2.9. Then in section 7.3 we prove various results which imply in 7.3.6 that $R_{T,\theta}$ is independent of the choice of B. The proofs in section 7.3 are independent of the results 7.2.3–7.2.9 where the independence of B is implicitly assumed.

Proposition 7.2.3. *Let $g \in G^F$. Then*

$$R_{T,\theta}(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \mathscr{L}((g, t), \tilde{X}).$$

Proof. Let

$$e = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) t.$$

Then we have

$$\begin{aligned} R_{T,\theta}(g) &= \sum_{i \geqslant 0} (-1)^i \operatorname{trace}(g, H_c^i(\tilde{X}, \bar{\mathbb{Q}}_l)_\theta) \\ &= \sum_{i \geqslant 0} (-1)^i \operatorname{trace}((g, e), H_c^i(\tilde{X}, \bar{\mathbb{Q}}_l)) \quad \text{by 7.2.2} \\ &= \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \sum_{i \geqslant 0} (-1)^i \operatorname{trace}((g, t), H_c^i(\tilde{X}, \bar{\mathbb{Q}}_l)) \\ &= \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \mathscr{L}((g, t), \tilde{X}). \end{aligned}$$

■

We now determine $R_{T,\theta}$ in the special case when the torus T is maximally split. In this case it turns out to be a character which is very familiar.

Proposition 7.2.4. *Let T be a maximally split F-stable torus and B an F-stable Borel subgroup of G containing T. Let $\theta \in \hat{T}^F$ and θ_{B^F} be the 1-dimensional representation of B^F which extends θ and has U^F in the kernel. Then*

$$R_{T,\theta} = {\theta_{B^F}}^{G^F}$$

is the character of G^F obtained by inducing the linear character θ_{B^F} of B^F.

Proof. Let $\mathfrak{B}^F$ be the set of F-stable Borel subgroups of G. This is the set of G^F-conjugates of B, so is finite. We show there is a map $\tilde{X} \to \mathfrak{B}^F$ given by $g \to gBg^{-1}$. If $g \in \tilde{X}$ then $g^{-1}F(g) \in U$. Let $g^{-1}F(g) = u$. Then we have

$$F(gBg^{-1}) = F(g)BF(g)^{-1} = guBu^{-1}g^{-1} = gBg^{-1}$$

and so $gBg^{-1} \in \mathfrak{B}^F$. This map from $\tilde{X}$ to $\mathfrak{B}^F$ is a morphism of varieties. It is surjective, since if $gBg^{-1} \in \mathfrak{B}^F$ then $gBg^{-1} = g'B(g')^{-1}$ for some $g' \in G^F$ and this is the image of $g' \in G^F \subset \tilde{X}$.

Now G^F acts on $\tilde{X}$ by left multiplication and on $\mathfrak{B}^F$ by $B' \underset{g}{\to} gB'g^{-1}$. The above morphism from $\tilde{X}$ to $\mathfrak{B}^F$ commutes with the G^F-actions. Let $\tilde{X}_1, \ldots, \tilde{X}_n$ be the fibres of this morphism, where $\tilde{X}_1$ is the fibre mapping to B. Then $\tilde{X}$ is the disjoint union of $\tilde{X}_1, \ldots, \tilde{X}_n$. Since G^F permutes the elements of $\mathfrak{B}^F$ transitively it follows that G^F permutes the fibres $\tilde{X}_j$ transitively. Moreover each $\tilde{X}_j$ is closed in $\tilde{X}$.

We now consider the action of T^F. T^F acts on $\tilde{X}$ by right multiplication and elements of $\tilde{X}$ in the same orbit under T^F are in the same fibre $\tilde{X}_j$. Thus T^F acts on each $\tilde{X}_j$ by right multiplication.

Now we have

$$R_{T,\theta}(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \mathscr{L}((g, t), \tilde{X}) \quad g \in G^F$$

by 7.2.3. However 7.1.7 shows that $\mathscr{L}((g, t), \tilde{X})$ can be expressed as an induced character. The stabilizer in G^F of the fibre $\tilde{X}_1$ is $G^F \cap B = B^F$. Thus the generalized character

$$(g, t) \to \mathscr{L}((g, t), \tilde{X})$$

of $G^F \times T^F$ is induced from the generalized character

$$(b, t) \to \mathscr{L}((b, t), \tilde{X}_1)$$

by 7.1.7. Hence the generalized character $R_{T,\theta}$ of G^F is induced from the generalized character of B^F given by

$$b \to \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \mathscr{L}((b, t), \tilde{X}_1).$$

Now $\tilde{X}_1 = \tilde{X} \cap B$. We show that this is equal to $T^F U$. Let $b = tu$ where $t \in T$, $u \in U$. Then

$$L(b) = u^{-1}t^{-1}F(t)F(u)$$

and so we have

$$b \in \tilde{X} \Leftrightarrow L(b) \in U \Leftrightarrow t^{-1}F(t) \in U \Leftrightarrow t \in T^F.$$

Thus $\tilde{X}_1 = T^F U$.

We now consider the natural map $T^F U \to T^F U/U$ which is a morphism of varieties. The right-hand side is finite and the fibres are all isomorphic to N-dimensional affine space K^N where $N = \dim U$. It follows by 7.1.5 that

$$\mathscr{L}((b, t), \tilde{X}_1) = \mathscr{L}((b, t), T^F U) = \mathscr{L}((b, t), T^F U/U) = \mathscr{L}((b, t), B^F/U^F)$$

since $T^F U/U$ is isomorphic to B^F/U^F. We now apply 7.1.11 and obtain

$$\mathscr{L}((b,t),\tilde{X}_1) = |(B^F/U^F)^{(b,t)}|$$

which is the number of fixed points of (b, t) on B^F/U^F. The elements of B^F/U^F have form sU^F where $s \in T^F$. Now sU^F lies in $(B^F/U^F)^{(b,t)}$ if and only if $bsU^Ft = sU^F$, which is equivalent to $bU^F = t^{-1}U^F$. Thus $|(B^F/U^F)^{(b,t)}| = 0$ unless $b \in t^{-1}U^F$. If $b \in t^{-1}U^F$ we have $|(B^F/U^F)^{(b,t)}| = |T^F|$. Given $b \in B^F$ there is a unique $t \in T^F$ for which this is so.

We therefore see that $R_{T,\theta}$ is induced from the generalized character of B^F given by $b \to \theta(t^{-1})$ where t is the unique element of T^F with $b \in t^{-1}U^F$. This generalized character is simply the lift θ_{B^F}. Hence $R_{T,\theta} = \theta_{B^F}{}^{G^F}$. ■

In the case when T is maximally split the generalized character $R_{T,\theta}$ is therefore the proper character obtained by lifting θ from T^F to B^F and then inducing θ_{B^F} to G^F. However such a process cannot work when T is not maximally split, since there will then be no F-stable Borel subgroup containing T. The $R_{T,\theta}$ were introduced precisely in order to give an analogue of the character $\theta_{B^F}{}^{G^F}$ when θ is a character of T^F and T is not necessarily maximally split.

We wish to calculate $R_{T,\theta}(g)$ by using the Jordan decomposition of $g \in G^F$. We have $g = su = us$ where $s, u \in G^F$ and s is semisimple, u is unipotent. Let $t \in T^F$ and consider the action of (g, t) on the (G^F, T^F)-bimodule $H_c^i(\tilde{X}, \bar{\mathbb{Q}}_l)$ given by $v \to gvt$. The maps $v \to vt$ and $v \to sv$ both have order prime to p and commute with one another so their composite $v \to svt$ also has order prime to p. The map $v \to uv$ has order a power of p. Moreover these two maps commute and their composite is $v \to suvt = gvt$. Thus the maps (s, t) and $(u, 1)$ are the p'- and p-parts of the map (g, t) on $H_c^i(\tilde{X}, \bar{\mathbb{Q}}_l)$.

Proposition 7.2.5. *Let $\tilde{X}^{(s,t)}$ be the subvariety of $\tilde{X}$ given by $\tilde{X}^{(s,t)} = \{x \in \tilde{X};\ sxt = x\}$. Then*

$$R_{T,\theta}(g) = \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\mathscr{L}(u,\tilde{X}^{(s,t)}).$$

Proof. This follows from 7.1.10. We have

$$R_{T,\theta}(g) = R_{T,\theta}(su) = \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\mathscr{L}((su,t),\tilde{X})$$

by 7.2.3, and this is equal to

$$\frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\mathscr{L}(u,\tilde{X}^{(s,t)})$$

by 7.1.10.

Proposition 7.2.6. *Let $t \in T^F$ and s be a semisimple element of G^F conjugate in G^F to t^{-1}. Let $(G^F)^{(s,t)} = \{k \in G^F;\ skt = k\}$, and $\tilde{Y}_t = \tilde{X} \cap C^0(t)$. Then we have:*

(i) *The map* $(k, z) \to kz$ *is a surjective morphism from* $(G^F)^{(s,t)} \times \tilde{Y}_t$ *to* $\tilde{X}^{(s,t)}$.

(ii) *There is a left action of* $C^0(t)^F$ *on* $(G^F)^{(s,t)} \times \tilde{Y}_t$ *given by* $(k, z) \underset{m}{\to} (km^{-1}, mz)$.

(iii) *The orbits of* $C^0(t)^F$ *on* $(G^F)^{(s,t)} \times \tilde{Y}_t$ *are the fibres of the morphism in* (i).

Proof. (i) Let $k \in (G^F)^{(s,t)}$ and $z \in \tilde{Y}_t$. Then $L(kz) = z^{-1}k^{-1}F(k)F(z) = z^{-1}F(z) \in U$. Hence $kz \in \tilde{X}$. Moreover we have $skzt = sktz = kz$. Thus $kz \in \tilde{X}^{(s,t)}$. Thus we have a morphism $(k, z) \to kz$ as required. We wish to show it is surjective. Let $g \in \tilde{X}^{(s,t)}$. Then $sgt = g$ and so $gt^{-1}g^{-1} = s$ and $F(g)t^{-1}F(g)^{-1} = s$. Hence $g^{-1}F(g) \in C(t)$. But $g^{-1}F(g)$ lies in U also and $C(t) \cap U \subseteq C^0(t)$ by section 1.14. Hence $g^{-1}F(g) \in C^0(t)$. By the Lang–Steinberg theorem there exists $z \in C^0(t)$ with $g^{-1}F(g) = z^{-1}F(z)$. Such a z lies in $C^0(t) \cap \tilde{X} = \tilde{Y}_t$. Let $gz^{-1} = k$. Then $k \in G^F$ and

$$skt = sgz^{-1}t = sgtz^{-1} = gz^{-1} = k$$

so that $k \in (G^F)^{(s,t)}$. Hence $g = kz$ as required.

(ii) Let $m \in C^0(t)^F$ and $k \in (G^F)^{(s,t)}$. Then $km^{-1} \in G^F$ and $skm^{-1}t = sktm^{-1} = km^{-1}$ so that $km^{-1} \in (G^F)^{(s,t)}$. Let $z \in \tilde{Y}_t$. Then $mz \in C^0(t)$ and

$$L(mz) = z^{-1}m^{-1}F(m)F(z) = z^{-1}F(z) \in U.$$

Hence $mz \in C^0(t) \cap \tilde{X} = \tilde{Y}_t$. Thus

$$(k, z) \to (km^{-1}, mz)$$

gives a left action of $C^0(t)^F$ on $(G^F)^{(s,t)} \times \tilde{Y}_t$.

(iii) If (k, z) and (km^{-1}, mz) are in the same orbit of $C^0(t)^F$ then they are clearly in the same fibre of the map in (i). Suppose conversely that (k_1, z_1) and (k_2, z_2) are in the same fibre. Then $k_1z_1 = k_2z_2$ and so

$$k_2{}^{-1}k_1 = z_2z_1{}^{-1} \in G^F \cap C^0(t) = C^0(t)^F.$$

Let $m = k_2{}^{-1}k_1 = z_2z_1{}^{-1}$. Then $(k_2, z_2) = (k_1m^{-1}, mz_1)$ and so (k_1, z_1) and (k_2, z_2) lie in the same $C^0(t)^F$-orbit. ■

Now $\tilde{X}^{(s,t)}$ and $\tilde{Y}_t$ are algebraic subsets of G and are therefore affine varieties.

Proposition 7.2.7. *Let s, t be as in 7.2.6. Then* $\tilde{X}^{(s,t)}$ *is the disjoint union of* $|C(t)^F : C^0(t)^F|$ *closed subvarieties each isomorphic to* $\tilde{Y}_t$.

Proof. Let $k \in (G^F)^{(s,t)}$. Then $(G^F)^{(s,t)} = kC(t)^F$. Let $z_1, \ldots, z_r$ be a set of coset representatives of $C^0(t)^F$ in $C(t)^F$. Then $(G^F)^{(s,t)}$ is the disjoint union of cosets $kz_iC^0(t)^F$. It follows from 7.2.6 that $\tilde{X}^{(s,t)}$ is the disjoint union of subsets $kz_i\tilde{Y}_t$. $kz_i\tilde{Y}_t$ is a closed subset of G, so also of $\tilde{X}^{(s,t)}$. Moreover left multiplication by kz_i gives an isomorphism of varieties $\tilde{Y}_t \to kz_i\tilde{Y}_t$. ■

We put $k_i = kz_i$. Thus $\tilde{X}^{(s,t)}$ is the disjoint union of closed subsets $k_i\tilde{Y}_t$.

We define $Q_T(u)$ for any unipotent element $u \in G^F$ by $Q_T(u) = R_{T,1}(u)$. We sometimes write $Q_T(u) = Q_T^G(u)$ when it is necessary to indicate the ambient group G being considered.

We are now able to prove an important character formula for the $R_{T,\theta}$'s.

Theorem 7.2.8. *Let $g \in G^F$ have Jordan decomposition $g = su = us$ where s is semisimple and u unipotent. Then*

$$R_{T,\theta}(g) = \frac{1}{|C^0(s)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx) Q^{C^0(s)}_{xTx^{-1}}(u).$$

Proof. We first observe that the expression $Q^{C^0(s)}_{xTx^{-1}}(u)$ is meaningful. Since $x^{-1}sx \in T^F$ we have $s \in xT^Fx^{-1} = (xTx^{-1})^F$. Thus xTx^{-1} is an F-stable maximal torus in $C^0(s)$. u is a unipotent element of $C^0(s)$ since each unipotent element of $C(s)$ lies in $C^0(s)$. Hence $Q^{C^0(s)}_{xTx^{-1}}(u)$ is well defined.

We now calculate $R_{T,\theta}(g)$, making use of 7.2.5. We have

$$\begin{aligned} R_{T,\theta}(g) &= \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \mathscr{L}(u, \tilde{X}^{(s,t)}) \\ &= \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \mathscr{L}(u, \bigcup k_i \tilde{Y}_t) \end{aligned}$$

by 7.2.7. Now each subset $k_i \tilde{Y}_t$ of $\tilde{X}^{(s,t)}$ is invariant under u. For $k_i \in (G^F)^{(s,t)}$ and so $sk_it = k_i$. Hence

$$t^{-1}k_i^{-1}uk_it = k_i^{-1}sus^{-1}k_i = k_i^{-1}uk_i$$

which shows that $k_i^{-1}uk_i \in C(t)$. In fact $k_i^{-1}uk_i \in C^0(t)$ since it is unipotent. Hence $k_i^{-1}uk_i \in C^0(t)^F$ and so $k_i^{-1}uk_i\tilde{Y}_t = \tilde{Y}_t$ and $uk_i\tilde{Y}_t = k_i\tilde{Y}_t$.

We may therefore apply 7.1.6 to this situation and obtain

$$R_{T,\theta}(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \sum_{i=1}^{r} \mathscr{L}(u, k_i \tilde{Y}_t).$$

Now u acts on $k_i\tilde{Y}_t$ as $k_i^{-1}uk_i$ acts on $\tilde{Y}_t$. Hence

$$\begin{aligned} R_{T,\theta}(g) &= \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \sum_{i=1}^{r} \mathscr{L}(k_i^{-1}uk_i, \tilde{Y}_t) \\ &= \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \frac{1}{|C^0(t)^F|} \sum_{k \in (G^F)^{(s,t)}} \mathscr{L}(k^{-1}uk, \tilde{Y}_t) \\ &= \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \frac{1}{|C^0(t)^F|} \sum_{\substack{k \in G^F \\ k^{-1}sk = t^{-1}}} \mathscr{L}(k^{-1}uk, \tilde{Y}_t) \\ &= \frac{1}{|T^F|} \frac{1}{|C^0(s)^F|} \sum_{\substack{k \in G^F \\ k^{-1}sk \in T^F}} \theta(k^{-1}sk) \mathscr{L}(k^{-1}uk, \tilde{Y}_{k^{-1}sk}) \end{aligned}$$

since $\tilde{Y}_{t^{-1}} = \tilde{Y}_t$

$$= \frac{1}{|T^F|}\frac{1}{|C^0(s)^F|}\sum_{\substack{k\in G^F\\ k^{-1}sk\in T^F}} \theta(k^{-1}sk)\mathscr{L}(u, \tilde{Y}_s).$$

We now put $s = 1$. This gives

$$R_{T,\theta}(u) = \frac{1}{|T^F|}\mathscr{L}(u, \tilde{X}).$$

Hence we have

$$Q_T(u) = R_{T,1}(u) = \frac{1}{|T^F|}\mathscr{L}(u, \tilde{X}).$$

We also have

$$\tilde{Y}_s = \tilde{X}\cap C^0(s) = L^{-1}(U)\cap C^0(s) = L^{-1}(U\cap C^0(s))\cap C^0(s)$$

and, by 3.5.5, $U\cap C^0(s)$ is a maximal unipotent subgroup of $C^0(s)$. It follows that $\tilde{Y}_s$ is the analogue for $C^0(s)$ of the variety $\tilde{X}$ in G. In particular we have

$$Q^{C^0(s)}_{kTk^{-1}}(u) = \frac{1}{|T^F|}\mathscr{L}(u, \tilde{Y}_s).$$

We thus obtain

$$R_{T,\theta}(g) = \frac{1}{|C^0(s)^F|}\sum_{\substack{k\in G^F\\ k^{-1}sk\in T^F}} \theta(k^{-1}sk)Q^{C^0(s)}_{kTk^{-1}}(u).$$

Corollary 7.2.9. *$R_{T,\theta}(u)$ is independent of θ if u is unipotent.*

The common value $Q_T(u)$ of the $R_{T,\theta}(u)$ for all characters θ of T^F is called a Green function. These functions were first investigated by J. A. Green in the case when $G^F = GL_n(q)$. We shall have more to say about the Green functions later in this chapter.

7.3 ORTHOGONALITY RELATIONS AND IRREDUCIBILITY

In the present section we shall obtain a formula for the scalar product $(R_{T,\theta}, R_{T',\theta'})$ of two of the Deligne–Lusztig generalized characters of G^F. This will enable us to show that under certain circumstances $R_{T,\theta}$ is, to within sign, an irreducible character of G^F.

Let T, T' be two F-stable maximal tori of G. We define $N(T, T')$ by

$$N(T, T') = \{g\in G;\ T^g = T'\}.$$

$N(T, T')$ is clearly a union of right cosets of T in G. We define $W(T, T')$ by

$$W(T, T') = \{Tg;\ g\in N(T, T')\}.$$

Since T, T' are F-stable $N(T, T')$ will also be F-stable and so there will be an induced action of F on $W(T, T')$.

Proposition 7.3.1. *There is a bijection between $W(T, T')^F$ and the set of right cosets of T^F in $N(T, T')^F$.*

Proof. $W(T, T')^F$ consists of the set of F-stable cosets of T in $N(T, T')$. Now every F-stable cost Tg contains an F-stable element. For $F(Tg) = Tg$ and so $F(g)g^{-1} \in T$. By the Lang–Steinberg theorem there exists $t \in T$ with $F(g)g^{-1} = t^{-1}F(t) = F(t)t^{-1}$. Thus $t^{-1}g$ is an F-stable element in Tg. The set of F-stable elements in Tg is a coset of T^F in $N(T, T')^F$. Conversely every right coset of T^F in $N(T, T')^F$ lies in a unique F-stable right coset of T in $N(T, T')$.

Proposition 7.3.2. *Let X be an affine variety acted on by a torus T. Let $X^T = \{x \in X; tx = x$ for all $t \in T\}$ and $X^t = \{x \in X; tx = x\}$. Then there exists $t \in T$ for which $X^T = X^t$.*

Proof. We use the linearization property described in section 1.5. There exists a finite-dimensional T-module V such that X is a closed subset of V and the T-action on X is the restriction of the T-action on V.

Now V is the direct sum of 1-dimensional T-submodules. Let $V = \oplus_i V_i$ be such a decomposition. Let $tv_i = \chi_i(t)v_i$ for $v_i \neq 0$ in V_i and $\chi_i \in X(T)$. Let $S_i = \{t \in T; \chi_i(t) = 1\}$. Let I denote the set of i for which $\chi_i = 1$. Then we have

$$V^T = \bigoplus_{i \in I} V_i.$$

If $i \notin I$ we have $\dim S_i = \dim T - 1$. Thus $\bigcup_{i \notin I} S_i$ is a proper subset of T. Let $t \in T$ be such that $t \notin \bigcup_{i \notin I} S_i$. Then $\chi_i(t) \neq 1$ for all $i \notin I$. Thus $V^t = \oplus_{i \in I} V_i = V^T$. It follows that $X^T = X \cap V^T = X \cap V^t = X^t$.

Proposition 7.3.3. *Let X be an affine variety which is acted on by a torus T and by an automorphism s of order prime to p which commutes with the action of all elements of T. Then s acts on X^T and $\mathscr{L}(s, X) = \mathscr{L}(s, X^T)$.*

Proof. Let $x \in X^T$. Then

$$t(sx) = s(tx) = sx \quad \text{for all } t \in T$$

and so $sx \in X^T$. Thus s acts on X^T. By 7.1.10 we have $\mathscr{L}(s, X) = \mathscr{L}(1, X^s)$ and $\mathscr{L}(s, X^T) = \mathscr{L}(1, (X^T)^s) = \mathscr{L}(1, (X^s)^T)$. Thus it is sufficient to prove the required proposition when $s = 1$. We must show that $\mathscr{L}(1, X) = \mathscr{L}(1, X^T)$. By 7.3.2 T contains an element t such that $X^T = X^t$. Thus we have

$$\mathscr{L}(1, X^T) = \mathscr{L}(1, X^t) = \mathscr{L}(t, X)$$

again by 7.1.10. But T, being connected, acts trivially on each $H_c^i(X, \bar{\mathbb{Q}}_l)$ by 7.1.12. Hence $\mathscr{L}(t, X) = \mathscr{L}(1, X)$ and the result follows. ■

We now come to the scalar product formula. Let T, T' be F-stable maximal tori of G. Then $T^g = T'$ for some $g \in G$. Conjugation by $g \in N(T, T')$ induces a map from X' to X given by

$${}^g\chi'(t) = \chi'(t^g) \qquad \chi' \in X', t \in T.$$

If g, g' are in the same right coset of T, so that $Tg = Tg'$, then ${}^g\chi' = {}^{g'}\chi'$. Thus we can define without ambiguity t^ω and ${}^\omega\chi'$ where $t \in T$, $\chi' \in X'$ and $\omega \in W(T, T')$.
Similarly we can define ${}^\omega\theta' \in \hat{T}^F$ for all $\theta' \in (\hat{T}')^F$ and $\omega \in W(T, T')^F$.

Theorem 7.3.4. $(R_{T,\theta}, R_{T',\theta'}) = |\{\omega \in W(T, T')^F; {}^\omega\theta' = \theta\}|$.

Proof. We recall from 7.2.3 that

$$R_{T,\theta}(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \mathscr{L}((g, t), L^{-1}(U)).$$

Hence we have

$$\begin{aligned}(R_{T,\theta}, R_{T',\theta'}) &= \frac{1}{|G^F|} \frac{1}{|T^F|} \frac{1}{|T'^F|} \sum_{g \in G^F} \sum_{t \in T^F} \sum_{t' \in T'^F} \\ &\qquad \theta(t^{-1})\overline{\theta'(t'^{-1})}\mathscr{L}((g, t), L^{-1}(U))\mathscr{L}((g, t'), L^{-1}(U')) \\ &= \frac{1}{|G^F|} \frac{1}{|T^F|} \frac{1}{|T'^F|} \sum_g \sum_t \sum_{t'} \\ &\qquad \theta(t^{-1})\theta'(t')\mathscr{L}((g, t) \times (g, t'); L^{-1}(U) \times L^{-1}(U'))\end{aligned}$$

by 7.1.9, where UT and $U'T'$ are Borel subgroups containing T, T' respectively. Now $L^{-1}(U)$ and $L^{-1}(U')$ are affine varieties, thus so is $L^{-1}(U) \times L^{-1}(U')$. This variety is acted on by the finite group G^F. By section 1.5 there is a strict quotient $(L^{-1}(U) \times L^{-1}(U'))/G^F$ which is itself an affine variety. Let $S = (L^{-1}(U) \times L^{-1}(U'))/G^F$. By 7.1.8 we have

$$(R_{T,\theta}, R_{T',\theta'}) = \frac{1}{|T^F|} \frac{1}{|T'^F|} \sum_t \sum_{t'} \theta(t^{-1})\theta'(t')\mathscr{L}((t, t'), S).$$

We must therefore investigate the affine variety S. We do so by comparing S with the variety S' given by

$$S' = \{(x, x', y) \in U \times U' \times G; xF(y) = yx'\}.$$

S' is an algebraic subset of $U \times U' \times G$ and is therefore an affine variety.
We assert that there is a morphism $L^{-1}(U) \times L^{-1}(U') \to S'$ given by

$$(g, g') \to (g^{-1}F(g), g'^{-1}F(g'), g^{-1}g').$$

The image lies in S' because

$$g^{-1}F(g) . F(g^{-1}g') = g^{-1}g' . g'^{-1}F(g').$$

Let $a \in G^F$. Then (ag, ag') has the same image as (g, g'). So, by section 1.5, the above morphism factors through the quotient S and we have a morphism $\psi: S \to S'$. We shall show that ψ is bijective.
Suppose $(x, x', y) \in S'$. Then $xF(y) = yx'$. We consider pairs $(g, g') \in L^{-1}(U) \times L^{-1}(U')$ satisfying $g^{-1}F(g) = x$, $g'^{-1}F(g') = x'$, $g^{-1}g' = y$. The equation $g^{-1}F(g) = x$ has a solution $g \in L^{-1}(U)$ by the Lang–Steinberg

theorem. Moreover any other solution g_1 satisfies $g_1 \in G^F g$. Having chosen g the element g' must be given by $g' = gy$. Such a g' will automatically be a solution of $g'^{-1}F(g') = x'$ since

$$g'^{-1}F(g') = y^{-1}g^{-1}F(g)F(y) = y^{-1}xF(y) = x'.$$

Thus there exists a solution (g, g') of these equations and any other solution will have the form (ag, ag') for $a \in G^F$. There will therefore be a unique element of S mapping to $(x, x', y) \in S'$ under ψ. Thus ψ is bijective.

Now $T^F \times T'^F$ acts on $L^{-1}(U) \times L^{-1}(U')$ by

$$(g, g') \underset{(t,t')}{\rightarrow} (gt, g't').$$

This induces an action of $T^F \times T'^F$ on the quotient variety S. The bijection $\psi\colon S \rightarrow S'$ then gives rises to an action of $T^F \times T'^F$ on S' which is given by

$$(x, x', y) \underset{(t,t')}{\rightarrow} (t^{-1}xt, t'^{-1}x't', t^{-1}yt').$$

By 7.1.5 we have $\mathscr{L}((t, t'), S) = \mathscr{L}((t, t'), S')$ and so

$$(R_{T,\theta}, R_{T',\theta'}) = \frac{1}{|T^F|}\frac{1}{|T'^F|}\sum_t\sum_{t'}\theta(t^{-1})\theta'(t')\mathscr{L}((t, t'), S').$$

We next show how to decompose S' into a disjoint union of locally closed subsets. This is based on a slightly unusual version of the Bruhat decomposition of G. We begin with the usual Bruhat decomposition $G = \bigcup_w B\dot{w}B$. Each double coset $B\dot{w}B$ is locally closed in G by section 1.10.

In fact we have

$$G = \bigcup_w UT\dot{w}U_w = \bigcup_w UT\dot{w}(U \cap (U^-)^w).$$

Now T is F-stable and since it normalizes U it will also normalize $F^{-1}(U)$. Thus $F^{-1}(U)T$ is also a Borel subgroup of G. With respect to this Borel subgroup the Bruhat decomposition of G is

$$G = \bigcup_w F^{-1}(U)T\dot{w}(F^{-1}(U) \cap (F^{-1}(U^-))^w).$$

Inverting the elements gives

$$G = \bigcup_w (F^{-1}(U) \cap (F^{-1}(U^-))^w)\dot{w}^{-1}TF^{-1}(U)$$
$$= \bigcup_w (F^{-1}(U) \cap {}^w(F^{-1}(U^-)))\dot{w}TF^{-1}(U).$$

Now $F^{-1}(U)T$ and $F^{-1}(U')T'$ are Borel subgroups of G containing T, T' respectively. Thus it is possible to find an element $z \in G$ satisfying

$$T^z = T' \quad (F^{-1}(U))^z = F^{-1}(U').$$

It follows that

$$G = \bigcup_w (F^{-1}(U) \cap {}^w(F^{-1}(U^-)))\dot{w}T(F^{-1}(U'))^{z^{-1}}$$

and so

$$G = Gz = \bigcup_{w} (F^{-1}(U) \cap {}^{w}(F^{-1}(U^{-})))\dot{w}TzF^{-1}(U').$$

Hence G is the disjoint union of subsets G_w where

$$G_w = (F^{-1}(U) \cap {}^{w}(F^{-1}(U^{-})))T\dot{w}zF^{-1}(U').$$

The subsets G_w are locally closed in G and their expressions as products in the above formula are expressions with uniqueness.

Having this decomposition of G we obtain a corresponding decomposition of S' given by

$$S' = \bigcup_{w} S'_w \quad \text{where} \quad S'_w = \{(x, x', y) \in S'; y \in G_w\}.$$

The S_w' are locally closed subsets of S'.

We observe that these subsets S_w' are invariant under the action of $T^F \times T'^F$ on S'. This is clear since

$$(x, x', y) \underset{(t,t')}{\rightarrow} (t^{-1}xt, t'^{-1}x't', t^{-1}yt')$$

and if $y \in G_w$ then $t^{-1}yt' \in G_w$ also. By 7.1.6 we have

$$(R_{T,\theta}, R_{T',\theta'}) = \frac{1}{|T^F|}\frac{1}{|T'^F|}\sum_{t}\sum_{t'} \theta(t^{-1})\theta'(t') \sum_{w \in W} \mathscr{L}((t, t'), S_w').$$

Now S_w' is isomorphic to the affine variety of all quintuples

$$(x, x', u, a, u') \in U \times U' \times (F^{-1}(U) \cap {}^{w}(F^{-1}(U^{-}))) \times T\dot{w}z \times F^{-1}(U')$$

satisfying

$$xF(u)F(a)F(u') = uau'x'.$$

We identify S_w' with this variety of quintuples. The action of $T^F \times T'^F$ on S_w' is given by

$$(x, x', u, a, u') \underset{(t,t')}{\rightarrow} (t^{-1}xt, t'^{-1}x't', t^{-1}ut, t^{-1}at', t'^{-1}u't').$$

It is convenient to pass from S_w' to another affine variety S_w'' isomorphic to it. Let

$$S_w'' = \{(\xi, \xi', u, a, u') \in U \times U' \times (F^{-1}(U) \cap {}^{w}(F^{-1}(U^{-}))) \times T\dot{w}z \times F^{-1}(U'); \xi F(a) = uau'\xi'\}.$$

Then the map $(x, x', u, a, u') \rightarrow (\xi, \xi', u, a, u')$ where $\xi = xF(u)$ and $\xi' = x'(F(u'))^{-1}$ is an isomorphism between S_w' and S_w''. The action of $T^F \times T'^F$ on S_w' translates under this isomorphism into an action on S_w'' given by

$$(\xi, \xi', u, a, u') \underset{(t,t')}{\rightarrow} (t^{-1}\xi t, t'^{-1}\xi' t', t^{-1}ut, t^{-1}at', t'^{-1}u't').$$

Let H_w be the subgroup of $T \times T'$ given by

$$H_w = \{(t, t') \in T \times T'; F(t')t'^{-1} = (F(\dot{w}z))^{-1}(F(t)t^{-1})F(\dot{w}z)\}.$$

H_w is a closed subgroup of $T \times T'$. It clearly contains $T^F \times T'^F$. We show that our action of $T^F \times T'^F$ on S_w'' extends to an action of H_w on S_w''. (This would not have been true on S_w'.) We must check that if $\xi F(a) = uau'\xi'$ and $(t, t') \in H_w$ then

$$t^{-1}\xi t \,.\, F(t^{-1})F(a)F(t') = t^{-1}ut \,.\, t^{-1}at' \,.\, t'^{-1}u't' \,.\, t'^{-1}\xi't'$$

i.e. $\xi t F(t^{-1})F(a) = uau'\xi' t'(F(t'))^{-1}$.

This is equivalent to

$$\xi t F(t^{-1})F(a) = \xi F(a)t'F((t')^{-1})$$

or to

$$F(t')t'^{-1} = F(a)^{-1}F(t)t^{-1}F(a).$$

Now $a = s\dot{w}z$ for some $s \in T$. Thus our condition becomes

$$F(t')t'^{-1} = (F(\dot{w}z))^{-1}F(t)t^{-1}F(\dot{w}z)$$

which is known to be true since $(t, t') \in H_w$.

Consider the connected component $H_w{}^0$ of H_w. $H_w{}^0$ is a closed connected subgroup of the torus $T \times T'$ so must itself be a torus. Let $(t, t') \in T^F \times T'^F$. Then the variety S_w'' is acted on by both $H_w{}^0$ and by (t, t'), and (t, t') commutes with every element of $H_w{}^0$. We may therefore apply 7.3.3 and obtain

$$\mathcal{L}((t, t'), S_w') = \mathcal{L}((t, t'), S_w'') = \mathcal{L}((t, t'), S_w''^{H_w{}^0}).$$

We must therefore consider which elements of S_w'' are fixed by all elements of $H_w{}^0$.

Now we have a projection homomorphism $H_w \to T$ given by $(t, t') \to t$. The Lang–Steinberg theorem shows that this map is surjective. We consider its restriction to $H_w{}^0$. The image of $H_w{}^0$ is a closed connected subgroup of T, so is a torus, but has finite index in the image of H_w which is T. Thus the projection map $H_w{}^0 \to T$ must also be surjective. Similarly the projection map $H_w{}^0 \to T'$ is surjective.

Now suppose that $(\xi, \xi', u, a, u') \in S_w''$ is fixed by all elements (t, t') of $H_w{}^0$. Then we have

$$\begin{aligned} t^{-1}\xi t &= \xi && \text{for all } t \in T\ . \\ t'^{-1}\xi' t' &= \xi' && \text{for all } t' \in T' \\ t^{-1}ut &= u && \text{for all } t \in T \\ t'^{-1}u't' &= u' && \text{for all } t' \in T'. \end{aligned}$$

However no non-identity element of U is fixed by all elements of T and no non-identity element of U' is fixed by all elements of T'. Thus we have $\xi = 1$, $\xi' = 1$, $u = 1$, $u' = 1$. Hence the given element of S_w'' has form $(1, 1, 1, a, 1)$ with $a \in T\dot{w}z$.

Now $(1, 1, 1, a, 1)$ lies in S_w'' if and only if $F(a) = a$. If this is so then $F(Ta) = Ta$ and so $F(T\dot{w}z) = T\dot{w}z$. The number of such elements $(1, 1, 1, a, 1) \in S_w''$ is therefore

$$\begin{cases} |(T\dot{w}z)^F| & \text{if } F(T\dot{w}z) = T\dot{w}z \\ 0 & \text{otherwise.} \end{cases}$$

We next show that if $F(T\dot{w}z) = T\dot{w}z$ then each $a \in (T\dot{w}z)^F$ will satisfy the condition that $(1, 1, 1, a, 1)$ is fixed by all elements of $H_w{}^0$. We must verify that $t^{-1}at' = a$ for all $(t, t') \in H_w{}^0$. We recall that H_w is given by

$$\begin{aligned} H_w &= \{(t, t') \in T \times T';\ F(t')t'^{-1} = F(a)^{-1}F(t)t^{-1}F(a)\} \\ &= \{(t, t') \in T \times T';\ t^{-1}at' \in G^F\} \\ &= \{(t, t') \in T \times T';\ t' \in a^{-1}tG^F \cap T'\}. \end{aligned}$$

Now we have

$$a^{-1}tG^F \cap T' = a^{-1}tG^F \cap a^{-1}Ta = a^{-1}t(G^F \cap T)a = a^{-1}tT^Fa.$$

Hence

$$\begin{aligned} H_w &= \{(t, t') \in T \times T';\ t' \in a^{-1}tT^Fa\} \\ &= \{(t, a^{-1}tsa);\ t \in T, s \in T^F\}. \end{aligned}$$

Thus the map $T \times T^F \to H_w$ given by $(t, s) \to (t, a^{-1}tsa)$ is an isomorphism. Using this isomorphism we can identify the connected component $H_w{}^0$. This is obtained by putting $s = 1$. Hence

$$H_w{}^0 = \{(t, a^{-1}ta);\ t \in T\}.$$

Thus for all $(t, t') \in H_w{}^0$ we have $t' = a^{-1}ta$, which gives $t^{-1}at' = a$ as required.

We have now proved that $S_w''^{H_w{}^0}$ is empty if $T\dot{w}z$ is not F-stable and consists of the elements $(1, 1, 1, a, 1)$ with $a \in (T\dot{w}z)^F$ if $T\dot{w}z$ is F-stable. In particular this set is finite. We now recall from 7.1.11 that $\mathscr{L}((t, t'), S_w''^{H_w{}^0})$ is the number of elements of $S_w''^{H_w{}^0}$ fixed by (t, t'). Thus we have

$$\begin{aligned} (R_{T,\theta}, R_{T',\theta'}) &= \frac{1}{|T^F|}\frac{1}{|T'^F|}\sum_{w \in W}\sum_{t \in T^F}\sum_{t' \in T'^F}\theta(t^{-1})\theta'(t')\mathscr{L}((t, t'), S_w''^{H_w{}^0}) \\ &= \frac{1}{|T^F|}\frac{1}{|T'^F|}\sum_{\substack{w \in W \\ F(T\dot{w}z) = T\dot{w}z}}\sum_{t}\sum_{t'}\sum_{\substack{a \in (T\dot{w}z)^F \\ t^{-1}at' = a}}\theta(t^{-1})\theta'(t') \\ &= \frac{1}{|T^F|}\frac{1}{|T'^F|}\sum_{\substack{w \in W \\ T\dot{w}z\ F\text{-stable}}}\sum_{t \in T^F}\sum_{a \in (T\dot{w}z)^F}\theta(t^{-1})\theta'(t^a) \\ &= \frac{1}{|T'^F|}\sum_{\substack{w \in W \\ T\dot{w}z\ F\text{-stable}}}\sum_{a \in (T\dot{w}z)^F}(\theta, {}^a\theta'). \end{aligned}$$

Now

$$W(T, T') = \{T\dot{w}z;\ w \in W\}$$
$$W(T, T')^F = \{T\dot{w}z;\ w \in W, (T\dot{w}z)^F = T\dot{w}z\}.$$

Moreover if $T\dot{w}z$ is F-stable then we have $|(T\dot{w}z)^F| = |T^F|$. Also if $T\dot{w}z$ is F-stable for any $w \in W$ it will follow that T and T' are G^F-conjugate and so $|T^F| = |T'^F|$. Bearing these facts in mind we have

$$\begin{aligned}(R_{T,\theta}, R_{T',\theta'}) &= \frac{1}{|T^F|} \sum_{\omega \in W(T,T')^F} |T^F|(\theta, {}^{\omega}\theta') \\ &= \sum_{\omega \in W(T,T')^F} (\theta, {}^{\omega}\theta') \\ &= |\{\omega \in W(T, T')^F; {}^{\omega}\theta' = \theta\}|.\end{aligned}$$

Definition. $\theta \in \hat{T}^F$ is said to be in general position if no non-identity element of $W(T)^F = (N(T)/T)^F$ fixes θ.

Corollary 7.3.5. *If $\theta \in \hat{T}^F$ is in general position then $\pm R_{T,\theta}$ is an irreducible character of G^F.*

Proof. We have $(R_{T,\theta}, R_{T,\theta}) = |\{\omega \in W(T)^F; {}^{\omega}\theta = \theta\}|$. If θ is in general position this gives $(R_{T,\theta}, R_{T,\theta}) = 1$. Since $R_{T,\theta}$ is a generalized character, i.e. an integral combination of irreducible characters, the orthogonality relations for irreducible characters imply that $\pm R_{T,\theta}$ is irreducible. ■

Another very useful consequence of the scalar product theorem is as follows. We recall that in defining the generalized character $R_{T,\theta}$ we chose a Borel subgroup B of G containing T. It is therefore conceivable that $R_{T,\theta}$ might depend upon the choice of B. In fact, however, this does not happen as we can now prove.

Proposition 7.3.6. *The generalized character $R_{T,\theta}$ of G^F is independent of the choice of Borel subgroup B containing T.*

Proof. Let B, B' be two Borel subgroups of G containing T. Let $R_{T,\theta,B}$ and $R_{T,\theta,B'}$ be the generalized characters of G^F which are constructed using B, B' respectively. We consider the scalar products of $R_{T,\theta,B}$, $R_{T,\theta,B'}$ with themselves and with one another. By 7.3.4 we have

$$(R_{T,\theta,B}, R_{T,\theta,B}) = (R_{T,\theta,B}, R_{T,\theta,B'}) = (R_{T,\theta,B'}, R_{T,\theta,B'}).$$

It follows that

$$(R_{T,\theta,B} - R_{T,\theta,B'}, R_{T,\theta,B} - R_{T,\theta,B'}) = 0$$

and so $R_{T,\theta,B} - R_{T,\theta,B'} = 0$. ■

A further corollary of the scalar product theorem is as follows.

Corollary 7.3.7. *If the F-stable maximal tori T, T' of G are not G^F-conjugate then $(R_{T,\theta}, R_{T',\theta'}) = 0$.*

Proof. If $(R_{T,\theta}, R_{T',\theta'}) \neq 0$ there exists $\omega \in W(T, T')^F$ such that ${}^{\omega}\theta' = \theta$. In particular $W(T, T')^F$ is non-empty. By 7.3.1 $N(T, T')^F$ is non-empty. Thus T, T' are G^F-conjugate. ■

We thus have a procedure for obtaining many irreducible characters of G^F. These have the form $\pm R_{T,\theta}$ where $\theta \in \hat{T}^F$ is in general position. Moreover if the tori T, T' are not G^F-conjugate the irreducible characters $\pm R_{T,\theta}$ and $\pm R_{T',\theta'}$ must be distinct.

We know by 7.3.7 that if T, T' are not G^F-conjugate then the generalized characters $R_{T,\theta}$, $R_{T',\theta'}$ are orthogonal. This does not necessarily mean, however, that they have no irreducible character in common when θ, θ' are not in general position. For instance one might have

$$R_{T,\theta} = \chi^i + \chi^j \qquad R_{T',\theta'} = \chi^i - \chi^j$$

where χ^i, χ^j are distinct irreducible characters of G^F. We would like to have a condition relating (T, θ) (T', θ') which will imply not merely that $(R_{T,\theta}, R_{T',\theta'}) = 0$ but that $R_{T,\theta}$, $R_{T',\theta'}$ have no common irreducible component. We recall that the pairs (T, θ) and (T', θ') are called geometrically conjugate if the equivalent conditions of 4.1.3 are satisfied.

Theorem 7.3.8. *Let T, T' be F-stable maximal tori of G and $\theta \in \hat{T}^F$, $\theta' \in \hat{T}'^F$. Suppose (T, θ) and (T', θ') are not geometrically conjugate. Then $R_{T,\theta}$ and $R_{T',\theta'}$ have no irreducible component in common.*

Proof. The proof is closely related to that of 7.3.4. Let $B = UT$, $B' = U'T'$ be Borel subgroups of G containing T, T'. Then we can consider the G^F-modules $H_c^j(L^{-1}(U), \bar{\mathbb{Q}}_l)_{\theta^{-1}}$ and $H_c^k(L^{-1}(U'), \bar{\mathbb{Q}}_l)_{\theta'}$. Their tensor product

$$H_c^j(L^{-1}(U), \bar{\mathbb{Q}}_l)_{\theta^{-1}} \otimes H_c^k(L^{-1}(U'), \bar{\mathbb{Q}}_l)_{\theta'}.$$

is also a G^F-module and we shall show that

$$(H_c^j(L^{-1}(U), \bar{\mathbb{Q}}_l)_{\theta^{-1}} \otimes H_c^k(L^{-1}(U'), \bar{\mathbb{Q}}_l)_{\theta'})^{G^F} = 0$$

for all j, k. This will be sufficient to prove our result. For it implies that

$$\mathrm{Hom}_{G^F}((H_c^j(L^{-1}(U), \bar{\mathbb{Q}}_l)_{\theta^{-1}} \otimes H_c^k(L^{-1}(U'), \bar{\mathbb{Q}}_l)_{\theta'}), \bar{\mathbb{Q}}_l) = 0$$

and therefore that

$$\mathrm{Hom}_{G^F}(H_c^k(L^{-1}(U'), \bar{\mathbb{Q}}_l)_{\theta'}, \quad (\mathrm{Hom}(H_c^j(L^{-1}(U), \bar{\mathbb{Q}}_l)_{\theta^{-1}}, \bar{\mathbb{Q}}_l)) = 0$$

for all j, k. Now we have

$$R_{T',\theta'}(g) = \sum_k (-1)^k \operatorname{trace}(g, H_c^k(L^{-1}(U'), \bar{\mathbb{Q}}_l)_{\theta'})$$

$$\overline{R_{T,\theta^{-1}}}(g) = \sum_j (-1)^j \operatorname{trace}(g, \mathrm{Hom}(H_c^j(L^{-1}(U), \bar{\mathbb{Q}}_l)_{\theta^{-1}}, \bar{\mathbb{Q}}_l)).$$

Hence $R_{T',\theta'}$ and $\overline{R_{T,\theta^{-1}}}$ have no common irreducible component. But 7.2.3 shows that $\overline{R_{T,\theta^{-1}}} = R_{T,\theta}$ since $\theta^{-1} = \bar{\theta}$. Thus $R_{T,\theta}$ and $R_{T',\theta'}$ have no common irreducible component and the result is proved.

Now by 7.1.9 we have

$$H_c^i(L^{-1}(U) \times L^{-1}(U'), \bar{\mathbb{Q}}_l) \cong \bigoplus_{\substack{j,k \\ j+k=i}} (H_c^j(L^{-1}(U), \bar{\mathbb{Q}}_l) \otimes H_c^k(L^{-1}(U'), \bar{\mathbb{Q}}_l)$$

and so, considering the $T^F \times T'^F$-action on both sides, we have

$$H_c^i(L^{-1}(U) \times L^{-1}(U'), \bar{\mathbb{Q}}_l)_{\theta^{-1},\theta'} \cong \bigoplus_{\substack{j,k \\ j+k=i}} (H_c^j(L^{-1}(U), \bar{\mathbb{Q}}_l)_{\theta^{-1}} \otimes H_c^k(L^{-1}(U'), \bar{\mathbb{Q}}_l)_{\theta'}).$$

We must therefore show that

$$(H_c^i(L^{-1}(U) \times L^{-1}(U'), \bar{\mathbb{Q}}_l)_{\theta^{-1},\theta'})^{G^F} = 0.$$

Now G^F acts on the affine variety $L^{-1}(U) \times L^{-1}(U')$ and, as in 7.3.4, we let $S = (L^{-1}(U) \times L^{-1}(U'))/G^F$. By 7.1.8 we have

$$H_c^i(L^{-1}(U) \times L^{-1}(U'), \bar{\mathbb{Q}}_l)^{G^F} \cong H_c^i(S, \bar{\mathbb{Q}}_l).$$

Considering the $T^F \times T'^F$ action on both sides we deduce

$$(H_c^i(L^{-1}(U) \times L^{-1}(U'), \bar{\mathbb{Q}}_l)_{\theta^{-1},\theta'})^{G^F} \cong H_c^i(S, \bar{\mathbb{Q}}_l)_{\theta^{-1},\theta'}.$$

Thus we must show that $H_c^i(S, \bar{\mathbb{Q}}_l)_{\theta^{-1},\theta'} = 0$. As in 7.3.4 we let S' be the affine variety given by

$$S' = \{(x, x', y) \in U \times U' \times G; xF(y) = yx'\}.$$

Then there is a bijective morphism $\psi: S \to S'$.

In the proof of 7.3.4 it was sufficient to know that ψ is a bijective morphism. However this time we need to know that ψ is actually an isomorphism, so that S' is isomorphic to the quotient $S = (L^{-1}(U) \times L^{-1}(U'))/G^F$. To prove this we first consider the variety X given by

$$X = \{(x, x', y) \in G \times G \times G; xF(y) = yx'\}.$$

The map

$$(x, y) \to (x, y^{-1}xF(y), y)$$

is an isomorphism between $G \times G$ and X. Consider the morphisms $\alpha: G \times G \to X$ and $\beta: X \to G \times G$ given by

$$(g, g') \xrightarrow{\alpha} (g^{-1}F(g), g'^{-1}F(g'), g^{-1}g')$$

$$(x, x', y) \xrightarrow{\beta} (x, x').$$

Their composite $\beta \circ \alpha\colon G \times G \to G \times G$ is given by

$$(g, g') \underset{\beta \circ \alpha}{\to} (g^{-1} F(g), g'^{-1} F(g')).$$

Now we know, just as in the proof of 7.3.4, that α is surjective and the fibres of α are the orbits of G^F acting on $G \times G$ by $(g, g') \to (ag, ag')$ $a \in G^F$. In order to show that the orbit map $\alpha\colon G \times G \to X$ induces an isomorphism between X and the strict quotient $(G \times G)/G^F$ it is sufficient by section 1.5 to know that X is smooth and α is separable. X is certainly smooth, being isomorphic to $G \times G$, and α will be separable provided its differential $(\mathrm{d}\alpha)_1$ is surjective, by section 1.3. Now we have

$$\mathrm{d}(\beta \circ \alpha)_1 = (\mathrm{d}\beta)_1 \circ (\mathrm{d}\alpha)_1.$$

However

$$\mathrm{d}(\beta \circ \alpha)_1 = -1 + (\mathrm{d}F)_1.$$

Since some power of F is the standard Frobenius map, some power of $(\mathrm{d}F)_1$ is multiplication by a positive power of p. In particular $(\mathrm{d}F)_1$ has no eigenvalue 1 and so $\mathrm{d}(\beta \circ \alpha)_1$ is bijective. This implies that $(\mathrm{d}\alpha)_1$ is also bijective, since $G \times G$ and X have the same dimension. Thus the orbit map $\alpha\colon G \times G \to X$ induces an isomorphism $\bar{\alpha}\colon (G \times G)/G^F \to X$.

We now restrict $\alpha\colon G \times G \to X$ to $L^{-1}(U) \times L^{-1}(U')$ and obtain a morphism whose image is S', just as in the proof of 7.3.4. Thus we have a surjective morphism

$$\alpha\colon L^{-1}(U) \times L^{-1}(U') \to S'$$

whose fibres are the G^F-orbits on $L^{-1}(U) \times L^{-1}(U')$, and this gives rise to an induced morphism

$$\psi\colon S = (L^{-1}(U) \times L^{-1}(U'))/G^F \to S'$$

which is bijective. Since the composite map

$$L^{-1}(U) \times L^{-1}(U') \underset{\text{inclusion}}{\to} G \times G \to (G \times G)/G^F$$

has the property that elements in the same G^F-orbit have the same image, we obtain an induced morphism

$$S = (L^{-1}(U) \times L^{-1}(U'))/G^F \to (G \times G)/G^F$$

which is injective. Moreover the image of the composite map

$$S \to (G \times G)/G^F \xrightarrow{\bar{\alpha}} X,$$

is S'. Thus the isomorphism $\bar{\alpha}\colon (G \times G)/G^F \to X$ restricts to an isomorphism $\psi\colon S \to S'$.

Since S is isomorphic to S' it is sufficient to show that

$$H_c^i(S', \bar{\mathbb{Q}}_l)_{\theta^{-1}, \theta'} = 0.$$

Again as in 7.3.4 we have a decomposition $S' = \bigcup_w S'_w$ of S' into a disjoint union of locally closed subsets. We recall from the proof of 7.3.4 that each S_w' is invariant under the action of $T^F \times T'^F$. Moreover we can choose a total ordering on W which extends the natural partial ordering, and will therefore have the property that $\bigcup_{w' \leqslant w} S'_{w'}$ is closed in S' for each $w \in W$. We may therefore apply property 7.1.6 of the l-adic cohomology to show it is sufficient to prove that

$$H_c^i(S_w', \bar{\mathbb{Q}}_l)_{\theta^{-1},\theta'} = 0 \quad \text{for all } w \in W.$$

We next recall from the proof of 7.3.4 that S_w' is isomorphic to a variety S_w'' which is acted on by the group H_w given by

$$H_w = \{(t, t') \in T \times T'; F(t')t'^{-1} = (F(\dot{w}z))^{-1}(F(t)t^{-1})F(\dot{w}z)\}$$

where z is an element of G satisfying $T^z = T'$ and $(F^{-1}(U))^z = F^{-1}(U')$. In particular $H_w{}^0$ acts on S_w'' and by 7.1.12 $H_w{}^0$ acts trivially on $H_c^i(S_w'', \bar{\mathbb{Q}}_l)$.

It will be sufficient to prove that $H_c^i(S_w'', \bar{\mathbb{Q}}_l)_{\theta^{-1},\theta'} = 0$ for all $w \in W$. We suppose if possible that $H_c^i(S_w'', \bar{\mathbb{Q}}_l)_{\theta^{-1},\theta'} \neq 0$. Elements of $T^F \times T'^F$ act on this space by the character (θ^{-1}, θ') and elements of $H_w{}^0$ act trivially on it. Hence $\theta^{-1} = 1$ and $\theta' = 1$ on $(T^F \times T'^F) \cap H_w{}^0$.

Let ϕ: $T \times T' \to T$ be defined by $\phi(t, t') = gt'g^{-1}t^{-1}$ where $g = F(\dot{w}z)$. Then ϕ is a homomorphism of algebraic groups. Consider the composite map

$$T \times T' \xrightarrow[L]{} T \times T' \xrightarrow[\phi]{} T.$$

Since we are dealing with abelian groups Lang's map L is a homomorphism, and so $\phi \circ L$ is a homomorphism also. We have

$$(\phi \circ L)(t, t') = gF(t')t'^{-1}g^{-1}(F(t)t^{-1})^{-1}.$$

Thus the kernel of $\phi \circ L$ is H_w.

Now given any two tori T_1, T_2 and a homomorphism α: $T_1 \to T_2$ we have a corresponding homomorphism $Y(\alpha)$: $Y(T_1) \to Y(T_2)$ defined by $\gamma_1 \to \alpha \circ \gamma_1$, $\gamma_1 \in Y(T_1)$. Moreover $\ker Y(\alpha) = Y(\ker \alpha)$ since

$$\gamma_1 \in \ker Y(\alpha) \Leftrightarrow \alpha \circ \gamma_1 = 0 \Leftrightarrow \gamma_1(\mathbf{G}_m) \subseteq \ker \alpha \Leftrightarrow \gamma_1 \in Y(\ker \alpha).$$

Thus we have homomorphisms

$$Y(T) \oplus Y(T') \to Y(T) \oplus Y(T') \to Y(T)$$
$$(\gamma, \gamma') \to (F(\gamma) - \gamma, F(\gamma') - \gamma') \to (F(\gamma') - \gamma')^{g^{-1}} - (F(\gamma) - \gamma)$$

and the kernel of the composite homomorphism is $Y(H_w) = Y(H_w{}^0)$.

Before proceeding further with the proof we need the following lemma.

Lemma 7.3.9. *Let T be an F-stable maximal torus of G and S be a subtorus of T. Consider the homomorphism*

$$Y(T) \to Y(T)/(F-1)Y(T) \cong T^F \quad \text{by 3.2.2.}$$

Then under this homomorphism elements of $Y(T) \cap (F-1)(Y(S) \otimes \mathbb{Q}_{p'})$ map into $T^F \cap S$.

Proof. We recall from 3.2.2 that we have a diagram

$$\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & T^F & & \\
 & & & & & & \downarrow & & \\
0 & \to & Y & \to & Y \otimes \mathbb{Q}_{p'} & \to & Y \otimes \mathbb{Q}_{p'}/\mathbb{Z} & \to & 0 \\
 & & \downarrow{\scriptstyle F-1} & & \downarrow{\scriptstyle F-1} & & \downarrow{\scriptstyle F-1} & & \\
0 & \to & Y & \to & Y \otimes \mathbb{Q}_{p'} & \to & Y \otimes \mathbb{Q}_{p'}/\mathbb{Z} & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & Y/(F-1)Y & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

with commutative squares and exact rows and columns.

Let $(F-1)(\sum_i \gamma_i \otimes q_i) \in Y(T) \cap (F-1)(Y(S) \otimes \mathbb{Q}_{p'})$ where $\gamma_i \in Y(S)$, $q_i \in \mathbb{Q}_{p'}$. Now the map

$$F - 1\colon Y(T) \otimes \mathbb{Q}_{p'} \to Y(T) \otimes \mathbb{Q}_{p'}$$

is injective by 3.2.2. (This is because F has no eigenvalue 1 on $Y(T) \otimes \mathbb{Q}$.) Thus the only element of $Y(T) \otimes \mathbb{Q}_{p'}$ mapping under $F - 1$ to $(F-1)(\sum_i \gamma_i \otimes q_i)$ is $\sum_i \gamma_i \otimes q_i$. Let $\bar{q}_i$ be the image of q_i in $\mathbb{Q}_{p'}/\mathbb{Z}$. Then we have the following elements related as in the diagram.

$$\begin{array}{ccc}
 & & \sum_i \gamma_i \otimes q_i \to \sum_i \gamma_i \otimes \bar{q}_i \\
 & & \downarrow{\scriptstyle F-1} \\
(F-1)\left(\sum_i \gamma_i \otimes q_i\right) & \to & (F-1)\left(\sum_i \gamma_i \otimes q_i\right)
\end{array}$$

and in the isomorphism between $Y/(F-1)Y$ and T^F given by the snake lemma the element $(F-1)Y + (F-1)(\sum_i \gamma_i \otimes q_i)$ of $Y/(F-1)Y$ corresponds to the image of $\sum_i \gamma_i \otimes \bar{q}_i \in Y \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ under the isomorphism $\psi\colon Y \otimes \mathbb{Q}_{p'}/\mathbb{Z} \to T$ of 3.1.2.

We must therefore show that $\psi(\sum_i \gamma_i \otimes \bar{q}_i) \in T^F \cap S$. We already know that this element lies in T^F. However we have

$$\sum_i \gamma_i \otimes \bar{q}_i \in Y(S) \otimes \mathbb{Q}_{p'}/\mathbb{Z}$$

and under the isomorphism ψ the subgroup $Y(S) \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ maps to S. Hence

$$\psi\left(\sum_i \gamma_i \otimes \bar{q}_i\right) \in T^F \cap S. \qquad \blacksquare$$

We now continue the proof of 7.3.8. We apply 7.3.9 to the subtorus $H_w{}^0$ of the torus $T \times T'$. Thus under the homomorphism

$$Y(T) \oplus Y(T') \to T^F \times T'^F$$

of 7.3.9, elements of $(Y(T) \oplus Y(T')) \cap (F-1)(Y(H_w{}^0) \otimes \mathbb{Q}_{p'})$ map into $(T^F \times T'^F) \cap H_w{}^0$.

Now the character (θ^{-1}, θ') of $T^F \times T'^F$ can be regarded as a character of $Y(T) \oplus Y(T')$ with $(F-1)(Y(T) \oplus Y(T'))$ in the kernel. However we have shown above that (θ^{-1}, θ') is the trivial character on $(T^F \times T'^F) \cap H_w{}^0$. (θ^{-1}, θ') must therefore also be the trivial character on

$$(Y(T) \oplus Y(T')) \cap (F-1)(Y(H_w{}^0) \otimes \mathbb{Q}_{p'}).$$

Consider the homomorphisms

$$Y(T) \oplus Y(T') \underset{F-1}{\rightarrow} Y(T) \oplus Y(T') \underset{\phi'}{\rightarrow} Y(T)$$

where $\phi'(\gamma, \gamma') = \gamma'^{g^{-1}} - \gamma$. We have seen above that $\ker(\phi' \circ (F-1)) = Y(H_w{}^0)$. Let $M = \ker \phi'$. We wish to show that (θ^{-1}, θ') is trivial on M. It is sufficient to prove that $M \subseteq (F-1)(Y(H_w{}^0) \otimes \mathbb{Q}_{p'})$. We note first that for $\xi \in Y(T) \oplus Y(T')$ we have

$$\xi \in Y(H_w{}^0) \Leftrightarrow \phi'(F-1)\xi = 0 \Leftrightarrow (F-1)\xi \in M.$$

Now let $m \in M$. Since $(Y(T) \oplus Y(T'))/(F-1)(Y(T) \oplus Y(T'))$ is finite of order prime to p there exists $n \in \mathbb{Z}$ with $(p, n) = 1$ such that $nm = (F-1)\xi$ for some $\xi \in Y(T) \oplus Y(T')$. Since $(F-1)\xi \in M$ we have $\xi \in Y(H_w{}^0)$. Thus $nm \in (F-1)Y(H_w{}^0)$ and so $m \in (F-1)(Y(H_w{}^0) \otimes \mathbb{Q}_{p'})$. Hence M lies in $(F-1)(Y(H_w{}^0) \otimes \mathbb{Q}_{p'})$. It follows that the character (θ^{-1}, θ') is trivial on M.

Let $\gamma \in Y(T)$. Then $\gamma^g \in Y(T')$ and $(\gamma, \gamma^g) \in M$. Hence (θ^{-1}, θ') is the identity at all such elements and so

$$\theta^{-1}(\gamma)\theta'(\gamma^g) = 1.$$

Thus $\theta(\gamma) = \theta'(\gamma^g) = {}^g\theta'(\gamma)$ and $\theta = {}^g\theta'$. We now have $T' = T^g$ and ${}^g\theta' = \theta$. This means by 4.1.3 that the pairs (T, θ) and (T', θ') are geometrically conjugate. However this contradicts our assumption. Hence our supposition that $H_c^i(S'_{w'}, \bar{\mathbb{Q}}_l)_{\theta^{-1},\theta'} \neq 0$ must be incorrect. Thus we have

$$H_c^i(S'_{w'}, \bar{\mathbb{Q}}_l)_{\theta^{-1},\theta'} = 0 \quad \text{for all } w \in W$$

and the theorem is proved.

7.4 FURTHER PROPERTIES OF THE $R_{T,\theta}$

We shall first determine the scalar product of $R_{T,\theta}$ with the principal character 1 of G^F.

Proposition 7.4.1.

$$(R_{T,\theta}, 1) = \begin{cases} 1 & \text{if } \theta = 1 \\ 0 & \text{if } \theta \neq 1 \end{cases}.$$

Proof. We have

$$\begin{aligned}(R_{T,\theta}, 1) &= \frac{1}{|G^F|}\sum_{g\in G^F} R_{T,\theta}(g)\\ &= \frac{1}{|G^F|}\sum_{g\in G^F}\frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\mathscr{L}((g,t),\tilde{X})\end{aligned}$$

by 7.2.3. Now the map $L: \tilde{X} \to U$ is a surjective morphism whose fibres are the orbits of G^F under its action on $\tilde{X}$ by left multiplication. Since $\tilde{X}$ is an affine variety and G^F is a finite group a strict quotient $\tilde{X}/G^F$ exists by section 1.5. Thus the morphism L factors through the quotient and there is a bijective morphism $\tilde{X}/G^F \to U$. By 7.1.8 we have

$$\frac{1}{|G^F|}\sum_{g\in G^F}\mathscr{L}((g,t),\tilde{X}) = \mathscr{L}(t, \tilde{X}/G^F)$$

and by 7.1.5 we have $\mathscr{L}(t, \tilde{X}/G^F) = \mathscr{L}(t, U)$. Thus we have

$$(R_{T,\theta}, 1) = \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\mathscr{L}(t, U)$$

where t acts on U by conjugation. Now consider a variety P with one point and trivial T^F-action. There is a surjective morphism from U to P with fibre isomorphic to affine space. Thus by 7.1.5 we have $\mathscr{L}(t, U) = \mathscr{L}(t, P)$ for all $t \in T^F$. Finally we have $\mathscr{L}(t, P) = 1$ by 7.1.11. It follows that $\mathscr{L}(t, U) = 1$ and so

$$(R_{T,\theta}, 1) = \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1}) = \begin{cases}1 & \text{if } \theta = 1\\ 0 & \text{if } \theta \neq 1\end{cases}$$

by the orthogonality relations for irreducible characters. ■

We show next that the principal character 1 of G^F can be expressed as a linear combination of the $R_{T,\theta}$ with $\theta = 1$.

Proposition 7.4.2.

$$1 = \frac{1}{|G^F|}\sum_{\substack{T\\ F(T)=T}} |T^F| R_{T,1}.$$

Proof. Let

$$\phi = \frac{1}{|G^F|}\sum_{\substack{T\\ F(T)=T}} |T^F| R_{T,1}$$

where the sum extends over all F-stable maximal tori T of G. ϕ is a rational combination of irreducible characters of G^F. We have

$$(\phi, 1) = \frac{1}{|G^F|} \sum_{\substack{T \\ F(T)=T}} |T^F|(R_{T,1}, 1)$$

$$= \frac{1}{|G^F|} \sum_{\substack{T \\ F(T)=T}} |T^F| \qquad \text{by 7.4.1.}$$

We now make use of the results on F-stable maximal tori proved in chapter 3. By 3.3.3 there is a bijection between the G^F-classes of F-stable maximal tori of G and the F-conjugacy classes in W. Let T_w be an F-stable maximal torus obtained from a maximally split torus by twisting by $w \in W$. Then the number of distinct F-stable tori in this G^F-conjugacy class is $|G^F: N_w{}^F|$ where $N_w = N(T_w)$. Thus we have

$$\frac{1}{|G^F|} \sum_{\substack{T \\ F(T)=T}} |T^F| = \frac{1}{|G^F|} \sum_{w} |T_w{}^F||G^F: N_w{}^F|$$

where the sum extends over one representative of each F-conjugacy class in W. Let $C_{W,F}(w)$ be the F-centralizer of w in W. By 3.3.6 we have $|N_w{}^F: T_w{}^F| = |C_{W,F}(w)|$ and we also know that $|W: C_{W,F}(w)|$ is the number of elements in the F-conjugacy class of W containing w. The above sum then becomes

$$\frac{1}{|G^F|} \sum_{\substack{T \\ F(T)=T}} |T^F| = \frac{1}{|G^F|} \sum_{w \in W} \frac{|T_w{}^F||G^F: N_w{}^F|}{|W: C_{W,F}(w)|} = \sum_{w \in W} \frac{1}{|W|} = 1.$$

Thus we have $(\phi, 1) = 1$.

We now calculate the scalar product (ϕ, ϕ). We have

$$(\phi, \phi) = \frac{1}{|G^F|^2} \left(\sum_{\substack{T \\ F(T)=T}} |T^F| R_{T,1}, \sum_{\substack{T' \\ F(T')=T'}} |T'^F| R_{T',1} \right)$$

$$= \frac{1}{|G^F|^2} \sum_T \sum_{T'} |T^F||T'^F|(R_{T,1}, R_{T',1})$$

$$= \frac{1}{|G^F|^2} \sum_T \sum_{T'} |T^F||T'^F||W(T, T')^F| \qquad \text{by 7.3.4.}$$

Now $|W(T, T')^F| = 0$ unless T, T' are G^F-conjugate. Moreover if T, T' are G^F-conjugate then $|T^F| = |T'^F|$ and there are $|G^F: N^F|$ distinct G^F-conjugates of T where $N = N(T)$. Also there are $|N^F|$ elements of G^F transforming T to each of them. It follows that

$$(\phi, \phi) = \frac{1}{|G^F|^2} \sum_T |G^F: N^F||T^F|^2 \frac{|N^F|}{|T^F|}$$

$$= \frac{1}{|G^F|} \sum_{\substack{T \\ F(T)=T}} |T^F| = 1$$

as above. Thus $(\phi, \phi) = 1$.

We now know that $(\phi, \phi) = (\phi, 1) = (1, 1) = 1$. Hence

$$(\phi - 1, \phi - 1) = (\phi, \phi) - 2(\phi, 1) + (1, 1) = 0.$$

It follows that $\phi - 1 = 0$ and so $\phi = 1$. ■

We shall next consider the situation when we have an F-stable maximal torus T which lies in an F-stable parabolic subgroup P of G. We shall show that there is then an F-stable Levi subgroup L of P containing T. We may then form the generalized characters $R_{T,\theta}{}^G$ and $R_{T,\theta}{}^L$ both in G^F and in L^F for any $\theta \in T^F$. We shall show that these two generalized characters are related in a simple way.

Proposition 7.4.3. *Let P be a parabolic subgroup of G and T be a maximal torus of P. Then there is a unique Levi subgroup L of P containing T. Thus if P and T are F-stable L will be F-stable also.*

Proof. Let B be a Borel subgroup of P containing T. B determines a positive system Φ^+ of roots of G with respect to T. Let Δ be the corresponding simple system of roots. Then there is a subset Δ_J of Δ such that

$$P = \langle T, X_\alpha, \alpha \in \Phi^+, X_{-\alpha}, \alpha \in \Delta_J \rangle.$$

Also $L = \langle T, X_\alpha, X_{-\alpha}, \alpha \in \Delta_J \rangle$ is a Levi subgroup of P containing T.

Let L' be any Levi subgroup of P containing T. Then $L' = L^p$ for some $p \in P$. Now T and $T^{p^{-1}}$ are both maximal tori of L so there exists $l \in L$ such that $T^{p^{-1}} = T^l$. Hence $lp \in N \cap P$, where $N = N_G(T)$. However $N \cap P \subseteq L$ and so $p \in L$ and $L' = L^p = L$.

Proposition 7.4.4. *Let T be an F-stable maximal torus of G which lies in the F-stable parabolic subgroup P of G. Let L be the Levi subgroup of P containing T. Let $\theta \in \hat{T}^F$. Let $R^G_{T,\theta}$ and $R^L_{T,\theta}$ be the generalized characters $R_{T,\theta}$ in G^F and L^F respectively.*

Let $(R^L_{T,\theta})_{P^F}$ be the generalized character of P^F which has $U_P{}^F$ in its kernel and agrees with $R^L_{T,\theta}$ on L^F. Let $(R^L_{T,\theta})_{P^F}{}^{G^F}$ be the induced character of G^F obtained from $(R^L_{T,\theta})_{P^F}$. Then we have

$$R^G_{T,\theta} = (R^L_{T,\theta})_{P^F}{}^{G^F}.$$

Proof. Let B be a Borel subgroup of G satisfying $T \subset B \subseteq P$, and let U be the unipotent radical of B. Let $P_1, P_2, \ldots, P_k$ be the distinct conjugates of P by elements of G^F, where $P_1 = P$. Let $\tilde{X} = L^{-1}(U)$ and $\tilde{X}_j = \{g \in \tilde{X}; {}^gP = P_j\}$ for $j = 1, \ldots, k$. Then $\tilde{X}$ is a disjoint union

$$\tilde{X} = \tilde{X}_1 \cup \ldots \cup \tilde{X}_k.$$

For let $g \in \tilde{X}$. Then $g^{-1}F(g) \in U \subseteq P$. By the Lang–Steinberg theorem there exists $p \in P$ with $g^{-1}F(g) = p^{-1}F(p)$. Thus $gp^{-1} \in G^F$. It follows that ${}^gP = {}^xP$ for some $x \in G^F$, hence that $g \in \tilde{X}_j$ for some j.

Now G^F acts on $\tilde{X}$ by left multiplication and permutes the closed subvarieties $\tilde{X}_1, \ldots, \tilde{X}_k$ transitively. Moreover T^F acts on $\tilde{X}$ by right multiplication and each $\tilde{X}_j$ is invariant under this T^F-action. By 7.2.3 we have

$$R^G_{T,\theta}(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \mathscr{L}((g, t), \tilde{X}) \quad g \in G^F.$$

However 7.1.7 shows that the generalized character

$$(g, t) \rightarrow \mathscr{L}((g, t), \tilde{X}) \quad g \in G^F, t \in T^F$$

of $G^F \times T^F$ is induced by the generalized character

$$(p, t) \rightarrow \mathscr{L}((p, t), \tilde{X}_1) \quad p \in P^F, t \in T^F$$

of $P^F \times T^F$. Hence the generalized character $R^G_{T,\theta}$ of G^F is induced from the generalized character of P^F given by

$$p \rightarrow \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \mathscr{L}((p, t), \tilde{X}_1).$$

In order to identify this generalized character we consider the morphism $\tilde{X}_1 \rightarrow \tilde{X}(P/U_P)$ given by $p \rightarrow pU_P$. Here $\tilde{X}(P/U_P) = L^{-1}(U/U_P)$ where L is Lang's map for P/U_P. If $p \in \tilde{X}_1 = \tilde{X} \cap P$ then we have

$$(pU_P)^{-1} F(pU_P) = p^{-1} F(p) U_P \in U/U_P$$

as required. This morphism is surjective and its fibres have the form pU_P where $p \in \tilde{X}_1$. Moreover this morphism is compatible with the left P^F-action and right T^F-action on both sides. Now each fibre pU_P is, as a variety, isomorphic to affine space. Thus by 7.1.5 we have

$$\mathscr{L}((p, t), \tilde{X}_1) = \mathscr{L}((p, t), \tilde{X}(P/U_P)).$$

The generalized character of P^F we require is therefore given by

$$p \rightarrow \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \mathscr{L}((p, t), \tilde{X}(P/U_P)).$$

Now $U_P{}^F$ acts trivially by left multiplication on $\tilde{X}(P/U_P)$ so lies in the kernel of this generalized character. Consider the restriction of this generalized character to L^F. Since we have an isomorphism $L \rightarrow P/U_P$ given by $l \rightarrow lU_P$ this generalized character of L^F is given by

$$\begin{aligned} l \rightarrow & \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \mathscr{L}((l, t), \tilde{X}(L)) \\ & = R^L_{T,\theta}(l) \qquad \text{by 7.2.3.} \end{aligned}$$

It follows that

$$R^G_{T,\theta} = (R^L_{T,\theta})_{P^F}{}^{G^F}.$$

Proposition 7.4.5. *Let T be an F-stable maximal torus of G and P be an F-stable parabolic subgroup of G which contains no G^F-conjugate of T. Let $\theta \in \hat{T}^F$. Then*

$$(R_{T,\theta}, 1_{P^F}{}^{G^F}) = 0.$$

Proof. Let L be an F-stable Levi subgroup of P. By 7.4.2 we have

$$1_{L^F} = \frac{1}{|L^F|} \sum_{\substack{T' \subseteq L \\ F(T') = T'}} |T'^F| R^L_{T',1}.$$

Hence

$$1_{P^F} = \frac{1}{|L^F|} \sum_{\substack{T' \subseteq L \\ F(T') = T'}} |T'^F| (R^L_{T',1})_{P^F}$$

and so

$$\begin{aligned} 1_{P^F}{}^{G^F} &= \frac{1}{|L^F|} \sum_{\substack{T' \subseteq L \\ F(T') = T'}} |T'^F| R^L_{T',1})_{P^F}{}^{G^F} \\ &= \frac{1}{|L^F|} \sum_{\substack{T' \subseteq L \\ F(T') = T'}} |T'^F| R^G_{T',1}. \end{aligned}$$

Now no torus T' occurring in this sum can be G^F-conjugate to T, since each such T' lies in P. Thus we have

$$(R_{T,\theta}, R_{T',1}) = 0 \qquad \text{by 7.3.7.}$$

It follows that $(R_{T,\theta}, 1_{P^F}{}^{G^F}) = 0$.

7.5 CHARACTER VALUES ON SEMISIMPLE ELEMENTS

In this section we shall determine $R_{T,\theta}(s)$ for any semisimple element s of G^F. We begin by determining $R_{T,\theta}(1)$.

Theorem 7.5.1. *Let T be an F-stable maximal torus of G and $\theta \in \hat{T}^F$. Then*

$$\varepsilon_G \varepsilon_T R_{T,\theta}(1) = |G^F : T^F|_{p'}.$$

(*We recall that* $\varepsilon_G = (-1)^{\text{rel.rank}\, G}$.)

Proof. $R_{T,\theta}$ is the character of the virtual G^F-module $\sum_i (-1)^i H^i_c(\tilde{X}, \bar{\mathbb{Q}}_l)_\theta$, so $\sum_{\theta \in \hat{T}^F} R_{T,\theta}$ is the character of the virtual G^F-module $\sum_i (-1)^i H^i_c(\tilde{X}, \bar{\mathbb{Q}}_l)$. Let $s \neq 1$ be a non-identity semisimple element of G^F. Then we have

$$\sum_\theta R_{T,\theta}(s) = \mathscr{L}(s, \tilde{X}) = \mathscr{L}(1, \tilde{X}^s)$$

by 7.1.10. However s acts on $\tilde{X}$ by left multiplication and so $\tilde{X}^s$ is empty. Thus $\mathscr{L}(1, \tilde{X}^s) = 0$ and $\sum_\theta R_{T,\theta}(s) = 0$.

We now consider the Steinberg character St of G^F and recall from 6.4.7 that $\mathrm{St}(g) = 0$ when $g \in G^F$ is not semisimple. Thus we have

$$\left(\sum_\theta R_{T,\theta}, \mathrm{St}\right) = \frac{1}{|G^F|}\left(\sum_\theta R_{T,\theta}(1)\right)\mathrm{St}(1).$$

Suppose first that T does not lie in any proper F-stable parabolic subgroup of G. Then we have

$$(R_{T,\theta}, \mathrm{St}) = \left(R_{T,\theta}, \sum_{\substack{J \subseteq I \\ \rho(J)=J}} (-1)^{|J'|} 1_{P_J{}^F}{}^{G^F}\right)$$

$$= \sum_{\substack{J \subseteq I \\ \rho(J)=J}} (-1)^{|J'|}(R_{T,\theta}, 1_{P_J{}^F}{}^{G^F}) = (-1)^{|I'|}(R_{T,\theta}, 1)$$

by 7.4.5, since all scalar products in the sum with $J \neq I$ are zero. Thus

$$\left(\sum_\theta R_{T,\theta}, \mathrm{St}\right) = (-1)^{|I'|}\sum_\theta (R_{T,\theta}, 1) = (-1)^{|I'|}$$

by 7.4.1. Let T be obtained from a maximally split torus T_0 by twisting by $w \in W$. By 6.5.3 w is not F-conjugate to an element of W_J for any proper F-stable subset J of I. By 6.5.5 we have

$$\text{rel. rank } T = \text{rel. rank } T_0 - |I'|.$$

By 6.5.7 we have

$$\text{rel. rank } T_0 = \text{rel. rank } G.$$

It follows that

$$\varepsilon_G \varepsilon_T = (-1)^{|I'|}$$

and so we have

$$\left(\sum_\theta R_{T,\theta}, \mathrm{St}\right) = \varepsilon_G \varepsilon_T.$$

Hence

$$\frac{1}{|G^F|}\left(\sum_\theta R_{T,\theta}(1)\right)\mathrm{St}(1) = \varepsilon_G \varepsilon_T.$$

By 7.2.9 $R_{T,\theta}(1)$ is independent of θ. Thus

$$\frac{1}{|G^F|}|T^F| R_{T,\theta}(1)\,\mathrm{St}(1) = \varepsilon_G \varepsilon_T.$$

Since $\mathrm{St}(1) = |G^F|_p$ we deduce that

$$\varepsilon_G \varepsilon_T R_{T,\theta}(1) = |G^F : T^F|_{p'}.$$

Now suppose that T lies in a proper F-stable parabolic subgroup P of G. By 7.4.3 there is an F-stable Levi subgroup L of P containing T. By 7.4.4 we have

$$R^G_{T,\theta} = (R^L_{T,\theta})_{P^F}{}^{G^F}$$

and so $R^G_{T,\theta}(1) = R^L_{T,\theta}(1).|G^F: P^F|$. We may assume inductively that

$$\varepsilon_L \varepsilon_T R^L_{T,\theta}(1) = |L^F: T^F|_{p'}.$$

It follows that

$$\varepsilon_G \varepsilon_T R_{T,\theta}(1) = \frac{\varepsilon_G \varepsilon_T}{\varepsilon_L \varepsilon_T} |G^F: P^F||L^F: T^F|_{p'} = \varepsilon_G \varepsilon_L |G^F: T^F|_{p'}.$$

Thus we must show finally that $\varepsilon_G = \varepsilon_L$. This can be seen as follows. P is conjugate in G to a unique standard parabolic subgroup P_J. Since P is F-stable P_J must be also. For if $F(P_J) = P_K$ then $P = F(P)$ is conjugate to P_K and so $J = K$. It follows that L_J is also F-stable.

Since P and P_J are both F-stable an application of the Lang–Steinberg theorem shows that P and P_J are conjugate by an element g of G^F. It follows that

$$\text{rel. rank } L = \text{rel. rank } P/U_P = \text{rel. rank } P_J/U_J = \text{rel. rank } L_J = \text{rel. rank } G$$

since L_J contains a maximally split torus of G. Hence $\varepsilon_L = \varepsilon_G$. ■

It follows from 7.3.5 and 7.5.1 that if $\theta \in \hat{T}^F$ is in general position then $\varepsilon_G \varepsilon_T R_{T,\theta}$ is an irreducible character of G^F. A convenient description of the sign $\varepsilon_G \varepsilon_T$ is given in the next proposition.

Proposition 7.5.2. *Let T be an F-stable maximal torus obtained from a maximally split torus T_0 by twisting by $w \in W$. Then $\varepsilon_G \varepsilon_T = \det w$. Thus $\varepsilon_G \varepsilon_T R_{T,\theta} = \det w R_{T,\theta}$ is an irreducible character of G^F if $\theta \in \hat{T}^F$ is in general position.*

Proof. We know from 6.5.7 that $\varepsilon_G = \varepsilon_{T_0}$. We also know from section 6.5 that

$$\varepsilon_T = (-1)^{\dim V^{F_0 w^{-1}}} \qquad \varepsilon_{T_0} = (-1)^{\dim V^{F_0}}$$

where $V = X_0 \otimes \mathbb{R}$. Also by 6.5.5 we have

$$\dim V^{F_0 w^{-1}} = \dim V^{F_0} - |J'|$$

where w is F-conjugate to an element of W_J, J F-stable, but not to an element of W_K for any proper F-stable subset K of J. Hence

$$\varepsilon_G \varepsilon_T \det w = \varepsilon_{T_0} \varepsilon_T \det w = (-1)^{|J'|} \det w.$$

Now we may assume without loss of generality that $w \in W_J$. Then by 6.5.6 $F_0 w^{-1}$ fixes no nonzero element of V_J. Now F_0 and w are real linear maps on V_J and $F_0 w^{-1}$ has finite order. Thus all eigenvalues of $F_0 w^{-1}$ on V_J are roots of unity. 1 does not occur as an eigenvalue, and the complex eigenvalues occur in conjugate pairs. Hence

$$\det_{V_J} F_0 w^{-1} = (-1)^k$$

where k is the multiplicity of -1 as an eigenvalue. Also $k \equiv \dim V_J \bmod 2$ and so

$$\det_{V_J} F_0 w^{-1} = (-1)^{\dim V_J}.$$

Next consider the eigenvalues of F_0 on V_J. 1 occurs as eigenvalue with multiplicity $|J'|$, -1 occurs with multiplicity k' say, and the remaining eigenvalues occur in complex conjugate pairs. Hence

$$\det_{V_J} F_0 = (-1)^{k'}$$

and $k' \equiv \dim V_J - |J'| \bmod 2$. Thus

$$\det_{V_J} F_0 = (-1)^{\dim V_J - |J'|}.$$

It follows that

$$\det_{V_J} w = (-1)^{|J'|}.$$

However we have $\det_{V_J} w = \det_V w$ since $w \in W_J$. Thus

$$\varepsilon_G \varepsilon_T \det w = (-1)^{|J'|} \det w = 1$$

and the result is proved. ■

We next derive a formula for the value of $R_{T,\theta}$ on any semisimple element.

Proposition 7.5.3. *Let s be a semisimple element of G^F. Then*

$$R_{T,\theta}(s) = \frac{\varepsilon_T \varepsilon_{C^0(s)}}{|T^F||C^0(s)^F|_p} \sum_{\substack{g \in G^F \\ g^{-1}sg \in T^F}} \theta(g^{-1}sg).$$

Proof. We use the character formula 7.2.8. This shows that

$$R_{T,\theta}(s) = \frac{1}{|C^0(s)^F|} \sum_{\substack{g \in G^F \\ g^{-1}sg \in T^F}} \theta(g^{-1}sg) Q^{C^0(s)}_{gTg^{-1}}(1).$$

By 7.5.1 we have

$$\begin{aligned} Q^{C^0(s)}_{gTg^{-1}}(1) = R^{C^0(s)}_{gTg^{-1},1}(1) &= \varepsilon_{C^0(s)} \varepsilon_T |C^0(s)^F : gT^F g^{-1}|_{p'} \\ &= \varepsilon_{C^0(s)} \varepsilon_T \frac{|C^0(s)^F|_{p'}}{|T^F|}. \end{aligned}$$

The result follows. ■

The formula in 7.5.3 can be expressed in a particularly simple way in terms of multiplication by the Steinberg character. Since the Steinberg character takes nonzero values on precisely the semisimple elements of G^F, a knowledge of the values of the product of the Steinberg character with any character is equivalent to the knowledge of the values of this character on all semisimple elements. 7.5.3 thus becomes equivalent to the following proposition.

Proposition 7.5.4. *For any $\theta \in \hat{T}^F$ we have*

$$\varepsilon_G \varepsilon_T R_{T,\theta} . \mathrm{St} = \theta_{T^F}{}^{G^F}.$$

Proof. We evaluate both sides at an element $s \in G^F$. Both sides are zero unless s is semisimple. If s is semisimple we have

$$\varepsilon_G \varepsilon_T R_{T,\theta}(s)\,\mathrm{St}(s) = \frac{\varepsilon_G \varepsilon_T \varepsilon_T \varepsilon_{C^0(s)} \varepsilon_G \varepsilon_{C^0(s)} |C^0(s)^F|_p}{|T^F||C^0(s)^F|_p} \sum_{\substack{g \in G^F \\ g^{-1}sg \in T^F}} \theta(g^{-1}sg)$$

by 6.5.9 and 7.5.3. Thus

$$\varepsilon_G \varepsilon_T R_{T,\theta}(s)\,\mathrm{St}(s) = \frac{1}{|T^F|} \sum_{\substack{g \in G^F \\ g^{-1}sg \in T^F}} \theta(g^{-1}sg) = \theta_{T^F}{}^{G^F}(s). \qquad \blacksquare$$

We show next that the characteristic function on each semisimple conjugacy class of G^F is a linear combination of generalized characters $R_{T,\theta}$.

Proposition 7.5.5. *Let $s \in G^F$ be semisimple. Then the function*

$$\frac{\varepsilon_{C^0}(s)}{|C(s)^F||C^0(s)^F|_p} \sum_{\substack{T \\ F(T)=T \\ s \in T}} \varepsilon_T \sum_{\theta \in \hat{T}^F} \theta(s)^{-1} R_{T,\theta}$$

takes the value 1 *on the G^F-conjugates of s and value* 0 *on all other elements of G^F.*

Proof. Let the above class function on G^F be χ and let the characteristic function on the G^F-conjugacy class of s be χ'. We shall prove that

$$(\chi, \chi) = (\chi, \chi') = (\chi', \chi').$$

It will follow that $(\chi - \chi', \chi - \chi') = 0$ and hence that $\chi - \chi' = 0$.

We begin by evaluating (χ', χ'). We have

$$(\chi', \chi') = \frac{1}{|G^F|} \sum_{\substack{x \in G^F \\ x \text{ conjugate to } s}} 1 = \frac{1}{|G^F|} |G^F : C(s)^F| = \frac{1}{|C(s)^F|}.$$

Next we consider (χ, χ'). We have

$$(\chi, \chi') = \frac{1}{|G^F|} |G^F : C(s)^F| \chi(s) = \frac{1}{|C(s)^F|} \chi(s)$$

$$= \frac{\varepsilon_{C^0(s)}}{|C(s)^F|^2 |C^0(s)^F|_p} \sum_{\substack{T \\ F(T)=T \\ s \in T}} \varepsilon_T \sum_{\theta \in \hat{T}^F} \theta(s)^{-1} R_{T,\theta}(s)$$

$$= \frac{\varepsilon_{C^0(s)}}{|C(s)^F|^2 |C^0(s)^F|_p} \sum_{\substack{T \\ F(T)=T \\ s \in T}} \varepsilon_T \sum_{\theta \in \hat{T}^F} \theta(s)^{-1}$$

$$\times \frac{\varepsilon_T \varepsilon_{C^0(s)}}{|T^F||C_0(s)^F|_p} \sum_{\substack{g \in G^F \\ g^{-1}sg \in T^F}} \theta(g^{-1}sg)$$

by 7.5.3

$$= \frac{1}{|C(s)^F|^2 |C^0(s)^F|_p{}^2} \sum_{\substack{T \\ F(T)=T \\ s\in T}} \frac{1}{|T^F|} \sum_{\theta\in\hat{T}^F} \sum_{\substack{g\in G^F \\ g^{-1}sg\in T^F}} \theta(s^{-1}g^{-1}sg).$$

Now $\sum_{\theta\in\hat{T}^F}\theta(s^{-1}g^{-1}sg) = 0$ unless $s^{-1}g^{-1}sg = 1$, in which case this sum is $|T^F|$. Thus we have

$$(\chi, \chi') = \frac{1}{|C(s)^F|^2 |C^0(s)^F|_p{}^2} \sum_{\substack{T \\ F(T)=T \\ s\in F}} \sum_{g\in C(s)^F} 1$$

$$= \frac{1}{|C(s)^F| |C^0(s)^F|_p{}^2} \sum_{\substack{T \\ F(T)=T \\ s\in T}} 1.$$

Now $s \in T$ if and only if $T \subseteq C^0(s)$ by 3.5.2. Thus we are summing over all F-stable maximal tori of $C^0(s)$. The number of F-stable maximal tori in $C^0(s)$ is $|C^0(s)^F|_p{}^2$ by 3.4.1. It follows that

$$(\chi, \chi') = \frac{1}{|C(s)^F|}.$$

Finally we consider the scalar product (χ, χ). We have

$$(\chi, \chi) = \frac{1}{|C(s)^F|^2 |C^0(s)^F|_p{}^2} \sum_{\substack{T \\ F(T)=T \\ s\in T}} \sum_{\theta\in\hat{T}^F} \sum_{\substack{T' \\ F(T')=T' \\ s\in T'}} \sum_{\theta'\in\hat{T}'^F}$$

$$\varepsilon_T \varepsilon_{T'} \theta(s)^{-1} \overline{\theta'(s)^{-1}} (R_{T,\theta}, R_{T',\theta'})$$

$$= \frac{1}{|C(s)^F|^2 |C^0(s)^F|_p{}^2} \sum_T \sum_\theta \sum_{T'} \sum_{\theta'}$$

$$\varepsilon_T \varepsilon_{T'} \theta(s)^{-1} \overline{\theta'(s)^{-1}} |\{\omega \in W(T, T')^F;\ {}^\omega\theta' = \theta\}|$$

by 7.3.4. Now we have

$$|\{\omega \in W(T, T')^F;\ {}^\omega\theta' = \theta\}| = \frac{1}{|T^F|} |\{g \in G^F;\ T^g = T', {}^g\theta' = \theta\}|.$$

Thus

$$(\chi, \chi) = \frac{1}{|C(s)^F|^2 |C^0(s)^F|_p{}^2} \sum_{\substack{T \\ F(T)=T \\ s\in T}} \sum_{\theta\in\hat{T}^F} \sum_{\substack{g\in G^F \\ s\in T^g}} \frac{1}{|T^F|} \varepsilon_T \varepsilon_{T^g} \theta(s)^{-1} \overline{({}^{g^{-1}}\theta(s))^{-1}}.$$

We have $\varepsilon_{T^g} = \varepsilon_T$ and ${}^{g^{-1}}\theta(s) = \theta(s^{g^{-1}})$, and so

$$
\begin{aligned}
(\chi, \chi) &= \frac{1}{|C(s)^F|^2 |C^0(s)^F|_p{}^2} \sum_{\substack{T \\ F(T)=T \\ s \in T}} \sum_{\theta \in \hat{T}^F} \sum_{\substack{g \in G^F \\ s \in T^g}} \frac{1}{|T^F|} \theta(s^{-1} gsg^{-1}) \\
&= \frac{1}{|C(s)^F|^2 |C^0(s)^F|_p{}^2} \sum_{\substack{T \\ F(T)=T \\ s \in T}} \frac{1}{|T^F|} \sum_{\substack{g \in G^F \\ s \in T^g}} \sum_{\theta \in \hat{T}^F} \theta(s^{-1} gsg^{-1}) \\
&= \frac{1}{|C(s)^F|^2 |C^0(s)^F|_p{}^2} \sum_{\substack{T \\ F(T)=T \\ s \in T}} \sum_{g \in C(s)^F} 1 \\
&= \frac{1}{|C(s)^F| |C^0(s)^F|_p{}^2} \sum_{\substack{T \\ F(T)=T \\ s \in T}} 1 \\
&= \frac{1}{|C(s)^F| |C^0(s)^F|_p{}^2} \sum_{\substack{T \subseteq C^0(s) \\ F(T)=T}} 1 = \frac{1}{|C(s)^F|}
\end{aligned}
$$

by 3.4.1. Hence $(\chi, \chi) = (\chi, \chi') = (\chi', \chi')$ as required. ∎

Proposition 7.5.5 has some very useful corollaries.

Corollary 7.5.6. *The character of the regular representation of G^F is given by*

$$\chi_{\text{reg}} = \frac{1}{|G^F|_p} \sum_{\substack{T \\ F(T)=T}} \sum_{\theta \in \hat{T}^F} \varepsilon_G \varepsilon_T R_{T,\theta}.$$

Proof. χ_{reg} takes value $|G^F|$ at the identity element and 0 elsewhere. We therefore put $s = 1$ in 7.5.5 and obtain

$$
\begin{aligned}
\chi_{\text{reg}} &= \frac{|G^F| \varepsilon_G}{|G^F| |G^F|_p} \sum_{\substack{T \\ F(T)=T}} \varepsilon_T \sum_{\theta \in \hat{T}^F} R_{T,\theta} \\
&= \frac{1}{|G^F|_p} \sum_{\substack{T \\ F(T)=T}} \sum_{\theta \in \hat{T}^F} \varepsilon_G \varepsilon_T R_{T,\theta}.
\end{aligned}
$$

Corollary 7.5.7. *Every class function on G^F which vanishes except on semisimple elements is a linear combination of generalized characters $R_{T,\theta}$.*

Corollary 7.5.8. *For every irreducible character χ^i of G^F there exists a generalized character $R_{T,\theta}$ such that $(R_{T,\theta}, \chi^i) \neq 0$.*

Proof. If $(R_{T,\theta}, \chi^i) = 0$ for all $R_{T,\theta}$ we would have $(\chi_{\text{reg}}, \chi^i) = 0$ by 7.5.6. However each irreducible character of G^F occurs as a component of the regular character with nonzero multiplicity, so we have a contradiction. ∎

The result of 7.5.8 shows the significance of the generalized characters $R_{T,\theta}$ for an understanding of all irreducible characters χ^i of G^F. Not only is $\pm R_{T,\theta}$ itself irreducible when θ is in general position, but if we take all possible $R_{T,\theta}$ and decompose them into irreducible characters then each irreducible character χ^i will occur as a component. We shall explore the implications of this in chapter 8.

7.6 FURTHER CHARACTER RELATIONS

We shall now prove some properties of the Green functions $Q_T(u)$ where T is an F-stable maximal torus of G and $u \in G^F$ is unipotent. We recall that $Q_T(u) = R_{T,1}(u)$. Since $Q_T(u)$ is the value at u of a generalized character of G^F it will be an algebraic integer. However we also know by 7.2.3 that

$$Q_T(u) = \frac{1}{|T^F|} \sum_{t \in T^F} \mathscr{L}((u,t), \tilde{X}).$$

Since the Lefschetz number $\mathscr{L}((u,t), \tilde{X})$ lies in $\mathbb{Z}$, $Q_T(u)$ lies in $\mathbb{Q}$. Thus $Q_T(u)$ is both an algebraic integer and a rational number, so must be a rational integer. Hence all the values $Q_T(u)$ lie in $\mathbb{Z}$.

We first prove a sum formula over the unipotent elements of G^F.

Proposition 7.6.1.

$$\sum_{\substack{u \in G^F \\ \text{unipotent}}} Q_T(u) = |G^F : T^F|.$$

Proof. By 7.4.1 we have $(R_{T,1}, 1) = 1$. Thus

$$\frac{1}{|G^F|} \sum_{g \in G^F} R_{T,1}(g) = 1.$$

Let $g = su = us$ be the Jordan decomposition of g. Then by 7.2.8 we have

$$\frac{1}{|G^F|} \sum_{g = su} \frac{1}{|C^0(s)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} Q^{C^0(s)}_{xTx^{-1}}(u) = 1.$$

This gives

$$\frac{1}{|G^F|} \sum_{\substack{s \in G^F \\ \text{semisimple}}} \frac{1}{|C^0(s)^F|} \sum_{\substack{u \in C^0(s)^F \\ \text{unipotent}}} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} Q^{C^0(s)}_{xTx^{-1}}(u) = 1$$

since each unipotent element of $C(s)$ lies in $C^0(s)$. Let Z be the centre of G. Then $s \in Z$ if and only if $C^0(s) = G$. Thus we obtain

$$\frac{1}{|G^F|} \sum_{s \in Z^F} \frac{1}{|G^F|} \sum_{\substack{u \in G^F \\ \text{unipotent}}} \sum_{x \in G^F} Q^G_{xTx^{-1}}(u)$$

$$+ \frac{1}{|G^F|} \sum_{\substack{s \in G^F \\ \text{semisimple} \\ C^0(s) \neq G}} \frac{1}{|C^0(s)^F|} \sum_{\substack{u \in C^0(s)^F \\ \text{unipotent}}} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} Q^{C^0(s)}_{xTx^{-1}}(u) = 1.$$

We argue by induction and assume that, if $C^0(s) \neq G$, then

$$\sum_{\substack{u \in C^0(s)^F \\ \text{unipotent}}} Q^{C^0(s)}_{xTx^{-1}}(u) = |C^0(s)^F : (xTx^{-1})^F|.$$

It follows that

$$\frac{|Z^F|}{|G^F|} \sum_{\substack{u \in G^F \\ \text{unipotent}}} Q^G_T(u) + \frac{1}{|G^F|} \sum_{\substack{s \in G^F \\ \text{semisimple} \\ C^0(s) \neq G}} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \frac{1}{|T^F|} = 1$$

and so

$$\frac{|Z^F|}{|G^F|} \sum_{\substack{u \in G^F \\ \text{unipotent}}} Q^G_T(u) - \frac{1}{|G^F|} \sum_{s \in Z^F} \sum_{x \in G^F} \frac{1}{|T^F|} + \frac{1}{|G^F|} \sum_{\substack{s \in G^F \\ \text{semisimple}}} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \frac{1}{|T^F|} = 1.$$

This gives

$$\frac{|Z^F|}{|G^F|} \sum_{\substack{u \in G^F \\ \text{unipotent}}} Q^G_T(u) - \frac{|Z^F|}{|T^F|} + 1 = 1$$

and so we have

$$\sum_{\substack{u \in G^F \\ \text{unipotent}}} Q^G_T(u) = |G^F : T^F|.$$

Proposition 7.6.2. (Orthogonality relations for Green functions) *Let T, T' be F-stable maximal tori of G. Then*

$$\frac{1}{|G^F|} \sum_{\substack{u \in G^F \\ \text{unipotent}}} Q_T(u) Q_{T'}(u^{-1}) = \frac{|N(T, T')^F|}{|T^F||T'^F|}.$$

In particular, if T, T' are not G^F-conjugate, we have

$$\sum_{\substack{u \in G^F \\ \text{unipotent}}} Q_T(u) Q_{T'}(u^{-1}) = 0.$$

Note that $Q_{T'}(u^{-1}) = \overline{Q_{T'}(u)} = Q_{T'}(u)$. For

$$Q_T(u) = \frac{1}{|T^F|} \sum_{t \in T^F} \mathscr{L}((u, t), \tilde{X})$$

is rational, and

$$\begin{aligned} Q_T(u^{-1}) &= \sum_i (-1)^i \operatorname{trace}(u^{-1}, H^i_c(\tilde{X}, \bar{\mathbb{Q}}_l)_1) \\ &= \sum_i (-1)^i \overline{\operatorname{trace}(u, H^i_c(\tilde{X}, \bar{\mathbb{Q}}_l)_1)} = \overline{Q_T(u)}. \end{aligned}$$

We have written $Q_{T'}(u^{-1})$ instead of $Q_{T'}(u)$ in 7.6.2 to emphasize the similarity to the usual orthogonality relations for characters, where the sum extends over all elements of G^F—not just the unipotent elements.

Proof. We use induction on dim G. The result is certainly true when $G = T$ since both sides are then equal to $1/|T^F|$. By 7.3.4 we have $(R_{T,1}, R_{T',1}) = |W(T, T')^F|$. We also have

$$(R_{T,1}, R_{T',1}) = \frac{1}{|G^F|} \sum_{g \in G^F} R_{T,1}(g) R_{T',1}(g^{-1})$$

$$= \frac{1}{|G^F|} \sum_{\substack{s \in G^F \\ \text{semisimple}}} \sum_{\substack{u \in C^0(s)^F \\ \text{unipotent}}} R_{T,1}(su) R_{T',1}(s^{-1}u^{-1})$$

$$= \frac{1}{|G^F|} \sum_{\substack{s \in G^F \\ \text{semisimple}}} \frac{1}{|C^0(s)^F|^2} \sum_{\substack{u \in C^0(s)^F \\ \text{unipotent}}} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}}$$

$$\times \sum_{\substack{x' \in G^F \\ x'^{-1}s^{-1}x' \in T'^F}} Q^{C^0(s)}_{xTx^{-1}}(u) Q^{C^0(s)}_{x'T'x'^{-1}}(u^{-1})$$

by 7.2.8. We know by induction that whenever $s \notin Z^F$ we have

$$\frac{1}{|C^0(s)^F|} \sum_{\substack{u \in C^0(s)^F \\ \text{unipotent}}} Q^{C^0(s)}_{xTx^{-1}}(u) Q^{C^0(s)}_{x'T'x'^{-1}}(u^{-1})$$

$$= \frac{|(N^{C^0(s)}(xTx^{-1}, x'T'x'^{-1}))^F|}{|(xTx^{-1})^F||(x'T'x'^{-1})^F|}.$$

It follows that

$$(R_{T,1}, R_{T',1}) = \frac{1}{|G^F|} \sum_{s \in Z^F} \sum_{\substack{u \in G^F \\ \text{unipotent}}} Q_T(u) Q_{T'}(u^{-1})$$

$$+ \frac{1}{|G^F|} \sum_{\substack{s \in G^F - Z^F \\ \text{semisimple}}} \frac{1}{|C^0(s)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \sum_{\substack{x' \in G^F \\ x'^{-1}s^{-1}x' \in T'^F}}$$

$$\frac{|(N^{C^0(s)}(xTx^{-1}, x'T'x'^{-1}))^F|}{|T^F||T'^F|}$$

$$= \frac{1}{|G^F|} \sum_{s \in Z^F} \sum_{\substack{u \in G^F \\ \text{unipotent}}} Q_T(u) Q_{T'}(u^{-1}) - \frac{|Z^F||N(T, T')^F|}{|T^F||T'^F|}$$

$$+ \frac{1}{|G^F|} \sum_{\substack{s \in G^F \\ \text{semisimple}}} \frac{1}{|C^0(s)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \sum_{\substack{x' \in G^F \\ x'^{-1}s^{-1}x' \in T'^F}}$$

$$\frac{|N^{C^0(s)}(xTx^{-1}, x'T'x'^{-1}))^F|}{|T^F||T'^F|}.$$

We now simplify the latter term. This is equal to

$$\frac{1}{|G^F|}\sum_{\substack{s\in G^F\\ \text{semisimple}}}\frac{1}{|C^0(s)^F|}\frac{1}{|T^F|}\frac{1}{|T'^F|}\sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}}\sum_{\substack{x'\in G^F\\ x'^{-1}s^{-1}x'\in T'^F}}\sum_{\substack{g\in C^0(s)^F\\ gxTx^{-1}g^{-1}=x'T'x'^{-1}}}1$$

$$=\frac{1}{|G^F|}\sum_{\substack{s\in G^F\\ \text{semisimple}}}\frac{1}{|C^0(s)^F|}\frac{1}{|T^F|}\frac{1}{|T'^F|}\sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}}$$

$$\times\sum_{g\in C^0(s)^F}\sum_{\substack{x'\in G^F\\ gxTx^{-1}g^{-1}=x'T'x'^{-1}}}1$$

$$=\frac{1}{|G^F|}\sum_{\substack{s\in G^F\\ \text{semisimple}}}\frac{1}{|C^0(s)^F|}\frac{1}{|T^F|}\frac{1}{|T'^F|}\sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}}\sum_{g\in C^0(s)^F}|N(T,T')^F|$$

$$=\frac{1}{|G^F|}\sum_{x\in G^F}\sum_{s\in xT^Fx^{-1}}\frac{|N(T,T')^F|}{|T^F||T'^F|}=\frac{|N(T,T')^F|}{|T'^F|}=|W(T,T')^F|.$$

Thus we have

$$(R_{T,1},R_{T',1})=|Z^F|\left(\frac{1}{|G^F|}\sum_{\substack{u\in G^F\\ \text{unipotent}}}Q_T(u)Q_{T'}(u^{-1})-\frac{|N(T,T')^F|}{|T^F||T'^F|}\right)$$
$$+|W(T,T')^F|.$$

However we know that $(R_{T,1},R_{T',1})=|W(T,T')^F|$ by 7.3.4. It follows that

$$\frac{1}{|G^F|}\sum_{\substack{u\in G^F\\ \text{unipotent}}}Q_T(u)Q_{T'}(u^{-1})=\frac{|N(T,T')^F|}{|T^F||T'^F|}.\qquad\blacksquare$$

We next prove some results on generalized characters ψ on G^F satisfying the condition that $\psi(g)=\psi(s)$ for all $g\in G^F$ where $g=su=us$ is the Jordan decomposition of g.

Proposition 7.6.3. *Let ψ be a generalized character on G^F such that $\psi(su)=\psi(s)$ whenever s is semisimple, u is unipotent, and $su=us$. Then for all F-stable maximal tori T of G and all $\theta\in\hat{T}^F$ we have*

$$(\psi,R_{T,\theta})_{G^F}=(\psi_{T^F},\theta)_{T^F}.$$

Proof.

$$(\psi, R_{T,\theta}) = \frac{1}{|G^F|} \sum_{g \in G^F} \psi(g) R_{T,\theta}(g^{-1})$$

$$= \frac{1}{|G^F|} \sum_{\substack{s \in G^F \\ \text{semisimple}}} \sum_{\substack{u \in C^0(s)^F \\ \text{unipotent}}} \psi(su) R_{T,\theta}(s^{-1}u^{-1})$$

$$= \frac{1}{|G^F|} \sum_{\substack{s \in G^F \\ \text{semisimple}}} \psi(s) \sum_{\substack{u \in C^0(s)^F \\ \text{unipotent}}} \frac{1}{|C^0(s)^F|}$$

$$\times \sum_{\substack{x \in G^F \\ x^{-1}s^{-1}x \in T^F}} \theta(x^{-1}s^{-1}x) Q^{C^0(s)}_{xTx^{-1}}(u^{-1})$$

by 7.2.8

$$= \frac{1}{|G^F|} \sum_{\substack{s \in G^F \\ \text{semisimple}}} \frac{1}{|C^0(s)^F|} \psi(s) \sum_{\substack{x \in G^F \\ x^{-1}s^{-1}x \in T^F}}$$

$$\theta(x^{-1}s^{-1}x) \sum_{\substack{u \in C^0(s)^F \\ \text{unipotent}}} Q^{C^0(s)}_{xTx^{-1}}(u^{-1}).$$

Now

$$\sum_{\substack{u \in C^0(s)^F \\ \text{unipotent}}} Q^{C^0(s)}_{xTx^{-1}}(u^{-1}) = |C^0(s)^F : (xTx^{-1})^F|$$

by 7.6.1. Thus

$$(\psi, R_{T,\theta}) = \frac{1}{|G^F|} \sum_{\substack{s \in G^F \\ \text{semisimple}}} \psi(s) \sum_{\substack{x \in G^F \\ x^{-1}s^{-1}x \in T^F}} \theta(x^{-1}s^{-1}x) \frac{1}{|T^F|}$$

$$= \frac{1}{|G^F|} \frac{1}{|T^F|} \sum_{x \in G^F} \sum_{\substack{s \in G^F \\ x^{-1}s^{-1}x \in T^F}} \psi(x^{-1}sx)\theta(x^{-1}s^{-1}x)$$

$$= \frac{1}{|T^F|} \sum_{t \in T^F} \psi(t)\theta(t^{-1}) = (\psi_{T^F}, \theta)_{T^F}.$$

■

We show next that any generalized character ψ of this special kind is a linear combination of generalized characters $R_{T,\theta}$.

Proposition 7.6.4. *Let ψ be a generalized character on G^F such that $\psi(su) = \psi(s)$ whenever s is semisimple, u is unipotent, and $su = us$. Then*

$$\psi = \sum_{\substack{(T,\theta) \\ \text{mod } G^F}} \frac{(\psi, \theta)_{T^F}}{|W(T)^F{}_\theta|} R_{T,\theta}$$

where the sum extends over pairs (T, θ) with $\theta \in \hat{T}^F$, one in each G^F-class of such pairs. Here $W(T)^F_\theta = \{\omega \in W(T, T)^F; {}^\omega\theta = \theta\}$.

Proof. (a) We first consider the class function $\psi \,.\, \mathrm{St}$. This vanishes except on semisimple elements. Thus by 7.5.7 it must be a linear combination of $R_{T,\theta}$'s. Since $R_{T,\theta} = R_{T',\theta'}$ if the pairs (T, θ) and (T', θ') are G^F-conjugate we have

$$\psi \,.\, \mathrm{St} = \sum_{\substack{(T,\theta) \\ \mathrm{mod}\, G^F}} C_{T,\theta} R_{T,\theta}.$$

We wish to determine the coefficients $C_{T,\theta}$. We have

$$\begin{aligned}(\psi \,.\, \mathrm{St}, R_{T',\theta'}) &= \sum_{\substack{(T,\theta) \\ \mathrm{mod}\, G^F}} C_{T,\theta}(R_{T,\theta}, R_{T',\theta'}) \\ &= C_{T',\theta'}(R_{T',\theta'}, R_{T',\theta'}) \quad \text{by 7.3.4} \\ &= C_{T',\theta'}|W(T')^F_{\theta'}| \quad \text{again by 7.3.4.}\end{aligned}$$

Hence

$$\begin{aligned}C_{T,\theta} &= \frac{(\psi \,.\, \mathrm{St}, R_{T,\theta})}{|W(T)^F_\theta|} = \frac{(\psi, \mathrm{St} \,.\, R_{T,\theta})}{|W(T)^F_\theta|} \\ &= \frac{\varepsilon_G \varepsilon_T(\psi, \theta_{T^F}{}^{G^F})}{|W(T)^F_\theta|} = \frac{\varepsilon_G \varepsilon_T(\psi, \theta)_{T^F}}{|W(T)^F_\theta|}\end{aligned}$$

using 7.5.4, Frobenius reciprocity, and the fact that the values of the Steinberg character are real. Thus we have

$$\psi \,.\, \mathrm{St} = \sum_{\substack{(T,\theta) \\ \mathrm{mod}\, G^F}} \frac{\varepsilon_G \varepsilon_T(\psi, \theta)_{T^F}}{|W(T)^F_\theta|} R_{T,\theta}.$$

(b) We show next that

$$(\psi, R_{T,\theta}) = \varepsilon_G \varepsilon_T(\psi \,.\, \mathrm{St}, R_{T,\theta}).$$

This follows from 7.6.3. We have

$$(\psi \,.\, \mathrm{St}, R_{T,\theta}) = (\psi, \mathrm{St} \,.\, R_{T,\theta}) = \varepsilon_G \varepsilon_T(\psi, \theta_{T^F}{}^{G^F}) = \varepsilon_G \varepsilon_T(\psi, \theta)_{T^F}$$

as in (a) above. Hence $(\psi, R_{T,\theta}) = \varepsilon_G \varepsilon_T(\psi \,.\, \mathrm{St}, R_{T,\theta})$ by 7.6.3.

(c) We now define the class function ψ' on G^F by

$$\psi' = \sum_{\substack{(T,\theta) \\ \mathrm{mod}\, G^F}} \frac{(\psi, \theta)_{T^F}}{|W(T)^F_\theta|} R_{T,\theta}.$$

We wish to show that $\psi = \psi'$. We shall show that $(\psi, \psi) = (\psi, \psi') = (\psi', \psi')$. It will follow that $(\psi - \psi', \psi - \psi') = 0$ and so $\psi - \psi' = 0$. We shall in fact show that

$$(\psi, \psi) = (\psi, \psi') = (\psi', \psi') = (\psi \,.\, \mathrm{St}, \psi \,.\, \mathrm{St}).$$

By (a) we have

$$(\psi \,.\, \mathrm{St}, \psi \,.\, \mathrm{St}) = \sum_{\substack{(T,\theta) \\ \mathrm{mod}\, G^F}} \frac{|(\psi, \theta)_{T^F}|^2}{|W(T)^F{}_\theta|^2} (R_{T,\theta}, R_{T,\theta}) = (\psi', \psi').$$

On the other hand we have

$$(\psi', \psi) = \sum_{\substack{(T,\theta)\\ \bmod G^F}} \frac{(\psi, \theta)_{T^F}}{|W(T)^F{}_\theta|} (R_{T,\theta}, \psi)$$

$$= \sum_{\substack{(T,\theta)\\ \bmod G^F}} \varepsilon_G \varepsilon_T \frac{(\psi, \theta)_{T^F}}{|W(T)^F{}_\theta|} (R_{T,\theta}, \psi\,.\,\mathrm{St}) \quad \text{by (b)}$$

$$= (\psi\,.\,\mathrm{St}, \psi\,.\,\mathrm{St}) \quad \text{by (a).}$$

Finally we must show that $(\psi, \psi) = (\psi\,.\,\mathrm{St}, \psi\,.\,\mathrm{St})$. We have

$$(\psi\,.\,\mathrm{St}, \psi\,.\,\mathrm{St}) = \frac{1}{|G^F|} \sum_{g \in G^F} \psi(g)\,\mathrm{St}(g)\overline{\psi(g)}\;\overline{\mathrm{St}(g)}$$

$$= \frac{1}{|G^F|} \sum_{\substack{s \in G^F\\ \text{semisimple}}} \psi(s)\overline{\psi(s)}\,\mathrm{St}(s)^2$$

since the Steinberg character vanishes except on semisimple elements, where it takes real values. We also have

$$(\psi, \psi) = \frac{1}{|G^F|} \sum_{g \in G^F} \psi(g)\overline{\psi(g)}$$

$$= \frac{1}{|G^F|} \sum_{\substack{s \in G^F\\ \text{semisimple}}} \sum_{\substack{u \in C^0(s)^F\\ \text{unipotent}}} \psi(s)\overline{\psi(s)}$$

$$= \frac{1}{|G^F|} \sum_{\substack{s \in G^F\\ \text{semisimple}}} \psi(s)\overline{\psi(s)} \sum_{\substack{u \in C^0(s)^F\\ \text{unipotent}}} 1.$$

But by 6.6.1 the number of unipotent elements of $C^0(s)^F$ is $|C^0(s)^F|_p{}^2$. Moreover by 6.4.7 we have

$$\mathrm{St}(s) = \pm |C^0(s)^F|_p.$$

It follows that

$$(\psi, \psi) = \frac{1}{|G^F|} \sum_{\substack{s \in G^F\\ \text{simisimple}}} \psi(s)\overline{\psi(s)}\,\mathrm{St}(s)^2 = (\psi\,.\,\mathrm{St}, \psi\,.\,\mathrm{St}).$$

We have now shown that

$$(\psi, \psi) = (\psi, \psi') = (\psi', \psi')$$

and the required result follows. ■

Proposition 7.6.4 has some useful corollaries. We first take $\psi = 1$ in 7.6.4.

Corollary 7.6.5.

$$1 = \sum_{\substack{T \\ \bmod G^F}} \frac{R_{T,1}}{|W(T)^F|}.$$

This result can also be deduced easily from 7.4.2.

Corollary 7.6.6. *The Steinberg character of G^F satisfies the condition*

$$\mathrm{St} = \sum_{\substack{T \\ \bmod G^F}} \frac{\varepsilon_G \varepsilon_T R_{T,1}}{|W(T)^F|}.$$

Proof. This follows from 7.6.4(a) by putting $\psi = 1$. ∎

An alternative formula for the Steinberg character, due to Srinivasan, can also be obtained from 7.6.5.

Corollary 7.6.7. (Srinivasan)

$$\mathrm{St} = \sum_{\substack{T \\ \bmod G^F}} \frac{\varepsilon_G \varepsilon_T 1_{T^F}{}^{G^F}}{|W(T)^F|}.$$

Proof. We multiply both sides of 7.6.5 by the Steinberg character. By 7.5.4 we have $R_{T,1}\,\mathrm{St} = \varepsilon_G \varepsilon_T 1_{T^F}{}^{G^F}$ and so the result follows. ∎

We next prove a result of Kawanaka.

Proposition 7.6.8. (Kawanaka) *Let ψ be a generalized character of G^F such that $\psi(su) = \psi(s)$ whenever s is semisimple, u is unipotent, and $su = us$. Then*

$$(\psi, \psi)_{G^F} = \sum_{\substack{T \\ \bmod G^F}} \frac{(\psi, \psi)_{T^F}}{|W(T)^F|}.$$

Proof. By 7.6.4 we have

$$(\psi, \psi)_{G^F} = \sum_{\substack{(T,\theta) \\ \bmod G^F}} \sum_{\substack{(T',\theta') \\ \bmod G^F}} \frac{(\psi, \theta)_{T^F}(\psi, \theta')_{T'^F}}{|W(T)^F{}_\theta||W(T')^F_{\theta'}|} (R_{T,\theta}, R_{T',\theta'})$$

$$= \sum_{\substack{(T,\theta) \\ \bmod G^F}} \frac{(\psi, \theta)_{T^F}{}^2}{|W(T)^F{}_\theta|} \quad \text{by 7.3.4}$$

$$= \sum_{\substack{T \\ \bmod G^F}} \sum_{\substack{\theta \in \hat{T}^F \\ \bmod W(T)^F}} \frac{(\psi, \theta)_{T^F}{}^2}{|W(T)^F{}_\theta|}.$$

Now two characters in $\hat{T}^F$ equivalent under $W(T)^F$ give the same value for

$(\psi, \theta)_{T^F}{}^2/|W(T)^F{}_\theta|$. Hence we have

$$(\psi, \psi)_{G^F} = \sum_{\substack{T \\ \text{mod } G^F}} \sum_{\theta \in \hat{T}^F} \frac{(\psi, \theta)_{T^F}{}^2}{|W(T)^F{}_\theta|} \frac{1}{|W(T)^F : W(T)^F{}_\theta|}$$

$$= \sum_{\substack{T \\ \text{mod } G^F}} \sum_{\theta \in \hat{T}^F} \frac{(\psi, \theta)_{T^F}{}^2}{|W(T)^F|}.$$

However $(\psi, \psi)_{T^F} = \sum_{\theta \in \hat{T}^F} (\psi, \theta)_{T^F}{}^2$. Thus we have

$$(\psi, \psi)_{G^F} = \sum_{\substack{T \\ \text{mod } G^F}} \frac{(\psi, \psi)_{T^F}}{|W(T)^F|}.$$

7.7 AN ALTERNATIVE DESCRIPTION OF $R_{T,1}$

We shall now show that the Deligne–Lusztig generalized character can be given an alternative description when $\theta = 1$ which is intuitively particularly simple.

Let $\mathfrak{B}$ be the set of all Borel subgroups of the connected reductive group G. Let B_0 be a fixed Borel subgroup. Then there is a bijective map between $\mathfrak{B}$ and the set G/B_0 of cosets gB_0 of B_0 in G given by ${}^gB_0 \rightarrow gB_0$. Now G/B_0 has the structure of a projective variety and so this bijection can be used to give $\mathfrak{B}$ such a structure also. $\mathfrak{B}$ is the variety of Borel subgroups. G acts as a group of automorphisms on G/B_0 by $gB_0 \rightarrow xgB_0$ and on $\mathfrak{B}$ by ${}^gB_0 \rightarrow {}^{xg}B_0$.

We shall consider the actions of G on $G/B_0 \times G/B_0$ and on $\mathfrak{B} \times \mathfrak{B}$.

Proposition 7.7.1. *G acts on $G/B_0 \times G/B_0$ by left multiplication as a group of automorphisms. The orbits are in bijective correspondence with elements of $W(T_0) = N(T_0)/T_0$ where T_0 is a maximal torus of B_0.*

Proof. Let $(g_1B_0, g_2B_0) \in G/B_0 \times G/B_0$. This element lies in the same G-orbit as $(B_0, g_1{}^{-1}g_2B_0)$. Now $g_1{}^{-1}g_2 \in B_0\dot{w}B_0$ for a unique $w \in W(T_0)$. Let $g_1{}^{-1}g_2 = b\dot{w}b'$. Then $(B_0, g_1{}^{-1}g_2B_0) = (B_0, b\dot{w}B_0)$, and this lies in the same G-orbit as $(B_0, \dot{w}B_0)$. Thus each orbit contains an element of the form $(B_0, \dot{w}B_0)$ for $w \in W(T_0)$.

Suppose $(B_0, \dot{w}B_0)$ lies in the same orbit as $(B_0, \dot{w}'B_0)$. Then $(B_0, \dot{w}'B_0) = (gB_0, g\dot{w}B_0)$ for some $g \in G$. Thus $B_0 = gB_0$ and $\dot{w}'B_0 = g\dot{w}B_0$. Hence $g \in B_0$ and $\dot{w}' \in B_0\dot{w}B_0$. It follows that $w' = w$. Hence each orbit contains a unique element of form $(B_0, \dot{w}B_0)$ for $w \in W(T_0)$. This gives a bijection between orbits and elements of $W(T_0)$.

Corollary 7.7.2. *G acts on $\mathfrak{B} \times \mathfrak{B}$ by conjugation as a group of automorphisms. The orbits are in bijective correspondence with elements of $W(T_0)$. Each orbit contains a unique element of form $(B_0, {}^{\dot{w}}B_0)$ for $w \in W(T_0)$.*

Definition. Let $B_1, B_2 \in \mathfrak{B}$. We say that B_1, B_2 are in relative position w if (B_1, B_2) is in the same G-orbit as $(B_0, {}^{\dot{w}}B_0)$.

We would like to make clear to what extent this definition depends upon the choice of Borel subgroup B_0 and maximal torus T_0 in B_0. This is the content of the next proposition.

Proposition 7.7.3. *Let B_0, B be Borel subgroups of G and T_0, T be maximal tori of G with $T_0 \subseteq B_0$ and $T \subseteq B$. Then:*

(i) *There exists $g \in G$ with ${}^gB_0 = B$ and ${}^gT_0 = T$.*

(ii) *The induced isomorphism α: $W(T_0) \to W(T)$ is independent of the choice of g.*

(iii) *B_1, B_2 are in relative position $w \in W(T_0)$ with respect to B_0, T_0 if and only if they are in relative position $\alpha(w) \in W(T)$ with respect to B, T.*

Proof. (i) This follows from the fact that any two Borel subgroups of G are conjugate and that any two maximal tori in a Borel subgroup are conjugate in that Borel subgroup.

(ii) It is sufficient to take $T = T_0$ and $B = B_0$. Then $g \in B_0 \cap N(T_0) = T_0$. Thus our isomorphism is conjugation by an element of T_0. Such an isomorphism acts trivially on $W(T_0)$.

(iii) Suppose B_1, B_2 are in relative position $w \in W(T_0)$ with respect to B_0, T_0. Then (B_1, B_2) is in the same G-orbit as $(B_0, {}^{\dot{w}}B_0)$. The definition of the isomorphism α: $W(T_0) \to W(T)$ shows that

$$(B, {}^{\alpha(\dot{w})}B) = ({}^gB_0, {}^{g\dot{w}}({}^gB_0))$$

and this lies in the same G-orbit as $(B_0, {}^{\dot{w}}B_0)$. Thus (B_1, B_2) lies in the same G-orbit as $(B, {}^{\alpha(\dot{w})}B)$. This shows that B_1, B_2 are in relative position $\alpha(w)$ with respect to B, T. ■

Let W be the Weyl group of G. We shall subsequently say that B_1, B_2 are in relative position $w \in W$ without identifying W with any particular $W(T_0)$. Proposition 7.7.3 shows that we can do this without ambiguity.

Definition. Let $w \in W$. Then $\mathfrak{O}_w$ is the set of all pairs $(B_1, B_2) \in \mathfrak{B} \times \mathfrak{B}$ such that B_1, B_2 are in relative position w.

We now consider the F-action on G and assume that B_0 is chosen to be F-stable and that T_0 is an F-stable maximal torus of B_0.

Proposition 7.7.4. $(B, F(B)) \in \mathfrak{O}_w$ *if and only if* $g^{-1}F(g) \in B_0\dot{w}B_0$ *where* $B = {}^gB_0$.

Proof. $(B, F(B)) = ({}^gB_0, {}^{F(g)}B_0)$. This lies in the same G-orbit as $(B_0, {}^{g^{-1}F(g)}B_0)$. Now $g^{-1}F(g) \in B_0\dot{w}B_0$ for a unique $w \in W$. Then $(B_0, {}^{g^{-1}F(g)}B_0)$ lies in the same G-orbit as $(B_0, {}^{\dot{w}}B_0)$. Hence $(B, F(B)) \in \mathfrak{O}_w$.

Definition. $\mathfrak{B}_w$ is the set of all $B \in \mathfrak{B}$ such that $(B, F(B)) \in \mathfrak{O}_w$.

$\mathfrak{B}_w$ is a locally closed subset of $\mathfrak{B}$ and so inherits from $\mathfrak{B}$ the structure of an algebraic variety. The result of 7.7.4 indicates that we should also consider

$L^{-1}(B_0\dot{w}B_0)$, which is a locally closed subset of G and so inherits from G the structure of an algebraic variety. The way in which $\mathfrak{B}_w$ and $L^{-1}(B_0\dot{w}B_0)$ are related is described in the next result.

Proposition 7.7.5. (i) *B_0 acts as a group of automorphisms of $L^{-1}(B_0\dot{w}B_0)$ by right multiplication.*

(ii) *The map $g \to {}^gB_0$ is a surjective morphism from $L^{-1}(B_0\dot{w}B_0)$ to $\mathfrak{B}_w$.*

(iii) *The fibres of this morphism are the orbits of B_0 on $L^{-1}(B_0\dot{w}B_0)$.*

Proof. (i) Let $g \in L^{-1}(B_0\dot{w}B_0)$ and $b \in B_0$. Consider the element gb. We have

$$L(gb) = (gb)^{-1}F(gb) = b^{-1}g^{-1}F(g)F(b) \in b^{-1}B_0\dot{w}B_0F(b) = B_0\dot{w}B_0$$

since b, $F(b)$ both lie in B_0. Hence $gb \in L^{-1}(B_0\dot{w}B_0)$.

(ii) Let $g \in L^{-1}(B_0\dot{w}B_0)$. We show that ${}^gB_0 \in \mathfrak{B}_w$. By 7.7.4 we have $({}^gB_0, F({}^gB_0)) \in \mathfrak{O}_w$ and so ${}^gB_0 \in \mathfrak{B}_w$. Hence $g \to {}^gB_0$ is a morphism from $L^{-1}(B_0\dot{w}B_0)$ into $\mathfrak{B}_w$.

Suppose $B \in \mathfrak{B}_w$. Then $(B, F(B)) \in \mathfrak{O}_w$ and so $B = {}^gB_0$ where $g^{-1}F(g) \in B_0\dot{w}B_0$ by 7.7.4. Hence $g \in L^{-}(B_0\dot{w}B_0)$ and so the morphism is surjective.

(iii) Let $g_1, g_2 \in L^{-1}(B_0\dot{w}B_0)$. Then g_1, g_2 lie in the same fibre of the above morphism if and only if ${}^{g_1}B_0 = {}^{g_2}B_0$. This is equivalent to $g_2 = g_1b$ for some $b \in B_0$. Thus the fibres are the orbits of B_0 on $L^{-1}(B_0\dot{w}B_0)$. ■

We next consider the set $L^{-1}(\dot{w}B_0)$. This is a closed subset of G so inherits from G the structure of an algebraic variety.

Proposition 7.7.6. (i) *$B_0 \cap {}^{\dot{w}}B_0$ acts on $L^{-1}(\dot{w}B_0)$ as a group of automorphisms by right multiplication.*

(ii) *The map $g \to {}^gB_0$ is a surjective morphism from $L^{-1}(\dot{w}B_0)$ to $\mathfrak{B}_w$.*

(iii) *The fibres of this morphism are the orbits of $B_0 \cap {}^{\dot{w}}B_0$ on $L^{-1}(\dot{w}B_0)$.*

Proof. (i) Let $g \in L^{-1}(\dot{w}B_0)$ and $b \in B_0 \cap {}^{\dot{w}}B_0$. Consider the element gb. We have

$$L(gb) = b^{-1}g^{-1}F(g)F(b) \in b^{-1}\dot{w}B_0F(b) = \dot{w}B_0$$

since $b^{-1}\dot{w} \in \dot{w}B_0$ and $F(b) \in B_0$. Hence $gb \in L^{-1}(\dot{w}B_0)$.

(ii) Let $g \in L^{-1}(\dot{w}B_0)$. We show that ${}^gB_0 \in \mathfrak{B}_w$. This follows from 7.7.5 since $L^{-1}(\dot{w}B_0) \subseteq L^{-1}(B_0\dot{w}B_0)$.

Suppose $B \in \mathfrak{B}_w$. Then $(B, F(B)) \in \mathfrak{O}_w$ and so there exists $g \in G$ with $(B, F(B)) = ({}^gB_0, {}^{g\dot{w}}B_0)$. Hence $B = {}^gB_0$ and $F({}^gB_0) = {}^{g\dot{w}}B_0$. It follows that ${}^{F(g)}B_0 = {}^{g\dot{w}}B_0$, so that $\dot{w}^{-1}g^{-1}F(g) \in B_0$. Hence $g \in L^{-1}(\dot{w}B_0)$. Hence the morphism $g \to {}^gB_0$ from $L^{-1}(\dot{w}B_0)$ to $\mathfrak{B}_w$ is surjective.

(iii) Let $g_1, g_2 \in L^{-1}(\dot{w}B_0)$. Then g_1, g_2 lie in the same fibre if and only if ${}^{g_1}B_0 = {}^{g_2}B_0$, and this is equivalent to $g_2 = g_1b$ for some $b \in B_0$. However such a b must lie in $B_0 \cap {}^{\dot{w}}B_0$. For we have

$$g_1{}^{-1}F(g_1) \in \dot{w}B_0 \qquad g_2{}^{-1}F(g_2) \in \dot{w}B_0$$

hence $b^{-1}g_1{}^{-1}F(g_1)F(b) \in \dot{w}B_0$, and so $b^{-1}g_1{}^{-1}F(g_1) \in \dot{w}B_0$. It follows that $b^{-1}\dot{w}B_0 = \dot{w}B_0$ and $b \in \dot{w}B_0\dot{w}^{-1} = {}^{\dot{w}}B_0$. Thus g_1, g_2 lie in the same fibre if and only if $g_2 = g_1 b$ for some $b \in B_0 \cap {}^{\dot{w}}B_0$. ∎

Let U_0 be the unipotent radical of B_0. We thus have $B_0 = U_0T_0$ where B_0, U_0, T_0 are all F-stable. We prove next a similar result to 7.7.6 using $L^{-1}(\dot{w}U_0)$ instead of $L^{-1}(\dot{w}B_0)$.

Proposition 7.7.7. *Let $T^{w^{-1}F} = \{t \in T; (F(t))^{w^{-1}} = t\}$. Then*

(i) *$(U_0 \cap {}^{\dot{w}}U_0)(T_0{}^{w^{-1}F})$ acts as a group of automorphisms on $L^{-1}(\dot{w}U_0)$ by right multiplication.*

(ii) *The map $g \to {}^gB_0$ is a surjective morphism from $L^{-1}(\dot{w}U_0)$ to $\mathfrak{B}_w$.*

(iii) *The fibres of this morphism are the orbits of $(U_0 \cap {}^{\dot{w}}U_0)(T_0{}^{w^{-1}F})$ on $L^{-1}(\dot{w}U_0)$.*

Proof. (i) Let $g \in L^{-1}(\dot{w}U_0)$ and $b \in (U_0 \cap {}^{\dot{w}}U_0)(T_0{}^{w^{-1}F})$. Then $b = ut$ where $u \in U_0 \cap {}^{\dot{w}}U_0$ and $t \in T_0{}^{w^{-1}F}$. Consider the element gb. We have

$$L(gb) = b^{-1}g^{-1}F(g)F(b) = t^{-1}u^{-1}F(g)F(u)F(t).$$

Now $g^{-1}F(g) = \dot{w}u_0$ for some $u_0 \in U_0$. Hence

$$L(gb) = t^{-1}u^{-1}\dot{w}u_0F(u)t^w = t^{-1}\dot{w}(u^{-1})^{\dot{w}}u_0F(u)t^w$$

$$= \dot{w}(t^w)^{-1}(u^{-1})^{\dot{w}}u_0F(u)t^w.$$

Now $(u^{-1})^{\dot{w}}u_0F(u) \in U_0$ thus $L(gb) \in \dot{w}U_0$. Hence $gb \in L^{-1}(\dot{w}U_0)$.

(ii) By 7.7.6 (ii) we see that $g \to {}^gB_0$ is a morphism from $L^{-1}(\dot{w}U_0)$ into $\mathfrak{B}_w$. We show it is surjective. Suppose $B \in \mathfrak{B}_w$. By 7.7.6 there exists $g' \in L^{-1}(\dot{w}B_0)$ such that $B = {}^{g'}B_0$. Let $g = g't$ where $t \in T_0$. Then $B = {}^gB_0$. We show t can be chosen so that $g \in L^{-1}(\dot{w}U_0)$.

Now $L(g) = t^{-1}g'^{-1}F(g')F(t)$ and so $g \in L^{-1}(\dot{w}U_0)$ if and only if $g'^{-1}F(g') \in t\dot{w}U_0F(t)^{-1}$. Now $g'^{-1}F(g') = \dot{w}u_0t_0$ for some $u_0 \in U_0$, $t_0 \in T_0$. Thus our required condition on t is

$$\dot{w}u_0t_0 \in t\dot{w}U_0F(t)^{-1}$$

which is equivalent to $u_0t_0 \in U_0t^wF(t)^{-1}$ or to $t_0 = t^wF(t)^{-1}$. Thus we seek an element $t \in T_0$ satisfying

$$t^{-1}(F(t))^{w^{-1}} = (t_0{}^{-1})^{w^{-1}}.$$

However the map $t \to t^{-1}(F(t))^{w^{-1}}$ of T_0 into itself is surjective. This follows from the Lang–Steinberg theorem applied to an F-stable maximal torus obtained from T_0 by twisting by w. Thus the element $t \in T_0$ can be chosen in the required way, and the morphism from $L^{-1}(\dot{w}U_0)$ to $\mathfrak{B}_w$ is surjective.

(iii) We now consider the fibres. Let $g_1, g_2 \in L^{-1}(\dot{w}U_0)$. Then g_1, g_2 are in the same fibre if and only if $g_2 = g_1b$ for some $b \in B_0$. Thus elements in the same orbit under $(U_0 \cap {}^{\dot{w}}U_0)(T_0{}^{w^{-1}F})$ certainly lie in the same fibre.

Suppose conversely that g_1, g_2 lie in the same fibre. Then $g_1{}^{-1}F(g_1) \in \dot{w}U_0$, $g_2{}^{-1}F(g_2) \in \dot{w}U_0$ and $g_2 = g_1 b$ with $b \in B_0$. By 7.7.6 we have $b \in B_0 \cap {}^{\dot{w}}B_0 = (U_0 \cap {}^{\dot{w}}U_0)T_0$. Let $b = u_0' t_0$ where $u_0' \in U_0 \cap {}^{\dot{w}}U_0$ and $t_0 \in T_0$. Let $g_1{}^{-1}F(g_1) = \dot{w}u_0$ where $u_0 \in U_0$. Then we have

$$g_2{}^{-1}F(g_2) = b^{-1}g_1{}^{-1}F(g_1)F(b) = t_0{}^{-1}u_0'^{-1}\dot{w}u_0F(u_0')F(t_0).$$

Since $g_2{}^{-1}F(g_2) \in \dot{w}U_0$ we have

$$\dot{w}^{-1}t_0{}^{-1}\dot{w}(u_0'^{-1})^{\dot{w}}u_0F(u_0')F(t_0) \in U_0$$

which gives

$$F(t_0)(t_0{}^{-1})^{w}(u_0'^{-1})^{\dot{w}}u_0F(u_0') \in U_0.$$

Now $(u_0'^{-1})^{\dot{w}}u_0F(u_0') \in U_0$ and so $F(t_0)(t_0{}^{-1})^{w}$ lies in $U_0 \cap T_0 = 1$. Thus $F(t_0) = t_0{}^{w}$ and so $t_0 \in T_0{}^{w^{-1}F}$. It follows that $b \in (U_0 \cap {}^{\dot{w}}U_0)(T_0{}^{w^{-1}F})$ as required. ■

Now let T be an F-stable maximal torus of G. Then T is obtained from the maximally split torus T_0 by twisting by an element $w \in W$. In other words we have $T = {}^{x}T_0$ where $x^{-1}F(x) = \dot{w}$. Let $U = {}^{x}U_0$.

Proposition 7.7.8. *There is an isomorphism between the varieties $L^{-1}(F(U))$ and $L^{-1}(\dot{w}U_0)$ given by right multiplication by x.*

Proof. Suppose $g \in L^{-1}(F(U))$ and consider gx. We have

$$L(gx) = x^{-1}g^{-1}F(g)F(x) \in x^{-1}F(U)F(x) = x^{-1}F(x)U_0 = \dot{w}U_0.$$

Thus $gx \in L^{-1}(\dot{w}U_0)$.

Conversely suppose $g \in L^{-1}(\dot{w}U_0)$ and consider gx^{-1}. We have

$$L(gx^{-1}) = xg^{-1}F(g)F(x^{-1}) \in x\dot{w}U_0F(x^{-1}) = F(x)U_0F(x)^{-1} = F(U).$$

Thus $gx^{-1} \in L^{-1}(F(U))$. ■

In 7.7.7 we considered the action of the group $(U_0 \cap {}^{\dot{w}}U_0)(T_0{}^{w^{-1}F})$ on $L^{-1}(\dot{w}U_0)$. Applying the isomorphism of 7.7.8 we obtain the following result.

Proposition 7.7.9. (i) *$(U \cap F(U))T^F$ acts on $L^{-1}(F(U))$ as a group of automorphisms by right multiplication.*

(ii) *Under the isomorphism of 7.7.8 the orbits of $L^{-1}(F(U))$ under $(U \cap F(U))T^F$ map to the orbits of $L^{-1}(\dot{w}U_0)$ under $(U_0 \cap {}^{\dot{w}}U_0)(T_0{}^{w^{-1}F})$.*

Proof. (i) Let $g \in L^{-1}(F(U))$ and $b \in (U \cap F(U))T^F$. Consider the element gb. We have $L(gb) = b^{-1}g^{-1}F(g)F(b)$. Let $b = ut$ where $u \in U \cap F(U)$ and $t \in T^F$. Then

$$L(gb) = t^{-1}u^{-1}g^{-1}F(g)F(u)t \in t^{-1}F(U)t = F(U)$$

and $gb \in L^{-1}(F(U))$.

(ii) Let $g_1, g_2 \in L^{-1}(F(U))$. Then $g_1x, g_2x \in L^{-1}(\dot{w}U_0)$. They are in the same orbit under $(U_0 \cap {}^{\dot{w}}U_0)(T_0{}^{w^{-1}F})$ if and only if

$$g_2x = g_1xut \qquad u \in U_0 \cap {}^{\dot{w}}U_0, t \in T_0{}^{w^{-1}F}.$$

This is equivalent to

$$g_1{}^{-1}g_2 = xutx^{-1} \in ({}^xU_0 \cap {}^{x\dot{w}}U_0)^x(T_0{}^{w^{-1}F}).$$

Now we have

$${}^xU_0 \cap {}^{x\dot{w}}U_0 = {}^xU_0 \cap {}^{F(x)}U_0 = U \cap F(U).$$

Also ${}^xT_0 = T$. Let $t = {}^xt_0$ where $t_0 \in T_0$. Then we have

$$t \in T^F \Leftrightarrow F({}^xt_0) = {}^xt_0 \Leftrightarrow {}^{x^{-1}F(x)}F(t_0) = t_0 \Leftrightarrow t_0 \in T_0{}^{w^{-1}F}.$$

Thus ${}^x(T_0{}^{w^{-1}F}) = T^F$. Hence g_1x, g_2x are in the same orbit under $(U_0 \cap {}^{\dot{w}}U_0)(T_0{}^{w^{-1}F})$ if and only if g_1, g_2 are in the same orbit under $(U \cap F(U))T^F$. ∎

We now combine the results of the last few propositions to obtain the following.

Proposition 7.7.10. *Let T be an F-stable maximal torus of G obtained from the maximally split torus T_0 by twisting by $w \in W$. Thus $T = {}^xT_0$ where $x^{-1}F(x) = \dot{w}$. Let $B_0 = U_0T_0$ be an F-stable Borel subgroup of G containing T_0 and let $U = {}^xU_0$. Then there is a surjective morphism from $L^{-1}(F(U))$ to $\mathfrak{B}_w$ for which the fibres are the orbits of $L^{-1}(F(U))$ under the group $(U \cap F(U))T^F$ acting by right multiplication.*

Proof. This follows from 7.7.7, 7.7.8 and 7.7.9. The morphism $L^{-1}(F(U)) \to \mathfrak{B}_w$ is given by $g \to {}^{gx}B_0$. ∎

We can now give our alternative description of the generalized character $R_{T,1}$ of G^F.

Theorem 7.7.11. *Let T be an F-stable maximal torus of G obtained from a maximally split torus by twisting by $w \in W$. Then*

$$R_{T,1}(g) = \mathscr{L}(g, \mathfrak{B}_w)$$

for all $g \in G^F$.

Proof. We first observe that $\mathfrak{B}_w$ is indeed invariant under the action of G^F. Let $B \in \mathfrak{B}_w$. Then $(B, F(B)) \in \mathfrak{O}_w$. Let $x \in G^F$. Then $({}^xB, F({}^xB)) = ({}^xB, {}^xF(B))$ lies in the same G-orbit as $(B, F(B))$. Thus $({}^xB, F({}^xB)) \in \mathfrak{O}_w$, and so ${}^xB \in \mathfrak{B}_w$. Thus $\mathfrak{B}_w$ is invariant under G^F.

Now choose $B = UT$ as in 7.7.10. Then $F(B) = F(U)T$ is also a Borel subgroup containing T. Since the choice of Borel subgroup containing T is irrelevant for the definition of $R_{T,1}$ we choose the latter and obtain

$$R_{T,1}(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \mathscr{L}((g, t), L^{-1}(F(U)))$$

by 7.2.3. Now $L^{-1}(F(U))$ is an affine variety acted on by the finite group T^F by right multiplication. Thus there is a strict quotient $L^{-1}(F(U))/T^F$ and 7.1.8 shows that

$$R_{T,1}(g) = \mathscr{L}(g, L^{-1}(F(U))/T^F).$$

Now the morphism $L^{-1}(F(U)) \to \mathfrak{B}_w$ of 7.7.10 has the property that elements in a given T^F-orbit all lie in the same fibre. It therefore factorizes through the quotient $L^{-1}(F(U))/T^F$ and we have morphisms

$$L^{-1}(F(U)) \to L^{-1}(F(U))/T^F \to \mathfrak{B}_w.$$

The latter morphism $L^{-1}(F(U))/T^F \to \mathfrak{B}_w$ is compatible with the G^F-actions on both. Moreover the fibres of this morphism are all isomorphic to $U \cap F(U)$ by 7.7.10. However

$$U \cap F(U) = {}^xU_0 \cap {}^{F(x)}U_0 = {}^x(U_0 \cap {}^{\dot{w}}U_0)$$

and $U_0 \cap {}^{\dot{w}}U_0$ is, as a variety, isomorphic to affine space of an appropriate dimension. The same is therefore true of $U \cap F(U)$ and so 7.1.5 shows that

$$\mathscr{L}(g, L^{-1}(F(U))/T^F) = \mathscr{L}(g, \mathfrak{B}_w) \qquad g \in G^F.$$

It follows that

$$R_{T,1}(g) = \mathscr{L}(g, \mathfrak{B}_w)$$

as required. ■

This result gives a good intuitive description of the generalized character $R_{T,1}$. If T is obtained from a maximally split torus by twisting by $w \in W$ then the value of $R_{T,1}$ at an element g in G^F is the Lefschetz number of g on the variety $\mathfrak{B}_w$ of Borel subgroups B with the property that B and $F(B)$ are in relative position w.

Chapter 8

FURTHER FAMILIES OF IRREDUCIBLE CHARACTERS

We shall show in this chapter how the Deligne–Lusztig generalized characters can be used to obtain further families of irreducible characters of G^F which are not themselves of the form $\varepsilon_G \varepsilon_T R_{T,\theta}$. G will continue to denote a connected reductive group. However we shall assume in addition in this chapter that the centre Z of G is connected. This condition is necessary to obtain the desired results.

8.1 THE GELFAND–GRAEV CHARACTER

Let B be an F-stable Borel subgroup of G and $B = UT$ where U is the unipotent radical of B and T is an F-stable maximal torus of B. Thus T is a maximally split torus of G. Now $U = \prod_{\alpha \in \Phi^+} X_\alpha$ with uniqueness and we define U^* by $U^* = \prod_{\substack{\alpha \in \Phi^+ \\ \alpha \notin \Delta}} X_\alpha$. Then the commutator relations show that U^* is a normal subgroup of U and that U/U^* is abelian. Since the Frobenius map F permutes the root subgroups X_α for $\alpha \in \Phi^+$ and also the root subgroups X_α for $\alpha \in \Delta$ we see that U^* is F-stable. Thus F acts on the quotient group U/U^*. Now U/U^* is isomorphic to the direct product $X_{\alpha_1} \times \ldots \times X_{\alpha_l}$ of the simple root subgroups and $U^F/(U^*)^F$ is isomorphic to the direct product of groups $X_J{}^F$, one for each ρ-orbit J on I, where X_J is the direct product of the X_{α_i} for $i \in J$.

Proposition 8.1.1. *There is a bijection between one-dimensional complex representations of U^F with $(U^*)^F$ in the kernel and sets (σ_J) where σ_J is a one-dimensional representation of $X_J{}^F$ and J runs over all ρ-orbits on I.*

Proof. This follows from the fact that $U^F/(U^*)^F$ is isomorphic to the direct product of the $X_J{}^F$.

Definition. A one-dimensional representation of U^F which contains $(U^*)^F$ in its kernel is called nondegenerate if the characters σ_J of 8.1.1 all satisfy $\sigma_J \neq 1$.

Now T^F acts on U^F by conjugation and $(U^*)^F$ is invariant under this T^F-action. Thus T^F acts on $U^F/(U^*)^F$, so also on the set of one-dimensional representations of $U^F/(U^*)^F$.

Proposition 8.1.2. (i) *The image of a nondegenerate character of* $U^F/(U^*)^F$ *under an element of* T^F *is nondegenerate.*

(ii) *Given any pair* σ, σ' *of nondegenerate characters of* $U^F/(U^*)^F$ *there is an element* $t \in T^F$ *which transforms* σ *into* σ'.

Proof. The assumption that Z is connected is essential for part (ii) of this proposition.

Since T^F leaves each subgroup $X_J{}^F$ of $U^F/(U^*)^F$ invariant it is clear that the image of a nondegenerate character of $U^F/(U^*)^F$ under an element of T^F is nondegenerate.

In order to prove (ii) we first consider the case when G is semisimple of adjoint type. In this case we know from section 2.9 rather explicitly how T^F acts on $U^F/(U^*)^F$. We recall that T^F is a direct product of subgroups $T_J{}^F$, one for each ρ-orbit J on I. $T_J{}^F$ acts trivially on $X_{J'}{}^F$ if $J \neq J'$. If $J = J'$ the action of $T_J{}^F$ on $X_J{}^F$ can be described as follows. There is an isomorphism $\lambda \to x_J(\lambda)$ from the finite field $F_{q^{|J|}}$ to $X_J{}^F$ and an isomorphism $\mu \to t_J(\mu)$ from $F_{q^{|J|}}{}^*$ to $T_J{}^F$ such that

$$t_J(\mu)x_J(\lambda)t_J(\mu)^{-1} = x_J(\lambda\mu).$$

We wish to show that given any pair σ, σ' of nondegenerate characters of $U^F/(U^*)^F$ there is an element of T^F transforming one to the other. Since $T_J{}^F$ acts trivially on $X_{J'}{}^F$ if $J \neq J'$ it is sufficient to show that given characters $\sigma_J \neq 1$, $\sigma'_J \neq 1$ of $X_J{}^F$ there is an element of $T_J{}^F$ transforming one to the other.

We show that if σ_J is a character of $X_J{}^F$ and $t \in T_J{}^F$ such that ${}^t\sigma_J = \sigma_J$ then either $t = 1$ or $\sigma_J = 1$. Let $t = t_J(\mu)$. Then we have

$${}^t\sigma_J(x_J(\lambda)) = \sigma_J(x_J(\lambda)) \quad \text{for all } \lambda \in F_{q^{|J|}}.$$

Thus

$$\sigma_J(x_J(\lambda)^t) = \sigma_J(x_J(\lambda))$$

and so

$$\sigma_J(x_J(\lambda\mu^{-1})) = \sigma_J(x_J(\lambda)) \quad \text{for all } \lambda \in F_{q^{|J|}}.$$

This gives

$$\sigma_J(x_J(\lambda(\mu^{-1} - 1))) = 1 \quad \text{for all } \lambda \in F_{q^{|J|}}.$$

If $t \neq 1$ then $\mu^{-1} - 1 \neq 0$ and so $\lambda(\mu^{-1} - 1)$ runs through all elements of $F_{q^{|J|}}$ as λ does. Hence $\sigma_J = 1$. Thus either $t = 1$ or $\sigma_J = 1$.

Let us now begin with a character $\sigma_J \neq 1$ of $X_J{}^F$ and transform it in turn by all the elements of $T_J{}^F$. Then we shall obtain $q^{|J|} - 1$ distinct non-identity characters of $X_J{}^F$. Thus each such character σ'_J has the form ${}^t\sigma_J$ for some $t \in T_J{}^F$. It follows that given any pair σ, σ' of nondegenerate characters of $U^F/(U^*)^F$ there is an element of T^F transforming one to the other.

Next let G be a group with $Z = 1$ which is not necessarily adjoint. Then there is a surjective map $G \xrightarrow{\theta} G_{\mathrm{ad}}$ from G into the adjoint group of the same type as G. The kernel of this map is the centre of G. Since $Z = 1$ the above map is a bijective homomorphism. There is a unique F-action on G_{ad} which is compatible with the given F-action on G under the above bijection θ. Let σ, σ' be nondegenerate characters of $U^F/(U^*)^F$. These map under θ to nondegenerate characters $\theta(\sigma)$, $\theta(\sigma')$ of $\theta(U)^F/(\theta(U^*))^F$. Since G_{ad} is of adjoint type there exists $\theta(t) \in \theta(T^F)$ which transforms $\theta(\sigma)$ to $\theta(\sigma')$. The element t then lies in T^F and transforms σ to σ'.

Finally let G be any connected reductive group for which Z is connected. Then G/Z is a group with trivial centre. Also we have

$$ZU/Z = U(G/Z), \qquad ZU^*/Z = U^*(G/Z)$$

and

$$U^F(G/Z)/U^{*F}(G/Z) \cong U^F/U^{*F}.$$

Since G/Z has trivial centre any two nondegenerate characters of U^F/U^{*F} will be conjugate by some element of $(T/Z)^F$. However $(T/Z)^F$ is isomorphic to T^F/Z^F since Z is connected. Thus there exists an element of T^F which transforms one of the given nondegenerate characters into the other.

Definition. Let σ be a nondegenerate character of U^F with U^{*F} in its kernel. Then the induced character $\Gamma = \sigma^{G^F}$ is independent of the choice of σ, by 8.1.2. Γ is called the Gelfand–Graev character of G^F.

The Gelfand–Graev character Γ is not an irreducible character of G^F, and we now prove a theorem of basic importance about the way in which Γ decomposes into irreducible components. This theorem was proved by Gelfand and Graev [1] in certain special cases, by Yokonuma [2], [3] for split groups G^F, and subsequently by Steinberg [15], theorem 49, in the general case.

Theorem 8.1.3. *All the irreducible components of the Gelfand–Graev character Γ of G^F occur with multiplicity 1.*

Proof. (i) $\Gamma = \sigma^{G^F}$ where σ is a nondegenerate character of U^F with $(U^*)^F$ in its kernel. Let $\mathbb{C}U^F$ be the group algebra of U^F over $\mathbb{C}$ and let $e \in \mathbb{C}U^F$ be the element

$$e = \frac{1}{|U^F|} \sum_{u \in U^F} \sigma(u^{-1})u.$$

Then $e^2 = e$ and $ue = \sigma(u)e$ for all $u \in U^F$. Thus e is idempotent and $\mathbb{C}U^F e$ is a one-dimensional left U^F-module affording the representation σ. The induced module

$$\mathbb{C}G^F \otimes_{\mathbb{C}U^F} \mathbb{C}U^F e \cong \mathbb{C}G^F e$$

is a G^F-module which affords the representation Γ. Let $\mathfrak{E}$ be the algebra of endomorphisms of this induced module. Then we have

$$\mathfrak{E} = \mathrm{End}_{\mathbb{C}G^F} \mathbb{C}G^F e \cong e\mathbb{C}G^F e$$

by Curtis and Reiner [1], p. 177. Now $\mathfrak{E}$ is isomorphic to a direct sum of complete matrix algebras over $\mathbb{C}$ of degrees equal to the multiplicities of the irreducible components of Γ (Curtis and Reiner [2], 11.25). Thus Γ will contain all its irreducible components with multiplicity 1 if and only if $\mathfrak{E}$ is commutative. It will therefore be sufficient to show that $\mathfrak{E}$ is commutative.

(ii) We shall prove that $\mathfrak{E}$ is commutative by showing the existence of a bijective map $\psi: G^F \to G^F$ satisfying the conditions

$$\begin{aligned} \psi(g_1 g_2) &= \psi(g_2)\psi(g_1) && g_1, g_2 \in G^F \\ \psi(U^F) &= U^F \\ \sigma(\psi(u)) &= \sigma(u) && \text{for all } u \in U^F \\ \psi(n) &= n && \text{for all } n \in N^F \text{ for which } ene \neq 0. \end{aligned}$$

Suppose that such a map ψ exists. It is an antiautomorphism of G^F. It can be extended by linearity to a map $\psi: \mathbb{C}G^F \to \mathbb{C}G^F$. It then satisfies $\psi(e) = e$ since we have

$$\begin{aligned} \psi(e) &= \frac{1}{|U^F|}\sum_{u \in U^F} \sigma(u^{-1})\psi(u) = \frac{1}{|U^F|}\sum_{u \in U^F} \sigma((\psi^{-1}(u))^{-1})u \\ &= \frac{1}{|U^F|}\sum_{u \in U^F} \sigma(\psi^{-1}(u^{-1}))u = \frac{1}{|U^F|}\sum_{u \in U^F} \sigma(u^{-1})u = e. \end{aligned}$$

Hence ψ can be restricted to $e\mathbb{C}G^F e$ to give a map $\psi: e\mathbb{C}G^F e \to e\mathbb{C}G^F e$. Now we have

$$G^F = B^F N^F B^F = U^F T^F N^F T^F U^F = U^F N^F U^F.$$

Moreover $eunu'e = \sigma(u)\sigma(u')ene$ for $u, u' \in U^F$, $n \in N^F$ and so $e\mathbb{C}G^F e$ is spanned by elements of the form ene for $n \in N^F$. Suppose $ene \neq 0$. Then $\psi(ene) = \psi(e)\psi(n)\psi(e) = ene$. Thus ψ fixes all nonzero elements of the form ene for $n \in N^F$. Since such elements span $e\mathbb{C}G^F e$ we see that ψ acts as the identity on $e\mathbb{C}G^F e$.

Let $a, b \in e\mathbb{C}G^F e$. Then we have

$$ab = \psi(ab) = \psi(b)\psi(a) = ba.$$

Thus $e\mathbb{C}G^F e$ is commutative as required.

(iii) We must therefore prove the existence of an antiautomorphism ψ of G^F which fixes U^F, fixes the character σ of U^F, and fixes each $n \in N^F$ for which $ene \neq 0$. We shall define ψ by

$$\psi(g) = \delta\gamma\iota(g) \qquad g \in G^F$$

where $\iota(g) = g^{-1}$, δ is conjugation by a certain element $t \in T^F$, and γ is the opposition graph automorphism of G^F. γ is obtained as follows. The map $\alpha \to -w_0(\alpha)$ is a permutation of the set of simple roots of G. Let $\bar{\alpha} = -w_0(\alpha)$. Then there is a graph automorphism of the semisimple group G' given by $x_\alpha(\lambda) \to x_{\bar{\alpha}}(\lambda)$ and $x_{-\alpha}(\lambda) \to x_{-\bar{\alpha}}(\lambda)$ for all $\alpha \in \Delta$ and $\lambda \in K$ (see Carter [3], p. 201 or Steinberg [15], p. 154). This automorphism acts as $z \to z^{-1}$ on the centre of G' (Tits [17], p. 37). It may therefore be extended to an automorphism of G satisfying $z \to z^{-1}$ for all $z \in Z$. G^F is invariant under this graph automorphism since w_0 is F-stable. γ is then defined as the restriction of this automorphism to G^F. γ fixes U^F and T^F. It acts on T^F by the rule

$$\gamma(x) = n_0 x^{-1} n_0^{-1} \qquad x \in T^F.$$

It follows that $\psi(g) = t\gamma(g^{-1})t^{-1}$ for all $g \in G^F$. In particular we see that $\psi(U^F) = U^F$ and that ψ is an antiautomorphism of G^F. We have not yet specified the conjugating element $t \in T^F$. We choose $t \in T^F$ so that $\psi(\sigma) = \sigma$. This is possible since by 8.1.2 any two nondegenerate characters of U^F with $(U^*)^F$ in the kernel are conjugate by an element of T^F. Both the maps $u \to \sigma(u)$ and $u \to \sigma(\gamma(u^{-1}))$ are nondegenerate characters of U^F and an element $t \in T^F$ transforming one to the other will give rise to a map ψ satisfying $\psi(\sigma) = \sigma$.

(iv) It remains to show that ψ fixes each $n \in N^F$ for which $ene \neq 0$. Let n be such an element and let $w = \pi(n) \in W^F$. Now we have

$$U^F = \prod_{A>0} X_A{}^F$$

with uniqueness where the root subgroups $X_A{}^F$ can be taken in any order, as in section 1.18. There is a corresponding factorization of the idempotent

$$e = \frac{1}{|U^F|} \sum_{u \in U^F} \sigma(u^{-1})u$$

as $e = \prod_{A>0} e_A$ where

$$e_A = \frac{1}{|X_A{}^F|} \sum_{u \in X_A{}^F} \sigma(u^{-1})u.$$

The equivalence classes A of roots of G are in bijective correspondence with the roots of G^F, as described in section 1.18, and W^F is the Weyl group of the root system of G^F.

Let us choose an equivalence class A which corresponds to a simple root of G^F, and suppose $w(A) > 0$ where $w \in W^F$ is the element defined above. Then we have

$$ene = \prod_{A'>0} e_{A'} . n . \prod_{A'>0} e_{A'} = \prod_{\substack{A'>0 \\ A' \neq w(A)}} e_{A'} . e_{w(A)} n e_A . \prod_{\substack{A'>0 \\ A' \neq A}} e_{A'} .$$

Since $ene \neq 0$ we must have $e_{w(A)} n e_A \neq 0$. Hence $e_{w(A)} n e_A n^{-1} \neq 0$. Since

$$e_A = \frac{1}{|X_A{}^F|} \sum_{u \in X_A{}^F} \sigma(u^{-1})u$$

we have

$$ne_A n^{-1} = \frac{1}{|X_A{}^F|} \sum_{u \in X_A{}^F} \sigma(u^{-1}) nun^{-1}$$
$$= \frac{1}{|X_{w(A)}{}^F|} \sum_{u \in X_{w(A)}{}^F} \sigma((n^{-1}un)^{-1})u.$$

Thus $ne_A n^{-1}$ is the idempotent giving rise to the one-dimensional representation ${}^n\sigma$ of $X_{w(A)}{}^F$. Also $e_{w(A)}$ is the idempotent giving rise to the one-dimensional representation σ of $X_{w(A)}{}^F$. Since

$$e_{w(A)} . ne_A n^{-1} \neq 0$$

we must have ${}^n\sigma = \sigma$ on $X_{w(A)}{}^F$, as distinct one-dimensional representations would have orthogonal idempotents (Curtis and Reiner [1], p. 234).

Now $\sigma \neq 1$ on $X_A{}^F$ since A corresponds to a simple root of G^F and σ is nondegenerate. Thus ${}^n\sigma \neq 1$ on $X_{w(A)}{}^F$. It follows that $\sigma \neq 1$ on $X_{w(A)}{}^F$. Since $(U^*)^F$ lies in the kernel of σ this implies that $w(A)$ corresponds to a simple root of G^F.

Thus the element $w \in W^F$ has the property that for all simple roots α of G^F for which $w(\alpha) > 0$ $w(\alpha)$ must be simple also. Let K' be the set of simple roots α of G^F such that $w(\alpha) > 0$. Let $(w_0)_{K'}$ be the element of maximal length in $(W^F)_{K'}$. Consider the element $w(w_0)_{K'} \in W^F$. We consider the effect of this element on the simple roots α of G^F. If $\alpha \in K'$ then $-(w_0)_{K'}(\alpha) \in K'$ and so $w(w_0)_{K'}(\alpha) < 0$. If $\alpha \notin K'$ then $(w_0)_{K'}(\alpha) = \alpha + \beta$ where β is a linear combination of roots in K'. Thus $w(w_0)_{K'}(\alpha) = w(\alpha) + \gamma$ where γ is a linear combination of roots in $w(K')$ and $w(\alpha)$ is a negative root not in $w(K')$. Thus $w(w_0)_{K'}(\alpha) < 0$ in this case also. Thus $w(w_0)_{K'}$ makes all simple roots of G^F negative. It follows that $w(w_0)_{K'}$ is equal to the element of maximal legnth in W^F, and this is equal to the element w_0 of maximal length in W. Hence $w(w_0)_{K'} = w_0$ and so $w = w_0(w_0)_{K'}$.

(v) We recall that W^F is generated as a Coxeter group by the elements s_J as J runs over the ρ-orbits on I. s_J is the unique element of W_J satisfying $s_J(\Phi_J{}^+) = \Phi_J{}^-$.

Now $w_0(\Delta) = -\Delta$ and so $w_0(\Delta_J) = -\Delta_{J^*}$ for some subset J^* of I. It follows that $w_0 W_J w_0{}^{-1} = W_{J^*}$ and that $w_0 s_J w_0 = s_{J^*}$. Let n_J be an element of N^F such that $\pi(n_J) = s_J$. We shall show that the coset representatives $n_J \in N^F$ can be chosen so that $\psi(n_J) = n_{J^*}$ for all ρ-orbits J on I.

For each ρ-orbit J on I we have a corresponding root subgroup $X_{A(J)}{}^F$ of G^F. Here $A(J)$ is the equivalence class of roots which contains J. (It may be larger than J). We introduce an equivalence relation on the ρ-orbits J of I as follows. This is the equivalence relation generated by the relations (a), (b):

(a) J_1 is equivalent to J_2 if there exists $n \in N^F$ with $nX_{A(J_1)}{}^F n^{-1} = X_{A(J_2)}{}^F$ and $\sigma(nxn^{-1}) = \sigma(x)$ for all $x \in X_{A(J_1)}{}^F$.

(b) J_1 is equivalent to J_2 if $\psi(X_{A(J_1)}{}^F) = X_{A(J_2)}{}^F$.

Considering the equivalence relation generated by (a), (b) we select one ρ-orbit J in each equivalence class. We then choose a coset representative $n_J \in N^F$ corresponding to it in the following way.

Suppose $J = J^*$. Then $\psi(X_{A(J)}{}^F) = X_{A(J)}{}^F$. Let $X_{A(J)}{}^* = X_{A(J)} \cap U^*$. Then $\psi(X_{A(J)}{}^{*F}) = X_{A(J)}{}^{*F}$. Now $X_{A(J)}{}^F/X_{A(J)}{}^{*F}$ is isomorphic to the group $X_J{}^F$ of 8.1.1, and as in the proof of 8.1.2 we know that $X_J{}^F$ is isomorphic to the additive group of the field $F_{q^{|J|}}$. Moreover the definition of ψ shows that ψ acts on $X_{A(J)}{}^F/X^*{}_{A(J)}{}^F$ by a map of $F_{q^{|J|}}$ into itself given by $\lambda \to \mu\lambda$ for some $\mu \in F_{q^{|J|}}{}^*$. We also know that ψ fixes a non-identity character σ_J of this group. This can only happen if $\mu = 1$. Thus we see that ψ acts trivially on $X_{A(J)}{}^F/X^*{}_{A(J)}{}^F$.

We wish to show that there is a non-identity element $x \in X_{A(J)}{}^F$ with $\psi(x) = x$. This is obvious if $X^*_{A(J)}{}^F = 1$. Otherwise $X_{A(J)}{}^F$ is a twisted root subgroup of G^F of type A_2, B_2 or G_2 and ψ is given by $\psi(g) = tg^{-1}t^{-1}$. It is then not difficult to check the existence of a non-identity element fixed by ψ in each case separately. (This is done by Steinberg in [15], p. 260.) Now we have

$$\langle X_{A(J)}{}^F, X_{-A(J)}{}^F\rangle \subseteq X_{-A(J)}{}^F S^F \cup X_{-A(J)}{}^F S^F n_J X_{-A(J)}{}^F$$

where $S = T \cap \langle X_{A(J)}, X_{-A(J)}\rangle$, by the Bruhat decomposition. Now x lies in $\langle X_{A(J)}{}^F, X_{-A(J)}{}^F\rangle$ but not in $X_{-A(J)}{}^F S^F$ since $x \neq 1$. So x lies in $X_{-A(J)}{}^F S^F n_J X_{-A(J)}{}^F$ and we may choose the coset representative $n_J \in N^F$ such that $x = x_1 n_J x_2$ where $x_1, x_2 \in X_{-A(J)}{}^F$. In fact n_J is uniquely determined by x in this way. We have

$$\psi(x) = x \qquad \psi(X_{-A(J)}{}^F) = X_{-A(J)}{}^F$$

and it follows that $\psi(n_J) = n_J$.

Now suppose $J \neq J^*$. We then choose any non-identity element $x \in X_{A(J)}{}^F$. We choose n_J to be the unique coset representative in N^F satisfying $x = x_1 n_J x_2$ with x_1, $x_2 \in X_{-A(J)}{}^F$.

We have now chosen representatives $n_J \in N^F$ for one ρ-orbit J in each equivalence class. For the remaining ρ-orbits J we choose n_J as follows. Given a ρ-orbit J there is a unique ρ-orbit J_1 in the equivalence class containing J for which n_{J_1} has already been chosen. We may pass from $\langle X_{A(J_1)}{}^F, X_{-A(J_1)}{}^F\rangle$ to $\langle X_{A(J)}{}^F, X_{-A(J)}{}^F\rangle$ by a succession of transformations of type (a) or (b) above and we define n_J to be the image of n_{J_1} under the composite map. We must, however, check that the result is independent of the choice of this composite map. So suppose we have such a composite map ϕ transforming $\langle X_{A(J)}{}^F, X_{-A(J)}{}^F\rangle$ into itself. ϕ is either an automorphism or an antiautomorphism. We have

$$\phi(X_{A(J)}{}^F) = X_{A(J)}{}^F, \qquad \phi(X_{-A(J)}{}^F) = X_{-A(J)}{}^F.$$

Now ϕ acts on $X_{A(J)}{}^F/X_{A(J)}{}^{*F}$ and $X_{A(J)}{}^F/X_{A(J)}{}^{*F}$ is isomorphic to the additive group of the field $F_{q^{|J|}}$. Moreover the definition of ϕ shows that ϕ acts on $X_{A(J)}{}^F/X_{A(J)}{}^{*F}$ by a map of $F_{q^{|J|}}$ into itself given by $\lambda \to \mu\lambda$ for some $\mu \in F_{q^{|J|}}{}^*$. Since ϕ fixes a nonprincipal character σ_J of $X_{A(J)}{}^F/X_{A(J)}{}^{*F}$ we must have $\mu = 1$, so ϕ induces the identity map on $X_{A(J)}{}^F/X_{A(J)}{}^{*F}$. If $X_{A(J)}{}^{*F} = 1$ then ϕ fixes each element of $X_{A(J)}{}^F$. Thus $\phi(x) = x$ where x is the non-identity element of $X_{A(J)}{}^F$ defined above. Since $x = x_1 n_J x_2$ with $x_1, x_2 \in X_{-A(J)}{}^F$ we have $\phi(n_J) = n_J$ as required. If, on the other hand, we have $X_{A(J)}{}^{*F} \neq 1$ then $\langle X_{A(J)}{}^F, X_{-A(J)}{}^F\rangle$ is a twisted group and we have $J = J^*$. Then $\psi(n_J) = n_J$ as above. Now either ϕ or $\phi\psi$ is an automorphism of $\langle X_{A(J)}{}^F, X_{-A(J)}{}^F\rangle$ which fixes $X_{A(J)}{}^F$ and $X_{-A(J)}{}^F$ and induces the identity map on $X_{A(J)}{}^F/X_{A(J)}{}^{*F}$. Such an automorphism must be the identity (see Steinberg [15], p. 198). Thus either $\phi(n_J) = n_J$ or $\phi\psi(n_J) = n_J$. Since $\psi(n_J) = n_J$ we have $\phi(n_J) = n_J$ in both cases.

Thus we have chosen a system of coset representatives $n_J \in N^F$ such that $\psi(n_J) = n_{J^*}$ for all ρ-orbits J of I.

(vi) We now choose representatives $n_w \in N^F$ for all $w \in W^F$. Each $w \in W^F$ has a reduced expression

$$w = s_{J_1}s_{J_2}\ldots s_{J_k}.$$

Let $n_w = n_{J_1}n_{J_2}\ldots n_{J_k}$. Then n_w is independent of the reduced expression chosen for w, by section 1.18. Consider $\psi(n_w)$. We have

$$\begin{aligned}\psi(n_w) = \psi(n_{J_1}n_{J_2}\ldots n_{J_k}) &= \psi(n_{J_k})\ldots\psi(n_{J_1})\\ &= n_{J_k{}^*}\ldots n_{J_1{}^*} = n_{w^*}\end{aligned}$$

where $w^* = s_{J_k{}^*}\ldots s_{J_1{}^*} = w_0 s_{J_k}\ldots s_{J_1} w_0^{-1} = w_0 w^{-1} w_0^{-1}$. Hence $\psi(n_w) = n_{w^*}$ with $w^* = w_0 w^{-1} w_0{}^{-1}$ and we know the effect of ψ on the elements n_w for all $w \in W^F$.

(vii) Now let $n \in N^F$ satisfy $ene \neq 0$. Let $w = \pi(n) \in W^F$. Then $w = w_0(w_0)_{K'}$ for some set K' of simple roots of G^F, by (iv). Thus

$$w^* = w_0 w^{-1} w_0{}^{-1} = w_0(w_0)_{K'} = w.$$

It follows that $\psi(n_w) = n_w$.

We wish to show that $\psi(n) = n$ and so we now compare n with n_w. We have $n = n_w t'$ for some $t' \in T^F$. Let J be a ρ-orbit on I corresponding to a simple root of G^F which lies in K'. Then $w(J) = J_1$ also corresponds to a simple root of G^F since $w = w_0(w_0)_{K'}$. We have

$$nX_{A(J)}{}^F n^{-1} = X_{A(J_1)}{}^F$$

$$\sigma(nxn^{-1}) = \sigma(x) \quad \text{for } x \in X_{A(J)}{}^F$$

from (iv) above. Thus J and J_1 are equivalent ρ-orbits in the sense of (v). It follows that $nn_Jn^{-1} = n_{J_1}$ as in (v).

Now $w_0(w_0)_{K'}w_0 = (w_0)_{L'}$ for some set L' of simple roots of G^F. Define $(n_0)_{K'}$, $(n_0)_{L'}$, $n_0 \in N^F$ by

$$(n_0)_{K'} = n_{(w_0)_{K'}}, \quad (n_0)_{L'} = n_{(w_0)_{L'}}, \quad n_0 = n_{w_0}.$$

Since

$$w_0 = w(w_0)_{K'} = (w_0)_{L'}w$$

we have

$$n_0 = n_w(n_0)_{K'} = (n_0)_{L'}n_w.$$

Thus

$$n_w(n_0)_{K'}n_w^{-1} = (n_0)_{L'}.$$

We compare this expression with $n(n_0)_{K'}n^{-1}$. Let

$$(w_0)_{K'} = s_{J_1}s_{J_2}\dots s_{J_k}$$

be a reduced expression for $(w_0)_{K'}$ in W^F. Then

$$(n_0)_{K'} = n_{J_1}n_{J_2}\dots n_{J_k}$$

and so

$$n(n_0)_{K'}n^{-1} = n_{w(J_1)}n_{w(J_2)}\dots n_{w(J_k)}$$

since $nn_{J_i}n^{-1} = n_{w(J_i)}$ as above. However

$$(w_0)_{L'} = s_{w(J_1)}s_{w(J_2)}\dots s_{w(J_k)}$$

is a reduced expression for $(w_0)_{L'}$, and so we have

$$(n_0)_{L'} = n_{w(J_1)}n_{w(J_2)}\dots n_{w(J_k)}.$$

It follows that $n(n_0)_{K'}n^{-1} = (n_0)_{L'}$. We have now shown that

$$n(n_0)_{K'}n^{-1} = n_w(n_0)_{K'}n_w^{-1},$$

and hence that

$$t'(n_0)_{K'}t'^{-1} = (n_0)_{K'}.$$

It follows that

$$\begin{aligned}\psi(n) = \psi(n_wt') = \psi(t')\psi(n_w) &= n_0t'n_0^{-1}n_w\\ &= n_w(n_0)_{K'}t'(n_0)_{K'}^{-1} = n_wt' = n.\end{aligned}$$

Thus $\psi(n) = n$ for all $n \in N^F$ satisfying $ene \neq 0$. This completes the proof of the theorem. ■

We observe now that the condition that the centre of G is connected passes on to each Levi subgroup of G.

Proposition 8.1.4. *Let G be connected, reductive. If the centre of G is connected then the centre of L_J is connected for each Levi subgroup L_J of G.*

Proof. By 4.5.1 we know that $Z(G)$ is connected if and only if $X/\mathbb{Z}\Phi$ has no p'-torsion. Here X is the character group of a maximal torus T of G, which we may take to lie in L_J. We must therefore show that $X/\mathbb{Z}\Phi_J$ has no p'-torsion also. Now $\mathbb{Z}\Phi/\mathbb{Z}\Phi_J$ has no torsion, being a free abelian group. Let $A/\mathbb{Z}\Phi_J$ be the torsion subgroup of $X/\mathbb{Z}\Phi_J$. Then $A \cap \mathbb{Z}\Phi = \mathbb{Z}\Phi_J$ and so $A/\mathbb{Z}\Phi_J$ is isomorphic to $(\mathbb{Z}\Phi + A)/\mathbb{Z}\Phi$. This lies in the torsion subgroup of $X/\mathbb{Z}\Phi$, so its order must be a power of the characteristic p (or must be trivial if the characteristic is 0). Thus $Z(L_J)$ is connected also. ■

We may therefore consider the Gelfand–Graev character $\Gamma_{L_J{}^F}$ of each Levi subgroup $L_J{}^F$ of G^F. The following proposition due to Rodier gives a very useful connection between the characters Γ_{G^F} and $\Gamma_{L_J{}^F}$.

Proposition 8.1.5. (Rodier) *Let P_J be an F-stable standard parabolic subgroup of G and L_J an F-stable standard Levi subgroup of P_J. Let f be a generalized character of $L_J{}^F$, $f_{P_J{}^F}$ be the generalized character of $P_J{}^F$ which has $U_J{}^F$ in the kernel and which agrees with f on $L_J{}^F$, and $f_{P_J{}^F}{}^{G^F}$ be the induced character of G^F. Then*

$$(f_{P_J{}^F}{}^{G^F}, \Gamma_{G^F}) = (f, \Gamma_{L_J{}^F}).$$

Proof. Let σ be a nondegenerate linear character of U^F, so that $\Gamma_{G^F} = \sigma^{G^F}$. Then

$$\begin{aligned}(f_{P_J{}^F}{}^{G^F}, \Gamma_{G^F}) &= (f_{P_J{}^F}{}^{G^F}, \sigma^{G^F})\\ &= \sum_{r \in R} (f_{P_J{}^F}, {}^r\sigma)_{P_J{}^F \cap {}^r(U^F)}\end{aligned}$$

by Mackey's formula where R is a set of double coset representatives of G^F with respect to $P_J{}^F$ and U^F. Now

$$P_J{}^F r U^F \subseteq P_J{}^F r B^F = P_J{}^F \dot{w} B^F$$

for some $w \in W^F$. Thus $r \in P_J{}^F \dot{w} T^F U^F$. Hence

$$P_J{}^F r U^F = P_J{}^F T^F r U^F = P_J{}^F \dot{w} T^F U^F = P_J{}^F \dot{w} B^F.$$

Thus R is a set of double coset representatives of G^F with respect to $P_J{}^F$ and B^F. Now there is a bijection between the set of double cosets of G^F with respect to $P_J{}^F$ and B^F and double cosets of W^F with respect to $W_J{}^F$ and 1, i.e. distinguished right coset representatives of $W_J{}^F$ in W^F. Thus $R = \{\dot{w}; w^{-1} \in D_J\}$. Now we have

$$P_J{}^F \cap {}^{\dot{w}}(U^F) = (U_J{}^F \cap {}^{\dot{w}}(U^F))(L_J{}^F \cap {}^{\dot{w}}(U^F))$$

with uniqueness, by 2.8.6. Thus we have

$$(f, {}^{\dot{w}}\sigma)_{P_J{}^F \cap {}^{\dot{w}}(U^F)} = \frac{1}{|P_J{}^F \cap {}^{\dot{w}}(U^F)|} \sum_{x,y} f(xy)\overline{({}^{\dot{w}}\sigma(xy))}$$

where $x \in U_J{}^F \cap {}^{\dot{w}}(U^F)$, $y \in L_J{}^F \cap {}^{\dot{w}}(U^F)$

$$\begin{aligned}&= \frac{1}{|P_J{}^F \cap {}^{\dot{w}}(U^F)|} \sum_{x,y} f(y)\overline{({}^{\dot{w}}\sigma(x))}\,\overline{({}^{\dot{w}}\sigma(y))}\\ &= \frac{1}{|P_J{}^F \cap {}^{\dot{w}}(U^F)|} \sum_{x} \overline{({}^{\dot{w}}\sigma(x))} \sum_{y} f(y)\overline{({}^{\dot{w}}\sigma(y))}.\end{aligned}$$

Now ${}^{\dot{w}}\sigma$ may be regarded as a linear character of $U_J{}^F \cap {}^{\dot{w}}(U^F)$. We have

$$\sum_{x} \overline{({}^{\dot{w}}\sigma(x))} = 0$$

unless ${}^{\dot{w}}\sigma$ is the principal character on $U_J{}^F \cap {}^{\dot{w}}(U^F)$. Suppose this is the case. Then $\sigma = 1$ on $U^F \cap (U_J{}^F)^{\dot{w}} = \prod_A X_A{}^F$ where the product extends over all equivalence classes A of roots satisfying

$$A > 0, \qquad w(A) > 0, \qquad w(A) \cap \Phi_J = \varnothing.$$

Now $\sigma = 1$ on $\prod_A X_A{}^F$ if and only if no simple root subgroup $X_A{}^F$ is involved in this product, and this is equivalent to the condition that

$$\{\alpha \in \Phi, \alpha > 0, w(\alpha) > 0, w(\alpha) \notin \Phi_J\} \cap \Delta = \varnothing.$$

Thus for each $\alpha \in \Delta$ we have either $w(\alpha) < 0$ or $w(\alpha) \in \Phi_J{}^+$.

Consider the element $(w_0)_J w \in W$. Let $\alpha \in \Delta$. If $w(\alpha) > 0$ then $w(\alpha) \in \Phi_J{}^+$ and $(w_0)_J w(\alpha) < 0$. If $w(\alpha) < 0$ then $w(\alpha) \notin \Phi_J$. For if $w(\alpha) \in \Phi_J$ then, since $w^{-1} \in D_J$, we have $\alpha = w^{-1}(w(\alpha)) < 0$ which is false. Thus $w(\alpha) \notin \Phi_J$ and so $(w_0)_J w(\alpha) < 0$. Hence $(w_0)_J w(\alpha) < 0$ for all $\alpha \in \Delta$. This implies that $(w_0)_J w = w_0$, and so $w = (w_0)_J w_0$.

Thus the only possible nonzero term in our sum occurs when $w = (w_0)_J w_0$. Hence

$$(f_{P_J{}^F}{}^{G^F}, \Gamma_{G^F}) = (f, {}^{\dot{w}}\sigma)_{P_J{}^F \cap {}^{\dot{w}}(U^F)} = (f, {}^{\dot{w}}\sigma)_{L_J{}^F \cap {}^{\dot{w}}(U^F)}$$

where $w = (w_0)_J w_0$. However by writing U^F as a product of root subgroups we see that

$$L_J{}^F \cap {}^{(w_0)_J w_0}(U^F) = L_J{}^F \cap U^F$$

and this is a maximal unipotent subgroup of $L_J{}^F$. Thus we have

$$\begin{aligned}(f_{P_J{}^F}{}^{G^F}, \Gamma_{G^F}) &= (f, {}^{(w_0)_J w_0}\sigma)_{L_J{}^F \cap U^F} \\ &= (f, ({}^{(w_0)_J w_0}\sigma)^{L_J{}^F})_{L_J{}^F}\end{aligned}$$

by Frobenius reciprocity. We verify finally that ${}^{(w_0)_J w_0}\sigma$ is a nondegenerate character on $L_J{}^F \cap U^F$.

Let $X_A{}^F$ be a root subgroup in $L_J{}^F \cap U^F$. Then

$${}^{(w_0)_J w_0}\sigma \neq 1 \text{ on } X_A{}^F \quad \text{if and only if} \quad \sigma \neq 1 \text{ on } X^F_{w_0(w_0)_J(A)}$$

and the condition for this is that $w_0(w_0)_J(A)$ is simple. However this is equivalent to the condition that A is simple, and so ${}^{(w_0)_J w_0}\sigma$ is nondegenerate on $L_J{}^F \cap U^F$. Hence

$$({}^{(w_0)_J w_0}\sigma)^{L_J{}^F} = \Gamma_{L_J{}^F}$$

and so

$$(f_{P_J{}^F}{}^{G^F}, \Gamma_{G^F}) = (f, \Gamma_{L_J{}^F})$$

as required. ■

We now define an operation on the generalized characters of any finite group called truncation with respect to a normal subgroup. Let G be a finite group and N be a normal subgroup of G. Let ξ be a generalized character of G. Then the map $T_{G/N}(\xi)$ defined by

$$T_{G/N}(\xi)(g) = \frac{1}{|N|} \sum_{n \in N} \xi(ng) \qquad g \in G$$

is also a generalized character of G, and is called the truncation of ξ with respect to N. In order to see that $T_{G/N}(\xi)$ is a generalized character it is sufficient to observe that if ξ is a character of G so is $T_{G/N}(\xi)$. If V is a G-module affording the character ξ then the subspace V^N of elements invariant under N is a G-module affording the character $T_{G/N}(\xi)$. For let

$$e = \frac{1}{|N|} \sum_{n \in N} n.$$

Then $V = Ve \oplus V(1-e)$ and $Ve = V^N$. We have

$$\begin{aligned} T_{G/N}(\xi)(g) &= \xi(eg) = \text{trace}(eg, V) = \text{trace}(eg, Ve) \\ &= \text{trace}(g, Ve) = \text{trace}(g, V^N). \end{aligned}$$

A special case of this situation arises when we have a generalized character ξ of a parabolic subgroup $P_J{}^F$ of G^F. We can then form $T_{P_J{}^F/U_J{}^F}(\xi)$, which is also a generalized character of $P_J{}^F$. Now $P_J{}^F = U_J{}^F L_J{}^F$ and $T_{P_J{}^F/U_J{}^F}(\xi)$ is constant on the cosets of $U_J{}^F$ in $P_J{}^F$. $T_{P_J{}^F/U_J{}^F}(\xi)$ is therefore determined by its restriction $(T_{P_J{}^F/U_J{}^F}(\xi))_{L_J{}^F}$ to $L_J{}^F$. We shall therefore write $T_{L_J{}^F}(\xi) = (T_{P_J{}^F/U_J{}^F}(\xi))_{L_J{}^F}$. $T_{L_J{}^F}(\xi)$ is called the truncation of ξ to $L_J{}^F$.

Lemma 8.1.6. *Let P_J be an F-stable standard parabolic subgroup of G and L_J be an F-stable standard Levi subgroup of P_J. Then the following three characters of $P_J{}^F$ are equal:*

(i) $T_{P_J{}^F/U_J{}^F}(\Gamma_{G^F})$.

(ii) $\Gamma_{L_J{}^F}$ *lifted to $P_J{}^F$ with $U_J{}^F$ in the kernel.*

(iii) $\sigma_{U^F}{}^{P_J{}^F}$, *where σ is a linear character of U^F with U^{*F} in the kernel such that $\sigma \neq 1$ on the simple root subgroups of U^F corresponding to roots in $J' = J/\rho$ and $\sigma = 1$ on the simple root subgroups of U^F corresponding to simple roots not in J'.*

Proof. Consider the restriction $\sigma_{L_J{}^F \cap U^F}$ of σ to $L_J{}^F \cap U^F$. This is a maximal unipotent subgroup of $L_J{}^F$ and $\sigma_{L_J{}^F \cap U^F}$ is a nondegenerate linear character of it. Thus $\sigma_{L_J{}^F \cap U^F}{}^{L_J{}^F} = \Gamma_{L_J{}^F}$. Now we have

$$U^F = U_J{}^F(L_J{}^F \cap U^F), \qquad P_J{}^F = U_J{}^F L_J{}^F$$

with uniqueness. We may lift $\sigma_{L_J{}^F \cap U^F}$ to U^F taking $U_J{}^F$ in the kernel, and the result is σ. It follows that $(\sigma_{L_J{}^F \cap U^F})^{L_J{}^F}$, when lifted to $P_J{}^F$ with $U_J{}^F$ in the kernel, is equal to $\sigma^{P_J{}^F}$. Thus $\Gamma_{L_J{}^F}$ lifted to $P_J{}^F$ is equal to $\sigma_{U^F}{}^{P_J{}^F}$.

We now consider the truncation $T_{P_J{}^F/U_J{}^F}(\Gamma_{G^F})$. This character of $P_J{}^F$ has $U_J{}^F$ in its kernel, so to show it is equal to $\Gamma_{L_J{}^F}$ lifted to $P_J{}^F$ it is sufficient to show that $T_{L_J{}^F}(\Gamma_{G^F})$ agrees with $\Gamma_{L_J{}^F}$. To see this, let ϕ be any irreducible character of $L_J{}^F$ and $\phi_{P_J{}^F}$ be the character of $P_J{}^F$ obtained by lifting ϕ. Then

$$\begin{aligned}(T_{L_J{}^F}(\Gamma_{G^F}), \phi) &= \frac{1}{|L_J{}^F|} \sum_{x \in L_J{}^F} \left(\frac{1}{|U_J{}^F|} \sum_{u \in U_J{}^F} \Gamma_{G^F}(ux) \right) \overline{\phi(x)} \\ &= \frac{1}{|P_J{}^F|} \sum_{p \in P_J{}^F} \Gamma_{G^F}(p) \overline{\phi_{P_J{}^F}(p)} \\ &= (\Gamma_{G^F}, \phi_{P_J{}^F})_{P_J{}^F} = (\Gamma_{G^F}, \phi_{P_J{}^F}{}^{G^F})\end{aligned}$$

by Frobenius reciprocity. We now apply 8.1.5. This gives

$$(\Gamma_{G^F}, \phi_{P_J{}^F}{}^{G^F}) = (\Gamma_{L_J{}^F}, \phi).$$

Hence we have $(T_{L_J{}^F}(\Gamma_{G^F}), \phi) = (\Gamma_{L_J{}^F}, \phi)$ for all irreducible characters ϕ of $L_J{}^F$. It follows that $T_{L_J{}^F}(\Gamma_{G^F}) = \Gamma_{L_J{}^F}$ as required.

Proposition 8.1.7. *The generalized character*

$$\sum_J (-1)^{|J'|} (T_{P_J{}^F/U_J{}^F}(\Gamma))^{G^F}$$

summed over all ρ-stable subsets J of I takes the value $|Z^F|q^l$ on regular unipotent elements of G^F and value 0 on all other elements of G^F. (Here $J' = J/\rho$.)

Proof. Let σ be a nondegenerate linear character of U^F, so that $\Gamma = \sigma^{G^F}$. Now U^F is a product of root subgroups $X_A{}^F$ with $A > 0$. For each ρ-stable subset J of I we define a linear character σ_J of U^F as follows.

$\sigma_J = \sigma$ on $X_A{}^F$, if $X_A{}^F$ is the root subgroup corresponding to a simple root in J'.

$\sigma_J = 1$ on $X_A{}^F$, for all other root subgroups of U^F. Then we have

$$T_{P_J{}^F/U_J{}^F}(\Gamma) = \sigma_J{}^{P_J{}^F} \quad \text{by 8.1.6.}$$

Hence

$$\sum_J (-1)^{|J'|} (T_{P_J{}^F/U_J{}^F}(\Gamma))^{G^F} = \sum_J (-1)^{|J'|} \sigma_J{}^{G^F}.$$

This function takes value 0 at all elements of G^F which are not unipotent. So suppose $u \in U^F$. Then $u = \prod_{A>0} u_A$ where $u_A \in X_A{}^F$. Now

$$\begin{aligned}\left(\sum_J (-1)^{|J'|} \sigma_J \right)(u) &= \sum_J (-1)^{|J'|} \sigma_J \left(\prod_{A>0} u_A \right) \\ &= \sum_J (-1)^{|J'|} \sigma_J \left(\prod_K u_{A(K)} \right)\end{aligned}$$

where K runs over all the ρ-orbits on I and $A(K)$ is the equivalence class of roots containing K. Hence

$$\left(\sum_J (-1)^{|J'|}\sigma_J\right)(u) = \sum_J (-1)^{|J'|} \prod_K \sigma_J(u_{A(K)})$$

$$= \sum_J (-1)^{|J'|} \prod_{K \subseteq J} \sigma_J(u_{A(K)})$$

$$= \sum_J (-1)^{|J'|} \prod_{K \subseteq J} \sigma_K(u_{A(K)})$$

$$= \prod_K (1 - \sigma_K(u_{A(K)})).$$

This takes value 0 unless all $u_{A(K)} \neq 1$. If all $u_{A(K)} \neq 1$ then the element u is regular unipotent. (See the remark at the end of section 2.9 and also 5.1.3.) Thus $\sum_J (-1)^{|J'|}\sigma_J$ vanishes on all unipotent elements which are not regular, and the same therefore applies to the induced character $\sum_J (-1)^{|J'|}{\sigma_J}^{G^F}$.

Now let u be a regular unipotent element of U^F and J be a fixed ρ-stable subset of I. Consider the set of all linear characters σ_J of U^F such that $\sigma_J \neq 1$ on the root subgroups ${X_{A(K)}}^F$ for $K \subseteq J$ and $\sigma_J = 1$ on all other root subgroups ${X_A}^F$ of U^F. Consider the sum

$$\sum_{\sigma_J} \sigma_J(u)$$

over all such linear characters σ_J. We have

$$\sigma_J(u) = \prod_{K \subseteq J} \sigma_J(u_K)$$

and so

$$\sum_{\sigma_J} \sigma_J(u) = \sum_{\sigma_J} \prod_{K \subseteq J} \sigma_J(u_K) = \sum_{\substack{\sigma_K \neq 1 \\ K \subseteq J}} \prod_{K \subseteq J} \sigma_K(u_K)$$

$$= \prod_{K \subseteq J} \sum_{\sigma_K \neq 1} \sigma_K(u_K) = \prod_{K \subseteq J} (-1) = (-1)^{|J'|}.$$

Thus we have

$$\sum_J (-1)^{|J'|}{\sigma_J}^{G^F}(u) = \sum_J \sum_{\sigma_J} \sigma_J(u){\sigma_J}^{G^F}(u)$$

$$= \sum_\sigma \sigma(u)\sigma^{G^F}(u)$$

summed over all linear characters σ of U^F with U^{*F} in the kernel

$$= \sum_\sigma \frac{1}{|U^F|} \sum_{\substack{g \in G^F \\ g^{-1}ug \in U^F}} \sigma(u)\sigma(g^{-1}ug)$$

$$= \frac{1}{|U^F|} \sum_{\substack{g \in G^F \\ g^{-1}ug \in U^F}} \sum_\sigma \sigma(ug^{-1}ug).$$

Now

$$\sum_{\sigma} \sigma(ug^{-1}ug) = \begin{cases} |U^F/U^{*F}| & \text{if } ug^{-1}ug \in U^{*F} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\sum_J (-1)^{|J'|} \sigma_J{}^{G^F}(u) = \frac{1}{|U^F|} |U^F : U^{*F}| \sum_{\substack{g \in G^F \\ g^{-1}ug \in U^F \\ ug^{-1}ug \in U^{*F}}} 1$$

$$= \frac{1}{|U^{*F}|} \times \text{the number of } g \in G^F \text{ such that } g^{-1}ug \in U^F \text{ and } ug^{-1}ug \in U^{*F}.$$

So let $g \in G^F$ be such that $g^{-1}ug \in U^F$ and $ug^{-1}ug \in U^{*F}$. By 5.1.3 u lies in a unique Borel subgroup of G. Since $u \in B \cap gBg^{-1}$ we have $gBg^{-1} = B$ and so $g \in B^F$. Thus we can write $g = tu'$ where $t \in T^F$ and $u' \in U^F$. Such an element automatically satisfies $g^{-1}ug \in U^F$. It satisfies $ug^{-1}ug \in U^{*F}$ if and only if

$$uu'^{-1}t^{-1}utu' \in U^{*F}$$

which is equivalent to

$$t^{-1}ut \in u^{-1}U^{*F}.$$

This condition is independent of u'. So consider which elements $t \in T^F$ satisfy $t^{-1}ut \in u^{-1}U^{*F}$. Now both u and u^{-1} are regular unipotent elements so, by 5.1.8, the cosets uU^{*F} and $u^{-1}U^{*F}$ are conjugate under the action of T^F on U^F/U^{*F}, since Z is connected. Thus there exists $t_0 \in T^F$ with $t_0{}^{-1}ut_0 \in u^{-1}U^{*F}$.

Let $t \in T^F$ be any other such element. Then by 5.1.8 applied to G/Z we see that $Zt_0 = Zt$. Thus $tt_0{}^{-1} \in Z \cap T^F = Z^F$. The element $t \in T^F$ can therefore be chosen in $|Z^F|$ ways. It follows that

$$\sum_J (-1)^{|J'|} \sigma_J{}^{G^F}(u) = \frac{1}{|U^{*F}|} \cdot |U^F| \cdot |Z^F| = |Z^F| q^l$$

since $|U^F : U^{*F}| = q^l$.

8.2 DUALITY OF GENERALIZED CHARACTERS

In this section we consider the map of generalized characters of G^F given by

$$\xi \to \sum_J (-1)^{|J'|} (T_{P_J{}^F/U_J{}^F}(\xi))^{G^F}$$

where the sum extends over all ρ-stable subsets J of I and where $J' = J/\rho$. Given any generalized character ξ of G^F we define ξ^* by

$$\xi^* = \sum_J (-1)^{|J'|} (T_{P_J{}^F/U_J{}^F}(\xi)^{G^F}.$$

Then ξ^* is also a generalized character of G^F. We aim to prove the following theorem.

Theorem 8.2.1. (Curtis, Alvis, Kawanaka) *The map $\xi \to \xi^*$ is an isometry of generalized characters of G^F which has order 2. Thus $\xi^{**} = \xi$ and $(\xi, \eta) = (\xi^*, \eta^*)$ for all generalized characters ξ, η of G^F.*

The map $\xi \to \xi^*$ was first considered by Curtis [13] and independently by Kawanaka [8]. Its properties have been worked out in detail by Alvis [1], [2], [3].

We have already encountered two examples of this map $\xi \to \xi^*$. If ξ is the principal character of G^F then ξ^* is the Steinberg character. If ξ is the Gelfand–Graev character of G^F then ξ^* is the generalized character taking value $|Z^F|q^l$ on the regular unipotent elements of G^F and value 0 elsewhere. ξ^* will be called the dual of ξ.

Before proving this result we must investigate the properties of the truncation map in more detail. It is convenient to work in the more general context of a finite group with a split BN-pair which also satisfies the commutator relations. For such a group G we may define the dual of a generalized character ξ of G to be

$$\xi^* = \sum_{J \subseteq I} (-1)^{|J|} (T_{P_J/U_J}(\xi))^G.$$

This agrees with the above definition of ξ^* when G is taken equal to G^F. The more general context has the advantage that the notation is somewhat simpler.

Proposition 8.2.2. *Let P_J be a standard parabolic subgroup of G with standard Levi subgroup L_J. Let ξ be a generalized character of G and ϕ a generalized character of L_J. Then*

$$(\xi, \phi_{P_J}{}^G) = (T_{L_J}(\xi), \phi).$$

Proof.
$$\begin{aligned}(T_{L_J}(\xi), \phi) &= \frac{1}{|L_J|} \sum_{x \in L_J} (T_{L_J}(\xi)(x))\overline{\phi(x)} \\ &= \frac{1}{|L_J|} \sum_{x \in L_J} \frac{1}{|U_J|} \sum_{u \in U_J} \xi(ux)\overline{\phi(x)} \\ &= \frac{1}{|P_J|} \sum_{p \in P_J} \xi(p)\overline{\phi_{P_J}(p)} = (\xi, \phi_{P_J})_{P_J} \\ &= (\xi, \phi_{P_J}{}^G)_G \quad \text{by Frobenius reciprocity.}\end{aligned}$$

Proposition 8.2.3. *Let ξ be a generalized character of G and L_J, L_K be two standard Levi subgroups of G where $K \subseteq J$. Then*

$$T_{L_K}(T_{L_J}(\xi)) = T_{L_K}(\xi).$$

Thus the operation of truncation to a standard Levi subgroup is transitive.

Proof. We recall from 2.6.6 and 2.6.7 that $P_K \cap L_J$ is a standard parabolic subgroup of L_J and that the Levi decomposition of this parabolic subgroup is $P_K \cap L_J = (U_K \cap L_J)L_K$. Hence, for $l \in L_K$, we have

$$(T_{L_K}(T_{L_J}(\xi)))(l) = \frac{1}{|U_K \cap L_J|} \sum_{u \in U_K \cap L_J} (T_{L_J}(\xi))(ul)$$

$$= \frac{1}{|U_K \cap L_J|} \sum_{u \in U_K \cap L_J} \frac{1}{|U_J|} \sum_{u' \in U_J} \xi(u'ul).$$

Now we have $U_K = U_J(U_K \cap L_J)$ with uniqueness. Hence

$$(T_{L_K}(T_{L_J}(\xi)))(l) = \frac{1}{|U_K|} \sum_{u'' \in U_K} \xi(u''l)$$

$$= (T_{L_K}(\xi))(l).$$

The result follows.

Proposition 8.2.4. *Let L_J, L_K be standard Levi subgroups of G with $K \subseteq J$. Let ϕ be a generalized character of L_K. Then*

$$(\phi_{L_J \cap P_K}{}^{L_J})_{P_J}{}^G = \phi_{P_K}{}^G.$$

Thus the operation of lifting and inducing is transitive.

Proof. Let ξ be any generalized character of G. Then we have

$$\begin{aligned}(\xi, (\phi_{L_J \cap P_K}{}^{L_J})_{P_J}{}^G) &= (T_{L_J}(\xi), \phi_{L_J \cap P_K}{}^{L_J}) \quad \text{by 8.2.2}\\ &= (T_{L_K}(T_{L_J}(\xi)), \phi) \quad \text{also by 8.2.2}\\ &= (T_{L_K}(\xi), \phi) \quad \text{by 8.2.3}\\ &= (\xi, \phi_{P_K}{}^G) \quad \text{by 8.2.2 again.}\end{aligned}$$

Thus

$$(\xi, (\phi_{L_J \cap P_K}{}^{L_J})_{P_J}{}^G) = (\xi, \phi_{P_K}{}^G)$$

for all generalized characters ξ of G. It follows that

$$(\phi_{L_J \cap P_K}{}^{L_J})_{P_J}{}^G = \phi_{P_K}{}^G. \qquad \blacksquare$$

We now prove some lemmas about truncation which are valid for any finite group.

Lemma 8.2.5. *Let G be a finite group and N be a normal subgroup of G. Let H be a subgroup of G containing N. Let ξ be a generalized character of H. Then*

$$T_{G/N}(\xi^G) = (T_{H/N}(\xi))^G.$$

Proof. We have

$$\xi^G(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ g^x \in H}} \xi(g^x) \qquad g \in G.$$

Thus

$$\begin{aligned}(T_{G/N}(\xi^G))(g) &= \frac{1}{|N|}\sum_{n\in N}\xi^G(ng)\\ &= \frac{1}{|N|}\sum_{n\in N}\frac{1}{|H|}\sum_{\substack{x\in G\\ (ng)^x\in H}}\xi((ng)^x)\\ &= \frac{1}{|H|}\sum_{\substack{x\in G\\ g^x\in H}}\frac{1}{|N|}\sum_{n\in N}\xi(n^x g^x)\\ &= \frac{1}{|H|}\sum_{\substack{x\in G\\ g^x\in H}}\frac{1}{|H|}\sum_{n\in N}\xi(ng^x)\\ &= \frac{1}{|H|}\sum_{\substack{x\in G\\ g^x\in H}}(T_{H/N}(\xi))(g^x)\\ &= ((T_{H/N}(\xi))^G)(g).\end{aligned}$$

Hence $T_{G/N}(\xi^G) = (T_{H/N}(\xi))^G$ as required.

Lemma 8.2.6. *Let G be a finite group, H be a subgroup of G and N a normal subgroup of G such that $G = NH$. Let ξ be a generalized character of H. Then*

$$T_{G/N}(\xi^G) = (T_{H/N\cap H}(\xi))_G.$$

Thus ξ induced to G and then truncated modulo N is the same as ξ truncated modulo $N \cap H$ and then lifted to G.

Proof. It is sufficient to show that the above generalized characters agree on H. We have

$$\xi^G(g) = \frac{1}{|H|}\sum_{\substack{x\in G\\ g^x\in H}}\xi(g^x) \quad g\in G$$

$$\begin{aligned}(T_{G/N}(\xi^G))(h) &= \frac{1}{|N|}\sum_{n\in N}\xi^G(nh) \quad h\in H\\ &= \frac{1}{|N|}\sum_{n\in N}\frac{1}{|H|}\sum_{\substack{x\in G\\ (nh)^x\in H}}\xi((nh)^x).\end{aligned}$$

Now each $x \in G$ has the form $x = h'n'$ with $h' \in H$, $n' \in N$ and can be expressed in this form in $|N \cap H|$ ways. Thus

$$(T_{G/N}(\xi^G))(h) = \frac{1}{|N|}\frac{1}{|H|}\frac{1}{|N\cap H|}\sum_{\substack{n\in N\\ (nh)^{h'n'}\in H}}\sum_{n'\in N}\sum_{h'\in H}\xi((nh)^{h'n'}).$$

Now

$$\begin{aligned}(nh)^{h'n'} &= n'^{-1}h'^{-1}nhh'n' \\ &= h'^{-1}.h'n'^{-1}h'^{-1}.n.hh'n'h'^{-1}h^{-1}.hh' \\ &= (mh)^{h'}\end{aligned}$$

where $m = h'n'^{-1}h'^{-1}.n.hh'n'h'^{-1}h^{-1} \in N$. Moreover $(nh)^{h'n'}$ lies in H if and only if $m \in N \cap H$. Thus we have

$$\begin{aligned}(T_{G/N}(\xi^G))(h) &= \frac{1}{|N|}\frac{1}{|H|}\frac{1}{|N \cap H|}\sum_{n\in N}\sum_{n'\in N}\sum_{h'\in H}\sum_{\substack{m\in N\cap H \\ h'n'^{-1}h'^{-1}.n.hh'n'h'^{-1}h^{-1}=m}} \xi(mh) \\ &= \frac{1}{|N|}\frac{1}{|N \cap H|}\sum_{n\in N}\sum_{n_1\in N}\sum_{\substack{m\in N\cap H \\ n_1^{-1}nhn_1h^{-1}=m}} \xi(mh) \\ &= \frac{1}{|N \cap H|}\sum_{m\in N\cap H} \xi(mh) \\ &= (T_{H/N\cap H}(\xi))(h).\end{aligned}$$

Thus $T_{G/N}(\xi^G)$ and $T_{H/N\cap H}(\xi)$ agree on H. It follows that

$$T_{G/N}(\xi^G) = (T_{H/N\cap H}(\xi))_G.$$

Proposition 8.2.7. *Let G be a finite group with a split BN-pair satisfying the commutator relations. Let L_{J_1} and L_{J_2} be standard Levi subgroups of G and suppose $L_{J_1} = {}^xL_{J_2}$ with $x \in N_{J_1,J_2}$. Then*

(i) *If ξ is a generalized character of G then*

$$T_{L_{J_1}}(\xi) = {}^x(T_{L_{J_2}}(\xi)).$$

(ii) *If ϕ is a generalized character of L_{J_1} then*

$$\phi_{P_{J_1}}{}^G = (\phi_{{}^xP_{J_2}})^G$$

where $\phi_{P_{J_1}}$, $\phi_{{}^xP_{J_2}}$ have U_{J_1}, ${}^xU_{J_2}$ respectively in their kernels.

Proof. (Howlett). (ii) implies (i) since we have

$$(T_{L_{J_1}}(\xi), \phi) = (\xi, \phi_{P_{J_1}}{}^G)$$

$$({}^x(T_{L_{J_2}}(\xi)), \phi) = (T_{L_{J_2}}(\xi), {}^{x^{-1}}\phi) = (\xi, ({}^{x^{-1}}\phi)_{P_{J_2}}{}^G) = (\xi, (\phi_{{}^xP_{J_2}})^G)$$

by 8.2.2. In order to prove (ii) we shall show

$$(\phi_{P_{J_1}}{}^G, \phi_{P_{J_1}}{}^G) = (\phi_{P_{J_1}}{}^G, (\phi_{{}^xP_{J_2}})^G) = ((\phi_{{}^xP_{J_2}})^G, (\phi_{{}^xP_{J_2}})^G).$$

Now N_{J_1,J_2} is a set of double coset representatives of G with respect to P_{J_1} and P_{J_2}. Thus $N_{J_1,J_2}x^{-1}$ is a set of double coset representatives of G with respect to P_{J_1} and ${}^xP_{J_2}$. By Mackey's formula we have

$$(\phi_{P_{J_1}}{}^G, (\phi_{{}^xP_{J_2}})^G) = \sum_{n' \in N_{J_1,J_2x^{-1}}} (\phi, {}^{n'}\phi)_{P_{J_1} \cap {}^{n'x}P_{J_2}}.$$

By 2.8.7 we have

$$P_{J_1} \cap {}^{n'x}P_{J_2} = (U_{J_1} \cap {}^{n'x}U_{J_2})(U_{J_1} \cap {}^{n'x}L_{J_2})(L_{J_1} \cap {}^{n'x}U_{J_2})L_K$$

with uniqueness, where $W_{J_1} \cap {}^{\pi(n'x)}W_{J_2} = W_K$. Hence we have

$$(\phi, {}^{n'}\phi)_{P_{J_1} \cap {}^{n'x}P_{J_2}} = \frac{1}{|P_{J_1} \cap {}^{n'x}P_{J_2}|} \sum_{x', y, z, t} \phi(x'yzt)\overline{{}^{n'}\phi(x'yzt)}$$

where $x' \in U_{J_1} \cap {}^{n'x}U_{J_2}$, $y \in U_{J_1} \cap {}^{n'x}L_{J_2}$, $z \in L_{J_1} \cap {}^{n'x}U_{J_2}$, $t \in L_K$. Since $U_{J_1} \cap {}^{n'x}U_{J_2}$ lies in the kernel of both ϕ and ${}^{n'}\phi$ we have

$$(\phi, {}^{n'}\phi)_{P_{J_1} \cap {}^{n'x}P_{J_2}} = \frac{|U_{J_1} \cap {}^{n'x}U_{J_2}|}{|P_{J_1} \cap {}^{n'x}P_{J_2}|} \sum_{y, z, t} \phi(yzt)\overline{{}^{n'}\phi(yzt)}.$$

We note also that $U_{J_1} \cap {}^{n'x}L_{J_1}$ lies in the kernel of ϕ and that $L_{J_1} \cap {}^{n'x}U_{J_2}$ lies in the kernel of ${}^{n'}\phi$. Thus we have

$$\begin{aligned}(\phi, {}^{n'}\phi)_{P_{J_1} \cap {}^{n'x}P_{J_2}} &= \frac{|U_{J_1} \cap {}^{n'x}U_{J_2}|}{|P_{J_1} \cap {}^{n'x}P_{J_2}|} \sum_{y, z, t} \phi(zt)\overline{{}^{n'}\phi(yt)} \\ &= \frac{|U_{J_1} \cap {}^{n'x}U_{J_2}|}{|P_{J_1} \cap {}^{n'x}P_{J_2}|} \sum_{t \in L_K} \left(\sum_{z \in L_{J_1} \cap {}^{n'x}U_{J_2}} \phi(zt) \right) \\ &\quad \times \overline{\left(\sum_{y \in U_{J_1} \cap {}^{n'x}L_{J_2}} {}^{n'}\phi(yt) \right)}.\end{aligned}$$

Now we know from 2.8.9 that $L_{J_1} \cap {}^{n'x}P_{J_2} = L_{J_1} \cap P_K$ is a standard parabolic subgroup of L_{J_1} with maximal normal p-subgroup $L_{J_1} \cap {}^{n'x}U_{J_2}$ and Levi subgroup L_K. Similarly $P_{J_1} \cap {}^{n'x}L_{J_2}$ is a parabolic subgroup of ${}^{n'x}L_{J_2}$ with maximal normal p-subgroup $U_{J_1} \cap {}^{n'x}L_{J_2}$ and Levi subgroup L_K. Thus we have

$$\begin{aligned}(\phi, {}^{n'}\phi)_{P_{J_1} \cap {}^{n'x}P_{J_2}} &= \frac{1}{|L_K|} \sum_{t \in L_K} (T_{L_K}(\phi))(t)\overline{(T_{L_K}({}^{n'}\phi))(t)} \\ &= (T_{L_K}(\phi), T_{L_K}({}^{n'}\phi)).\end{aligned}$$

It follows that

$$\begin{aligned}(\phi_{P_{J_1}}{}^G, (\phi_{{}^xP_{J_2}})^G) &= \sum_{n' \in N_{J_1,J_2x^{-1}}} (T_{L_K}(\phi), T_{L_K}({}^{n'}\phi)) \\ &= \sum_{w' \in D_{J_1,J_2\bar{x}^{-1}}} (T_{L_K}(\phi), T_{L_K}({}^{w'}\phi))\end{aligned}$$

where $\bar{x} = \pi(x) \in W$.

By a similar argument, replacing J_2 by J_1 and x by 1, we have

$$(\phi_{P_{J_1}}{}^G, \phi_{P_{J_1}}{}^G) = \sum_{w \in D_{J_1, J_1}} (T_{L_{K'}}(\phi), T_{L_{K'}}({}^w\phi))$$

where $K' = J_1 \cap w(J_1)$.

We first observe that the summands in the two above expressions are in bijective correspondence. For $L_{J_1} = {}^xL_{J_2}$ with $x \in N_{J_1, J_2}$ and this implies by 2.8.8 that $J_1 = \bar{x}(J_2)$ and $W_{J_1} = {}^{\bar{x}}W_{J_2}$. Now D_{J_1, J_2} is a set of double coset representatives of W with respect to W_{J_1} and W_{J_2}, and so $D_{J_1, J_2}\bar{x}^{-1}$ is a set of double coset representatives for W with respect to W_{J_1} and W_{J_1}. However D_{J_1, J_1} is also a set of such double coset representatives. We consider how these two sets of representatives are related.

Let $w' \in D_{J_1, J_2}\bar{x}^{-1}$ with $w' = d\bar{x}^{-1}$ and $d \in D_{J_1, J_2}$. Then $w'(J_1) = d(J_2) \subseteq \Phi^+$ since $D_{J_1, J_2} = D_{J_1}{}^{-1} \cap D_{J_2}$. We may write

$$w'^{-1} = w^{-1}w_1{}^{-1}$$

with $w^{-1} \in D_{J_1}$, $w_1{}^{-1} \in W_{J_1}$ and $l(w'^{-1}) = l(w^{-1}) + l(w_1{}^{-1})$. Then $w_1 w(J_1) \subseteq \Phi^+$. Since $l(w_1 w) = l(w_1) + l(w)$ this implies that $w(J_1) \subseteq \Phi^+$. Thus $w \in D_{J_1}$ and so $w \in D_{J_1} \cap D_{J_1}{}^{-1} = D_{J_1, J_1}$, Hence w' can be factorized as

$$w' = w_1 w \qquad w_1 \in W_{J_1}, w \in D_{J_1, J_1}.$$

We now compare K and K'. We have $K = J_1 \cap w'\bar{x}(J_2)$ and $K' = J_1 \cap w(J_1)$. We shall show that $w_1(K') = K$. We have

$$\Phi_K = \Phi_{J_1} \cap w'\bar{x}(\Phi_{J_2})$$
$$\Phi_{K'} = \Phi_{J_1} \cap w(\Phi_{J_1})$$

by Kilmoyer's theorem 2.7.4, since $w'\bar{x} \in D_{J_1, J_2}$ and $w \in D_{J_1, J_1}$. Hence

$$\Phi_K = \Phi_{J_1} \cap w_1 w\bar{x}(\Phi_{J_2}) = \Phi_{J_1} \cap w_1 w(\Phi_{J_1}) = w_1(\Phi_{J_1} \cap w(\Phi_{J_1})) = w_1(\Phi_{K'}).$$

We also have

$$\begin{aligned} w_1(\Delta_{K'}) &= w_1(\Delta_{J_1} \cap w(\Delta_{J_1})) = d\bar{x}^{-1}w^{-1}(\Delta_{J_1} \cap w(\Delta_{J_1})) \\ &= d\bar{x}^{-1}(w^{-1}(\Delta_{J_1}) \cap \Delta_{J_1}) \subseteq dx^{-1}(\Delta_{J_1}) = d(\Delta_{J_2}) \subseteq \Phi^+. \end{aligned}$$

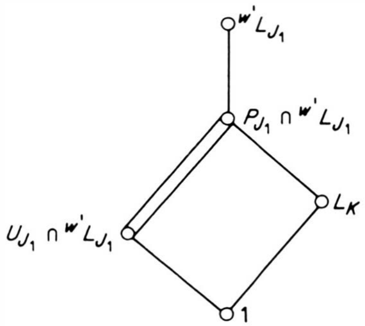
This $w_1(\Delta_{K'}) \subseteq \Phi^+$ and it follows that $w_1(\Delta_{K'}) = \Delta_K$, i.e. $w_1(K') = K$. We see in particular that $w_1 \in D_{K'}$ and $w_1{}^{-1} \in D_K$, so that $w_1 \in D_{K, K'}$.

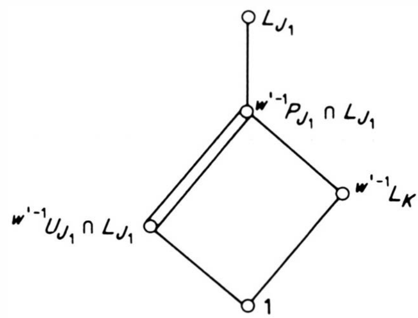
We now use this information to show that

$$(T_{L_K}(\phi), T_{L_K}({}^{w'}\phi)) = (T_{L_{K'}}(\phi), T_{L_{K'}}({}^w\phi))$$

where $w' = w_1 w$ as above. $T_{L_K}({}^{w'}\phi)$ is the character of L_K obtained by truncating, as in the first figure, the character ${}^{w'}\phi$ of ${}^{w'}L_{J_1}$. It is therefore conjugate to the character of ${}^{w'^{-1}}L_K$ obtained by truncating the character ϕ of L_{J_1} as in the second figure. However we have

$${}^{w'^{-1}}L_K = {}^{w^{-1}w_1{}^{-1}}L_K = {}^{w^{-1}}L_{K'} = L_{K''}$$





where $K'' = w^{-1}(K') = J_1 \cap w^{-1}(J_1)$. Also

$$ {}^{w'^{-1}}P_{J_1} \cap L_{J_1} = {}^{w^{-1}}P_{J_1} \cap L_{J_1} = P_{K''} \cap L_{J_1} $$

is a standard parabolic subgroup of L_{J_1} with maximal normal p-subgroup

$$ {}^{w'^{-1}}U_{J_1} \cap L_{J_1} = {}^{w^{-1}}U_{J_1} \cap L_{J_1} = U_{K''} \cap L_{J_1}. $$

This follows from 2.8.9 since $w^{-1} \in D_{J_1,J_1}$. Thus we have

$$ T_{L_K}({}^{w'}\phi) = {}^{w'}(T_{L_{K''}}(\phi)). $$

In an exactly similar way we obtain

$$ T_{L_{K'}}({}^{w}\phi) = {}^{w}(T_{L_{K''}}(\phi)). $$

Thus we have to show

$$ (T_{L_K}(\phi), {}^{w'}(T_{L_{K''}}(\phi))) = (T_{L_{K'}}(\phi), {}^{w}(T_{L_{K''}}(\phi))). $$

Now $w' = w_1 w$ and so we have

$$ (T_{L_{K'}}(\phi), {}^{w}(T_{L_{K''}}(\phi))) = ({}^{w_1}(T_{L_{K'}}(\phi)), {}^{w'}(T_{L_{K''}}(\phi))). $$

It will therefore be sufficient to show that

$$ T_{L_K}(\phi) = {}^{w_1}(T_{L_{K'}}(\phi)). $$

Since $w_1 \in D_{K,K'}$ and $L_K = {}^{w_1}L_{K'}$ we may argue by induction that this is true, unless $J_1 = I$. However if $J_1 = I$ the result is trivial. Thus we have

$$ (T_{L_K}(\phi), T_{L_K}({}^{w'}\phi)) = (T_{L_{K'}}(\phi), T_{L_{K'}}({}^{w}\phi)) $$

and so also

$$ (\phi_{P_{J_1}}{}^G, (\phi_{{}^*P_{J_1}})^G) = (\phi_{P_{J_1}}{}^G, \phi_{P_{J_1}}{}^G). $$

By symmetry we also have

$$ (\phi_{P_{J_1}}{}^G, (\phi_{{}^*P_{J_1}})^G) = ((\phi_{{}^*P_{J_1}})^G, (\phi_{{}^*P_{J_1}})^G) $$

and so the proposition is proved. ∎

Note. For groups of type G^F a shorter proof of this result has been given by Deligne. An outline of Deligne's proof can be found in Lusztig and Spaltenstein [1].

Proposition 8.2.8. *Let G be a finite group with a split BN-pair satisfying the commutator relations. Let L_J be a standard Levi subgroup of G and let ξ be a generalized character of G. Then*

$$(T_{L_J}(\xi))^* = T_{L_J}(\xi^*).$$

Thus the operations of duality and truncation to a Levi subgroup commute.

Proof. We have

$$\xi^* = \sum_{J_2 \subseteq I} (-1)^{|J_2|}(T_{P_{J_2}/U_{J_2}}(\xi))^G.$$

Hence

$$T_{L_{J_1}}(\xi^*) = \sum_{J_2} (-1)^{|J_2|} T_{L_{J_1}}((T_{P_{J_2}/U_{J_2}}(\xi))^G).$$

Now

$$(T_{P_{J_2}/U_{J_2}}(\xi)^G)_{P_{J_1}} = \sum_{n \in N_{J_1,J_2}} ((^n(T_{P_{J_2}/U_{J_2}}(\xi))_{P_{J_1} \cap {}^nP_{J_2}})^{P_{J_1}}$$

by Mackey's formula, since N_{J_1,J_2} is a set of double coset representatives of G with respect to P_{J_1}, P_{J_2}. Let $n \in N_{J_1,J_2}$, $\pi(n) = w$, and $W_{J_1} \cap {}^wW_{J_2} = W_K$ as in Kilmoyer's theorem 2.7.4. Then $P_{J_1} \cap {}^nP_{J_2} \subseteq P_K \subseteq P_{J_1}$ by 2.8.3 and so

$$(T_{P_{J_2}/U_{J_2}}(\xi)^G)_{P_{J_1}} = \sum_{n \in N_{J_1,J_2}} (((^n(T_{P_{J_2}/U_{J_2}}(\xi)))_{P_{J_1} \cap {}^nP_{J_2}})^{P_K})^{P_{J_1}}.$$

Thus

$$T_{L_{J_1}}(\xi^*) = \sum_{J_2} (-1)^{|J_2|} T_{L_{J_1}}\left(\sum_{n \in N_{J_1,J_2}} (((^nT_{P_{J_2}/U_{J_2}}(\xi)))_{P_{J_1} \cap {}^nP_{J_2}})^{P_K})^{P_{J_1}}\right).$$

Now $U_{J_1} \subseteq P_K \subseteq P_{J_1}$. So we may apply 8.2.5 and obtain

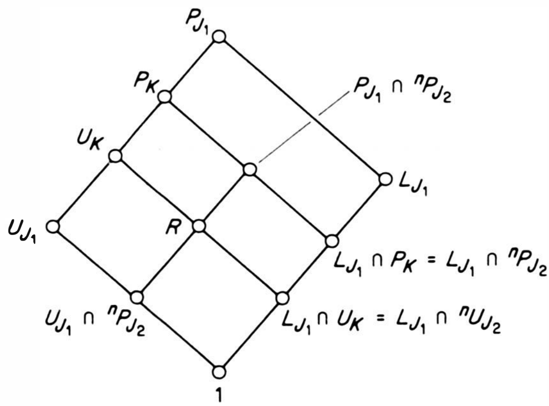
$$T_{L_{J_1}}(\xi^*) = \sum_{J_2} (-1)^{|J_2|} \sum_{n \in N_{J_1,J_2}} (\eta^{P_{J_1}})_{L_{J_1}}$$

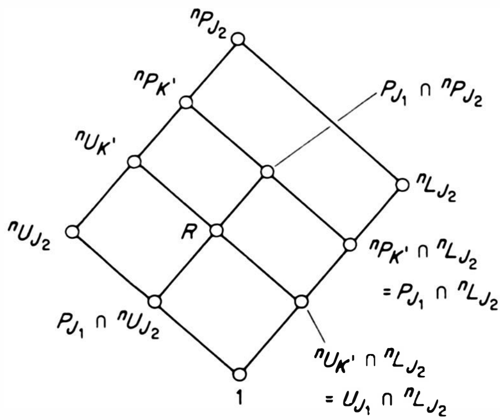
where η is the generalized character of P_K given by

$$\eta = T_{P_K/U_{J_1}}(((^n(T_{P_{J_2}/U_{J_2}}(\xi)))_{P_{J_1} \cap {}^nP_{J_2}})^{P_K}).$$

We now recall from section 2.8 that the subgroups under consideration are related as in the following diagrams. Here K' is defined by $W_{J_1} \cap {}^wW_{J_2} = {}^wW_{K'}$ and R is the largest normal p-subgroup of $P_{J_1} \cap {}^nP_{J_2}$. We now apply 8.2.6 with P_K, $P_{J_1} \cap {}^nP_{J_2}$, replacing G, H, N respectively. This gives

$$\begin{aligned}\eta &= (T_{P_{J_1} \cap {}^nP_{J_2}/U_{J_1} \cap {}^nP_{J_2}}((^n(T_{P_{J_2}/U_{J_2}}(\xi)))_{P_{J_1} \cap {}^nP_{J_2}}))_{P_K} \\ &= (T_{P_{J_1} \cap {}^nP_{J_2}/U_{J_1} \cap {}^nP_{J_2}}((T_{{}^nP_{J_2}/{}^nU_{J_2}}(\xi))_{P_{J_1} \cap {}^nP_{J_2}}))_{P_K}.\end{aligned}$$





Now we have ${}^nU_{J_2}(U_{J_1} \cap {}^nP_{J_2}) = {}^nU_{K'}$. Hence

$$T_{P_{J_1} \cap {}^nP_{J_2}/U_{J_1} \cap {}^nP_{J_2}}((T_{{}^nP_{J_2}/{}^nU_{J_2}}(\xi))_{P_{J_1} \cap {}^nP_{J_2}}) = (T_{{}^nP_{K'}/{}^nU_{K'}}(\xi))_{P_{J_1} \cap {}^nP_{J_2}}.$$

It follows that

$$\eta = ((T_{{}^nP_{K'}/{}^nU_{K'}}(\xi))_{P_{J_1} \cap {}^nP_{J_2}})_{P_K}.$$

Now ${}^nL_{K'} = L_K$ and so by 8.2.7 we have

$$T_{P_K/U_K}(\xi) = T_{{}^nP_{K'}/{}^nU_{K'}}(\xi) \quad \text{on } L_K.$$

Since $P_{J_1} \cap {}^nP_{J_2} = RL_K$ and $R \subseteq U_K \cap {}^nU_{K'}$ it follows that

$$T_{P_K/U_K}(\xi) = T_{{}^nP_{K'}/{}^nU_{K'}}(\xi) \quad \text{on } P_{J_1} \cap {}^nP_{J_2}.$$

Hence

$$\eta = ((T_{P_K/U_K}(\xi))_{P_{J_1} \cap {}^nP_{J_2}})_{P_K} = T_{P_K/U_K}(\xi).$$

We may now substitute for η in $T_{L_{J_1}}(\xi^*)$. We have

$$\begin{aligned} T_{L_{J_1}}(\xi^*) &= \sum_{J_2} (-1)^{|J_2|} \sum_{n \in N_{J_1, J_2}} (\eta^{P_{J_1}})_{L_{J_1}} \\ &= \sum_{J_2} (-1)^{|J_2|} \sum_{n \in N_{J_1, J_2}} ((T_{P_K/U_K}(\xi))^{P_{J_1}})_{L_{J_1}} \\ &= \sum_{J_2} (-1)^{|J_2|} \sum_{n \in N_{J_1, J_2}} ((T_{P_K/U_K}(\xi))_{L_{J_1} \cap P_K})^{L_{J_1}} \\ &= \sum_{J_2} (-1)^{|J_2|} \sum_{n \in N_{J_1, J_2}} ((T_{L_K}(\xi))_{L_{J_1} \cap P_K})^{L_{J_1}} \\ &= \sum_{J_2} (-1)^{|J_2|} \sum_{n \in N_{J_1, J_2}} ((T_{L_K}(T_{L_{J_1}}(\xi)))_{L_{J_1} \cap P_K})^{L_{J_1}} \end{aligned}$$

by 8.2.3

$$\begin{aligned} &= \sum_{J_2} (-1)^{|J_2|} \sum_{n \in N_{J_1, J_2}} (T_{L_{J_1} \cap P_K / L_{J_1} \cap U_K}(T_{L_{J_1}}(\xi)))^{L_{J_1}} \\ &= \sum_{K \subseteq J_1} \sum_{J_2} (-1)^{|J_2|} \sum_{\substack{n \in N_{J_1, J_2} \\ W_{J_1} \cap {}^w W_{J_2} = W_K}} (T_{L_{J_1} \cap P_K / L_{J_1} \cap U_K}(T_{L_{J_1}}(\xi)))^{L_{J_1}} \\ &= \sum_{K \subseteq J_1} \sum_{J_2} (-1)^{|J_2|} a_{J_1 J_2 K} (T_{L_{J_1} \cap P_K / L_{J_1} \cap U_K}(T_{L_{J_1}}(\xi)))^{L_{J_1}} \end{aligned}$$

where $a_{J_1 J_2 K}$ is the number of elements $w \in D_{J_1, J_2}$ with $W_{J_1} \cap {}^w W_{J_2} = W_K$. By 2.7.6 we have

$$\sum_{J_2} (-1)^{|J_2|} a_{J_1 J_2 K} = (-1)^{|K|}$$

since $K \subseteq J_1$. Hence

$$\begin{aligned} T_{L_{J_1}}(\xi^*) &= \sum_{K \subseteq J_1} (-1)^{|K|} (T_{L_{J_1} \cap P_K / L_{J_1} \cap U_K}(T_{L_{J_1}}(\xi))^{L_{J_1}} \\ &= (T_{L_{J_1}}(\xi))^* \end{aligned}$$

and the proof is complete. ■

We now have all the material necessary for the proof of Theorem 8.2.1.

Proof of 8.2.1. Let ξ, η be generalized characters of G. Then

$$\xi^* = \sum_{J \subseteq I} (-1)^{|J|} (T_{P_J/U_J}(\xi))^G.$$

Thus we have

$$\begin{aligned}
(\xi^*, \eta) &= \sum_J (-1)^{|J|}((T_{P_J/U_J}(\xi))^G, \eta) \\
&= \sum_J (-1)^{|J|}((T_{L_J}(\xi))_{P_J}{}^G, \eta) \\
&= \sum_J (-1)^{|J|}(T_{L_J}(\xi), T_{L_J}(\eta)) \quad \text{by 8.2.2} \\
&= \sum_J (-1)^{|J|}(\xi, (T_{L_J}(\eta))_{P_J}{}^G) \quad \text{again by 8.2.2} \\
&= \sum_J (-1)^{|J|}(\xi, (T_{P_J/U_J}(\eta))^G) \\
&= (\xi, \eta^*).
\end{aligned}$$

Thus $(\xi^*, \eta) = (\xi, \eta^*)$. We now consider ξ^{**}. We have

$$\begin{aligned}
\xi^{**} &= \sum_J (-1)^{|J|}(T_{P_J/U_J}(\xi^*))^G \\
&= \sum_J (-1)^{|J|}(T_{L_J}(\xi^*))_{P_J}{}^G \\
&= \sum_J (-1)^{|J|}((T_{L_J}(\xi))^*)_{P_J}{}^G \quad \text{by 8.2.8} \\
&= \sum_J (-1)^{|J|} \sum_{K \subseteq J} (-1)^{|K|}((T_{L_K}(T_{L_J}(\xi)))_{P_K \cap L_J}{}^{L_J})_{P_J}{}^G \\
&= \sum_J (-1)^{|J|} \sum_{K \subseteq J} (-1)^{|K|}((T_{L_K}(\xi))_{P_K \cap L_J}{}^{L_J})_{P_J}{}^G \quad \text{by 8.2.3} \\
&= \sum_J (-1)^{|J|} \sum_{K \subseteq J} (-1)^{|K|}(T_{L_K}(\xi))_{P_K}{}^G \quad \text{by 8.2.4} \\
&= \sum_J (-1)^{|J|} \sum_{K \subseteq J} (-1)^{|K|}(T_{P_K/U_K}(\xi))^G \\
&= \sum_K (T_{P_K/U_K}(\xi))^G \sum_{J \supseteq K} (-1)^{|J-K|}
\end{aligned}$$

Now

$$\sum_{\substack{J \\ K \subseteq J \subseteq I}} (-1)^{|J-K|} = \begin{cases} 0 & \text{if } K \neq I \\ 1 & \text{if } K = I \end{cases}$$

Thus

$$\xi^{**} = (T_{G/1}(\xi))^G = \xi^G = \xi.$$

Finally we show that the map $\xi \to \xi^*$ is an isometry. We have

$$(\xi^*, \eta^*) = (\xi, \eta^{**}) = (\xi, \eta)$$

for all generalized characters ξ, η of G. Thus the map $\xi \to \xi^*$ is an involutary isometry of generalized characters.

8.3 CHARACTER VALUES ON REGULAR UNIPOTENT ELEMENTS

We now return to the situation where G is a connected reductive group with connected centre, with Frobenius map F. We have seen in 8.1.3 that all the irreducible components of the Gelfand–Graev character Γ occur with multiplicity 1. We shall now determine the number of such irreducible components. Let Ξ be the class function on G^F which takes value $|Z^F|q^l$ on the regular unipotent elements of G^F and value 0 on all other elements. By 8.1.7 we have $\Gamma^* = \Xi$. Thus Ξ is in fact a generalized character of G^F.

Proposition 8.3.1. $(\Gamma, \Gamma) = (\Xi, \Xi) = |Z^F|q^l$. *Thus Γ has $|Z^F|q^l$ irreducible components.*

Proof. We use the properties of the duality map established in section 8.2. Since $\Xi = \Gamma^*$ we have

$$(\Gamma, \Gamma) = (\Xi, \Xi) = \frac{1}{|G^F|} \sum_{\substack{u \in G^F \\ \text{regular} \\ \text{unipotent}}} \Xi(u)\overline{\Xi(u)}$$

$$= \frac{1}{|G^F|} |Z^F|q^l|Z^F|q^l m$$

where m is the number of regular unipotent elements of G^F. By 5.1.9 we know that

$$m = \frac{|G^F|}{|Z^F|q^l}.$$

Hence $(\Gamma, \Gamma) = |Z^F|q^l$.

Proposition 8.3.2. *Let χ^i be an irreducible character of G^F. Then (Ξ, χ^i) is either* 1, -1 *or* 0. *There are $|Z^F|q^l$ irreducible characters χ^i for which $(\Xi, \chi^i) = \pm 1$.*

Proof. Since the duality map $\xi \to \xi^*$ is an isometry we have

$$(\Xi, \chi^i) = (\Xi^*, (\chi^i)^*) = (\Gamma, (\chi^i)^*).$$

Moreover

$$1 = (\chi^i, \chi^i) = ((\chi^i)^*, (\chi^i)^*).$$

Thus $(\chi^i)^*$ is a generalized character with $((\chi^i)^*, (\chi^i)^*) = 1$. This means that

$(\chi^i)^* = \pm\chi^j$ for some irreducible character χ^j. Since all irreducible components of Γ have multiplicity 1 we have

$$(\Gamma, (\chi^i)^*) = \pm(\Gamma, \chi^j) = \begin{cases} \pm 1 & \text{if } \chi^j \text{ occurs in } \Gamma \\ 0 & \text{otherwise} \end{cases}.$$

Thus $(\Xi, \chi^i) \in \{1, -1, 0\}$. Since there are $|Z^F|q^l$ irreducible characters χ^j with $(\Gamma, \chi^j) = 1$ there are $|Z^F|q^l$ irreducible characters χ^i with $(\Xi, \chi^i) = \pm 1$.

Proposition 8.3.3. (i) *Let χ^i be an irreducible character of G^F. Then the average value of χ^i on the regular unipotent elements of G^F is* 1, -1, *or* 0.

(ii) *If p is a good prime for G then $\chi^i(u) = 1, -1, 0$ for any regular unipotent element u of G^F. (A prime p is good for a reductive group G if and only if p is good for the semisimple group G/Z. p is good for a semisimple group if and only if p is good for each simple component.)*

Proof. Consider the scalar product (χ^i, Ξ). We have

$$(\chi^i, \Xi) = \frac{1}{|G^F|} \sum_{\substack{u \\ \text{regular} \\ \text{unipotent}}} \chi^i(u)|Z^F|q^l$$

$$= \frac{|Z^F|}{|G^F|} q^l \sum_{\substack{u \\ \text{regular} \\ \text{unipotent}}} \chi^i(u).$$

Since the number of regular unipotent elements of G^F is $|G^F|/|Z^F|q^l$ this is the average value of χ^i on the regular unipotent elements. By 8.3.2 its value is 1, -1 or 0.

If p is a good prime for G then any two regular unipotent elements of G^F are conjugate in G^F, by 5.1.7. Thus if p is a good prime for G $\chi^i(u)$ is constant for all regular unipotent elements u of G^F. Thus $\chi^i(u) = 1$, -1 or 0. ■

We shall next prove a result of Green, Lehrer and Lusztig relating the value of χ^i on the regular unipotent elements to the degree $\chi^i(1)$ when p is a good prime for G.

Proposition 8.3.4. (Green, Lehrer, Lusztig) *Let χ^i be an irreducible character of G^F and $u \in G^F$ be regular unipotent. Let p be a good prime for G. Then*

$$\chi^i(1) \equiv \chi^i(u) \bmod p.$$

Proof. Let V be a $\mathbb{C}G^F$-module affording the character χ^i. Let $\chi(t)$ be the characteristic polynomial of u on V. The roots of $\chi(t)$ are the eigenvalues of u, and these are all p^sth roots of unity for some positive integer s. Thus the coefficients of $\chi(t)$ are algebraic integers which lie in $\mathbb{Q}(\sqrt[p^s]{1})$.

Now $\mathbb{Q}(\sqrt[p^s]{1})$ is a Galois extension of $\mathbb{Q}$. In fact $\mathbb{Q}(\sqrt[p^s]{1}) = \mathbb{Q}(\varepsilon)$ where $\varepsilon = e^{2\pi i/p^s}$ is a primitive p^sth root of unity. The other primitive p^sth roots of unity are the numbers ε^k where k is not divisible by p. Thus the elements of the Galois group of $\mathbb{Q}(\varepsilon)/\mathbb{Q}$ fix $\mathbb{Q}$ and map ε to ε^k for some k not divisible by p. Let $\gamma_k \in \operatorname{Gal} \mathbb{Q}(\varepsilon)/\mathbb{Q}$ satisfy $\gamma_k(\varepsilon) = \varepsilon^k$.

Now the eigenvalues of u^k are the kth powers of the eigenvalues of u. Thus the characteristic polynomial of u^k when k is not divisible by p is the image under γ_k of the characteristic polynomial of u. But if u is regular unipotent and k is not divisible by p then u^k is regular unipotent, by 5.1.4. Since p is a good prime for G u and u^k are therefore conjugate in G^F, so have the same characteristic polynomial. Hence $\chi(t)$ is fixed by every element of the Galois group of $\mathbb{Q}(\varepsilon)/\mathbb{Q}$. It follows that the coefficients of $\chi(t)$ lies in $\mathbb{Q}$ and, since they are algebraic integers, they must therefore lie in $\mathbb{Z}$. Hence $\chi(t) \in \mathbb{Z}[t]$.

Since $\chi(t) \in \mathbb{Z}[t]$ we may, using an appropriate canonical form, choose a basis of V with respect to which the matrix of u has coefficients in $\mathbb{Z}$. Let $V_{\mathbb{Z}}$ be the set of $\mathbb{Z}$-combinations of elements of this basis. Then u maps $V_{\mathbb{Z}}$ into itself. Let

$$\bar{u}\colon V_{\mathbb{Z}}/pV_{\mathbb{Z}} \to V_{\mathbb{Z}}/pV_{\mathbb{Z}}$$

be the induced map on $V_{\mathbb{Z}}/pV_{\mathbb{Z}}$. This is a vector space over the prime field F_p. $\bar{u}$ is a unipotent linear map of $V_{\mathbb{Z}}/pV_{\mathbb{Z}}$ into itself, so all its eigenvalues are 1. Hence

$$\operatorname{trace}_{V_{\mathbb{Z}}/pV_{\mathbb{Z}}} \bar{u} \equiv \dim_{F_p}(V_{\mathbb{Z}}/pV_{\mathbb{Z}}) \bmod p.$$

Now $\dim_{F_p}(V_{\mathbb{Z}}/pV_{\mathbb{Z}}) = \dim_{\mathbb{C}} V$. Moreover $\operatorname{trace}_V u \in \mathbb{Z}$ and $\operatorname{trace}_V u \equiv \operatorname{trace}_{V_{\mathbb{Z}}/pV_{\mathbb{Z}}} \bar{u} \bmod p$. It follows that

$$\chi^i(u) = \operatorname{trace}_V(u) \equiv \dim_{\mathbb{C}} V \bmod p.$$

Hence $\chi^i(u) \equiv \chi^i(1) \bmod p$. ■

This result has the following remarkable corollary.

Corollary 8.3.5. *Let p be a good prime for G. Then the degree of each irreducible character of G^F is congruent to* 1, -1 *or* 0 mod p.

Proof. This follows from 8.3.3 and 8.3.4. ■

Corollary 8.3.6. *Let p be a good prime for G. Then G^F has exactly $|Z^F|q^l$ irreducible characters of degree prime to p. These characters take value ± 1 on the class of regular unipotent elements of G^F. All the other irreducible characters of G^F take value* 0 *on this class.*

Proof. This follows from 8.3.2, 8.3.3 and 8.3.4. ■

The results in this section suggest that we should concentrate on two particular types of irreducible character of G^F.

Definition. An irreducible character χ^i of G^F is semisimple if the average value of χ^i on the regular unipotent elements of G^F is nonzero. (This average value must then be ± 1 by 8.3.3.)

If p is a good prime for G we see that an irreducible character χ^i of G^F is semisimple if and only if the degree of χ^i is not divisible by p, although this is not always valid if p is a bad prime. The terminology 'semisimple' is used to suggest an analogy with the semisimple conjugacy classes of G^F, which are the classes of elements of order not divisible by p.

Definition. The irreducible characters of G^F which occur as components of the Gelfand–Graev character Γ will be called the regular characters of G^F.

This terminology is used to suggest an analogy with the conjugacy classes of regular elements in G^F. We emphasize however that the regular characters of G^F are irreducible, and are not to be confused with the character of the regular representation of G^F.

Proposition 8.3.7. (i) *The image of a semisimple character χ^i of G^F under the duality map is given by $(\chi^i)^* = \pm\chi^j$ where χ^j is a regular character of G^F.*
(ii) *The image of a regular character χ^i of G^F is given $(\chi^i)^* = \pm\chi^j$ where χ^j is a semisimple character of G^F.*

Proof. This follows from the facts that χ^i is semisimple if and only if $(\chi^i, \Xi) \neq 0$, χ^i is regular if and only if $(\chi^i, \Gamma) \neq 0$, and $\Gamma^* = \Xi$. ∎

We investigate further the properties of the semisimple and regular characters in the next section.

8.4 SEMISIMPLE AND REGULAR CHARACTERS

Proposition 8.4.1. *Let T be an F-stable maximal torus of G and $\theta \in \hat{T}^F$. Let u be a regular unipotent element of G^F. Then $R_{T,\theta}(u) = 1$.*

Note. Although this result is true in general it is considerably easier to prove when $p \neq 2$, p is a good prime for G and q is sufficiently large. The proof we give will make these assumptions. For a proof which is valid without these assumptions the reader is referred to Deligne and Lusztig [1], 9.16.

We need the following lemma.

Lemma 8.4.2. *If q is sufficiently large T^F has a character θ in general position.*

Proof. We recall that $\theta \in \hat{T}^F$ is in general position if no non-identity element of $W(T)^F$ fixes θ. Let $w \in W(T)^F$ with $w \neq 1$. Then $C_T(w)$ is a proper subgroup of T which is F-stable. dim $T = r$ is the rank of G, and dim $C_T(w) \leqslant r - 1$. By 3.3.5 $|T^F|$ is a polynomial in q of degree r and $|C_T(w)^F| = |C_{T^F}(w)|$ is a polynomial of degree at most $r - 1$. *Hence* $\sum_{w \in W(T)^F, w \neq 1} |C_{T^F}(w)|$ is a polynomial in q of degree at most $r - 1$. Thus

$$|T^F| > \sum_{\substack{w \in W(T)^F \\ w \neq 1}} |C_{T^F}(w)|$$

if q is sufficiently large. Thus T^F contains an element not fixed by any $w \in W(T)^F$ with $w \neq 1$. By applying this fact to the dual group and using 4.4.1 we see that there is an element $\theta \in \hat{T}^F$ not fixed by any $w \neq 1$ in $W(T)^F$.

Proof of 8.4.1. We know by 7.2.9 that $R_{T,\theta}(u)$ is independent of θ. We suppose that q is sufficiently large that T^F has a character θ in general position. Choosing θ in this way we know that $\varepsilon_G\varepsilon_T R_{T,\theta}$ is an irreducible character of G^F. Moreover we have

$$\varepsilon_G\varepsilon_T R_{T,\theta}(1) = |G^F : T^F|_{p'} \quad \text{by 7.5.1.}$$

We suppose that p is a good prime for G. Then by 8.3.3 and 8.3.4 we see that

$$\varepsilon_G\varepsilon_T R_{T,\theta}(u) = \pm 1.$$

Thus we need only determine the sign. We know by 8.3.4 that

$$\varepsilon_G\varepsilon_T R_{T,\theta}(u) \equiv \frac{|G^F|_{p'}}{|T^F|} \bmod p.$$

Let T_0 be a maximally split torus of G and suppose T is obtained from T_0 by twisting with the element $w \in W$. Then by 3.3.5 we have

$$|T^F| = \det_{Y_0 \otimes \mathbb{R}}(q1 - F_0{}^{-1}w)$$
$$|T_0{}^F| = \det_{Y_0 \otimes \mathbb{R}}(q1 - F_0{}^{-1}).$$

We now show that if ψ is any linear map of $Y_0 \otimes \mathbb{R}$ into itself such that ψ has finite order and $\det(1 - q\psi) \in \mathbb{Z}$, then $\det(1 - q\psi) \equiv 1 \bmod p$.

Let $\omega_1, \ldots, \omega_r$ be the eigenvalues of ψ. $\omega_1, \ldots, \omega_r$ are roots of unity since ψ has finite order. Let $\det(1 - q\psi) = a$. Then we have

$$(1 - q\omega_1)(1 - q\omega_2)\ldots(1 - q\omega_l) = a.$$

Now q itself need not lie in $\mathbb{Z}$ but q^δ is a positive power of p for some positive integer δ. Thus $q, \omega_1, \ldots, \omega_l$ are all algebraic integers. Thus there is an algebraic integer ξ such that

$$1 + q\xi = a.$$

Then $q^\delta\xi^\delta = (a - 1)^\delta$ and so ξ^δ is both an algebraic integer and a rational number. Hence $\xi^\delta \in \mathbb{Z}$ and so q^δ divides $(a - 1)^\delta$ in $\mathbb{Z}$. Thus p divides $(a - 1)^\delta$, so p divides $a - 1$ and $a \equiv 1 \bmod p$.

In particular we have

$$\det(1 - qF_0w^{-1}) \equiv 1 \bmod p$$
$$\det(1 - qF_0) \equiv 1 \bmod p.$$

Thus we have

$$\frac{|T^F|}{|T_0{}^F|} = \frac{\det(qF_0 - w)}{\det(qF_0 - 1)} \equiv \det w \bmod p.$$

Also we have

$$|G^F|_{p'} = |T_0{}^F| \sum_{w \in W^F} q^{l(w)} \equiv |T_0{}^F| \bmod p.$$

It follows that

$$\varepsilon_G \varepsilon_T R_{T,\theta}(u) \equiv \frac{|G^F|_{p'}}{|T^F|} \equiv \frac{|T_0{}^F|}{|T^F|} \equiv \frac{1}{\det w} \bmod p.$$

Thus

$$R_{T,\theta} = \frac{1}{\varepsilon_G \varepsilon_T \det w}.$$

But $\varepsilon_G \varepsilon_T = \det w$ by 7.5.2 and so the result is proved.

Proposition 8.4.3. *Let T be an F-stable maximal torus of G and $\theta \in \hat{T}^F$. Then* $(\Xi, R_{T,\theta}) = 1$.

Proof.

$$(\Xi, R_{T,\theta}) = \frac{1}{|G^F|} \sum_{\substack{u \in G^F \\ \text{regular} \\ \text{unipotent}}} \Xi(u)\overline{R_{T,\theta}(u)}$$

$$= \frac{1}{|G^F|} |Z^F| q^l m$$

where m is the number of regular unipotent elements of G^F, by 8.4.1. Now 5.1.9 shows that $m = |G^F|/|Z^F|q^l$. Thus $(\Xi, R_{T,\theta}) = 1$.

Proposition 8.4.4. *Let ξ be a generalized character of G^F which satisfies*

$$(\xi, \xi) = |Z^F|q^l, \qquad (\xi, R_{T,\theta}) = \varepsilon_{T,\theta} = \pm 1$$

for all F-stable maximal tori T of G and all $\theta \in \hat{T}^F$. Then there exist distinct irreducible characters χ^κ of G^F, one associated with each geometric conjugacy class κ of pairs (T, θ), such that

$$\xi = \sum_\kappa \varepsilon_\kappa \chi^\kappa \quad \textit{where } \varepsilon_\kappa = \pm 1.$$

Moreover we have

$$\varepsilon_\kappa \chi^\kappa = \sum_{\substack{(T,\theta) \in \kappa \\ \bmod G^F}} \frac{\varepsilon_{T,\theta} R_{T,\theta}}{(R_{T,\theta}, R_{T,\theta})}$$

where the sum extends over one representative (T, θ) in each G^F-orbit on κ.

Note. Ξ satisfies the hypothesis of 8.4.4 with $\varepsilon_{T,\theta} = 1$ for all (T, θ). This follows from 8.3.1 and 8.4.3.

Proof. The number of distinct irreducible characters χ^i of G^F appearing in ξ is at most $|Z^F|q^l$, since $(\xi, \xi) = |Z^F|q^l$. However there are $|Z^F|q^l$ geometric conjugacy classes of pairs (T, θ) by 4.4.6 and the generalized characters $R_{T,\theta}$ corresponding to (T, θ) in different geometric conjugacy classes have no irreducible characters in common by 7.3.8. Since $(\xi, R_{T,\theta}) = \pm 1$ for all (T, θ) ξ must contain an irreducible component in common with each $R_{T,\theta}$. Hence ξ contains at least $|Z^F|q^l$ irreducible components. It follows that ξ contains exactly $|Z^F|q^l$ irreducible components, and that each component of ξ is a component of some $R_{T,\theta}$ for (T, θ) in a uniquely determined geometric conjugacy class κ. Let this component be denoted by χ^κ. Thus

$$\xi = \sum_{\kappa} \varepsilon_\kappa \chi^\kappa \qquad \varepsilon_\kappa \in \mathbb{Z}$$

with $\sum_\kappa \varepsilon_\kappa{}^2 = |Z^F|q^l$. Since there are $|Z^F|q^l$ summands, each ε_κ must be ± 1.

We now wish to show that $\varepsilon_\kappa \chi^\kappa$ is a linear combination of generalized characters of the form $R_{T,\theta}$. Since G^F-conjugate pairs (T, θ) give the same $R_{T,\theta}$ we may write

$$\varepsilon_\kappa \chi^\kappa = \sum_{\substack{(T,\theta) \\ \mathrm{mod}\, G^F}} \lambda_{T,\theta} R_{T,\theta} + \psi$$

where $(\psi, R_{T,\theta}) = 0$ for all $R_{T,\theta}$. Suppose $(T', \theta') \notin \kappa$. Then $(\chi^\kappa, R_{T',\theta'}) = 0$ and so

$$\lambda_{T',\theta'}(R_{T',\theta'}, R_{T',\theta'}) = 0$$

since $(R_{T,\theta}, R_{T',\theta'}) = 0$ when (T, θ) and (T', θ') are not G^F-conjugate. Hence $\lambda_{T',\theta'} = 0$ when $(T', \theta') \notin \kappa$. Thus we have

$$\varepsilon_\kappa \chi^\kappa = \sum_{\substack{(T,\theta) \in \kappa \\ \mathrm{mod}\, G^F}} \lambda_{T,\theta} R_{T,\theta} + \psi.$$

Now suppose $(T, \theta) \in \kappa$. Then we have

$$\varepsilon_{T,\theta} = (\xi, R_{T,\theta}) = (\varepsilon_\kappa \chi^\kappa, R_{T,\theta}) = \lambda_{T,\theta}(R_{T,\theta}, R_{T,\theta}).$$

Hence

$$\lambda_{T,\theta} = \frac{\varepsilon_{T,\theta}}{(R_{T,\theta}, R_{T,\theta})}.$$

It follows that

$$\varepsilon_\kappa \chi^\kappa = \sum_{\substack{(T,\theta) \in \kappa \\ \mathrm{mod}\, G^F}} \frac{\varepsilon_{T,\theta} R_{T,\theta}}{(R_{T,\theta}, R_{T,\theta})} + \psi$$

where $(\psi, R_{T,\theta}) = 0$ for all $R_{T,\theta}$. It remains to show that $\psi = 0$.

Now $(\varepsilon_\kappa \chi^\kappa, \varepsilon_\kappa \chi^\kappa) = 1$. Thus we have

$$\left(\sum_{\substack{(T,\theta) \in \kappa \\ \mathrm{mod}\, G^F}} \frac{\varepsilon_{T,\theta} R_{T,\theta}}{(R_{T,\theta}, R_{T,\theta})}, \sum_{\substack{(T,\theta) \in \kappa \\ \mathrm{mod}\, G^F}} \frac{\varepsilon_{T,\theta} R_{T,\theta}}{(R_{T,\theta}, R_{T,\theta})} \right) + (\psi, \psi) = 1.$$

Using the orthogonality relations between the $R_{T,\theta}$ for (T, θ) which are not G^F-conjugate we obtain

$$\sum_{\substack{(T,\theta)\in\kappa \\ \bmod G^F}} \frac{1}{(R_{T,\theta}, R_{T,\theta})} + (\psi, \psi) = 1.$$

Now

$$(\psi, \psi) \geqslant 0 \quad \text{and} \quad \frac{1}{(R_{T,\theta}, R_{T,\theta})} > 0.$$

Thus it will be sufficient to show that

$$\sum_{\substack{(T,\theta)\in\kappa \\ \bmod G^F}} \frac{1}{(R_{T,\theta}, R_{T,\theta})} = 1.$$

For then we shall have $(\psi, \psi) = 0$, which implies that $\psi = 0$.

Now we know that

$$\sum_{\substack{(T,\theta)\in\kappa \\ \bmod G^F}} \frac{1}{(R_{T,\theta}, R_{T,\theta})} \leqslant 1$$

and so it will be sufficient to show that

$$\sum_{\substack{(T,\theta) \\ \bmod G^F}} \frac{1}{(R_{T,\theta}, R_{T,\theta})} = |Z^F|q^l$$

since there are $|Z^F|q^l$ geometric conjugacy classes of pairs (T, θ). Now we have

$$(R_{T,\theta}, R_{T,\theta}) = |W(T)^{F,\theta}|$$

by 7.3.4, where $W(T)^{F,\theta} = \{\omega \in W(T)^F; {}^{\omega}\theta = \theta\}$. This we must show

$$\sum_{\substack{(T,\theta) \\ \bmod G^F}} \frac{1}{|W(T)^{F,\theta}|} = |Z^F|q^l.$$

We have

$$\sum_{\substack{(T,\theta) \\ \bmod G^F}} \frac{1}{|W(T)^{F,\theta}|} = \sum_{\substack{T \\ \bmod G^F}} \frac{1}{|W(T)^F|} \sum_{\substack{\theta \\ \bmod W(T)^F}} |WT)^F: W(T)^{F,\theta}|.$$

Now $|W(T)^F: W(T)^{F,\theta}|$ is the number of conjugates of $\theta \in \hat{T}^F$ under $W(T)^F$ and so

$$\sum_{\theta \bmod W(T)^F} |W(T)^F: W(T)^{F,\theta}| = |T^F|.$$

Thus we have

$$\sum_{\substack{(T,\theta) \\ \bmod G^F}} \frac{1}{|W(T)^{F,\theta}|} \quad \sum_{\substack{T \\ \bmod G^F}} \frac{|T^F|}{|W(T)^F|}.$$

Now each F-stable maximal torus T is obtained from a maximally split torus T_0 by twisting with an element $w \in W$. Two such tori are G^F-conjugate if and only if the corresponding elements of W are F-conjugate. Moreover the number of F-conjugates of w is $|W: C_{W,F}(w)|$where

$$C_{W,F}(w) = \{x \in W; x^{-1}wF(x) = w\}$$

is the F-centralizer of w. It follows that

$$\sum_{\substack{T \\ \mathrm{mod}\, G^F}} \frac{|T^F|}{|W(T)^F|} = \sum_{w \in W} \frac{|T_w{}^F|}{|W(T_w)^F||W: C_{W,F}(w)|}.$$

But by 3.3.6 we have $|W(T_w)^F| = |C_{W,F}(w)|$. Thus

$$\sum_{\substack{T \\ \mathrm{mod}\, G^F}} \frac{|T^F|}{|W(T)^F|} = \sum_{w \in W} \frac{|T_w{}^F|}{|W|}.$$

Now $|T_w{}^F| = |T_0{}^{w^{-1} \circ F}|$ and so

$$\sum_{\substack{T \\ \mathrm{mod}\, G^F}} \frac{|T^F|}{|W(T)^F|} = \frac{1}{|W|} \sum_{w \in W} |T_0{}^{w^{-1} \circ F}| = \left| \left(\frac{T_0}{W} \right)^F \right|$$

by 3.7.4. This is the number of F-stable semisimple classes of G by 3.7.2, so is equal to $|Z^F|q^l$ by 3.7.6. Thus

$$\sum_{\substack{(T,\theta) \\ \mathrm{mod}\, G^F}} \frac{1}{|W(T)^{F,\theta}|} = |Z^F|q^l \quad \text{as required.}$$

■

We wish to show next that the Gelfand–Graev character Γ satisfies the hypothesis of 8.4.4.

Proposition 8.4.5. $(\Gamma, R_{T,\theta}) = \varepsilon_G \varepsilon_T$ *for any F-stable maximal torus T of G and any $\theta \in \hat{T}^F$.*

Proof. We prove this by induction, assuming the result for proper F-stable Levi subgroups of G. To begin the induction we assume that G is a torus. Then $G = T$ and Γ is the regular representation of T. Also $R_{T,\theta} = \theta$. Since θ occurs with multiplicity 1 in the regular representation we have $(\Gamma, R_{T,\theta}) = 1$.

Now suppose that G is not a torus and that T lies in some proper F-stable parabolic subgroup of G. Then T lies in an F-stable Levi subgroup of this parabolic subgroup, by 7.4.3, and the proof of 7.5.1 shows that this is G^F-conjugate to an F-stable standard Levi subgroup L_J of G. Thus we may assume without loss of generality that $T \subseteq L_J \subset G$. We apply Rodier's lemma 8.1.5 to the generalized character $R_{T,\theta}^{L_J}$ of $L_J{}^F$. We have

$$((R_{T,\theta}^{L_J})_{P_J{}^F}{}^{G^F}, \Gamma_{G^F}) = (R_{T,\theta}^{L_J}, \Gamma_{L_J{}^F}).$$

However $(R^{L_J}_{T,\theta})_{P_J{}^F}{}^{G^F} = R^G_{T,\theta}$ by 7.4.4. Thus $(R^G_{T,\theta}, \Gamma_{G^F}) = (R^{L_J}_{T,\theta}, \Gamma_{L_J{}^F})$. By induction we have $(R^{L_J}_{T,\theta}, \Gamma_{L_J{}^F}) = \varepsilon_{L_J}\varepsilon_T$. Thus

$$(R^G_{T,\theta}, \Gamma_{G^F}) = \varepsilon_{L_J}\varepsilon_T = \varepsilon_G\varepsilon_T$$

since $\varepsilon_{L_J} = \varepsilon_G$.

Suppose now that T lies in no proper F-stable parabolic subgroup of G. By 8.1.7 we have

$$\sum_J (-1)^{|J'|}(T_{P_J{}^F/U_J{}^F}(\Gamma))^{G^F} = \Xi$$

where the sum extends over all ρ-stable subsets J of I. By 8.4.3 we have $(\Xi, R_{T,\theta}) = 1$. We shall show that

$$((T_{P_J{}^F/U_J{}^F}(\Gamma))^{G^F}, R_{T,\theta}) = 0 \quad \text{if } J \neq I.$$

It will follow that

$$(-1)^{|I'|}(\Gamma, R_{T,\theta}) = 1.$$

However by 6.5.3 and 6.5.5 we have $\varepsilon_G\varepsilon_T = (-1)^{|I'|}$. Thus

$$(\Gamma, R_{T,\theta}) = \varepsilon_G\varepsilon_T$$

as required.

Thus suppose J is a proper ρ-stable subset of I. By induction the Gelfand–Graev character $\Gamma_{L_J{}^F}$ satisfies the hypotheses of 8.4.4 and so is a linear combination of generalized characters of the form $R^{L_J}_{T',\theta'}$. Also $T_{P_J{}^F/U_J{}^F}(\Gamma)$ is equal to $\Gamma_{L_J{}^F}$ lifted to $P_J{}^F$, by 8.1.6. Thus $(T_{P_J{}^F/U_J{}^F}(\Gamma))^{G^F} = (\Gamma_{L_J{}^F})_{P_J{}^F}{}^{G^F}$ is a linear combination of generalized characters of the form $(R^{L_J}_{T',\theta'})_{P_J{}^F}{}^{G^F} = R^G_{T',\theta'}$ by 7.4.4. All the maximal tori T' occurring in the sum lie in L_J. Since T does not lie in any G^F-conjugate of L_J we see that T, T' cannot be G^F-conjugate. Thus $(R_{T,\theta}, R_{T',\theta'}) = 0$ for all (T', θ') occurring in the sum. Hence

$$((T_{P_J{}^F/U_J{}^F}(\Gamma))^{G^F}, R_{T,\theta}) = 0 \quad \text{if } J \neq I$$

and the result follows.

Proposition 8.4.6. *For each geometric conjugacy class κ of pairs (T, θ) the class function*

$$\sum_{\substack{(T,\theta)\in\kappa \\ \text{mod } G^F}} \frac{R_{T,\theta}}{(R_{T,\theta}, R_{T,\theta})}$$

is, to within sign, an irreducible character of G^F. The $|Z^F|q^l$ irreducible characters of G^F obtained in this way are all distinct, and are the semisimple characters of G^F.

Proof. This follows from 8.4.4 by putting $\xi = \Xi$. We have $(\Xi, R_{T,\theta}) = 1$ by 8.4.3. The irreducible characters of G^F obtainable in this way are precisely those which are components of Ξ. These are the semisimple characters of G^F, by the proof of 8.3.3.

Proposition 8.4.7. *For each geometric conjugacy class κ of pairs (T, θ) the class function*

$$\sum_{\substack{(T,\theta)\in\kappa \\ \text{mod } G^F}} \frac{\varepsilon_G \varepsilon_T R_{T,\theta}}{(R_{T,\theta}, R_{T,\theta})}$$

is an irreducible character of G^F. The $|Z^F|q^l$ irreducible characters of G^F obtained in this way are all distinct, and are the regular characters of G^F.

Proof. This follows from 8.4.4 by putting $\zeta = \Gamma$. By 8.4.5 we have $\varepsilon_{T,\theta} = \varepsilon_G \varepsilon_T$. Thus the above class function is, to within sign, an irreducible character of G^F. However its value at the identity element is positive, thus the character is itself irreducible. The irreducible characters obtained in this way are the components of the Gelfand–Graev character Γ. These are the regular characters of G^F. ∎

For each geometric conjugacy class κ of pairs (T, θ) we define

$$\chi_\kappa^{ss} = \pm \sum_{\substack{(T,\theta)\in\kappa \\ \text{mod } G^F}} \frac{R_{T,\theta}}{(R_{T,\theta}, R_{T,\theta})}$$

where the sign is chosen so that $\chi_\kappa^{ss}(1) > 0$. We also define

$$\chi_\kappa^{reg} = \sum_{\substack{(T,\theta)\in\kappa \\ \text{mod } G^F}} \frac{\varepsilon_G \varepsilon_T R_{T,\theta}}{(R_{T,\theta}, R_{T,\theta})}.$$

χ_κ^{ss}, χ_κ^{reg} are respectively the semisimple and regular characters of G^F associated with the geometric conjugacy class κ.

We now determine the degrees of the semisimple and regular characters. We recall from 4.4.6 that there is a bijective correspondence between the geometric conjugacy classes of pairs (T, θ) in G and the F^*-stable semisimple conjugacy classes in the dual group G^*. Now the centre $Z(G)$ is connected so, by 4.5.9, the dual group G^* satisfies the condition that the centralizer of each semisimple element is connected. The Lang–Steinberg theorem thus implies that each F^*-stable semisimple conjugacy class of G^* intersects G^{*F^*} in a semisimple conjugacy class of G^{*F^*}.

Theorem 8.4.8. *Let κ be a geometric conjugacy class of pairs (T, θ) in G. Let $s^* \in G^{*F^*}$ be an element of the semisimple class of the dual group corresponding to the geometric conjugacy class κ. Then the degree of χ_κ^{ss} is given by*

$$\chi_\kappa{}^{ss}(1) = |G^{*F^*} : C_{G^{*F^*}}(s^*)|_{p'}.$$

Proof. We have

$$\chi_\kappa^{ss} = \varepsilon \sum_{\substack{(T,\theta)\in\kappa \\ \text{mod } G^F}} \frac{R_{T,\theta}}{(R_{T,\theta}, R_{T,\theta})} \quad \text{where } \varepsilon = \pm 1.$$

We multiply both sides by the Steinberg character and obtain

$$\chi_\kappa^{ss} . \mathrm{St} = \varepsilon \sum_{\substack{(T,\theta)\in\kappa \\ \mathrm{mod}\, G^F}} \frac{R_{T,\theta} . \mathrm{St}}{(R_{T,\theta}, R_{T,\theta})}$$

$$= \varepsilon \sum_{\substack{(T,\theta)\in\kappa \\ \mathrm{mod}\, G^F}} \frac{\varepsilon_G \varepsilon_T {\theta_{T^F}}^{G^F}}{(R_{T,\theta}, R_{T,\theta})}, \quad \text{by 7.5.4}$$

$$= \varepsilon\varepsilon_G \sum_{\substack{(T,\theta)\in\kappa \\ \mathrm{mod}\, G^F}} \frac{\varepsilon_T {\theta_{T^F}}^{G^F}}{|W(T)^{F,\theta}|} \quad \text{by 7.3.4.}$$

In particular we have

$$\chi_\kappa^{ss}(1)\, \mathrm{St}(1) = \varepsilon\varepsilon_G \sum_{\substack{(T,\theta)\in\kappa \\ \mathrm{mod}\, G^F}} \frac{\varepsilon_T |G^F : T^F|}{|W(T)^{F,\theta}|}.$$

We now recall some results on the duality of reductive groups. Corresponding to the F-stable maximal torus T of G we can find an F^*-stable maximal torus T^* of G^* which is in duality with T, by 4.3.4. Moreover we have an isomorphism between the character group $\hat{T}^F$ and the torus T^{*F^*}, by 4.4.1. Let $s^* \in T^{*F^*}$ correspond to $\theta \in \hat{T}^F$ under this isomorphism. We also have an isomorphism between $W(T)$ and $W(T^*)$ in which the F-action on $W(T)$ is transformed into the inverse of the F^*-action on $W(T^*)$ (see 4.3.2 and 4.3.4). Thus $W(T)^F$ maps to $W(T^*)^{F^*}$ under this isomorphism and $W(T)^{F,\theta}$ maps to $W(T^*)^{F^*,s^*}$ where

$$W(T)^{F,\theta} = \{w \in W(T)^F;\ {}^w\theta = \theta\}$$
$$W(T^*)^{F^*,s^*} = \{w^* \in W(T^*)^{F^*};\ s^{*w^*} = s^*\}$$
$$= (W^{C(s^*)}(T^*))^{F^*}.$$

In particular we have

$$|W(T)^{F,\theta}| = |(W^{C(s^*)}(T^*))^{F^*}|.$$

By 4.5.9 $C_{G^*}(s^*)$ is connected. Let $S^* = C_{G^*}(s^*)$. Then we have

$$\chi_\kappa^{ss}(1)\, \mathrm{St}(1) = \varepsilon\varepsilon_G \sum_{\substack{(T,\theta)\in\kappa \\ \mathrm{mod}\, G^F}} \frac{\varepsilon_{T^*} |G^{*F^*} : T^{*F^*}|}{|(W^{S^*}(T^*))^{F^*}|}$$

since $|G^{*F^*}| = |G^F|$, $|T^{*F^*}| = |T^F|$ and $\varepsilon_{T^*} = \varepsilon_T$ by 4.4.2, 4.4.4 and the definition of the relative rank. Now to sum over all the G^F-classes of pairs $(T, \theta) \in \kappa$ is equivalent to summing in the dual group over all G^{*F^*}-classes of pairs $s^* \in T^*$ with $s^* \in C^*$, where s^* is an F^*-stable semisimple element and C^* is the semisimple class of G^* corresponding to κ. It follows that

$$\chi_\kappa^{ss}(1)\, \mathrm{St}(1) = \varepsilon\varepsilon_G \sum_{\substack{(s^*\in T^*)\,\mathrm{mod}\, G^{*F^*} \\ s^*\in C^{*F^*}}} \frac{\varepsilon_{T^*} |G^{*F^*} : T^{*F^*}|}{|(W^{S^*}(T^*))^{F^*}|}.$$

Now by 3.5.2 we know $s^* \in T^*$ if and only if $T^* \subseteq S^*$. We may fix an F^*-stable element $s^* \in C^*$ since all such elements are conjugate in G^{*F^*}. The sum then extends over all F^*-stable maximal tori T^* with $T^* \subseteq S^*$, taken modulo $C_{G^{*F^*}}(s^*) = S^{*F^*}$. Thus we have

$$\chi_\kappa^{ss}(1)\,\mathrm{St}(1) = \varepsilon\varepsilon_G \sum_{\substack{T^* \subseteq S^* \\ \bmod S^{*F^*}}} \frac{\varepsilon_{T^*}|G^{*F^*} : T^{*F^*}|}{|(W^{S^*}(T^*))^{F^*}|}$$

$$= \varepsilon\varepsilon_G |G^{*F^*} : S^{*F^*}| \sum_{\substack{T^* \subseteq S^* \\ \bmod S^{*F^*}}} \frac{\varepsilon_{T^*}|S^{*F^*} : T^{*F^*}|}{|W^{S^*}(T^*)^{F^*}|}.$$

We now recall from 7.6.5 that

$$\sum_{\substack{T \\ \bmod G^F}} \frac{R_{T,1}}{|W(T)^F|} = 1_{G^F}.$$

Multiplying by the Steinberg character of G^F we obtain

$$\sum_{\substack{T \\ \bmod G^F}} \frac{R_{T,1}\,.\,\mathrm{St}}{|W(T)^F|} = \mathrm{St}.$$

By 7.5.4 this is equivalent to

$$\sum_{\substack{T \\ \bmod G^F}} \frac{\varepsilon_G\varepsilon_T 1_{T^F}{}^{G^F}}{|W(T)^F|} = \mathrm{St}.$$

We evaluate both sides of the identity and obtain

$$\varepsilon_G \sum_{\substack{T \\ \bmod G^F}} \frac{\varepsilon_T|G^F : T^F|}{|W(T)^F|} = |G^F|_p.$$

We now apply this formula to the group S^*. Thus

$$\varepsilon_{S^*} \sum_{\substack{T^* \subseteq S^* \\ \bmod S^{*F^*}}} \frac{\varepsilon_{T^*}|S^{*F^*} : T^{*F^*}|}{|W^{S^*}(T^*)^{F^*}|} = |S^{*F^*}|_p.$$

Hence

$$\chi_\kappa^{ss}(1)\,\mathrm{St}(1) = \varepsilon\varepsilon_G|G^{*F^*} : S^{*F^*}|\varepsilon_{S^*}|S^{*F^*}|_p.$$

Since $\mathrm{St}(1) = |G^F|_p = |G^{*F^*}|_p$ we obtain

$$\chi_\kappa^{ss}(1) = \varepsilon\varepsilon_G\varepsilon_{S^*}|G^{*F^*} : S^{*F^*}|_{p'}.$$

Since $\chi_\kappa^{ss}(1) > 0$ we have $\varepsilon = \varepsilon_G\varepsilon_{S^*}$. Thus

$$\chi_\kappa^{ss}(1) = |G^{*F^*} : S^{*F^*}|_{p'}$$

and the theorem is proved. ∎

This result exhibits a striking connection between the semisimple characters of G^F and the semisimple conjugacy classes of the dual group G^{*F^*}. There is a bijective correspondence between semisimple characters of G^F and semisimple classes of G^{*F^*} such that the degree of a semisimple character of G^F is the p'-part of the index of the centralizer of a semisimple element of G^{*F^*} in the corresponding semisimple class.

We now prove a similar result for the degrees of the regular characters of G^F.

Theorem 8.4.9. *Let κ be a geometric conjugacy class of pairs (T, θ) in G. Let $s^* \in G^{*F^*}$ be an element in the semisimple class of the dual group corresponding to κ. Then*

$$\chi_\kappa^{\mathrm{reg}}(1) = |G^{*F^*}: C_{G^*F^*}(s^*)|_{p'} |C_{G^*F^*}(s^*)|_p.$$

Proof. As before we let $S^* = C_{G^*}(s^*)$. We have

$$\chi_\kappa^{\mathrm{reg}} = \sum_{\substack{(T,\theta)\in\kappa \\ \mathrm{mod}\, G^F}} \frac{\varepsilon_G \varepsilon_T R_{T,\theta}}{(R_{T,\theta}, R_{T,\theta})}.$$

It follows that

$$\begin{aligned}\chi_\kappa^{\mathrm{reg}}\,.\,\mathrm{St} &= \sum_{\substack{(T,\theta)\in\kappa \\ \mathrm{mod}\, G^F}} \frac{\varepsilon_G \varepsilon_T R_{T,\theta}\,.\,\mathrm{St}}{(R_{T,\theta}, R_{T,\theta})} \\ &= \sum_{\substack{(T,\theta)\in\kappa \\ \mathrm{mod}\, G^F}} \frac{\theta_{T^F}{}^{G^F}}{(R_{T,\theta}, R_{T,\theta})} \quad \text{by 7.5.4.}\end{aligned}$$

Evaluating at the identity we obtain

$$\chi_\kappa^{\mathrm{reg}}(1)\,.\,\mathrm{St}(1) = \sum_{\substack{(T,\theta)\in\kappa \\ \mathrm{mod}\, G^F}} \frac{|G^F: T^F|}{(R_{T,\theta}, R_{T,\theta})} = \sum_{\substack{(T,\theta)\in\kappa \\ \mathrm{mod}\, G^F}} \frac{|G^F: T^F|}{|W(T)^{F,\theta}|}$$

by 7.3.4. We now express the right-hand side in terms of the dual group. As in the proof of 8.4.8 we have

$$\chi_\kappa^{\mathrm{reg}}(1)\,.\,\mathrm{St}(1) = \sum_{\substack{(s^*\in T^*)\,\mathrm{mod}\, G^{*F^*} \\ s^*\in C^{*F^*}}} \frac{|G^{*F^*}: T^{*F^*}|}{|(W^{S^*}(T^*))^{F^*}|}$$

where C^* is the semisimple class in G^* corresponding to κ. As in 8.4.8 we can choose a fixed $s^* \in C^{*F^*}$ and sum over all $T^* \subseteq S^*$ mod S^{*F^*}. Thus

$$\begin{aligned}\chi_\kappa^{\mathrm{reg}}(1)\,.\,\mathrm{St}(1) &= \sum_{\substack{T^*\subseteq S^* \\ \mathrm{mod}\, S^{*F^*}}} \frac{|G^{*F^*}: T^{*F^*}|}{|(W^{S^*}(T^*))^{F^*}|} \\ &= |G^{*F^*}: S^{*F^*}| \sum_{\substack{T^*\subseteq S^* \\ \mathrm{mod}\, S^{*F^*}}} \frac{|S^{*F^*}: T^{*F^*}|}{|(W^{S^*}(T^*))^{F^*}|}.\end{aligned}$$

We now recall from 7.6.6 that

$$\sum_{\substack{T\\ \mathrm{mod}\, G^F}} \frac{\varepsilon_G \varepsilon_T R_{T,1}}{|W(T)^F|} = \mathrm{St}.$$

Multiplying both sides by St we obtain

$$\sum_{\substack{T\\ \mathrm{mod}\, G^F}} \frac{\varepsilon_G \varepsilon_T R_{T,1} \,.\, \mathrm{St}}{|W(T)^F|} = \mathrm{St} \,.\, \mathrm{St}$$

which by 7.5.4 gives

$$\sum_{\substack{T\\ \mathrm{mod}\, G^F}} \frac{1_{T^F}{}^{G^F}}{|W(T)^F|} = \mathrm{St} \,.\, \mathrm{St}.$$

Evaluating at the identity we obtain

$$\sum_{\substack{T\\ \mathrm{mod}\, G^F}} \frac{|G^F : T^F|}{|W(T)^F|} = \mathrm{St}(1)^2 = |\mathrm{G}^F|_p{}^2.$$

We now apply this result to the group S^{*F^*}. This gives

$$\sum_{\substack{T^* \subseteq S^*\\ \mathrm{mod}\, S^{*F^*}}} \frac{|S^{*F^*} : T^{*F^*}|}{|(W^{S^*}(T^*))^{F^*}|} = |S^{*F^*}|_p^2.$$

Hence

$$\chi_\kappa^{\mathrm{reg}}(1) \,.\, \mathrm{St}(1) = |G^{*F^*} : S^{*F^*}| \,.\, |S^{*F^*}|_p{}^2.$$

Since $\mathrm{St}(1) = |G^F|_p = |G^{*F^*}|_p$ we have

$$\chi_\kappa^{\mathrm{reg}}(1) = |G^{*F^*} : S^{*F^*}|_{p'} \,.\, |S^{*F^*}|_p$$

and the theorem is proved. ∎

Thus we now know the degrees of all the irreducible components of the Gelfand–Graev character of G^F.

We emphasize finally that the results in this chapter are valid only when the centre of G is connected. The situation when the centre is not connected can be more complicated. Information about the case when the centre is not connected can be found in Lusztig [21], chapter 14.

CHAPTER 9

CUSPIDAL REPRESENTATIONS

In the present chapter we introduce a class of irreducible characters of a finite group with split BN-pair called cuspidal characters. We shall show that, in a certain sense, the study of all irreducible characters of such groups can be reduced to the study of the cuspidal characters and the way in which such cuspidal characters decompose when induced from proper parabolic subgroups. At a later stage we shall specialize to the case when the finite group has the form G^F where G is a connected reductive group and F is a Frobenius map.

9.1 CUSPIDAL CHARACTERS INDUCED FROM PARABOLIC SUBGROUPS

Let G be a finite group with a split BN-pair which satisfies the commutator relations. Let χ be an irreducible character of G. Let P_J be a standard parabolic subgroup of G and U_J be the maximal normal unipotent subgroup of P_J. We recall that the truncation $T_{P_J/U_J}(\chi)$ is the character of P_J defined by

$$(T_{P_J/U_J}(\chi))(p) = \frac{1}{|U_J|} \sum_{u \in U_J} \chi(up)$$

for $p \in P_J$.

Definition. The irreducible character χ of G is called cuspidal if $T_{P_J/U_J}(\chi) = 0$ for all standard parabolic subgroups $P_J \neq G$.

It follows from this definition that χ is cuspidal if and only if $T_{P/U_P}(\chi) = 0$ for all proper parabolic subgroups P of G, where U_P is the maximal normal unipotent subgroup of P. For each such parabolic subgroup is conjugate to one of the standard ones P_J.

If G has no proper parabolic subgroups at all then the truncation condition is vacuously satisfied and so all the irreducible characters are cuspidal. In this case we have $G = B = N$ and $B = UH$ with $U = 1$. Thus $G = H$, and G is an abelian group of order prime to p where p is the characteristic of the split BN-pair. Conversely an abelian group of order prime to p has a split BN-pair of characteristic p in which $G = B = N$. In such a group all the irreducible characters are cuspidal.

Proposition 9.1.1. *Let χ be an irreducible character of G. Then χ is cuspidal if and only if $(\chi, 1_{U_J}{}^G) = 0$ for all $J \neq I$.*

Proof. Suppose χ is cuspidal. Then for any $J \neq I$ we have

$$(T_{P_J/U_J}(\chi))(1) = \frac{1}{|U_J|} \sum_{u \in U_J} \chi(u) = 0$$

and so

$$(\chi, 1_{U_J}{}^G) = (\chi, 1)_{U_J} = 0$$

by Frobenius reciprocity.

Conversely suppose that $(\chi, 1_{U_J}{}^G) = 0$ for all $J \neq I$. Then $(\chi, 1)_{U_J} = 0$ for all $J \neq I$. Let ρ be a representation of G with character χ. Then, for $J \neq I$, ρ decomposes on restriction to U_J into irreducible components, none of which is 1_{U_J}. Let ρ' be one of these irreducible components. Then we have

$$(\rho'_{ij}, (1_{U_J})_{11}) = 0 \qquad \text{for all } i, j$$

by the orthogonality relations mentioned in section 6.1. Thus

$$\sum_{u \in U_J} \rho'_{ij}(u) = 0 \qquad \text{for all } i, j.$$

It follows that

$$\sum_{u \in U_J} \rho'(u) = 0$$

for all components ρ' of ρ_{U_J}. Hence

$$\sum_{u \in U_J} \rho(u) = 0.$$

It follows that, for any $g \in G$, we have

$$\left(\sum_{u \in U_J} \rho(u)\right)\rho(g) = 0.$$

Thus

$$\sum_{u \in U_J} \rho(ug) = 0$$

and so, taking traces, we have

$$\sum_{u \in U_J} \chi(ug) = 0.$$

In particular we have $(T_{P_J/U_J}(\chi))(p) = 0$ for all $p \in P_J$ and so χ is cuspidal.

Corollary 9.1.2. *Let χ be an irreducible character of G. Then the following conditions on χ are equivalent:*

(i) *χ is cuspidal.*
(ii) *$(\chi, 1_{U_J}{}^G) = 0$ for all $J \neq I$.*
(iii) *$\sum_{u \in U_J} \chi(ug) = 0$ for all $g \in G$ and all $J \neq I$.*
(iv) *$\sum_{u \in U_J} \chi(gu) = 0$ for all $g \in G$ and all $J \neq I$.*

Proof. This is an easy consequence of the proof of 9.1.1. ■

Now let $P_J = U_J L_J$ be the standard Levi decomposition of P_J. For each irreducible character ϕ of L_J we have a corresponding irreducible character ϕ_{P_J} of P_J defined by

$$\phi_{P_J}(ul) = \phi(l) \qquad u \in U_J, l \in L_J.$$

ϕ_{P_J} is called the lift of ϕ to P_J. Our next proposition asserts that each irreducible character of G is a component of some induced character $\phi_{P_J}{}^G$ for some J and some cuspidal character ϕ of L_J.

Proposition 9.1.3. *Let χ be any irreducible character of G. Then there exists a subset $J \subseteq I$ and a cuspidal irreducible character ϕ of L_J such that $(\chi, \phi_{P_J}{}^G) \neq 0$.*

Proof. Let $\mathscr{S} = \{J \subseteq I; (\chi_{U_J}, 1_{U_J} \neq 0\}$. Then $\mathscr{S}$ is non-empty since $I \in \mathscr{S}$. Let J be a minimal element of $\mathscr{S}$. Let V be an irreducible G-module affording the character χ and let V' be given by

$$V' = \{v \in V; vu = v \text{ for all } u \in U_J\}.$$

Then $V' \neq 0$ since $(\chi_{U_J}, 1_{U_J}) \neq 0$. Also V' is a P_J-module, since if $v \in V'$, $p \in P_J$, $u \in U_J$ we have

$$(vp)u = vpup^{-1}p = vp$$

and so $vp \in V'$.

We consider V' as an L_J-module and let its character be ϕ. Let $\phi = \sum_i \phi_i$ where each ϕ_i is an irreducible character of L_J. Now V' affords the character $\phi_{P_J} = \sum_i (\phi_i)_{P_J}$ of P_J and V affords the character χ_{P_J} of P_J. Since V' is a P_J-submodule of V each $(\phi_i)_{P_J}$ is a component of χ_{P_J}. Thus we have

$$((\phi_i)_{P_J}{}^G, \chi) = ((\phi_i)_{P_J}, \chi_{P_J}) \neq 0.$$

Thus χ occurs as component of the induced character $(\phi_i)_{P_J}{}^G$. We shall now show that each ϕ_i is a cuspidal character of L_J.

Suppose that some ϕ_i is not cuspidal. We recall that the standard parabolic subgroups of L_J are the subgroups $P_K \cap L_J$ for $K \subseteq J$ and that the Levi decomposition of $P_K \cap L_J$ is

$$P_K \cap L_J = (U_K \cap L_J)L_K.$$

Thus for some proper subset $K \subset J$ we have

$$(\phi_i, 1_{U_K \cap L_J}) \neq 0.$$

Hence there exists $v \neq 0$ in V' with $vx = v$ for all $x \in U_K \cap L_J$. But $vx = v$ for all $x \in U_J$ and we have $U_K = U_J(U_K \cap L_J)$. Thus $vx = v$ for all $x \in U_K$. It follows that $(\chi_{U_K}, 1_{U_K}) \neq 0$. Hence $K \in \mathcal{S}$. However K is a proper subset of J and J is a minimal element of $\mathcal{S}$, so we have a contradiction. Thus ϕ_i must be cuspidal and the result is proved. ■

We next consider when two induced characters $(\phi_1)_{P_{J_1}}{}^G, (\phi_2)_{P_{J_2}}{}^G$ can have a common irreducible component, when ϕ_1, ϕ_2 are cuspidal irreducible characters of L_{J_1}, L_{J_2} respectively. We first prove a lemma.

Lemma 9.1.4. *Suppose $J_1, J_2 \subseteq I$ and let $n \in N_{J_1,J_2}$ satisfy $L_{J_1} = {}^nL_{J_2}$. Let ϕ_1, ϕ_2 be irreducible characters of L_{J_1}, L_{J_2} respectively with $\phi_1 = {}^n\phi_2$. Then ϕ_1 is cuspidal if and only if ϕ_2 is cuspidal.*

Proof. Let $w = \pi(n)$. Then $w \in D_{J_1,J_2}$ and we have $w(J_2) = J_1$ by 2.8.8. The proper standard parabolic subgroups of L_{J_2} are the subgroups $P_J \cap L_{J_2}$ for $J \subset J_2$. We have

$$^n(P_J \cap L_{J_2}) = {}^nP_J \cap L_{J_1} = P_K \cap L_{J_1}$$

where $K = J_1 \cap w(J)$, by 2.8.9, since $w \in D_{J_1,J}$. However $w(J) \subseteq J_1$ and so $K = w(J)$. Thus $P_K \cap L_{J_1}$ is a standard parabolic subgroup of L_{J_1}.

We see therefore that conjugation by n transforms standard parabolic subgroups of L_{J_2} to standard parabolic subgroups of L_{J_1}. It follows from the definition of a cuspidal character that conjugation by n transforms cuspidal characters of L_{J_2} to cuspidal characters of L_{J_1}.

Proposition 9.1.5. *Let ϕ_1, ϕ_2 be cuspidal irreducible characters of the standard Levi subgroups L_{J_1}, L_{J_2} of G respectively. Suppose*

$$((\phi_1)_{P_{J_1}}{}^G, (\phi_2)_{P_{J_2}}{}^G) \neq 0.$$

Then there exists $n \in N_{J_1,J_2}$ such that $L_{J_1} = {}^nL_{J_2}$ and $\phi_1 = {}^n\phi_2$. Moreover $(\phi_1)_{P_{J_1}}{}^G = (\phi_2)_{P_{J_2}}^G$.

Proof. Since N_{J_1,J_2} is a set of double coset representatives for G with respect to P_{J_1} and P_{J_2} we have

$$((\phi_1)_{P_{J_1}}{}^G, (\phi_2)_{P_{J_2}}{}^G) = \sum_{n \in N_{J_1,J_2}} ((\phi_1)_{P_{J_1}}, {}^n((\phi_2)_{P_{J_2}}))_{P_{J_1} \cap {}^nP_{J_2}}$$

by Mackey's formula. By 2.8.7 we have a decomposition of $P_{J_1} \cap {}^nP_{J_2}$ given by

$$P_{J_1} \cap {}^nP_{J_2} = (U_{J_1} \cap {}^nU_{J_2})(U_{J_1} \cap {}^nL_{J_2})(L_{J_1} \cap {}^nU_{J_2})L_K$$

where $K = J_1 \cap w(J_2)$ and $w = \pi(n)$. Each element of $P_{J_1} \cap {}^nP_{J_2}$ has a unique expression of the form $xyzt$ where $x \in U_{J_1} \cap {}^nU_{J_2}$, $y \in U_{J_1} \cap {}^nL_{J_2}$, $z \in L_{J_1} \cap {}^nU_{J_2}$ and $t \in L_K$. Thus we have

$$((\phi_1)_{P_{J_1}}, {}^n((\phi_2)_{P_{J_2}}))_{P_{J_1} \cap {}^nP_{J_2}} = \frac{1}{|P_{J_1} \cap {}^nP_{J_2}|} \sum_{x,y,z,t} ((\phi_1)_{P_{J_1}}(xyzt))\overline{({}^n((\phi_2)_{P_{J_2}})(xyzt)}$$

Now $U_{J_1} \cap {}^nU_{J_2}$ lies in the kernel of both $(\phi_1)_{P_{J_1}}$ and ${}^n((\phi_2)_{P_{J_2}})$. Also $U_{J_1} \cap {}^nL_{J_2}$ lies in the kernel of $(\phi_1)_{P_{J_1}}$ whereas $L_{J_1} \cap {}^nU_{J_2}$ lies in the kernel of ${}^n((\phi_2)_{P_{J_2}})$. Thus we have

$$((\phi_1)_{P_{J_1}}, {}^n((\phi_2)_{P_{J_2}}))_{P_{J_1} \cap {}^nP_{J_2}} = \frac{|U_{J_1} \cap {}^nU_{J_2}|}{|P_{J_1} \cap {}^nP_{J_2}|} \sum_t \left(\sum_z \phi_1(zt)\right)\overline{\left(\sum_y {}^n\phi_2(yt)\right)}.$$

Now suppose that $((\phi_1)_{P_{J_1}}{}^G, (\phi_2)_{P_{J_2}}{}^G) \neq 0$. Then there exists $n \in N_{J_1,J_2}$ such that $((\phi_1)_{P_{J_1}}, {}^n((\phi_2)_{P_{J_2}}))_{P_{J_1} \cap {}^nP_{J_2}} \neq 0$. That there exists $t \in L_K$ such that

$$\sum_{z \in L_{J_1} \cap {}^nU_{J_2}} \phi_1(zt) \neq 0 \quad \text{and} \quad \sum_{y \in U_{J_1} \cap {}^nL_{J_2}} {}^n\phi_2(yt) \neq 0.$$

However $L_{J_1} \cap {}^nU_{J_2}$ is the maximal normal unipotent subgroup of the parabolic subgroup $L_{J_1} \cap {}^nP_{J_2}$ of L_{J_1} and $U_{J_1} \cap {}^nL_{J_2}$ is the maximal normal unipotent subgroup of the parabolic subgroup $P_{J_1} \cap {}^nL_{J_2}$ of ${}^nL_{J_2}$. But ϕ_1 and ${}^n\phi_2$ are cuspidal characters of L_{J_1}, ${}^nL_{J_2}$ respectively and so by 9.1.2 we must have

$$L_{J_1} \cap {}^nU_{J_2} = U_{J_1} \cap {}^nL_{J_2} = 1.$$

By 2.8.8 we see that $L_{J_1} \subseteq {}^nL_{J_2}$ and $L_{J_2} \subseteq {}^{n^{-1}}L_{J_1}$. Thus $L_{J_1} = {}^nL_{J_2}$.

Finally we show that $\phi_1 = {}^n\phi_2$. We have

$$\begin{aligned} ((\phi_1)_{P_{J_1}}, {}^n((\phi_2)_{P_{J_2}}))_{P_{J_1} \cap {}^nP_{J_2}} &= \frac{|U_{J_1} \cap {}^nU_{J_2}|}{|P_{J_1} \cap {}^nP_{J_2}|} \sum_{t \in L_K} \phi_1(t)\overline{({}^n\phi_2)(t)} \\ &= \frac{|U_{J_1} \cap {}^nU_{J_2}|}{|P_{J_1} \cap {}^nP_{J_2}|} |L_K|(\phi_1, {}^n\phi_2)_{L_K} \\ &= (\phi_1, {}^n\phi_2)_{L_K}. \end{aligned}$$

Since $((\phi_1)_{P_{J_1}}, {}^n((\phi_2)_{P_{J_2}}))_{P_{J_1} \cap {}^nP_{J_2}} \neq 0$ for the given element $n \in N_{J_1,J_2}$ we have $(\phi_1, {}^n\phi_2)_{L_K} \neq 0$. However since $L_{J_1} = {}^nL_{J_2}$ we have $K = J_1 \cap w(J_2) = J_1$ and $\phi_1, {}^n\phi_2$ are both irreducible characters of $L_{J_1} = L_K$. Since $(\phi_1, {}^n\phi_2)_{L_K} \neq 0$ we must therefore have $\phi_1 = {}^n\phi_2$.

Finally by 8.2.7 we have

$$(\phi_1)_{P_{J_1}}{}^G = ({}^n\phi_2)_{{}^nP_{J_2}}{}^G = (\phi_2)_{P_{J_2}}{}^G.$$

9.2 THE DECOMPOSITION OF THE SET OF IRREDUCIBLE CHARACTERS INTO SERIES

The results of section 9.1 indicate that there is a natural way of dividing the set of irreducible characters of a group G with split BN-pair into equivalence classes, which will be called series. We shall explore this relation in the present section.

For any subset $J \subseteq I$ we define C_J by $C_J = \{w \in W; w(J) = J\}$.

Lemma 9.2.1. *For any subset $J \subseteq I$, N_J is a normal subgroup of $N_G(L_J) \cap N$, and*

$$N_G(L_J) \cap N/N_J \cong N_W(W_J)/W_J = C_J W_J/W_J \cong C_J.$$

Proof. $N_J = N \cap L_J$ by 2.6.3 and so $N_J \subseteq N_G(L_J) \cap N$. Let $n \in N_G(L_J) \cap N$. Then we have

$$N_J{}^n = N \cap L_J{}^n = N \cap L_J = N_J.$$

Thus N_J is normal in $N_G(L_J) \cap N$. Now we have

$$N_G(L_J) \cap N/N_J \cong N_W(L_J)/W_J.$$

Moreover, for $w \in W$ we have

$$w \in N_W(L_J) \Leftrightarrow w(\Phi_J) = \Phi_J \Leftrightarrow w \in N_W(W_J).$$

Hence

$$N_G(L_J) \cap N/N_J \cong N_W(W_J)/W_J.$$

Now let $w \in N_W(W_J)$. Then $w = d_J w_J$ with $d_J \in D_J$, $w_J \in W_J$. But $d_J(J) \subseteq \Phi^+$ and since $d_J \in N(W_J)$ we have $d_J(J) \in \Phi_J{}^+$. Hence $d_J(\Phi_J{}^+) = \Phi_J{}^+$ and so $d_J(J) = J$. Thus $d_J \in C_J$ and $N_W(W_J) = C_J W_J$. However we also have $C_J \cap W_J = 1$. It follows that

$$N_W(W_J)/W_J = C_J W_J/W_J \cong C_J.$$

Proposition 9.2.2. *Let J_1, J_2 be subsets of I. Then the following conditions are equivalent:*

(i) *$J_1 = w(J_2)$ for some $w \in W$.*
(ii) *$\Phi_{J_1} = w(\Phi_{J_2})$ for some $w \in W$.*
(iii) *$L_{J_1} = {}^nL_{J_2}$ for some $n \in N$.*
(iv) *$L_{J_1} = {}^nL_{J_2}$ for some $n \in N_{J_1,J_2}$.*

Proof. It is evident that the first three conditions are equivalent. To see that (iii) implies (iv) let $L_{J_1} = {}^nL_{J_2}$ for $n \in N$. Then $n \in N_{J_1} n' N_{J_2}$ for some $n' \in N_{J_1,J_2}$. Thus $L_{J_1} = {}^{n'}L_{J_2}$ as required. ∎

If the conditions of 9.2.2 are satisfied by J_1, J_2 we say that J_1, J_2 are associated.

We now come to the main theorem in this section.

Theorem 9.2.3 (i) *For each $J \subseteq I$ C_J acts by conjugation on the set of irreducible cuspidal characters of L_J.*

(ii) *If we choose one subset J from each class of associated subsets of I and one cuspidal character ϕ from each orbit of C_J on the cuspidal characters of L_J, and then take the irreducible components of $\phi_{P_J}{}^G$ for all such J, ϕ, we obtain each irreducible character of G just once.*

Proof. By 9.1.3 each irreducible character of G occurs as component of some $\phi_{P_J}{}^G$ where ϕ is cuspidal. By 9.1.5 we see that if J_1, J_2 are not associated then $(\phi_1)_{P_{J_1}}{}^G$, $(\phi_2)_{P_{J_2}}{}^G$ have no common component. We may therefore assume that $J_1 = J_2$ and must consider when $(\phi_1)_{P_J}{}^G$, $(\phi_2)_{P_J}{}^G$ have a common component. By 9.1.5 and 9.2.1 we see that ϕ_1 and ϕ_2 lie in the same orbit of C_J. Conversely if ϕ_1 and ϕ_2 lie in the same orbit of C_J then $\phi_1 = {}^n\phi_2$ for some $n \in N_G(L_J) \cap N$ and this n may be chosen to lie in $N_{J,J}$. 9.1.5 then shows that $(\phi_1)_{P_J}{}^G = (\phi_2)_{P_J}{}^G$. Thus we do not lose any irreducible characters of G by taking one cuspidal character ϕ of L_J in each C_J-orbit. This completes the proof. ∎

We say that an irreducible character χ of G lies in the J-series if χ occurs as component of $\phi_{P_J}{}^G$ for some cuspidal irreducible character of L_J. The irreducible characters of G thus fall into equivalence classes called series, with one series for each class of associated subsets J of I. When $J = \phi$ the J-series is called the principal series. The principal series consists of the components of $\phi_B{}^G$ where ϕ is a character of H of degree 1. When $J = I$ the J-series is called the discrete series. Thus an irreducible character χ of G lies in the discrete series if and only if χ is cuspidal.

In chapter 10 we shall consider in detail how an induced character $\phi_{P_J}{}^G$ decomposes when ϕ is a cuspidal irreducible character of L_J. Using the methods available to us so far we can determine the scalar product $(\phi_{P_J}{}^G, \phi_{P_J}{}^G)$.

Proposition 9.2.4. *Let ϕ be a cuspidal irreducible character of L_J. Then*

$$(\phi_{P_J}{}^G, \phi_{P_J}{}^G) = |\{w \in C_J;\ {}^w\phi = \phi\}|.$$

Proof. By Mackey's formula we have

$$(\phi_{P_J}{}^G, \phi_{P_J}{}^G) = \sum_{n \in N_{J,J}} (\phi_{P_J}, {}^n(\phi_{P_J}))_{P_J \cap {}^nP_J}.$$

If $(\phi_{P_J}, {}^n(\phi_{P_J}))_{P_J \cap {}^nP_J} \neq 0$ then we must have $L_J = {}^nL_J$ as in the proof of 9.1.5. Thus $n \in N_G(L_J) \cap N$. Then, again as in the proof of 9.1.5, we have

$$(\phi_{P_J}, {}^n(\phi_{P_J}))_{P_J \cap {}^nP_J} = (\phi, {}^n\phi)_{L_K}$$

where $K = J \cap w(J)$ with $w = \pi(n)$. However we have

$$K = J \cap w(J) = J$$

and so

$$(\phi, {}^n\phi)_{L_J} = \begin{cases} 1 & \text{if } {}^n\phi = \phi \\ 0 & \text{otherwise} \end{cases}.$$

Thus $(\phi_{P_J}{}^G, \phi_{P_J}{}^G)$ is the number of double coset representatives $n \in N_{J,J}$ which lie in $N_G(L_J) \cap N$ and satisfy ${}^n\phi = \phi$. By 9.2.1 this is the number of double coset representatives $w \in D_{J,J}$ which lie in $N_W(W_J)$ and satisfy ${}^w\phi = \phi$.

Now if w lies in $N_W(W_J)$ the double coset $W_J w W_J$ is equal to the coset $W_J w$. Thus the number required is the number of cosets in $N_W(W_J)/W_J$ for which ${}^w\phi = \phi$. By 9.2.1 again this is equal to

$$|\{w \in C_J;\ {}^w\phi = \phi\}|.$$

9.3 APPLICATIONS TO THE DELIGNE–LUSZTIG CHARACTERS

We now suppose that G is a connected reductive group with Frobenius map F. Then G^F has a split BN-pair and satisfies the commutator relations, and so the considerations of sections 9.1 and 9.2 can be applied to the characters of G^F. We know in particular that if T is an F-stable maximal torus of G and θ is a character of T^F in general position then $\varepsilon_G\varepsilon_T R_{T,\theta}$ is an irreducible character of G^F. We wish to determine when this irreducible character is cuspidal.

We begin with a preliminary result.

Proposition 9.3.1. *Let P be an F-stable parabolic subgroup of G and L be an F-stable Levi subgroup contained in P. Then there exists $g \in G^F$ which transforms P into an F-stable standard parabolic subgroup P_J and L into the F-stable standard Levi subgroup L_J.*

Proof. P contains an F-stable Borel subgroup B'. Both B and B' are F-stable Borel subgroups of G and so we can find an element of G^F transforming B' to B. It will then transform P to an F-stable standard parabolic subgroup P_J.

We may therefore assume that $P = P_J$. P_J then contains F-stable Levi subgroups L and L_J. Now $B \cap L$ is an F-stable Borel subgroup of L. Let T' be an F-stable maximal torus of $B \cap L$. Then T' is an F-stable maximal torus of G which is maximally split. Thus there exists $g \in B^F$ such that $T'^g = T$. Then L^g is a Levi subgroup of P_J containing T. However the only Levi subgroup of P_J containing T is L_J. Thus the element g of $P_J{}^F$ transforms L into L_J.

Theorem 9.3.2. *Let T be an F-stable maximal torus of G and $\theta \in \hat{T}^F$ be a character of T^F in general position. Then the irreducible character $\varepsilon_G\varepsilon_T R_{T,\theta}$ of G^F is cuspidal if and only if T lies in no proper F-stable parabolic subgroup of G.*

Proof. Suppose first that T lies in no proper F-stable parabolic subgroup of G. To show that $\varepsilon_G\varepsilon_T R_{T,\theta}$ is cuspidal it will be sufficient to prove that

$$(\varepsilon_G\varepsilon_T R_{T,\theta}, 1_{U_J{}^F}{}^{G^F}) = 0$$

for any proper F-stable standard parabolic subgroup P_J with unipotent radical U_J. We have $P_J = U_J L_J$ where U_J, L_J are F-stable. Now

$$1_{U_J{}^F}{}^{G^F} = (1_{U_J{}^F}{}^{P_J{}^F})^{G^F} \quad \text{and} \quad 1_{U_J{}^F}{}^{P_J{}^F} = (\chi_{\text{reg}}{}^{L_J{}^F})_{P_J{}^F}$$

where $\chi_{\text{reg}}{}^{L_J{}^F}$ is the character of the regular representation of $L_J{}^F$. By 7.5.6 we have

$$\chi_{\text{reg}}{}^{L_J{}^F} = \frac{1}{|L_J{}^F|_p} \sum_{\substack{T' \subseteq L_J \\ F(T') = T'}} \sum_{\theta' \in \hat{T}'^F} \varepsilon_{L_J}\varepsilon_{T'} R^{L_J}_{T',\theta'}.$$

It follows that

$$1_{U_J{}^F}{}^{G^F} = ((\chi_{\text{reg}}{}^{L_J{}^F})_{P_J{}^F})^{G^F} = \frac{1}{|L_J{}^F|_p} \sum_{T'} \sum_{\theta'} \varepsilon_{L_J}\varepsilon_{T'} (R^{L_J}_{T',\theta'})_{P_J{}^F}{}^{G^F}.$$

By 7.4.4 we know that $(R^{L_J}_{T',\theta'})_{P_J{}^F}{}^{G^F} = R_{T',\theta'}$. Thus we have

$$1_{U_J{}^F}{}^{G^F} = \frac{1}{|L_J{}^F|_p} \sum_{T'} \sum_{\theta'} \varepsilon_{L_J}\varepsilon_{T'} R_{T',\theta'}.$$

Now T, T' are not G^F-conjugate when T' occurs in the above sum. For T' lies in a proper F-stable parabolic subgroup of G but T does not. Thus $(R_{T,\theta}, R_{T',\theta'}) = 0$ for all (T', θ') in the above sum, by 7.3.7. It follows that

$$(\varepsilon_G \varepsilon_T R_{T,\theta}, 1_{U_J{}^F}{}^{G^F}) = 0$$

and so $\varepsilon_G \varepsilon_T R_{T,\theta}$ is cuspidal.

Now suppose that T lies in a proper F-stable parabolic subgroup P of G. We show that $\varepsilon_G \varepsilon_T R_{T,\theta}$ is not cuspidal in this case. There is a unique Levi subgroup L of P containing T and L is F-stable. Then we have

$$R_{T,\theta} = (R^L_{T,\theta})_{P^F}{}^{G^F} \qquad \text{by 7.4.4.}$$

Hence $\varepsilon_G \varepsilon_T R_{T,\theta} = (\varepsilon_L \varepsilon_T R^L_{T,\theta})_{P^F}{}^{G^F}$ since $\varepsilon_L = \varepsilon_G$.

Let $\phi = \varepsilon_L \varepsilon_T R^L_{T,\theta}$. Then ϕ is an irreducible character of L^F. Now P^F has a Levi decomposition $P^F = U_P{}^F L^F$. Thus the irreducible character ϕ_{P^F} of P^F occurs as a component of $1_{U_P{}^F}{}^{P^F}$. Thus $\phi_{P^F}{}^{G^F}$ occurs inside $1_{U_P{}^F}{}^{G^F}$ and so no component of $\phi_{P^F}{}^{G^F}$ can be cuspidal. Thus $\varepsilon_G \varepsilon_T R_{T,\theta}$ is not cuspidal. ■

We wish next to determine the effect of the duality of generalized characters on the irreducible characters of the form $\varepsilon_G \varepsilon_T R_{T,\theta}$. We first prove a general result about the effect of this duality on cuspidal characters.

Proposition 9.3.3. *Let G be a finite group with split BN-pair satisfying the commutator relations. Let χ be a cuspidal character of G. Then the dual χ^* is given by $\chi^* = (-1)^{|\Delta|}\chi$.*

Proof. We have

$$\chi^* = \sum_{J \subseteq I} (-1)^{|J|} (T_{P_J/U_J}(\chi))^G.$$

Since χ is cuspidal we have $T_{P_J/U_J}(\chi) = 0$ for all proper subsets J of I. Thus

$$\chi^* = (-1)^{|I|} (T_{G/1}(\chi))^G = (-1)^{|\Delta|}\chi. \qquad ■$$

We now return to the case of a group G^F where G is connected reductive.

Proposition 9.3.4. *Let T be an F-stable maximal torus of G and $\theta \in \hat{T}^F$ be a character in general position. Then the dual of the irreducible character $\varepsilon_G \varepsilon_T R_{T,\theta}$ is given by*

$$(\varepsilon_G \varepsilon_T R_{T,\theta})^* = R_{T,\theta}.$$

Proof. Let P be an F-stable parabolic subgroup of G which is minimal with respect to the property of containing T. There is a unique Levi subgroup L of P containing T. By 9.3.1 there is an element of G^F which transforms the pair P, L into P_J, L_J for some $J \subseteq I$. We may therefore assume without loss of generality that $P = P_J$ and $L = L_J$.

Now we know by 7.4.4 that

$$(R^{L_J}_{T,\theta})_{P_J{}^F}{}^{G^F} = R_{T,\theta}.$$

We also know that T lies in no proper F-stable parabolic subgroup of L_J, as in the proof of 9.3.2. Thus by 9.3.2 the irreducible character $\varepsilon_{L_J}\varepsilon_T R_{T,\theta}^{L_J}$ of $L_J{}^F$ is cuspidal.

Thus by 9.3.3 we have

$$(\varepsilon_{L_J}\varepsilon_T R_{T,\theta}^{L_J})^* = (-1)^{|J'|}\varepsilon_{L_J}\varepsilon_T R_{T,\theta}^{L_J}.$$

We next observe that if ξ is any generalized character of $L_J{}^F$ then

$$(\xi^*)_{P_J{}^F}{}^{G^F} = (\xi_{P_J{}^F}{}^{G^F})^*.$$

To see this let η be any generalized character of G^F. Then

$$((\xi^*)_{P_J{}^F}{}^{G^F}, \eta) = (\xi^*, T_{L_J{}^F}(\eta))_{L_J{}^F}$$

by 8.2.2

$$= (\xi, (T_{L_J{}^F}(\eta))^*)_{L_J{}^F}$$

since * is an isometry

$$\begin{aligned} &= (\xi, T_{L_J{}^F}(\eta^*)) && \text{by 8.2.5} \\ &= (\xi_{P_J{}^F}{}^{G^F}, \eta^*) && \text{by 8.2.2} \\ &= ((\xi_{P_J{}^F}{}^{G^F})^*, \eta) \end{aligned}$$

since * is an isometry. It follows that

$$(\xi^*)_{P_J{}^F}{}^{G^F} = (\xi_{P_J{}^F}{}^{G^F})^*.$$

Thus we have

$$\begin{aligned} R_{T,\theta}{}^* &= ((R_{T,\theta}^{L_J})_{P_J{}^F}{}^{G^F})^* = ((R_{T,\theta}^{L_J})^*)_{P_J{}^F}{}^{G^F} \\ &= ((-1)^{|J'|}R_{T,\theta}^{L_J})_{P_J{}^F}{}^{G^F} = (-1)^{|J'|}R_{T,\theta}. \end{aligned}$$

It remains to determine the sign $(-1)^{|J'|}$. We recall from 6.5.2 that an F-stable Levi subgroup L_J contains an F-stable maximal torus T of type w if and only if w is F-conjugate to an element of W_J. Thus, if our given torus T has type w, then w is F-conjugate to an element of W_J but not to an element of W_K for any F-stable $K \subset J$. We then see by 6.5.5. that

$$\dim V^{F_0 w^{-1}} = \dim V^{F_0} - |J'|.$$

Now $\dim V^{F_0 w^{-1}}$ is the relative rank of T and $\dim V^{F_0}$ is the relative rank of G. Thus

$$(-1)^{|J'|} = \varepsilon_G \varepsilon_T.$$

It follows that

$$(\varepsilon_G\varepsilon_T R_{T,\theta})^* = R_{T,\theta}$$

and the result is proved. ■

Note. The result of 9.3.4 is true even if θ is not in general position, but the proof is more complicated. A proof can be found in Deligne and Lusztig [3].

The idea of cuspidal characters originally appeared in the theory of Lie groups, where it was used extensively by Harish-Chandra. Harish-Chandra adapted this idea to the theory of groups of Lie type over finite fields. A summary of his results can be found in Harish-Chandra [3], and more details in Harish-Chandra [2]. An exposition of the theory of cusp forms for finite groups was given by Springer in Borel *et al.* [1], part C.

CHAPTER 10

THE DECOMPOSITION OF INDUCED CUSPIDAL CHARACTERS

We have seen that every irreducible character χ of a finite group G with split BN-pair satisfying the commutator relations occurs as a component of $\phi_{P_J}{}^G$ for some irreducible cuspidal character ϕ of some standard Levi subgroup L_J of G. Moreover J and ϕ are uniquely determined up to conjugacy by χ. In order to understand the irreducible characters of such a group G one must therefore know the cuspidal characters and also know how to decompose the induced characters $\phi_{P_J}{}^G$ into irreducible components when ϕ is cuspidal. We discuss this latter problem in the present chapter, and describe a theory due to Howlett and Lehrer which gives information about the decomposition of $\phi_{P_J}{}^G$ (Howlett and Lehrer [1]).

We are as usual interested in applying these results to the groups G^F where G is a connected reductive group and $F:G \to G$ is a Frobenius map. It is, however, convenient to present the theory in a more general context. In this chapter we shall assume that G is a finite group satisfying the following conditions:

(i) G has a split BN-pair.
(ii) G satisfies the commutator relations.
(iii) For each $w \in W$ a coset representative $n_w = \dot{w} \in N$ can be chosen such that, for each reduced expression $w = s_{i_1}s_{i_2}\dots s_{i_r}$, we have $\dot{w} = \dot{s}_{i_1}\dot{s}_{i_2}\dots\dot{s}_{i_r}$.

We know from section 1.18 that the groups G^F satisfy these three conditions.

We shall assume throughout this chapter that G is a finite group satisfying conditions (i) (ii), (iii).

10.1 INDUCED MODULES AND THEIR ENDOMORPHISM ALGEBRAS

Let G be a finite group satisfying conditions (i), (ii), (iii) and let ρ be an irreducible cuspidal representation of the standard Levi subgroup L_J of G. Let ϕ

be the character of ρ. Let M be a left L_J-module which affords the representation ρ. Then M may also be regarded as a P_J-module on which U_J acts trivially, and which affords an irreducible representation of P_J which will be denoted by ρ_{P_J}, and which has character ϕ_{P_J}. We can then form the induced representation, which has character $\phi_{P_J}{}^G$. Our aim is to describe the irreducible components of $\phi_{P_J}{}^G$ and to find their degrees.

We recall from 9.2.3 that

$$(\phi_{P_J}{}^G, \phi_{P_J}{}^G) = |W^{J,\phi}|$$

where $W^{J,\phi} = \{w \in W; w(J) = J, {}^w\phi = \phi\}$. $W^{J,\phi}$ will be called the ramification group of $\phi_{P_J}{}^G$.

We may construct a module which affords the character $\phi_{P_J}{}^G$ as follows. Let $\mathfrak{F}$ be the set of all maps from G into M. $\mathfrak{F}$ is then a finite-dimensional vector space over $\mathbb{C}$. $\mathfrak{F}$ may be made into a left $\mathbb{C}G$-module by defining gf, $g \in G, f \in \mathfrak{F}$ to satisfy

$$(gf)x = f(xg) \qquad x \in G.$$

For then we have

$$(g_1(g_2f))x = (g_2f)(xg_1) = f(xg_1g_2) = ((g_1g_2f)x$$

and so $g_1(g_2f) = (g_1g_2f)$ for all $g_1, g_2 \in G$.

Let $\mathfrak{F}(J, \rho)$ be the subset of $\mathfrak{F}$ defined by

$$\mathfrak{F}(J, \rho) = \{f \in \mathfrak{F}; f(pg) = \rho(p)f(g) \qquad \text{for all } p \in P_J, g \in G\}.$$

$\mathfrak{F}(J, \rho)$ is clearly a subspace of $\mathfrak{F}$. It is, moreover, a submodule of $\mathfrak{F}$. For if $f \in \mathfrak{F}(J, \rho)$, $x \in G$ we have

$$(xf)(pg) = f(pgx) = \rho(p)f(gx) = \rho(p).((xf)(g)) \qquad \text{for all } p \in P_J, g \in G.$$

Thus $xf \in \mathfrak{F}(J, \rho)$.

Proposition 10.1.1. *$\mathfrak{F}(J, \rho)$ affords the representation $\rho_{P_J}{}^G$ of G.*

Proof. Let $g_1 = 1, g_2, \ldots, g_m$ be a set of right coset representatives of P_J in G. Thus each element of G has a unique expression as pg_i where $p \in P_J$ and $i \in \{1, 2, \ldots, m\}$. We have

$$f(pg_i) = \rho(p)f(g_i) \qquad f \in \mathfrak{F}(J, \rho).$$

Let $\mathfrak{F}(J, \rho)_i = \{f \in \mathfrak{F}(J, \rho); f(g_j) = 0 \text{ for all } j \neq i\}$. Then we have

$$\mathfrak{F}(J, \rho) = \bigoplus_{i=1}^{m} \mathfrak{F}(J, \rho)_i.$$

We show that $\mathfrak{F}(J, \rho)_1$ is a P_J-submodule giving the representation ρ_{P_J}. Consider the map $M \to \mathfrak{F}(J, \rho)_1$ given by $v \to f_v$ where $f_v(1) = v$. Then

$$f_v(p) = \rho(p)v \qquad \text{if } p \in P_J$$

$$f_v(x) = 0 \qquad \text{if } x \notin P_J.$$

The map $v \to f_v$ is bijective. Moreover we have

$$(pf_v)(1) = f_v(p) = \rho(p)f_v(1) = \rho(p)v.$$

Thus if v maps to f_v then $\rho(p)v$ maps to pf_v, and we have an isomorphism of P_J-modules. Thus $\mathfrak{F}(J, \rho)_1$ affords the representation ρ_{P_J} of P_J.

We show next that $g_i^{-1}\mathfrak{F}(J, \rho)_1 = \mathfrak{F}(J, \rho)_i$. For we have

$$\begin{aligned} f \in g_i^{-1}\mathfrak{F}(J, \rho)_1 &\Leftrightarrow (g_i f)x = 0 \qquad \text{if } x \notin P_J \\ &\Leftrightarrow f(xg_i) = 0 \qquad \text{if } x \notin P_J \\ &\Leftrightarrow f \in \mathfrak{F}(J, \rho)_i. \end{aligned}$$

Thus

$$\mathfrak{F}(J, \rho) = \mathfrak{F}(J, \rho)_1 \oplus g_2^{-1}\mathfrak{F}(J, \rho)_1 \oplus \dots \oplus g_m^{-1}\mathfrak{F}(J, \rho)_1.$$

Since $G = P_J \cup g_2^{-1}P_J \cup \dots \cup g_m^{-1}P_J$ it follows that $\mathfrak{F}(J, \rho)$ affords the induced representation $\rho_{P_J}{}^G$.

Let $\mathfrak{E} = \mathrm{End}_G\, \mathfrak{F}(J, \rho)$ be the algebra of endomorphisms of this induced module. Then the modules M and $\mathfrak{F}(J, \rho)$ and the endomorphism algebra $\mathfrak{E}$ can be described in terms of a primitive idempotent $e \in \mathbb{C}P_J$. For M, being an irreducible P_J-module, is isomorphic to $\mathbb{C}P_J e$ for some primitive idempotent $e \in \mathbb{C}P_J$. The induced module $\mathfrak{F}(J, \rho)$ is then isomorphic to $\mathbb{C}Ge$ and the endomorphism algebra $\mathfrak{E}$ is isomorphic to $e\mathbb{C}Ge$. The identity element of $e\mathbb{C}Ge$ is e.

Now each irreducible $\mathfrak{E}$-module gives rise to a trace function from $\mathfrak{E}$ to $\mathbb{C}$ called the character of the module, and the functions obtained in this way are called the irreducible characters of $\mathfrak{E}$.

Proposition 10.1.2. *There is a bijective correspondence between irreducible characters χ of G satisfying $(\phi_{P_J}{}^G, \chi) \neq 0$ and irreducible characters $\chi_{\mathfrak{E}}$ of $\mathfrak{E}$. $\chi_{\mathfrak{E}}$ is the restriction of χ on $\mathbb{C}G$ to $e\mathbb{C}Ge$. The degree of $\chi_{\mathfrak{E}}$ is given by $\chi_{\mathfrak{E}}(e) = (\phi_{P_J}{}^G, \chi)$.*

Proof. Let M_0 be the $\mathbb{C}G$-module affording the character χ and consider $\mathrm{Hom}_G(\mathbb{C}Ge, M_0)$. If such a homomorphism maps e to $m \in M_0$ we have $em = m$ and so $m \in eM_0$. Conversely given any $m \in eM_0$ there is a G-homomorphism $\mathbb{C}Ge \to M_0$ mapping e to m. For if $a \in \mathbb{C}Ge$ then $ae = 0$ implies $am = aem = 0$. Thus we have

$$\mathrm{Hom}_G(\mathbb{C}Ge, M_0) \cong eM_0.$$

Thus

$$\dim eM_0 = \dim \mathrm{Hom}_G(\mathbb{C}Ge, M_0) = (\phi_{P_J}{}^G, \chi).$$

Now eM_0 is a left $e\mathbb{C}Ge$-module. We show it is an irreducible $e\mathbb{C}Ge$-module. Suppose $m \in eM_0$ with $m \neq 0$. Then

$$e\mathbb{C}Gem = e\mathbb{C}Gm = eM_0$$

since M_0 is an irreducible $\mathbb{C}G$-module. Thus eM_0 is an irreducible $e\mathbb{C}Ge$-module. Consider the character of $e\mathbb{C}Ge$ afforded by eM_0. Let $x \in e\mathbb{C}Ge$. Then $xM_0 \subseteq eM_0$. Hence

$$\chi(x) = \operatorname{trace}_{M_0} x = \operatorname{trace}_{eM_0} x.$$

Thus the character of $e\mathbb{C}Ge$ afforded by eM_0 is χ restricted to $e\mathbb{C}Ge$.

We have therefore shown that each irreducible $\mathbb{C}G$-module M_0 with character χ satisfying $(\phi_{P_J}{}^G, \chi) \neq 0$ gives rise to an irreducible $e\mathbb{C}Ge$-module eM_0 with character $\chi_{\mathfrak{E}}$, which is χ restricted to $e\mathbb{C}Ge$.

Conversely let N be any irreducible $e\mathbb{C}Ge$-module and suppose its character is ψ. Then $N = e\mathbb{C}Gen$ where n is a primitive idempotent in $e\mathbb{C}Ge$. Then n is also a primitive idempotent in $\mathbb{C}G$. For if $n = n_1 + n_2$ where $n_1, n_2 \in \mathbb{C}G$ satisfy

$$n_1{}^2 = n_1 \neq 0, \quad n_2{}^2 = n_2 \neq 0, \quad n_1 n_2 = n_2 n_1 = 0$$

then we have

$$n_1 = nn_1 = nn_1n = enn_1ne \in e\mathbb{C}Ge$$

$$n_2 = nn_2 = nn_2n = enn_2ne \in e\mathbb{C}Ge$$

which is a contradiction. Thus n is a primitive idempotent in $\mathbb{C}G$ and so $\mathbb{C}Gn$ is an irreducible $\mathbb{C}G$-module. Let its character be χ. Then the character of $e\mathbb{C}Gn = N$ is $\chi_{\mathfrak{E}}$. Thus $\psi = \chi_{\mathfrak{E}}$ for some χ.

Finally we prove that χ is the only irreducible character of G such that $\chi_{\mathfrak{E}} = \psi$. Let M_0' be any irreducible $\mathbb{C}G$-module whose character χ' satisfies $\chi'_{\mathfrak{E}} = \psi$. Consider the map

$$N \to ne\mathbb{C}Gen$$

given by left multiplication by n. Since n is a primitive idempotent in $e\mathbb{C}Ge$ we have $ne\mathbb{C}Gen = \mathbb{C}n$. Thus left multiplication by n maps N into $\mathbb{C}n$ and so $\psi(n) = 1$. Hence $\chi'(n) = 1$ also. Thus $nM_0' \neq 0$. Let $m' \in M_0'$ satisfy $nm' \neq 0$. Consider the map $\mathbb{C}Gn \to M_0'$ given by $x \to xm'$. This is a homomorphism of $\mathbb{C}G$-modules. It is nonzero sinze $nm' \neq 0$. Since both $\mathbb{C}Gn$ and M_0' are irreducible it is an isomorphism. Hence $\chi' = \chi$. ■

In view of this result we shall concentrate on investigating the endomorphism algebra $\mathfrak{E} = \operatorname{End}_G \mathfrak{F}(J, \rho)$. We first try to find a basis of $\mathfrak{E}$.

Suppose $w \in W$ has the property that $w(\Delta_J) \subseteq \Delta$. Then $w(\Delta_J) = \Delta_K$ for some $K \subseteq I$ and J, K are associated subsets of I. Let $n_w = \dot{w} \in N$ satisfy $\pi(\dot{w}) = w$. Then ${}^{\dot{w}}L_J = L_K$. We may therefore define a representation ${}^{\dot{w}}\rho$ of L_K by

$${}^{\dot{w}}\rho(x) = \rho(x^{\dot{w}}) \qquad x \in L_K.$$

Proposition 10.1.3. *Let $w \in W$ satisfy $w(J) = K$. Then there is a map $\theta_w : \mathfrak{F}(J, \rho) \to \mathfrak{F}(K, {}^{\dot{w}}\rho)$ defined by*

$$(\theta_w f)g = \frac{1}{|U_K|} \sum_{u \in U_K} f(\dot{w}^{-1}ug) \qquad g \in G$$

which is a homomorphism of G-modules.

Proof. Let $f \in \mathfrak{F}(J, \rho)$. We show that $\theta_w f \in \mathfrak{F}(K, {}^{\dot{w}}\rho)$. Let $l \in L_K$ and $g \in G$. Then

$$\begin{aligned}(\theta_w f)(lg) &= \frac{1}{|U_K|} \sum_{u \in U_K} f(\dot{w}^{-1} ulg) \\ &= \frac{1}{|U_K|} \sum_{u \in U_K} f(\dot{w}^{-1} l(l^{-1} ul)g) \\ &= \frac{1}{|U_K|} \sum_{u \in U_K} f(\dot{w}^{-1} l\dot{w}\dot{w}^{-1} ug) \\ &= \rho(\dot{w}^{-1} l\dot{w}) \frac{1}{|U_K|} \sum_{u \in U_K} f(\dot{w}^{-1} ug) \\ &= {}^{\dot{w}}\rho(l)((\theta_w f)(g)).\end{aligned}$$

Moreover, for $u \in U_K$, we have

$$(\theta_w f)(ulg) = (\theta_w f)(lg) = {}^{\dot{w}}\rho(l)((\theta_w f)(g)) = {}^{\dot{w}}\rho(ul)((\theta_w f)g).$$

Thus $(\theta_w f)(pg) = {}^{\dot{w}}\rho(p)((\theta_w f)g)$ for all $p \in P_K$. This shows that $\theta_w f \in \mathfrak{F}(K, {}^{\dot{w}}\rho)$.

We show next that θ_w is a homomorphism of G-modules. Let $x \in G$. Then

$$\begin{aligned}(\theta_w(xf))g &= \frac{1}{|U_K|} \sum_{u \in U_K} (xf)(\dot{w}^{-1} ug) \\ &= \frac{1}{|U_K|} \sum_{u \in U_K} f(\dot{w}^{-1} ugx) = (\theta_w f)(gx) = (x(\theta_w f))g.\end{aligned}$$

Thus $\theta_w(xf) = x(\theta_w f)$. ■

We now consider in particular the case $J = K$. Then $w \in C_J$ and θ_w is a G-homomorphism from $\mathfrak{F}(J, \rho)$ into $\mathfrak{F}(J, {}^{\dot{w}}\rho)$. Suppose more particularly that $w \in W^{J,\phi}$. Then ${}^{w}\phi = \phi$ and so ρ and ${}^{\dot{w}}\rho$ are equivalent representations of L_J. They are also equivalent when lifted to P_J. Thus the module M which affords ρ can be made to afford ${}^{\dot{w}}\rho$ with respect to a different basis. Thus there exists a nonsingular linear map $\tilde{\rho}(\dot{w}): M \to M$ such that

$${}^{\dot{w}}\rho(p) = \tilde{\rho}(\dot{w})^{-1} \rho(p) \tilde{\rho}(\dot{w})$$

for all $p \in P_J$. Since ρ is irreducible $\tilde{\rho}(\dot{w})$ is determined up to a nonzero scalar multiple, by Schur's lemma (Curtis and Reiner [1], p. 181).

Proposition 10.1.4. *Let $w \in W^{J,\phi}$. Then there is an element $B_w \in \operatorname{End} \mathfrak{F}(J, \rho)$ given by*

$$(B_w f)x = \tilde{\rho}(\dot{w})((\theta_w f)x) \qquad x \in G.$$

Proof. Let $f \in \mathfrak{F}(J, \rho)$. We must show first that $B_w f \in \mathfrak{F}(J, \rho)$ also. We know that $\theta_w f \in \mathfrak{F}(J, {}^{\dot{w}}\rho)$. Let $p \in P_J$ and $g \in G$.

Then

$$\begin{aligned}(B_w f)(pg) &= \tilde{\rho}(\dot{w})((\theta_w f)(pg)) \\ &= \tilde{\rho}(\dot{w})({}^{\dot{w}}\rho(p)((\theta_w f)(g))) \\ &= \tilde{\rho}(\dot{w})\tilde{\rho}(\dot{w})^{-1}\rho(p)\tilde{\rho}(\dot{w})((\theta_w f)g) \\ &= \rho(p)\tilde{\rho}(\dot{w})((\theta_w f)g) = \rho(p)((B_w f)g).\end{aligned}$$

Thus $B_w f \in \mathfrak{F}(J, \rho)$.

We now show that B_w is a homomorphism of G-modules. Let $g \in G$. Then

$$\begin{aligned}(B_w(gf))x &= \tilde{\rho}(\dot{w})(\theta_w(gf)\cdot x) \\ &= \tilde{\rho}(\dot{w})((g(\theta_w f))x) = \tilde{\rho}(\dot{w})((\theta_w f)(xg)) \\ &= (B_w f)(xg) = (g(B_w f))x.\end{aligned}$$

Hence $B_w(gf) = g(B_w f)$ and B_w is a homomorphism of G-modules. Thus $B_w \in \operatorname{End} \mathfrak{F}(J, \rho)$. ■

We can now obtain our required basis of $\mathfrak{E}$.

Theorem 10.1.5. *The elements B_w, $w \in W^{J,\phi}$, form a basis for $\mathfrak{E}$.*

Proof. We know that $\dim \mathfrak{E} = (\phi_{P_J}{}^G, \phi_{P_J}{}^G)$ by section 6.1. Also $(\phi_{P_J}{}^G, \phi_{P_J}{}^G) = |W^{J,\phi}|$ by 9.2.3. Hence $\dim \mathfrak{E} = |W^{J,\phi}|$. It is therefore sufficient to show that the elements B_w, $w \in W^{J,\phi}$ are linearly independent.

Suppose $\sum_w \xi_w B_w = 0$ where $\xi_w \in \mathbb{C}$. Then $\sum_w \xi_w(B_w f) = 0$ for all $f \in \mathfrak{F}(J, \rho)$. Hence $\sum_w \xi_w(B_w f)g = 0$ for all $f \in \mathfrak{F}(J, \rho)$, $g \in G$. Let v be a nonzero vector in M and let $f_v \in \mathfrak{F}$ satisfy

$$f_v(g) = \begin{cases} \rho(g)v & \text{if } g \in P_J \\ 0 & \text{if } g \notin P_J \end{cases}.$$

Then $f_v \in \mathfrak{F}(J, \rho)$ as in the proof of 10.1.1. Thus we have

$$\sum_{w' \in W^{J,\phi}} \xi_{w'}(B_{w'} f_v)\dot{w} = 0 \qquad \text{for all } w \in W^{J,\phi}.$$

This gives

$$\sum_{w' \in W^{J,\phi}} \xi_{w'}\tilde{\rho}(\dot{w}')((\theta_{w'} f_v)\dot{w}) = 0$$

so

$$\sum_{w' \in W^{J,\phi}} \xi_{w'}\tilde{\rho}(\dot{w}') \frac{1}{|U_J|} \sum_{u \in U_J} f_v(\dot{w}'^{-1}u\dot{w}) = 0.$$

Since $f_v(g) = 0$ if $g \notin P_J$ we may assume that $\dot{w}'^{-1}u\dot{w} \in P_J$. Thus $\dot{w} \in P_J\dot{w}'P_J$. However this implies that $w \in W_J w' W_J$. Since both w and w' lie in C_J and $C_J \subseteq N_W(W_J)$ we have $wW_J = w'W_J$. Since $C_J \cap W_J = 1$ we have $w = w'$. Thus the only nonzero term in the above sum occurs when $w' = w$. Hence

$$\xi_w \tilde{\rho}(\dot{w}) \sum_{u \in U_J} f_v(\dot{w}^{-1} u \dot{w}) = 0$$

or

$$\xi_w \sum_{u \in U_J \cap {}^w P_J} \tilde{\rho}(\dot{w}) \rho(\dot{w}^{-1} u \dot{w}) v = 0.$$

Now w lies in C_J so $w(J) = J$. Hence $w \in D_J \cap D_J^{-1} = D_{J,J}$. Thus by 2.8.6 we have

$$U_J \cap {}^w P_J = (U_J \cap {}^w U_J)(U_J \cap {}^w L_J) = U_J \cap {}^w U_J$$

since

$$U_J \cap {}^w L_J = U_J \cap L_J = 1.$$

Thus we have $u \in U_J \cap {}^w U_J$ and $\dot{w}^{-1} u \dot{w} \in U_J$, so that $\rho(\dot{w}^{-1} u \dot{w})$ is the identity. It follows that

$$\xi_w |U_J \cap {}^w U_J| \tilde{\rho}(\dot{w}) v = 0$$

and so $\xi_w \tilde{\rho}(\dot{w}) v = 0$. But $v \neq 0$ and $\tilde{\rho}(\dot{w})$ is nonsingular. Thus $\tilde{\rho}(\dot{w}) v \neq 0$. Hence $\xi_w = 0$. Thus the elements B_w are linearly independent and so form a basis for $\mathfrak{E}$.

10.2 THE COMPOSITION OF INTERTWINING OPERATORS

In order to understand the structure of the endomorphism algebra $\mathfrak{E}$ we must know how its basis elements B_w multiply together. In the present section we shall show that if $w, w' \in W^{J,\phi}$ satisfy the condition that $l(ww') = l(w) + l(w')$ then $B_w B_{w'}$ is a scalar multiple of $B_{ww'}$.

We first need some preliminary results about the intertwining operators θ_w.

Lemma 10.2.1. *Suppose $w \in W$ satisfies $w(\Delta_J) = \Delta_K$ where $J, K \subseteq I$. Then the G-module homomorphism $\theta_w : \mathfrak{F}(J, \rho) \to \mathfrak{F}(K, {}^{\dot{w}}\rho)$ satisfies*

$$(\theta_w f) g = \frac{1}{|U_K \cap U_{w^{-1}}|} \sum_{u \in U_K \cap U_{w^{-1}}} f(\dot{w}^{-1} u g).$$

Proof. We have $U = U_{w_0 w^{-1}} U_{w^{-1}}$ and $U_{w_0 w^{-1}} \cap U_{w^{-1}} = 1$. Since U_K is a product of certain root subgroups we have

$$U_K = (U_K \cap U_{w_0 w^{-1}})(U_K \cap U_{w^{-1}}).$$

Let $u \in U_K$ factorize as $u = u_1 u_2$ where $u_1 \in U_K \cap U_{w_0 w^{-1}}$ and $u_2 \in U_K \cap U_{w^{-1}}$. Then

$$\begin{aligned}(\theta_w f) g &= \frac{1}{|U_K|} \sum_{u \in U_K} f(\dot{w}^{-1} u g) \\ &= \frac{1}{|U_K \cap U_{w_0 w^{-1}}|} \frac{1}{|U_K \cap U_{w^{-1}}|} \sum_{u_1} \sum_{u_2} f(\dot{w}^{-1} u_1 \dot{w} \dot{w}^{-1} u_2 g).\end{aligned}$$

We show that $\dot{w}^{-1}u_1\dot{w} \in U_J$. We know that $U_K \cap U_{w_0 w^{-1}}$ is the product of root subgroups X_α for roots satisfying $\alpha > 0$, $w_0 w^{-1}(\alpha) < 0$, $\alpha \notin \Phi_K$. Thus $(U_K \cap U_{w_0 w^{-1}})^{\dot{w}}$ is the product of root subgroups $X_{w^{-1}(\alpha)}$ with α as above. Since $w^{-1}(\alpha) > 0$ and $w^{-1}(\alpha) \notin \Phi_J$ all such root subgroups will lie in U_J. Hence $\dot{w}^{-1}u_1\dot{w} \in U_J$. It follows that

$$f(\dot{w}^{-1}u_1\dot{w}\dot{w}^{-1}u_2 g) = f(\dot{w}^{-1}u_2 g)$$

and so we have

$$(\theta_w f)g = \frac{1}{|U_K \cap U_{w^{-1}}|} \sum_{u_2 \in U_K \cap U_{w^{-1}}} f(\dot{w}^{-1}u_2 g).$$

Proposition 10.2.2. *Suppose $w, w' \in W$ satisfy $w'(\Delta_J) \subseteq \Delta$ and $ww'(\Delta_J) \subseteq \Delta$. Suppose also that $l(ww') = l(w) + l(w')$. Then the G-module homomorphisms*

$$\mathfrak{F}(J, \rho) \xrightarrow{\theta_{w'}} \mathfrak{F}(w'J, {}^{\dot{w}'}\rho) \xrightarrow{\theta_w} \mathfrak{F}(ww'J, {}^{\dot{w}\dot{w}'}\rho)$$

satisfy $\theta_w\theta_{w'} = \theta_{ww'}$.

Proof. Let $f \in \mathfrak{F}(J, \rho)$. Then, by 10.2.1, we have

$$(\theta_w\theta_{w'}f)g = \frac{1}{|U_{ww'J} \cap U_{w^{-1}}|} \sum_{u \in U_{ww'J} \cap U_{w^{-1}}} (\theta_{w'}f)(\dot{w}^{-1}ug)$$

$$= \frac{1}{|U_{ww'J} \cap U_{w^{-1}}|} \frac{1}{|U_{w'J} \cap U_{w'^{-1}}|} \sum_u \sum_{u' \in U_{w'J} \cap U_{w'^{-1}}} f(\dot{w}'^{-1}u'\dot{w}^{-1}ug)$$

$$= \frac{1}{|U_{ww'J} \cap U_{w^{-1}}|} \frac{1}{|U_{w'J} \cap U_{w'^{-1}}|} \sum_u \sum_{u'} f(\dot{w}'^{-1}\dot{w}^{-1}({}^{\dot{w}}u')ug).$$

We shall show in the following lemma 10.2.3 that

$$({}^{\dot{w}}(U_{w'J} \cap U_{w'^{-1}}))(U_{ww'J} \cap U_{w^{-1}}) = U_{ww'J} \cap U_{(ww')^{-1}}$$

with uniqueness. It then follows that

$$(\theta_w\theta_{w'}f)g = \frac{1}{|U_{ww'J} \cap U_{(ww')^{-1}}|} \sum_{u'' \in U_{ww'J} \cap U_{(ww)^{-1}}} f((\dot{w}\dot{w}')^{-1}u''g)$$

$$= (\theta_{ww'}f)g.$$

Thus $\theta_w\theta_{w'} = \theta_{ww'}$ as required. ■

To complete the proof we must therefore establish the following lemma.

Lemma 10.2.3. *Let $w, w' \in W$ satisfy $l(ww') = l(w) + l(w')$. Then*

$$({}^{\dot{w}}(U_{w'J} \cap U_{w'^{-1}})) \cdot (U_{ww'J} \cap U_{w^{-1}}) = U_{ww'J} \cap U_{(ww')^{-1}}$$

with uniqueness.

Proof. Both sides are products of root subgroups x_α. We consider which roots α appear in the three subgroups concerned. We show that the roots appearing in the two subgroups on the left-hand side are disjoint and that their union is the set of roots appearing on the right-hand side. This will establish the result.

Consider first the set Ψ of roots appearing in $U_{ww'J} \cap U_{(ww')^{-1}}$. These are the roots α satisfying

$$\alpha > 0, \qquad w'^{-1}w^{-1}(\alpha) < 0, \qquad w'^{-1}w^{-1}(\alpha) \notin \Phi_J.$$

Next consider the set Ψ_1 of roots appearing in $U_{ww'J} \cap U_{w^{-1}}$. These are the roots α satisfying

$$\alpha > 0, \qquad w^{-1}(\alpha) < 0, \qquad w'^{-1}w^{-1}(\alpha) \notin \Phi_J.$$

Finally consider the set Ψ_2 of roots appearing in ${}^{w}(U_{w'J} \cap U_{w'^{-1}})$. These are the roots of the form $\alpha = w(\beta)$ where

$$\beta > 0, \qquad w'^{-1}(\beta) < 0, \qquad w'^{-1}(\beta) \notin \Phi_J.$$

This may be stated also as

$$w^{-1}(\alpha) > 0, \qquad w'^{-1}w^{-1}(\alpha) < 0, \qquad w'^{-1}w^{-1}(\alpha) \notin \Phi_J.$$

We shall show that $\Psi_1 = \{\alpha \in \Psi; w^{-1}(\alpha) < 0\}$ and $\Psi_2 = \{\alpha \in \Psi; w^{-1}(\alpha) > 0\}$. In order to prove this we must show that every root $\alpha \in \Psi_1$ satisfies $w'^{-1}w^{-1}(\alpha) < 0$ and that every root $\alpha \in \Psi_2$ satisfies $\alpha > 0$.

Since $l(ww') = l(w) + l(w')$ we have $l(w'^{-1}w^{-1}) = l(w'^{-1}) + l(w^{-1})$ and so every root $\alpha > 0$ with $w^{-1}(\alpha) < 0$ also satisfies $w'^{-1}w^{-1}(\alpha) < 0$. Also every root satisfying $w^{-1}(\alpha) > 0$ and $w'^{-1}w^{-1}(\alpha) < 0$ must satisfy $\alpha > 0$. For is $\alpha < 0$ and $\alpha = -\beta$ then we have $\beta > 0$, $w^{-1}(\beta) < 0$ and $w'^{-1}w^{-1}(\beta) > 0$ which is a contradiction.

This proves the required facts about Ψ_1 and Ψ_2. Thus Ψ is the disjoint union of Ψ_1 and Ψ_2 and the lemma is proved. ∎

We next wish to prove a similar result for the basis elements B_w of $\mathfrak{E}$. We recall that

$$(B_w f)x = \tilde{\rho}(\dot{w})((\theta_w f)x) \qquad x \in G.$$

Here $\tilde{\rho}(\dot{w})$ is a nonsingular map of M into itself. However $\tilde{\rho}(\dot{w})$ induces a linear map from $\mathfrak{F}(J, \rho)$ into $\mathfrak{F}(J, {}^{\dot{w}^{-1}}\rho)$ (which will also be denoted by $\tilde{\rho}(\dot{w})$) defined by

$$(\tilde{\rho}(\dot{w})f)x = \tilde{\rho}(\dot{w})(f(x)) \qquad f \in \mathfrak{F}(J, \rho), \qquad x \in G.$$

In order to show that $\tilde{\rho}(\dot{w})f \in F(J, {}^{\dot{w}^{-1}}\rho)$ let $l \in L_J$, $g \in G$. Then we have

$$\begin{aligned}(\tilde{\rho}(\dot{w})f)(lg) &= \tilde{\rho}(\dot{w})(f(lg)) = \tilde{\rho}(\dot{w})(\rho(l)f(g)) \\ &= \tilde{\rho}(\dot{w})\rho(l)\tilde{\rho}(\dot{w})^{-1}\tilde{\rho}(\dot{w})f(g) \\ &= \rho(\dot{w}l\dot{w}^{-1})((\tilde{\rho}(\dot{w})f)g) \\ &= ({}^{\dot{w}^{-1}}\rho(l))((\tilde{\rho}(\dot{w})f)g).\end{aligned}$$

Now let $u \in U_J$. Then

$$(\tilde{\rho}(\dot{w})f)(ulg) = \tilde{\rho}(\dot{w})(f(ulg)) = \tilde{\rho}(\dot{w})(f(lg))$$
$$= (\tilde{\rho}(\dot{w})f)(lg).$$

Thus $\tilde{\rho}(\dot{w})f \in \mathfrak{F}(J, {}^{\dot{w}^{-1}}\rho)$. Moreover the map $\tilde{\rho}(w)$: $\mathfrak{F}(J, \rho) \to \mathfrak{F}(J, {}^{\dot{w}^{-1}}\rho)$ is a homomorphism of G-modules. For if $g \in G$ we have

$$(\tilde{\rho}(\dot{w})(gf))x = \tilde{\rho}(\dot{w})((gf)x) = \tilde{\rho}(w)(f(xg))$$
$$= (\tilde{\rho}(\dot{w})f)(xg) = (g(\tilde{\rho}(\dot{w})f))x.$$

Thus

$$\tilde{\rho}(\dot{w})(gf) = g(\tilde{\rho}(\dot{w})f).$$

A similar calculation shows that $\tilde{\rho}(\dot{w})$ induces a G-module homomorphism $\mathfrak{F}(J, {}^{\dot{w}'}\rho) \to \mathfrak{F}(J, {}^{\dot{w}'\dot{w}^{-1}}\rho)$ for each $w' \in C_J$.

Using these homomorphisms we can express B_w as a composite map, where $w \in W^{J,\phi}$. For $f \in \mathfrak{F}(J, \rho)$ we have

$$(B_w f)x = \tilde{\rho}(\dot{w})((\theta_w f)x) = (\tilde{\rho}(\dot{w})(\theta_w f))x$$

for all $x \in G$. Thus

$$B_w f = \tilde{\rho}(\dot{w})(\theta_w f) \qquad \text{for all } f \in \mathfrak{F}(J, \rho).$$

Hence

$$B_w = \tilde{\rho}(\dot{w}) \circ \theta_w.$$

Thus we have G-module homomorphisms

$$\mathfrak{F}(J, \rho) \xrightarrow{\theta_w} \mathfrak{F}(J, {}^{\dot{w}}\rho) \xrightarrow{\tilde{\rho}(\dot{w})} \mathfrak{F}(J, \rho)$$

whose composite is B_w.

Proposition 10.2.4. *Let w, $w' \in W^{J,\phi}$. Then the following diagram of G-module homomorphisms is commutative:*

$$\begin{array}{ccc} \mathfrak{F}(J, \rho) & \xrightarrow{\tilde{\rho}(\dot{w})} & \mathfrak{F}(J, {}^{\dot{w}^{-1}}\rho) \\ {\scriptstyle \theta_{w'}}\downarrow & & \downarrow{\scriptstyle \theta_{w'}} \\ \mathfrak{F}(J, {}^{\dot{w}'}\rho) & \xrightarrow{\tilde{\rho}(\dot{w})} & \mathfrak{F}(J, {}^{\dot{w}'\dot{w}^{-1}}\rho) \end{array}$$

Proof. Let $f \in \mathfrak{F}(J, \rho)$ and $x \in G$. Then

$$\begin{aligned}(\theta_{w'}\tilde{\rho}(\dot{w})f)x &= \frac{1}{|U_J|}\sum_{u \in U_J}(\tilde{\rho}(\dot{w})f)(\dot{w}'^{-1}ux)\\ &= \frac{1}{|U_J|}\sum_{u \in U_J}\tilde{\rho}(\dot{w})f(\dot{w}'^{-1}ux)\\ &= \tilde{\rho}(\dot{w})\frac{1}{|U_J|}\sum_{u \in U_J}f(\dot{w}'^{-1}ux)\\ &= \tilde{\rho}(\dot{w})((\theta_{w'}f)x) = (\tilde{\rho}(\dot{w})(\theta_{w'}f))x.\end{aligned}$$

Thus

$$\theta_{w'}\tilde{\rho}(\dot{w})f = \tilde{\rho}(\dot{w})\theta_{w'}f. \qquad \blacksquare$$

We recall that $\tilde{\rho}(\dot{w}): M \to M$ was defined up to a nonzero scalar by the equation

$$^{\dot{w}}\rho(l) = \tilde{\rho}(\dot{w})^{-1}\rho(l)\tilde{\rho}(\dot{w}) \qquad l \in L_J.$$

Suppose $w, w' \in W^{J,\phi}$ satisfy $l(ww') = l(w) + l(w')$. The coset representatives $\dot{w}, \dot{w}', (\dot{ww'})$ then satisfy $(\dot{ww'}) = \dot{w}\dot{w}'$. The map $\tilde{\rho}(\dot{w}\dot{w}'): M \to M$ is defined up to a nonzero scalar by the equation

$$^{\dot{w}\dot{w}'}\rho(l) = \tilde{\rho}(\dot{w}\dot{w}')^{-1}\rho(l)\tilde{\rho}(\dot{w}\dot{w}') \qquad l \in L_J.$$

Now we have

$$\begin{aligned}^{\dot{w}\dot{w}'}\rho(l) &= {}^{\dot{w}'}\rho(l^{\dot{w}}) = \tilde{\rho}(\dot{w}')^{-1}\rho(l^{\dot{w}})\tilde{\rho}(\dot{w}')\\ &= \tilde{\rho}(\dot{w}')^{-1}\,{}^{\dot{w}}\rho(l)\tilde{\rho}(\dot{w}') = \tilde{\rho}(\dot{w}')^{-1}\tilde{\rho}(\dot{w})^{-1}\rho(l)\tilde{\rho}(\dot{w})\tilde{\rho}(\dot{w}').\end{aligned}$$

It follows that

$$\tilde{\rho}(\dot{w}\dot{w}')^{-1}\rho(l)\tilde{\rho}(\dot{w}\dot{w}') = \tilde{\rho}(\dot{w}')^{-1}\tilde{\rho}(\dot{w})^{-1}\rho(l)\tilde{\rho}(\dot{w})\tilde{\rho}(\dot{w}')$$

for all $l \in L_J$. Thus $\tilde{\rho}(\dot{w})\tilde{\rho}(\dot{w}')$ is a nonzero scalar multiple of $\tilde{\rho}(\dot{w}\dot{w}')$. Let

$$\tilde{\rho}(\dot{w})\tilde{\rho}(\dot{w}') = \lambda(w, w')\tilde{\rho}(\dot{w}\dot{w}')$$

where $\lambda(w, w') \in \mathbb{C}$, $\lambda(w, w') \neq 0$.

Proposition 10.2.5. *Suppose $w, w' \in W^{J,\phi}$ satisfy $l(ww') = l(w) + l(w')$. Then $B_w B_{w'} = \lambda(w, w')B_{ww'}$.*

Proof. We have a sequence of G-homomorphisms

$$\mathfrak{F}(J, \rho) \xrightarrow{\theta_{w'}} \mathfrak{F}(J, {}^{\dot{w}'}\rho) \xrightarrow{\tilde{\rho}(\dot{w}')} \mathfrak{F}(J, \rho) \xrightarrow{\theta_w} \mathfrak{F}(J, {}^{\dot{w}}\rho) \xrightarrow{\tilde{\rho}(\dot{w})} \mathfrak{F}(J, \rho)$$

and $B_w = \tilde{\rho}(\dot{w}) \circ \theta_w$, $B_{w'} = \tilde{\rho}(\dot{w}') \circ \theta_{w'}$. Hence

$$\begin{aligned} B_w B_{w'} &= \tilde{\rho}(\dot{w}) \circ \theta_w \circ \tilde{\rho}(\dot{w}') \circ \theta_{w'} \\ &= \tilde{\rho}(\dot{w}) \circ \tilde{\rho}(\dot{w}') \circ \theta_w \circ \theta_{w'} \quad \text{by 10.2.4} \\ &= \lambda(w, w')\tilde{\rho}(\dot{w}\dot{w}') \circ \theta_{ww'} \quad \text{by 10.2.2} \\ &= \lambda(w, w') B_{ww'}. \end{aligned}$$

10.3 A PROJECTIVE REPRESENTATION OF $K_{J,\Phi}$

Let $K_{J,\Phi}$ be the subgroup of G generated by L_J and the elements $\dot{w}$ for $w \in W^{J,\Phi}$. Since $w(J) = J$ for all $w \in W^{J,\Phi}$ we see that $\dot{w} \in N_G(L_J)$. Thus $K_{J,\Phi}$ is a subgroup of $N_G(L_J)$ containing L_J.

Lemma 10.3.1. *$K_{J,\Phi}/L_J$ is isomorphic to $W^{J,\Phi}$.*

Proof. We know by 9.2.2 that

$$N_G(L_J) \cap N/N_J \cong N_W(W_J)/W_J \cong C_J.$$

It follows that $\langle N_J, \dot{w}; w \in W^{J,\Phi}\rangle/N_J \cong W^{J,\Phi}$. Since $N_J = N \cap L_J$ we have

$$\begin{aligned} K_{J,\Phi}/L_J &= \langle L_J, \dot{w}; w \in W^{J,\Phi}\rangle/L_J \\ &\cong \langle N_J, \dot{w}; w \in W^{J,\Phi}\rangle/N_J \cong W^{J,\Phi}. \end{aligned}$$

Thus $K_{J,\Phi}/L_J \cong W^{J,\Phi}$ as required. ■

We now construct a projective representation of $K_{J,\Phi}$ on the module M. M is a left L_J-module affording the representation ρ of L_J. Our projective representation of $K_{J,\Phi}$ will agree with ρ on restriction to L_J. It will therefore be denoted by $\tilde{\rho}$.

Proposition 10.3.2. *There is a projective representation $\tilde{\rho}$ of $K_{J,\Phi}$ on the module M which extends the ordinary representation ρ of L_J and is given by*

$$\tilde{\rho}(l\dot{w}) = \rho(l)\tilde{\rho}(\dot{w}) \qquad l \in L_J, w \in W^{J,\Phi}.$$

Proof. It follows from 10.3.1 that each element of $K_{J,\Phi}$ is uniquely expressible in the form $l\dot{w}$ for $l \in L_J$ and $w \in W^{J,\Phi}$. Thus $\tilde{\rho}(l\dot{w})$: $M \to M$ is well defined by the above formula. To show we have a projective representation we must verify that $\tilde{\rho}(k_1k_2)$ is a scalar multiple of $\tilde{\rho}(k_1)\tilde{\rho}(k_2)$ for all $k_1, k_2 \in K_{J,\Phi}$.

Let $k_1 = l_1\dot{w}_1$, $k_2 = l_2\dot{w}_2$ with $l_1, l_2 \in L_J$ and $w_1, w_2 \in W^{J,\Phi}$. Then we have

$$\begin{aligned} k_1k_2 = l_1\dot{w}_1 l_2 \dot{w}_2 &= l_1 \dot{w}_1 l_2 \dot{w}_1^{-1} \dot{w}_1 \dot{w}_2 \\ &= l_1 \dot{w}_1 l_2 \dot{w}_1^{-1} \dot{w}_1 \dot{w}_2 (w_1 \dot{w}_2)^{-1} (w_1 \dot{w}_2). \end{aligned}$$

Now l_1, $\dot{w}_1 l_2 \dot{w}_1^{-1}$ and $\dot{w}_1\dot{w}_2(w_1 \dot{w}_2)^{-1}$ lie in L_J. Thus we have

$$\tilde{\rho}(k_1k_2) = \rho(l_1)\rho(\dot{w}_1 l_2 \dot{w}_1^{-1})\rho(\dot{w}_1\dot{w}_2(w_1\dot{w}_2)^{-1})\tilde{\rho}(w_1\dot{w}_2).$$

On the other hand

$$\begin{aligned}\tilde{\rho}(k_1)\tilde{\rho}(k_2) &= \rho(l_1)\tilde{\rho}(\dot{w}_1)\rho(l_2)\tilde{\rho}(\dot{w}_2)\\ &= \rho(l_1)\tilde{\rho}(\dot{w}_1)\rho(l_2)\tilde{\rho}(\dot{w}_1)^{-1}\tilde{\rho}(\dot{w}_1)\tilde{\rho}(\dot{w}_2)\\ &= \rho(l_1)\rho(\dot{w}_1 l_2 \dot{w}_1{}^{-1})\tilde{\rho}(\dot{w}_1)\tilde{\rho}(\dot{w}_2)\tilde{\rho}(w_1\dot{}w_2)^{-1}\tilde{\rho}(w_1\dot{}w_2).\end{aligned}$$

Comparing $\tilde{\rho}(k_1k_2)$ with $\tilde{\rho}(k_1)\tilde{\rho}(k_2)$ we see that it is sufficient to show that $\tilde{\rho}(\dot{w}_1)\tilde{\rho}(\dot{w}_2)\tilde{\rho}(w_1\dot{}w_2)^{-1}$ is a scalar multiple of $\rho(\dot{w}_1\dot{w}_2(w_1\dot{}w_2)^{-1})$.

Now we have, for all $l \in L_J$,

$$\begin{aligned}&\tilde{\rho}((w_1\dot{}w_2))\tilde{\rho}(\dot{w}_2)^{-1}\tilde{\rho}(\dot{w}_1)^{-1}\rho(l)\tilde{\rho}(\dot{w}_1)\tilde{\rho}(\dot{w}_2)\tilde{\rho}((w_1\dot{}w_2))^{-1}\\ &\qquad\qquad = \rho((w_1\dot{}w_2)\dot{w}_2{}^{-1}\dot{w}_1{}^{-1}l\dot{w}_1\dot{w}_2(w_1\dot{}w_2)^{-1}).\end{aligned}$$

However, since $\dot{w}_1\dot{w}_2(w_1\dot{}w_2)^{-1} \in L_J$ we have

$$\rho((w_1\dot{}w_2)\dot{w}_2{}^{-1}\dot{w}_1{}^{-1}l\dot{w}_1\dot{w}_2(w_1\dot{}w_2)^{-1}) = \rho(\dot{w}_1\dot{w}_2(w_1\dot{}w_2)^{-1})\rho(l)\rho(\dot{w}_1\dot{w}_2(w_1\dot{}w_2)^{-1}.$$

Thus $\tilde{\rho}(\dot{w}_1)\tilde{\rho}(\dot{w}_2)\tilde{\rho}((w_1\dot{}w_2))^{-1}\rho(\dot{w}_1\dot{w}_2(w_1\dot{}w_2)^{-1})^{-1}$ commutes with $\rho(l)$ for all $l \in L_J$. Since ρ is irreducible this must be a scalar multiple of the identity, by Schur's lemma. Thus $\tilde{\rho}(\dot{w}_1)\tilde{\rho}(\dot{w}_2)\tilde{\rho}(w_1\dot{}w_2))^{-1}$ is a scalar multiple of $\rho(\dot{w}_1\dot{w}_2(w_1\dot{}w_2)^{-1})$ and the proof is complete. ∎

We now define a function $\lambda: W^{J,\phi} \times W^{J,\phi} \to \mathbb{C}^*$ by the equation

$$\tilde{\rho}(\dot{w}_1)\tilde{\rho}(\dot{w}_2) = \lambda(w_1, w_2)\tilde{\rho}(\dot{w}_1\dot{w}_2)$$

for $w_1, w_2 \in W^{J,\phi}$. We observe that this definition is consistent with the notation of 10.2.5 when $\lambda(w_1, w_2)$ was defined for elements $w_1, w_2 \in W^{J,\phi}$ satisfying $l(w_1w_2) = l(w_1) + l(w_2)$. We now have a definition of $\lambda(w_1, w_2)$ for all pairs of elements $w_1, w_2 \in W^{J,\phi}$.

Proposition 10.3.3. *The function $\lambda: W^{J,\phi} \times W^{J,\phi} \to \mathbb{C}^*$ satisfies the condition*

$$\lambda(w_1, w_2)\lambda(w_1w_2, w_3) = \lambda(w_1, w_2w_3)\lambda(w_2, w_3)$$

for all $w_1, w_2, w_3 \in W^{J,\phi}$.

(Such a function is called a 2-cocycle.)

Proof. We have

$$\begin{aligned}\tilde{\rho}(\dot{w}_1)\tilde{\rho}(\dot{w}_2) &= \lambda(w_1, w_2)\tilde{\rho}(\dot{w}_1\dot{w}_2)\\ &= \lambda(w_1, w_2)\tilde{\rho}(\dot{w}_1\dot{w}_2(w_1\dot{}w_2)^{-1}(w_1\dot{}w_2))\\ &= \lambda(w_1, w_2)\rho(\dot{w}_1\dot{w}_2(w_1\dot{}w_2)^{-1})\tilde{\rho}((w_1\dot{}w_2)).\end{aligned}$$

Thus

$$\begin{aligned}
&\tilde{\rho}(\dot{w}_1)\tilde{\rho}(\dot{w}_2)\tilde{\rho}(\dot{w}_3)\\
&\quad = \lambda(w_1, w_2)\rho(\dot{w}_1\dot{w}_2(w_1\dot{}w_2)^{-1})\tilde{\rho}((w_1\dot{}w_2))\tilde{\rho}(\dot{w}_3)\\
&\quad = \lambda(w_1, w_2)\lambda(w_1w_2, w_3)\rho(\dot{w}_1\dot{w}_2(w_1\dot{}w_2)^{-1})\tilde{\rho}((w_1\dot{}w_2)\dot{w}_3)\\
&\quad = \lambda(w_1, w_2)\lambda(w_1w_2, w_3)\rho(\dot{w}_1\dot{w}_2(w_1\dot{}w_2)^{-1})\rho((w_1\dot{}w_2)\dot{w}_3(w_1\dot{w}_2w_3)^{-1})\tilde{\rho}((w_1\dot{w}_2w_3))\\
&\quad = \lambda(w_1, w_2)\lambda(w_1w_2, w_3)\rho(\dot{w}_1\dot{w}_2\dot{w}_3(w_1\dot{w}_2w_3)^{-1}\tilde{\rho}((w_1\dot{w}_2w_3)).
\end{aligned}$$

Similarly we have

$$\tilde{\rho}(\dot{w}_2)\tilde{\rho}(\dot{w}_3) = \lambda(w_2, w_3)\tilde{\rho}(\dot{w}_2\dot{w}_3) = \lambda(w_2, w_3)\rho(\dot{w}_2\dot{w}_3(w_2\dot{}w_3)^{-1})\tilde{\rho}((w_2\dot{}w_3)).$$

Thus

$$\begin{aligned}
\tilde{\rho}(\dot{w}_1)\tilde{\rho}(\dot{w}_2)\tilde{\rho}(\dot{w}_3) &= \lambda(w_2, w_3)\tilde{\rho}(\dot{w}_1)\rho(\dot{w}_2\dot{w}_3(w_2\dot{}w_3)^{-1})\tilde{\rho}((w_2\dot{}w_3))\\
&= \lambda(w_2, w_3)\rho(\dot{w}_1\dot{w}_2\dot{w}_3(w_2\dot{}w_3)^{-1}\dot{w}_1{}^{-1})\tilde{\rho}(\dot{w}_1)\tilde{\rho}((w_2\dot{}w_3))\\
&= \lambda(w_2, w_3)\lambda(w_1, w_2w_3)\rho(\dot{w}_1\dot{w}_2\dot{w}_3(w_2\dot{}w_3)^{-1}\dot{w}_1{}^{-1})\tilde{\rho}(\dot{w}_1(w_2\dot{}w_3))\\
&= \lambda(w_2, w_3)\lambda(w_1, w_2w_3)\rho(\dot{w}_1\dot{w}_2\dot{w}_3(w_1\dot{w}_2w_3)^{-1})\tilde{\rho}((w_1\dot{w}_2w_3)).
\end{aligned}$$

Comparing these two expressions we obtain

$$\lambda(w_1, w_2)\lambda(w_1w_2, w_3) = \lambda(w_2, w_3)\lambda(w_1, w_2w_3)$$

as required ■

Now the cocycle λ is not uniquely determined, since each $\tilde{\rho}(\dot{w})$ is defined only up to a nonzero scalar. The next result shows that λ can be chosen to have some favourable properties.

Proposition 10.3.4. *The function $\lambda: W^{J,\phi} \times W^{J,\phi} \to \mathbb{C}^*$ can be chosen to satisfy the conditions:*

(i) $\lambda(w, 1) = \lambda(1, w) = 1$ *for all* w.
(ii) $\lambda(w, w^{-1}) = 1$ *for all* w.
(iii) $\lambda(w_1, w_2) = \dfrac{1}{\lambda(w_2{}^{-1}, w_1{}^{-1})} = \lambda(w_2{}^{-1}w_1{}^{-1}, w_1)$ *for all* w_1, w_2.
(iv) $\lambda(w_1, w_2)$ *is a root of unity for all* w_1, w_2.

Proof. If we replace $\tilde{\rho}(\dot{w})$ by $\xi_w\tilde{\rho}(\dot{w})$ then $\lambda(w_1, w_2)$ will be replaced by the equivalent cocycle $\lambda'(w_1, w_2) = \xi_{w_1}\xi_{w_2}\xi_{w_1w_2}{}^{-1}\lambda(w_1, w_2)$. We show that the ξ_w can be chosen so that λ' satisfies (i), (ii), (iii), (iv).

For each $w \in W^{J,\phi}$ let $\mathscr{R}(w)$ be the square matrix with rows and columns indexed by $W^{J,\phi}$ given by

$$(\mathscr{R}(w))_{x,y} = \begin{cases} \lambda(w, y) & \text{if } x = wy \\ 0 & \text{if } x \neq wy. \end{cases}$$

Then $\mathscr{R}(w)$ is a nonsingular monomial matrix. Moreover, if $w_1, w_2, x, y \in W^{J,\phi}$, we have

$$\begin{aligned}
(\mathscr{R}(w_1)\mathscr{R}(w_2))_{x,y} &= \sum_z (\mathscr{R}(w_1))_{x,z}(\mathscr{R}(w_2))_{z,y} \\
&= \lambda(w_2, y)(\mathscr{R}(w_1))_{x,w_2y} \\
&= \begin{cases} \lambda(w_1, w_2y)\lambda(w_2, y) & \text{if } x = w_1w_2y \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \lambda(w_1, w_2)\lambda(w_1w_2, y) & \text{if } x = w_1w_2y \\ 0 & \text{otherwise} \end{cases} \\
&= \lambda(w_1, w_2)(\mathscr{R}(w_1w_2))_{x,y}.
\end{aligned}$$

Thus $\mathscr{R}(w_1)\mathscr{R}(w_2) = \lambda(w_1, w_2)\mathscr{R}(w_1w_2)$.

We see in particular that $\mathscr{R}(1)$ is a scalar multiple of the identity. We choose ξ_1 so that $\xi_1\mathscr{R}(1) = I$. We see also that $\mathscr{R}(w^{-1})$ is a scalar multiple of $\mathscr{R}(w)^{-1}$. For each $w \in W^{J,\phi}$ of order 2 we choose ξ_w so that $(\xi_w\mathscr{R}(w))^2 = I$. For each pair of inverse elements $w, w^{-1} \in W^{J,\phi}$ with $w^{-1} \neq w$ we choose ξ_w to satisfy $\det(\xi_w\mathscr{R}(w)) = 1$ and $\xi_{w^{-1}}$ to satisfy $\xi_{w^{-1}}\mathscr{R}(w^{-1}) = (\xi_w\mathscr{R}(w))^{-1}$.

We now define $\mathscr{R}'(w) = \xi_w\mathscr{R}(w)$. We then have

$$\begin{aligned}
\mathscr{R}'(w_1)\mathscr{R}'(w_2) &= \lambda'(w_1, w_2)\mathscr{R}'(w_1w_2) \\
\mathscr{R}'(1) &= I \\
\mathscr{R}'(w^{-1}) &= (\mathscr{R}'(w))^{-1} \\
\det \mathscr{R}'(w) &= \pm 1
\end{aligned}$$

for all $w_1, w_2, w \in W^{J,\phi}$.

Putting $w_1 = 1$ or $w_2 = 1$ we see that $\lambda'(w, 1) = \lambda'(1, w) = 1$ for all w. Putting $w_2 = w_1^{-1}$ we see that $\lambda(w, w^{-1}) = 1$ for all w. Thus λ' satisfies conditions (i), (ii). We also have

$$\mathscr{R}'(w_2)^{-1}\mathscr{R}'(w_1)^{-1} = \frac{1}{\lambda'(w_1, w_2)}\mathscr{R}'(w_1w_2)^{-1}$$

so

$$\lambda'(w_2^{-1}, w_1^{-1}) = \frac{1}{\lambda'(w_1, w_2)}.$$

We also have

$$\mathscr{R}'(w_1w_2)^{-1}\mathscr{R}'(w_1) = \lambda'(w_1, w_2)\mathscr{R}'(w_2)^{-1}$$

and so

$$\lambda'(w_2^{-1}w_1^{-1}, w_1) = \lambda'(w_1, w_2).$$

Thus λ' satisfies condition (iii). Finally we have

$$\det \mathscr{R}'(w_1)\cdot\det \mathscr{R}'(w_2) = \lambda'(w_1, w_2)^k \det \mathscr{R}'(w_1 w_2)$$

where $k = |W^{J,\phi}|$. Hence $\lambda'(w_1, w_2)^k = \pm 1$ and so λ' satisfies condition (iv). ∎

We shall assume subsequently that the cocycle λ is chosen as in 10.3.4.

10.4 THE QUOTIENT ROOT SYSTEM

We have seen that the structure of the endomorphism algebra $\mathfrak{E} = \mathrm{End}_G\, \mathfrak{F}(J, \rho)$ is closely connected to that of the ramification group $W^{J,\phi}$. We shall show that $W^{J,\phi}$, although not in general a reflection group, nevertheless contains a normal reflection subgroup which is a large part of it. The root system of this reflection subgroup is obtained as a quotient root system from the root system of W in a way which we shall now describe.

Suppose the Coxeter group W acts in its natural representation on the real vector space V. Let V_J be the subspace spanned by the roots in Δ_J, where $J \subseteq I$. We shall introduce a quotient root system in the space V/V_J. Since $V = V_J \oplus V_J^\perp$ we have $V/V_J \cong V_J^\perp$ so we could equally well work in $V_J^\perp$.

Suppose $\alpha \in \Phi - \Delta_J$ is a root such that $\Delta_J \cup \{\alpha\} \subseteq w(\Delta)$ for some $w \in W$. Thus $\Delta_J \cup \{\alpha\}$ lies in some simple system of roots. We then define $w_{\bar{\alpha}} \in W$ by

$$w_{\bar{\alpha}} = (w_0)_{J\cup\{\alpha\}}(w_0)_J$$

where $(w_0)_J$ and $(w_0)_{J\cup\{\alpha\}}$ are the elements of maximal length in the Coxeter groups with simple root systems Δ_J and $\Delta_J \cup \{\alpha\}$ respectively.

Lemma 10.4.1. *$w_{\bar{\alpha}}(\Delta_J) = \Delta_J$ if and only if $w_{\bar{\alpha}}^2 = 1$.*

Proof. $w_{\bar{\alpha}}(\Delta_J) = \Delta_J$ is equivalent to $(w_0)_{J\cup\{\alpha\}}(\Delta_J) = -\Delta_J$. However $(w_0)_{J\cup\{\alpha\}}$ transforms Δ_J into the negative of the subset of $\Delta_J \cup \{\alpha\}$ opposed to Δ_J. Thus $w_{\bar{\alpha}}(\Delta_J) = \Delta_J$ is equivalent to the condition that Δ_J is self-opposed in $\Delta_J \cup \{\alpha\}$. $w_{\bar{\alpha}}^2 = 1$ is equivalent to

$$(w_0)_{J\cup\{\alpha\}}(w_0)_J(w_0)_{J\cup\{\alpha\}} = (w_0)_J.$$

This is again equivalent to the condition that Δ_J is self-opposed in $\Delta_J \cup \{\alpha\}$. Thus the two given conditions are equivalent. ∎

Let $\Omega = \{\alpha \in \Phi - \Delta_J\colon \Delta_J \cup \{\alpha\} \subseteq w(\Delta)$ for some $w \in W$, $w_{\bar{\alpha}}^2 = 1\}$. Thus Ω is the set of roots α not in Δ_J which, together with Δ_J, form a simple system $\Delta_J \cup \{\alpha\}$ in which Δ_J is self-opposed.

Let $R_J = \langle w_{\bar{\alpha}} : \alpha \in \Omega\rangle$ be the subgroup of W generated by the elements $w_{\bar{\alpha}}$ for all $\alpha \in \Omega$.

Let $\bar{V} = V/V_J$, $\bar{v} = V_J + v$ for $v \in V$, and $\bar{\Omega} = \{\bar{\alpha}; \alpha \in \Omega\}$. The isomorphism $\bar{V} \cong V_J^\perp$ enables us to define a $W^{J,\phi}$-invariant scalar product on $\bar{V}$.

Theorem 10.4.2. (i) *The elements $\bar{\alpha} \in \bar{\Omega}$ for $\alpha \in \Omega$ are all distinct.*

(ii) *The elements of $\bar{\Omega}$, normalized by positive scalars to make them into unit vectors, form the root system of a group with a split BN-pair. (However the elements of $\bar{\Omega}$ need not span $\bar{V}$.)*

(iii) *$w_{\bar{\alpha}}$ acts on $\bar{V}$ as the reflection in the hyperplane orthogonal to $\bar{\alpha}$.*

(iv) *R_J acts faithfully on $\bar{V}$ and R_J is the Weyl group of the root system $\bar{\Omega}$.*

Proof. Let $\alpha \in \Omega$. We show first that $w_{\bar{\alpha}}$ acts on $\bar{V}$ as the reflection in the hyperplane orthogonal to $\bar{\alpha}$. We have $\bar{V} \cong V_J{}^{\perp}$ and we make use of the decomposition $V = V_J \oplus V_J{}^{\perp}$. Let $v \in V$ satisfy $v = v_J + \bar{v}$ where $v_J \in V_J$ and $\bar{v} \in V_J{}^{\perp}$. Suppose $\bar{v}$ satisfies $(\bar{\alpha}, \bar{v}) = 0$. Then

$$w_{\bar{\alpha}}\bar{v} = (w_0)_{J\cup\{\alpha\}}(w_0)_J\bar{v} = (w_0)_{J\cup\{\alpha\}}\bar{v} = \bar{v}$$

since $\bar{v} \in V_J{}^{\perp}$ and $(\bar{\alpha}, \bar{v}) = 0$. Moreover we have

$$w_{\bar{\alpha}}\bar{\alpha} = (w_0)_{J\cup\{\alpha\}}(w_0)_J\bar{\alpha} = (w_0)_{J\cup\{\alpha\}}\bar{\alpha}.$$

Now $(w_0)_{J\cup\{\alpha\}}(\Delta_J \cup \{\alpha\}) = -(\Delta_J \cup \{\alpha\})$ and $(w_0)_{J\cup\{\alpha\}}(\Delta_J) = -\Delta_J$ by 10.4.1. Thus $(w_0)_{J\cup\{\alpha\}}\alpha = -\alpha$ and so $(w_0)_{J\cup\{\alpha\}}\bar{\alpha} = -\bar{\alpha}$. Thus $w_{\bar{\alpha}}\bar{\alpha} = -\bar{\alpha}$. We have therefore seen that $w_{\bar{\alpha}}\bar{v} = \bar{v}$ when $(\bar{\alpha}, \bar{v}) = 0$ and $w_{\bar{\alpha}}\bar{\alpha} = -\bar{\alpha}$. Thus $w_{\bar{\alpha}}$ acts on $\bar{V}$ as the reflection in the hyperplane orthogonal to $\bar{\alpha}$.

We now wish to show that $\bar{\Omega}$ gives rise to a root system. To do so we may assume without loss of generality that the root system Φ is irreducible. W is then either the Weyl group of a simple algebraic group or is isomorphic to the dihedral group of order 16 (see secion 2.10). In the latter case we have $\dim V = 2$ and the only nontrivial possibility for $\bar{V}$ arises when $|J| = 1$. Then $\dim \bar{V} = 1$ and $\bar{\Omega}$ is a root system of type A_1.

We may therefore assume that Φ arises from the root system Φ' of a simple algebraic group by replacing all roots in Φ' by unit vectors in the same direction. We then know that the elements of Φ' satisfy:

$$\alpha, \beta \in \Phi' \quad \text{implies } w_\alpha(\beta) \in \Phi'$$

$$\alpha, \beta \in \Phi' \quad \text{implies } \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$$

$$\alpha, \lambda\alpha \in \Phi' \quad \text{implies } \lambda = \pm 1.$$

Let $\bar{\Omega}'$ be the set of vectors obtained from Φ' in the same way that $\bar{\Omega}$ is obtained from Φ. We shall show that the elements of $\bar{\Omega}'$ also satisfy the above three conditions. $\bar{\Omega}'$ will then be the root system of some semisimple algebraic group. The unit vectors in the directions of the elements of $\bar{\Omega}'$ will then form the root system of a group with split BN-pair. These are also the unit vectors in the directions of the elements of $\bar{\Omega}$, so we shall obtain the required result.

Suppose $\alpha, \beta \in \Omega'$ satisfy $\bar{\alpha} = \lambda\bar{\beta}$ where $\lambda > 0$. We wish to show that $\lambda = 1$. The group $\langle w_\gamma; \gamma \in \Delta_J \cup \{\alpha\}\rangle$ is a Weyl group whose roots are all the roots in Φ which are linear combinations of $\Delta_J \cup \{\alpha\}$. The group $\langle w_\gamma; \gamma \in \Delta_J \cup \{\beta\}\rangle$ is a Weyl group whose roots are all roots in Φ which are linear combinations of

$\Delta_J \cup \{\beta\}$. However $V_{J\cup\{\alpha\}} = V_{J\cup\{\beta\}}$ since $\alpha - \lambda\beta \in V_J$. Thus these two root systems are the same. Moreover the two positive systems determined by the simple systems $\Delta_J \cup \{\alpha\}$, $\Delta_J \cup \{\beta\}$ are the same, since $\alpha \in V_J + \lambda\beta$ and $\beta \in V_J + \lambda^{-1}\alpha$ where $\lambda > 0$. It follows that the two simple systems $\Delta_J \cup \{\alpha\}$, $\Delta_J \cup \{\beta\}$ must be the same. Hence $\alpha = \beta$ and so $\bar{\alpha} = \bar{\beta}$ and $\lambda = 1$.

We show next that $2(\bar{\alpha}, \bar{\beta})/(\bar{\alpha}, \bar{\alpha}) \in \mathbb{Z}$ for all $\alpha, \beta \in \Omega'$. We have

$$\begin{aligned} w_{\bar{\alpha}}\beta &= (w_0)_{J\cup\{\alpha\}}(w_0)_J\beta \\ &\equiv (w_0)_{J\cup\{\alpha\}}\beta \bmod V_J \\ &\equiv \beta + \lambda\alpha \bmod V_J \qquad \text{for some } \lambda \in \mathbb{Z} \end{aligned}$$

since $2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}$. It follows that

$$w_{\bar{\alpha}}\bar{\beta} = \bar{\beta} + \lambda\bar{\alpha} = \bar{\beta} - \frac{2(\bar{\alpha}, \bar{\beta})}{(\bar{\alpha}, \bar{\alpha})}\bar{\alpha}.$$

Thus $2(\bar{\alpha}, \bar{\beta})/(\bar{\alpha}, \bar{\alpha}) \in \mathbb{Z}$. (*Note* $2(\bar{\alpha}, \bar{\beta})/(\bar{\alpha}, \bar{\alpha})$ need not be equal to $2(\alpha, \beta)/(\alpha, \alpha)$.)

We show next that if $\bar{\alpha}, \bar{\beta} \in \bar{\Omega}$, then $w_{\bar{\alpha}}(\bar{\beta}) \in \bar{\Omega}'$. Let $\alpha, \beta \in \Omega'$ and let $\gamma = w_{\bar{\alpha}}(\beta)$. We shall show that $\gamma \in \Omega'$. It will follow that $w_{\bar{\alpha}}(\bar{\beta}) = \bar{\gamma} \in \bar{\Omega}'$.

Now $\Delta_J \cup \{\gamma\} = \Delta_J \cup \{w_{\bar{\alpha}}(\beta)\} = w_{\bar{\alpha}}(\Delta_J \cup \{\beta\})$ and this lies in some simple system of roots in Φ'. Thus $w_{\bar{\gamma}}$ is well defined. In order to prove that $\gamma \in \Omega'$ we must show that $w_{\bar{\gamma}}(\Delta_J) = \Delta_J$. We have

$$\begin{aligned} w_{\bar{\gamma}}(\Delta_J) &= (w_0)_{J\cup\{\gamma\}}(w_0)_J(\Delta_J) \\ &= (w_0)_{w_{\bar{\alpha}}(J\cup\{\beta\})}(w_0)_J(\Delta_J) \\ &= w_{\bar{\alpha}}(w_0)_{J\cup\{\beta\}}w_{\bar{\alpha}}^{-1}(w_0)_J(\Delta_J) \\ &= w_{\bar{\alpha}}w_{\bar{\beta}}(w_0)_J^{-1}w_{\bar{\alpha}}^{-1}(w_0)_J(\Delta_J) \\ &= \Delta_J \end{aligned}$$

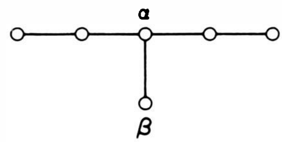
since $(w_0)_J(\Delta_J) = -\Delta_J$, $w_{\bar{\alpha}}(\Delta_J) = \Delta_J$, $w_{\bar{\beta}}(\Delta_J) = \Delta_J$.

We have now checked the three conditions which show that $\bar{\Omega}'$ is the root system of a semisimple algebraic group. The unit vectors in the directions of the elements in $\bar{\Omega}$ thus form the root system of a group with split BN-pair.

The Weyl group of this root system is the group generated by the elements $w_{\bar{\alpha}}$, $\bar{\alpha} \in \bar{\Omega}$, acting on $\bar{V}$. Now the group generated by the $w_{\bar{\alpha}}$, $\bar{\alpha} \in \bar{\Omega}$, is R_J. We show that R_J acts faithfully on $\bar{V}$. Suppose $w \in R_J$ acts trivially on $\bar{V}$. Then for all $\alpha \in \Delta - \Delta_J$ we have $w(\alpha) \equiv \alpha \bmod V_J$. Thus $w(\alpha) > 0$. Also for all $\beta \in \Delta_J$ we have $w(\beta) > 0$ since $w(\Delta_J) = \Delta_J$. Thus $w(\gamma) > 0$ for all $\gamma \in \Delta$, and this implies that $w = 1$. Hence R_J acts faithfully on $\bar{V}$.

It remains only to show that if $\alpha, \beta \in \Omega$ satisfy $\bar{\alpha} = \bar{\beta}$ then $\alpha = \beta$. This follows from what has been proved above. For the roots in Φ which lie in $V_{J\cup\{\alpha\}}$ are the same as those in $V_{J\cup\{\beta\}}$ since $V_{J\cup\{\alpha\}} = V_{J\cup\{\beta\}}$. The positive system determined by $\Delta_J \cup \{\alpha\}$ is the same as the positive system determined by $\Delta_J \cup \{\beta\}$ since $\alpha \in V_J + \beta$ and $\beta \in V_J + \alpha$. Thus the simple systems $\Delta_J \cup \{\alpha\}, \Delta_J \cup \{\beta\}$ must be equal, and so $\alpha = \beta$. ■

We now give an example of a quotient root system. Suppose that W is of type E_6 and that W_J is the unique parabolic subgroup of W of type $A_2 \times A_2$. The simple roots not in J are the roots α, β shown in the diagram.



Thus $\bar{\Omega} = \{\bar{\alpha}, \bar{\beta}\}$ and $2(\bar{\alpha}, \bar{\beta})/(\bar{\alpha}, \bar{\alpha}) = -3, 2(\bar{\beta}, \bar{\alpha})/(\bar{\beta}, \bar{\beta}) = -1$. Thus the quotient root system has type G_2.

10.5 THE QUADRATIC RELATIONS

We now wish to calculate $B^2_{w_{\bar{\alpha}}}$ as a linear combinaton of the basis elements B_w of $\mathfrak{E}$, when $\alpha \in \Delta - \Delta_J$ satisfies the condition that $w_{\bar{\alpha}} \in W^{J,\phi}$. We recall that $w_{\bar{\alpha}} \in W$ is defined for each $\alpha \in \Delta - \Delta_J$ and that if $w_{\bar{\alpha}} \in W^{J,\phi}$ then $w_{\bar{\alpha}}{}^2 = 1$ and $w_{\bar{\alpha}} \in R_J$.

We recall also that, for any $w \in W^{J,\phi}$, $B_w = \tilde{\rho}(\dot{w}) \circ \theta_w$ where θ_w maps $\mathfrak{F}(J, \rho)$ into $\mathfrak{F}(J, {}^{\dot{w}}\rho)$ and $\tilde{\rho}(\dot{w})$ maps $\mathfrak{F}(J, {}^{\dot{w}}\rho)$ back into $\mathfrak{F}(J, \rho)$. θ_w was defined by

$$(\theta_w f)g = \frac{1}{|U_{w(J)}|} \sum_{u \in U_{w(J)}} f(\dot{w}^{-1}ug) \qquad g \in G.$$

We now define a map θ_n: $\mathfrak{F}(J, \rho) \to \mathfrak{F}(w(J), {}^n\rho)$ for each $n \in N$ for which $w = \pi(n)$ satisfies $w(J) \subseteq \Delta$. We define $\theta_n f$ for $f \in \mathfrak{F}(J, \rho)$ by

$$(\theta_n f)g = \frac{1}{|U_{w(J)}|} \sum_{u \in U_{w(J)}} f(n^{-1}ug) \qquad g \in G.$$

It is clear that when $n = \dot{w}$ we have $\theta_n = \theta_w$. However it is useful to have this definition of θ_n for other values of n also.

Let $\alpha \in \Delta - \Delta_J$. Then $w_{\bar{\alpha}} \in W$ and we write $n_{\bar{\alpha}} = \dot{w}_{\bar{\alpha}}$. We have maps

$$\mathfrak{F}(J, \rho) \xrightarrow{\theta_{n_{\bar{\alpha}}}} \mathfrak{F}(w_{\bar{\alpha}}(J), {}^{n_{\bar{\alpha}}}\rho) \xrightarrow{\theta_{(n_{\bar{\alpha}}^{-1})}} \mathfrak{F}(J, \rho).$$

Thus $\theta_{n_{\bar{\alpha}}^{-1}}\theta_{n_{\bar{\alpha}}} \in \operatorname{End} \mathfrak{F}(J, \rho) = \mathfrak{E}$. It follows that $\theta_{n_{\bar{\alpha}}^{-1}}\theta_{n_{\bar{\alpha}}}$ can be expressed in terms of the basis elements B_w of $\mathfrak{E}$.

Proposition 10.5.1. *$\theta_{n_{\bar{\alpha}}^{-1}}\theta_{n_{\bar{\alpha}}}$ has the form $\xi 1 + \eta B_{w_{\bar{\alpha}}}$ if $w_{\bar{\alpha}} \in W^{J,\phi}$ and has the form $\xi 1$ if $w_{\bar{\alpha}} \notin W^{J,\phi}$, where $\xi, \eta \in \mathbb{C}$.*

Proof. Let $K \subseteq I$ be defined by $\Delta_K = \Delta_J \cup \{\alpha\}$. Then $P_J \cap L_K$ is a maximal parabolic subgroup of L_K, and its Levi decomposition is

$$P_J \cap L_K = (U_J \cap L_K)L_J.$$

We recall that $\mathfrak{F}(J, \rho)$ is a G-module affording the representation $\rho_{P_J}{}^G$. We consider the subspace $\mathfrak{F}(J, \rho)_K$ of $\mathfrak{F}(J, \rho)$ defined by

$$\mathfrak{F}(J, \rho)_K = \{f \in \mathfrak{F}(J, \rho); \operatorname{Supp} f \subseteq P_K\}.$$

The support of f lies in P_K implies that $f(g) = 0$ for all $f \in \mathfrak{F}(J, \rho)_K$ and $g \notin P_K$. It is easy to see that $\mathfrak{F}(J, \rho)_K$ is a P_K-module affording the representation $\rho_{P_J}{}^{P_K}$ of P_K.

Now $\rho_{P_J}{}^{P_K}$ is obtained from ρ_{L_J} by first lifting to P_J and then inducing to P_K. The lifting from L_J to P_J can be done in two steps—first lift ρ_{L_J} to $\rho_{P_J \cap L_K}$ and then lift $\rho_{P_J \cap L_K}$ to ρ_{P_J}.

However the process of lifting $\rho_{P_J \cap L_K}$ to ρ_{P_J} and then inducing to $\rho_{P_J}{}^{P_K}$ is equivalent to first inducing $\rho_{P_J \cap L_K}$ to $\rho_{P_J \cap L_K}{}^{L_K}$ and then lifting to $(\rho_{P_J \cap L_K}{}^{L_K})_{P_K}^{P_J \cap L_K}$. It follows that $\rho_{P_J}{}^{P_K} = (\rho_{P_J \cap L_K}{}^{L_K})_{P_K}$. Thus every endomorphism of $\mathfrak{F}(J, \rho)_K$ as an L_K-module is also an endomorphism of $\mathfrak{F}(J, \rho)_K$ as a P_K-module. Hence

$$\operatorname{End}_{P_K} \mathfrak{F}(J, \rho)_K = \operatorname{End}_{L_K} \mathfrak{F}(J, \rho)_K.$$

Now $\mathfrak{F}(J, \rho)_K$, considered as an L_K-module, is the analogue of $\mathfrak{F}(J, \rho)$ in L_K. Thus $\dim \operatorname{End}_{L_K} \mathfrak{F}(J, \rho)_K$ is $|W_K{}^{J,\phi}|$ where

$$W_K{}^{J,\phi} = \{w \in W_K; w(J) = J, {}^w\phi = \phi\}.$$

However the only elements of W_K which could possibly satisfy $w(J) = J$ are 1 and $w_{\bar{\alpha}}$. For suppose $w \in W_K$, $w \neq 1$ and $w(J) = J$. Since $\Delta_K = \Delta_J \cup \{\alpha\}$, w transforms α into a negative root. In fact w transforms each positive root involving α into a negative root, since α occurs in the transform with a negative coefficient. On the other hand w transforms every positive root in Φ_K not involving α into a positive root. Exactly the same is true of the element $w_{\bar{\alpha}} = (w_0)_{J \cup \{\alpha\}}(w_0)_J$. Thus w and $w_{\bar{\alpha}}$ transform the same positive roots into negative roots and so $w = w_{\bar{\alpha}}$. We have therefore shown that $W_K{}^{J,\phi}$ is either $\{1\}$ or $\{1, w_{\bar{\alpha}}\}$. In fact

$$W_K{}^{J,\phi} = \begin{cases} \{1, w_{\bar{\alpha}}\} & \text{if } w_{\bar{\alpha}} \in W^{J,\phi} \\ \{1\} & \text{if } w_{\bar{\alpha}} \notin W^{J,\phi}. \end{cases}$$

Let $w \in W_K{}^{J,\phi}$ and consider $B_w \in \operatorname{End}_G \mathfrak{F}(J, \rho)$. Then B_w gives rise, on restriction to $\mathfrak{F}(J, \rho)_K$, to an endomorphism $B_w|_{\mathfrak{F}(J,\rho)_K} \in \operatorname{End}_{L_K} \mathfrak{F}(J, \rho)_K$. For if $f(g) = 0$ for all $g \notin P_K$ the same is true of $B_w f$ when $w \in W_K$. Moreover the action of B_w on $\mathfrak{F}(J, \rho)$ is determined by its action on $\mathfrak{F}(J, \rho)_K$. For if $\operatorname{Supp} f \subseteq P_K$ and $g \in G$ then $\operatorname{Supp} gf \subseteq P_K g^{-1}$. Each $f \in \mathfrak{F}(J, \rho)$ can be expressed as a sum of functions with support on one coset $P_K g^{-1}$ and the effect of B_w on these functions is determined by its effect on $\mathfrak{F}(J, \rho)_K$. Since we know that the elements B_w are linearly independent on $\mathfrak{F}(J, \rho)$ they must therefore be linearly independent on $\mathfrak{F}(J, \rho)_K$ also. Thus $\operatorname{End}_{L_K} \mathfrak{F}(J, \rho)_K$ has basis

$$\begin{cases} \{1, B_{w_{\bar{\alpha}}}|_{\mathfrak{F}(J,\rho)_K}\} & \text{if } w_{\bar{\alpha}} \in W^{J,\phi} \\ \{1\} & \text{if } w_{\bar{\alpha}} \notin W^{J,\phi}. \end{cases}$$

Now $\theta_{n_{\bar{\alpha}}^{-1}}\theta_{n_{\bar{\alpha}}}$ lies in $\operatorname{End}_G \mathfrak{F}(J, \rho)$ and stabilizes $\mathfrak{F}(J. \rho)_K$. Thus $\theta_{n_{\bar{\alpha}}^{-1}}\theta_{n_{\bar{\alpha}}}|_{\mathfrak{F}(J,\rho)_K}$ is a linear combination of the elements $\{1, B_{w_{\bar{\alpha}}}|_{\mathfrak{F}(J,\rho)_K}\}$ if $w_{\bar{\alpha}} \in W^{J,\phi}$ and a scalar multiple of 1 if $w_{\bar{\alpha}} \notin W^{J,\phi}$. Moreover the action of $\theta_{n_{\bar{\alpha}}^{-1}}\theta_{n_{\bar{\alpha}}}$ on $\mathfrak{F}(J, \rho)$ is determined by its action on $\mathfrak{F}(J, \rho)_K$, as pointed out above. It follows that

$$\theta_{n_{\bar{\alpha}}^{-1}}\theta_{n_{\bar{\alpha}}} = \begin{cases} \xi 1 + \eta B_{w_{\bar{\alpha}}} & \text{if } w_{\bar{\alpha}} \in W^{J,\phi} \\ \xi 1 & \text{if } w_{\bar{\alpha}} \notin W^{J,\phi} \end{cases}$$

for numbers $\xi, \eta \in \mathbb{C}$.

Proposition 10.5.2. *Suppose $\alpha \in \Delta - \Delta_J$ and $w_{\bar{\alpha}} \in W^{J,\phi}$. Then there exist $\xi, \eta \in \mathbb{C}$ such that*

$$B_{w_{\bar{\alpha}}}{}^2 = \xi 1 + \eta B_{w_{\bar{\alpha}}}.$$

(Note that $B_1 = 1$.)

Proof. We shall use the result of 10.5.1. We have $B_w = \tilde{\rho}(\dot{w}) \circ \theta_w$ and so

$$\begin{aligned} B_{w_{\bar{\alpha}}}{}^2 &= \tilde{\rho}(n_{\bar{\alpha}}) \circ \theta_{n_{\bar{\alpha}}} \circ \tilde{\rho}(n_{\bar{\alpha}}) \circ \theta_{n_{\bar{\alpha}}} \\ &= \tilde{\rho}(n_{\bar{\alpha}}) \circ \tilde{\rho}(n_{\bar{\alpha}}) \circ \theta_{n_{\bar{\alpha}}} \circ \theta_{n_{\bar{\alpha}}} && \text{by 10.2.4} \\ &= \lambda(w_{\bar{\alpha}}, w_{\bar{\alpha}})\tilde{\rho}(n_{\bar{\alpha}})^2\theta_{n_{\bar{\alpha}}}\theta_{n_{\bar{\alpha}}} \\ &= \tilde{\rho}(n_{\bar{\alpha}})^2\theta_{n_{\bar{\alpha}}}\theta_{n_{\bar{\alpha}}} && \text{by 10.3.4(ii).} \end{aligned}$$

Now $w_{\bar{\alpha}}{}^2 = 1$ and so $n_{\bar{\alpha}}{}^2 \in H = B \cap N$. Let $h = n_{\bar{\alpha}}{}^2$. We consider $\rho(h)\theta_{n_{\bar{\alpha}}}: \mathfrak{F}(J, {}^{n_{\bar{\alpha}}}\rho) \to \mathfrak{F}(J, \rho)$. Let $f \in \mathfrak{F}(J, {}^{n_{\bar{\alpha}}}\rho)$. Then we have

$$(\theta_{n_{\bar{\alpha}}} f)g = \frac{1}{|U_J|} \sum_{u \in U_J} f(n_{\bar{\alpha}}{}^{-1}ug) \qquad g \in G$$

$$\begin{aligned} (\rho(h)\theta_{n_{\bar{\alpha}}} f)g &= \frac{1}{|U_J|} \sum_{u \in U_J} \rho(h) f(n_{\bar{\alpha}}{}^{-1}ug) \\ &= \frac{1}{|U_J|} \sum_{u \in U_J} f(hn_{\bar{\alpha}}{}^{-1}ug) = \theta_{n_{\bar{\alpha}}h^{-1}} f(g). \end{aligned}$$

Thus $\rho(h)\theta_{n_{\bar{\alpha}}} = \theta_{n_{\bar{\alpha}}h^{-1}}$. It follows that

$$B_{w_{\bar{\alpha}}}{}^2 = \rho(h)\theta_{n_{\bar{\alpha}}}\theta_{n_{\bar{\alpha}}} = \theta_{n_{\bar{\alpha}}h^{-1}}\theta_{n_{\bar{\alpha}}} = \theta_{n_{\bar{\alpha}}^{-1}}\theta_{n_{\bar{\alpha}}}.$$

The required result now follows from 10.5.1. ■

We now discuss the determination of ξ and η. For each $w \in W$ we write $\text{ind } w = |U_{w'}|$.

Proposition 10.5.3.

$$\xi = \frac{1}{\text{ind } w_{\bar{\alpha}}}.$$

Proof. We have $\theta_{n_{\bar{\alpha}}^{-1}}\theta_{n_{\bar{\alpha}}}f = \xi f + \eta B_{w_{\bar{\alpha}}}f$ for all $f \in \mathfrak{F}(J, \rho)$. Now

$$\theta_{n_{\bar{\alpha}}^{-1}}\theta_{n_{\bar{\alpha}}}f(g) = \frac{1}{|U_J \cap U_{w_{\bar{\alpha}}}|} \sum_{u \in U_J \cap U_{w_{\bar{\alpha}}}} (\theta_{n_{\bar{\alpha}}}f)(n_{\bar{\alpha}}ug) \qquad \text{by 10.2.1}$$

$$= \frac{1}{|U_J \cap U_{w_{\bar{\alpha}}}|} \frac{1}{|U_{w_{\bar{\alpha}}(J)} \cap U_{w_{\bar{\alpha}}^{-1}}|} \sum_u \sum_{u' \in U_{w_{\bar{\alpha}}(J)} \cap U_{w_{\bar{\alpha}}^{-1}}} f(n_{\bar{\alpha}}^{-1}u'n_{\bar{\alpha}}ug)$$

again by 10.2.1.

Let $m \neq 0 \in M$. Then there is a unique element $f_m \in \mathfrak{F}(J, \rho)_J$ with $f_M(1) = m$. In fact $\mathfrak{F}(J, \rho)_J$ is isomorphic to M under the map $m \to f_m$ where $f_m(l) = \rho(l)m$ for all $l \in L_J$. We shall calculate ξ by taking $f = f_m$ and $g = 1$. Now $f_m(x) = 0$ unless $x \in P_J$. We therefore need to know when an element of the form $n_{\bar{\alpha}}^{-1}u'n_{\bar{\alpha}}u$ lies in P_J, where $u \in U_J \cap U_{w_{\bar{\alpha}}}$ and $u' \in U_{w_{\bar{\alpha}}(J)} \cap U_{w_{\bar{\alpha}}^{-1}}$. Since $u \in U_J$ we have

$$n_{\bar{\alpha}}^{-1}u'n_{\bar{\alpha}}u \in P_J \Leftrightarrow n_{\bar{\alpha}}^{-1}u'n_{\bar{\alpha}} \in P_J \Leftrightarrow u' \in n_{\bar{\alpha}}P_Jn_{\bar{\alpha}}^{-1}.$$

We shall show that $U_{w_{\bar{\alpha}}(J)} \cap U_{w_{\bar{\alpha}}^{-1}} \cap n_{\bar{\alpha}}P_Jn_{\bar{\alpha}}^{-1} = 1$. To see this it is sufficient to show that no root subgroup X_β lies in this intersection. Such a root β would have to satisfy

$$\beta > 0, \quad w_{\bar{\alpha}}^{-1}(\beta) < 0, \quad w_{\bar{\alpha}}^{-1}(\beta) \notin \Phi_J, \quad X_{w_{\bar{\alpha}}^{-1}(\beta)} \subseteq P_J$$

and the latter three conditions cannot be satisfied simultaneously. Hence

$$n_{\bar{\alpha}}^{-1}u'n_{\bar{\alpha}}u \in P_J \Leftrightarrow u' = 1.$$

We therefore obtain

$$\theta_{n_{\bar{\alpha}}^{-1}}\theta_{n_{\bar{\alpha}}}f_m(1) = \frac{1}{|U_J \cap U_{w_{\bar{\alpha}}}|} \frac{1}{|U_{w_{\bar{\alpha}}(J)} \cap U_{w_{\bar{\alpha}}^{-1}}|} \sum_{u \in U_J \cap U_{w_{\bar{\alpha}}}} f_m(u).$$

But $f_m(u) = f_m(1) = m$. Thus

$$\theta_{n_{\bar{\alpha}}^{-1}}\theta_{n_{\bar{\alpha}}}f_m(1) = \frac{1}{|U_{w_{\bar{\alpha}}(J)} \cap U_{w_{\bar{\alpha}}^{-1}}|} m.$$

On the other hand we have

$$\xi f_m(1) + \eta B_{w_{\bar{\alpha}}}f_m(1) = \xi m + \eta\tilde{\rho}(n_{\bar{\alpha}}) \frac{1}{|U_{w_{\bar{\alpha}}(J)} \cap U_{w_{\bar{\alpha}}^{-1}}|} \sum_{u \in U_{w_{\bar{\alpha}}(J)} \cap U_{w_{\bar{\alpha}}^{-1}}} f_m(n_{\bar{\alpha}}^{-1}u)$$

by 10.2.1.

Now $u \in U \subseteq P_J$ and $n_{\bar{\alpha}} \notin P_J$ so $n_{\bar{\alpha}}^{-1}u \notin P_J$. Thus $f_m(n_{\bar{\alpha}}^{-1}u) = 0$ and so the above expression reduces to ξm. Hence

$$\xi = \frac{1}{|U_{w_{\bar{\alpha}}(J)} \cap U_{w_{\bar{\alpha}}^{-1}}|}.$$

Now $U_{w_{\bar{\alpha}}(J)} \cap U_{w_{\bar{\alpha}}^{-1}}$ is the product of all root subgroups X_β satisfying $\beta > 0$, $w_{\bar{\alpha}}^{-1}(\beta) < 0$, $w_{\bar{\alpha}}^{-1}(\beta) \notin \Phi_J$. Let $\gamma = -w_{\bar{\alpha}}^{-1}(\beta)$. Then γ satisfies

$$\gamma > 0, \quad w_{\bar{\alpha}}(\gamma) < 0, \quad \gamma \notin \Phi_J.$$

However if $\gamma > 0$ and $\gamma \in \Phi_J$ then $w_{\bar{\alpha}}(\gamma) > 0$ since $w_{\bar{\alpha}}(\Delta_J) = \Delta_J$. Thus $\gamma > 0$ and $w_{\bar{\alpha}}(\gamma) < 0$ imply $\gamma \notin \Phi_J$. Thus the root subgroups X_γ are just those in $U_{w_{\bar{\alpha}}}$. Since ${}^{n_{\bar{\alpha}}}X_\gamma = X_{-\beta}$ we obtain

$$|U_{w_{\bar{\alpha}}(J)} \cap U_{w_{\bar{\alpha}}^{-1}}| = |U_{w_{\bar{\alpha}}}|.$$

Hence

$$\xi = \frac{1}{|U_{w_{\bar{\alpha}}}|} = \frac{1}{\operatorname{ind} w_{\bar{\alpha}}}$$

as required.

Proposition 10.5.4. (i) *η is given by the equation*

$$\eta 1_M = \frac{1}{\operatorname{ind} w_{\bar{\alpha}}} \tilde{\rho}(n_{\bar{\alpha}})^{-1} \sum_{\substack{u, u' \in U_{w_{\bar{\alpha}}} \\ n_{\bar{\alpha}}^{-1}u'n_{\bar{\alpha}}un_{\bar{\alpha}} \in P_J}} \rho(n_{\bar{\alpha}}^{-1}u'n_{\bar{\alpha}}un_{\bar{\alpha}}).$$

(ii) *$\operatorname{ind} w_{\bar{\alpha}} \cdot \eta$ is an algebraic integer.*

Proof. (i) We determine η by taking $f = f_m$ and $g = n_{\bar{\alpha}}$. We have

$$\theta_{n_{\bar{\alpha}}^{-1}}\theta_{n_{\bar{\alpha}}}f_m(n_{\bar{\alpha}}) = \frac{1}{|U_J \cap U_{w_{\bar{\alpha}}}|} \frac{1}{|U_{w_{\bar{\alpha}}(J)} \cap U_{w_{\bar{\alpha}}^{-1}}|} \sum_{u \in U_J \cap U_{w_{\bar{\alpha}}}} \sum_{u' \in U_{w_{\bar{\alpha}}(J)} \cap U_{w_{\bar{\alpha}}^{-1}}} f_m(n_{\bar{\alpha}}^{-1}u'n_{\bar{\alpha}}un_{\bar{\alpha}})$$

$$= \frac{1}{|U_J \cap U_{w_{\bar{\alpha}}}|^2} \sum_{u, u' \in U_J \cap U_{w_{\bar{\alpha}}}} f_m(n_{\bar{\alpha}}^{-1}u'n_{\bar{\alpha}}un_{\bar{\alpha}})$$

since $w_{\bar{\alpha}}(J) = J$ and $w_{\bar{\alpha}}^2 = 1$. This expression is also equal to $\xi f_m(n_{\bar{\alpha}}) + \eta B_{w_{\bar{\alpha}}} f_m(n_{\bar{\alpha}})$

$$= \xi f_m(n_{\bar{\alpha}}) + \eta \tilde{\rho}(n_{\bar{\alpha}}) \frac{1}{|U_{w_{\bar{\alpha}}(J)} \cap U_{w_{\bar{\alpha}}^{-1}}|} \sum_{u \in U_{w_{\bar{\alpha}}(J)} \cap U_{w_{\bar{\alpha}}^{-1}}} f_m(n_{\bar{\alpha}}^{-1}un_{\bar{\alpha}})$$

$$= \eta \tilde{\rho}(n_{\bar{\alpha}}) \frac{1}{|U_J \cap U_{w_{\bar{\alpha}}}|} \sum_{u \in U_J \cap U_{w_{\bar{\alpha}}}} f_m(n_{\bar{\alpha}}^{-1}un_{\bar{\alpha}})$$

since $f_m(n_{\bar{\alpha}}) = 0$ as $n_{\bar{\alpha}} \notin P_J$.

Suppose $n_{\bar{\alpha}}^{-1}un_{\bar{\alpha}} \in P_J$. Then $u \in U_J \cap U_{w_{\bar{\alpha}}} \cap n_{\bar{\alpha}}P_Jn_{\bar{\alpha}}^{-1} = 1$, as in 10.5.3. Thus the above sum is

$$\eta \tilde{\rho}(n_{\bar{\alpha}}) \frac{1}{|U_J \cap U_{w_{\bar{\alpha}}}|} m.$$

It follows that

$$\eta \tilde{\rho}(n_{\bar{\alpha}})m = \frac{1}{|U_J \cap U_{w_{\bar{\alpha}}}|} \sum_{u, u' \in U_J \cap U_{w_{\bar{\alpha}}}} f_m(n_{\bar{\alpha}}^{-1}u'n_{\bar{\alpha}}un_{\bar{\alpha}}).$$

Now $|U_J \cap U_{w_{\bar{\alpha}}}| = |U_{w_{\bar{\alpha}}(J)} \cap U_{w_{\bar{\alpha}}^{-1}}| = |U_{w_{\bar{\alpha}}}|$ as in 10.5.3. Thus we have $U_J \cap U_{w_{\bar{\alpha}}} = U_{w_{\bar{\alpha}}}$ and

$$\eta \tilde{\rho}(n_{\bar{\alpha}})m = \frac{1}{\operatorname{ind} w_{\bar{\alpha}}} \sum_{\substack{u, u' \in U_{w_{\bar{\alpha}}} \\ n_{\bar{\alpha}}^{-1}u'n_{\bar{\alpha}}un_{\bar{\alpha}} \in P_J}} \rho(n_{\bar{\alpha}}^{-1}u'n_{\bar{\alpha}}un_{\bar{\alpha}})m.$$

Hence

$$\eta m = \frac{1}{\operatorname{ind} w_{\bar{\alpha}}} \tilde{\rho}(n_{\bar{\alpha}})^{-1} \sum_{\substack{u, u' \in U_{w_{\bar{\alpha}}} \\ n_{\bar{\alpha}}^{-1} u' n_{\bar{\alpha}} u n_{\bar{\alpha}} \in P_J}} \rho(n_{\bar{\alpha}}^{-1} u' n_{\bar{\alpha}} u n_{\bar{\alpha}}) m.$$

Since this holds for all $m \in M$ we obtain the required result.

(ii) The projective representation $\tilde{\rho}$ of $K_{J,\phi}$ is equivalent to a projective representation whose matrix entries are all algebraic integers (cf. Curtis and Reiner [1], 53.7 and 29.16). Since

$$\tilde{\rho}(n_{\bar{\alpha}})^{-1} = \tilde{\rho}((\dot{w}_{\bar{\alpha}}^{-1}))\rho(h^{-1}) \qquad \text{where } h = \dot{w}_{\bar{\alpha}}(\dot{w}_{\bar{\alpha}}^{-1})$$

we see that the expression giving $\operatorname{ind} w_{\bar{\alpha}} \cdot \eta 1_M$ in (i) is a matrix whose coefficients are algebraic integers. Thus $\operatorname{ind} w_{\bar{\alpha}} \cdot \eta$ is an algebraic integer. ■

The formula we have obtained for η is somewhat complicated! We can, however, obtain another interpretation of the number η which has a more intuitive meaning. We have seen that the module $\mathfrak{F}(J, \rho)_K$ affords the representation $\rho_{P_J \cap L_K}{}^{L_K}$ of L_K and that

$$\dim \operatorname{End}_{L_K} \mathfrak{F}(J, \rho)_K = |W_K{}^{J,\phi}| = 2$$

since $W_K{}^{J,\phi} = \{1, w_{\bar{\alpha}}\}$.

We shall now change our notation, for this section only, for the sake of simplification. Let G be a finite group satisfying the conditions (i), (ii), (iii) of the introduction to this chapter. Let P_J be a standard maximal parabolic subgroup of G. Suppose ρ is an irreducible cuspidal representation of L_J such that $\dim \operatorname{End}(\rho_{P_J}{}^G) = 2$. Let ϕ be the character of ρ. Then $(\phi_{P_J}{}^G, \phi_{P_J}{}^G) = 2$. Thus $\phi_{P_J}{}^G = \chi + \chi'$ where χ, χ' are distinct irreducible characters of G.

Since $\mathfrak{F}(J, \rho)$ is a module affording the induced representation $\rho_{P_J}{}^G$ there is a projection map

$$p_\chi : \mathfrak{F}(J, \rho) \to \mathfrak{F}(J, \rho)$$

whose image is the submodule of $\mathfrak{F}(J, \rho)$ affording the character χ. p_χ is given by

$$p_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1}) g$$

by Curtis and Reiner [1], p. 236. p_χ lies in $\operatorname{End}_G \mathfrak{F}(J, \rho)$. Now $|W^{J,\phi}| = 2$ and $W^{J,\phi} = \{1, w_{\bar{\alpha}}\}$ where $\Delta = \Delta_J \cup \{\alpha\}$. The elements $1, B_{w_{\bar{\alpha}}}$ for a basis for $\operatorname{End}_G \mathfrak{F}(J, \rho)$ and so we have

$$p_\chi = \lambda 1 + \mu B_{w_{\bar{\alpha}}} \qquad \text{for some } \lambda, \mu \in \mathbb{C}.$$

We calculate λ by applying p_χ to $f_m \in \mathfrak{F}(J, \rho)$ and evaluating at 1. We have

$$(p_\chi f_m)(1) = \lambda f_m(1) + \mu B_{w_{\bar{\alpha}}} f_m(1).$$

This gives

$$\frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})(g f_m)(1) = \lambda m + \frac{\mu}{\operatorname{ind} w_{\bar{\alpha}}} \tilde{\rho}(n_{\bar{\alpha}}) \sum_{u \in U_{w_{\bar{\alpha}}}} f_m(n_{\bar{\alpha}}^{-1} u)$$

since $U_{w_{\dot{x}}(J)} = U_J = U_{w_{\dot{x}}}$ and $|U_{w_{\dot{x}}}| = \text{ind } w_{\dot{x}}$. However $n_{\dot{x}}^{-1}u \notin P_J$ since $u \in P_J$ and $n_{\dot{x}} \notin P_J$. Thus $f_m(n_{\dot{x}}^{-1}u) = 0$ and we have

$$\frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1}) f_m(g) = \lambda m$$

which gives

$$\frac{\chi(1)}{|G|} \sum_{g \in P_J} \chi(g^{-1}) \rho(g) m = \lambda m.$$

Thus

$$\frac{\chi(1)}{|G|} \sum_{g \in P_J} \chi(g^{-1}) g$$

acts on M as $\lambda 1$.

Taking traces we obtain

$$\frac{\chi(1)}{|G|} \sum_{g \in P_J} \chi(g^{-1}) \phi(g) = \lambda \dim M.$$

Hence

$$\begin{aligned} \lambda &= \frac{1}{\dim M} \frac{\chi(1)}{|G|} |P_J| (\phi, \chi)_{P_J} \\ &= \frac{1}{\dim M} \frac{\chi(1)}{|G|} |P_J| (\phi_{P_J}{}^G, \chi)_G \qquad \text{by Frobenius reciprocity} \\ &= \frac{1}{\dim M} \frac{\chi(1)}{|G|} |P_J|. \end{aligned}$$

However we also have

$$\dim \mathfrak{F}(J, \rho) = \dim M . |G : P_J| = \chi(1) + \chi'(1).$$

Thus

$$\lambda = \frac{\chi(1)}{\chi(1) + \chi'(1)}.$$

Now $B_{w_{\dot{x}}}$ satisfies the quadratic relation $B_{w_{\dot{x}}}{}^2 = \xi 1 + \eta B_{w_{\dot{x}}}$ where $\xi = 1/\text{ind } w_{\dot{x}}$ and we wish to determine η. Let v_1, v_2 be the roots of the equation $x^2 = \xi + \eta x$. These will be the eigenvalues of $B_{w_{\dot{x}}}$ on $\text{End}_G \mathfrak{F}(J, \rho)$. So the eigenvalues of $p_\chi = \lambda 1 + \mu B_{w_{\dot{x}}}$ on $\text{End}_G \mathfrak{F}(J, \rho)$ are $\lambda + \mu v_1, \lambda + \mu v_2$. But p_χ is simply a projection onto a 1-dimensional subspace of the 2-dimensional space $\text{End}_G \mathfrak{F}(J, \rho)$, so its eigenvalues are 0, 1. Thus we have (with suitable numbering)

$$\lambda + \mu v_1 = 1 \qquad \lambda + \mu v_2 = 0.$$

Thus $\mu \neq 0$ and

$$v_1 = \frac{1 - \lambda}{\mu} \qquad v_2 = \frac{-\lambda}{\mu}.$$

In particular we have

$$\frac{\nu_1}{\nu_2} = \frac{\lambda - 1}{\lambda} = 1 - \frac{1}{\lambda} = -\frac{\chi'(1)}{\chi(1)}.$$

Thus the ratio of the eigenvalues of $B_{w_{\bar{\alpha}}}$ on $\mathrm{End}_G \mathfrak{F}(J, \rho)$ is the negative of the ratio of the degrees of the irreducible components of $\rho_{P_J}{}^G$.

We wish to show next that this ratio is a power of p. We have

$$\nu_i^2 = \xi + \eta\nu_i \qquad \text{where } \xi = \frac{1}{\text{ind } w_{\bar{\alpha}}}.$$

Thus

$$(\nu_i \text{ ind } w_{\bar{\alpha}})^2 = \text{ind } w_{\bar{\alpha}} + (\nu_i \text{ ind } w_{\bar{\alpha}}).\eta \text{ ind } w_{\bar{\alpha}}.$$

So ν_i ind $w_{\bar{\alpha}}$ satisfies the equation

$$x^2 = \text{ind } w_{\bar{\alpha}} + (\text{ind } w_{\bar{\alpha}}.\eta)x.$$

Now ind $w_{\bar{\alpha}}$ is a positive integer and ind $w_{\bar{\alpha}}.\eta$ is an algebraic integer, by 10.5.4. Thus ν_i ind $w_{\bar{\alpha}}$ must also be an algebraic integer for $i = 1, 2$. Now $\nu_1\nu_2 = -\xi$. Thus we have

$$\frac{\nu_1}{\nu_2} \text{ ind } w_{\bar{\alpha}} = \frac{-\nu_1{}^2}{\xi} \text{ ind } w_{\bar{\alpha}} = -(\nu_1 \text{ ind } w_{\bar{\alpha}})^2.$$

Similarly

$$\frac{\nu_2}{\nu_1} \text{ ind } w_{\bar{\alpha}} = -(\nu_2 \text{ ind } w_{\bar{\alpha}})^2.$$

Thus

$$\frac{\nu_1}{\nu_2} \text{ ind } w_{\bar{\alpha}} \quad \text{and} \quad \frac{\nu_2}{\nu_1} \text{ ind } w_{\bar{\alpha}}$$

are algebraic integers. They are also rational numbers, since ν_1/ν_2 is rational and ind $w_{\bar{\alpha}}$ is an integer. Thus they are rational integers.

Let $\nu_1/\nu_2 = a/b$ where $a, b \in \mathbb{Z}$ satisfy $(a, b) = 1$. Then we have (a/b) ind $w_{\bar{\alpha}} \in \mathbb{Z}$ and (b/a) ind $w_{\bar{\alpha}} \in \mathbb{Z}$. Since $(a, b) = 1$ we see that a and b both divide ind $w_{\bar{\alpha}}$. But ind $w_{\bar{\alpha}}$ is a power of p, where p is the characteristic of the split BN-pair. Thus a and b are both, to within sign, powers of p. Since $(a, b) = 1$, one of a, b must therefore be ± 1.

Suppose that the irreducible characters χ, χ' satisfy $\chi'(1) \geqslant \chi(1)$. Then

$$\frac{\chi'(1)}{\chi(1)} = -\frac{a}{b}$$

must be a power of p. Let $\chi'(1)/\chi(1) = p^c$ where $c \geqslant 0$. Then $\nu_1/\nu_2 = -p^c$ and

$v_1 v_2 = -1/\text{ind } w_{\bar{\alpha}}$. Also $\eta = v_1 + v_2$, so

$$\eta^2 = v_1{}^2 + v_2{}^2 + 2v_1 v_2 = \frac{p^c}{\text{ind } w_{\bar{\alpha}}} + \frac{1}{p^c \text{ ind } w_{\bar{\alpha}}} - \frac{2}{\text{ind } w_{\bar{\alpha}}}$$

$$= \frac{(p^c - 1)^2}{p^c \text{ ind } w_{\bar{\alpha}}}.$$

Thus

$$\eta = \pm \frac{(p^c - 1)}{(p^c \text{ ind } w_{\bar{\alpha}})^{\frac{1}{2}}}.$$

We now revert to our former notation, and have proved the following result.

Proposition 10.5.5. *Suppose $\alpha \in \Delta - \Delta_J$ and $w_{\bar{\alpha}} \in W^{J,\phi}$. Let $K = J \cup \{\alpha\}$. Then the induced character $\phi_{P_J \cap L_K}{}^{L_K}$ splits into exactly two irreducible components χ, χ' whose degrees are related by $\chi'(1)/\chi(1) = p^c$ for some integer $c \geqslant 0$. Moreover the number η of* 10.5.2 *is given up to sign by*

$$\eta = \pm \frac{(p^c - 1)}{(p^c \text{ ind } w_{\bar{\alpha}})^{\frac{1}{2}}}.$$

In particular $\eta = 0$ if and only if the two irreducible components χ, χ' of $\phi_{P_J \cap L_K}{}^{L_K}$ have the same degree.

10.6 A DECOMPOSITION OF $W^{J,\phi}$

We shall now obtain a decomposition of $W^{J,\phi}$ as a semi-direct product which will be useful in the subsequent work.

Let $\alpha \in \Omega$. Then $\Delta_J \cup \{\alpha\} \subseteq w(\Delta)$ for some $w \in W$ and we have $w_{\bar{\alpha}}{}^2 = 1$. Let L_K be defined by

$$L_K = \langle H, X_\beta;\ \beta \text{ lies in the root subsystem of } \Phi \text{ generated by } \Delta_J \cup \{\alpha\}\rangle.$$

Then L_K is a standard Levi subgroup of G with respect to some simple system of roots (not necessarily Δ) and L_J is a Levi subgroup of a maximal parabolic subgroup of L_K. The irreducible representation ρ can be lifted from L_J to such a parabolic subgroup $P_J \cap L_K$ and then induced to L_K to give $\rho_{P_J \cap L_K}{}^{L_K}$. This induced representation has two inequivalent irreducible components. Let the ratio of their degrees be p_α. Then we have seen in section 10.5 that p_α is a non-negative integral power of p (possibly $p^0 = 1$).

Lemma 10.6.1. *Let $w \in W^{J,\phi}$ and let $\alpha \in \Omega$ be such that $w_{\bar{\alpha}} \in W^{J,\phi}$. Then $w(\alpha) \in \Omega$ also and we have $w_{\overline{w(\alpha)}} \in W^{J,\phi}$.*

Proof. We know that $\Delta_J \cup \{\alpha\}$ lies in some simple system of roots in Φ. Since $w(\Delta_J) = \Delta_J$ we have $w(\Delta_J \cup \{\alpha\}) = \Delta_J \cup \{w(\alpha)\}$. Thus $\Delta_J \cup \{w(\alpha)\}$ lies in some simple system of roots. Moreover we have

$$ww_{\bar{\alpha}}w^{-1} = w(w_0)_{J\cup\{\alpha\}}(w_0)_J w^{-1} = (w_0)_{J\cup\{w(\alpha)\}}(w_0)_J = w_{\overline{w(\alpha)}}.$$

This shows that $w_{\overline{w(\alpha)}}^{\,2} = 1$ and so $w(\alpha) \in \Omega$. It also shows that $w_{\overline{w(\alpha)}} \in W^{J,\phi}$. ■

It follows from 10.6.1 that if p_α is defined and $w \in W^{J,\phi}$ then $p_{w(\alpha)}$ is also defined.

Lemma 10.6.2. *If $\alpha \in \Omega$ and $w \in W^{J,\phi}$ then $p_{w(\alpha)} = p_\alpha$.*

Proof. The operation of conjugation by w fixes L_J, transforms $L_{K(\alpha)}$ into $L_{K(w(\alpha))}$ and transforms the induced character $\phi_{P_J\cap L_{K(\alpha)}}{}^{L_{K(\alpha)}}$ into $\phi_{P_J\cap L_{K(w(\alpha))}}{}^{L_{K(w(\alpha))}}$. Both induced characters have two inequivalent irreducible components and the ratio of the degrees of these components must be the same in both cases. Thus $p_{w(\alpha)} = p_\alpha$. ■

We now define a subset Γ of Ω by

$$\Gamma = \{\alpha \in \Omega;\ w_{\bar{\alpha}} \in W^{J,\phi},\ p_\alpha \neq 1\}.$$

Thus a root $\alpha \in \Phi$ lies in Γ if and only if the following conditions are all satisfied:

$\alpha \notin \Delta_J$
$\Delta_J \cup \{\alpha\}$ lies in some simple root system in Φ
$w_{\bar{\alpha}}^{\,2} = 1$ (This is equivalent to $w_{\bar{\alpha}}(\Delta_J) = \Delta_J$)
$w_{\bar{\alpha}} \in W^{J,\phi}$
$p_\alpha \neq 1$.

Lemmas 10.6.1 and 10.6.2 show that $w(\Gamma) = \Gamma$ for all $w \in W^{J,\phi}$. Let $\Gamma^+ = \Gamma \cap \Phi^+$. Let $R_{J,\phi}$ be the subgroup of W generated by the elements $w_{\bar{\alpha}}$ for all $\alpha \in \Gamma$. Let

$$C_{J,\phi} = \{w \in W^{J,\phi};\ w(\Gamma^+) = \Gamma^+\}.$$

Proposition 10.6.3. (i) $W^{J,\phi} = R_{J,\phi}C_{J,\phi}$.
(ii) $R_{J,\phi} \cap C_{J,\phi} = 1$.
(iii) *$R_{J,\phi}$ is normal in $W^{J,\phi}$.*
(iv) *$R_{J,\phi}$ is a reflection subgroup of $W^{J,\phi}$ whose root system in $\bar{V} = V/V_J$ is*

$$\bar{\Gamma} = \{\bar{\alpha};\ \alpha \in \Gamma\}.$$

Proof. We know from 10.4.2 that $\bar{\Omega}$ determines a root system in $\bar{V}$ or a subspace of $\bar{V}$. We show that $\bar{\Gamma}$ gives a subsystem of this root system. It is sufficient to show that if $\alpha, \beta \in \Gamma$ then $w_{\bar{\alpha}}(\bar{\beta}) \in \bar{\Gamma}$. Since $\beta \in \Gamma$ and $w_{\bar{\alpha}} \in W^{J,\phi}$ we have $w_{\bar{\alpha}}(\beta) \in \Gamma$. This maps to $w_{\bar{\alpha}}(\bar{\beta}) \in \bar{\Gamma}$. Thus $\bar{\Gamma}$ gives a subsystem of $\bar{\Omega}$.

The Weyl group of the root system given by $\bar{\Omega}$ is R_J, by 10.4.2. The Weyl group of the subsystem $\bar{\Gamma}$ is the subgroup of R_J generated by the elements $w_{\bar{\alpha}}$ for all $\alpha \in \Gamma$, and this is $R_{J,\phi}$. Thus $R_{J,\phi}$ is a reflection subgroup of $W^{J,\phi}$ with root system $\bar{\Gamma}$.

Let $\alpha \in \Gamma$ and $w \in W^{J,\phi}$. Then

$$ww_{\bar{\alpha}}w^{-1} = w_{\overline{w(\alpha)}} \qquad \text{by 10.6.1}$$

and $w(\alpha) \in \Gamma$. This shows that $R_{J,\phi}$ is normal in $W^{J,\phi}$.

Let $w \in R_{J,\phi} \cap C_{J,\phi}$. Then w lies in the Weyl group of the root system $\bar{\Gamma}$ but transforms every positive root to a positive root. Thus $w = 1$. Hence $R_{J,\phi} \cap C_{J,\phi} = 1$.

Now let w be any element of $W^{J,\phi}$. Since $w(\Gamma) = \Gamma$ we have $w(\bar{\Gamma}) = \bar{\Gamma}$. If $w \notin C_{J,\phi}$ we can find a positive root in $\bar{\Gamma}$ made negative by w. So we can find a simple root $\bar{\alpha} \in \bar{\Gamma}$ which w makes negative. But then $ww_{\bar{\alpha}}$ makes fewer positive roots in $\bar{\Gamma}$ negative than w. Continuing in this way we eventually find an element $c \in wR_{J,\phi}$ such that $c(\overline{\Gamma^+}) = \bar{\Gamma}^+$. Then $c \in C_{J,\phi}$ and so $w \in C_{J,\phi}R_{J,\phi}$. Hence $W^{J,\phi} = R_{J,\phi}C_{J,\phi}$ as required.

10.7 FURTHER MULTIPLICATION FORMULAE

We wish to find an expression for $B_w B_{w'}$ as a linear combination of elements $B_{w''}$ for arbitrary $w, w' \in W^{J,\phi}$. The multiplication formulae we have proved so far in 10.2.5 and 10.5.2 are not sufficient to give this. In proving the additional relations which are needed we must take into account the fact that there may be subsets of Δ which are equivalent to Δ_J under W without being equal to Δ_J. Let

$$\mathbf{J} = \{ J_i \subseteq I; \Delta_{J_i} = w(\Delta_J) \text{ for some } w \in W \}.$$

$\mathbf{J}$ is called the class of subsets of I which are associated to J. For each $J_i \in \mathbf{J}$ and each root $\alpha \in \Phi$ with $\alpha \notin \Delta_{J_i}$ such that $\Delta_{J_i} \cup \{\alpha\}$ lies in some simple system in Φ we define w_{α, J_i} by

$$w_{\bar{\alpha}, J_i} = (w_0)_{J_i \cup \{\alpha\}}(w_0)_{J_i}.$$

Thus $w_{\alpha, J} = w_\alpha$ as defined earlier. We note that if $\alpha \in \Delta$ $w_{\bar{\alpha}, J_i}$ transforms J_i into an element of $\mathbf{J}$.

Proposition 10.7.1. *Let $J_1, J_2 \in \mathbf{J}$ and suppose $w \in W$ satisfies $w(\Delta_{J_1}) = \Delta_{J_2}$.*

(i) *Let $\alpha \in \Delta$, $\alpha \notin \Delta_{J_2}$. Then*

$$l(w_{\alpha, J_2} w) = \begin{cases} l(w) + l(w_{\alpha, J_2}) & \text{if } w^{-1}(\alpha) > 0 \\ l(w) - l(w_{\alpha, J_2}) & \text{if } w^{-1}(\alpha) < 0. \end{cases}$$

(ii) *Let $\alpha \in \Delta$, $\alpha \notin \Delta_{J_1}$. Let Δ_{J_0} be the subset of $\Delta_{J_1} \cup \{\alpha\}$ given by $(w_0)_{J_1 \cup \{\alpha\}}(\Delta_{J_1}) = -\Delta_{J_0}$. Let $\beta \in \Delta$ satisfy $\Delta_{J_1} \cup \{\alpha\} = \Delta_{J_0} \cup \{\beta\}$. Then*

$$l(ww_{\beta, J_0}) = \begin{cases} l(w) + l(w_{\beta, J_0}) & \text{if } w(\alpha) > 0 \\ l(w) - l(w_{\beta, J_0}) & \text{if } w(\alpha) < 0. \end{cases}$$

(*Note that $w_{\beta, J_0} = w_{\alpha, J_1}^{-1}$.*)

Proof. (i) Consider the positive roots in Φ made negative by $w_{\bar{\alpha},J_2}$. These are the roots in $\Phi_{J_2\cup\{\alpha\}}$ which involve α. The roots mapped to these by w are those which are positive combinations of $w^{-1}(\alpha)$ with roots in Δ_{J_1}. If $w^{-1}(\alpha)>0$ all these roots are positive. It follows that

$$l(w_{\bar{\alpha},J_2}w)=l(w_{\bar{\alpha},J_2})+l(w).$$

If $w^{-1}(\alpha)<0$ all these roots are negative. In this case we have

$$l(w_{\bar{\alpha},J_2}w)=l(w)-l(w_{\bar{\alpha},J_2}).$$

(ii) This is obtained from (i) by taking inverses. We have

$$l(w^{-1}w_{\bar{\alpha},J_2}{}^{-1})=\begin{cases}l(w^{-1})+l(w_{\bar{\alpha},J_2}{}^{-1}) & \textit{if } w^{-1}(\alpha)>0\\ l(w^{-1})-l(w_{\bar{\alpha},J_2}{}^{-1}) & \textit{if } w^{-1}(\alpha)<0.\end{cases}$$

We now replace w by w^{-1} and interchange J_1, J_2. This gives

$$l(ww_{\bar{\alpha},J_1}{}^{-1})=\begin{cases}l(w)+l(w_{\bar{\alpha},J_1}{}^{-1}) & \text{if } w(\alpha)>0\\ l(w)-l(w_{\bar{\alpha},J_1}{}^{-1}) & \text{if } w(\alpha)<0.\end{cases}$$

Finally we note that $w_{\bar{\alpha},J_1}{}^{-1}=w_{\bar{\beta},J_0}$ and so the result follows. ∎

We show next that any element $w\in W$ which has the property that it transforms some element of $\mathbf{J}$ into an element of $\mathbf{J}$ can be expressed as a product of elements of the form $w_{\bar{\alpha},J_i}$ in such a way that the lengths are additive.

Proposition 10.7.2. *Let $J_1, J'\in\mathbf{J}$ and let $w\in W$ satisfy $w(\Delta_{J_1})=\Delta_{J'}$. Then w can be expressed in the form*

$$w=w_{\bar{\alpha}_k,J_k}\dots w_{\bar{\alpha}_1,J_1}$$

where $l(w)=l(w_{\bar{\alpha}_k,J_k})+\dots+l(w_{\bar{\alpha}_1,J_1})$ with $\alpha_1,\dots,\alpha_k\in\Delta$, $J_1,\dots,J_k\in\mathbf{J}$, such that $w_{\bar{\alpha}_i,J_i}(\Delta_{J_i})=\Delta_{J_{i+1}}$ and $J_{k+1}=J'$.

Proof. If $w=1$ the result is trivial. So suppose $w\neq 1$. Then there exists $\alpha_1\in\Delta$ with $w(\alpha_1)<0$. By 10.7.1 we have

$$l(ww_{\bar{\alpha}_1,J_1}{}^{-1})=l(w)-l(w_{\bar{\alpha}_1,J_1}{}^{-1}).$$

Thus we have a factorization

$$w=(ww_{\bar{\alpha}_1,J_1}{}^{-1}).w_{\bar{\alpha}_1,J_1}$$

in which the lengths are additive. Let $w_{\bar{\alpha}_1,J_1}(\Delta_{J_1})=\Delta_{J_2}$. Then $J_2\in\mathbf{J}$ and $ww_{\bar{\alpha}_1,J_1}{}^{-1}(\Delta_{J_2})=\Delta_{J'}$. Since $l(ww_{\bar{\alpha}_1,J_1}{}^{-1})<l(w)$ we may apply induction on $l(w)$ to obtain a factorization

$$ww_{\bar{\alpha}_1,J_1}{}^{-1}=w_{\bar{\alpha}_k,J_k}\dots w_{\bar{\alpha}_2,J_2}$$

of the required type. The result follows.

Proposition 10.7.3. *Suppose $w \in W$ satisfies $w(\Delta_J) \subseteq \Delta$. Let $\alpha \in \Delta$, $\alpha \notin w(\Delta_J)$. Write $n_{\bar{\alpha}} = (w_{\bar{\alpha}, w(J)})$. Suppose that either $w^{-1}(\alpha) > 0$ or that $w^{-1}(\alpha) < 0$ and $w^{-1}(\alpha) \notin \Gamma$. Then*

$$\theta_{n_{\bar{\alpha}}}\theta_{\dot{w}} = \left(\frac{\operatorname{ind} w_{\bar{\alpha}, w(J)}w}{\operatorname{ind} w_{\bar{\alpha}, w(J)} \operatorname{ind} w}\right)^{\frac{1}{2}} \theta_{n_{\bar{\alpha}}\dot{w}}$$

on $\mathfrak{F}(J, \rho)$.

Proof. We recall that the maps $\theta_{n_{\bar{\alpha}}}, \theta_{\dot{w}}$ operate as follows:

$$\mathfrak{F}(J, \rho) \xrightarrow{\theta_{\dot{w}}} \mathfrak{F}(wJ, {}^{\dot{w}}\rho) \xrightarrow{\theta_{n_{\bar{\alpha}}}} \mathfrak{F}(w_{\bar{\alpha}, wJ}wJ, {}^{n_{\bar{\alpha}}\dot{w}}\rho).$$

First suppose that $w^{-1}(\alpha) > 0$. Then by 10.7.1 we have $l(w_{\bar{\alpha}, w(J)}w) = l(w_{\bar{\alpha}, w(J)}) + l(w)$. It follows that

$$\theta_{n_{\bar{\alpha}}}\theta_{\dot{w}} = \theta_{n_{\bar{\alpha}}\dot{w}} \qquad \text{by 10.2.2.}$$

Moreover

$$\operatorname{ind}(w_{\bar{\alpha}, w(J)}w) = \operatorname{ind}(w_{\bar{\alpha}, w(J)}) \operatorname{ind} w.$$

Thus the required result follows.

Now suppose that $w^{-1}(\alpha) < 0$ and $w^{-1}(\alpha) \notin \Gamma$. Then 10.7.1 shows that

$$l(w_{\bar{\alpha}, w(J)}w) = l(w) - l(w_{\bar{\alpha}, w(J)}).$$

Let $w_1 = w_{\bar{\alpha}, w(J)}w$. Then we have

$$l(w) = l(w_{\bar{\alpha}, w(J)}{}^{-1}) + l(w_1).$$

Consider the subset of $w(\Delta_J) \cup \{\alpha\}$ opposed to $w(\Delta_J)$. This is $w_{\bar{\alpha}, w(J)}w(\Delta_J) = w_1(\Delta_J)$. Let $w(\Delta_J) \cup \{\alpha\} = w_1(\Delta_J) \cup \{\beta\}$. Then $w_{\bar{\alpha}, w(J)}{}^{-1} = w_{\beta, w_1(J)}$. Let $n_\beta = (w_{\beta, \dot{w}_1(J)})$. Then $w = w_{\beta, w_1(J)}w_1$ with $l(w) = l(w_{\beta, w_1(J)}) + l(w_1)$. Thus $\dot{w} = n_\beta \dot{w}_1$. Also we have

$$\theta_{\dot{w}} = \theta_{n_\beta}\theta_{\dot{w}_1} \qquad \text{by 10.2.2.}$$

It follows that

$$\theta_{n_{\bar{\alpha}}}\theta_{\dot{w}} = \theta_{n_{\bar{\alpha}}}\theta_{n_\beta}\theta_{\dot{w}_1}.$$

We now have maps acting as follows:

$$\mathfrak{F}(J, \rho) \xrightarrow{\theta_{\dot{w}_1}} \mathfrak{F}(w_1 J, {}^{\dot{w}_1}\rho) \xrightarrow{\theta_{n_\beta}} \mathfrak{F}(w_{\beta, w_1(J)}w_1 J, {}^{n_\beta\dot{w}_1}\rho) \xrightarrow{\theta_{n_{\bar{\alpha}}}} \mathfrak{F}(w_1 J, {}^{n_{\bar{\alpha}}n_\beta\dot{w}_1}\rho).$$

We now apply 10.5.1 to $\theta_{n_\beta^{-1}}\theta_{n_\beta}$ acting on $\mathfrak{F}(w_1 J, {}^{\dot{w}_1}\rho)$. This gives

$$\theta_{n_\beta^{-1}}\theta_{n_\beta} = \begin{cases} \xi 1 + \eta B_{w_{\beta, w_1(J)}} & \text{if } w_{\beta, w_1(J)} \in W^{w_1 J, w_1\phi} \\ \xi 1 & \text{otherwise} \end{cases}$$

where $\xi = 1/\operatorname{ind} w_{\beta, w_1(J)}$. We shall show that under the given assumptions $\eta = 0$ even in the first case. So suppose $w_{\beta, w_1(J)} \in W^{w_1 J, w_1\phi}$. Then

$w_{\bar{\alpha}, w(J)} = w_{\beta, w_1(J)}^{-1} \in W^{w_1 J, w_1 \phi}$. Hence $w_1^{-1} w_{\bar{\alpha}, w(J)} w_1 \in W^{J, \phi}$. This gives $w^{-1} w_{\bar{\alpha}, w(J)} w \in W^{J, \phi}$ thus $w_{\overline{w^{-1}(\alpha)}} \in W^{J, \phi}$. Now $w^{-1}(\alpha) \notin \Delta_J$ and $\Delta_J \cup \{w^{-1}(\alpha)\}$ lies in a simple system in Φ. Since $w_{\overline{w^{-1}(\alpha)}}$ lies in $W^{J, \phi}$ the only reason for $w^{-1}(\alpha)$ to fail to lie in Γ can be that $p_{w^{-1}(\alpha)} = 1$. Thus the extension $\Delta_J \cup \{w^{-1}(\alpha)\}$ of Δ_J gives rise to the parameter $p_{w^{-1}(\alpha)} = 1$. So the extension $w(\Delta_J) \cup \{\alpha\}$ of $w(\Delta_J)$ also gives rise to a parameter equal to 1. By conjugation it follows that the extension $w_1(\Delta_J) \cup \{\beta\}$ of $w_1(\Delta_J)$ gives rise to a parameter equal to 1. It now follows from 10.5.5 that $\eta = 0$. Thus we have

$$\theta_{n_\beta^{-1}} \theta_{n_\beta} = \xi 1 \quad \text{where} \quad \xi = \frac{1}{\operatorname{ind} w_{\beta, w_1(J)}}.$$

We must next compare $\theta_{n_{\bar{\alpha}}}$ with $\theta_{n_\beta^{-1}}$ acting on $\mathfrak{F}(w_{\beta, w_1(J)}, {}^{n_\beta \dot{w}_1}\rho)$. We have $n_{\bar{\alpha}} = n_\beta^{-1}(n_\beta n_{\bar{\alpha}})$ with $n_\beta n_{\bar{\alpha}} \in H$. Thus

$$\theta_{n_{\bar{\alpha}}} = \theta_{n_\beta^{-1}(n_\beta n_{\bar{\alpha}})} = {}^{n_\beta \dot{w}_1}\rho(n_{\bar{\alpha}}^{-1} n_\beta^{-1}) \theta_{n_\beta^{-1}} = {}^{\dot{w}_1}\rho(n_\beta^{-1} n_{\bar{\alpha}}^{-1}) \theta_{n_\beta^{-1}}.$$

It follows that

$$\begin{aligned} \theta_{n_{\bar{\alpha}}} \theta_{n_\beta} &= {}^{\dot{w}_1}\rho(n_\beta^{-1} n_{\bar{\alpha}}^{-1}) \theta_{n_\beta^{-1}} \theta_{n_\beta} = \xi {}^{\dot{w}_1}\rho(n_\beta^{-1} n_{\bar{\alpha}}^{-1}) \\ &= \xi \rho(\dot{w}_1^{-1} n_\beta^{-1} n_{\bar{\alpha}}^{-1} \dot{w}_1) = \xi \rho(\dot{w}^{-1} n_{\bar{\alpha}}^{-1} \dot{w}_1). \end{aligned}$$

We thus have

$$\begin{aligned} \theta_{n_{\bar{\alpha}}} \theta_{n_\beta} \theta_{\dot{w}_1} &= \xi \rho(\dot{w}^{-1} n_{\bar{\alpha}}^{-1} \dot{w}_1) \theta_{\dot{w}_1} \\ &= \xi \rho(\dot{w}^{-1} n_{\bar{\alpha}}^{-1} \dot{w}_1) \theta_{n_{\bar{\alpha}} \dot{w} (\dot{w}^{-1} n_{\bar{\alpha}}^{-1} \dot{w}_1)}. \end{aligned}$$

Now $\dot{w}^{-1} n_{\bar{\alpha}}^{-1} \dot{w}_1 \in H$. Hence

$$\theta_{n_{\bar{\alpha}}} \theta_{n_\beta} \theta_{\dot{w}_1} = \xi \rho(\dot{w}^{-1} n_{\bar{\alpha}}^{-1} \dot{w}_1) \rho(\dot{w}_1^{-1} n_{\bar{\alpha}} \dot{w}) \theta_{n_{\bar{\alpha}} \dot{w}} = \xi \theta_{n_{\bar{\alpha}} \dot{w}}.$$

Thus $\theta_{n_{\bar{\alpha}}} \theta_{\dot{w}} = \xi \theta_{n_{\bar{\alpha}} \dot{w}}$ on $\mathfrak{F}(J, \rho)$.

Finally we have

$$\xi = \frac{1}{\operatorname{ind} w_{\beta, w_1(J)}} = \frac{1}{\operatorname{ind} w_{\bar{\alpha}, w(J)}} = \left(\frac{\operatorname{ind} w_1}{\operatorname{ind} w_{\bar{\alpha}, w(J)} \operatorname{ind} w} \right)^{\frac{1}{2}}$$

since $w_{\bar{\alpha}, w(J)} = w_{\beta, w_1(J)}^{-1}$ and $\operatorname{ind} w = \operatorname{ind} w_1 \,.\, \operatorname{ind} w_{\bar{\alpha}, w(J)}$. This gives the required result.

Corollary 10.7.4. *With the same assumptions as in* 10.7.3, *let* $n \in N$ *satisfy* $\pi(n) = w$. *Then*

$$\theta_{n_{\bar{\alpha}}} \theta_n = \left(\frac{\operatorname{ind} w_{\bar{\alpha}, w(J)} w}{\operatorname{ind} w_{\bar{\alpha}, w(J)} \operatorname{ind} w} \right)^{\frac{1}{2}} \theta_{n_{\bar{\alpha}} n}$$

on $\mathfrak{F}(J, \rho)$.

Proof. We have $n = \dot{w} h$ for some $h \in H$. Thus, given $f \in \mathfrak{F}(J, \rho)$, we have

$$\theta_{n_{\bar{\alpha}}} \theta_n f = \theta_{n_{\bar{\alpha}}} \theta_{\dot{w} h} f = \theta_{n_{\bar{\alpha}}} \rho(h^{-1}) \theta_{\dot{w}} f = \rho(h^{-1}) \theta_{n_{\bar{\alpha}}} \theta_{\dot{w}} f.$$

We also have

$$\theta_{n_\alpha n} f = \theta_{n_\alpha \dot{w} h} f = \rho(h^{-1})\theta_{n_\alpha \dot{w}} f.$$

The result then follows from 10.7.3.

Proposition 10.7.5. *Suppose $w \in W$ satisfies $w(\Delta_J) \subseteq \Delta$ and let $w' \in W^{J,\phi}$. Suppose there is no positive root in Γ^+ which is transformed into negative roots by both w and w'^{-1}. Then*

$$\theta_{\dot{w}}\theta_{\dot{w}'} = \left(\frac{\text{ind } ww'}{\text{ind } w \text{ ind } w'}\right)^{\frac{1}{2}} \theta_{\dot{w}\dot{w}'}$$

on $\mathfrak{F}(J, \rho)$.

Proof. We shall use induction on $l(w)$. If $w = 1$ the result is trivial. So suppose $l(w) > 0$. Then by 10.7.2 we have $w = w_{\alpha,J_1} w_1$ with $l(w) = l(w_{\alpha,J_1}) + l(w_1)$ and $w_1(\Delta_J) = \Delta_{J_1}$. Let $n_\alpha = (\dot{w}_{\alpha,J_1})$. Then $\theta_{\dot{w}} = \theta_{n_\alpha}\theta_{\dot{w}_1}$ by 10.2.2.

Since every positive root made negative by w_1 is also made negative by w there is no positive root in Γ^+ made negative by both w_1 and w'^{-1}. Thus we have by induction

$$\theta_{\dot{w}_1}\theta_{\dot{w}'} = \left(\frac{\text{ind } w_1 w'}{\text{ind } w_1 \text{ ind } w'}\right)^{\frac{1}{2}} \theta_{\dot{w}_1\dot{w}'}.$$

Thus

$$\theta_{\dot{w}}\theta_{\dot{w}'} = \theta_{n_\alpha}\theta_{\dot{w}_1}\theta_{\dot{w}'} = \left(\frac{\text{ind } w_1 w'}{\text{ind } w_1 \text{ ind } w'}\right)^{\frac{1}{2}} \theta_{n_\alpha}\theta_{\dot{w}_1\dot{w}'}.$$

We wish to use 10.7.4. to evaluate $\theta_{n_\alpha}\theta_{\dot{w}_1\dot{w}'}$. To see that the conditions of 10.7.4 are satisfied we must check that either $w'^{-1}w_1^{-1}(\alpha) > 0$ or $w'^{-1}w_1^{-1}(\alpha) < 0$ and $w'^{-1}w_1^{-1}(\alpha) \notin \Gamma$. So suppose $w'^{-1}w_1^{-1}(\alpha) < 0$. Since $l(w) = l(w_{\alpha,J_1}) + l(w_1)$ we see that $w_1^{-1}(\alpha) > 0$. Thus $w_1^{-1}(\alpha)$ is a positive root made negative by w'^{-1}. It is also made negative by w since $ww_1^{-1}(\alpha) = w_{\alpha,J_1}(\alpha) < 0$. It follows that $w_1^{-1}(\alpha) \notin \Gamma$. Since $w' \in W^{J,\phi}$ this implies $w'^{-1}w_1^{-1}(\alpha) \notin \Gamma$. We may therefore apply 10.7.4 and obtain

$$\theta_{n_\alpha}\theta_{\dot{w}_1\dot{w}'} = \left(\frac{\text{ind}(w_{\alpha,J_1} w_1 w')}{\text{ind } w_{\alpha,J_1} \text{ ind}(w_1 w')}\right)^{\frac{1}{2}} \theta_{n_\alpha \dot{w}_1 \dot{w}'}.$$

Hence

$$\begin{aligned}\theta_{\dot{w}}\theta_{\dot{w}'} &= \left(\frac{\text{ind } w_1 w'}{\text{ind } w_1 \text{ ind } w'}\right)^{\frac{1}{2}} \left(\frac{\text{ind}(w_{\alpha,J_1} w_1 w')}{\text{ind } w_{\alpha,J_1} \text{ ind}(w_1 w')}\right)^{\frac{1}{2}} \theta_{n_\alpha \dot{w}_1 \dot{w}'} \\ &= \left(\frac{\text{ind } ww'}{\text{ind } w \text{ ind } w'}\right)^{\frac{1}{2}} \theta_{\dot{w}\dot{w}'}\end{aligned}$$

since $w = w_{\alpha,J_1} w_1$ and $l(w) = l(w_{\alpha,J_1}) + l(w_1)$. ∎

Let $\bar{\Lambda}$ be the simple system of roots in $\bar{\Gamma}$ corresponding to the positive system $\bar{\Gamma}^+$. Let $\Lambda = \{\alpha \in \Gamma; \bar{\alpha} \in \bar{\Lambda}\}$.

Lemma 10.7.6. *Λ is the set of all $\alpha \in \Gamma^+$ such that the only positive root in Γ made negative by $w_{\bar\alpha}$ is α.*

Proof. If α has the given property then certainly $\bar\alpha \in \bar\Lambda$. Suppose conversely that $\bar\alpha \in \bar\Lambda$. Consider positive roots in Γ made negative by $w_{\bar\alpha}$. Let $\beta \in \Gamma^+$ with $w_{\bar\alpha}(\beta) < 0$. *Then* $\bar\beta > 0$ and $w_{\bar\alpha}(\bar\beta) < 0$, so $\bar\beta = \bar\alpha$. It follows that $\beta = \alpha$ by 10.4.2(i). Thus the only positive root in Γ made negative by $w_{\bar\alpha}$ is α. ■

We now proceed to the additional multiplication formulae for the B_w which we need. These will be proved in 10.7.7, 10.7.8 and 10.7.9.

Proposition 10.7.7. *Let $w \in W^{J,\phi}$ and $w' \in C_{J,\phi}$. Then*

$$B_w B_{w'} = \left(\frac{\text{ind } ww'}{\text{ind } w \text{ ind } w'}\right)^{\frac{1}{2}} \lambda(w, w') B_{ww'}$$

$$B_{w'} B_w = \left(\frac{\text{ind } w'w}{\text{ind } w' \text{ ind } w}\right)^{\frac{1}{2}} \lambda(w', w) B_{w'w}.$$

Proof.
$$\begin{aligned} B_w B_{w'} &= \tilde\rho(\dot w)\theta_{\dot w}\tilde\rho(\dot w')\theta_{\dot w'} \\ &= \tilde\rho(\dot w)\tilde\rho(\dot w')\theta_{\dot w}\theta_{\dot w'} \qquad \text{by 10.2.4.} \end{aligned}$$

Since $w'^{-1} \in C_{J,\phi}$ w'^{-1} transforms no positive root in Γ^+ into a negative root. We may therefore apply 10.7.5 and obtain

$$\begin{aligned} B_w B_{w'} &= \tilde\rho(\dot w)\tilde\rho(\dot w')\left(\frac{\text{ind } ww'}{\text{ind } w \text{ ind } w'}\right)^{\frac{1}{2}} \theta_{\dot w\dot w'} \\ &= \lambda(w, w')\tilde\rho(\dot w\dot w')\left(\frac{\text{ind } ww'}{\text{ind } w \text{ ind } w'}\right)^{\frac{1}{2}} \theta_{\dot w\dot w'} \\ &= \left(\frac{\text{ind } ww'}{\text{ind } w \text{ ind } w'}\right)^{\frac{1}{2}} \lambda(w, w')\tilde\rho(h(\dot{ww'}))\theta_{h(\dot{ww'})} \end{aligned}$$

where $\dot w\dot w' = h(\dot{ww'})$ with $h \in H$. Let $ww' = w''$. Then we have, for $f \in \mathfrak{F}(J, \rho)$ and $g \in G$,

$$\begin{aligned} (\tilde\rho(h\dot w'')\theta_{h\dot w''}f)g &= \rho(h)\tilde\rho(\dot w'')\frac{1}{|U_{w''(J)}|}\sum_{u \in U_{w''(J)}} f((\dot w'')^{-1}h^{-1}ug) \\ &= \rho(h)\tilde\rho(\dot w'')\frac{1}{|U_{w''(J)}|}\sum_{u \in U_{w''(J)}} \rho((\dot w'')^{-1}h^{-1}\dot w'')f((\dot w'')^{-1}ug) \\ &= \tilde\rho(h\dot w''(\dot w'')^{-1}h^{-1}\dot w'')\theta_{\dot w''}f(g) \\ &= (\tilde\rho(\dot w'')\theta_{\dot w''}f)g = B_{w''}f(g). \end{aligned}$$

Thus $\tilde\rho(h\dot w'')\theta_{h\dot w''} = B_{w''}$ and the required result follows.

The second formula is proved similarly.

Proposition 10.7.8. *Let $\alpha \in \Lambda$ and $w \in W^{J,\Phi}$. Then*

$$B_w B_{w_{\bar{\alpha}}} = \left(\frac{\operatorname{ind} w w_{\bar{\alpha}}}{\operatorname{ind} w \operatorname{ind} w_{\bar{\alpha}}}\right)^{\frac{1}{2}} \lambda(w, w_{\bar{\alpha}}) B_{w w_{\bar{\alpha}}}$$

if $w(\alpha) > 0$. *Also*

$$B_{w_{\bar{\alpha}}} B_w = \left(\frac{\operatorname{ind} w_{\bar{\alpha}} w}{\operatorname{ind} w_{\bar{\alpha}} \operatorname{ind} w}\right)^{\frac{1}{2}} \lambda(w_{\bar{\alpha}}, w) B_{w_{\bar{\alpha}} w}$$

if $w^{-1}(\alpha) > 0$.

Proof. $B_w B_{w_{\bar{\alpha}}} = \tilde{\rho}(\dot{w})\theta_{\dot{w}}\tilde{\rho}(n_{\bar{\alpha}})\theta_{n_{\bar{\alpha}}} = \tilde{\rho}(\dot{w})\tilde{\rho}(n_{\bar{\alpha}})\theta_{\dot{w}}\theta_{n_{\bar{\alpha}}}$ by 10.2.4. Since $\alpha \in \Lambda$ we know by 10.7.6 that α is the only root in Γ^+ made negative by $w_{\bar{\alpha}}$. Since $w(\alpha) > 0$ and $w_{\bar{\alpha}} = w_{\bar{\alpha}}^{-1}$ we may apply 10.7.5 and obtain

$$B_w B_{w_{\bar{\alpha}}} = \tilde{\rho}(\dot{w})\tilde{\rho}(n_{\bar{\alpha}})\left(\frac{\operatorname{ind} w w_{\bar{\alpha}}}{\operatorname{ind} w \operatorname{ind} w_{\bar{\alpha}}}\right)^{\frac{1}{2}} \theta_{\dot{w} n_{\bar{\alpha}}}$$

$$= \left(\frac{\operatorname{ind} w w_{\bar{\alpha}}}{\operatorname{ind} w \operatorname{ind} w_{\bar{\alpha}}}\right)^{\frac{1}{2}} \lambda(w, w_{\bar{\alpha}})\tilde{\rho}(\dot{w} n_{\bar{\alpha}})\theta_{\dot{w} n_{\bar{\alpha}}}.$$

But $\dot{w} n_{\bar{\alpha}} = h(w w_{\bar{\alpha}})^{\cdot}$ for some $h \in H$ and $\tilde{\rho}(h(w w_{\bar{\alpha}})^{\cdot})\theta_{h(w w_{\bar{\alpha}})^{\cdot}} = B_{w w_{\bar{\alpha}}}$ as in the proof of 10.7.7. Thus the result follows.

The second formula is proved similarly. ∎

The results of 10.7.7 and 10.7.8 arise from situations in which $\eta = 0$ in the quadratic relations. However we also need a result which is valid when $\eta \neq 0$.

Proposition 10.7.9. *Let $\alpha \in \Lambda$. Then*

$$B_{w_{\bar{\alpha}}}^{\ 2} = \frac{1}{\operatorname{ind} w_{\bar{\alpha}}} 1 \pm \frac{(p_\alpha - 1)}{(p_\alpha \operatorname{ind} w_{\bar{\alpha}})^{\frac{1}{2}}} B_{w_{\bar{\alpha}}}.$$

Note If $\alpha \in \Delta$ we have proved this already in 10.5.2, 10.5.3 and 10.5.5. However in general a root α in Λ need not lie in Δ. We know merely that $\Delta_J \cup \{\alpha\}$ lies in some simple system of Φ, but this need not be Δ.

Proof. We show that the result follows from 10.5.2, 10.5.3 and 10.5.5. This is clear if $\alpha \in \Delta$. So suppose $\alpha \notin \Delta$. There exists $\beta \in \Delta$ with $w_{\bar{\alpha}}(\beta) < 0$. Then we have

$$l(w_\beta w_{\bar{\alpha}}) = l(w_{\bar{\alpha}}) - l(w_\beta) \qquad \text{by 10.7.1.}$$

Now $w_{\bar{\alpha}}(\beta) = \beta - \lambda\alpha + v_J$ for some $\lambda \in \mathbb{R}$ and some $v_J \in V_J$. Since $w_{\bar{\alpha}}(\beta) < 0$ we must have $\lambda > 0$. Thus

$$w_\beta w_{\bar{\alpha}}(\beta) = -\lambda\alpha + \mu\beta + v_J' \qquad \text{for some } \mu \in \mathbb{R}$$

and some $v_J' \in V_J$. Since $\lambda > 0$ we see that $w_\beta w_{\bar{\alpha}}(\beta) < 0$ since this root has at least one negative coefficient when expressed in terms of simple roots. It follows that

$$l(w_\beta w_{\bar{\alpha}} w_\beta^{-1}) = l(w_\beta w_{\bar{\alpha}}) - l(w_\beta^{-1}) \qquad \text{by 10.7.1.}$$

Hence $l(w_\beta w_{\bar\alpha} w_\beta^{-1}) = l(w_{\bar\alpha}) - 2l(w_\beta)$ and $w_\beta w_{\bar\alpha} w_\beta^{-1} = w_{\overline{w_\beta(\alpha)}, w_\beta(J)}$. If $w_\beta(\alpha) \notin \Delta$ we repeat the process. Continuing in this way we see that there exists $u \in W$ such that $uw_{\bar\alpha}u^{-1} = w_{\bar\gamma, J'}$ where $\gamma \in \Delta$ and $l(w_{\bar\alpha}) = l(w_{\bar\gamma, J'}) + 2l(u)$, with $u(\alpha) = \gamma$ and $u(\Delta_J) = \Delta_{J'}$.

We now apply the results of 10.5.2, 10.5.3 and 10.5.5 to $B_{w_{\bar\gamma}}$ with J' replacing J. Thus we have

$$B_{w_{\bar\gamma}}{}^2 = \frac{1}{\operatorname{ind} w_{\bar\gamma}} 1 \pm \frac{(p_\gamma - 1)}{(p_\gamma \operatorname{ind} w_{\bar\gamma})^{\frac{1}{2}}} B_{w_{\bar\gamma}}.$$

Now $w_{\bar\alpha} = u^{-1}w_{\bar\gamma, J'}u$ and $l(w_{\bar\alpha}) = l(u^{-1}) + l(w_{\bar\alpha, J'}) + l(u)$. Thus each positive root made negative by u is made negative by $w_{\bar\alpha}$. But the only positive root in Γ^+ made negative by $w_{\bar\alpha}$ is α by 10.7.6, and $u(\alpha) > 0$. Thus u makes no root in Γ^+ negative. Hence we may apply 10.7.5 and obtain

$$\theta_{\dot u}\theta_{n_{\bar\alpha}} = \left(\frac{\operatorname{ind} uw_{\bar\alpha}}{\operatorname{ind} u \operatorname{ind} w_{\bar\alpha}}\right)^{\frac{1}{2}} \theta_{\dot u n_{\bar\alpha}}.$$

Let $n_{\bar\gamma} = (w_{\bar\gamma, J'}\dot{})$. Then we have

$$\theta_{n_{\bar\gamma}}\theta_{\dot u} = \theta_{(w_{\bar\gamma, J'}u)\dot{}} = \theta_{(u w_{\bar\alpha})\dot{}}$$

since $l(w_{\bar\gamma, J'}u) = l(w_{\bar\gamma, J'}) + l(u)$. Also we have

$$\theta_{\dot u n_{\bar\alpha}} = \theta_{(\dot{uw_{\bar\alpha}})(\dot{uw_{\bar\alpha}})^{-1}\dot u n_{\bar\alpha}} = \rho(n_{\bar\alpha}{}^{-1}\dot u^{-1}(\dot{uw_{\bar\alpha}}))\theta_{(\dot{uw_{\bar\alpha}})}$$

since $(\dot{uw_{\bar\alpha}})^{-1}\dot u n_{\bar\alpha} \in H$. It follows that

$$\theta_{\dot u}\theta_{n_{\bar\alpha}} = \left(\frac{\operatorname{ind} uw_{\bar\alpha}}{\operatorname{ind} u \operatorname{ind} w_{\bar\alpha})}\right)^{\frac{1}{2}} \rho(n_{\bar\alpha}{}^{-1}\dot u^{-1}(\dot{uw_{\bar\alpha}}))\phi_{n_{\bar\gamma}}\theta_{\dot u}.$$

Now $\operatorname{ind} uw_{\bar\alpha} = \operatorname{ind} w_{\bar\gamma, J'}u = \operatorname{ind} w_{\bar\gamma, J'} \operatorname{ind} u$. Thus

$$\theta_{\dot u}\theta_{n_{\bar\alpha}} = \left(\frac{\operatorname{ind} w_{\bar\gamma, J'}}{\operatorname{ind} w_{\bar\alpha}}\right)^{\frac{1}{2}} \rho(n_{\bar\alpha}{}^{-1}\dot u^{-1}(\dot{uw_{\bar\alpha}}))\theta_{n_{\bar\gamma}}\theta_{\dot u}.$$

Hence we have

$$\begin{aligned}
\theta_{\dot u}B_{w_{\bar\alpha}} &= \theta_{\dot u}\tilde\rho(n_{\bar\alpha})\theta_{n_{\bar\alpha}} = \tilde\rho(n_{\bar\alpha})\theta_{\dot u}\theta_{n_{\bar\alpha}} \qquad \text{by 10.2.4}\\
&= \left(\frac{\operatorname{ind} w_{\bar\gamma, J'}}{\operatorname{ind} w_{\bar\alpha}}\right)^{\frac{1}{2}} \tilde\rho(\dot u^{-1}(\dot{uw_{\bar\alpha}}))\theta_{n_{\bar\gamma}}\theta_{\dot u}\\
&= \left(\frac{\operatorname{ind} w_{\bar\gamma, J'}}{\operatorname{ind} w_{\bar\alpha}}\right)^{\frac{1}{2}} \tilde\rho(\dot u^{-1}n_{\bar\gamma}\dot u)\theta_{n_{\bar\gamma}}\theta_{\dot u}\\
&= \left(\frac{\operatorname{ind} w_{\bar\gamma, J'}}{\operatorname{ind} w_{\bar\alpha}}\right)^{\frac{1}{2}} {}^{\dot u}\tilde\rho(n_{\bar\gamma})\theta_{n_{\bar\gamma}}\theta_{\dot u}.
\end{aligned}$$

Now $\theta_{\dot u}$ maps $\mathfrak{F}(J, \rho)$ into $\mathfrak{F}(J', {}^{\dot u}\rho)$ and ${}^{\dot u}\tilde\rho(n_{\bar\gamma})\theta_{n_{\bar\gamma}} = B_{w_{\bar\gamma}}$ on $\mathfrak{F}(J', {}^{\dot u}\rho)$. Thus

$$\theta_{\dot u}B_{w_{\bar\alpha}} = \left(\frac{\operatorname{ind} w_{\bar\gamma, J'}}{\operatorname{ind} w_{\bar\alpha}}\right)^{\frac{1}{2}} B_{w_{\bar\gamma}}\theta_{\dot u} \qquad \text{on } \mathfrak{F}(J, \rho).$$

It follows that

$$\theta_{\dot{u}} B_{w_{\alpha}}{}^2 = \frac{\operatorname{ind} w_{\gamma, J'}}{\operatorname{ind} w_{\alpha}} B_{w_{\gamma}}{}^2 \theta_{\dot{u}} \qquad \text{on } \mathfrak{F}(J, \rho).$$

Hence

$$\begin{aligned} \theta_{\dot{u}} B_{w_{\alpha}}{}^2 &= \frac{\operatorname{ind} w_{\gamma, J'}}{\operatorname{ind} w_{\alpha}} \left(\frac{1}{\operatorname{ind} w_{\gamma, J'}} 1 \pm \frac{(p_{\gamma} - 1)}{(p_{\gamma} \operatorname{ind} w_{\gamma, J'})^{\frac{1}{2}}} B_{w_{\gamma}} \right) \theta_{\dot{u}} \\ &= \frac{1}{\operatorname{ind} w_{\alpha}} \theta_{\dot{u}} \pm \frac{(p_{\alpha} - 1)}{(p_{\alpha} \operatorname{ind} w_{\alpha})^{\frac{1}{2}}} B_{w_{\gamma}} \theta_{\dot{u}} \left(\frac{\operatorname{ind} w_{\gamma, J'}}{\operatorname{ind} w_{\alpha}} \right)^{\frac{1}{2}} \end{aligned}$$

since $p_{\alpha} = p_{\gamma}$

$$\begin{aligned} &= \frac{1}{\operatorname{ind} w_{\alpha}} \theta_{\dot{u}} \pm \frac{(p_{\alpha} - 1)}{(p_{\alpha} \operatorname{ind} w_{\alpha})^{\frac{1}{2}}} \theta_{\dot{u}} B_{w_{\alpha}} \\ &= \theta_{\dot{u}} \left(\frac{1}{\operatorname{ind} w_{\alpha}} 1 \pm \frac{(p_{\alpha} - 1)}{(p_{\alpha} \operatorname{ind} w_{\alpha})^{\frac{1}{2}}} B_{w_{\alpha}} \right). \end{aligned}$$

Finally we must show that $\theta_{\dot{u}}$ is invertible. Since $u(\Delta_J) = \Delta_{J'}$ we can apply 10.7.2 and write

$$u = w_{\alpha_k, J_k} \dots w_{\alpha_1, J_1}$$

with $\alpha_1, \dots, \alpha_k \in \Delta$, $J_1, \dots, J_k \in \mathbf{J}$ and $l(u) = l(w_{\alpha_k, J_k}) + \dots + l(w_{\alpha_1, J_1})$. Hence $\theta_{\dot{u}} = \theta_u = \theta_{w_{\alpha_k, J_k}} \dots \theta_{w_{\alpha_1, J_1}}$ by 10.2.2. It is therefore sufficient to show that each factor is invertible. It suffices in fact to show that $\theta_{w_{\alpha}, J} = \theta_n$ is invertible for each $\alpha \in \Delta - \Delta_J$. By 10.5.1 we have

$$\theta_{n_{\alpha}^{-1}} \theta_{n_{\alpha}} = \begin{cases} \xi 1 + \eta B_{w_{\alpha}} & \text{if } w_{\alpha} \in W^{J, \Phi} \\ \xi 1 & \text{otherwise.} \end{cases}$$

We know also that $\xi \neq 0$. In the latter case this shows directly that $\theta_{n_{\alpha}}$ is invertible. In the former case we know by 10.5.2 that

$$\xi 1 + \eta B_{w_{\alpha}} = B_{w_{\alpha}}{}^2.$$

Since $\xi 1 = B_{w_{\alpha}}(B_{w_{\alpha}} - \eta 1)$ is invertible $B_{w_{\alpha}}$ must be invertible and so $B_{w_{\alpha}}{}^2$ is invertible. Since $\theta_{n_{\alpha}^{-1}} \theta_{n_{\alpha}} = B_{w_{\alpha}}^2$ this implies that $\theta_{n_{\alpha}}$ is invertible as required.

Hence we can multiply on the left by $(\theta_{\dot{u}})^{-1}$ in the above equation to obtain the desired result.

10.8 THE BASIS T_w OF THE ENDOMORPHISM ALGEBRA

The multiplication formulae in $\mathfrak{E} = \operatorname{End}_G \mathfrak{F}(J, \rho)$ can be simplified if we replace the basis B_w, $w \in W^{J, \Phi}$, of $\mathfrak{E}$ by a basis T_w where T_w is a certain scalar multiple of B_w. We shall define the elements T_w for $w \in W^{J, \Phi}$ in a number of stages. We begin with the case when $w = w_{\bar{\alpha}}$ with $\alpha \in \Lambda$. By 10.7.9 we have

$$B_{w_{\bar{\alpha}}}{}^2 = \frac{1}{\operatorname{ind} w_{\bar{\alpha}}} 1 + \frac{\varepsilon_\alpha(p_\alpha - 1)}{(p_\alpha \operatorname{ind} w_{\bar{\alpha}})} B_{w_{\bar{\alpha}}} \qquad \alpha \in \Lambda$$

where $\varepsilon_\alpha = \pm 1$. For $\alpha \in \Lambda$ we define $T_{w_{\bar{\alpha}}}$ by

$$T_{w_{\bar{\alpha}}} = \varepsilon_\alpha(p_\alpha \operatorname{ind} w_{\bar{\alpha}})^{\frac{1}{2}} B_{w_{\bar{\alpha}}}.$$

Then we have

$$\begin{aligned} T_{w_{\bar{\alpha}}}{}^2 &= p_\alpha \operatorname{ind} w_{\bar{\alpha}} \left(\frac{1}{\operatorname{ind} w_{\bar{\alpha}}} 1 + \frac{\varepsilon_\alpha(p_\alpha - 1)}{(p_\alpha \operatorname{ind} w_{\bar{\alpha}})^{\frac{1}{2}}} B_{w_{\bar{\alpha}}} \right) \\ &= p_\alpha 1 + (p_\alpha - 1) T_{w_{\bar{\alpha}}}. \end{aligned}$$

We next wish to define T_w for all $w \in R_{J,\phi}$. We first need a lemma.

Lemma 10.8.1. *Let $\alpha, \beta \in \Lambda$ and $w \in W^{J,\phi}$ satisfy $w(\alpha) = \beta$. Then*

$$B_w T_{w_{\bar{\alpha}}} = T_{w_{\bar{\beta}}} B_w.$$

Proof. By 10.7.8 we have

$$B_w B_{w_{\bar{\alpha}}} = \left(\frac{\operatorname{ind} w w_{\bar{\alpha}}}{\operatorname{ind} w \operatorname{ind} w_{\bar{\alpha}}} \right)^{\frac{1}{2}} \lambda(w, w_{\bar{\alpha}}) B_{w w_{\bar{\alpha}}}$$

since $w(\alpha) > 0$. Thus

$$\begin{aligned} B_w T_{w_{\bar{\alpha}}} &= \varepsilon_\alpha(p_\alpha \operatorname{ind} w_{\bar{\alpha}})^{\frac{1}{2}} \left(\frac{\operatorname{ind} w w_{\bar{\alpha}}}{\operatorname{ind} w \operatorname{ind} w_{\bar{\alpha}}} \right)^{\frac{1}{2}} \lambda(w, w_{\bar{\alpha}}) B_{w w_{\bar{\alpha}}} \\ &= \varepsilon_\alpha p_\alpha{}^{\frac{1}{2}} \left(\frac{\operatorname{ind} w w_{\bar{\alpha}}}{\operatorname{ind} w} \right)^{\frac{1}{2}} \lambda(w, w_{\bar{\alpha}}) B_{w w_{\bar{\alpha}}}. \end{aligned}$$

We also have

$$B_{w_{\bar{\beta}}} B_w = \left(\frac{\operatorname{ind} w_{\bar{\beta}} w}{\operatorname{ind} w_{\bar{\beta}} \operatorname{ind} w} \right)^{\frac{1}{2}} \lambda(w_{\bar{\beta}}, w) B_{w_{\bar{\beta}} w}$$

since $w^{-1}(\beta) > 0$. Thus

$$T_{w_{\bar{\beta}}} B_w = \varepsilon_\beta p_\beta{}^{\frac{1}{2}} \left(\frac{\operatorname{ind} w_{\bar{\beta}} w}{\operatorname{ind} w} \right)^{\frac{1}{2}} \lambda(w_{\bar{\beta}}, w) B_{w_{\bar{\beta}} w}.$$

Now $w(\alpha) = \beta$ and so $w_{\bar{\beta}} = w w_{\bar{\alpha}} w^{-1}$ and $w w_{\bar{\alpha}} = w_{\bar{\beta}} w$. Moreover $p_\alpha = p_\beta$. We therefore need to compare $\lambda(w, w_{\bar{\alpha}})$ with $\lambda(w_{\bar{\beta}}, w)$. By 10.3.3 we have

$$\lambda(w_{\bar{\beta}}, w) \lambda(w_{\bar{\beta}} w, w_{\bar{\alpha}}) = \lambda(w_{\bar{\beta}}, w w_{\bar{\alpha}}) \lambda(w, w_{\bar{\alpha}}).$$

Also by 10.3.4 we have

$$\begin{aligned}\lambda(w_\beta w, w_{\bar\alpha}) &= \lambda(w_{\bar\alpha}^{-1}(w_\beta w)^{-1}, w_\beta w)\\ &= \lambda(w_{\bar\alpha}^{-1}w^{-1}w_\beta, w_\beta w) = \lambda(w^{-1}, w_\beta w) = \lambda(w^{-1}w_\beta w, w^{-1})\\ &= \lambda(w_{\bar\alpha}, w^{-1}) = \frac{1}{\lambda(w, w_{\bar\alpha})}.\end{aligned}$$

Similarly we have $\lambda(w_\beta, ww_{\bar\alpha}) = 1/\lambda(w_\beta, w)$. It follows that $\lambda(w, w_{\bar\alpha})^2 = \lambda(w_\beta, w)^2$ so that $\lambda(w, w_{\bar\alpha}) = \pm\lambda(w_\beta, w)$. This gives $B_w T_{w_{\bar\alpha}} = \pm T_{w_\beta} B_w$. It follows that $B_w T_{w_{\bar\alpha}}{}^2 = T_{w_\beta}{}^2 B_w$ and so

$$B_w(p_\alpha 1 + (p_\alpha - 1)T_{w_{\bar\alpha}}) = (p_\beta 1 + (p_\beta - 1)T_{w_\beta})B_w.$$

We recall that $p_\alpha = p_\beta \neq 1$ since $\alpha, \beta \in \Lambda$. Thus we have $B_w T_{w_{\bar\alpha}} = T_{w_\beta} B_w$ as required. ∎

We write $T_{\bar\alpha} = T_{w_{\bar\alpha}}$ when $\alpha \in \Lambda$.

Proposition 10.8.2. *Let $w \in R_{J,\phi}$ and let*

$$w = w_{\bar\alpha_1} \dots w_{\bar\alpha_n} = w_{\beta_1} \dots w_{\beta_n} \qquad \alpha_i, \beta_i \in \Lambda$$

be two reduced expressions for w. Then we have

$$T_{\bar\alpha_1} \dots T_{\bar\alpha_n} = T_{\beta_1} \dots T_{\beta_n}.$$

Proof. We use induction on n. If $n = 1$ the result is clear, so we suppose $n > 1$. Then $w_{\bar\alpha_1} w_{\beta_1} \dots w_{\beta_n}$ is not a reduced expression. Thus there exists k such that $w_{\bar\alpha_1} w_{\beta_1} \dots w_{\beta_k}$ is reduced but $w_{\bar\alpha_1} w_{\beta_1} \dots w_{\beta_{k+1}}$ is not. It follows that $w_{\bar\alpha_1} w_{\beta_1} \dots w_{\beta_k} = w_{\beta_1} \dots w_{\beta_{k+1}}$.

Consider the reduced expression $w_{\beta_1} \dots w_{\beta_k}$ in $R_{J,\phi}$. It follows from 10.7.1 that

$$l(w_{\beta_1} \dots w_{\beta_k}) = l(w_{\beta_1}) + \dots + l(w_{\beta_k})$$

where l is, as usual, the length function on W. Thus $T_{\beta_1} \dots T_{\beta_k}$ is a nonzero scalar multiple of $B_{w_{\beta_1}} \dots B_{w_{\beta_k}}$ which is in turn a nonzero scalar multiple of $B_{w'}$ where $w' = w_{\beta_1} \dots w_{\beta_k}$, by 10.2.5. We have $w_{\bar\alpha_1} w' = w' w_{\beta_{k+1}}$ and so we may apply 10.8.1. This gives

$$B_{w'} T_{\beta_{k+1}} = T_{\bar\alpha_1} B_{w'}.$$

Hence

$$T_{\beta_1} \dots T_{\beta_k} T_{\beta_{k+1}} = T_{\bar\alpha_1} T_{\beta_1} \dots T_{\beta_k}.$$

It follows that

$$T_{\beta_1} \dots T_{\beta_n} = T_{\bar\alpha_1} T_{\beta_1} \dots T_{\beta_k} T_{\beta_{k+2}} \dots T_{\beta_n}.$$

However we have

$$w_{\beta_1} \dots w_{\beta_k} w_{\beta_{k+2}} \dots w_{\beta_n} = w_{\bar\alpha_2} \dots w_{\bar\alpha_n}.$$

These are two reduced expressions for an element of $R_{J,\phi}$ of length $n-1$. By induction we have

$$T_{\beta_1}\dots T_{\beta_k}T_{\beta_{k+2}}\dots T_{\beta_n} = T_{\bar\alpha_2}\dots T_{\bar\alpha_n}.$$

It follows that

$$T_{\bar\alpha_1}\dots T_{\bar\alpha_n} = T_{\bar\alpha_1}T_{\beta_1}\dots T_{\beta_k}T_{\beta_{k+2}}\dots T_{\beta_n} = T_{\beta_1}\dots T_{\beta_n}$$

and the proposition is proved. ∎

This proposition shows that we may define T_w unambiguously for any $w \in R_{J,\phi}$ by

$$T_w = T_{\bar\alpha_1}\dots T_{\bar\alpha_n}$$

where $w = w_{\bar\alpha_1}\dots w_{\bar\alpha_n}$ is any reduced expression for w with $\alpha_i \in \Lambda$.

We next define T_w for $w \in C_{J,\phi}$. This is given by

$$T_w = (\operatorname{ind} w)^{\frac{1}{2}}B_w \qquad w \in C_{J,\phi}.$$

Finally we define T_w for all $w \in W^{J,\phi}$. By 10.6.3 each $w \in W^{J,\phi}$ is uniquely expressible in the form $w = w_1w_2$ where $w_1 \in C_{J,\phi}$ and $w_2 \in R_{J,\phi}$. We then define T_w by

$$T_w = T_{w_1}T_{w_2}.$$

It is also convenient to define p_w for all $w \in W^{J,\phi}$ by

$$p_w = \prod_{\substack{\alpha\in\Gamma^+\\ w(\alpha)<0}} p_\alpha.$$

We observe that if $w = w_1w_2$ with $w_1 \in C_{J,\phi}$ and $w_2 \in R_{J,\phi}$ then $p_w = p_{w_2}$. For $w_1(\Gamma^+) = \Gamma^+$ and so $w_1w_2(\alpha) < 0$ if and only if $w_2(\alpha) < 0$ for all $\alpha \in \Gamma^+$.

We show next that T_w is a nonzero scalar multiple of B_w for all $w \in W^{J,\phi}$.

Proposition 10.8.3. *Let $w \in W^{J,\phi}$. Then*

$$T_w = \varepsilon_w(p_w \operatorname{ind} w)^{\frac{1}{2}}B_w$$

for some root of unity ε_w.

Proof. We use induction on the number of positive roots in Γ made negative by w. First suppose this is 0. Then $w \in C_{J,\phi}$ and $p_w = 1$. In this case we have $T_w = (\operatorname{ind} w)^{\frac{1}{2}}B_w$ which has the required form with $\varepsilon_w = 1$.

Now suppose there is a positive root in Γ made negative by w. Then $w(\alpha) < 0$ for some $\alpha \in \Lambda$. Let $w = w_1w_2$ with $w_1 \in C_{J,\phi}$ and $w_2 \in R_{J,\phi}$. Then $w_2(\alpha) < 0$. By 10.7.1 we have

$$l(w_2w_{\bar\alpha}) = l(w_2) - l(w_{\bar\alpha}).$$

Hence

$$w_2 = (w_2w_{\bar\alpha})w_{\bar\alpha} \quad \text{with} \quad l(w_2) = l(w_2w_{\bar\alpha}) + l(w_{\bar\alpha}).$$

Let $u = ww_{\bar{\alpha}} = w_1(w_2 w_{\bar{\alpha}})$. Then the number of positive roots in Γ made negative by u is less than for w. By induction we have

$$T_u = \varepsilon_u (p_u \operatorname{ind} u)^{\frac{1}{2}} B_u$$

where ε_u is a root of unity. Now we have

$$T_{w_2} = T_{w_2 w_{\bar{\alpha}}} T_{w_{\bar{\alpha}}}$$

and so

$$T_w = T_{w_1} T_{w_2} = T_{w_1} T_{w_2 w_{\bar{\alpha}}} T_{w_{\bar{\alpha}}} = T_u T_{w_{\bar{\alpha}}}.$$

It follows that

$$\begin{aligned} T_w &= \varepsilon_u (p_u \operatorname{ind} u)^{\frac{1}{2}} \varepsilon_\alpha (p_\alpha \operatorname{ind} w_{\bar{\alpha}})^{\frac{1}{2}} B_u B_{w_{\bar{\alpha}}} \\ &= \varepsilon_u (p_u \operatorname{ind} u)^{\frac{1}{2}} \varepsilon_\alpha (p_\alpha \operatorname{ind} w_{\bar{\alpha}})^{\frac{1}{2}} \left(\frac{\operatorname{ind} u w_{\bar{\alpha}}}{\operatorname{ind} u \operatorname{ind} w_{\bar{\alpha}}} \right)^{\frac{1}{2}} \lambda(u, w_{\bar{\alpha}}) B_{u w_{\bar{\alpha}}} \end{aligned}$$

by 10.7.8, since $u(\alpha) > 0$. Hence

$$T_w = \varepsilon_u \varepsilon_\alpha (p_u p_\alpha)^{\frac{1}{2}} (\operatorname{ind} w)^{\frac{1}{2}} \lambda(u, w_{\bar{\alpha}}) B_w.$$

Let $N(w) = \{\alpha \in \Phi^+, w(\alpha) \in \Phi^-\}$. Then since $w = uw_{\bar{\alpha}}$ we have

$$N(w) \cap \Gamma = (N(w_{\bar{\alpha}}) \cap \Gamma) \cup w_{\bar{\alpha}}(N(u) \cap \Gamma)$$

(a disjoint union). It follows that

$$p_w = p_\alpha p_u.$$

Hence $T_w = \varepsilon_w (p_w \operatorname{ind} w)^{\frac{1}{2}} B_w$ where $\varepsilon_w = \varepsilon_u \varepsilon_\alpha \lambda(u, w_{\bar{\alpha}})$. By induction ε_w is a root of unity, since this is true of ε_u, ε_α and $\lambda(u, w_{\bar{\alpha}})$ by 10.3.4. ■

In order to give the multiplication formulae for the new basis elements T_w it is convenient to introduce a new 2-cocycle cohomologous to λ. For $w, w' \in W^{J,\phi}$ we define $\mu(w, w')$ by

$$\mu(w, w') = \varepsilon_w \varepsilon_{w'} \varepsilon_{ww'}^{-1} \lambda(w, w').$$

The next result describes the properties of this cocycle μ.

Proposition 10.8.4. *Let $x, x' \in C_{J,\phi}$ and $w, w' \in R_{J,\phi}$. Then*

$$\mu(xw, x'w') = \mu(x, x') = \lambda(x, x').$$

In particular we have

$$\mu(w, x'w') = \mu(xw, w') = 1.$$

Proof. We shall prove this result in a number of stages

(i) We know that $T_x T_w = T_{xw}$. Thus

$$\varepsilon_x (p_x \operatorname{ind} x)^{\frac{1}{2}} \varepsilon_w (p_w \operatorname{ind} w)^{\frac{1}{2}} B_x B_w = \varepsilon_{xw} (p_{xw} \operatorname{ind} xw)^{\frac{1}{2}} B_{xw}.$$

By 10.7.7 this gives

$$\varepsilon_x \varepsilon_w \varepsilon_{xw}^{-1} \lambda(x, w) = 1$$

since $p_x = 1$ and $p_{xw} = p_w$. Hence $\mu(x, w) = 1$ for $x \in C_{J,\phi}$, $w \in R_{J,\phi}$. We also have

$$\begin{aligned} T_w T_x &= \varepsilon_w (p_w \operatorname{ind} w)^{\frac{1}{2}} \varepsilon_x (p_x \operatorname{ind} x)^{\frac{1}{2}} B_w B_x \\ &= \varepsilon_w \varepsilon_x \lambda(w, x)(p_w p_x)^{\frac{1}{2}} (\operatorname{ind} wx)^{\frac{1}{2}} B_{wx} \qquad \text{by 10.7.7} \\ &= \varepsilon_w \varepsilon_x \varepsilon_{wx}^{-1} \lambda(w, x) T_{wx} \\ &= \mu(x, w) T_{x, x^{-1}wx} \\ &= \mu(w, x) T_x T_{x^{-1}wx}. \end{aligned}$$

Thus $T_w B_x = \mu(w, x) B_x T_{x^{-1}wx}$. Let $w = w_{\bar{\alpha}_1} \dots w_{\bar{\alpha}_k}$ be a reduced expression for $w \in R_{J,\phi}$. Then $x^{-1}wx = w_{\beta_1} \dots w_{\beta_k}$ where $x^{-1}(\alpha_i) = \beta_i$. Thus

$$T_w = T_{\bar{\alpha}_1} \dots T_{\bar{\alpha}_k}, \qquad T_{x^{-1}wx} = T_{\beta_1} \dots T_{\beta_k}.$$

We now have

$$T_w B_x = T_{\bar{\alpha}_1} \dots T_{\bar{\alpha}_k} B_x = B_x T_{\beta_1} \dots T_{\beta_k} = B_x T_{x^{-1}wx}$$

by 10.8.1. Thus we have $\mu(w, x) = 1$ for all $x \in C_{J,\phi}$, $w \in R_{J,\phi}$.

(ii) We next consider $\mu(w_{\bar{\alpha}}, w)$ when $\alpha \in \Lambda$ and $w \in R_{J,\phi}$.

First suppose $w^{-1}(\alpha) > 0$. Then $T_{w_{\bar{\alpha}}} T_w = T_{w_{\bar{\alpha}} w}$ and so

$$\varepsilon_\alpha (p_\alpha \operatorname{ind} w_{\bar{\alpha}})^{\frac{1}{2}} \varepsilon_w (p_w \operatorname{ind} w)^{\frac{1}{2}} B_{w_{\bar{\alpha}}} B_w = \varepsilon_{w_{\bar{\alpha}} w} (p_{w_{\bar{\alpha}} w} \operatorname{ind} w_{\bar{\alpha}} w)^{\frac{1}{2}} B_{w_{\bar{\alpha}} w}.$$

But $B_{w_{\bar{\alpha}}} B_w = (\operatorname{ind} w_{\bar{\alpha}} w / \operatorname{ind} w_{\bar{\alpha}} \operatorname{ind} w)^{\frac{1}{2}} \lambda(w_{\bar{\alpha}}, w) B_{w_{\bar{\alpha}} w}$ by 10.7.8. Since $p_\alpha p_w = p_{w_{\bar{\alpha}} w}$ we obtain $\mu(w_{\bar{\alpha}}, w) = 1$.

Now suppose $w^{-1}(\alpha) < 0$. Then $w^{-1} w_{\bar{\alpha}}(\alpha) > 0$ and so $(w_{\bar{\alpha}} w)^{-1}(\alpha) > 0$. Thus we have $\mu(w_{\bar{\alpha}}, w_{\bar{\alpha}} w) = 1$. But

$$\mu(w_{\bar{\alpha}}, w_{\bar{\alpha}}) \mu(w_{\bar{\alpha}} w_{\bar{\alpha}}, w) = \mu(w_{\bar{\alpha}}, w_{\bar{\alpha}} w) \mu(w_{\bar{\alpha}}, w)$$

and

$$\mu(w_{\bar{\alpha}}, w_{\bar{\alpha}}) = \varepsilon_\alpha \varepsilon_\alpha \varepsilon_1^{-1} \lambda(w_{\bar{\alpha}}, w_{\bar{\alpha}}) = 1$$

since $\varepsilon_\alpha = \pm 1$, $\varepsilon_1 = 1$ and $\lambda(w_{\bar{\alpha}}, w_{\bar{\alpha}}) = 1$ by 10.3.4. We also have

$$\mu(1, w) = \varepsilon_1 \varepsilon_w \varepsilon_w^{-1} \lambda(1, w) = 1.$$

Thus we deduce that $\mu(w_{\bar{\alpha}}, w) = 1$ in this case also.

(iii) We now consider $\mu(w, w')$ for $w, w' \in R_{J,\phi}$. We shall show that $\mu(w, w') = 1$ by induction on $l(w)$. If $l(w) = 0$ we have $w = 1$ and

$$\mu(1, w') = \varepsilon_1 \varepsilon_{w'} \varepsilon_{w'}^{-1} \lambda(1, w') = 1.$$

So suppose $l(w) > 0$. Then there exists $\alpha \in \Lambda$ with $w = w'' w_{\bar{\alpha}}$ and $l(w'') < l(w)$. We have

$$\mu(w'', w_{\bar{\alpha}}) \mu(w'' w_{\bar{\alpha}}, w') = \mu(w'', w_{\bar{\alpha}} w') \mu(w_{\bar{\alpha}}, w').$$

Now $\mu(w'', w_{\bar{\alpha}}) = \mu(w'', w_{\bar{\alpha}}w') = 1$ by induction, and $\mu(w_{\bar{\alpha}}, w') = 1$ by (ii). Thus we obtain $\mu(w, w') = 1$ also.

(iv) We next consider $\mu(xw, w')$ with $x \in C_{J,\phi}$ and $w, w' \in R_{J,\phi}$. We have

$$\mu(x, w)\mu(xw, w') = \mu(x, ww')\mu(w, w').$$

Now $\mu(x, w) = \mu(x, ww') = 1$ by (i) and $\mu(w, w') = 1$ by (iii). Thus we deduce that $\mu(xw, w') = 1$.

(v) Finally we consider the general case $\mu(xw, x'w')$. We have

$$\mu(x. x')\mu(xx', w') = \mu(x, x'w')\mu(x', w').$$

Now $\mu(xx', w') = \mu(x', w') = 1$ by (i). Thus $\mu(x, x'w') = \mu(x, x')$. We also have

$$\mu(x, w)\mu(xw, x') = \mu(x, wx')\mu(w, x').$$

Now $\mu(x, w) = \mu(w, x') = 1$ by (i). Thus

$$\mu(xw, x') = \mu(x, wx') = \mu(x, x' . x'^{-1}wx') = \mu(x, x').$$

Finally we have

$$\mu(xw, x')\mu(xwx', w') = \mu(xw, x'w')\mu(x', w').$$

Now $\mu(x', w') = 1$ by (i) and $\mu(xwx', w') = 1$ by (iv). Thus

$$\mu(xw, x'w') = \mu(xw, x') = \mu(x, x')$$

and the result is proved. ■

We can now give the multiplication formulae for the basis elements T_w of $\mathfrak{E}$.

Theorem 10.8.5. (i) *Suppose* $w \in W^{J,\phi}$ *and* $w' \in C_{J,\phi}$. *Then*

$$T_w T_{w'} = \mu(w, w')T_{ww'}.$$
$$T_{w'} T_w = \mu(w', w)T_{w'w}.$$

(ii) *Let* $w \in W^{J,\phi}$ *and* $\alpha \in \Lambda$. *Then*

$$T_{\bar{\alpha}}T_w = \begin{cases} T_{w_{\bar{\alpha}}w} & \text{if } w^{-1}(\alpha) > 0 \\ p_\alpha T_{w_{\bar{\alpha}}w} + (p_\alpha - 1)T_w & \text{if } w^{-1}(\alpha) < 0 \end{cases}$$

$$T_w T_{\bar{\alpha}} = \begin{cases} T_{ww_{\bar{\alpha}}} & \text{if } w(\alpha) > 0 \\ p_\alpha T_{ww_{\bar{\alpha}}} + (p_\alpha - 1)T_w & \text{if } w(\alpha) < 0. \end{cases}$$

Proof. (i) We have

$$\begin{aligned} T_w T_{w'} &= \varepsilon_w (p_w \operatorname{ind} w)^{\frac{1}{2}} \varepsilon_{w'} (p_{w'} \operatorname{ind} w')^{\frac{1}{2}} B_w B_{w'} \\ &= \varepsilon_w \varepsilon_{w'} (p_w p_{w'})^{\frac{1}{2}} (\operatorname{ind} ww')^{\frac{1}{2}} \lambda(w, w') B_{ww'} \end{aligned}$$

by 10.7.7

$$= \varepsilon_w \varepsilon_{w'} \varepsilon_{ww'}{}^{-1} \lambda(w, w') \left(\frac{p_w p_{w'}}{p_{ww'}} \right)^{\frac{1}{2}} T_{ww'}$$

$$= \mu(w, w') T_{ww'}$$

since $p_{w'} = 1$ and $p_{ww'} = p_w$.

We see similarly that $T_{w'} T_w = \mu(w', w) T_{w'w}$.

(ii) We have

$$T_{\bar{\alpha}} T_w = \varepsilon_\alpha (p_\alpha \text{ ind } w_{\bar{\alpha}})^{\frac{1}{2}} \varepsilon_w (p_w \text{ ind } w)^{\frac{1}{2}} B_{w_{\bar{\alpha}}} B_w.$$

Suppose $w^{-1}(\alpha) > 0$. Then we may apply 10.7.8 and obtain

$$T_{\bar{\alpha}} T_w = \varepsilon_\alpha \varepsilon_w (p_\alpha p_w)^{\frac{1}{2}} (\text{ind } w_{\bar{\alpha}} w)^{\frac{1}{2}} \lambda(w_{\bar{\alpha}}, w) B_{w_{\bar{\alpha}} w}$$

$$= \varepsilon_\alpha \varepsilon_w \varepsilon_{w_{\bar{\alpha}} w}{}^{-1} \lambda(w_{\bar{\alpha}}, w) \left(\frac{p_\alpha p_w}{p_{w_{\bar{\alpha}} w}} \right)^{\frac{1}{2}} T_{w_{\bar{\alpha}} w}$$

$$= \mu(w_{\bar{\alpha}}, w) T_{w_{\bar{\alpha}} w} \qquad \text{since } p_{w_{\bar{\alpha}} w} = p_\alpha p_w$$

$$= T_{w_{\bar{\alpha}} w} \qquad \text{since } \mu(w_{\bar{\alpha}}, w) = 1 \text{ by 10.8.4.}$$

Now suppose $w^{-1}(\alpha) < 0$. Then $w^{-1} w_{\bar{\alpha}}(\alpha) > 0$. Thus $T_{\bar{\alpha}} T_{w_{\bar{\alpha}} w} = T_w$. It follows that

$$T_{\bar{\alpha}} T_w = T_{\bar{\alpha}}{}^2 T_{w_{\bar{\alpha}} w} = (p_\alpha 1 + (p_\alpha - 1) T_{\bar{\alpha}}) T_{w_{\bar{\alpha}} w}$$

$$= p_\alpha T_{w_{\bar{\alpha}} w} + (p_\alpha - 1) T_w.$$

The formula for $T_w T_{\bar{\alpha}}$ can be proved similarly. ■

It is clear that by iterating the formulae in 10.8.5 we can express the product $T_w T_{w'}$ for any pair of elements $w, w' \in W^{J,\phi}$ as a linear combination of basis elements $T_{w''}$. Thus we have a description of the structure of the algebra $\mathfrak{E} = \mathrm{End}_G\, \mathfrak{F}(J, \rho)$.

10.9 THE ENDOMORPHISM ALGEBRA AS A SYMMETRIC ALGEBRA

We have seen in 10.1.2 that there is a bijection between irreducible characters χ of G satisfying $(\Phi_{P_J}{}^G, \chi) \neq 0$ and irreducible characters $\chi_{\mathfrak{E}}$ of $\mathfrak{E}$. We wish to obtain a formula for the multiplicity with which an irreducible character $\chi_{\mathfrak{E}}$ of $\mathfrak{E}$ occurs in $\mathfrak{F}(J, \rho)$. This will also give us a formula for the degree $\chi(1)$ of χ. In order to obtain such a formula we must use the fact that $\mathfrak{E}$ can be considered as a symmetric algebra.

Proposition 10.9.1. *Let $w, w' \in W^{J,\phi}$. Then $T_w T_{w'}$ contains T_1 with nonzero coefficient if and only if $w' = w^{-1}$. Moreover the coefficient of T_1 in $T_w T_{w^{-1}}$ is p_w.*

Proof. (i) We first prove this for $w, w' \in R_{J,\phi}$. The multiplication laws of

10.8.5 show that if $T_{w''}$ occurs with nonzero coefficient in $T_w T_{w'}$ then $l'(w'') \geqslant l'(w) - l'(w')$ and $l'(w'') \geqslant l'(w') - l'(w)$, where l' is the length function for the reflection group $R_{J,\phi}$. Thus if $T_w T_{w'}$ contains T_1 with nonzero coefficient we must have $l'(w) = l'(w')$. So let $l'(w) = l'(w') = k$ and

$$T_{w'} = T_{\bar{\alpha}_1} \dots T_{\bar{\alpha}_k} \qquad \alpha_i \in \Lambda.$$

The multiplication laws show that the only component of $T_w T_{\bar{\alpha}_1}$ of length at most $k-1$ can be $T_{ww_{\bar{\alpha}_1}}$. The only component of $T_w T_{\bar{\alpha}_1} T_{\bar{\alpha}_2}$ which could have length at most $k-2$ is then $T_{ww_{\bar{\alpha}_1}w_{\bar{\alpha}_2}}$. Continuing in this way we see that the only component of $T_w T_{\bar{\alpha}_1} \dots T_{\bar{\alpha}_k}$ which could have length 0 is $T_{ww_{\bar{\alpha}_1}\dots w_{\bar{\alpha}_k}}$. Thus if T_1 occurs as a component of $T_w T_{w'}$ we must have $ww_{\bar{\alpha}_1} \dots w_{\bar{\alpha}_k} = 1$, i.e. $ww' = 1$.

Now consider whether $T_w T_{w^{-1}}$ contains T_1 as a component. The multiplication laws show that $T_w T_{\bar{\alpha}_1}$ contains $T_{ww_{\bar{\alpha}_1}}$ with multiplicity p_{α_1}. Also $T_w T_{\bar{\alpha}_1} T_{\bar{\alpha}_2}$ contains $T_{ww_{\bar{\alpha}_1}w_{\bar{\alpha}_2}}$ with multiplicity $p_{\alpha_1} p_{\alpha_2}$. Eventually we see that $T_w T_{\bar{\alpha}_1} \dots T_{\bar{\alpha}_k}$ contains T_1 with multiplicity $p_{\alpha_1} \dots p_{\alpha_k}$. Now $w = w_{\bar{\alpha}_1} \dots w_{\bar{\alpha}_k}$ is a reduced expression for w in $R_{J,\phi}$ and by 10.7.6 each $w_{\bar{\alpha}_i}$ makes only one positive root in Γ negative. Thus the positive roots in Γ made negative by w are

$$\alpha_k,\ w_{\bar{\alpha}_k}(\alpha_{k-1}),\ w_{\bar{\alpha}_k} w_{\bar{\alpha}_{k-1}}(\alpha_{k-2}), \dots, w_{\bar{\alpha}_k} \dots w_{\bar{\alpha}_2}(\alpha_1).$$

Hence

$$p_{w'} = \prod_{\substack{\alpha \in \Gamma^+ \\ w'(\alpha) < 0}} p_\alpha = p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_k}$$

since $p_{w(\alpha)} = p_\alpha$ for $\alpha \in \Omega$ and $w \in W^{J,\phi}$. Thus $T_w T_{w^{-1}}$ contains T_1 as component with coefficient p_w.

(ii) We now consider elements $xw,\ x'w' \in W^{J,\phi}$ where $x, x' \in C_{J,\phi}$ and $w, w' \in R_{J,\phi}$. We have

$$\begin{aligned} T_{xw} T_{x'w'} &= T_x T_w T_{x'} T_{w'} = T_x T_{wx'} T_{w'} \\ &= T_x T_{x'.x'^{-1}wx'} T_{w'} = T_x T_{x'} T_{x'^{-1}wx'} T_{w'} \\ &= \mu(x, x') T_{xx'} T_{x'^{-1}wx'} T_{w'}. \end{aligned}$$

This shows that $T_{xw} T_{x'w'}$ can only contain T_1 as a component if $xx' = 1$ and $x'^{-1}wx' = w'^{-1}$. Then

$$x'w' = x^{-1}x'^{-1}w^{-1}x' = w^{-1}x^{-1} = (xw)^{-1}.$$

Conversely suppose that $x'w' = (xw)^{-1}$ and consider the coefficient of T_1 in $T_{xw}T_{x'w'}$. We have

$$T_{xw} T_{x'w'} = \mu(x, x') T_{w'^{-1}} T_{w'} = T_{w'^{-1}} T_{w'}$$

since $\mu(x, x') = \mu(x, x^{-1}) = \lambda(x, x^{-1}) = 1$. Thus the coefficient of T_1 in $T_{xw}T_{x'w'}$ is $p_{w'^{-1}} = p_{x'^{-1}wx'} = p_w = p_{xw}$. ■

We now define a scalar product on the endomorphism algebra $\mathfrak{E}$. Let $w, w' \in W^{J,\phi}$. We define

$$(T_w, T_{w'}) = \begin{cases} p_w & \text{if } w' = w^{-1} \\ 0 & \text{if } w' \neq w^{-1} \end{cases}$$

and extend this definition by linearity to give the scalar product of any two elements of $\mathfrak{E}$.

Proposition 10.9.2. *This scalar product makes $\mathfrak{E}$ into a symmetric algebra.*

Proof. The scalar product is certainly nondegenerate. So we must verify that $(T_w, T_{w'}) = (T_{w'}, T_w)$ and $(T_w T_{w'}, T_{w''}) = (T_w, T_{w'} T_{w''})$ for all $w, w', w'' \in W^{J,\phi}$. The former relation holds because $p_w = p_{w^{-1}}$ for all $w \in W^{J,\phi}$. We consider the latter relation.

Now $(T_w, T_{w'})$ is the coefficient of T_1 in $T_w T_{w'}$. Thus for all $U_1, U_2 \in \mathfrak{E}$ (U_1, U_2) is the coefficient of T_1 in $U_1 U_2$. Hence the scalar products $(T_w T_{w'}, T_{w''})$ and $(T_w, T_{w'} T_{w''})$ are both equal to the coefficient of T_1 in $T_w T_{w'} T_{w''}$.

Proposition 10.9.3. (Fossum) *If A is a symmetric algebra over $\mathbb{C}$ with a pair of dual bases $e_1, \ldots, e_n; f_1, \ldots, f_n$ (so that $(e_i, f_j) = \delta_{ij}$) and if χ_A, χ'_A are two distinct irreducible characters of A then*

$$\sum_{i=1}^{n} \chi_A(e_i)\chi'_A(f_i) = 0.$$

Proof. See Fossum [1], p. 11. ∎

In our symmetric algebra $\mathfrak{E}$ we can easily identify a pair of dual bases. One can take $\{T_w; w \in W^{J,\phi}\}$ and

$$\left\{\frac{1}{p_w} T_{w^{-1}}; w \in W^{J,\phi}\right\}.$$

Proposition 10.9.4. *$T_w : \mathfrak{F}(J, \rho) \to \mathfrak{F}(J, \rho)$ has trace 0 if $w \in W^{J,\phi}$ and $w \neq 1$.*

Proof. Let $g_1 = 1, g_2, \ldots, g_m$ be a set of right coset representatives of P_J in G. Let $\mathfrak{F}(J, \rho)_i$ be given by

$$\mathfrak{F}(J, \rho)_i = \{f \in \mathfrak{F}(J, \rho); f(g_j) = 0 \qquad \text{for all } j \neq i\}.$$

Then $\mathfrak{F}(J, \rho) = \oplus_{i=1}^{m} \mathfrak{F}(J, \rho)_i$ as in 10.1.1. Suppose $f \in \mathfrak{F}(J, \rho)_i$ and consider $T_w f$. This is a scalar multiple of $B_w f$. We have

$$(B_w f)x = \tilde{\rho}(\dot{w}) \frac{1}{|U_J|} \sum_{u \in U_J} f(\dot{w}^{-1} u x).$$

Suppose $(B_w f)g_i \neq 0$. Then there exists $u \in U_J$ for which $f(\dot{w}^{-1} u g_i) \neq 0$. This implies that $\dot{w}^{-1} u g_i \in P_J g_i$ and so that $\dot{w}^{-1} u \in P_J$. Hence $\dot{w} \in P_J$. But $w \in W^{J,\phi}$ and so $\dot{w} \in P_J$ can only hold if $w = 1$. Thus if $w \neq 1$ we have $(B_w f)g_i = 0$. Thus the summand of $B_w f \in \mathfrak{F}(J, \rho)$ which lies in $\mathfrak{F}(J, \rho)_i$ is zero. Hence trace $B_w = 0$ if $w \neq 1$. It follows that trace $T_w = 0$ also.

Proposition 10.9.5. *Suppose $\chi_{\mathfrak{E}}$ is an irreducible character of $\mathfrak{E}$ which occurs with multiplicity m in the $\mathfrak{E}$-module $\mathscr{F}(J,\rho)$. Then*

$$m = \frac{\dim \mathfrak{F}(J,\rho)\,.\,\chi_{\mathfrak{E}}(T_1)}{\sum_{w\in W^{J,\phi}} \chi_{\mathfrak{E}}(T_w)\chi_{\mathfrak{E}}\left(\frac{1}{p_w}T_{w^{-1}}\right)}.$$

Proof. Let $\chi_{\mathfrak{E}}^1,\dots,\chi_{\mathfrak{E}}^n$ be the irreducible characters of $\mathfrak{E}$ and let $m_1,\dots,m_n$ be their multiplicities in $\mathfrak{F}(J,\rho)$. Then

$$\text{trace}\,(T_w;\mathfrak{F}(J,\rho)) = \sum_{j=1}^{n} m_j\chi_{\mathfrak{E}}^j(T_w).$$

Now

$$\text{trace}\; T_w = \begin{cases} 0 & \text{if } w\neq 1 \\ \dim \mathfrak{F}(J,\rho) & \text{if } w = 1. \end{cases} \qquad \text{by 10.9.4}$$

Thus

$$\sum_{i=1}^{n} m_j\chi_{\mathfrak{E}}^j(T_w) = \begin{cases} 0 & \text{if } w\neq 1 \\ \dim \mathscr{F}(J,\rho) & \text{if } w = 1. \end{cases}$$

It follows that

$$\sum_{w\in W^{J,\phi}}\left(\sum_{j=1}^{n} m_j\chi_{\mathfrak{E}}^j(T_w)\right)\chi_{\mathfrak{E}}^i\left(\frac{1}{p_w}T_{w^{-1}}\right) = \dim \mathfrak{F}(J,\rho)\chi_{\mathfrak{E}}^i(T_1).$$

On the other hand we have

$$\begin{aligned}\sum_{w\in W^{J,\phi}}&\left(\sum_{j=1}^{n} m_j\chi_{\mathfrak{E}}^j(T_w)\right)\chi_{\mathfrak{E}}^i\left(\frac{1}{p_w}T_{w^{-1}}\right)\\ &= \sum_{j=1}^{n} m_j \sum_{w\in W^{J,\phi}} \chi_{\mathfrak{E}}^j(T_w)\chi_{\mathfrak{E}}^i\left(\frac{1}{p_w}T_{w^{-1}}\right)\\ &= m_i \sum_{w\in W^{J,\phi}} \chi_{\mathfrak{E}}^i(T_w)\chi_{\mathfrak{E}}^i\left(\frac{1}{p_w}T_{w^{-1}}\right) \qquad \text{by 10.9.3.}\end{aligned}$$

Hence we have

$$m_i = \frac{\dim \mathfrak{F}(J,\rho)\chi_{\mathfrak{E}}^i(T_1)}{\sum_{w\in W^{J,\phi}} \chi_{\mathfrak{E}}^i(T_w)\chi_{\mathfrak{E}}^i\left(\frac{1}{p_w}T_{w^{-1}}\right)}.$$

Proposition 10.9.6. *Let χ be an irreducible character of G satisfying $(\phi_{P_J}{}^G,\chi)\neq 0$. Then the degree of χ is given by*

$$\chi(1) = \frac{\dim \mathfrak{F}(J,\rho)\chi_{\mathfrak{E}}(T_1)}{\sum_{w\in W^{J,\phi}} \chi_{\mathfrak{E}}(T_w)\chi_{\mathfrak{E}}\left(\frac{1}{p_w}T_{w^{-1}}\right)}$$

where $\chi_{\mathfrak{E}}$ is the irreducible character of $\mathfrak{E}$ corresponding to χ as in 10.1.2.

Proof. Let $\alpha(\mathbb{C}G)$ be the subalgebra of End $\mathfrak{F}(J, \rho)$ induced by $\mathbb{C}G$. Then the subalgebras $\alpha(\mathbb{C}G)$ and $\mathfrak{E}$ of End $\mathfrak{F}(J, \rho)$ have the double centralizer property, i.e. each is the centralizer of the other, by Curtis and Reiner [1], p. 403. In these circumstances there is a bijective correspondence between the irreducible $\mathbb{C}G$-submodules and the irreducible $\mathfrak{E}$-submodules of $\mathfrak{F}(J, \rho)$ such that the degree of one of these submodules is equal to the multiplicity of the other in $\mathfrak{F}(J, \rho)$ (see Brauer [1], p. 10). Thus the degree of χ is equal to the multiplicity of $\chi_{\mathfrak{E}}$ *in* $\mathfrak{F}(J, \rho)$. By 10.9.5 this multiplicity is given by the above formula.

10.10 SOME SPECIAL CASES

We now consider the special case in which Δ_J is self-opposed in Δ_K for all K satisfying $J \subseteq K \subseteq I$. This means that $(w_0)_K(\Delta_J) = -\Delta_J$ for all such K. This situation occurs, for example, when Δ_J is the only subset of Δ of its type. (For example, there is only one subset of a simple system of type E_6 which has type D_4.)

Lemma 10.10.1. *Suppose Δ_J is self-opposed in $\Delta_J \cup \{\alpha\}$ for all $\alpha \in \Delta - \Delta_J$. Then the set $\bar{\alpha}$ for $\alpha \in \Delta - \Delta_J$ forms a simple system in $\bar{\Omega}$.*

Proof. For each $\alpha \in \Delta - \Delta_J$ we have $\alpha \in \Omega$. For

$$w_{\bar{\alpha}}(\Delta_J) = (w_0)_{J \cup \{\alpha\}}(w_0)_J(\Delta_J) = (w_0)_{J \cup \{\alpha\}}(-\Delta_J) = \Delta_J$$

since Δ_J is self-opposed in $\Delta_J \cup \{\alpha\}$. Now the $\bar{\alpha}$ for $\alpha \in \Delta - \Delta_J$ form a basis for $\bar{V} = V/V_J$. Moreover for any root $\beta \in \Omega$ we have

$$\beta \equiv \sum_{i \in I - J} \lambda_i \alpha_i \bmod V_J \qquad \text{with all } \lambda_i \geqslant 0 \text{ or all } \lambda_i \leqslant 0.$$

It follows that $\bar{\beta} = \sum_{i \in I - J} \lambda_i \bar{\alpha}_i$ with all $\lambda_i \geqslant 0$ or all $\lambda_i \leqslant 0$. Thus we have a simple system in $\bar{\Omega}$.

Proposition 10.10.2. *Suppose Δ_J is self-opposed in Δ_K for all K with $J \subseteq K \subseteq I$. Then we have*

$$\{w \in W; w(J) = J\} = \langle w_{\bar{\alpha}}; \alpha \in \Omega \rangle.$$

Thus $C_J = R_J$.

Proof. In general we have $R_J \subseteq C_J$. So suppose $w \in C_J$. By 10.7.2 w can be expressed in the form

$$w = w_{\bar{\alpha}_k, J_k} \cdots w_{\bar{\alpha}_1, J_1}$$

where $\alpha_1, \ldots, \alpha_k \in \Delta$, $J_1 = J$ and $w_{\bar{\alpha}_i, J_i}(\Delta_{J_i}) = \Delta_{J_{i+1}}$. However under the given assumptions all J_i are equal to J. For $J_1 = J$, so assume inductively that $J_i = J$. Then

$$\Delta_{J_{i+1}} = w_{\bar{\alpha}_i, J}(\Delta_J) = (w_0)_{J \cup \{\alpha_i\}}(w_0)_J(\Delta_J)$$
$$= (w_0)_{J \cup \{\alpha_i\}}(-\Delta_J) = \Delta_J$$

since Δ_J is self-opposed in $\Delta_J \cup \{\alpha_i\}$. Thus $J_{i+1} = J$. Thus $w = w_{\bar{\alpha}_k} \dots w_{\bar{\alpha}_1} \in R_J$ and so $C_J \subseteq R_J$.

Proposition 10.10.3. *Suppose Δ_J is self-opposed in Δ_K for all K with $J \subseteq K \subseteq I$. Let $\alpha, \beta \in \Delta - \Delta_J$ satisfy $\alpha \neq \beta$. Then the order of the element $w_{\bar{\alpha}} w_{\bar{\beta}}$ is given by*

$$\frac{2(|\Phi^+_{J \cup \{\alpha, \beta\}}| - |\Phi_J{}^+|)}{|\Phi^+_{J \cup \{\alpha\}}| + |\Phi^+_{J \cup \{\beta\}}| - 2|\Phi_J{}^+|}.$$

Proof. Consider the element $(w_0)_{J \cup \{\alpha, \beta\}}(w_0)_J$ of W. We have $(w_0)_{J \cup \{\alpha, \beta\}}(w_0)_J(\Delta_J) = \Delta_J$ since Δ_J is self-opposed in $\Delta_J \cup \{\alpha, \beta\}$. Thus $(w_0)_{J \cup \{\alpha, \beta\}}(w_0)_J$ lies in $C_J = R_J$. Consider the positive roots in $\bar{V} = V/V_J$ made negative by this element. These are just the positive combinations of $\bar{\alpha}$ and $\bar{\beta}$. Thus we have $(w_0)_{J \cup \{\alpha, \beta\}}(w_0)_J = (w_0)_{\{\bar{\alpha}, \bar{\beta}\}}$.

Let $w_{\bar{\alpha}} w_{\bar{\beta}}$ have order m. Then

$$\underset{(2m \text{ factors})}{w_{\bar{\alpha}} w_{\bar{\beta}} w_{\bar{\alpha}} w_{\bar{\beta}} \dots w_{\bar{\alpha}} w_{\bar{\beta}}} = 1.$$

and

$$\underset{(m \text{ factors})}{w_{\bar{\alpha}} w_{\bar{\beta}} w_{\bar{\alpha}} \dots} = (w_0)_{\{\bar{\alpha}, \bar{\beta}\}}.$$

Thus we have

$$(w_0)_{J \cup \{\alpha, \beta\}}(w_0)_J = \underset{(m \text{ factors})}{w_{\bar{\alpha}} w_{\bar{\beta}} w_{\bar{\alpha}} \dots} = \underset{(m \text{ factors})}{w_{\bar{\beta}} w_{\bar{\alpha}} w_{\bar{\beta}} \dots}.$$

We next observe that

$$l(w_{\bar{\alpha}} w_{\bar{\beta}} w_{\bar{\alpha}} \dots) = l(w_{\bar{\alpha}}) + l(w_{\bar{\beta}}) + l(w_{\bar{\alpha}}) + \dots.$$

This follows from 10.7.1. Thus we have

$$\begin{aligned} 2l((w_0)_{J \cup \{\alpha, \beta\}}(w_0)_J) &= l(w_{\bar{\alpha}} w_{\bar{\beta}} w_{\bar{\alpha}} \dots) + l(w_{\bar{\beta}} w_{\bar{\alpha}} w_{\bar{\beta}} \dots) \\ &= m(l(w_{\bar{\alpha}}) + l(w_{\bar{\beta}})). \end{aligned}$$

But we also have

$$\begin{aligned} l((w_0)_{J \cup \{\alpha, \beta\}}(w_0)_J) &= |\Phi^+_{J \cup \{\alpha, \beta\}}| - |\Phi_J{}^+| \\ l(w_{\bar{\alpha}}) &= |\Phi^+_{J \cup \{\alpha\}}| - |\Phi_J{}^+| \\ l(w_{\bar{\beta}}) &= |\Phi^+_{J \cup \{\beta\}}| - |\Phi_J{}^+|. \end{aligned}$$

It follows that

$$m = \frac{2(|\Phi^+_{J \cup \{\alpha, \beta\}}| - |\Phi_J{}^+|}{|\Phi^+_{J \cup \{\bar{\alpha}\}}| + |\Phi^+_{J \cup \{\bar{\beta}\}}| - 2|\Phi_J{}^+|}. \qquad \blacksquare$$

We now consider a still more special case. Suppose that Δ_J is self-opposed in Δ_K for all K with $J \subseteq K \subseteq I$ and that the character ϕ of L_J is invariant under all automorphisms of L_J. Then we have $W^{J,\phi} = R_J$. For

$$\begin{aligned} W^{J,\phi} &= \{w \in W;\ w(J) = J,\ {}^{w}\phi = \phi\} \\ &= \{w \in C_J;\ {}^{w}\phi = \phi\} \\ &= \{w \in R_J:{}^{w}\phi = \phi\} \\ &= R_J \end{aligned}$$

since conjugation by $w \in R_J$ gives an automorphism of L_J. Thus in this case $W^{J,\phi}$ is a reflection group R_J whose type can be determined simply from 10.10.3.

Finally we specialize still further to the case when $P_J = B$ and $\phi = 1$. Then J is empty and so $W^{J,\phi} = W$. We calculate the numbers p_α in this case for $\alpha \in \Delta$. We recall from 10.5.2, 10.5.3 and 10.5.4 that

$$B_{w_{\bar{\alpha}}}^2 = \xi 1 + \eta B_{w_{\bar{\alpha}}}$$

where $\xi = 1/\text{ind}\ w_{\bar{\alpha}}$ and η is given by

$$\eta 1_M = \frac{1}{\text{ind}\ w_{\bar{\alpha}}} \tilde{\rho}(n_{\bar{\alpha}})^{-1} \sum_{\substack{u, u' \in U_{w_{\bar{\alpha}}} \\ n_{\bar{\alpha}}^{-1} u' n_{\bar{\alpha}} u n_{\bar{\alpha}} \in P_J}} \rho(n_{\bar{\alpha}}^{-1} u' n_{\bar{\alpha}} u n_{\bar{\alpha}}).$$

In our special case $w_{\bar{\alpha}} = s_\alpha$, $\tilde{\rho}(x) = 1$ for all x, $U_{w_{\bar{\alpha}}} = X_\alpha$ and $P_J = B$. Thus we have

$$\eta = \frac{1}{\text{ind}\ s_\alpha} \times \text{ the number of } (u, u') \in X_\alpha \times X_\alpha \text{ such that } u' n_\alpha u \in n_\alpha B n_\alpha^{-1}.$$

Now $u' n_\alpha u \in \langle X_\alpha, X_{-\alpha}, H\rangle = X_\alpha H \cup X_\alpha H n_\alpha X_\alpha$. Also

$$n_\alpha B n_\alpha^{-1} = n_\alpha U H n_\alpha^{-1} = n_\alpha U n_\alpha^{-1} H = \prod_{\substack{\beta > 0 \\ \beta \neq \alpha}} X_\beta . X_{-\alpha} H.$$

Thus

$$n_\alpha B n_\alpha^{-1} \cap \langle X_\alpha, X_{-\alpha}, H\rangle = X_{-\alpha} H.$$

Now the elements $u' n_\alpha u$ for $u, u' \in X_\alpha$ are all distinct, and we must decide which lie in $X_{-\alpha}H$. We have

$$X_{-\alpha} H \cap X_\alpha H = H$$

and so

$$X_{-\alpha} H \cap X_\alpha H n_\alpha X_\alpha = X_{-\alpha} H - H.$$

Now all elements of form vh for $v \in X_{-\alpha}$, $v \neq 1$ and $h \in H$ lie in $X_\alpha H n_\alpha X_\alpha = X_\alpha n_\alpha X_\alpha H$. Moreover for a given $v \neq 1$ there is a unique $h \in H$ such that $vh \in X_\alpha n_\alpha X_\alpha$. It follows that

$$|X_\alpha n_\alpha X_\alpha \cap X_{-\alpha} H| = |X_{-\alpha}| - 1 = \text{ind}\ s_\alpha - 1.$$

Hence

$$\eta = \frac{\text{ind}\ s_\alpha - 1}{\text{ind}\ s_\alpha}.$$

Now the parameter p_α is given by the equation

$$\eta^2 = \frac{(p_\alpha - 1)^2}{p_\alpha \operatorname{ind} s_\alpha}.$$

Thus

$$(p_\alpha - \operatorname{ind} s_\alpha)\left(p_\alpha - \frac{1}{\operatorname{ind} s_\alpha}\right) = 0.$$

Since $p_\alpha \geqslant 1$ we have $p_\alpha = \operatorname{ind} s_\alpha$. Thus we have proved the following proposition.

Proposition 10.10.4. *The algebra* $\operatorname{End}(1_B{}^G)$ *has a basis* $T_w, w \in W$, *with multiplication relations*

$$T_{s_\alpha} T_w = \begin{cases} T_{s_\alpha w} & \text{if } w^{-1}(\alpha) > 0 \\ p_\alpha T_{s_\alpha w} + (p_\alpha - 1) T_w & \text{if } w^{-1}(\alpha) < 0 \end{cases}$$

$$T_w T_{s_\alpha} = \begin{cases} T_{w s_\alpha} & \text{if } w(\alpha) > 0 \\ p_\alpha T_{w s_\alpha} + (p_\alpha - 1) T_w & \text{if } w(\alpha) < 0 \end{cases}$$

where $p_\alpha = \operatorname{ind} s_\alpha$.

Proof. 10.8.5 reduces to this when $P_J = B$ and $\phi = 1$. ∎

The algebra $\operatorname{End}(1_B{}^G)$ is called the Hecke algebra. The multiplication relations for the Hecke algebra are well known, and were derived originally by Iwahori [1] when all p_α are equal to q and by Matsumoto [1] in general. In this chapter we have derived the multiplication relations for the Hecke algebra from a situation which is very much more general. These relations for the Hecke algebra could of course be derived in a more direct manner which would avoid many of the complications which appear in our general situation but which are absent in the Hecke algebra case.

10.11 THE GENERIC ALGEBRA AND ITS SPECIALIZATIONS

We now wish to compare the endomorphism algebra $\mathfrak{E} = \operatorname{End}_G \mathfrak{F}(J, \rho)$ with the group algebra of $W^{J,\phi}$ twisted by the cocycle μ. To do this we introduce the generic algebra. We introduce for each $\alpha \in \Lambda$ an indeterminate t_α over $\mathbb{C}$, assuming that $t_\alpha = t_\beta$ if and only if $\beta = w(\alpha)$ for some $w \in W^{J,\phi}$.

Proposition 10.11.1 *There is a unique associative algebra* $A(t_\alpha)$ *over the* *polynomial ring* $\mathbb{C}[t_\alpha, \alpha \in \Lambda]$ *with* $\mathbb{C}[t_\alpha]$*-basis* a_w, $w \in W^{J,\phi}$, *and multiplication relations*

$$a_w a_{w'} = \mu(w, w')a_{ww'} \qquad w \in W^{J,\phi},\ w' \in C_{J,\phi}$$

$$a_{w'}a_w = \mu(w', w)a_{w'w}$$

$$a_{w_{\bar\alpha}}a_w = \begin{cases} a_{w_{\bar\alpha}w} & \text{if } w^{-1}(\alpha) > 0, \alpha \in \Lambda \\ t_\alpha a_{w_{\bar\alpha}w} + (t_\alpha - 1)a_w & \text{if } w^{-1}(\alpha) < 0 \end{cases}$$

$$a_w a_{w_{\bar\alpha}} = \begin{cases} a_{ww_{\bar\alpha}} & \text{if } w(\alpha) > 0, \alpha \in \Lambda \\ t_\alpha a_{ww_{\bar\alpha}} + (t_\alpha - 1)a_w & \text{if } w(\alpha) < 0. \end{cases}$$

Proof. The idea of constructing algebras of this type is due to Tits. It is clear that if there is an associative algebra over $\mathbb{C}[t_\alpha]$ satisfying the above relations it is uniquely determined by these relations. For the product of any two basis elements can be determined by using these relations. We must therefore show the existence of such an algebra.

We begin with a free $\mathbb{C}[t_\alpha]$-module M with basis a_w, $w \in W^{J,\phi}$. We then define endomorphisms $\rho_{w'}, \rho_{w_{\bar\alpha}}, \lambda_{w'}, \lambda_{w_{\bar\alpha}}$ of M where $w' \in C_{J,\phi}$, $\alpha \in \Lambda$ as follows:

$$\rho_{w'}(a_w) = \mu(w, w')a_{ww'}$$

$$\rho_{w_{\bar\alpha}}(a_w) = \begin{cases} a_{ww_{\bar\alpha}} & \text{if } w(\alpha) > 0 \\ t_\alpha a_{ww_{\bar\alpha}} + (t_\alpha - 1)a_w & \text{if } w(\alpha) < 0 \end{cases}$$

$$\lambda_{w'}(a_w) = \mu(w', w)a_{w'w}$$

$$\lambda_{w_{\bar\alpha}}(a_w) = \begin{cases} a_{w_{\bar\alpha}w} & \text{if } w^{-1}(\alpha) > 0 \\ t_\alpha a_{w_{\bar\alpha}w} + (t_\alpha - 1)a_w & \text{if } w^{-1}(\alpha) < 0. \end{cases}$$

The definition of these endomorphisms is motivated by the actions of right and left multiplication in the algebra whose existence we are trying to prove. Let $\mathscr{R}$ be the algebra of endomorphisms of M generated by $\rho_{w'}$ and $\rho_{w_{\bar\alpha}}$ for all $w' \in C_{J,\phi}$ and all $\alpha \in \Lambda$ and $\mathscr{L}$ be the algebra generated by $\lambda_{w'}$ and $\lambda_{w_{\bar\alpha}}$ for all $w' \in C_{J,\phi}$ and all $\alpha \in \Lambda$.

We shall show that the subalgebras $\mathscr{R}$ and $\mathscr{L}$ of End M centralize one another. Suppose this is done. We consider the maps $\rho: \mathscr{R} \to M$ and $\lambda: \mathscr{L} \to M$ defined by $\rho(r) = r(a_1)$, $\lambda(l) = l(a_1)$ for $r \in \mathscr{R}$, $l \in \mathscr{L}$. We see that ρ and λ are surjective. For given any element $w \in W^{J,\phi}$ we have

$$w = w_{\bar\alpha_1} \ldots w_{\bar\alpha_k} w'$$

with $\alpha_1, \ldots, \alpha_k \in \Lambda$ and $w' \in C_{J,\phi}$ such that the expression $w_{\bar\alpha_1} \ldots w_{\bar\alpha_k}$ in $R_{J,\phi}$ is reduced. We then have

$$\rho_{w'}\rho_{w_{\bar\alpha_k}} \cdots \rho_{w_{\bar\alpha_1}}(a_1) = \mu a_w \qquad \text{for some } \mu \neq 0 \text{ in } \mathbb{C}$$

$$\lambda_{w_{\bar\alpha_1}} \ldots \lambda_{w_{\bar\alpha_k}}\lambda_{w'}(a_1) = \mu a_w \qquad \text{for some } \mu \neq 0 \text{ in } \mathbb{C}.$$

Thus each a_w lies in the image of ρ and λ and ρ, λ are surjective. We observe next that ρ, λ are injective. For let $r \in \mathscr{R}$ satisfy $\rho(r) = 0$. Then $r(a_1) = 0$ and so $lr(a_1) = 0$ for all $l \in \mathscr{L}$. Since $lr = rl$ we have $rl(a_1) = 0$. Since λ is surjective we have $rm = 0$ for all $m \in M$, and so $r = 0$. Thus ρ is injective and a similar argument shows that λ is injective.

We thus have a bijective map $\rho: \mathscr{R} \to M$. Now $\mathscr{R}$ is an algebra whereas M is just a $\mathbb{C}[t_\alpha]$-module. We may therefore use the map ρ to define an algebra structure on M. We define $mm' = \rho(\rho^{-1}(m)\rho^{-1}(m'))$ for $m, m' \in M$. The definition of ρ shows that this algebra satisfies our required relations.

It remains to show that the subalgebras $\mathscr{R}$ and $\mathscr{L}$ of End M centralize one another. Thus we must show

$$\rho_{w'}\lambda_{w''} = \lambda_{w''}\rho_{w'} \qquad \text{for } w', w'' \in C_{J,\phi}$$

$$\rho_{w'}\lambda_{w_\alpha} = \lambda_{w_\alpha}\rho_{w'} \qquad \text{for } w' \in C_{J,\phi}, \alpha \in \Lambda$$

$$\lambda_{w'}\rho_{w_\alpha} = \rho_{w_\alpha}\lambda_{w'}$$

$$\rho_{w_\alpha}\lambda_{w_\beta} = \lambda_{w_\beta}\rho_{w_\alpha} \qquad \text{for } \alpha, \beta \in \Lambda.$$

Let $w \in W^{J,\phi}$. Then

$$\begin{aligned}\rho_{w'}\lambda_{w''}(a_w) &= \rho_{w'}(\mu(w'', w)a_{w''w}) \\ &= \mu(w'', w)\mu(w''w, w')a_{w''ww'} \\ \lambda_{w''}\rho_{w'}(a_w) &= \lambda_{w''}(\mu(w, w')a_{ww'}) \\ &= \mu(w, w')\mu(w'', ww')a_{w''ww'}.\end{aligned}$$

Since $\mu(w'', w)\mu(w''w, w') = \mu(w'', ww')\mu(w, w')$ we see that $\rho_{w'}\lambda_{w''}(a_w) = \lambda_{w''}\rho_{w'}(a_w)$.

The relations $\rho_{w'}\lambda_{w_\alpha}(a_w) = \lambda_{w_\alpha}\rho_{w'}(a_w)$ and $\lambda_{w'}\rho_{w_\alpha}(a_w) = \rho_{w_\alpha}\lambda_{w'}(a_w)$ for $w \in W^{J,\phi}$, $w' \in C_{J,\phi}$, $\alpha \in \Lambda$ may also be checked without difficulty.

Finally we show $\rho_{w_\alpha}\lambda_{w_\beta}(a_w) = \lambda_{w_\beta}\rho_{w_\alpha}(a_w)$ for $\alpha, \beta \in \Lambda$ and $w \in W^{J,\phi}$. Here it is best to consider a number of cases separately. We consider the following possible situations:

(i) $w(\alpha) > 0$, $w^{-1}(\beta) > 0$, $w(\alpha) \neq \beta$
(ii) $w(\alpha) = \beta$
(iii) $w(\alpha) > 0$, $w^{-1}(\beta) < 0$
(iv) $w(\alpha) < 0$, $w^{-1}(\beta) > 0$
(v) $w(\alpha) < 0$, $w^{-1}(\beta) < 0$, $w(\alpha) \neq -\beta$
(vi) $w(\alpha) = -\beta$.

In cases (i), (iii), (iv), (v) the relation $\rho_{w_\alpha}\lambda_{w_\beta}(a_w) = \lambda'_{w_\beta}\rho_{w_\alpha}(a_w)$ follows directly from the definitions. Now suppose we are in case (ii), so that $w(\alpha) = \beta$. Then we have

$$\rho_{w_\alpha}\lambda_{w_\beta}(a_w) = \rho_{w_\alpha}(a_{w_\beta w}) = t_\alpha a_{w_\beta ww_\alpha} + (t_\alpha - 1)a_{w_\beta w}$$

$$\lambda_{w_\beta}\rho_{w_\alpha}(a_w) = \lambda_{w_\beta}(a_{ww_\alpha}) = t_\beta a_{w_\beta ww_\alpha} + (t_\beta - 1)a_{ww_\alpha}.$$

However $w_\beta w = w w_{\bar\alpha}$ since $w(\alpha) = \beta$ and $t_\alpha = t_\beta$ also. Thus we obtain

$$\rho_{w_{\bar\alpha}}\lambda_{w_\beta}(a_w) = \lambda_{w_\beta}\rho_{w_{\bar\alpha}}(a_w).$$

The relation in case (vi), when $w(\alpha) = -\beta$, may be checked in a similar manner and so the theorem is proved. ■

The algebra $A(t_\alpha; \alpha \in \Lambda)$ whose existence has been proved in 10.11.1 is called the generic algebra. An algebra homomorphism $\mathbb{C}[t_\alpha] \xrightarrow{\sigma} \mathbb{C}$ is called a specialization. Given a specialization σ with $\sigma(t_\alpha) = \lambda_\alpha$ we have a corresponding specialized algebra $A(\lambda_\alpha)$. This is an algebra over $\mathbb{C}$ with basis a_w, $w \in W^{J,\phi}$, and multiplication relations

$$a_w a_{w'} = \mu(w, w')a_{ww'} \qquad w \in W^{J,\phi},\ w' \in C_{J,\phi}$$

$$a_{w'} a_w = \mu(w', w)a_{w'w}$$

$$a_{w_{\bar\alpha}} a_w = \begin{cases} a_{w_{\bar\alpha} w} & \text{if } w^{-1}(\alpha) > 0,\ \alpha \in \Lambda \\ \lambda_\alpha a_{w_{\bar\alpha} w} + (\lambda_\alpha - 1)a_w & \text{if } w^{-1}(\alpha) < 0 \end{cases}$$

$$a_w a_{w_{\bar\alpha}} = \begin{cases} a_{w w_{\bar\alpha}} & \text{if } w(\alpha) > 0,\ \alpha \in \Lambda \\ \lambda_\alpha a_{w w_{\bar\alpha}} + (\lambda_\alpha - 1)a_w & \text{if } w(\alpha) < 0. \end{cases}$$

In particular we have $A(p_\alpha) \cong \mathfrak{E}$ and $A(1) \cong (\mathbb{C}W^{J,\phi})_\mu$. For putting $\lambda_\alpha = p_\alpha$ we obtain the relations of 10.8.5 and putting $\lambda_\alpha = 1$ for all α we obtain the relations for the group algebra $(\mathbb{C}W^{J,\phi})_\mu$ of $W^{J,\phi}$ twisted by the cocycle μ. We note that the twisted group algebra $(\mathbb{C}W^{J,\phi})_\mu$ is semisimple. For it is isomorphic to a factor ring of the ordinary group ring $\mathbb{C}X$ of some group X, and $\mathbb{C}X$ is always semisimple (see Passman [1]).

The algebra $A(t_\alpha) \otimes_{\mathbb{C}[t_\alpha]} \mathbb{C}(t_\alpha)$ is a finite-dimensional associative algebra over the field of fractions $\mathbb{C}(t_\alpha)$ of the polynomial ring $\mathbb{C}[t_\alpha]$. We recall from Bourbaki [5] that a finite-dimensional algebra with identity over a field of characteristic 0 is semisimple if and only if its trace-form is nondegenerate, and this will be true if and only if the discriminant of the algebra with respect to a given basis is nonzero. (The discriminant is the determinant of the matrix representing the trace-form with respect to the given basis.)

Now the discriminant of the algebra $A(t_\alpha) \otimes_{\mathbb{C}[t_\alpha]} \mathbb{C}(t_\alpha)$ with respect to its natural basis $\{a_w\}$ is a polynomial in the t_α. Moreover its image under the specialization $t_\alpha \to p_\alpha$ gives the discriminant of the algebra $\mathfrak{E}$ with respect to its basis $\{T_w\}$, and this is nonzero since $\mathfrak{E}$ is semisimple. (Recall that $\mathfrak{E}$ is the algebra of endomorphisms of a semisimple module.) Thus the discriminant of $A(t_\alpha) \otimes_{\mathbb{C}[t_\alpha]} \mathbb{C}(t_\alpha)$ is a nonzero polynomial in the t_α, and so this algebra is semisimple. Moreover it remains semisimple over any extension field.

Proposition 10.11.2. (Tits) *If $A(\lambda_\alpha)$ and $A(\lambda'_\alpha)$ are two specializations of the generic algebra $A(t_\alpha)$ which are both semisimple, with $\lambda_\alpha, \lambda'_\alpha \in \mathbb{C}$, then $A(\lambda_\alpha)$ is isomorphic to $A(\lambda'_\alpha)$.*

Proof. We follow the argument given by Steinberg [15]. Let $R = \mathbb{C}[t_\alpha]$ and

$F = \mathbb{C}(t_\alpha)$ be the field of fractions of R. Let $\bar{F}$ be the algebraic closure of F. Let $\sigma: R \to \mathbb{C}$ be a ring-homomorphism and A_σ be the $\mathbb{C}$-algebra obtained from the R-algebra $A = A(t_\alpha)$ by means of the specialization σ. Let $A_{\bar{F}} = A \otimes_R \bar{F}$. We know that $A_{\bar{F}}$ is semisimple and assume that σ is such that A is semisimple also. Thus both $A_{\bar{F}}$ and A_σ are direct sums of complete matrix algebras over $\bar{F}$, $\mathbb{C}$ respectively. The degrees of the matrix algebras are called the numerical invariants of the semisimple algebras. We shall show that $A_{\bar{F}}$, A_σ have the same numerical invariants. By choosing σ to satisfy firstly $\sigma(t_\alpha) = \lambda_\alpha$ and secondly $\sigma(t_\alpha) = \lambda'_\alpha$ it follows that the $\mathbb{C}$-algebras $A(\lambda_\alpha)$, $A(\lambda'_\alpha)$ have the same numerical invariants and are therefore isomorphic.

We have a basis $\{a_w\}$ of $A_{\bar{F}}$ and we choose indeterminates x_w over $\bar{F}$, one for each basis element. Let $a = \sum_w x_w a_w$ be the 'generic element' of $A_{\bar{F}} \otimes_{\bar{F}} \bar{F}(x_w)$. Let $P(t)$ be the characteristic polynomial of the endomorphism of $A_{\bar{F}} \otimes_{\bar{F}} \bar{F}(x_w)$ given by right multiplication by a. $P(t)$ will factorize into a product

$$P(t) = \prod_i P_i(t)^{d_i}$$

of distinct monic irreducible polynomials $P_i(t)$ over $\bar{F}(x_w)$.

We show that d_i is the degree of $P_i(t)$ and that the numbers d_i are the numerical invariants of the algebra $A_{\bar{F}}$. We use the fact that $A_{\bar{F}}$ is a direct sum of complete matrix algebras over $\bar{F}$ so has a basis of matrix units E^k_{ij}, $(1 \leqslant i, j \leqslant d_k)$, satisfying

$$E^k_{ij} E^{k'}_{i'j'} = \delta_{kk'}\, \delta_{ji'} E^k_{ij'}.$$

Let $a = \sum_{i,j,k} y_{ij}{}^k E_{ij}{}^k$. Then $\bar{F}(x_w) = \bar{F}(y_{ij}{}^k)$ since the matrix giving the change of basis has coefficients in $\bar{F}$. However the characteristic polynomial of a acting by right multiplication, and calculated with respect to the basis of matrix units, is clearly

$$\prod_k \det(t1 - (y_{ij}{}^k))^{d_k}.$$

Also the polynomial $\det(t1 - (y_{ij}{}^k))$ has degree d_k and is irreducible over $\bar{F}(y_{ij}{}^k)$. To see this we observe that the $y_{ij}{}^k$ are algebraically independent over $\bar{F}$ (since the x_w are) and so we may specialize by replacing $(y_{ij}{}^k)$ by an arbitrary $d_k \times d_k$ matrix over $\bar{F}$. However by taking matrices of the form

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ c_1 & c_2 & \cdots & c_{d_k} \end{pmatrix}$$

we can obtain any monic polynomial of degree d_k over $\bar{F}$. Thus $\det(t1 - (y_{ij}{}^k))$ is irreducible over $\bar{F}(y_{ij}{}^k) = \bar{F}(x_w)$.

Hence

$$P(t) = \prod_i P_i(t)^{d_i}$$

where d_i is the degree of $P_i(t)$ and the d_i are the numerical invariants of A_F.

We consider the coefficients of $P_i(t)$. These lie in $\bar{F}(x_w)$ and also in the field generated by the roots of $P_i(t)$. However each root of $P_i(t)$ is a root of $P(t)$, so is integral over the field generated by the coefficients of $P(t)$. Now all coefficients of $P(t)$ lie in $R[x_w]$. Thus the coefficients of $P_i(t)$ lie in the integral closure of $R[x_{w'}]$ in $\bar{F}(x_w)$.

Let I be the integral closure of R in $\bar{F}$. Then $I[x_{w'}]$ is the integral closure of $R[x_{w'}]$ in $\bar{F}(x_w)$ by Bourbaki [6], chapter V. Thus the coefficients of $P_i(t)$ lie in $I[x_w]$.

We next observe that our homomorphism $\sigma: R \to \mathbb{C}$ can be extended to a homomorphism $\sigma: I \to \mathbb{C}$. By Zorn's lemma it is sufficient to adjoin one element at a time which will be integral over R. Since $\mathbb{C}$ is algebraically closed we can choose the image of this element to be a root of the appropriate equation over $\mathbb{C}$.

Consider the specialized algebra A_σ over $\mathbb{C}$ and the 'generic element' $\sigma(a) = \sum_{w'} x_{w'} a_{w'}$ in $A_\sigma \otimes_{\mathbb{C}} \mathbb{C}(x_w)$. Let $P_\sigma(t)$ be the characteristic polynomial of $\sigma(a)$ acting by right multiplication in $A_\sigma \otimes_{\mathbb{C}} \mathbb{C}(x_w)$. The coefficients of $P_\sigma(t)$ will lie in $\mathbb{C}[x_w]$ and $P_\sigma(t)$ is the specialization of $P(t) \in R[x_w]$. Now $P(t)$ factorizes as

$$P(t) = \prod_i P_i(t)^{d_i}$$

in $I[x_w]$ and so $P_\sigma(t)$ factorizes as

$$P_\sigma(t) = \sum_i P_{i,\sigma}(t)^{d_i}$$

in $\mathbb{C}[x_{w'}]$ where $P_{i,\sigma}(t)$ is the specialization of $P_i(t)$. We have $d_i = \deg P_{i,\sigma}(t)$.

We now claim that the numbers d_i are the numerical invariants of A_σ. This is certainly true if the $P_{i,\sigma}(t)$ are irreducible in $\mathbb{C}[x_w]$ and all distinct, as we have seen above by the argument using matrix units. However if some $P_{i,\sigma}(t)$ were not irreducible or if $P_{i,\sigma}(t) = P_{j,\sigma}(t)$ for some $i \neq j$ then $P_\sigma(t)$ would have an irreducible factor over $\mathbb{C}[x_w]$ of multiplicity greater than its degree, which again contradicts what we have seen above. Thus A_F, A_σ have the same numerical invariants and the proof is complete.

Corollary 10.11.3. *$\mathfrak{E}$ is isomorphic to $(\mathbb{C}W^{J,\phi})_\mu$.*

Proof. $\mathfrak{E} \cong A(p_\alpha)$ and $(\mathbb{C}W^{J,\phi})_\mu \cong A(1)$. Both algebras are semisimple, and so the result follows from 10.11.2. ■

We next derive a connection between the irreducible characters of the generic algebra and the irreducible characters of its specializations. As before $F = \mathbb{C}(t_\alpha)$ is the field of fractions of $R = \mathbb{C}[t_\alpha]$ and $A_F = A(t_\alpha) \otimes_R \bar{F}$. The following result gives a bijective correspondence between irreducible characters of A_F and those of a specialization $A(\lambda_\alpha)$.

Proposition 10.11.4. *Let $\sigma:\mathbb{C}[t_\alpha] \to \mathbb{C}$ be a specialization and let $\sigma(t_\alpha) = \lambda_\alpha$ for $\alpha \in \Lambda$. Suppose the specialized algebra $A(\lambda_\alpha)$ is semisimple. Let ψ be the character of an irreducible representation of A_F. Then $\psi(a_w)$ lies in the integral closure I of $\mathbb{C}[t_\alpha]$ in $\bar{F}$. Moreover the ring homomorphism $\sigma:\mathbb{C}[t_\alpha] \to \mathbb{C}$ can be extended to a ring homomorphism $\sigma: I \to \mathbb{C}$. The linear map $\psi^\sigma: A(\lambda_\sigma) \to \mathbb{C}$ given by $a_w \to \sigma(\psi(a_w))$ is then the character of an irreducible representation of $A(\lambda_\alpha)$. Moreover this gives a bijective correspondence between irreducible characters of A_F and $A(\lambda_\alpha)$.*

Proof. We use the description of A_F as a direct sum of complete matrix algebras given in the proof of 10.11.2. A_F has a basis of matrix units E_{ij}^k. Let $a_w = \sum_{i,j,k} c_{ij}{}^k E_{ij}{}^k$ with $c_{ij}{}^k \in \bar{F}$. Then the irreducible representations of A_F are given by $a_w \overset{\rho_k}{\to} (c_{ij}{}^k)$ for the various values of k. The character ψ_k of ρ_k is given by

$$\psi_k(a_w) = \sum_i c_{ii}^k.$$

Now ρ_k extends to a representation of $A_F \otimes_F \bar{F}(x_w)$ which satisfies

$$\psi_k(\sum x_w a_w) = \sum_w x_w \sum_i c_{ii}^k.$$

The proof of 10.11.2 shows that the characteristic polynomial of $\sum x_w a_w$ on this irreducible module is $P_k(t) \in I[x_w][t]$. The character of this element is the negative of one of the coefficients of this polynomial. Thus we have

$$\psi_k(\sum x_w a_w) \in I[x_w].$$

Let $\theta_w:\bar{F}[x_w] \to \bar{F}$ be the ring homomorphism satisfying $\theta_w(x_w) = 1$, $\theta_w(x_{w'}) = 0$ if $w' \neq w$. Then

$$\psi_k(a_w) = \theta_w(\psi_k(\sum x_w a_w)) \in I.$$

Thus $\psi_k(a_w)$ lies in I as required.

We have already seen in 10.11.2 that each ring homomorphism $\sigma:\mathbb{C}[t_\alpha] \to \mathbb{C}$ can be extended to a ring homomorphism $\sigma: I \to \mathbb{C}$. We know from the proof of 10.11.2 that $P_{k,\sigma}(t) \in \mathbb{C}[x_w][t]$ is the characteristic polynomial of $\sum x_w a_w$ on one of the irreducible modules of $A_\sigma \otimes_{\mathbb{C}} \mathbb{C}(x_w)$. The trace of $\sum x_w a_w$ on this module is therefore $\sigma(\psi_k(\sum x_w a_w)) \in \mathbb{C}[x_w]$. The trace of a_w on this module is obtained from this by applying $\theta_w:\mathbb{C}[x_w] \to \mathbb{C}$. However the operators θ_w, σ commute and so the trace of a_w on this module is $\sigma(\psi_k(a_w))$. Moreover we know from the proof of 10.11.2 that all irreducible characters of A_σ have this form, so the proof is complete. ∎

We now apply these ideas in the situation in which $C_{J,\phi} = 1$, so that $W^{J,\phi} = R_{J,\phi}$ is a reflection group. Let G' be a group with split BN-pair (B', N') with Weyl group $W^{J,\phi}$. Then the endomorphism algebras $\mathrm{End}(\phi_{P_J}{}^G)$ and $\mathrm{End}(1_{B'}{}^{G'})$ are specializations of the same generic algebra. This generic algebra has basis a_w, $w \in W^{J,\phi}$, over $\mathbb{C}[t_\alpha; \alpha \in \Lambda]$ with multiplication relations

$$a_{w_{\bar{\alpha}}} a_w = \begin{cases} a_{w_{\bar{\alpha}} w} & \text{if } w^{-1}(\alpha) > 0 \\ t_\alpha a_{w_{\bar{\alpha}} w} + (t_\alpha - 1) a_w & \text{if } w^{-1}(\alpha) < 0 \end{cases} \qquad \alpha \in \Lambda$$

$$a_w a_{w_{\bar{\alpha}}} = \begin{cases} a_{w w_{\bar{\alpha}}} & \text{if } w(\alpha) > 0 \\ t_\alpha a_{w w_{\bar{\alpha}}} + (t_\alpha - 1) a_w & \text{if } w(\alpha) < 0 \end{cases} \qquad \alpha \in \Lambda.$$

In this situation the generic algebra has three specializations

$$\sigma: t_\alpha \to p_\alpha$$

$$\sigma': t_\alpha \to \operatorname{ind} s_\alpha$$

$$\sigma'': t_\alpha \to 1$$

which lead to specialized algebras

$$A(p_\alpha) \cong \operatorname{End}(\phi_{P_J}{}^G)$$

$$A(\operatorname{ind} s_\alpha) \cong \operatorname{End}(1_{B'}{}^{G'})$$

$$A(1) \cong \mathbb{C} W^{J,\phi}.$$

Thus the irreducible characters of these three algebras are in natural bijection.

Let ψ be an irreducible character of $A_{\bar{F}}$. Let χ_ψ be the irreducible component of $\phi_{P_J}{}^G$ corresponding to the specialized character ψ^σ of $\operatorname{End}(\phi_{P_J}{}^G)$ as in 10.1.2. Let χ'_ψ be the irreducible component of $1_{B'}{}^{G'}$ corresponding to the specialized character $\psi^{\sigma'}$ of $\operatorname{End}(1_{B'}{}^{G'})$ as in 10.1.2. We wish to relate the degree of χ_ψ to the degree of χ'_ψ. By 10.9.6 we have

$$\deg \chi_\psi = \frac{\dim \mathfrak{F}(J, \rho) \deg \psi}{\sum_{w \in W^{J,\phi}} \psi^\sigma(T_w) \psi^\sigma\left(\dfrac{1}{p_w} T_{w^{-1}}\right)}$$

$$= \frac{|G:P_J| . \deg \phi . \deg \psi}{\sum_{w \in W^{J,\phi}} \dfrac{1}{p_w} \psi^\sigma(T_w) \psi^\sigma(T_{w^{-1}})}.$$

Similarly we have

$$\deg \chi'_\psi = \frac{|G':B'| . \deg \psi}{\sum_{w \in W^{J,\phi}} \dfrac{1}{\operatorname{ind} w} \psi^{\sigma'}(T_w) \psi^{\sigma'}(T_{w^{-1}})}.$$

Thus $\deg \chi'_\psi$ is obtained by specializing the element $d_\psi \in \bar{F}$ given by

$$d_\psi = \frac{(\sum_{w \in W^{J,\phi}} t_w) \deg \psi}{\sum_{w \in W^{J,\phi}} \dfrac{1}{t_w} \psi(a_w) \psi(a_{w^{-1}})}$$

where $t_w = \prod_{\alpha_i} t_{\alpha_i}$ over all α_i occurring in a reduced expression $w = w_{\bar{\alpha}_1} w_{\bar{\alpha}_2} \dots w_{\bar{\alpha}_k}$

of w. This is because $|G':B'| = \sum_{w \in W^{J.\phi}} \operatorname{ind} w$. The element $d_\psi \in \bar{F}$ is called the generic degree of the character ψ of A_F. All the generic degrees are known explicitly—we have one for each irreducible character of the reflection group $W^{J.\phi}$.

We now show how deg χ_ψ can be obtained in terms of the generic degree d_ψ.

Theorem 10.11.5. *Suppose ϕ is an irreducible cuspidal character of L_J which has the property that $W^{J.\phi} = R_{J.\phi}$ (i.e. $C_{J.\phi} = 1$). Let ψ be an irreducible character of A_F with generic degree $d_\psi \in \bar{F}$. Let $\sigma : I \to \mathbb{C}$ be a specialization as in* 10.11.4 *with $\sigma(t_\alpha) = p_\alpha$. Then the degree of the irreducible component χ_ψ of $\phi_{P_J}{}^G$ is given by*

$$deg\ \chi_\psi = \frac{|G:P_J| . \deg \phi . \sigma(d_\psi)}{\sum_{w \in W^{J.\phi}} p_w}.$$

Proof. We have

$$\deg \chi_\psi = \frac{|G:P_J| . \deg \phi . \deg \psi}{\sum_{w \in W^{J.\phi}} \frac{1}{p_w} \psi^\sigma(T_w)\psi^\sigma(T_{w^{-1}})}.$$

and

$$\sigma(d_\psi) = \frac{(\sum_{w \in W^{J.\phi}} p_w) \deg \psi}{\sum_{w \in W^{J.\phi}} \frac{1}{p_w} \psi^\sigma(T_w)\psi^\sigma(T_{w^{-1}})}.$$

The result follows. ∎

Theorem 10.11.5 gives a most useful way of calculating the degrees of irreducible characters of G once the degrees of irreducible cuspidal characters of the Levi subgroups L_J are known.

A slightly more complicated formula for deg χ_ψ has been obtained by Howlett which is valid even if $C_{J.\phi} \neq 1$.

Chapter 11

REPRESENTATIONS OF FINITE COXETER GROUPS

In the present chapter we shall introduce certain ideas on representations of Coxeter groups which will be needed in our subsequent discussion on the unipotent representations of the groups G^F.

11.1 MULTIPLICITIES IN THE REGULAR REPRESENTATION

We have seen in section 2.4 how one can construct a graded $\mathbb{R}W$-module U which affords the regular representation of W. Let

$$U = \bigoplus_i U_i$$

be the decomposition of U into its graded components. Each irreducible character ϕ of W appears as component in the regular character with multiplicity equal to its degree $\phi(1)$. Let $n_i(\phi)$ be the multiplicity with which ϕ occurs in the W-module U_i. It will be convenient to consider all the $n_i(\phi)$ simultaneously by forming the polynomial

$$P_U^{\phi}(t) = \sum_i n_i(\phi)t^i.$$

We next establish a formula which is useful for calculating this polynomial.

Proposition 11.1.1.

$$P_U^{\phi}(t) = \prod_{i=1}^{l} (1 - t^{d_i}).\frac{1}{|W|} \sum_{w \in W} \frac{\phi(w)}{\det(1 - tw)}$$

where $d_1, \ldots, d_l$ are the degrees of the basic polynomial invariants of W.

Proof. We shall consider the relevant vector spaces over $\mathbb{C}$ rather than over $\mathbb{R}$ in this proof, to ensure that all the eigenvalues of an element $w \in W$ lie in the space.

Let V be a module affording the natural representation of W and $\mathfrak{P}$ be the algebra of polynomial functions on V. Let $w \in W$ and suppose the eigenvalues of w on V are $\lambda_1, \ldots, \lambda_l$. Then the eigenvalues of w on the homogeneous component $\mathfrak{P}_n$ of $\mathfrak{P}$ have the form $\lambda_1^{k_1} \ldots \lambda_l^{k_l}$ for all sets $k_1, \ldots, k_l$ of non-negative integers for which $k_1 + \ldots + k_l = n$. Now $1 - tw$ acts on V with eigenvalues $1 - \lambda_1 t$, $1 - \lambda_2 t, \ldots, 1 - \lambda_l t$ and so

$$\det(1 - tw) = (1 - \lambda_1 t) \ldots (1 - \lambda_l t).$$

Thus

$$\begin{aligned}\frac{1}{\det(1 - tw)} &= \prod_{i=1}^{l} \frac{1}{(1 - \lambda_i t)} = \prod_{i=1}^{l} (1 + \lambda_i t + \lambda_i^2 t^2 + \ldots) \\ &= \sum_{n \geq 0} \Bigg(\sum_{\substack{k_1, \ldots, k_l \\ k_1 + \ldots + k_l = n}} \lambda_1^{k_1} \ldots \lambda_l^{k_l} \Bigg) t^n \\ &= \sum_{n \geq 0} (\operatorname{trace}_{\mathfrak{P}_n} w) t^n.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{|W|} \sum_{w \in W} \frac{\phi(w)}{\det(1 - tw)} &= \frac{1}{|W|} \sum_{w \in W} \phi(w) \sum_{n \geq 0} (\operatorname{trace}_{\mathfrak{P}_n} w) t^n \\ &= \sum_{n \geq 0} \left(\frac{1}{|W|} \sum_{w \in W} \phi(w) \operatorname{trace}_{\mathfrak{P}_n} w \right) t^n \\ &= \sum_{n \geq 0} (\phi, \phi_{\mathfrak{P}_n}) t^n\end{aligned}$$

where $\phi_{\mathfrak{P}_n}$ is the character of W on $\mathfrak{P}_n$.

Let $\mathfrak{J}$ be the set of W-invariants in $\mathfrak{P}$. We recall from section 2.4 that U and $\mathfrak{J}$ are graded W-modules and that $U \otimes \mathfrak{J}$ is W-isomorphic to $\mathfrak{P}$. In particular we have

$$\sum_{\substack{k, l \\ k + l = n}} (U_k \otimes \mathfrak{J}_l) \cong \mathfrak{P}_n.$$

Now

$$U_k \otimes \mathfrak{J}_l \cong U_k \oplus \ldots \oplus U_k$$

as a W-module, where the right-hand side has $\dim \mathfrak{J}_l$ terms. Let ϕ_{U_k} be the character of W on U_k. Then we have

$$\phi_{\mathfrak{P}_n} = \sum_{\substack{k, l \\ k + l = n}} \dim \mathfrak{J}_l \,.\, \phi_{U_k}.$$

Thus

$$(\phi, \phi_{\mathfrak{P}_n}) = \sum_{\substack{k, l \\ k + l = n}} (\phi, \phi_{U_k}) \dim \mathfrak{J}_l.$$

Now we have

$$P_{\mathfrak{V}}^{\phi}(t) = \sum_{k \geqslant 0} (\phi, \phi_{V_k}) t^k$$

and we define $P_{\mathfrak{P}}^{\phi}(t)$, $P_{\mathfrak{J}}(t)$ by

$$P_{\mathfrak{P}}^{\phi}(t) = \sum_{n \geqslant 0} (\phi, \phi_{\mathfrak{P}_n}) t^n$$

$$P_{\mathfrak{J}}(t) = \sum_{l \geqslant 0} \dim \mathfrak{J}_l . t^l.$$

The above formula shows that $P_{\mathfrak{P}}^{\phi}(t) = P_{\mathfrak{V}}^{\phi}(t) P_{\mathfrak{J}}(t)$.

We have also shown above that

$$P_{\mathfrak{P}}^{\phi}(t) = \frac{1}{|W|} \sum_{w \in W} \frac{\phi(w)}{\det(1 - tw)}.$$

Thus the required result will follow once we know that

$$P_{\mathfrak{J}}(t) = \prod_{i=1}^{l} \frac{1}{(1 - t^{d_i})}.$$

This is, however, a well known result. In order to prove it we let $I_1, \ldots, I_l$ be polynomial generators of $\mathfrak{J}$ of degrees $d_1, \ldots, d_l$ respectively. (Recall from 2.4.1 that $\mathfrak{J}$ is isomorphic to a polynomial ring in l variables.) Then the homogeneous component $\mathfrak{J}_n$ has as basis all polynomials $I_1^{e_1} \ldots I_l^{e_l}$ for which $\sum_{i \geqslant 1} d_i e_i = n$. The number of such polynomials is the coefficient of t^n in the power series

$$(1 + t^{d_1} + t^{2d_1} + \ldots)(1 + t^{d_2} + t^{2d_2} + \ldots) \ldots (1 + t^{d_l} + t^{2d_l} + \ldots) = \prod_{i=1}^{l} \frac{1}{(1 - t^{d_i})}.$$

Hence

$$P_{\mathfrak{J}}(t) = \sum_{i=1}^{l} \frac{1}{(1 - t^{d_i})}$$

and our result follows. ■

For any irreducible character ϕ of W we shall subsequently write $P_\phi(t) = P_{\mathfrak{V}}^{\phi}(t)$.

Proposition 11.1.2. *Let ε be the sign character of W. Then, given any irreducible character ϕ of W, $\varepsilon\phi$ is also an irreducible character and*

$$P_{\varepsilon\phi}(t) = t^N P_\phi(t^{-1}).$$

Proof. It is clear that if ϕ is irreducible so is $\varepsilon\phi$. In order to evaluate $P_{\varepsilon\phi}(t)$ we use 11.1.1. We have

$$P_\phi(t) = \frac{1}{|W|} \sum_{w \in W} \phi(w) \frac{\prod_{i=1}^{l} (1 - t^{d_i})}{\det(1 - tw)}.$$

It follows that

$$P_\phi(t^{-1}) = \frac{1}{|W|} \sum_{w \in W} \phi(w) \frac{\prod_{i=1}^{l} (1 - t^{-d_i})}{\det(1 - t^{-1}w)}$$

$$= \frac{1}{|W|} \sum_{w \in W} \phi(w) \frac{\prod_{i=1}^{l} (t^{d_i} - 1) . \det(t1)}{\det(t1 - w) . t^{\Sigma d_i}}.$$

Now we have $\det(t1) = t^l$ and $d_1 + \ldots + d_l = N + l$ by 2.4.1. Thus

$$t^N P_\phi(t^{-1}) = \frac{1}{|W|} \sum_{w \in W} \frac{\phi(w) \prod_{i=1}^{l} (t^{d_i} - 1)}{\det(t1 - w)}$$

$$= \frac{1}{|W|} \sum_{w \in W} \frac{\phi(w) \prod_{i=1}^{l} (1 - t^{d_i})}{\det(w - t1)}$$

$$= \frac{1}{|W|} \sum_{w \in W} \frac{\phi(w) \prod_{i=1}^{l} (1 - t^{d_i})}{\det(w^{-1} - t1)}$$

since w, w^{-1} have the same characteristic polynomial on V

$$= \frac{1}{|W|} \sum_{w \in W} \frac{\phi(w) \prod_{i=1}^{l} (1 - t^{d_i})}{\det w^{-1} \det(1 - tw)}$$

$$= \frac{1}{|W|} \sum_{w \in W} \frac{\phi(w)\varepsilon(w) \prod_{i=1}^{l} (1 - t^{d_i})}{\det(1 - tw)} = P_{\varepsilon\phi}(t).$$

11.2 THE j-OPERATION

Let W be a finite Coxeter group and V a vector space over $\mathbb{R}$ on which W acts as a group generated by reflections. Let W' be a subgroup of W which is generated by reflections and let $V^{W'}$ be the subspace of V given by

$$V^{W'} = \{v \in V;\ w'(v) = v \text{ for all } w' \in W'\}.$$

$V^{W'}$ is a W'-submodule of V and so there is a decomposition $V = V' \oplus V^{W'}$ where V' is a W'-module which has no nonzero W'-invariants.

Theorem 11.2.1. (Macdonald, Lusztig and Spaltenstein) *Let $\mathfrak{P}_e(V')$ be the space of homogeneous polynomial functions on V' of degree e. Let U' be an absolutely irreducible W'-submodule of $\mathfrak{P}_e(V')$ which occurs with multiplicity* 1 *in $\mathfrak{P}_e(V')$ and which does not occur in $\mathfrak{P}_i(V')$ if $0 \leqslant i < e$. We may regard U' as a subspace of $\mathfrak{P}_e(V)$ and consider the W-submodule U of $\mathfrak{P}_e(V)$ generated by U'. Then:*

(i) *U is an irreducible W-module.*
(ii) *U occurs with multiplicity* 1 *in $\mathfrak{P}_e(V)$.*
(iii) *U does not occur in $\mathfrak{P}_i(V)$ if $0 \leqslant i < e$.*

Proof. Let $\phi: U \to \mathfrak{P}_i(V)$ be a homomorphism of W-modules and $\phi': U' \to \mathfrak{P}_i(V)$ be the restriction of ϕ to U'. Then ϕ' is a homomorphism of W'-modules. Now as a W'-module $\mathfrak{P}_i(V)$ is isomorphic to

$$\bigotimes_{0 \leqslant j \leqslant i} (\mathfrak{P}_j(V') \otimes \mathfrak{P}_{i-j}(V^{W'})).$$

Suppose first that $i < e$. Then U' does not occur in $\mathfrak{P}_j(V')$ for $0 \leqslant j \leqslant i$. Since W' acts trivially on $\mathfrak{P}_{i-j}(V^{W'})$, U' does not occur in $\mathfrak{P}_j(V') \otimes \mathfrak{P}_{i-j}(V^{W'})$ for $0 \leqslant j \leqslant i$. Hence U' does not occur in $\mathfrak{P}_i(V)$ and so $\phi' = 0$. But ϕ is a homomorphism of W-modules and U' generates U as a W-module. Hence $\phi = 0$.

Suppose now that $i = e$. Then U' does not occur in $\mathfrak{P}_j(V') \otimes \mathfrak{P}_{i-j}(V^{W'})$ for $0 \leqslant j < e$ and so ϕ' maps U' into $\mathfrak{P}_e(V') \otimes \mathfrak{P}_0(V^{W'}) = \mathfrak{P}_e(V')$. But U' occurs with multiplicity 1 in $\mathfrak{P}_e(V')$ and so ϕ' must be a scalar multiple of the identity. Since ϕ is a homomorphism of W-modules and U' generates U as a W-module ϕ must also be a scalar multiple of the identity. Thus U must be irreducible, and occurs with multiplicity one in $\mathfrak{P}_e(V)$. ■

We note that an irreducible character ϕ of W occurs with multiplicity 1 in $\mathfrak{P}_e(V)$ and does not occur in $\mathfrak{P}_i(V)$ for $i < e$ if and only if the polynomial $P_{\phi}^{\mathfrak{P}}(t)$ has the form

$$P_{\phi}^{\mathfrak{P}}(t) = t^e + \text{terms involving higher powers of } t.$$

If U is the W-module obtained from the W'-module U' by the method of 11.2.1 we write $U = j_{W'}{}^{W}(U')$, or simply $U = j(U')$. We shall now describe a special case of this operation, discovered by Macdonald.

Proposition 11.2.2. *For each root $\alpha \in \Phi$ let H_α be the element in the dual space of V given by $H_\alpha(v) = (\alpha, v)$. Let f be the element of $\mathfrak{P}_N(V)$ given by $f = \prod_{\alpha \in \Phi^+} H_\alpha$, where $N = |\Phi^+|$. Then $U = \mathbb{R}f$ is a W-submodule of $\mathfrak{P}_N(V)$. It occurs with multiplicity 1 in $\mathfrak{P}_N(V)$ and does not occur in $\mathfrak{P}_i(V)$ if $i < N$. U is a W-module which affords the sign representation $w \to \det w$.*

Proof. We first observe that $w_\beta(H_\alpha) = H_{w_\beta(\alpha)}$ for all $\beta \in \Phi$. For we have

$$(w_\beta(H_\alpha))v = H_\alpha(w_\beta v) = (\alpha, w_\beta v) = (w_\beta(\alpha), v) = H_{w_\beta(\alpha)}(v).$$

If $\beta \in \Delta$ is a simple root it follows that $w_\beta(f) = -f$. For $w_\beta = s_\beta$ transforms β to $-\beta$ and all other positive roots to positive roots. Thus for all $w \in W$ we have $w(f) = \det w . f$. Hence $U = \mathbb{R}f$ is a W-module which affords the sign representation.

Now a polynomial function p which satisfies $wp = \det w . p$ for all $w \in W$ must be the product of f with a W-invariant polynomial (see, for example, Carter [3], lemma 9.4.6). Thus there can be no such polynomial $p \in \mathfrak{P}_i(V)$ for $i < N$, and the only such polynomials $p \in \mathfrak{P}_N(V)$ are those in $\mathbb{R}f$. Hence the sign representation occurs with multiplicity 1 in $\mathfrak{P}_N(V)$ and does not occur in $\mathfrak{P}_i(V)$ for $i < N$. ■

We thus have a situation in which 11.2.1 can be applied.

Proposition 11.2.3. (Macdonald) *Let W be a finite Coxeter group acting as a reflection group on V and let W' be a reflection subgroup of W. Let V' be a W'-submodule of V such that $V = V' \oplus V^{W'}$. Let U' be the W'-submodule of $\mathfrak{P}_{N'}(V')$ giving the sign representation of W', where N' is the number of positive roots of W'. Then the W-submodule U of $\mathfrak{P}_{N'}(V)$ generated by U' is irreducible, occurs with multiplicity 1 in $\mathfrak{P}_{N'}(V)$, and does not occur in $\mathfrak{P}_i(V)$ if $i < N'$.*

The irreducible representations obtained by the method of 11.2.3 are called the Macdonald representations of W. These are the representations $U = j(U')$ where U' affords the sign representation of some reflection subgroup of W.

Our next result shows that the j-operation is transitive.

Proposition 11.2.4. *Let W be a finite Coxeter group acting as a reflection group on V. Let $W \supseteq W' \supseteq W''$ where W', W'' are subgroups of W generated by reflections. Let*

$$V = V' \oplus V^{W'} \text{ where } V' \text{ is a } W'\text{-submodule}$$

$$V' = V'' \oplus V'^{W''} \text{ where } V'' \text{ is a } W''\text{-submodule.}$$

Then $V = V'' \oplus V^{W''}$. Let U'' be an irreducible W''-submodule of $\mathfrak{P}_e(V'')$ which occurs with multiplicity 1 in $\mathfrak{P}_e(V'')$ and does not occur in $\mathfrak{P}_i(V'')$ if $i < e$. Then

$$j_{W'}{}^{W}(j_{W''}{}^{W'}(U'')) = j_{W''}{}^{W}(U'').$$

Proof. We have

$$U'' \subseteq \mathfrak{P}_e(V'') \subseteq \mathfrak{P}_e(V') \subseteq \mathfrak{P}_e(V).$$

$j_{W''}{}^{W'}(U'')$ is the W'-submodule of $\mathfrak{P}_e(V')$ generated by U''. $j_{W'}{}^{W}(j_{W''}{}^{W'}(U''))$ is the W-submodule of $\mathfrak{P}_e(V)$ generated by the W'-submodule of $\mathfrak{P}_e(V')$ generated by U'', which is the W-submodule of $\mathfrak{P}_e(V)$ generated by U'', viz. $j_{W''}{}^{W}(U'')$.

Proposition 11.2.5. *Let U' be an irreducible W'-submodule of $\mathfrak{P}_e(V')$ which occurs with multiplicity 1 in $\mathfrak{P}_e(V')$ and does not occur in $\mathfrak{P}_i(V')$ if $i < e$. Let $U = j(U')$. Then:*

(i) *U occurs as component of the induced module $U' \otimes_{\mathbb{R}W'} \mathbb{R}W$ with multiplicity one.*

(ii) *All irreducible components of the induced module which are not isomorphic to U do not occur in $\mathfrak{P}_i(V)$ if $0 \leqslant i \leqslant e$.*

Proof. We see that U occurs as component of the induced module by Frobenius reciprocity, since U' is a W'-submodule of U.

Now let M be an irreducible W-submodule of the induced module, and suppose M is isomorphic to a submodule of $\mathfrak{P}_i(V)$. Now $V = V' \oplus V^{W'}$ and we consider $\mathfrak{P}_i(V)$ as a W'-module. Since W' acts trivially on $V^{W'}$ we see that $\mathfrak{P}_i(V)$ decomposes into the direct sum of $\mathfrak{P}_i(V')$ with other submodules isomorphic to

$\mathfrak{P}_j(V')$ with $j < i$. Now M contains a W'-submodule isomorphic to U', by Frobenius reciprocity. Since U' does not appear in $\mathfrak{P}_i(V')$ for $0 \leqslant i < e$ we must have $i \geqslant e$.

Now suppose that $i = e$. Then M is an irreducible W-submodule of $\mathfrak{P}_e(V)$ which contains a W'-submodule equal to U'. Hence M is equal to U.

Thus if M is not isomorphic to U M cannot occur in $\mathfrak{P}_i(V)$ when $0 \leqslant i \leqslant e$. ■

This proposition gives an alternative way of describing the j-operation. $j(U)$ is determined up to isomorphism as the irreducible component of the induced module $U' \otimes_{\mathbb{R}W'} \mathbb{R}W$ which occurs in $\mathfrak{P}_e(V)$ for e as small as possible. For this reason the operation j has been called truncated induction.

11.3 GENERIC DEGREES, FAKE DEGREES AND SPECIAL REPRESENTATIONS

In the present section W will be a Weyl group. We will associate with each irreducible character ϕ of W a polynomial $\tilde{P}_\phi(t) \in \mathbb{Q}[t]$. This polynomial is related to the generic degree considered in section 10.11. We recall from section 10.11 that for each irreducible character ϕ of W there is an element $d_\phi \in \bar{F}$ where F is the field of fractions of $\mathbb{C}[t_\alpha: \alpha \in \Delta]$. Here we have introduced one indeterminate t_α for each simple root $\alpha \in \Delta$, subject to the condition that $t_\alpha = t_\beta$ if there exists $w \in W$ with $w(\alpha) = \beta$. The significance of the element d_ϕ is as follows. If G' is a finite group with split BN-pair (B', N') with Weyl group W there is a bijective correspondence between irreducible components of $1_{B'}{}^{G'}$ and irreducible characters of W. Let χ'_ϕ be the irreducible character of G' corresponding to the character ϕ of W. The group G' has certain parameters $p_{\alpha,\alpha} \in \Delta$, such that $p_\alpha = p_\beta$ whenever there exists $w \in W$ with $w(\alpha) = \beta$. Then the degree of χ'_ϕ can be obtained from the element d_ϕ by replacing t_α by p_α whenever it occurs.

We now consider a special case of this situation. Suppose G' is a finite Chevalley group over the field of q elements. Then G' has the form G^F where F is a Frobenius map, and G^F has a split BN-pair (B^F, N^F) such that for each $\alpha \in \Delta$ the parameter p_α is equal to q. Let $\tilde{P}_\phi(t)$ be the element obtained from d_ϕ by replacing each indeterminate t_α by the indeterminate t. Then $\tilde{P}_\phi(t)$ is a function of one variable t which, when t is replaced by any prime power q, gives the degree of the irreducible component χ_ϕ of $1_{B^F}{}^{G^F}$ corresponding to the character ϕ of W. The functions $\tilde{P}_\phi(t)$ are known for all irreducible characters of all Weyl groups and turn out in all cases to be polynomials with rational coefficients. (This is not, however, true of the elements d_ϕ themselves.) The polynomial $\tilde{P}_\phi(t) \in \mathbb{Q}[t]$ will be called the generic degree of ϕ.

The endomorphism algebra $\mathrm{End}(1_{B^F}{}^{G^F})$ when G^F is a finite Chevalley group is called the Hecke algebra. One can give an abstract description of the Hecke algebra by taking a particular case of the more general results obtained in chapter 10. The Hecke algebra has a basis T_w, $w \in W$, over $\mathbb{C}$ and satisfies the multiplication relations

$$T_{s_\alpha}T_w = \begin{cases} T_{s_\alpha w} & \text{if } w^{-1}(\alpha) > 0 \\ qT_{s_\alpha w} + (q-1)T_w & \text{if } w^{-1}(\alpha) < 0 \end{cases}$$

$$T_wT_{s_\alpha} = \begin{cases} T_{ws_\alpha} & \text{if } w(\alpha) > 0 \\ qT_{ws_\alpha} + (q-1)T_w & \text{if } w(\alpha) < 0. \end{cases}$$

It is a specialization of the generic algebra in one variable t which has basis a_w, $w \in W$, over $\mathbb{C}[t]$ and multiplication relations

$$a_{s_\alpha}a_w = \begin{cases} a_{s_\alpha w} & \text{if } w^{-1}(\alpha) > 0 \\ ta_{s_\alpha w} + (t-1)a_w & \text{if } w^{-1}(\alpha) < 0 \end{cases}$$

$$a_wa_{s_\alpha} = \begin{cases} a_{ws_\alpha} & \text{if } w(\alpha) > 0 \\ ta_{ws_\alpha} + (t-1)a_w & \text{if } w(\alpha) < 0. \end{cases}$$

If G^F is a Chevalley group of type A_l over F_q it can be shown that for each irreducible character ϕ of W the generic degree $\tilde{P}_\phi(t)$ is equal to the polynomial $P_\phi(t)$ introduced in section 11.1. This is not in general true for Chevalley groups of type other than A_l. Nevertheless the polynomial $P_\phi(t)$ does give what one might regard as a 'first approximation' to the generic degree $\tilde{P}_\phi(t)$. For this reason the polynomial $P_\phi(t)$ has been called by Lusztig the fake degree corresponding to ϕ. A comparison of the generic degrees and the fake degrees turns out to be of considerable importance in understanding the irreducible representations of the Weyl group.

Proposition 11.3.1. *Let W' be a reflection subgroup of W and ψ be an irreducible character of W'. Suppose*

$$P_\psi(t) = t^e + \textit{terms involving higher powers of } t.$$

For any irreducible character ϕ of W let t^{a_ϕ} be the highest power of t dividing $P_\phi(t)$. Then for any irreducible component ϕ of ψ^W we have $a_\phi \geqslant e$ and there is just one component with $a_\phi = e$.

Proof. This follows from 11.2.5. ■

We next prove a somewhat similar result for generic degrees. For each irreducible character ϕ of W let its generic degree $\tilde{P}_\phi(t)$ have the form

$\tilde{P}_\phi(t) = \tilde{\gamma}_\phi t^{\tilde{a}_\phi} + \ldots + \tilde{\delta}_\phi t^{\tilde{b}_\phi}$ where $\tilde{a}_\phi \leqslant \tilde{b}_\phi$ and $\tilde{\gamma}_\phi, \tilde{\delta}_\phi \neq 0$. Thus $t^{\tilde{a}_\phi}$ is the highest power of t dividing $\tilde{P}_\phi(t)$.

Lemma 11.3.2. *Let ϕ be an irreducible character of W and ε be the sign character of W. Then*

$$\tilde{P}_{\phi\varepsilon}(t) = t^N\tilde{P}_\phi(t^{-1}) \quad \textit{where } N = |\Phi^+|.$$

Proof. Let ψ be the character of the generic algebra from which ϕ is obtained by the specialization which replaces t by 1. Using the definition of the generic degree d_ψ given in section 10.11, and recalling that t_w is equal to $t^{l(w)}$ in the present situation, we have

$$\tilde{P}_\phi(t) = \frac{(\sum_{w \in W} t^{l(w)}) \deg \psi}{\sum_{w \in W} \frac{1}{t^{l(w)}} \psi(a_w)\psi(a_w{}^{-1})}.$$

Let $\rho(a_{s_\alpha}, t)$ be the matrix representing a_{s_α} in a representation of the generic algebra with character ψ, where $\alpha \in \Delta$. Then the map

$$a_{s_\alpha} \to (-t)\rho\left(a_{s_\alpha}, \frac{1}{t}\right) \qquad \alpha \in \Delta$$

also extends to a representation of the generic algebra, as is readily verified by checking the defining relations. Moreover the representation of W corresponding to this is $\varepsilon\phi$, as one sees by replacing t by 1. Let the character of this representation of the generic algebra be ψ^*. Then we have

$$\psi^*(a_w, t) = (-t)^{l(w)}\psi\left(a_w, \frac{1}{t}\right) \qquad w \in W.$$

It follows that

$$\begin{aligned}\tilde{P}_{\varepsilon\phi}(t) &= \frac{(\sum_{w \in W} t^{l(w)}) \deg \psi^*}{\sum_{w \in W} \frac{1}{t^{l(w)}} \psi^*(a_w, t)\psi^*(a_{w^{-1}}, t)} \\ &= \frac{(\sum_{w \in W} t^{l(w)}) \deg \psi^*}{\sum_{w \in W} \frac{1}{t^{l(w)}} (-t)^{l(w)} . \psi\left(a_w, \frac{1}{t}\right) . (-t)^{l(w^{-1})}\psi\left(a_{w^{-1}}, \frac{1}{t}\right)} \\ &= \frac{(\sum_{w \in W} t^{l(w)}) \deg \psi}{\sum_{w \in W} t^{l(w)}\psi\left(a_w, \frac{1}{t}\right)\psi\left(a_{w^{-1}}, \frac{1}{t}\right)}\end{aligned}$$

since $\deg \psi^* = \deg \psi = \deg \phi$. However we also have

$$t^N \tilde{P}_\phi(t^{-1}) = \frac{t^N(\sum_{w \in W} (t^{-1})^{l(w)}) \deg \psi}{\sum_{w \in W} t^{l(w)}\psi\left(a_w, \frac{1}{t}\right)\psi\left(a_{w^{-1}}, \frac{1}{t}\right)}.$$

Hence $t^N \tilde{P}_\phi(t^{-1}) = \tilde{P}_{\phi\varepsilon}(t)$, since

$$t^N\left(\sum_{w \in W} t^{-l(w)}\right) = \sum_{w \in W} t^{l(w_0 w)} = \sum_{w \in W} t^{l(w)}.$$

Proposition 11.3.3. *Let J be a subset of I. Let ψ be an irreducible character of W_J and ϕ be an irreducible component of ψ^W. Let the generic degrees of ϕ, ψ be*

$$\tilde{P}_\phi(t) = \tilde{\gamma}_\phi t^{\tilde{a}_\phi} + \ldots + \tilde{\delta}_\phi t^{\tilde{b}_\phi}$$
$$\tilde{P}_\psi(t) = \tilde{\gamma}_\psi t^{\tilde{a}_\psi} + \ldots + \tilde{\delta}_\psi t^{\tilde{b}_\psi}.$$

Then $\tilde{a}_\phi \geqslant \tilde{a}_\psi$. Moreover if $\tilde{a}_\phi = \tilde{a}_\psi$ then $\tilde{\gamma}_\phi \leqslant \tilde{\gamma}_\psi$

Proof. Let ε_W, ε_{W_J} be the sign characters of W, W_J respectively. Let $\phi' = \phi\varepsilon_W$, $\psi' = \psi\varepsilon_{W_J}$. Then ϕ', ψ' are irreducible characters of W, W_J respectively. Since ψ appears as a component of ϕ restricted to W_J, by Frobenius reciprocity, $\psi\varepsilon_{W_J}$ appears as component of $\phi\varepsilon_W$ restricted to W_J. Thus $(\psi'^W, \phi') \neq 0$.

Let $G(q)$ be a finite Chevalley group over F_q with Weyl group W and let $L_J(q)$ be its standard Levi subgroup corresponding to J. Consider the principal series representations of $G(q)$, $L_J(q)$ respectively corresponding to ϕ', ψ'. Let them be χ, ξ. Then we have

$$(\xi_{P_J(q)}{}^{G(q)}, \chi) = (\psi'^W, \phi') \neq 0$$

by Benson and Curtis [1], 4.4. Thus χ occurs as a component of $\xi_{P_J(q)}{}^{G(q)}$. In particular we have

$$\deg \xi_{P_J(q)}{}^{G(q)} \geqslant \deg \chi.$$

Thus

$$\deg \xi \,.\, |G(q)\colon P_J(q)| \geqslant \deg \chi \quad \text{and so} \quad \tilde{P}_{\psi'}(q)|G(q)\colon P_J(q)| \geqslant \tilde{P}_{\phi'}(q).$$

Since this holds for all prime powers q we may compare the degrees of the polynomials in q appearing on both sides. The degree of $|G(q)\colon P_J(q)|$ is $N - N_J$ where $N = |\Phi^+|$ and $N_J = |\Phi_J{}^+|$. The degree of $\tilde{P}_{\psi'}(q)$ is $\tilde{b}_{\psi'}$ and that of $\tilde{P}_{\phi'}(q)$ is $\tilde{b}_{\phi'}$. It follows that

$$\tilde{b}_{\psi'} + N - N_J \geqslant \tilde{b}_{\phi'}.$$

However we have

$$\tilde{P}_{\phi\varepsilon}(t) = t^N \tilde{P}_\phi(t^{-1})$$

by 11.3.2. Hence $\tilde{b}_{\phi'} = N - \tilde{a}_\phi$ and similarly we have $\tilde{b}_{\psi'} = N_J - \tilde{a}_\psi$. Hence $\tilde{a}_\phi \geqslant \tilde{a}_\psi$.

Now suppose that $\tilde{a}_\phi = \tilde{a}_\psi$. Then the two polynomials we are comparing have the same degree. We therefore compare their leading coefficients. This gives $\tilde{\delta}_{\psi'} \geqslant \tilde{\delta}_{\phi'}$. However the equation

$$\tilde{P}_{\phi'}(t) = t^N \tilde{P}_\phi(t^{-1})$$

shows that $\tilde{\delta}_{\phi'} = \tilde{\gamma}_\phi$ and we similarly have $\tilde{\delta}_{\psi'} = \tilde{\gamma}_\psi$. Hence $\tilde{\gamma}_\phi \leqslant \tilde{\gamma}_\psi$. ∎

We next state some empirical facts about the polynomials

$$P_\phi(t) = \gamma_\phi t^{a_\phi} + \ldots + \delta_\phi t^{b_\phi} \quad \text{and} \quad \tilde{P}_\phi(t) = \tilde{\gamma}_\phi t^{\tilde{a}_\phi} + \ldots + \tilde{\delta}_\phi t^{\tilde{b}_\phi}.$$

These facts may be verified by observation in the case of each Weyl group individually, but no general proof seems to be known.

Proposition 11.3.4. $\tilde{a}_\phi \leqslant a_\phi$.

We observe that $\tilde{b}_\phi \geqslant b_\phi$ is a consequence of this. For we have

$$\tilde{b}_\phi = N - \tilde{a}_{\phi\varepsilon} \geqslant N - a_{\phi\varepsilon} = b_\phi.$$

Proposition 11.3.5. *For almost all irreducible characters ϕ of W we have*

$$P_\phi(t^{-1}) = t^{-c} P_\phi(t)$$

for some positive integer c. This fails for indecomposable Weyl groups only in the following cases. There are two irreducible characters of $W(E_7)$ of degree 512 and four irreducible characters of $W(E_8)$ of degree 4096, two of which take value 512 on the class of reflections and two take value -512 on this class. These characters are the only ones which do not satisfy $P_\phi(t^{-1}) = t^{-c} P_\phi(t)$.

We may obtain a formula for $P_\phi(t^{-1})$ which has no exceptions at all as follows. We define an involution i on the set of irreducible characters of W in the following way. $i(\phi) = \phi$ if ϕ is not one of the characters of $W(E_7)$ or $W(E_8)$ described above. i interchanges the two characters of $W(E_7)$ of degree 512. i interchanges the two characters of $W(E_8)$ of degree 4096 which take value 512 on the reflections. i also interchanges the two characters of $W(E_8)$ of degree 4096 which take value -512 on the reflections. If i is defined in this way then we have

$$P_\phi(t^{-1}) = t^{-c} P_{i(\phi)}(t)$$

for some positive integer c.

The condition $P_\phi(t^{-1}) = t^{-c} P_\phi(t)$ may be thought of as asserting that the coefficients of the polynomial $P_\phi(t)$ are palindromic. It may appear odd that this is so for all irreducible characters ϕ with only the above exceptions. It is interesting to note, however, that the characters ϕ of W for which $i(\phi) \neq \phi$ are precisely the characters corresponding to irreducible representations of the Hecke ring of W which are not rational (see Curtis [8]).

Proposition 11.3.6. *The positive integer c_ϕ in the formula*

$$P_\phi(t^{-1}) = t^{-c_\phi} P_{i(\phi)}(t)$$

is given by $c_\phi = N - t_\phi$ where $N = |\Phi^+|$ and

$$t_\phi = \frac{1}{\phi(1)} \sum_{\substack{s \in W \\ s\,\text{reflection}}} \phi(s).$$

Proposition 11.3.7. *If ϕ is an irreducible character of W for which $i(\phi) = \phi$ then $\tilde{a}_\phi + \tilde{b}_\phi = a_\phi + b_\phi$.*

If $i(\phi) \neq \phi$ then the integers $(\tilde{a}_\phi, a_\phi, b_\phi, \tilde{b}_\phi)$ do not satisfy the above condition. They are

$(11, 11, 51, 52)$ $(11, 12, 52, 52)$ in E_7

$(11, 11, 93, 94)$ $(11, 12, 94, 94)$ $(26, 27, 109, 109)$ $(26, 26, 108, 109)$ in E_8.

Proposition 11.3.8. *Suppose W is an indecomposable Weyl group. Suppose ϕ is an irreducible character of W with $a_\phi = \tilde{a}_\phi$. Then $\gamma_\phi = 1$ and $\tilde{\gamma}_\phi$ is one of the numbers* $1, \frac{1}{2^c}, \frac{1}{6}, \frac{1}{24}, \frac{1}{60}$.

Definition. An irreducible character of W will be called special if $a_\phi = \tilde{a}_\phi$. This means that the generic degree $\tilde{P}_\phi(t)$ and the fake degree $P_\phi(t)$ are divisible by the same power of t.

Lemma 11.3.9. *If ϕ is a special character and $i(\phi) = \phi$ then $\phi\varepsilon$ is also special.*

Proof. We have $a_{\phi\varepsilon} = N - b_\phi$ by 11.1.2 and $\tilde{a}_{\phi\varepsilon} = N - \tilde{b}_\phi$ by 11.3.2. It is therefore sufficient to show that $b_\phi = \tilde{b}_\phi$. But $\tilde{a}_\phi + \tilde{b}_\phi = a_\phi + b_\phi$ by 11.3.7 since $i(\phi) = \phi$. Since $a_\phi = \tilde{a}_\phi$ we have $b_\phi = \tilde{b}_\phi$ and the result follows.

Note. If ϕ is special but $i(\phi) \neq \phi$ then $\phi\varepsilon$ is not special. However in this case $i(\phi\varepsilon)$ is special. We thus obtain:

Corollary 11.3.10. *If ϕ is a special character of W then $i(\phi\varepsilon)$ is also special. Thus the map $\phi \to i(\phi\varepsilon)$ is an involution on the set of special characters of W.*

Proposition 11.3.11. *Let J be a subset of I and ψ be a special irreducible character of W_J. Then $\phi = j(\psi)$ is a special irreducible character of W.*

Proof. Since ψ is special we have $a_\psi = \tilde{a}_\psi$. Since $\phi = j(\psi)$ we have $a_\phi = a_\psi$ by 11.2.1. Moreover ϕ is an irreducible component of ψ^W by 11.2.5. Thus $\tilde{a}_\phi \geqslant \tilde{a}_\psi$ by 11.3.3. Also $\tilde{a}_\phi \leqslant a_\phi$ by 11.3.4. Hence we have

$$\tilde{a}_\phi \geqslant \tilde{a}_\psi = a_\psi = a_\phi \geqslant \tilde{a}_\phi$$

and so we have equality throughout. Thus the character ϕ is also special. ■

The special characters of the Weyl group play an important rôle in understanding the unipotent characters of the finite groups of Lie type.

11.4 REPRESENTATIONS OF WEYL GROUPS OF CLASSICAL TYPE

In this section we shall describe briefly the irreducible representations of the classical Weyl groups $W(A_l)$. $W(B_l)$, $W(C_l)$, $W(D_l)$ indicating which of the irreducible representations are special.

Proposition 11.4.1. (Type A_l) *The Weyl group $W(A_l)$ is isomorphic to the symmetric group S_{l+1}. All the irreducible representations of $W(A_l)$ may be obtained as Macdonald representations. Let α be a partition of $l+1$ with $\alpha = (\alpha_1, \alpha_2, \ldots)$. Let α^* be the dual partition of α and let Φ' be a subsystem of $\Phi(A_l)$ of type*

$$A_{\alpha_1^*-1} + A_{\alpha_2^*-1} + \cdots.$$

Let W' be the Weyl group of Φ' and $j_{W'}^{W}(\varepsilon_{W'})$ be the Macdonald representation of W obtained from W'. Let $\phi_\alpha = j_{W'}^{W}(\varepsilon_{W'})$. Then each irreducible character of $W(A_l)$ has the form ϕ_α for just one partition α of $l+1$. For example $\phi_{(l+1)}$ is the unit character and $\phi_{(1^{l+1})}$ is the sign character. All the irreducible characters of W are special. The number a_{ϕ_α} is the number of positive roots of Φ'.

Proof. See Macdonald [1].

Proposition 11.4.2. (Type B_l) *The Weyl group $W(B_l)$ has order $2^l l!$ and is isomorphic to the wreath product $C_2 \wr S_l$. All its irreducible representations may be obtained as Macdonald representations. There is one irreducible representation of $W(B_1)$ for each ordered pair of partitions (α, β) with $|\alpha| + |\beta| = l$. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\beta = (\beta_1, \beta_2, \ldots)$ with*

$$0 \leqslant \alpha_1 \leqslant \alpha_2 \ldots \qquad 0 \leqslant \beta_1 \leqslant \beta_2 \ldots .$$

(We allow the possibility that certain parts α_i, β_i are 0, for a reason which will become apparent shortly.) Let α^*, β^* be the dual partitions of α, β. Then $\Phi(B_l)$ has a subsystem Φ' of type

$$D_{\alpha_1^*} + D_{\alpha_2^*} + \ldots + B_{\beta_1^*} + B_{\beta_2^*} + \ldots .$$

(We use the convention that D_1 is the empty root system.) Let W' be the Weyl group of Φ' and $j_{W'}{}^{W}(\varepsilon_{W'})$ be the Macdonald representation of W obtained from W'. Let $\phi_{\alpha,\beta} = j_{W'}{}^{W}(\varepsilon_{W'})$. Then each irreducible character of $W(B_l)$ has the form $\phi_{\alpha,\beta}$ for just one ordered pair of partitions (α, β) with $|\alpha| + |\beta| = l$. For example $\phi_{(l,-)}$ is the unit representation and $\phi_{(-,1^l)}$ is the sign representation.

For each such ordered pair (α, β) we choose an appropriate number of zeros as parts of α or β so that α has one more part than β. We then define the symbol of (α, β) to be the array

$$\begin{pmatrix} \alpha_1 & & \alpha_2 + 1 & & \alpha_3 + 2 & & \ldots & & \alpha_m + (m-1) & & \alpha_{m+1} + m \\ & \beta_1 & & \beta_2 + 1 & & \beta_3 + 2 & & \ldots & & \beta_m + (m-1) & \end{pmatrix}$$

We consider the equivalence relation on the symbols generated by

$$\begin{pmatrix} 0, & & \lambda_1 + 1, & & \lambda_2 + 1, & & \ldots, & & \lambda_m + 1 & & \lambda_{m+1} + 1 \\ & 0, & & \mu_1 + 1, & & \mu_2 + 1, & & \ldots, & & \mu_m + 1 & \end{pmatrix}$$
$$\sim \begin{pmatrix} \lambda_1, & & \lambda_2, & & \ldots, & & \lambda_m, & & \lambda_{m+1} \\ & \mu_1, & & \mu_2, & & \ldots, & & \mu_m & \end{pmatrix}.$$

Each ordered pair (α, β) of partitions then defines a unique equivalence class of symbols. Let

$$\begin{pmatrix} \lambda_1, & & \lambda_2, & & \ldots, & & \lambda_m, & & \lambda_{m+1} \\ & \mu_1, & & \mu_2, & & \ldots, & & \mu_m & \end{pmatrix}$$

be a symbol in this class. Then the positive integers $a_{\alpha,\beta} = a_{\phi_{\alpha,\beta}}$ and $\tilde{a}_{\alpha,\beta} = \tilde{a}_{\phi_{\alpha,\beta}}$ are given by

$$a_{\alpha,\beta} = 2 \sum_{\substack{i,j \\ 1 \leqslant i < j \leqslant m+1}} \inf(\lambda_i, \lambda_j) + 2 \sum_{\substack{i,j \\ 1 \leqslant i < j \leqslant m}} \inf(\mu_i, \mu_j)$$

$$+ \sum_{i=1}^{m} \mu_i - \binom{2m-1}{2} - \binom{2m-3}{2} - \dots$$

$$\tilde{a}_{\alpha,\beta} = \sum_{\substack{i,j \\ 1 \leqslant i < j \leqslant m+1}} \inf(\lambda_i, \lambda_j) + \sum_{\substack{i,j \\ 1 \leqslant i < j \leqslant m}} \inf(\mu_i, \mu_j)$$

$$+ \sum_{\substack{i,j \\ 1 \leqslant i \leqslant m+1 \\ 1 \leqslant j \leqslant m}} \inf(\lambda_i, \mu_j) - \binom{2m-1}{2} - \binom{2m-3}{2} - \dots.$$

The condition $a_{\alpha,\beta} = \tilde{a}_{\alpha,\beta}$ for $\phi_{\alpha,\beta}$ to be a special character is satisfied if and only if

$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \mu_2 \leqslant \dots \leqslant \mu_m \leqslant \lambda_{m+1}.$$

Proof. See Lusztig [12].

Proposition 11.4.3. (Type C_l) *The Weyl group $W(C_l)$ is isomorphic to $W(B_l)$. All its irreducible representations may be obtained as Macdonald representations. There is one irreducible representation of $W(C_l)$ for each ordered pair of partitions (α, β) with $|\alpha| + |\beta| = l$. $\Phi(C_l)$ has a subsystem Φ' of type*

$$D_{\alpha_1{}^*} + D_{\alpha_2{}^*} + \dots + C_{\beta_1{}^*} + C_{\beta_2{}^*} + \dots.$$

Let W' be the Weyl group of Φ'. Then $\phi_{\alpha,\beta} = j_{W'}{}^{W}(\varepsilon_{W'})$ is an irreducible character of W and each irreducible character is obtained just once in this way. The integers $a_{\alpha,\beta}$, $\tilde{a}_{\alpha,\beta}$ and the condition $a_{\alpha,\beta} = \tilde{a}_{\alpha,\beta}$ for $\phi_{\alpha,\beta}$ to be special are the same as for $W(B_l)$. $\phi_{(l,-)}$ is the unit representation and $\phi_{(-,1^l)}$ is the sign representation.

Proof. See Lusztig [12].

Proposition 11.4.4. (Type D_l) *The Weyl group $W(D_l)$ has order $2^{l-1}l!$ It is a subgroup of $W(B_l)$ of index 2. The representation $\phi_{\alpha,\beta}$ of $W(B_l)$ remains irreducible on restriction to $W(D_l)$ if $\alpha \neq \beta$ and $\phi_{\alpha,\beta}$, $\phi_{\beta,\alpha}$ coincide on restriction to $W(D_l)$. If $\alpha = \beta$ then $\phi_{\alpha,\alpha}$ decomposes into two irreducible components $\phi_{\alpha,\alpha}'$, $\phi_{\alpha,\alpha}''$ of $W(D_l)$. We obtain all irreducible representations of $W(D_l)$ in this way. $\phi_{(l,-)}$ is the unit representation and $\phi_{(1^l,-)}$ is the sign representation.*

Let us write $\alpha = (\alpha_1, \alpha_2, \dots)$, $\beta = (\beta_1, \beta_2, \dots)$ where $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \dots$, $0 \leqslant \beta_1 \leqslant \beta_2 \leqslant \dots$. We choose the number of parts equal to zero so that α, β have the same number of parts. We then define the symbol of (α, β) to be the array

$$\begin{pmatrix} \alpha_1 & \alpha_2 + 1 & \dots & \alpha_m + m - 1 \\ \beta_1 & \beta_2 + 1 & \dots & \beta_m + m - 1 \end{pmatrix}.$$

In this case the arrays

$$\begin{pmatrix} \alpha_1 & \alpha_2+1 & \dots & \alpha_m+m-1 \\ \beta_1 & \beta_2+1 & \dots & \beta_m+m-1 \end{pmatrix}, \quad \begin{pmatrix} \beta_1 & \beta_2+1 & \dots & \beta_m+m-1 \\ \alpha_1 & \alpha_2+1 & \dots & \alpha_m+m-1 \end{pmatrix}$$

are regarded as being the same symbol. We consider the equivalence relation on symbols generated by

$$\begin{pmatrix} 0 & \lambda_1+1 & \lambda_2+1 & \dots & \lambda_m+1 \\ 0 & \mu_1+1 & \mu_2+1 & \dots & \mu_m+1 \end{pmatrix} \sim \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_m \\ \mu_1 & \mu_2 & \dots & \mu_m \end{pmatrix}.$$

Each pair of partitions (α, β) determines in this way a unique equivalence class of symbols.

The numbers $a_{\alpha,\beta}$, $\tilde{a}_{\alpha,\beta}$ if $\alpha \neq \beta$ are given by the formulae

$$\begin{aligned} a_{\alpha,\beta} = 2 \sum_{\substack{i,j \\ 1 \leqslant i<j \leqslant m}} \inf(\lambda_i, \lambda_j) + 2 \sum_{\substack{i,j \\ 1 \leqslant i<j \leqslant m}} \inf(\mu_i, \mu_j) \\ + \inf\left(\sum \lambda_i, \sum \mu_j\right) - \binom{2m-2}{2} - \binom{2m-4}{2} - \dots \end{aligned}$$

$$\begin{aligned} \tilde{a}_{\alpha,\beta} = \sum_{\substack{i,j \\ 1 \leqslant i<j \leqslant m}} \inf(\lambda_i, \lambda_j) + \sum_{\substack{i,j \\ 1 \leqslant i<j \leqslant m}} \inf(\mu_i, \mu_j) \\ + \sum_{\substack{i,j \\ 1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant m}} \inf(\lambda_i, \mu_j) - \binom{2m-2}{2} - \binom{2m-4}{2} - \dots. \end{aligned}$$

The numbers $a_{\alpha,\alpha}'$, $a_{\alpha,\alpha}''$, $\tilde{a}_{\alpha,\alpha}'$, $\tilde{a}_{\alpha,\alpha}''$ are given by

$$\begin{aligned} a_{\alpha,\alpha}' &= a_{\alpha,\alpha}'' = \tilde{a}_{\alpha,\alpha}' = \tilde{a}_{\alpha,\alpha}'' \\ &= 4 \sum_{\substack{i,j \\ 1 \leqslant i<j \leqslant m}} \inf(\lambda_i, \lambda_j) + \sum_{\substack{i \\ 1 \leqslant i \leqslant m}} \lambda_i - \binom{2m-2}{2} - \binom{2m-4}{2} - \dots. \end{aligned}$$

The condition $a_{\alpha,\beta} = \tilde{a}_{\alpha,\beta}$ is satisfied for $\phi_{\alpha,\beta}$ if and only if we have either

$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \mu_2 \leqslant \dots \leqslant \lambda_m \leqslant \mu_m$$

or

$$\mu_1 \leqslant \lambda_1 \leqslant \mu_2 \leqslant \lambda_2 \leqslant \dots \leqslant \mu_m \leqslant \lambda_m.$$

The representations $\phi_{\alpha,\alpha}'$, $\phi_{\alpha,\alpha}''$ are always special.

Proof. See Lusztig [12].

Chapter 12

UNIPOTENT CHARACTERS

12.1 GEOMETRIC CONJUGACY CLASSES OF IRREDUCIBLE CHARACTERS

Let G be a connected reductive group and $F: G \to G$ a Frobenius map. Suppose the centre Z of G is connected. We show how the results of previous chapters enable us to define an equivalence relation on the set of all irreducible characters of G^F, with one equivalence class for each geometric conjugacy class of pairs (T, θ). These equivalence classes will then be called geometric conjugacy classes of irreducible characters of G^F.

We first recall from 7.5.8 that, given any irreducible character χ^i of G^F, there exists a pair (T, θ) for which $(R_{T,\theta}, \chi^i) \neq 0$. There may be several such pairs (T, θ). However we know from 7.3.8 that if the pairs (T, θ) and (T', θ') are not geometrically conjugate then $R_{T,\theta}$ and $R_{T',\theta'}$ have no irreducible component in common. Thus if $(R_{T,\theta}, \chi^i) \neq 0$ and $(R_{T',\theta'}, \chi^i) \neq 0$ then (T, θ) and (T', θ') must be geometrically conjugate. In this way each irreducible character χ^i determines a geometric conjugacy class κ of pairs (T, θ). We say that χ^i, χ^j are geometrically conjugate if the class κ obtained from each of them is the same. Thus χ^i, χ^j are geometrically conjugate if and only if there exist pairs (T, θ) and (T', θ') such that $(R_{T,\theta}, \chi^i) \neq 0$, $(R_{T',\theta'}, \chi^j) \neq 0$ and (T, θ), (T', θ') are geometrically conjugate.

We next consider how the semisimple characters χ^i fall into geometric conjugacy classes. We recall that an irreducible character χ^i of G^F is semisimple if $\sum_{u \text{ regular unipotent}} \chi^i(u) \neq 0$. This is equivalent to the condition $(\chi^i, \Xi) \neq 0$ where Ξ is the class function defined in section 8.3. By 8.4.3 we have $(\Xi, R_{T,\theta}) = 1$ for all pairs (T, θ). Thus by 8.4.4 applied to the generalized character Ξ we see that

$$\Xi = \sum_{\kappa} \varepsilon_\kappa \chi_\kappa^{ss}$$

where $\varepsilon_\kappa = \pm 1$ and χ_κ^{ss} is the irreducible character of G^F given by

$$\varepsilon_\kappa \chi_\kappa^{ss} = \sum_{\substack{(T,\theta) \in \kappa \\ \bmod G^F}} \frac{R_{T,\theta}}{(R_{T,\theta}, R_{T,\theta})}.$$

The irreducible characters of G^F of the form χ_κ^{ss} are therefore precisely the semisimple characters. Moreover the semisimple character χ_κ^{ss} lies in the geometric conjugacy class κ. For if $(T, \theta) \in \kappa$ we have $(\varepsilon_\kappa \chi_\kappa^{ss}, R_{T,\theta}) = 1$ by 7.3.7. Thus $(\chi_\kappa^{ss}, R_{T,\theta}) = \pm 1$. We therefore see that each geometric conjugacy class of irreducible characters of G^F contains exactly one semisimple character.

In a similar way we can show that each geometric conjugacy class of irreducible characters contains a unique regular character. We recall that an irreducible character χ^i of G^F is regular if and only if $(\Gamma, \chi^i) \neq 0$ where Γ is the Gelfand–Graev character. Moreover we know by 8.4.5 that $(\Gamma, R_{T,\theta}) = \varepsilon_G \varepsilon_T$ and by 8.3.1 that $(\Gamma, \Gamma) = |Z^F| q^l$. Thus by 8.4.4 applied to Γ we see that

$$\Gamma = \sum_\kappa \chi_\kappa^{reg}$$

where χ_κ^{reg} is the irreducible character of G^F given by

$$\chi_\kappa^{reg} = \sum_{\substack{(T,\theta) \in \kappa \\ \operatorname{mod} G^F}} \frac{\varepsilon_G \varepsilon_T R_{T,\theta}}{(R_{T,\theta}, R_{T,\theta})}.$$

The irreducible characters of G^F of the form χ_κ^{reg} are therefore precisely the regular characters. Moreover the regular character χ_κ^{reg} lies in the geometric conjugacy class κ. For if $(T, \theta) \in \kappa$ then we have $(\chi_\kappa^{reg}, R_{T,\theta}) = \varepsilon_G \varepsilon_T \neq 0$. Thus each geometric conjugacy class of irreducible characters of G^F contains exactly one regular character.

It is natural to consider in detail the geometric conjugacy class containing the principal character 1. Since $(R_{T,1}, 1) = 1$ by 7.4.1 this geometric conjugacy class is the one containing $(T, 1)$. The definition of geometric conjugacy shows that, for any two F-stable maximal tori T, T', $(T, 1)$ is geometrically conjugate to $(T', 1)$ but not to (T', θ') if $\theta' \neq 1$. Thus an irreducible character χ^i of G^F is geometrically conjugate to the principal character 1 if and only if χ^i occurs as a component of $R_{T,1}$ for some F-stable maximal torus T.

Definition. An irreducible character χ^i of G^F is called unipotent if χ^i occurs as a component of $R_{T,1}$ for some F-stable maximal torus T of G.

It follows from this definition that the unipotent characters form a single geometric conjugacy class. The semisimple character in this class is the principal character. The regular character in this class is the Steinberg character, since $(\mathrm{St}, R_{T,1}) \neq 0$ for all T by 7.6.6. Thus the only irreducible character of G^F which is both semisimple and unipotent is the principal character, and the only irreducible character of G^F which is both regular and unipotent is the Steinberg character.

Considerable progress in understanding the unipotent characters has been made by Lusztig. In particular the degrees of all the irreducible unipotent characters have been determined. We shall give in this chapter an outline of what has been proved about the unipotent characters. The technical details of the proofs are beyond the scope of the present volume, and we refer the reader to the recent book of Lusztig in which these proofs can be found [21].

The properties of the unipotent representations and their characters are closely connected to various other aspects of the study of G and of W, in particular the theory of unipotent conjugacy classes in G and the representation theory of Weyl groups. We shall therefore give more information about these subjects than has been given in earlier chapters.

The study of unipotent characters of G^F can be reduced to the case when G is simple of adjoint type. For let G be connected reductive with connected centre Z. Then Z^F lies in the kernel of every unipotent representation of G^F. For every unipotent character of G^F occurs as component of some generalized character $R_{T,1}$. However $R_{T,1}(g) = \mathscr{L}(g, \mathfrak{B}_w)$ for $g \in G^F$, by 7.7.12, so each unipotent representation occurs as component of some module of the form $H_c^i(\mathfrak{B}_w, \bar{\mathbb{Q}}_l)$. Now G^F acts on $\mathfrak{B}_w$ by $B \to {}^xB$, $x \in G^F$, $B \in \mathfrak{B}_w$, thus Z^F acts trivially on $\mathfrak{B}_w$. Hence Z^F acts trivially on each cohomology module $H_c^i(\mathfrak{B}_w, \bar{\mathbb{Q}}_l)$, and so lies in the kernel of each unipotent representation. Thus there is a bijective correspondence between unipotent representations of G^F and $G^F/Z^F \cong (G/Z)^F$. We may therefore assume that G has trivial centre. G is thus a connected semisimple group and there is a bijective morphism $G \to G_{\text{ad}}$ from G to the corresponding semisimple group of adjoint type. This leads to an isomorphism between G^F and $G_{\text{ad}}{}^F$. We may therefore assume that G is semisimple of adjoint type. G is then a direct product of simple groups of adjoint type. The simple components may be permuted by the Frobenius map F. However if G is isomorphic to $G_1 \times G_2$ where G_1, G_2 are F-stable semisimple groups then there is a bijection $(\rho_1, \rho_2) \to \rho_1 \otimes \rho_2$ between pairs of irreducible unipotent representations of $G_1{}^F$, $G_2{}^F$ and irreducible unipotent representations of G^F. Also if $G = G_1 \times G_2 \times \ldots \times G_m$ where the G_i are simple groups permuted transitively by F then G^F is isomorphic to $(G_1)^{F^m}$. Thus we may assume that our group G is simple of adjoint type. We shall make this assumption in the remainder of the chapter.

12.2 CUSPIDAL UNIPOTENT CHARACTERS

We begin by describing the way in which the degrees of the irreducible unipotent characters of G^F were first determined by Lusztig. This uses the theory of cuspidal characters. We know from chapter 9 that every irreducible character appears as component in an induced character $\phi_{P_J{}^F}{}^{G^F}$ where ϕ is a cuspidal irreducible character of $L_J{}^F$. In the case of unipotent characters one can strengthen this result. Each irreducible unipotent character of G^F appears as component in $\phi_{P_J{}^F}{}^{G^F}$ for some irreducible cuspidal unipotent character ϕ of $L_J{}^F$. One must therefore first determine the cuspidal unipotent characters and then consider how to decompose them when induced from a parabolic subgroup.

The unipotent characters of the groups G^F of classical type were investigated in Lusztig's paper [9]. One can assume by induction a knowledge of the unipotent characters ϕ of the proper Levi subgroups $L_J{}^F$ of G^F and one knows from the Howlett–Lehrer theory how to decompose the induced characters $\phi_{P_J{}^F}{}^{G^F}$. All the components will also be unipotent. One can determine the total number of irreducible unipotent characters of G^F and then calculate how many are not contained as components of $\phi_{P_J F}{}^{G^F}$ for any proper Levi subgroup $L_J{}^F$ of G^F. This will give the number of cuspidal unipotent characters of G^F. This number turns out to be either 0 or 1 in each group G^F of classical type. Lusztig [10], p. 28, also obtained a formula for the sum of the squares of the degrees of the irreducible unipotent representations of G^F. This is as follows:

$$\sum_{\chi \text{ unipotent}} (\deg \chi)^2 = \frac{1}{|W|} \sum_{w \in W} \frac{|G^F|_{p'}{}^2}{\det(Fw - 1)^2}.$$

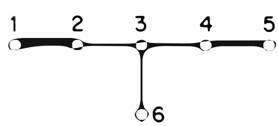
Thus, knowing inductively the degrees of all but one of the unipotent characters, one can calculate the degree of the remaining cuspidal unipotent character. These characters and their degrees are shown in section 13.7.

Now if G^F has classical type and $L_J{}^F$ is a Levi subgroup which has a cuspidal unipotent character then J is the unique F-stable subgraph of its type in the Dynkin diagram of G^F. The unique cuspidal unipotent character ϕ of $L_J{}^F$ will be invariant under all automorphisms of $L_J{}^F$. We are therefore in the particularly favourable case of the Howlett–Lehrer theory described in section 10.10. We have $W^{J.\phi} = R_{J.\phi} = R_J$ and the cocycle μ is trivial. The structure of the reflection group R_J is obtained from 10.10.3. It remains, however, to determine the parameters p_α. This was done by Lusztig by combinatorial methods. The results are given in section 13.7. The Howlett–Lehrer theory will then give the degrees of all the irreducible components of $\phi_{P_J F}{}^{G^F}$, using 10.11.5. In this way one obtains the degrees of all unipotent characters of G^F.

We now turn to a discussion of the groups of exceptional type. In such groups G^F there may be more than one cuspidal unipotent character. Thus one cannot use the same argument as in the case of the classical groups. A way of obtaining certain cuspidal unipotent characters in this case was described by Lusztig in his paper 'Coxeter orbits and eigenspaces of Frobenius' [8]. We first define an F-Coxeter element of W. We recall that F acts on W in such a way as to permute the simple reflections s_i, $i \in I$. An F-Coxeter element of W is one of the form $s_{i_1}s_{i_2}\dots s_{i_k}$ where $i_1, \dots, i_k \in I$ contains just one representative from each F-orbit on I. The F-Coxeter elements of W form an F-conjugacy class as described in section 3.3. If w is an F-Coxeter element then the F-centralizer of w is $\{x \in W; x^{-1}wF(x) = w\}$. The order of the F-centralizer of w is independent of the choice of the F-Coxeter element w, and is denoted by h_0. h_0 is called the F-Coxeter number. Let δ be the smallest positive integer such that F^δ acts trivially on I. We know from 7.7.12 that $H_c^i(\mathfrak{B}_w; \bar{\mathbb{Q}}_l)$ is a G^F-module, all of whose components are unipotent. Its structure as a module for G^F and for F^δ is independent of the choice of the F-Coxeter element w. It is shown by Lusztig that when w is an F-Coxeter element $H_c^i(\mathfrak{B}_w; \bar{\mathbb{Q}}_l)$ is the direct sum of its nonzero eigenspaces under F^δ and that each of these eigenspaces is irreducible. Also no two of these nonzero eigenspaces within a given $H_c^i(\mathfrak{B}_w, \bar{\mathbb{Q}}_l)$ are isomorphic G^F-modules. Furthermore no nonzero eigenspace within $H_c^i(\mathfrak{B}_w, \bar{\mathbb{Q}}_l)$ is isomorphic to one within $H_c^j(\mathfrak{B}_w, \bar{\mathbb{Q}}_l)$ if $i \neq j$. Certain of these nonzero eigenspaces give cuspidal unipotent representations of G^F.

However these cuspidal unipotent characters obtained from the eigenspaces of F^δ on $H_c^i(\mathfrak{B}_{w'}, \bar{\mathbb{Q}}_l)$ do not in general exhaust the cuspidal unipotent characters of the groups of exceptional type. The remainder have been determined by Lusztig in [10] and [11] by somewhat complicated arguments. The cuspidal unipotent characters and their degrees for all types are given in section 13.7.

These cuspidal unipotent characters have some agreeable properties which make them particularly suitable for the application of the Howlett–Lehrer theory. If $L_J{}^F$ is a Levi subgroup of G^F which has a cuspidal unipotent character then in each case J is the unique F-stable subgraph of its type in the Dynkin diagram of G. Thus $C_{J,\phi} = 1$ in the notation of chapter 10 and so the cocycle μ is trivial. Hence $W^{J,\phi} = R_{J,\phi}$. Moreover one even has $R_{J,\phi} = R_J$. For if ϕ is the only cuspidal unipotent character of $L_J{}^F$ of its degree then clearly $R_{J,\phi} = R_J$. This is in fact the case unless L_J is of type E_6 or E_7 when there are just two cuspidal unipotent characters of $L_J{}^F$, both of the same degree, which come from the Coxeter elements by the method described above. Let $w \in R_J$. Since $w(J) = J$, w acts on Δ_J by a graph automorphism. If w acts as the identity graph automorphism then w fixes a Coxeter element of W_J, and so w will fix ϕ. This must certainly be the case if J has type E_7, which has no non-identity graph automorphisms. Finally suppose J is of type E_6 and w acts nontrivially on Δ_J. Then w fixes the Coxeter element $s_1 s_3 s_5 s_2 s_4 s_6$ corresponding to the diagram



and so w fixes ϕ also. Thus $W^{J,\phi} = R_{J,\phi} = R_J$ in each case. The structure of R_J can be obtained from 10.10.3. Finally the parameters p_α have been calculated by Lusztig in each case. They are also shown in section 13.7. This makes possible the application of the Howlett–Lehrer theory to find the degrees of the components of $\phi_{P_J{}^F}{}^{G^F}$ for any cuspidal unipotent character ϕ of a Levi subgroup $L_J{}^F$ of G^F. The degrees are calculated from 10.11.5. By this method the degrees of all irreducible unipotent characters can be obtained. They are listed in sections 13.8 and 13.9. These degrees were originally obtained by Lusztig before the appearance of the results by Howlett and Lehrer. He obtained the relevant results independently in the cases he was considering.

12.3 FAMILIES OF UNIPOTENT CHARACTERS

Having obtained the degrees of the irreducible unipotent representations of G^F from the theory of cuspidal characters, Lusztig observed that these representations fall into families in a remarkable way. To explain this we first assume that G^F is split, so that F acts trivially on W. Let $w \in W$ and T_w be an F-stable maximal torus of G obtained from a maximally split torus by twisting with w. We write $R_w = R_{T_w,1}$. R_w is a generalized character of G^F which depends only upon the conjugacy class of w. Let χ be an irreducible unipotent character of G^F. We observe that if (χ, R_w) is known for all $w \in W$ then $\chi(s)$ will be known for each semisimple element $s \in G^F$. For the characteristic function on the G^F-conjugacy class of s is a linear combination of $R_{T,\theta}$'s by 7.5.5, and $(R_{T,\theta}, \chi) = 0$ unless $\theta = 1$ by 7.3.8. For each irreducible character ϕ of W we define R_ϕ by

$$R_\phi = \frac{1}{|W|} \sum_{w \in W} \phi(w) R_w.$$

R_ϕ is a rational combination of irreducible characters of G^F. We observe that each R_w can be expressed in terms of R_ϕ's by

$$R_w = \sum_{\phi \in \hat{W}} \phi(w) R_\phi$$

where ϕ runs over the set $\hat{W}$ of irreducible characters of W. Thus a knowledge of the scalar products (χ, R_w) is equivalent to a knowledge of the scalar products (χ, R_ϕ). However we shall concentrate attention on the (χ, R_ϕ) since it is known that in the simplest case when G has type A_l the irreducible unipotent characters of G^F are precisely the functions of the form R_ϕ for $\phi \in \hat{W}$. We say that two irreducible unipotent characters χ, χ' of G^F lie in the same family if there exists a sequence

$$\chi = \chi^1, \chi^2, \ldots, \chi^r = \chi'$$

of irreducible unipotent characters of G^F such that consecutive characters χ^{i-1}, χ^i satisfy

$$(\chi^{i-1}, R_{\phi_i}) \neq 0 \qquad (\chi^i, R_{\phi_i}) \neq 0$$

for some irreducible character ϕ_i of W.

Lusztig observed that the unipotent characters in a given family **F** could be parametrized in terms of a certain finite group Γ. For each family Γ is one of the groups

$$1, C_2 \times \ldots \times C_2, S_3, S_4, S_5.$$

Given such a group Γ we define $M(\Gamma)$ to be the set of pairs (x, σ) where $x \in \Gamma$ is determined up to conjugacy and σ is an irreducible character of $C_\Gamma(x)$. The number of elements in $M(\Gamma)$ is given by:

Γ	: 1	$C_2 \times \ldots \times C_2$ (e factors)	S_3	S_4	S_5
$\lvert M(\Gamma)\rvert$	: 1	2^{2e}	8	21	39

For each family **F** there is a corresponding group Γ and a bijection between **F** and $M(\Gamma)$. Thus each family **F** has a number of elements which is one of 1, 2^{2e}, 8, 21, 39. Moreover it is possible to describe the scalar products (χ, R_ϕ) in terms of $M(\Gamma)$. This is done as follows. For any pair of elements $(x, \sigma), (y, \tau) \in M(\Gamma)$ we define a complex number $\{(x, \sigma), (y, \tau)\}$ by

$$\{(x, \sigma), (y, \tau)\} = \frac{1}{|C_\Gamma(x)|} \frac{1}{|C_\Gamma(y)|} \sum_{\substack{g \in \Gamma \\ x.gyg^{-1} = gyg^{-1}.x}} \sigma(gyg^{-1}) \overline{\tau(g^{-1}xg)}.$$

We observe that $gyg^{-1} \in C_\Gamma(x)$ so that $\sigma(gyg^{-1})$ is defined, and $g^{-1}xg \in C_\Gamma(y)$ so that $\tau(g^{-1}xg)$ is defined. The matrix with $|M(\Gamma)|$ rows and columns given by the numbers $\{(x, \sigma), (y, \tau)\}$ turns out to be Hermitian, unitary and self-inverse. It is called by Lusztig a non-abelian Fourier transform matrix. It is shown by Lusztig that there is a bijection

$$\begin{aligned} M(\Gamma) &\to \mathbf{F} \\ (x, \sigma) &\to \chi^{\mathbf{F}}_{(x,\sigma)} \end{aligned}$$

such that

$$(\chi^{\mathbf{F}}_{(x,\sigma)}, R_\phi) = \begin{cases} \{(x, \sigma), (y, \tau)\} & \text{if } \chi_\phi = \chi^{\mathbf{F}}_{(y,\tau)} \\ 0 & \text{if } \chi_\phi \notin \mathbf{F} \end{cases}$$

where χ_ϕ is the component of $1_{B^F}{}^{G^F}$ corresponding to the irreducible character ϕ of W. This was proved for the exceptional groups in [15] and for the classical groups in [18], [20]. This formula therefore permits us to obtain the values of the unipotent characters on the semisimple classes. In particular it gives the degrees of the irreducible unipotent characters, which are given by the formula

$$\deg \chi^{\mathbf{F}}_{(x,\phi)} = \sum_{\substack{(y,\tau)\in M(\Gamma) \\ \chi^{\mathbf{F}}_{(y,\tau)} = \chi_\phi \text{ for some } \phi \in W}} \{(x, \sigma), (y, \tau)\} . \left(\frac{1}{|W|} \sum_{w\in W} \phi(w)\varepsilon(w)|G^F : T^F{}_w|_{p'}\right).$$

Now the bijection between the family $\mathbf{F}$ and the set $M(\Gamma)$ has the property that $\chi^{\mathbf{F}}_{(1,1)} = \chi_\phi$ for some irreducible character ϕ of W. This character ϕ of W satisfies the condition $(\chi, R_\phi) > 0$ for all $\chi \in \mathbf{F}$. Moreover it is the only irreducible character of W satisfying this condition, as can be seen by using the properties of the above Fourier transform matrix. It also follows from the definition of the families that distinct families give rise to distinct characters ϕ of W in this way. Thus we see that the families of irreducible unipotent characters of G^F are in bijective correspondence with a certain set of irreducible characters of W. These are the irreducible characters ϕ of W which satisfy $(\chi, R_\phi) \geqslant 0$ for all unipotent characters χ. They turn out to be precisely the special characters of W defined in section 11.3. This explains the importance of the special characters of W in the theory of unipotent representations of G^F. The special character ϕ of W coming from a family $\mathbf{F}$ of unipotent characters of G^F has the property that the coefficient $\tilde{\gamma}_\phi$ of the smallest power of t appearing in the generic degree $\tilde{P}_\phi(t)$ is given by $\tilde{\gamma}_\phi = 1/|\Gamma|$. Moreover this condition determines the group Γ uniquely, since we know that Γ is one of the groups $1, C_2 \times \ldots \times C_2, S_3, S_4, S_5$.

In the above discussion we have assumed that G^F is split. However if G^F is not split the unipotent characters of G^F may again be divided into families using

$$R_\phi = \frac{1}{|W|} \sum_{w\in W} \phi(F_0 w) R_w$$

instead of the previous definition of R_ϕ. This time ϕ is an irreducible character of $\langle W, F_0\rangle$. The families and the representations within a given family may again be parametrized in a simple way. The details are given in Lusztig and Srinivasan

[1], Lusztig [15] and Lusztig [20]. In the groups ${}^2A_l$ and ${}^2E_6$ the number of unipotent characters in the twisted group is the same as that for the untwisted group, and the same holds for the numbers of characters in the individual families. Moreover the degrees of the unipotent characters of the twisted groups are obtained from those of the untwisted groups by replacing q by $-q$. The situation in the twisted groups ${}^2D_l$ and ${}^3D_4$ is somewhat more complicated.

12.4 CELL REPRESENTATIONS OF THE WEYL GROUP

The division of the irreducible unipotent characters of G^F into families focuses attention on the irreducible characters of the Weyl group. We have an equivalence relation on the set $\hat{W}$ of irreducible characters of W given by $\phi \sim \phi'$ if and only if the corresponding components $\chi_\phi, \chi_{\phi'}$ of $1_{B^F}{}^{G^F}$ lie in the same family. Here G^F is the split group with Weyl group W. Each equivalence class of $\hat{W}$ then contains a unique special irreducible character.

A more direct way of describing this equivalence relation on $\hat{W}$ has been given by Lusztig in [19]. He defines a certain set of representations of W, not necessarily irreducible, called cells. For each parabolic subgroup W_J of W and each irreducible character ψ of W_J let $\psi^W = \sum n_i\phi_i$ with $\phi_i \in \hat{W}$. Let $\tilde{j}(\psi)$ be given by

$$\tilde{j}(\psi) = \sum_{\tilde{a}_{\phi_i} = \tilde{a}_\psi} n_i\phi_i.$$

We recall that $t^{\tilde{a}_\phi}$ is the highest power of t dividing $\tilde{P}_\phi(t)$ and that $\tilde{a}_{\phi_i} \geqslant \tilde{a}_\psi$ for all $\phi_i \in \hat{W}$ occurring in ψ^W. To obtain $\tilde{j}(\psi)$ from ψ we therefore truncate the induced character ψ^W to include just those components for which the value of $\tilde{a}$ is the same as for ψ. We may also define $\tilde{j}(\psi)$ for a character ψ of W_J which is not necessarily irreducible, extending the original definition by linearity. The operation $\tilde{j}$ is transitive.

The representations of W called cells can now be defined. The set of cells is the smallest set of representations of W satisfying the following conditions:

(i) The unit representation of $W = 1$ is a cell.
(ii) If ϕ is a cell of W_J then $\tilde{j}(\phi)$ and $\tilde{j}(\phi) \otimes \varepsilon$ are cells of W. (ε is the sign representation.)

The cells can be described for all the Weyl groups individually. They are given explicitly in section 13.2. They have the property that each irreducible character of W is a component of at least one cell; that each cell contains a unique special character as component and contains it with multiplicity 1; and that two cells for which the special components are distinct have no common irreducible components. Thus two irreducible characters ϕ, ϕ' of W may be said to be equivalent if there exist cells c, c' such that ϕ occurs in c, ϕ' occurs in c', and c, c' have the same special component. This equivalence relation on $\hat{W}$ defined in terms of cells turns out to be the same as the equivalence relation defined above in terms of families of unipotent characters in G^F (Lusztig [21]).

The cells can be described in an intuitive way in terms of their generic degrees. The cells are the minimal combinations of irreducible characters of W which have the property that the generic degree has integral coefficients. Each cell c satisfies the condition that

$$\tilde{P}_c(t) = P_c(t) = t^a + \dots + t^b, \quad a \leqslant b$$

where the intermediate terms have coefficients in $\mathbb{Z}$, whereas each irreducible component ϕ of c satisfies

$$\tilde{P}_\phi(t) = \frac{1}{v} t^a + \dots + \frac{1}{v} t^b$$

for some integer $v \geqslant 1$. Thus each character which is properly contained in the cell c has generic degree with non-integral coefficients.

12.5 THE KAZHDAN–LUSZTIG POLYNOMIALS

It is conjectured that the cells of representations of W are obtained from certain W-modules constructed in a paper of Kazhdan and Lusztig [1]. In this paper are defined certain polynomials $P_{y,w}(t)$ where y, w are elements of a Coxeter group W satisfying $y \leqslant w$. Here y, w are related by the Bruhat partial order on W described in section 1.9. The polynomials are defined in terms of the generic ring of W in one variable. However one considers this generic ring over the base ring $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$. For each $w \in W$ there is a unique element C_w of this generic ring such that

$$\begin{aligned} C_w &= \sum_{\substack{y \in W \\ y \leqslant w}} (-1)^{l(w)-l(y)} t^{\frac{1}{2}(l(w)-l(y))} P_{y,w}(t^{-1}) a_y \\ &= \sum_{\substack{y \in W \\ y \leqslant w}} (-1)^{l(w)-l(y)} t^{-\frac{1}{2}(l(w)-l(y))} P_{y,w}(t)(a_{y^{-1}})^{-1} \end{aligned}$$

where $P_{y,w}(t) \in \mathbb{Z}[t]$ has degree at most $\frac{1}{2}(l(w) - l(y) - 1)$ if $y < w$ and $P_{w,w}(t) = 1$. The polynomials $P_{y,w}(t)$ uniquely defined in this way may be calculated explicitly by induction on $l(w)$, starting with $P_{1,1}(t) = 1$. These polynomials are of great interest, and are relevant to the solution of a number of problems in representation theory which had formerly seemed unrelated. They are used by Kazhdan and Lusztig [1] to define certain equivalence relations on W as follows.

We write $y \prec w$ to mean that $P_{y,w}(t)$ exists and has degree equal to its maximum possible value $\frac{1}{2}(l(w) - l(y) - 1)$. We then define a pre-order relation $w \leqslant_L w'$ on W to mean that there exist elements $x_1, x_2, \dots, x_k \in W$ such that $w = x_1$, $x_k = w'$ and for each i we have either $x_{i-1} \prec x_i$ or $x_i \prec x_{i-1}$ and there exists a Coxeter generator $s \in S$ such that $l(sx_{i-1}) < l(x_{i-1})$ and $l(sx_i) > l(x_i)$. We may then define an equivalence relation $w \sim_L w'$ to mean $w \leqslant_L w'$ and $w' \leqslant_L w$. The equivalence classes with respect to the relation $\sim_L$ are called left cells. One can similarly define right cells by replacing the condition on s by $l(x_{i-1}s) < l(x_{i-1})$ and $l(x_i s) > l(x_i)$. One can also define two-sided cells by replacing the condition on s by:

$$\text{either} \quad l(sx_{i-1}) < l(x_{i-1}) \quad \text{and} \quad l(sx_i) > l(x_i)$$
$$\text{or} \quad l(x_{i-1}s) < l(x_{i-1}) \quad \text{and} \quad l(x_i s) > l(x_i).$$

It is clear that every left cell will lie in a unique two-sided cell, and the same will be true of any right cell.

Now every left cell gives rise to a representation of the Coxeter group W for which a basis of the representation space consists of the elements w in the left cell. The Coxeter generator $s \in S$ acts on this space by the linear map

$$w \xrightarrow{s} \begin{cases} -w & \text{if } l(sw) < l(w) \\ w + sw + \sum\limits_{\substack{y < w \\ l(sy) < l(y)}} \mu(y, w)y & \text{if } l(sw) > l(w) \end{cases}$$

where $\mu(y, w)$ is the coefficient of the leading term $t^{\frac{1}{2}(l(w) - l(y) - 1)}$ in $P_{y,w}(t)$.

The representations of W obtained in this way from the left cells are not in general irreducible. However, if W is a Weyl group, Lusztig has conjectured that the representations of W obtained from the left cells are the cell representations, and two left cells lie in the same two-sided cell if and only if their cell representations contain the same special representation of W as component. This conjecture would give a bijective correspondence between special representations of W and two-sided cells of W.† The families of irreducible unipotent characters of G^F would then be in natural bijective correspondence with the two-sided cells of W.

12.6 SPRINGER'S CONSTRUCTION OF REPRESENTATIONS OF THE WEYL GROUP

A very useful method of constructing the irreducible modules for a Weyl group W has been obtained by T. A. Springer. We describe Springer's construction briefly and consider the way it is related to our previous observations on the representations of W. Let $G_{\mathbb{C}}$ be the simple adjoint algebraic group over $\mathbb{C}$ whose Weyl group is W. For each unipotent element $u \in G_{\mathbb{C}}$ we consider $\mathfrak{B}_u$, the variety of all Borel subgroups of $G_{\mathbb{C}}$ containing u. This is a projective variety of dimension $e(u) = \frac{1}{2}(\dim C(u) - l)$ by 5.10.1.

Springer shows in [13] how it is possible to define an action of W on the cohomology groups $H^i(\mathfrak{B}_u, \mathbb{Q})$. The top nonvanishing cohomology group is $H^{2e(u)}(\mathfrak{B}_u, \mathbb{Q})$. We know by a theorem of Spaltenstein [1] that all the irreducible components of $\mathfrak{B}_u$ have the same dimension $e(u)$ and that $\dim_{\mathbb{Q}} H^{2e(u)}(\mathfrak{B}_u, \mathbb{Q})$ is the number of such components. There is in fact a basis of this space in natural bijective correspondence with the irreducible components of $\mathfrak{B}_u$.

† The existence of a bijection between special representations of W and two-sided cells of W has been proved by Barbasch and Vogan.

Now the centralizer $C(u)$ acts on $\mathfrak{B}_u$ by conjugation and the connected centralizer $C(u)^0$ fixes each irreducible component of $\mathfrak{B}_u$, as otherwise the connected group $C(u)^0$ would have a proper normal subgroup of finite index. Thus the finite group $A(u) = C(u)/C(u)^0$ acts on the set of components. $C(u)$ will have an induced action on $H^{2e(u)}(\mathfrak{B}_u, \mathbb{Q})$ with $C(u)^0$ acting trivially, and the quotient $A(u)$ acts on the natural basis of $H^{2e(u)}(\mathfrak{B}_u, \mathbb{Q})$ as it acts on the components of $\mathfrak{B}_u$. Thus we get a representation of $A(u)$ on $V_u = H^{2e(u)}(\mathfrak{B}_u, \mathbb{Q})$.

Now the W-action on V_u defined by Springer commutes with the action of $A(u)$. Thus we obtain an action of $A(u) \times W$ on V_u. For each irreducible character ψ of $A(u)$ let $V_{u,\psi}$ be the sum of all $A(u)$-submodules of V_u affording the character ψ. Then $V_{u,\psi}$ is a W-module. It can in fact be shown to be a direct sum of isomorphic irreducible W-modules whenever it is nonzero. Let $\phi_{u,\psi}$ be the irreducible character of W obtained from $V_{u,\psi}$ if $V_{u,\psi} \neq 0$. It is shown by Springer that each irreducible character of W has the form $\phi_{u,\psi}$ for some unipotent element $u \in G_{\mathbb{C}}$ and some irreducible character ψ of $A(u)$ for which $V_{u,\psi} \neq 0$. Moreover $\phi_{u,\psi} = \phi_{u',\psi'}$ if and only if u, u' are conjugate and $\psi = \psi'$. Thus the irreducible characters of W can be parametrized by pairs (C, ψ) where C is a unipotent conjugacy class of $G_{\mathbb{C}}$ and ψ is an irreducible character of $A(u)$ for $u \in C$. However not all characters ψ of $A(u)$ will appear in general, since for certain ψ one may have $V_{u,\psi} = 0$.

It has been shown by Alexeevski [1] and independently by Mizuno that $A(u)$ must be isomorphic to one of the groups

$$1, C_2 \times \ldots \times C_2, S_3, S_4, S_5$$

and consequently that all irreducible representations of $A(u)$ can be written over $\mathbb{Q}$. Thus the discussion of representations of W by Springer's method can be carried out over $\mathbb{Q}$ rather than over some field extension of $\mathbb{Q}$.

An alternative approach to the action of W on $H^i(\mathfrak{B}_u, \mathbb{Q})$ has been given by Slodowy [1]. This approach, using the theory of singularities, gives what is perhaps a more intuitive explanation of the reason that W acts on the cohomology of $\mathfrak{B}_u$.

We next consider how the special irreducible characters of W fit into the above description due to Springer. This has been investigated by Shoji [1], [2] for classical groups and for type F_4, by Springer for G_2 ([10], p. 205), and by Alvis and Lusztig [1] for E_6, E_7, E_8. The results are as follows. We write $\phi_u = \phi_{u,1} \otimes \varepsilon$. For example, when $u = 1$, ϕ_u is the sign representation ε of W. The map $u \to \phi_u$ then determines an injective map from the set of unipotent conjugacy classes of $G_{\mathbb{C}}$ into the set of irreducible characters of W. The image contains the set $\mathcal{S}_W$ of special irreducible characters but is in general larger than $\mathcal{S}_W$. In fact the image is the set $\bar{\mathcal{S}}_W$ which is defined in the following way. We have $W = \langle s_\alpha : \alpha \in \Delta \rangle$. Let α_0 be the highest short root in Φ. (α_0 is the highest root if all roots have the same length.) For each proper subset J of I we have a subgroup $W_{\tilde{J}}$ defined by

$$W_{\tilde{J}} = \langle s_\alpha, \alpha \in J; w_{\alpha_0} \rangle.$$

W_J is a reflection subgroup of W. So for each special irreducible character $\psi \in \mathcal{S}_{W_J}$ we may consider the irreducible character $j_{W_J}{}^W(\psi)$ of W defined in section 11.2 by truncated induction. This character will not in general be special. We define $\bar{\mathcal{S}}_W$ by

$$\bar{\mathcal{S}}_W = \{j_{W_J}{}^W(\psi);\ J \subset I,\ \psi \in \mathcal{S}_{W_J}\}.$$

We may in fact assume that $|J| = |I| - 1$ in this definition of $\bar{\mathcal{S}}_W$, since the set of $j_{W_J}{}^W(\psi)$ for such J, ψ will give the whole of $\bar{\mathcal{S}}_W$. The map $u \to \phi_u$ then induces a bijection between the set of unipotent conjugacy classes of $G_{\mathbb{C}}$ and the set $\bar{\mathcal{S}}_W$. Moreover the invariant a_{ϕ_u} giving the highest power of t dividing $P_{\phi_u}(t)$ is equal to $\dim \mathfrak{B}_u$.

12.7 SPECIAL UNIPOTENT CLASSES

Knowing the bijection between the set $\bar{\mathcal{S}}_W$ of characters of W and the set of all unipotent classes of $G_{\mathbb{C}}$, it is natural to consider those unipotent classes which correspond to characters in $\mathcal{S}_W$. These are called special unipotent classes. We know from chapter 5 that if p is not too small the unipotent classes of G are in natural bijection with the unipotent classes of $G_{\mathbb{C}}$, and so we may consider the special unipotent classes of G also. The number of special unipotent classes of G is equal to the number of families of unipotent characters of G^F.

There is a conjectural relationship between the families of unipotent characters of G^F and the special unipotent classes of G. It is conjectured by Lusztig [14] that for each irreducible unipotent character χ of the split group G^F there is a unique unipotent class C of G of maximal dimension subject to the condition $\sum_{x \in C^F} \chi(x) \neq 0$.† Moreover two unipotent characters χ, χ' of G^F will conjecturally give rise to the same unipotent class C in this way if and only if χ, χ' lie in the same family.† The set of unipotent classes C of G arising in this way should be precisely the set of special unipotent classes. Also the degree of the unipotent characters $\chi^{\mathbf{F}}_{(x,\sigma)}$ in a family $\mathbf{F}$ should have the form

$$\deg \chi^{\mathbf{F}}_{(x,\sigma)} = \pm \frac{\deg \sigma}{|C_\Gamma(x)|} \cdot q^{\dim \mathfrak{B}_u} + \text{terms involving higher powers of } q$$

where $u \in C$.† If these conjectures are true we should then have a particularly simple description of the partition of the irreducible unipotent characters of G^F into families.

Now the set of all unipotent classes of G is a partially ordered set under the relation $C \leqslant C'$ if and only if $C \subseteq \bar{C}'$. We consider in particular the partially ordered set of all special unipotent classes. This set admits an involution $C \to C^*$ defined by

$$\phi_{u^*} = i(\phi_u \otimes \varepsilon) \qquad u \in C,\ u^* \in C^*.$$

† This is known to be true if the characteristic is sufficiently large.

This comes from the involution $\phi \to i(\phi \otimes \varepsilon)$ on $\mathscr{S}_W$ described in 11.3.10. The involution $C \to C^*$ is known to be order reversing. The partially ordered sets of special unipotent classes are described in section 13.4. The set of all unipotent classes of G does not in general admit an order-reversing involution.

We observe also that the set of unipotent classes of G itself falls into families, where each family contains just one special unipotent class. For the unipotent classes of G correspond to elements of $\bar{\mathscr{S}}_W$ which, as we have seen, fall into families in which each family contains just one element of $\mathscr{S}_W$.

The bijection between unipotent classes of $G_{\mathbb{C}}$ and characters in $\bar{\mathscr{S}}_W$ leads to an interesting conjecture relating the unipotent classes to a certain affine Weyl group. The affine Weyl group W_a of a root system Φ is the Coxeter group whose diagram is the extended Dynkin diagram of Φ (Bourbaki [2], p. 173). The nodes of the extended Dynkin diagram correspond to the simple roots Δ of Φ together with the negative of the highest root. The affine Weyl group W_a^* of the dual root system Φ^* is the Coxeter group whose diagram is the extended Dynkin diagram of Φ^*. The nodes of the extended Dynkin diagram of Φ^* may be regarded as corresponding to the simple roots Δ of Φ together with the negative of the highest short root of Φ.

We now recall the definition of $\bar{\mathscr{S}}_W$. This is the set of all irreducible characters of W of the form $j_{W_J}{}^W(\psi)$ for all $\psi \in \mathscr{S}_{W_J}$ and all maximal parabolic subgroups W_J of W_a^*. Thus the definition of $\bar{\mathscr{S}}_W$ is closely linked to the affine Weyl group W_a^*. We mentioned above the fact that the elements of $\mathscr{S}_W$ are in natural bijection with the two-sided cells of W. Lusztig has further conjectured that the elements of $\bar{\mathscr{S}}_W$ are in natural bijection with the two-sided cells of W_a^*. Thus the number of two-sided cells of W_a^* should be finite and each should intersect some maximal parabolic subgroup W_J of W_a^*. It would follow that there is a natural bijection between the unipotent conjugacy classes of $G_{\mathbb{C}}$ and the two-sided cells of W_a^*.

Although we have allowed ourselves to be somewhat speculative in this chapter, the reader will see that, if the above conjectures are true, they will provide remarkable connections between the unipotent conjugacy classes, the unipotent characters, the representations of the Weyl group, and the structure of the Weyl group and the affine Weyl group.

12.8 MIDDLE INTERSECTION COHOMOLOGY

The Kazhdan–Lusztig polynomials $P_{y,w}(t)$ play a significant rôle not only in the context of unipotent representations of the groups G^F but also in problems concerned with Verma modules, with primitive ideals in universal enveloping algebras of semisimple Lie algebras and with modular representations of algebraic groups. We cannot go into detail on these subjects in the present volume. However the reason that these polynomials are so significant seems to be that they are related to a type of cohomology called middle intersection cohomology. This was introduced in the context of topological spaces by Goresky and Macpherson and in an alternative way in the context of algebraic varieties by Deligne. Given any algebraic variety X it is possible to construct for each $i \in \mathbb{Z}$ a cohomology sheaf $\mathscr{H}^i(X, \bar{\mathbb{Q}}_l)$ called the middle intersection cohomology sheaf. For each $x \in X$ the stalk of this sheaf at x is denoted by $\mathscr{H}^i_x(X, \bar{\mathbb{Q}}_l)$. If X is nonsingular then $\mathscr{H}^i_x(X, \bar{\mathbb{Q}}_l)$ coincides with the l-adic cohomology $H^i(X, \bar{\mathbb{Q}}_l)$.

The connection between the middle intersection cohomology sheaf and the Kazhdan–Lusztig polynomials for the Weyl group is as follows. Let $\mathfrak{B}_w = BwB/B$ and $\bar{\mathfrak{B}}_w$ be the closure of $\mathfrak{B}_w$ in $\mathfrak{B} = G/B$. One knows that $\mathfrak{B}_y \subseteq \bar{\mathfrak{B}}_w$ if and only if $y \leqslant w$. Suppose this is so and let $x \in \mathfrak{B}_y$. Then we have

$$P_{y,w}(t) = \sum_i \dim \mathscr{H}^{2i}_x(\bar{\mathfrak{B}}_w, \bar{\mathbb{Q}}_l)t^i$$

and $\mathscr{H}^{2i+1}_x(\bar{\mathfrak{B}}_w, \bar{\mathbb{Q}}_l) = 0$ for all i. Thus the polynomial $P_{y,w}(t)$ contains information about the dimensions of the stalks of the middle intersection cohomology sheaf of $\bar{\mathfrak{B}}_w$ at points in $\mathfrak{B}_y$.

There are indications that middle intersection cohomology will also be relevant in determining the values of the unipotent characters of G^F on the unipotent classes. The variety to be considered here is not the variety $\mathfrak{B}$ of all Borel subgroups but rather the variety of all unipotent elements of G. This is, at least, the case for groups of type A, as is shown by Lusztig in [17].

The middle intersection cohomology sheaves can best be understood in the context of the category of perverse sheaves. This category is described in detail in the paper by Beilinson, Bernstein and Deligne [1].

12.9 A JORDAN DECOMPOSITION FOR CHARACTERS

Having concluded our remarks on the unipotent characters of G^F we consider to what extent a knowledge of the semisimple characters and the unipotent characters of G^F gives a knowledge of all irreducible characters of G^F. Lusztig has proved an analogue for irreducible characters of the Jordan decomposition for elements of G^F. Suppose as usual that G is a connected reductive group with connected centre. We recall from Section 12.1 that the irreducible characters of G^F fall into geometric conjugacy classes, and that each geometric conjugacy class contains a unique semisimple character. The irreducible characters in the geometric conjugacy class containing the principal character 1 are the unipotent characters of G^F.

Lusztig's idea is to relate the characters of G^F in a given geometric conjugacy class to the unipotent characters of a smaller reductive group. Let χ be any irreducible character of G^F. Then χ lies in the same geometric conjugacy class as some semisimple character χ_s of G^F. χ_s determines a semisimple conjugacy class of the dual group G^{*F^*} as in 4.4.6. Let s^* be an element in this conjugacy class and consider $C_{G^{*F^*}}(s^*)$. It has been shown by Lusztig that there is a bijection $\chi \to \chi_u$ between irreducible characters χ of G^F geometrically conjugate to χ_s and unipotent characters χ_u of $C_{G^{*F^*}}(s^*)$. Moreover this bijection can be chosen to satisfy the condition that

$$(\chi, \varepsilon_G \varepsilon_T R_{T,\theta}) = (\chi_u, \varepsilon_{C_{G^*}(s^*)} \varepsilon_{T^*} R_{T^*,1})$$

for all pairs (T, θ) in the given geometric conjugacy class, where T^* is a maximal torus of $C_{G^*}(s^*)$ in duality with T. This property does not determine the bijection $\chi \to \chi_u$ uniquely in general. It does however imply that

$$\chi(1) = \chi_s(1)\chi_u(1).$$

The Jordan decomposition for irreducible characters of G^F can therefore be stated in the following way. There is a bijective map $\chi \to (\chi_s, \chi_u)$ between the irreducible characters of G^F and pairs (χ_s, χ_u) where χ_s is a semisimple character of G^F and χ_u is a unipotent character of $C_{G^{*F^*}}(s^*)$. The degrees of these characters satisfy the condition $\chi(1) = \chi_s(1)\chi_u(1)$.

The exposition of the topics mentioned in this chapter has necessarily been very sketchy. For further details on these subjects the reader is advised to consult the book [21] by Lusztig.

Chapter 13

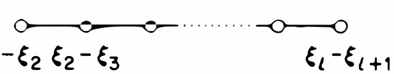
EXPLICIT RESULTS ON SIMPLE GROUPS

We shall now give a variety of explicit information about some of the subjects discussed in this book. It will be valid for simple groups and will be presented partly in the form of tables.

13.1 UNIPOTENT CLASSES OF $G_{\mathrm{ad}}(\mathbb{C})$

Let $G_{\mathrm{ad}}(\mathbb{C})$ be a simple algebraic group of adjoint type over $\mathbb{C}$. We shall describe the unipotent conjugacy classes of this group in each case. We recall from chapter 5 that this will also give us a description of the unipotent classes of $G_{\mathrm{ad}}(K)$ for an algebraically closed field K of characteristic p, provided p is sufficiently large. It will also give a description of the orbits of nilpotent elements of $\mathfrak{g}_{\mathbb{C}}$ under the action of $G_{\mathrm{ad}}(\mathbb{C})$, and of the orbits of nilpotent elements of $\mathfrak{g}_K$ under the action of $G_{\mathrm{ad}}(K)$ when p is sufficiently large.

Type A_l The unipotent classes of $G_{\mathrm{ad}}(\mathbb{C}) = PGL_{l+1}(\mathbb{C})$ are parametrized by partitions α of $l + 1$. A unipotent element of type $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ has elementary divisors in the natural matrix representation of degree $l + 1$ equal to $(t - 1)^{\alpha_1}, (t - 1)^{\alpha_2}, \ldots, (t - 1)^{\alpha_k}$. For each elementary divisor α_i we take the set of integers $\alpha_i - 1, \alpha_i - 3, \ldots, 3 - \alpha_i, 1 - \alpha_i$. We take the union of these sets for all elementary divisors and write this union as $(\xi_1, \xi_2, \ldots, \xi_{l+1})$ with $\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_{l+1}$. Then the weighted Dynkin diagram of this unipotent class (cf. section 5.6) is



Consider, for example, the group of type A_3. We are then considering partitions of 4 and the weighted Dynkin diagrams are as follows:

α	ξ_i				Diagram
4	3	1	-1	-3	2 2 2
31	2	0	0	-2	2 0 2
22	1	1	-1	-1	0 2 0
211	1	0	0	-1	1 0 1
1111	0	0	0	0	0 0 0

Type C_l The unipotent classes of $G_{\mathrm{ad}}(\mathbb{C}) = PSp_{2l}(\mathbb{C})$ are parametrized by pairs of partitions (α, β) with $|\alpha| + |\beta| = l$ where β has distinct parts. A unipotent element of type (α, β) has elementary divisors in the natural matrix representation of degree $2l$ given by $(t-1)^{\alpha_1}$, $(t-1)^{\alpha_1}, \ldots, (t-1)^{\alpha_k}$, $(t-1)^{\alpha_k}$, $(t-1)^{2\beta_1}, \ldots, (t-1)^{2\beta_h}$ where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_h)$. For each elementary divisor $(t-1)^d$ we take the set of integers $d-1, d-3, \ldots, 3-d, 1-d$. We then take the union of these sets for all elementary divisors and write this union as $(\xi_1, \xi_2, \ldots, \xi_{2l})$ with $\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_{2l}$. We then have $\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_l \geqslant 0$ and the weighted Dynkin diagram of this unipotent class is

Consider, for example, the group of type C_2. Then we have the following unipotent classes:

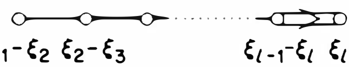
(α, β)	Elementary divisors	ξ_i	Diagram
2, –	2, 2	$1, 1, -1, -1$	0 2
11, –	1, 1, 1, 1	0, 0, 0, 0	0 0
1, 1	1, 1, 2	$1, 0, 0, -1$	1 0
–, 2	4	$3, 1, -1, -3$	2 2

Type B_l The unipotent classes of $G_{\mathrm{ad}}(\mathbb{C}) = SO_{2l+1}(\mathbb{C})$ are parametrized by pairs of partitions (α, β) with $2|\alpha| + |\beta| = 2l + 1$ where β has distinct odd parts.

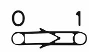
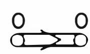
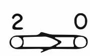
An element of type (α, β) has elementary divisors in the natural matrix representation of degree $2l + 1$ given by

$$(t-1)^{\alpha_1}, (t-1)^{\alpha_1}, \ldots, (t-1)^{\alpha_k}, (t-1)^{\alpha_k}, (t-1)^{\beta_1}, \ldots, (t-1)^{\beta_h}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_h)$. For each elementary divisor $(t-1)^d$ we take the set of integers $d-1, d-3, \ldots, 3-d, 1-d$. We then take the union of these sets for all elementary divisors and write this union as $(\xi_1, \xi_2, \ldots, \xi_{2l+1})$ with $\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_{2l+1}$. We then have $\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_l \geqslant 0$ and the weighted Dynkin diagram of this unipotent class is



Consider, for example, the group of type B_2. Then we have the following unipotent classes:

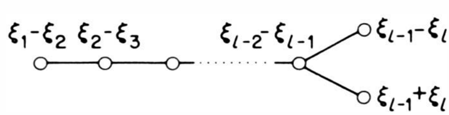
(α, β)	Elementary divisors	ξ_i	Diagram
2, 1	2, 2, 1	1, 1, 0, -1, -1	
11, 1	1, 1, 1, 1, 1	0, 0, 0, 0, 0	
1, 3	3, 1, 1	2, 0, 0, 0, -2	
$-$, 5	5	4, 2, 0, -2, -4	

Type D_l The unipotent classes of $G_{\mathrm{ad}}(\mathbb{C}) = PSO_{2l}(\mathbb{C})$ are parametrized by pairs of partitions (α, β) with $2|\alpha| + |\beta| = 2l$ where β has distinct odd parts. There are two classes of this kind when β is empty and all parts of α are even. An element of type (α, β) has elementary divisors in the natural matrix representation of degree $2l$ given by

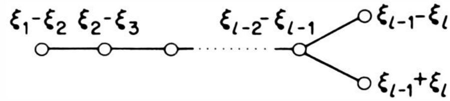
$$(t-1)^{\alpha_1}, (t-1)^{\alpha_1}, \ldots, (t-1)^{\alpha_k}, (t-1)^{\alpha_k}, (t-1)^{\beta_1}, \ldots, (t-1)^{\beta_h}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_h)$.

For each elementary divisor $(t-1)^d$ we take the set of integers $d-1, d-3, \ldots, 3-d, 1-d$. We then take the union of these sets for all elementary divisors and write this union as $(\xi_1, \xi_2, \ldots, \xi_{2l})$ with $\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_{2l}$. We then have $\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_l \geqslant 0$ and the weighted Dynkin diagram of this unipotent class is

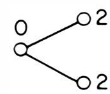
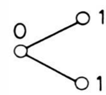
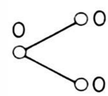
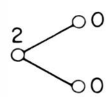
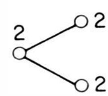


or



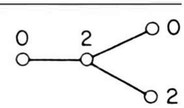
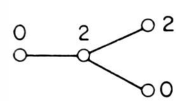
If there is some odd part in (α, β) then $\xi_l = 0$ and so these diagrams are the same. If β is empty and all parts of α are even then $\xi_l \neq 0$ and we obtain two weighted Dynkin diagrams corresponding to the two unipotent classes of this type.

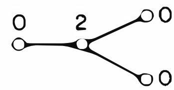
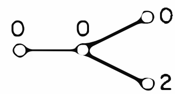
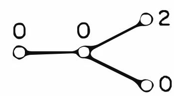
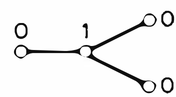
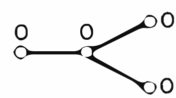
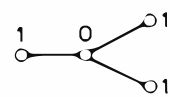
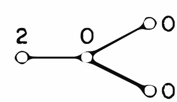
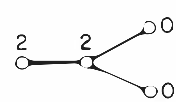
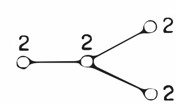
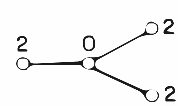
Consider, for example, the group of type D_3. Then we have the following unipotent classes:

(α, β)	Elementary divisors	ξ_i	Diagram
3, –	3, 3	2, 2, 0, 0, -2, -2	
21, –	2, 2, 1, 1	1, 1, 0, 0, -1, -1	
111, –	1, 1, 1, 1, 1, 1	0, 0, 0, 0, 0, 0	
1, 31	3, 1, 1, 1	2, 0, 0, 0, 0, -2	
–, 51	5, 1	4, 2, 0, 0, -2, -4	

Of course D_3 is the same as A_3 so we obtain the same weighted Dynkin diagrams as in type A_3 above.

Next consider type D_4. Then we obtain the following unipotent classes:

(α, β)	Elementary divisors	ξ_i	Diagram
4, –	4, 4	3, 3, 1, 1, -1, -1, -3, -3	 

(α, β)	Elementary divisors	ξ_i	Diagram
31, –	3, 3, 1, 1	2, 2, 0, 0, 0, 0, -2, -2	
22, –	2, 2, 2, 2	1, 1, 1, 1, -1, -1, -1, -1	
			
211, –	2, 2, 1, 1, 1, 1	1, 1, 0, 0, 0, 0, -1, -1	
1111, –	1, 1, 1, 1, 1, 1, 1, 1	0, 0, 0, 0, 0, 0, 0, 0	
2, 31	3, 2, 2, 1	2, 1, 1, 0, 0, -1, -1, -2	
11, 31	3, 1, 1, 1, 1, 1	2, 0, 0, 0, 0, 0, 0, -2	
1, 51	5, 1, 1, 1	4, 2, 0, 0, 0, 0, -2, -4	
–, 71	7, 1	6, 4, 2, 0, 0, -2, -4, -6	
–, 53	5, 3	4, 2, 2, 0, 0, -2, -2, -4	

We now consider the structure of the connected centralizer $C(u)^0$ of a unipotent element $u \in G_{ad}(\mathbb{C})$. We again concentrate for the moment on the groups of classical type. We have

$$C(u)^0 = RC, \qquad R \cap C = 1$$

where R is the unipotent radical of $C(u)^0$ and C is a connected reductive group. Following Springer–Steinberg in Borel *et al.* [1], chapter IV, we give for each unipotent element u the type of the reductive group C, the dimension of the unipotent group R and the dimension of the centralizer $C(u)$.

For each unipotent element u in a group $G_{ad}(\mathbb{C})$ of classical type we denote by r_i the number of elementary divisors of u in the natural matrix representation which are equal to $(t-1)^i$. The results may then be tabulated as follows.

Type A_l

Type of C

$$\left(\prod_i A_{r_i - 1}\right) \times T_k$$

where $k = (\text{No. of } r_i) - 1$ and T_k is a torus of dimension k.

dim R

$$\sum_i (r_i + r_{i+1} + \dots)^2 - \sum_i r_i^2$$

dim $C(u)$

$$\sum_i (r_i + r_{i+1} + \dots)^2 - 1.$$

Type C_l

Type of C

$$\prod_{i \text{ odd}} C_{r_i/2} \times \prod_{\substack{i \text{ even} \\ r_i \text{ even}}} D_{r_i/2} \times \prod_{\substack{i \text{ even} \\ r_i \text{ odd}}} B_{(r_i - 1)/2}$$

dim R

$$\tfrac{1}{2}\sum_i (r_i + r_{i+1} + \dots)^2 - \tfrac{1}{2}\sum_i r_i^2 + \tfrac{1}{2}\sum_{i \text{ even}} r_i$$

dim $C(u)$

$$\tfrac{1}{2}\sum_i (r_i + r_{i+1} + \dots)^2 + \tfrac{1}{2}\sum_{i \text{ odd}} r_i.$$

Types B_l and D_l

Type of C

$$\prod_{i \text{ even}} C_{r_i/2} \times \prod_{\substack{i \text{ odd} \\ r_i \text{ even}}} D_{r_i/2} \times \prod_{\substack{i \text{ odd} \\ r_i \text{ odd}}} B_{(r_i - 1)/2}$$

dim R

$$\tfrac{1}{2}\sum_i (r_i + r_{i+1} + \ldots)^2 - \tfrac{1}{2}\sum_i r_i^2 - \tfrac{1}{2}\sum_{i \text{ even}} r_i$$

dim $C(u)$

$$\tfrac{1}{2}\sum_i (r_i + r_{i+1} + \ldots)^2 - \tfrac{1}{2}\sum_{i \text{ odd}} r_i.$$

Note that in these tables D_1 must be interpreted as a 1-dimensional torus T_1 wherever it occurs.

We next consider the group of components $C(u)/C(u)^0$ of the centralizer of u. This group is isomorphic to $C_2 \times \ldots \times C_2$ ($a(u)$ factors) in all groups $G_{ad}(\mathbb{C})$ of classical type, for suitable $a(u)$. The integers $a(u)$ are determined by Springer–Steinberg in Borel *et al.* [1], chapter IV. A minor modification is needed in obtaining our results from theirs since the groups we are considering here are connected of adjoint type. The results are as follows.

Type A_l $a(u) = 1$ for all u.

Type C_l $a(u) = n(u) - \delta(u)$ where $n(u)$ is the number of even i such that $r_i > 0$ and $\delta(u)$ is 1 if there is an even i with r_i odd and 0 otherwise.

Type B_l

$$a(u) = \begin{cases} n(u) - 1 & \text{if } n(u) \geq 1 \\ 0 & \text{if } n(u) = 0 \end{cases}$$

where $n(u)$ is the number of odd i such that $r_i > 0$.

Type D_l

$$a(u) = \begin{cases} n(u) - 1 - \delta(u) & \text{if } n(u) \geq 1 \\ 0 & \text{if } n(u) = 0 \end{cases}$$

where $n(u)$ is the number of odd i such that $r_i > 0$ and $\delta(u)$ is 1 if there is an odd i with r_i odd and 0 otherwise.

We give a few examples to illustrate these results. We describe the type of C, the dimension of R, the dimension of $C(u)$, and the group $C(u)/C(u)^0$ for all unipotent elements in groups of type A_3, C_2, C_3, B_2, D_3. Note that since $B_2 = C_2$ and $A_3 = D_3$ the results are the same for these two pairs of groups.

Type A_3

Elementary divisors	Type of C	dim R	dim $C(u)$	$C(u)/C(u)^0$
4	1	3	3	1
3, 1	T_1	4	5	1
2, 2	A_1	4	7	1
2, 1, 1	$A_1 \times T_1$	5	9	1
1, 1, 1, 1	A_3	0	15	1

Type C_2

Elementary divisors	Type of C	dim R	dim $C(u)$	$C(u)/C(u)^0$
2,2	T_1	3	4	C_2
1, 1, 1, 1	C_2	0	10	1
2, 1, 1	C_1	3	6	1
4	1	2	2	1

Type B_2

Elementary divisors	Type of C	dim R	dim $C(u)$	$C(u)/C(u)^0$
2, 2, 1	C_1	3	6	1
1, 1, 1, 1, 1	B_2	0	10	1
3, 1, 1	T_1	3	4	C_2
5	1	2	2	1

Type C_3

Elementary divisors	Type of C	dim R	dim $C(u)$	$C(u)/C(u)^0$
3, 3	C_1	4	7	1
2, 2, 1, 1	$C_1 + T_1$	7	11	C_2
1, 1, 1, 1, 1, 1	C_3	0	21	1
2, 2, 2	B_1	6	9	1
2, 1, 1, 1, 1	C_2	5	15	1
4, 1, 1	C_1	4	7	1
6	1	3	3	1
4, 2	1	5	5	C_2

Type D_3

Elementary divisors	Type of C	dim R	dim $C(u)$	$C(u)/C(u)^0$
3, 3	T_1	4	5	1
2, 2, 1, 1	$C_1 \times T_1$	5	9	1
1, 1, 1. 1, 1, 1	D_3	0	15	1
3, 1, 1, 1	B_1	4	7	1
5, 1	1	3	3	1

We now turn to the groups $G_{ad}(\mathbb{C})$ of exceptional type. In these groups we label the unipotent classes by the distinguished parabolic subgroup P_L of the Levi subgroup L corresponding to the given unipotent class by the Bala–Carter theorem 5.9.5(ii). The labelling for the distinguished parabolic subgroups is given at the end of chapter 5. For each unipotent class in $G_{ad}(\mathbb{C})$ we give its weighted Dynkin diagram, the type of C, the dimension of R, the dimension of $C(u)$, the dimension of $\mathfrak{B}_u$, and the structure of $C(u)/C(u)^0$. The dimension of the variety $\mathfrak{B}_u$ of Borel subgroups containing u can be obtained from 5.10.1. The structure of the groups $C(u)/C(u)^0$ was obtained by Alexeevski [1] and independently by Mizuno. Information about $C(u)$, C and R was obtained by Elashvili [1]. The results for the exceptional groups are as follows. Note that in these tables we sometimes distinguish between types A_i and $\tilde{A}_i$. The reason for this is that in groups for which there are roots of different lengths a Levi subgroup of type A_i can either correspond to a subdiagram of the Dynkin diagram representing long roots or a subdiagram representing short roots. These will give nonconjugate Levi subgroups. If the subdiagram represents long roots we write A_i and if it represents short roots we write $\tilde{A}_i$.

Type G_2

Unipotent class	Diagram	Type of C	dim R	dim $C(u)$	dim $\mathfrak{B}_u$	$C(u)/C(u)^0$
1	0 0	G_2	0	14	6	1
A_1	1 0	A_1	5	8	3	1
$\tilde{A}_1$	0 1	A_1	3	6	2	1
$G_2(a_1)$	2 0	1	4	4	1	S_3
G_2	2 2	1	2	2	0	1

Type F_4

Unipotent class	Diagram	Type of C	dim R	dim $C(u)$	dim $\mathfrak{B}_u$	$C(u)/C(u)^0$
1	0 0 0 0	F_4	0	52	24	1
A_1	1 0 0 0	C_3	15	36	16	1
$\tilde{A}_1$	0 0 0 1	A_3	15	30	13	S_2
$A_1 + \tilde{A}_1$	0 1 0 0	$A_1 + A_1$	18	24	10	1
A_2	2 0 0 0	A_2	14	22	9	S_2
$\tilde{A}_2$	0 0 0 2	G_2	8	22	9	1
$A_2 + \tilde{A}_1$	0 0 1 0	A_1	15	18	7	1
B_2	2 0 0 1	$A_1 + A_1$	10	16	6	S_2
$\tilde{A}_2 + A_1$	0 1 0 1	A_1	13	16	6	1
$C_3(a_1)$	1 0 1 0	A_1	11	14	5	S_2
$F_4(a_3)$	0 2 0 0	1	12	12	4	S_4
B_3	2 2 0 0	A_1	7	10	3	1
C_3	1 0 1 2	A_1	7	10	3	1
$F_4(a_2)$	0 2 0 2	1	8	8	2	S_2
$F_4(a_1)$	2 2 0 2	1	6	6	1	S_2
F_4	2 2 2 2	1	4	4	0	1

Type E_6

Unipotent class	Diagram	Type of C	dim R	dim $C(u)$	dim $\mathfrak{B}_u$	$C(u)/C(u)^0$
1	$\begin{smallmatrix} 0&0&0&0&0\\ &&0&& \end{smallmatrix}$	E_6	0	78	36	1
A_1	$\begin{smallmatrix} 0&0&0&0&0\\ &&1&& \end{smallmatrix}$	A_5	21	56	25	1
$2A_1$	$\begin{smallmatrix} 1&0&0&0&1\\ &&0&& \end{smallmatrix}$	$B_3 + T_1$	24	46	20	1
$3A_1$	$\begin{smallmatrix} 0&0&1&0&0\\ &&0&& \end{smallmatrix}$	$A_2 + A_1$	27	38	16	1
A_2	$\begin{smallmatrix} 0&0&0&0&0\\ &&2&& \end{smallmatrix}$	$A_2 + A_2$	20	36	15	S_2
$A_2 + A_1$	$\begin{smallmatrix} 1&0&0&0&1\\ &&1&& \end{smallmatrix}$	$A_2 + T_1$	23	32	13	1
$2A_2$	$\begin{smallmatrix} 2&0&0&0&2\\ &&0&& \end{smallmatrix}$	G_2	16	30	12	1
$A_2 + 2A_1$	$\begin{smallmatrix} 0&1&0&1&0\\ &&0&& \end{smallmatrix}$	$A_1 + T_1$	24	28	11	1
A_3	$\begin{smallmatrix} 1&0&0&0&1\\ &&2&& \end{smallmatrix}$	$B_2 + T_1$	15	26	10	1
$2A_2 + A_1$	$\begin{smallmatrix} 1&0&1&0&1\\ &&0&& \end{smallmatrix}$	A_1	21	24	9	1
$A_3 + A_1$	$\begin{smallmatrix} 0&1&0&1&0\\ &&1&& \end{smallmatrix}$	$A_1 + T_1$	18	22	8	1
$D_4(a_1)$	$\begin{smallmatrix} 0&0&2&0&0\\ &&0&& \end{smallmatrix}$	T_2	18	20	7	S_3
A_4	$\begin{smallmatrix} 2&0&0&0&2\\ &&2&& \end{smallmatrix}$	$A_1 + T_1$	14	18	6	1
D_4	$\begin{smallmatrix} 0&0&2&0&0\\ &&2&& \end{smallmatrix}$	A_2	10	18	6	1
$A_4 + A_1$	$\begin{smallmatrix} 1&1&0&1&1\\ &&1&& \end{smallmatrix}$	T_1	15	16	5	1
A_5	$\begin{smallmatrix} 2&1&0&1&2\\ &&1&& \end{smallmatrix}$	A_1	11	14	4	1
$D_5(a_1)$	$\begin{smallmatrix} 1&1&0&1&1\\ &&2&& \end{smallmatrix}$	T_1	13	14	4	1
$E_6(a_3)$	$\begin{smallmatrix} 2&0&2&0&2\\ &&0&& \end{smallmatrix}$	1	12	12	3	S_2
D_5	$\begin{smallmatrix} 2&0&2&0&2\\ &&2&& \end{smallmatrix}$	T_1	9	10	2	1
$E_6(a_1)$	$\begin{smallmatrix} 2&2&0&2&2\\ &&2&& \end{smallmatrix}$	1	8	8	1	1
E_6	$\begin{smallmatrix} 2&2&2&2&2\\ &&2&& \end{smallmatrix}$	1	6	6	0	1

Type E_7

Unipotent class	Diagram	Type of C	dim R	dim $C(u)$	dim $\mathfrak{B}_u$	$C(u)/C(u)^0$
1	0 0 0 0 0 0 0	E_7	0	133	63	1
A_1	0 0 0 0 0 1 0	D_6	33	99	46	1
$2A_1$	0 1 0 0 0 0 0	$B_4 + A_1$	42	81	37	1
$(3A_1)''$	2 0 0 0 0 0 0	F_4	27	79	36	1
$(3A_1)'$	0 0 0 0 1 0 0	$C_3 + A_1$	45	69	31	1
A_2	0 0 0 0 0 2 0	A_5	32	67	30	S_2
$4A_1$	1 0 0 0 0 0 1	C_3	42	63	28	1
$A_2 + A_1$	0 1 0 0 0 1 0	$A_3 + T_1$	41	57	25	S_2
$A_2 + 2A_1$	0 0 0 1 0 0 0	$3A_1$	42	51	22	1
A_3	0 1 0 0 0 2 0	$B_3 + A_1$	25	49	21	1
$2A_2$	0 2 0 0 0 0 0	$G_2 + A_1$	32	49	21	1
$A_2 + 3A_1$	0 0 0 0 0 0 2	G_2	35	49	21	1
$(A_3 + A_1)''$	2 0 0 0 0 2 0	B_3	26	47	20	1
$2A_2 + A_1$	0 1 0 0 1 0 0	$2A_1$	37	43	18	1
$(A_3 + A_1)'$	0 0 0 1 0 1 0	$3A_1$	32	41	17	1
$D_4(a_1)$	0 0 0 0 2 0 0	$3A_1$	30	39	16	S_3
$A_3 + 2A_1$	1 0 1 0 0 1 0	$2A_1$	33	39	16	1
D_4	0 0 0 0 2 2 0	C_3	16	37	15	1
$D_4(a_1) + A_1$	1 0 0 0 1 0 1	$2A_1$	31	37	15	S_2
$A_3 + A_2$	0 1 0 1 0 0 0	$A_1 + T_1$	31	35	14	S_2
A_4	0 2 0 0 0 2 0	$A_2 + T_1$	24	33	13	S_2
$A_3 + A_2 + A_1$	0 0 2 0 0 0 0	A_1	30	33	13	1
$(A_5)''$	2 2 0 0 0 2 0	G_2	17	31	12	1

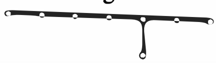
Type E_7—continued

Unipotent class	Diagram	Type of C	dim R	dim $C(u)$	dim $\mathfrak{B}_u$	$C(u)/C(u)^0$
$D_4 + A_1$	1 0 0 0 1 2 / 1	B_2	21	31	12	1
$A_4 + A_1$	0 1 0 1 0 1 / 0	T_2	27	29	11	S_2
$D_5(a_1)$	0 1 0 1 0 2 / 0	$A_1 + T_1$	23	27	10	S_2
$A_4 + A_2$	0 0 0 2 0 0 / 0	A_1	24	27	10	1
$(A_5)'$	0 2 0 1 0 1 / 0	$2A_1$	19	25	9	1
$A_5 + A_1$	2 1 0 1 0 1 / 0	A_1	22	25	9	1
$D_5(a_1) + A_1$	0 0 2 0 0 2 / 0	A_1	22	25	9	1
$D_6(a_2)$	2 0 1 0 1 0 / 1	A_1	20	23	8	1
$E_6(a_3)$	0 2 0 0 2 0 / 0	A_1	20	23	8	S_2
D_5	0 2 0 0 2 2 / 0	$2A_1$	15	21	7	1
$E_7(a_5)$	2 0 0 2 0 0 / 0	1	21	21	7	S_3
A_6	0 2 0 2 0 0 / 0	A_1	16	19	6	1
$D_5 + A_1$	0 1 1 0 1 2 / 1	A_1	16	19	6	1
$D_6(a_1)$	2 0 1 0 1 2 / 1	A_1	16	19	6	1
$E_7(a_4)$	2 0 0 2 0 2 / 0	1	17	17	5	S_2
D_6	2 2 1 0 1 2 / 1	A_1	12	15	4	1
$E_6(a_1)$	0 2 0 2 0 2 / 0	T_1	14	15	4	S_2
E_6	0 2 0 2 2 2 / 0	A_1	10	13	3	1
$E_7(a_3)$	2 2 0 2 0 2 / 0	1	13	13	3	S_2
$E_7(a_2)$	2 0 2 0 2 2 / 2	1	11	11	2	1
$E_7(a_1)$	2 2 2 0 2 2 / 2	1	9	9	1	1
E_7	2 2 2 2 2 2 / 2	1	7	7	0	1

Type E_8

Unipotent class	Diagram	Type of C	dim R	dim $C(u)$	dim $\mathfrak{B}_u$	$C(u)/C(u)^0$
1	0 0 0 0 0 0 0 0	E_8	0	248	120	1
A_1	1 0 0 0 0 0 0 0	E_7	57	190	91	1
$2A_1$	0 0 0 0 0 0 1 0	B_6	78	156	74	1
$3A_1$	0 1 0 0 0 0 0 0	$F_4 + A_1$	81	136	64	1
A_2	2 0 0 0 0 0 0 0	E_6	56	134	63	S_2
$4A_1$	0 0 0 0 0 0 0 1	C_4	84	120	56	1
$A_2 + A_1$	1 0 0 0 0 0 1 0	A_5	77	112	52	S_2
$A_2 + 2A_1$	0 0 1 0 0 0 0 0	$B_3 + A_1$	78	102	47	1
A_3	2 0 0 0 0 0 1 0	B_5	45	100	46	1
$A_2 + 3A_1$	0 0 0 0 0 1 0 0	$G_2 + A_1$	77	94	43	1
$2A_2$	0 0 0 0 0 0 2 0	$2G_2$	64	92	42	S_2
$2A_2 + A_1$	0 1 0 0 0 0 1 0	$G_2 + A_1$	69	86	39	1
$A_3 + A_1$	1 0 1 0 0 0 0 0	$B_3 + A_1$	60	84	38	1
$D_4(a_1)$	0 2 0 0 0 0 0 0	D_4	54	82	37	S_3
D_4	2 2 0 0 0 0 0 0	F_4	28	80	36	1
$2A_2 + 2A_1$	0 0 0 1 0 0 0 0	B_2	70	80	36	1
$A_3 + 2A_1$	1 0 0 0 0 1 0 0	$B_2 + A_1$	63	76	34	1
$D_4(a_1) + A_1$	0 1 0 0 0 0 0 1	$3A_1$	63	72	32	S_3
$A_3 + A_2$	0 0 1 0 0 0 1 0	$B_2 + T_1$	59	70	31	S_2
A_4	2 0 0 0 0 0 2 0	A_4	44	68	30	S_2
$A_3 + A_2 + A_1$	0 0 0 0 1 0 0 0	$2A_1$	60	66	29	1
$D_4 + A_1$	2 1 0 0 0 0 0 1	C_3	43	64	28	1
$D_4(a_1) + A_2$	0 0 0 0 0 0 0 2	A_2	56	64	28	S_2

Type E_8—continued

Unipotent class	Diagram 	Type of C	dim R	dim $C(u)$	dim $\mathfrak{B}_u$	$C(u)/C(u)^0$
$A_4 + A_1$	1 0 1 0 0 0 1 0	$A_2 + T_1$	51	60	26	S_2
$2A_3$	0 0 0 1 0 0 1 0	B_2	50	60	26	1
$D_5(a_1)$	2 0 1 0 0 0 1 0	A_3	43	58	25	S_2
$A_4 + 2A_1$	1 0 0 0 1 0 0 0	$A_1 + T_1$	52	56	24	S_2
$A_4 + A_2$	0 0 2 0 0 0 0 0	$2A_1$	48	54	23	1
A_5	1 0 1 0 0 0 2 0	$G_2 + A_1$	35	52	22	1
$D_5(a_1) + A_1$	2 0 0 0 1 0 0 0	$2A_1$	46	52	22	1
$A_4 + A_2 + A_1$	0 0 1 0 0 1 0 0	A_1	49	52	22	1
$D_4 + A_2$	2 0 0 0 0 0 0 2	A_2	42	50	21	S_2
$E_6(a_3)$	0 2 0 0 0 0 2 0	G_2	36	50	21	S_2
D_5	2 2 0 0 0 0 2 0	B_3	27	48	20	1
$A_4 + A_3$	0 1 0 0 1 0 0 0	A_1	45	48	20	1
$A_5 + A_1$	1 0 0 0 1 0 1 0	$2A_1$	40	46	19	1
$D_5(a_1) + A_2$	1 0 1 0 0 1 0 0	A_1	43	46	19	1
$D_6(a_2)$	0 1 0 0 0 1 0 1	$2A_1$	38	44	18	S_2
$E_6(a_3) + A_1$	0 1 0 1 0 0 1 0	A_1	41	44	18	S_2
$E_7(a_5)$	0 0 1 0 1 0 0 0	A_1	39	42	17	S_3
$D_5 + A_1$	2 1 0 1 0 0 1 0	$2A_1$	34	40	16	1
$E_8(a_7)$	0 0 0 2 0 0 0 0	1	40	40	16	S_5
A_6	0 0 2 0 0 0 2 0	$2A_1$	32	38	15	1
$D_6(a_1)$	2 1 0 0 0 1 0 1	$2A_1$	32	38	15	S_2
$A_6 + A_1$	0 0 1 0 1 0 1 0	A_1	33	36	14	1
$E_7(a_4)$	2 0 1 0 1 0 0 0	A_1	33	36	14	S_2
$E_6(a_1)$	2 0 2 0 0 0 2 0	A_2	26	34	13	S_2

Type E_8—continued

Unipotent class	Diagram 	Type of C	dim R	dim $C(u)$	dim $\mathfrak{B}_u$	$C(u)/C(u)^0$
$D_5 + A_2$	2 0 0 2 0 0 0 0	T_1	33	34	13	S_2
D_6	2 1 0 0 0 1 2 1	B_2	22	32	12	1
E_6	2 2 2 0 0 0 2 0	G_2	18	32	12	1
$D_7(a_2)$	1 0 1 0 1 0 1 0	T_1	31	32	12	S_2
A_7	0 1 1 0 1 0 1 0	A_1	27	30	11	1
$E_6(a_1) + A_1$	2 0 1 0 1 0 1 0	T_1	29	30	11	S_2
$E_7(a_3)$	2 0 1 0 1 0 2 0	A_1	25	28	10	S_2
$E_8(b_6)$	2 0 0 0 2 0 0 0	1	28	28	10	S_3
$D_7(a_1)$	2 0 0 2 0 0 2 0	T_1	25	26	9	S_2
$E_6 + A_1$	2 2 1 0 1 0 1 0	A_1	23	26	9	1
$E_7(a_2)$	2 2 0 1 0 1 0 1	A_1	21	24	8	1
$E_8(a_6)$	0 2 0 0 2 0 0 0	1	24	24	8	S_3
D_7	1 0 1 1 0 1 2 1	A_1	19	22	7	1
$E_8(b_5)$	2 2 0 0 2 0 0 0	1	22	22	7	S_3
$E_7(a_1)$	2 2 0 1 0 1 2 1	A_1	17	20	6	1
$E_8(a_5)$	0 2 0 0 2 0 2 0	1	20	20	6	S_2
$E_8(b_4)$	2 2 0 0 2 0 2 0	1	18	18	5	S_2
E_7	2 2 2 1 0 1 2 1	A_1	13	16	4	1
$E_8(a_4)$	2 0 2 0 2 0 2 0	1	16	16	4	S_2
$E_8(a_3)$	2 2 2 0 2 0 2 0	1	14	14	3	S_2
$E_8(a_2)$	2 2 0 2 0 2 2 2	1	12	12	2	1
$E_8(a_1)$	2 2 2 2 0 2 2 2	1	10	10	1	1
E_8	2 2 2 2 2 2 2 2	1	8	8	0	1

13.2 FAMILIES OF IRREDUCIBLE CHARACTERS OF THE WEYL GROUP

We have described in section 12.4 how the irreducible characters of the Weyl group W can be divided into families in such a way that each family contains a unique special representation. The families are defined by means of certain representations of W, not necessarily irreducible, called cells. We shall now describe explicitly the families of irreducible characters of each indecomposable Weyl group, and also describe the cells as integral combinations of irreducible characters. The results are due to Lusztig [19].

Type A_l We recall from 11.4.1 that the irreducible characters of $W(A_l)$ have the form ϕ_α where α is a partition of $l + 1$. The decomposition into families is trivial in this case, since each ϕ_α lies in a family by itself. The cell representations are simply the irreducible characters ϕ_α of W.

Type B_l, C_l We recall from 11.4.2 and 11.4.3 that the irreducible characters of $W(B_l) \cong W(C_l)$ have the form $\phi_{\alpha,\beta}$ where α, β are partitions such that $|\alpha| + |\beta| = l$. To each such pair (α, β) of partitions we can associate a symbol class as in 11.4.2. Let

$$\begin{pmatrix} \lambda_1, & & \lambda_2, & & \ldots, & & \lambda_{m+1} \\ & \mu_1, & & \mu_2, \ldots, & \mu_m & & \end{pmatrix}$$

be a symbol in this class. Then two characters lie in the same family if and only if they possess symbols for which the unordered sets $\{\lambda_1, \ldots, \lambda_{m+1}, \mu_1, \ldots, \mu_m\}$ are the same. This unordered set will have the property that no number occurs more than twice. Each family contains a unique character whose symbol has the property that

$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \mu_m \leqslant \lambda_{m+1}.$$

The characters with this property are the special characters of W.

We now describe the cell representations for $W(B_l)$. Each family of irreducible characters of W determines a set of symbols

$$\begin{pmatrix} \lambda_1, \lambda_2, & \ldots, & \lambda_{m+1} \\ \mu_1, & \mu_2, \ldots, \mu_m & \end{pmatrix}$$

with the same unordered set $\{\lambda_1, \ldots, \lambda_{m+1}, \mu_1, \ldots, \mu_m\}$. Let this set be $\{z_1, \ldots, z_{2m+1}\}$ where $z_1 \leqslant z_2 \leqslant \ldots \leqslant z_{2m+1}$. Each number occurs at most twice as a z_i. A pairing on $\{z_1, \ldots, z_{2m+1}\}$ is an arrangement of these numbers into m pairs and one isolated element. A pairing is called admissible if it satisfies the following two conditions:

(a) whenever $z_i = z_{i+1}$, (z_i, z_{i+1}) is a pair.

(b) If all numbers occurring twice are removed then some pair consists of consecutive z's remaining, and the same is true when this pair is removed also, and so on until just one z is left.

Each set $\{z_1, \ldots, z_{2m+1}\}$ gives rise to one cell representation for each admissible pairing. The cell corresponding to a given admissible pairing is the sum of all the $\phi_{\alpha,\beta}$ with symbol

$$\begin{pmatrix} \lambda_1, & & \ldots, \lambda_{m+1} \\ & \mu_1, \ldots, \mu_m & \end{pmatrix}$$

satisfying $\{\lambda_1, \ldots, \lambda_{m+1}, \mu_1, \ldots, \mu_m\} = \{z_1, \ldots, z_{2m+1}\}$ and such that $\mu_1, \ldots, \mu_m$ contains one number from each of the m pairs.

A typical cell will be a sum of 2^r distinct irreducible characters $\phi_{\alpha,\beta}$ where there are $2r + 1$ z_i's occurring without repetition in $\{z_1, \ldots, z_{2m+1}\}$.

We illustrate this by considering the example of type B_2. We list the set of all ordered pairs (α, β) of partitions with $|\alpha| + |\beta| = 2$ and give the symbol corresponding to each for which m is as small as possible:

(α, β)	$(2, -)$	$(11, -)$	$(1, 1)$	$(-, 2)$	$(-, 11)$
Symbol	$\begin{pmatrix} 2 \\ - \end{pmatrix}$	$\begin{pmatrix} 1 & & 2 \\ & 0 & \end{pmatrix}$	$\begin{pmatrix} 0 & & 2 \\ & 1 & \end{pmatrix}$	$\begin{pmatrix} 0 & & 1 \\ & 2 & \end{pmatrix}$	$\begin{pmatrix} 0 & & 1 & & 2 \\ & 1 & & 2 & \end{pmatrix}$

The irreducible characters $\phi_{\alpha,\beta}$ thus fall into families as follows:

$$\phi_{(11,-)} \sim \phi_{(1,1)} \sim \phi_{(-,2)}$$

$$\phi_{(2,-)}$$

$$\phi_{(-,11)}$$

We now describe the cells corresponding to these three families. Corresponding to the family $\{\phi_{(2,-)}\}$ we have a single cell $\phi_{(2,-)}$. Corresponding to the family $\{\phi_{(-,11)}\}$ we have a single cell $\phi_{(-,11)}$. Corresponding to the family $\{\phi_{(11,-)}, \phi_{(1,1)}, \phi_{(-,2)}\}$ there are two cells. They are:

$\phi_{(11,-)} + \phi_{(1,1)}$	corresponding to the pairing (0 1)(2)
$\phi_{(1,1)} + \phi_{(-,2)}$	corresponding to the pairing (1 2)(0).

We note that the pairing (0 2)(1) is not admissible.

Type D_l We recall from 11.4.4 that the irreducible characters of $W(D_l)$ have the form $\phi_{\alpha,\beta}$ where α, β are partitions such that $|\alpha| + |\beta| = l$. One has $\phi_{\alpha,\beta} = \phi_{\beta,\alpha}$. If $\alpha = \beta$ one has two irreducible characters $\phi_{\alpha,\alpha}'$, $\phi_{\alpha,\alpha}''$. To each pair (α, β) of partitions we associate a symbol class as in 11.4.4. Let

$$\begin{pmatrix} \lambda_1, \ldots, \lambda_m \\ \mu_1, \ldots, \mu_m \end{pmatrix} = \begin{pmatrix} \mu_1, \ldots, \mu_m \\ \lambda_1, \ldots, \lambda_m \end{pmatrix}$$

be a symbol in this class. Two irreducible characters $\phi_{\alpha,\beta}$ with $\alpha \neq \beta$ lie in the same family if and only if they possess symbols for which the unordered sets

$\{\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m\}$ are the same. Characters of the form $\phi_{\alpha,\alpha}'$ and $\phi_{\alpha,\alpha}''$ lie in a family by themselves. Each family contains a unique character whose symbol has the property that

$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_m \leqslant \mu_m \quad \text{or} \quad \mu_1 \leqslant \lambda_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_m \leqslant \lambda_m.$$

The characters with this property are the special characters of $W(D_l)$. In particular $\phi_{\alpha,\alpha}'$ and $\phi_{\alpha,\alpha}''$ are special characters.

Having described the families of irreducible characters we now describe the cell representations. Each family of irreducible characters of $W(D_l)$ determines a set of symbols

$$\begin{pmatrix} \lambda_1, \ldots, \lambda_m \\ \mu_1, \ldots, \mu_m \end{pmatrix}$$

with the same unordered set $\{\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m\}$. Let this set be $\{z_1, \ldots, z_{2m}\}$ with $z_1 \leqslant z_2 \leqslant \ldots \leqslant z_{2m}$. Each number occurs at most twice as a z_i. A pairing on $\{z_1, \ldots, z_{2m}\}$ is an arrangement of these numbers into m pairs. A pairing is called admissible if:

(a) Whenever $z_i = z_{i+1}$ then (z_i, z_{i+1}) is a pair.

(b) If all numbers occurring twice are removed then some pair consists of consecutive z's remaining, and the same is true when this pair is removed also, and so on until no z is left.

Each set $\{z_1, \ldots, z_{2m}\}$ gives rise to one cell for each admissible pairing. The cell corresponding to a given admissible pairing is the sum of all the $\phi_{\alpha,\beta}$ with symbol

$$\begin{pmatrix} \lambda_1, \ldots, \lambda_m \\ \mu_1, \ldots, \mu_m \end{pmatrix}$$

satisfying $\{\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m\} = \{z_1, \ldots, z_{2m}\}$ such that $\mu_1, \ldots, \mu_m$ contains one number from each of the m pairs.

A typical cell will be a sum of 2^{r-1} distinct irreducible characters $\phi_{\alpha,\beta}$ where there are $2r$ z_i's occurring without repetition in $\{z_1, \ldots, z_{2m}\}$. If $\alpha = \beta$, $\phi_{\alpha,\alpha}'$ and $\phi_{\alpha,\alpha}''$ will form cells by themselves.

We illustrate this by considering the example $W(D_4)$. We list the set of all unordered pairs (α, β) of partitions with $|\alpha| + |\beta| = 4$ and give the symbol corresponding to each for which m is as small as possible (see table opposite).

The irreducible characters $\phi_{\alpha,\beta}$, $\alpha \neq \beta$, $\phi_{\alpha,\alpha}'$, $\phi_{\alpha,\alpha}''$ thus fall into families as follows:

$$(\phi_{4,-})\ (\phi_{31,-})\ (\phi_{211,-})\ (\phi_{1111,-})$$

$$(\phi_{3,1})\ (\phi_{111,1})\ (\phi_{2,2}')\ (\phi_{2,2}'')$$

$$(\phi_{11,11}')\ (\phi_{11,11}'')\ (\phi_{22,-}, \phi_{21,1}, \phi_{2,11}).$$

(α, β)	Symbol
$(4, -)$	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$
$(31, -)$	$\begin{pmatrix} 1 & & 4 \\ & 0 & & 1 \end{pmatrix}$
$(22, -)$	$\begin{pmatrix} 2 & & 3 \\ & 0 & & 1 \end{pmatrix}$
$(211, -)$	$\begin{pmatrix} 1 & & 2 & & 4 \\ & 0 & & 1 & & 2 \end{pmatrix}$
$(1111, -)$	$\begin{pmatrix} 1 & & 2 & & 3 & & 4 \\ & 0 & & 1 & & 2 & & 3 \end{pmatrix}$
$(3, 1)$	$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$
$(21, 1)$	$\begin{pmatrix} 1 & & 3 \\ & 0 & & 2 \end{pmatrix}$
$(111, 1)$	$\begin{pmatrix} 1 & & 2 & & 3 \\ & 0 & & 1 & & 3 \end{pmatrix}$
$(2, 2)$	$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$
$(2, 11)$	$\begin{pmatrix} 0 & & 3 \\ & 1 & & 2 \end{pmatrix}$
$(11, 11)$	$\begin{pmatrix} 1 & & 2 \\ & 1 & & 2 \end{pmatrix}$

Thus there are 10 families containing 1 character and 1 family containing 3 characters. Each character ϕ in a family by itself gives rise to a unique cell which is simply ϕ itself. The family containing 3 characters has $\{z_1, \ldots, z_{2m}\} = \{0 \quad 1 \quad 2 \quad 3\}$. The admissible pairings on this set are (01)(23) and (03)(12). The cells corresponding to these admissible pairings are

$$\phi_{21,1} + \phi_{2,11}$$

$$\phi_{22,-} + \phi_{21,1}.$$

Type G_2 We shall describe the irreducible characters of the Weyl groups of exceptional type by means of two integers d, e. $\phi_{d,e}$ denotes an irreducible character of degree d which occurs as component of $\mathfrak{P}_e(V)$ but not of $\mathfrak{P}_i(V)$ for $i < e$. These two integers are sufficient to specify the character $\phi_{d,e}$ uniquely when W has type E_6, E_7, E_8. However in types G_2 and F_4 we must be more precise.

The irreducible characters of $W(G_2)$ have the form

$$\phi_{1,0},\ \phi_{2,1},\ \phi_{2,2},\ \phi_{1,3}',\ \phi_{1,3}'',\ \phi_{1,6}.$$

In order to distinguish between the characters $\phi_{1,3}'$ and $\phi_{1,3}''$ we give the character table of $W(G_2)$. $W(G_2)$ is generated by elements s_1, s_2 which are reflections in the hyperplanes orthogonal to the simple roots α_1, α_2 respectively, where α_1 is short and α_2 is long. Thus G_2 has Dynkin diagram

The character table of $W(G_2)$ is:

		$\phi_{1,0}$	$\phi_{1,6}$	$\phi_{1,3}'$	$\phi_{1,3}''$	$\phi_{2,1}$	$\phi_{2,2}$
1	(1)	1	1	1	1	2	2
$\tilde{A}_1$	(s_1)	1	-1	1	-1	0	0
A_1	(s_2)	1	-1	-1	1	0	0
G_2	(s_1s_2)	1	1	-1	-1	1	-1
A_2	$(s_1s_2s_1s_2)$	1	1	1	1	-1	-1
$A_1 + \tilde{A}_1$	(w_0)	1	1	-1	-1	-2	2

We have given a representative element in each conjugacy class. Each such element is a Coxeter element of some reflection subgroup, and the type of this reflection subgroup can be used to parametrize the conjugacy class, as in Carter [2]. As before A_1 denotes a subsystem with long roots and $\tilde{A}_1$ with short roots.

The families of irreducible characters of G_2 are given by

$$(\phi_{1,0})\ (\phi_{2,1},\ \phi_{2,2},\ \phi_{1,3}',\ \phi_{1,3}'')\ (\phi_{1,6}).$$

The special characters are $\phi_{1,0}$, $\phi_{2,1}$, $\phi_{1,6}$.

The cells are given by the following combinations of irreducible characters:

$$\phi_{1,0}$$

$$\phi_{2,1} + \phi_{2,2} + \phi_{1,3}'$$

$$\phi_{2,1} + \phi_{2,2} + \phi_{1,3}''$$

$$\phi_{1,6}.$$

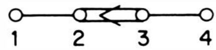
Type F_4 The irreducible characters of $W(F_4)$ have the form

$$\phi_{1,0}\ \phi_{4,1}\ \phi_{9,2}\ \phi_{8,3}'\ \phi_{8,3}''\ \phi_{2,4}'\ \phi_{2,4}''\ \phi_{12,4}\ \phi_{16,5}\ \phi_{9,6}'\ \phi_{9,6}''\ \phi_{6,6}'\ \phi_{6,6}''\ \phi_{4,7}'$$
$$\phi_{4,7}''\ \phi_{4,8}\ \phi_{8,9}'\ \phi_{8,9}''\ \phi_{9,10}\ \phi_{1,12}'\ \phi_{1,12}''\ \phi_{4,13}\ \phi_{2,16}'\ \phi_{2,16}''\ \phi_{1,24}.$$

In order to distinguish clearly between the pairs of characters $\phi_{d,e}'$ $\phi_{d,e}''$ with the same integers d, e we give the character table of $W(F_4)$. This character table was calculated by Kondo and we shall specify the relation between our notation and Kondo's.

		$\phi_{1,0}$ $\chi_{1.1}$	$\phi_{1,12}''$ $\chi_{1.2}$	$\phi_{1,12}'$ $\chi_{1.3}$	$\phi_{1,24}$ $\chi_{1.4}$	$\phi_{2,4}''$ $\chi_{2.1}$	$\phi_{2,16}'$ $\chi_{2.2}$	$\phi_{2,4}'$ $\chi_{2.3}$	$\phi_{2,16}''$ $\chi_{2.4}$	$\phi_{4,8}$ $\chi_{4.1}$	$\phi_{9,2}$ $\chi_{9.1}$	$\phi_{9,6}''$ $\chi_{9.2}$	$\phi_{9,6}'$ $\chi_{9.3}$	$\phi_{9,10}$ $\chi_{9.4}$	$\phi_{6,6}'$ $\chi_{6.1}$	$\phi_{6,6}''$ $\chi_{6.2}$	$\phi_{12,4}$ $\chi_{12.1}$	$\phi_{4,1}$ $\chi_{4.2}$	$\phi_{4,7}''$ $\chi_{4.3}$	$\phi_{4,7}'$ $\chi_{4.4}$	$\phi_{4,13}$ $\chi_{4.5}$	$\phi_{8,3}''$ $\chi_{8.1}$	$\phi_{8,9}'$ $\chi_{8.2}$	$\phi_{8,3}'$ $\chi_{8.3}$	$\phi_{8,9}''$ $\chi_{8.4}$	$\phi_{16,5}$ $\chi_{16.1}$
1	1	1	1	1	1	2	2	2	2	4	9	9	9	9	6	6	12	4	4	4	4	8	8	8	8	16
z	$4A_1$	1	1	1	1	2	2	2	2	4	9	9	9	9	6	6	12	−4	−4	−4	−4	−8	−8	−8	−8	−16
ab	$2A_1$	1	1	1	1	2	2	2	2	4	1	1	1	1	−2	−2	−4	0	0	0	0	0	0	0	0	0
e	A_2	1	1	1	1	2	2	−1	−1	−2	0	0	0	0	0	0	0	1	1	1	1	2	2	−1	−1	−2
ez	D_4	1	1	1	1	2	2	−1	−1	−2	0	0	0	0	0	0	0	−1	−1	−1	−1	−2	−2	1	1	2
$adbdcd$	$D_4(a_1)$	1	1	1	1	2	2	2	2	4	−3	−3	−3	−3	2	2	4	0	0	0	0	0	0	0	0	0
σ	$\tilde{A}_2$	1	1	1	1	−1	−1	2	2	−2	0	0	0	0	0	0	0	1	1	1	1	−1	−1	2	2	−2
σz	$C_3 + A_1$	1	1	1	1	−1	−1	2	2	−2	0	0	0	0	0	0	0	−1	−1	−1	−1	1	1	−2	−2	2
σe	$A_2 + \tilde{A}_2$	1	1	1	1	−1	−1	−1	−1	1	0	0	0	0	3	3	−3	−2	−2	−2	−2	2	2	2	2	−2
σez	$F_4(a_1)$	1	1	1	1	−1	−1	−1	−1	1	0	0	0	0	3	3	−3	2	2	2	2	−2	−2	−2	−2	2
$cd\sigma$	F_4	1	1	1	1	−1	−1	−1	−1	1	0	0	0	0	−1	−1	1	0	0	0	0	0	0	0	0	0
d	A_1	1	1	−1	−1	2	−2	0	0	0	3	3	−3	−3	0	0	0	2	2	−2	−2	4	−4	0	0	0
dz	$3A_1$	1	1	−1	−1	2	−2	0	0	0	3	3	−3	−3	0	0	0	−2	−2	2	2	−4	4	0	0	0
σd	$\tilde{A}_2 + A_1$	1	1	−1	−1	−1	1	0	0	0	0	0	0	0	0	0	0	−1	−1	1	1	1	−1	0	0	0
σdz	C_3	1	1	−1	−1	−1	1	0	0	0	0	0	0	0	0	0	0	1	1	−1	−1	−1	1	0	0	0
adb	A_3	1	1	−1	−1	2	−2	0	0	0	−1	−1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
τ	$\tilde{A}_1$	1	−1	1	−1	0	0	2	−2	0	3	−3	3	−3	0	0	0	2	−2	2	−2	0	0	4	−4	0
τz	$2A_1 + \tilde{A}_1$	1	−1	1	−1	0	0	2	−2	0	3	−3	3	−3	0	0	0	−2	2	−2	2	0	0	−4	4	0
$e\tau$	$A_2 + \tilde{A}_1$	1	−1	1	−1	0	0	−1	1	0	0	0	0	0	0	0	0	−1	1	−1	1	0	0	1	−1	0
$e\tau z$	B_3	1	−1	1	−1	0	0	−1	1	0	0	0	0	0	0	0	0	1	−1	1	−1	0	0	−1	1	0
$ca\tau$	$B_2 + A_1$	1	−1	1	−1	0	0	2	−2	0	−1	1	−1	1	0	0	0	0	0	0	0	0	0	0	0	0
τd	$A_1 + \tilde{A}_1$	1	−1	−1	1	0	0	0	0	0	1	−1	−1	1	2	−2	0	0	0	0	0	0	0	0	0	0
$a\tau$	B_2	1	−1	−1	1	0	0	0	0	0	1	−1	−1	1	−2	2	0	2	−2	−2	2	0	0	0	0	0
$a\tau z$	$A_3 + \tilde{A}_1$	1	−1	−1	1	0	0	0	0	0	1	−1	−1	1	−2	2	0	−2	2	2	−2	0	0	0	0	0
$cdb\tau$	B_4	1	−1	−1	1	0	0	0	0	0	−1	1	1	−1	0	0	0	0	0	0	0	0	0	0	0	0

$W(F_4)$ is generated by simple reflections s_1, s_2, s_3, s_4 corresponding to simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. We suppose that α_1, α_2 are short roots and α_3, α_4 are long roots so that the Dynkin diagram of F_4 is



In order to conform to Kondo's notation [1] for the conjugacy class representatives we define elements $a, b, c, d, e, \sigma, \tau, z$ by $a = s_3$, $\tau = s_2$, $\sigma = s_2 s_1$, $d = s_4$, $b = s_2 s_3 s_2$, $c = \sigma b \sigma^{-1}$, $e = (abcd)^2$, $z = (abcd)^3$. The conjugacy class representatives can be given in terms of these elements. We shall also describe the conjugacy classes by the type of the reflection subgroup in which the elements of the class occur as Coxeter elements, as in Carter [2]. (There are two conjugacy classes which are not Coxeter classes of reflection subgroups.)

Each irreducible character has been given two labels—the one called $\chi_{i,j}$ is Kondo's notation and the one called $\phi_{d,e}$ is ours. The reason for choosing $\phi_{d,e}'$ and $\phi_{d,e}''$ in the way we have done will become clear when we discuss generic degrees in detail.

The families of irreducible characters of $W(F_4)$ are as follows. The special character in each family is the first one shown.

$$(\phi_{1,0})\ (\phi_{9,2})\ (\phi_{8,3}')\ (\phi_{8,3}'')\ (\phi_{8,9}')\ (\phi_{8,9}'')\ (\phi_{9,10})\ (\phi_{1,24})$$

$$(\phi_{4,1}\ \phi_{2,4}'\ \phi_{2,4}'') \qquad (\phi_{4,13}\ \phi_{2,16}'\ \phi_{2,16}'')$$

$$(\phi_{12,4}\ \phi_{16,5}\ \phi_{6,6}'\ \phi_{6,6}''\ \phi_{9,6}'\ \phi_{9,6}''\ \phi_{4,7}'\ \phi_{4,7}''\ \phi_{4,8}\ \phi_{1,12}'\ \phi_{1,12}'')$$

Thus we have 8 families with 1 character, 2 families with 3 characters, and 1 family with 11 characters.

The cells are given by the following combinations of irreducible characters:

$\phi_{1,0}$

$\phi_{9,2}$

$\phi_{8,3}'$

$\phi_{8,3}''$

$\phi_{8,9}'$

$\phi_{8,9}''$

$\phi_{9,10}$

$\phi_{1,24}$

$\phi_{4,1} + \phi_{2,4}'$, $\qquad \phi_{4,1} + \phi_{2,4}''$

$\phi_{4,13} + \phi_{2,16}'$, $\qquad \phi_{4,13} + \phi_{2,16}''$

$$\phi_{12,4} + \phi_{16,5} + \phi_{9,6}'' + \phi_{6,6}' + \phi_{4,7}''$$

$$\phi_{12,4} + \phi_{16,5} + \phi_{9,6}' + \phi_{6,6}' + \phi_{4,7}'$$

$$\phi_{12,4} + \phi_{16,5} + 2\phi_{9,6}'' + \phi_{6,6}'' + \phi_{4,7}'' + \phi_{1,12}''$$

$$\phi_{12,4} + \phi_{16,5} + 2\phi_{9,6}' + \phi_{6,6}'' + \phi_{4,7}' + \phi_{1,12}'$$

$$\phi_{12,4} + 2\phi_{16,5} + \phi_{9,6}' + \phi_{9,6}'' + \phi_{6,6}'' + \phi_{4,8}$$

Type E_6 $W(E_6)$ has 25 irreducible characters and each of them is uniquely determined by the two integers d, e. These characters fall into families as follows. The special character in each family is the first one shown.

$(\phi_{1,0})$ $(\phi_{6,1})$ $(\phi_{20,2})$ $(\phi_{64,4})$ $(\phi_{60,5})$ $(\phi_{81,6})$ $(\phi_{24,6})$ $(\phi_{81,10})$ $(\phi_{60,11})$ $(\phi_{24,12})$ $(\phi_{64,13})$ $(\phi_{20,20})$ $(\phi_{6,25})$ $(\phi_{1,36})$

$$(\phi_{30,3}\ \phi_{15,4}\ \phi_{15,5}) \qquad (\phi_{30,15}\ \phi_{15,16}\ \phi_{15,17})$$

$$(\phi_{80,7}\ \phi_{60,8}\ \phi_{90,8}\ \phi_{10,9}\ \phi_{20,10}).$$

Thus $W(E_6)$ has 14 families with 1 character, 2 families with 3 characters and 1 family with 5 characters.

The cells are given by the following combinations of irreducible characters:

$$\phi_{1,0} \qquad \phi_{1,36}$$

$$\phi_{6,1} \qquad \phi_{6,25}$$

$$\phi_{20,2} \qquad \phi_{20,20}$$

$$\phi_{64,4} \qquad \phi_{64,13}$$

$$\phi_{60,5} \qquad \phi_{60,11}$$

$$\phi_{81,6} \qquad \phi_{81,10}$$

$$\phi_{24,6} \qquad \phi_{24,12}$$

$$\phi_{30,3} + \phi_{15,4} \qquad \phi_{30,3} + \phi_{15,5}$$

$$\phi_{30,15} + \phi_{15,16} \qquad \phi_{30,15} + \phi_{15,17}$$

$$\phi_{80,7} + 2\phi_{90,8} + \phi_{20,10}$$

$$\phi_{80,7} + \phi_{60,8} + \phi_{10,9}$$

$$\phi_{80,7} + \phi_{90,8} + \phi_{60,8}.$$

Type E_7 $W(E_7)$ has 60 irreducible characters and each of them is uniquely determined by the two integers d, e. These characters fall into families as follows. The special character in each family is the first one shown.

$(\phi_{1,0})$ $(\phi_{7,1})$ $(\phi_{27,2})$ $(\phi_{21,3})$ $(\phi_{189,5})$ $(\phi_{210,6})$ $(\phi_{105,6})$ $(\phi_{168,6})$ $(\phi_{189,7})$ $(\phi_{378,9})$ $(\phi_{210,10})$ $(\phi_{105,12})$ $(\phi_{210,13})$ $(\phi_{378,14})$ $(\phi_{105,15})$ $(\phi_{189,20})$ $(\phi_{210,21})$ $(\phi_{105,21})$

$(\phi_{168,21})$ $(\phi_{189,22})$ $(\phi_{21,36})$ $(\phi_{27,37})$ $(\phi_{7,46})$ $(\phi_{1,63})$ $(\phi_{56,3}\ \phi_{35,4}\ \phi_{21,6})$ $(\phi_{120,4}\ \phi_{105,5}\ \phi_{15,7})$ $(\phi_{405,8}\ \phi_{216,9}\ \phi_{189,10})$ $(\phi_{420,10}\ \phi_{336,11}\ \phi_{84,12})$ $(\phi_{512,11}\ \phi_{512,12})$ $(\phi_{420,13}\ \phi_{336,14}\ \phi_{84,15})$ $(\phi_{405,15}\ \phi_{216,16}\ \phi_{189,17})$ $(\phi_{120,25}\ \phi_{105,26}\ \phi_{15,28})$ $(\phi_{56,30}\ \phi_{35,31}\ \phi_{21,33})$ $(\phi_{315,7}\ \phi_{280,8}\ \phi_{70,9}\ \phi_{280,9}\ \phi_{35,13})$ $(\phi_{315,16}\ \phi_{280,17}\ \phi_{70,18}\ \phi_{280,18}\ \phi_{35,22})$.

Thus $W(E_7)$ has 24 families with 1 character, 1 family with 2 characters, 8 families with 3 characters and 2 families with 5 characters.

The cells are given by the following combinations of irreducible characters:

$\phi_{1,0}$ $\phi_{7,1}$ $\phi_{27,2}$ $\phi_{21,3}$ $\phi_{189,5}$ $\phi_{210,6}$ $\phi_{105,6}$ $\phi_{168,6}$

$\phi_{189,7}$ $\phi_{378,9}$ $\phi_{210,10}$ $\phi_{105,12}$ $\phi_{210,13}$ $\phi_{378,14}$ $\phi_{105,15}$ $\phi_{189,20}$

$\phi_{210,21}$ $\phi_{105,21}$ $\phi_{168,21}$ $\phi_{189,22}$ $\phi_{21,36}$ $\phi_{27,37}$ $\phi_{7,46}$ $\phi_{1,63}$

$\phi_{56,3}+\phi_{35,4}$ $\qquad$ $\phi_{56,3}+\phi_{21,6}$

$\phi_{56,30}+\phi_{35,31}$ $\qquad$ $\phi_{56,30}+\phi_{21,33}$

$\phi_{120,4}+\phi_{105,5}$ $\qquad$ $\phi_{120,4}+\phi_{15,7}$

$\phi_{120,25}+\phi_{105,26}$ $\qquad$ $\phi_{120,25}+\phi_{15,28}$

$\phi_{405,8}+\phi_{216,9}$ $\qquad$ $\phi_{405,8}+\phi_{189,10}$

$\phi_{405,15}+\phi_{216,16}$ $\qquad$ $\phi_{405,15}+\phi_{189,17}$

$\phi_{420,10}+\phi_{336,11}$ $\qquad$ $\phi_{420,10}+\phi_{84,12}$

$\phi_{420,13}+\phi_{336,14}$ $\qquad$ $\phi_{420,13}+\phi_{84,15}$

$\phi_{512,11}+\phi_{512,12}$

$\phi_{315,7}+2\phi_{280,9}+\phi_{35,13}$

$\phi_{315,7}+\phi_{280,8}+\phi_{70,9}$

$\phi_{315,7}+\phi_{280,8}+\phi_{280,9}$

$\phi_{315,16}+2\phi_{280,18}+\phi_{35,22}$

$\phi_{315,16}+\phi_{280,17}+\phi_{70,18}$

$\phi_{315,16}+\phi_{280,17}+\phi_{280,18}$.

Type E_8 $\quad W(E_8)$ has 112 irreducible characters and each of them is uniquely determined by the two integers d, e. These characters fall into families as follows. The special character in each family is the first one shown.

$(\phi_{1,0})$ $(\phi_{8,1})$ $(\phi_{35,2})$ $(\phi_{560,5})$ $(\phi_{567,6})$ $(\phi_{3240,9})$ $(\phi_{525,12})$ $(\phi_{4536,13})$ $(\phi_{2835,14})$ $(\phi_{6075,14})$ $(\phi_{4200,15})$ $(\phi_{2100,20})$ $(\phi_{4200,21})$ $(\phi_{2835,22})$ $(\phi_{6075,22})$ $(\phi_{4536,23})$ $(\phi_{3240,31})$ $(\phi_{525,36})$ $(\phi_{567,46})$ $(\phi_{560,47})$ $(\phi_{35,74})$ $(\phi_{8,91})$ $(\phi_{1,120})$ $(\phi_{112,3}\ \phi_{84,4}\ \phi_{28,8})$ $(\phi_{210,4}\ \phi_{160,7}\ \phi_{50,8})$ $(\phi_{700,6}\ \phi_{400,7}\ \phi_{300,8})$ $(\phi_{2268,10}\ \phi_{972,12}\ \phi_{1296,13})$

$(\phi_{2240,10}\ \phi_{1400,11}\ \phi_{840,13})$ $(\phi_{4096,11}\ \phi_{4096,12})$ $(\phi_{4200,12}\ \phi_{3360,13}\ \phi_{840,14})$ $(\phi_{2800,13}\ \phi_{700,16}\ \phi_{2100,16})$ $(\phi_{5600,15}\ \phi_{3200,16}\ \phi_{2400,17})$ $(\phi_{5600,21}\ \phi_{3200,22}\ \phi_{2400,23})$ $(\phi_{4200,24}\ \phi_{3360,25}\ \phi_{840,26})$ $(\phi_{2800,25}\ \phi_{700,28}\ \phi_{2100,28})$ $(\phi_{4096,26}\ \phi_{4096,27})$ $(\phi_{2240,28}\ \phi_{1400,29}\ \phi_{840,31})$ $(\phi_{2268,30}\ \phi_{972,32}\ \phi_{1296,33})$ $(\phi_{700,42}\ \phi_{400,43}\ \phi_{300,44})$ $(\phi_{210,52}\ \phi_{160,55}\ \phi_{50,56})$ $(\phi_{112,63}\ \phi_{84,64}\ \phi_{28,68})$ $(\phi_{1400,7}\ \phi_{1344,8}\ \phi_{448,9}\ \phi_{1008,9}\ \phi_{56,19})$ $(\phi_{1400,8}\ \phi_{1050,10}\ \phi_{1575,10}\ \phi_{175,12}\ \phi_{350,14})$ $(\phi_{1400,32}\ \phi_{1050,34}\ \phi_{1575,34}\ \phi_{175,36}\ \phi_{350,38})$ $(\phi_{1400,37}\ \phi_{1344,38}\ \phi_{448,39}\ \phi_{1008,39}\ \phi_{56,49})$ $(\phi_{4480,16}\ \phi_{7168,17}\ \phi_{3150,18}\ \phi_{4200,18}\ \phi_{4536,18}\ \phi_{5670,18}\ \phi_{1344,19}\ \phi_{2016,19}\ \phi_{5600,19}\ \phi_{2688,20}\ \phi_{420,20}\ \phi_{1134,20}\ \phi_{1400,20}\ \phi_{1680,22}\ \phi_{168,24}\ \phi_{448,25}\ \phi_{70,32})$.

Thus $W(E_8)$ has 23 families with 1 character, 2 families with 2 characters, 16 families with 3 characters, 4 families with 5 characters, and 1 family with 17 characters.

The cells are given by the following combinations of irreducible characters:

$\phi_{1,0}\ \phi_{8,1}\ \phi_{35,2}\ \phi_{560,5}\ \phi_{567,6}\ \phi_{3240,9}\ \phi_{525,12}\ \phi_{4536,13}\ \phi_{2835,14}\ \phi_{6075,14}\ \phi_{4200,15}\ \phi_{2100,20}\ \phi_{4200,21}\ \phi_{2835,22}\ \phi_{6075,22}\ \phi_{4536,23}\ \phi_{3240,31}\ \phi_{525,36}\ \phi_{567,46}\ \phi_{560,47}\ \phi_{35,74}\ \phi_{8,91}\ \phi_{1,120}$

$\phi_{112,3} + \phi_{84,4}$	$\phi_{112,3} + \phi_{28,8}$
$\phi_{112,63} + \phi_{84,64}$	$\phi_{112,63} + \phi_{28,68}$
$\phi_{210,4} + \phi_{160,7}$	$\phi_{210,4} + \phi_{50,8}$
$\phi_{210,52} + \phi_{160,55}$	$\phi_{210,52} + \phi_{50,56}$
$\phi_{700,6} + \phi_{400,7}$	$\phi_{700,6} + \phi_{300,8}$
$\phi_{700,42} + \phi_{400,43}$	$\phi_{700,42} + \phi_{300,44}$
$\phi_{2268,10} + \phi_{972,12}$	$\phi_{2268,10} + \phi_{1296,13}$
$\phi_{2268,30} + \phi_{972,32}$	$\phi_{2268,30} + \phi_{1296,33}$
$\phi_{2240,10} + \phi_{1400,11}$	$\phi_{2240,10} + \phi_{840,13}$
$\phi_{2240,28} + \phi_{1400,29}$	$\phi_{2240,28} + \phi_{840,31}$
$\phi_{4200,12} + \phi_{3360,13}$	$\phi_{4200,12} + \phi_{840,14}$
$\phi_{4200,24} + \phi_{3360,25}$	$\phi_{4200,24} + \phi_{840,26}$
$\phi_{2800,13} + \phi_{700,16}$	$\phi_{2800,13} + \phi_{2100,16}$
$\phi_{2800,25} + \phi_{700,28}$	$\phi_{2800,25} + \phi_{2100,28}$
$\phi_{5600,15} + \phi_{3200,16}$	$\phi_{5600,15} + \phi_{2400,17}$
$\phi_{5600,21} + \phi_{3200,22}$	$\phi_{5600,21} + \phi_{2400,23}$
$\phi_{4096,11} + \phi_{4096,12}$	$\phi_{4096,26} + \phi_{4096,27}$

$\phi_{1400,7} + 2\phi_{1008,9} + \phi_{56,19}$

$\phi_{1400,7} + \phi_{1344,8} + \phi_{448,9}$

$\phi_{1400,7} + \phi_{1344,8} + \phi_{1008,9}$

$\phi_{1400,37} + 2\phi_{1008,39} + \phi_{56,49}$

$\phi_{1400,37} + \phi_{1344,38} + \phi_{448,39}$

$\phi_{1440,37} + \phi_{1344,38} + \phi_{1008,39}$

$\phi_{1400,8} + 2\phi_{1575,10} + \phi_{350,14}$

$\phi_{1400,8} + \phi_{1050,10} + \phi_{175,12}$

$\phi_{1400,8} + \phi_{1575,10} + \phi_{1050,10}$

$\phi_{1400,32} + 2\phi_{1575,34} + \phi_{350,38}$

$\phi_{1400,32} + \phi_{1050,34} + \phi_{175,36}$

$\phi_{1400,32} + \phi_{1575,34} + \phi_{1050,34}$

$\phi_{4480,16} + 2\phi_{7168,17} + \phi_{4200,18} + 2\phi_{4536,18} + 2\phi_{5670,18} + 2\phi_{5600,19} + \phi_{2688,20} + \phi_{1400,20} + \phi_{1680,22}$

$\phi_{4480,16} + 2\phi_{7168,17} + \phi_{3150,18} + \phi_{4200,18} + \phi_{4536,18} + \phi_{5670,18} + \phi_{2016,19} + \phi_{5600,19} + \phi_{2688,20}$

$\phi_{4480,16} + \phi_{7168,17} + 3\phi_{4536,18} + 3\phi_{5670,18} + 2\phi_{5600,19} + 2\phi_{1400,20} + 3\phi_{1680,22} + \phi_{448,25} + \phi_{70,32}$

$\phi_{4480,16} + \phi_{7168,17} + \phi_{3150,18} + \phi_{4536,18} + 2\phi_{5670,18} + 2\phi_{5600,19} + \phi_{1134,20} + \phi_{1680,22} + \phi_{448,25}$

$\phi_{4480,16} + \phi_{7168,17} + 2\phi_{4200,18} + \phi_{4536,18} + \phi_{5670,18} + \phi_{1344,19} + \phi_{5600,19} + \phi_{1400,20} + \phi_{168,24}$

$\phi_{4480,16} + \phi_{7168,17} + \phi_{3150,18} + \phi_{4200,18} + \phi_{5670,18} + \phi_{1344,19} + \phi_{5600,19} + \phi_{1134,20}$

$\phi_{4480,16} + \phi_{7168,17} + \phi_{3150,18} + \phi_{4200,18} + \phi_{1344,19} + \phi_{2016,19} + \phi_{420,20}.$

13.3 THE RELATION BETWEEN UNIPOTENT CLASSES AND CHARACTERS OF THE WEYL GROUP

We recall from section 12.6 that there is an injective map from the set of irreducible characters of W into the set of pairs (C, ψ) where C is a unipotent conjugacy class of $G_{ad}(\mathbb{C})$ and ψ is an irreducible character of $A(u) = C(u)/C(u)^0$ where $u \in C$. This injective map is obtained from Springer's construction of representations of the Weyl group. We shall give an explicit description of this injection in the present section.

In the case of groups of type A the correspondence is particularly simple since $A(u) = 1$ for all unipotent elements u. There is in this case a bijective map between unipotent classes of $G_{ad}(\mathbb{C})$ and irreducible characters of W. We know from section 13.1 that the unipotent classes are described by partitions $\xi = (\xi_1\ \xi_2 \ldots)$ of $l + 1$ where the elementary divisors of a unipotent element in this class are $(t-1)^{\xi_1}, (t-1)^{\xi_2}, \ldots$. From 11.4.1 we know that the irreducible characters of W are also described by partitions of $l + 1$. So for each partition ξ of $l + 1$ we have a unipotent class C_ξ and an irreducible character ϕ_ξ of W. The Springer correspondence is then given by $C_\xi \Leftrightarrow \phi_\xi$. Note in particular that when $\xi = (l + 1)$ C_ξ is the class of regular unipotent elements and ϕ_ξ is the unit representation of W. If u is a regular unipotent element $\mathfrak{B}_u$ consists of a single Borel subgroup and W acts trivially on $H^0(\mathfrak{B}_u, \mathbb{Q})$. On the other hand when $\xi = (1^{l+1})$ C_ξ is the class containing the unit element of $G_{ad}(\mathbb{C})$ and ϕ_ξ is the sign character of W.

In the case of groups of classical type B_l, C_l, D_l the Springer correspondence has been determined explicitly by Shoji [2], and an alternative description has been given by Lusztig [23]. We describe Lusztig's version.

Type C_l We know from 11.4.3 that the irreducible characters of $W(C_l)$ are described by pairs of partitions (ξ, η) with $|\xi| + |\eta| = l$. The classes of unipotent elements in $G_{ad}(\mathbb{C})$ may be described in terms of the partition λ of $2l$ given by the elementary divisors in the natural representation. There is a bijective correspondence between unipotent classes and partitions λ of $2l$ in which each odd part occurs with even multiplicity.

We now describe an injective map $\lambda \to (\xi, \eta)$ from partitions λ of $2l$ in which each odd part has even multiplicity to pairs of partitions with $|\xi| + |\eta| = l$. We write the parts of λ in increasing order and ensure that λ has an even number of parts by calling the first part 0 if necessary. Thus

$$\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_{2k}.$$

We now define $\lambda_i^* = \lambda_i + i - 1$. We have

$$\lambda_1^* < \lambda_2^* < \ldots < \lambda_{2k}^*.$$

We divide λ^* into its odd and even parts. It has the same number of even parts as odd parts. Let the odd parts be

$$2\xi_1^* + 1 < 2\xi_2^* + 1 < \ldots < 2\xi_k^* + 1$$

and the even parts be

$$2\eta_1^* < 2\eta_2^* < \ldots < 2\eta_k^*.$$

Then we have

$$0 \leqslant \xi_1^* < \xi_2^* < \ldots < \xi_k^*$$
$$0 \leqslant \eta_1^* < \eta_2^* < \ldots < \eta_k^*.$$

We then define $\xi_i = \xi_i^* - (i-1)$ and $\eta_i = \eta_i^* - (i-1)$. We then have

$$0 \leqslant \xi_1 \leqslant \xi_2 \leqslant \ldots \leqslant \xi_k$$
$$0 \leqslant \eta_1 \leqslant \eta_2 \leqslant \ldots \leqslant \eta_k$$

and $|\xi| + |\eta| = l$. Thus we have defined a map $\lambda \to (\xi, \eta)$ which is injective. (We omit the parts equal to 0 to describe the partitions (ξ, η) in their usual form.)

For example, if λ is the partition 222 of 6, we first write $\lambda = 0222$. We then have $\lambda^* = 0345$, $\xi^* = 12$, $\eta^* = 02$, $\xi = 11$, $\eta = 01$. Thus λ corresponds to the pair $(\xi, \eta) = (11, 1)$.

The image of this map $\lambda \to (\xi, \eta)$ may be described as follows. Given any pair (ξ, η) of partitions with $|\xi| + |\eta| = l$ we define a symbol associated with (ξ, η) as follows. (*Note* This is *not* the same procedure as was used in section 11.4 to construct a symbol from a pair of partitions.) We first ensure that ξ has exactly one more part than η by adding zeros as parts where necessary. We then write the parts of ξ, η in increasing order and consider the symbol

$$\begin{pmatrix} \xi_1 & & \xi_2 + 2 & & \xi_3 + 4 & & \xi_4 + 6 \ldots \\ & \eta_1 + 1 & & \eta_2 + 3 & & \eta_3 + 5 \ldots & \end{pmatrix}$$

(ξ, η) lies in the image of the above map $\lambda \to (\xi, \eta)$ if and only if its symbol satisfies

$$\xi_1 \leqslant \eta_1 + 1 \leqslant \xi_2 + 2 \leqslant \eta_2 + 3 \leqslant \ldots.$$

If (ξ, η) has this property then the Springer correspondence maps the character $\phi_{(\xi,\eta)}$ of W into the pair $(C_\lambda, 1)$ where C_λ is the unipotent class whose elementary divisors give the partition λ with $\lambda \to (\xi, \eta)$, and 1 is the unit character of $A(u)$ for $u \in C_\lambda$.

If (ξ, η) is not in the image of the above map there will be a unique (ξ', η') in the image whose symbol contains the same entries with the same multiplicities as the symbol of (ξ, η). Let $\lambda \to (\xi', \eta')$. Then the Springer correspondence maps $\phi_{(\xi,\eta)}$ into the pair (C_λ, ψ) for some character ψ of $A(u)$. We must therefore describe ψ. The group $A(u) = C(u)/C(u)^0$ is isomorphic to $C_2 \times \ldots \times C_2$. It is generated by elements a_i, one for each even i for which the number r_i of elementary divisors $(t-1)^i$ is nonzero. In addition to the relations $2a_i = 0$ there is one further relation $\sum_{r_i \text{ odd}}^{i \text{ even}} a_i = 0$ which arises from the fact that we are working in the adjoint group rather than in the symplectic group itself.

We have $\psi(a_i) = \pm 1$ for each generator a_i. In order to decide which we consider the symbols associated with (ξ, η) and (ξ', η'). These symbols contain the same entries with the same multiplicities. Let S be the set of entries which occur exactly once. A subset

$$(i+1, i+2, \ldots, j) \qquad 0 \leqslant i < j$$

of S is called an interval if $i \notin S$ and $j + 1 \notin S$. The number of intervals in S turns out to be equal to the number of generators a_i where i is even and $r_i \neq 0$. We associate the intervals to the generators a_i using the natural ordering on both.

We take an interval I and compare the entries in I in the symbols of (ξ, η) and (ξ', η'). There are two possibilities. Either the entries from I in the first row of the symbol of (ξ', η') remain in the first row of the symbol of (ξ, η) and similarly for the second row, or the entries from I in the first row of the symbol of (ξ', η') are in the second row of the symbol of (ξ, η) and vice-versa. In the first case we have $\psi(a_i) = 1$ and in the second case we have $\psi(a_i) = -1$.

Type B_l We know from 11.4.2 that the irreducible characters of $W(B_l)$ are described by pairs of partitions (ξ, η) with $|\xi| + |\eta| = l$. The classes of unipotent elements in $G_{\mathrm{ad}}(\mathbb{C})$ may be described in terms of the partition λ of $2l + 1$ given by the elementary divisors in the natural representation. There is a bijective correspondence between unipotent classes and partitions λ of $2l + 1$ in which each even part occurs with even multiplicity.

We now describe an injective map $\lambda \to (\xi, \eta)$ from such partitions λ into pairs of partitions (ξ, η) with $|\xi| + |\eta| = l$. λ has an odd number of parts. We write them in increasing order and then define $\lambda^*, \xi^*, \eta^*, \xi, \eta$ as in type C_l above. (This time ξ^* will have one more part than η^*.) Then $\lambda \to (\xi, \eta)$ is the required map.

The image of this map may be described as follows. Given a pair (ξ, η) of partitions with $|\xi| + |\eta| = l$ we define a symbol associated with ξ, η as follows. First ensure that ξ has exactly one more part than η by adding zeros as parts where necessary. Then write the parts of ξ and η in increasing order and consider the symbol

$$\begin{pmatrix} \xi_1 & & \xi_2 + 2 & & \xi_3 + 4 & & \xi_4 + 6 \ldots \\ & \eta_1 & & \eta_2 + 2 & & \eta_3 + 4 \ldots & \end{pmatrix}.$$

(ξ, η) lies in the image of the above map $\lambda \to (\xi, \eta)$ if and only if its symbol satisfies

$$\xi_1 \leqslant \eta_1 \leqslant \xi_2 + 2 \leqslant \eta_2 + 2 \leqslant \xi_3 + 4 \leqslant \ldots.$$

If (ξ, η) has this property then the Springer correspondence maps the character $\phi_{(\xi, \eta)}$ of W into the pair $(C_\lambda, 1)$. If (ξ, η) is not in the image there will be a unique (ξ', η') which does lie in the image whose symbol contains the same entries with the same multiplicities as the symbol of (ξ, η). Let $\lambda \to (\xi', \eta')$. Then the Springer correspondence maps $\phi_{(\xi, \eta)}$ into the pair (C_λ, ψ) for some character ψ of $A(u)$. The group $A(u) = C(u)/C(u)^0$ is isomorphic to $C_2 \times \ldots \times C_2$. This time we have one generator a_i for each odd i for which $r_i > 0$, and relations $2a_i = 0$. Moreover we must take the subgroup consisting of the elements $\{\sum_{r_i > 0}^{i \text{ odd}} n_i a_i;\ \sum n_i \text{ even}\}$ to obtain $A(u)$, since we are working in the special orthogonal group $SO_{2l+1}(\mathbb{C})$ rather than the full orthogonal group. The character ψ of $A(u)$ is obtained just as in type C_l above. We have $\psi(a_i) = \pm 1$. The number of generators a_i is equal to the number of intervals in the set S of entries which occur exactly once in the symbol of (ξ, η). However the definition of an interval is slightly different in this case. An interval is now defined as a subset of S of the form

$$(i, i + 1, \ldots, j) \qquad 0 \leqslant i \leqslant j$$

where $i-1 \notin S$ and $j+1 \notin S$. We decide whether $\psi(a_i)$ is 1 or -1 by comparing the entries from the corresponding interval in the symbols of (ξ, η) and (ξ', η') as before. The restriction of ψ to the subgroup $\{\sum n_i a_i; \sum n_i \text{ even}\}$ then gives the required character of $A(u)$.

Type D_l We know from 11.4.4 that the irreducible characters of $W(D_l)$ are described by unordered pairs of partitions (ξ, η) with $|\xi| + |\eta| = l$, and that one obtains two irreducible characters $\phi_{(\xi,\xi)}'$, $\phi_{(\xi,\xi)}''$ when $\xi = \eta$.

The classes of unipotent elements in $G_{\mathrm{ad}}(\mathbb{C})$ may be described in terms of the partition λ of $2l$ given by the elementary divisors in the natural representation. There is a bijective correspondence between unipotent classes and partitions λ of $2l$ in which each even part occurs with even multiplicity, except that each such partition in which all parts are even gives rise to two unipotent classes.

As before we describe an injective map $\lambda \to (\xi, \eta)$ from partitions λ of $2l$ in which each even part occurs an even number of times to pairs of partitions (ξ, η) with $|\xi| + |\eta| = l$. λ has an even number of parts. We write them in increasing order and then define λ^*, ξ^*, η^*, ξ, η as in type C_l. ξ^*, η^* will have the same number of parts. Then $\lambda \to (\xi, \eta)$ is the required map.

The image of this map may be described as follows. Given a pair (ξ, η) of partitions with $|\xi| + |\eta| = l$ we define a symbol associated with (ξ, η) as follows. First ensure that ξ, η have the same number of parts by adding zeros as parts where necessary. Then write the parts of ξ, η in increasing order and consider the symbol

$$\begin{pmatrix} \xi_1 & \xi_2 + 2 & \xi_3 + 4 \ldots \\ \eta_1 & \eta_2 + 2 & \eta_3 + 4 \ldots \end{pmatrix}.$$

The rows of this symbol are regarded as being unordered. Then (ξ, η) lies in the image of the above map $\lambda \to (\xi, \eta)$ if and only if its symbol satisfies

$$\xi_1 \leqslant \eta_1 \leqslant \xi_2 + 2 \leqslant \eta_2 + 2 \leqslant \xi_3 + 4 \leqslant \ldots$$

or

$$\eta_1 \leqslant \xi_1 \leqslant \eta_2 + 2 \leqslant \xi_2 + 2 \leqslant \eta_3 + 4 \leqslant \ldots.$$

If (ξ, η) has this property and $\xi \neq \eta$ then the Springer correspondence maps the character $\phi_{(\xi,\eta)}$ of W into the pair $(C_\lambda, 1)$.

If (ξ, η) is not in the image there will be a unique (ξ', η') which does lie in the image whose symbol contains the same entries with the same multiplicities as the symbol of (ξ, η). Let $\lambda \to (\xi', \eta')$. Then the Springer correspondence maps $\phi_{(\xi,\eta)}$ into the pair (C_λ, ψ) for some character ψ of $A(u)$. The group $A(u) = C(u)/C(u)^0$ is isomorphic to $C_2 \times \ldots \times C_2$. This time we have one generator a_i for each odd i for which $r_i > 0$, and relations $2a_i = 0$. We then descend to the subgroup $\{\sum_{r_i > 0}^{i \text{ odd}} n_i a_i;\ n_i \text{ even}\}$. We then impose one additional relation $\sum_{r_i \text{ odd}}^{i \text{ odd}} a_i = 0$ since we are working in the adjoint group rather than the special orthogonal group. (The number of odd i for which r_i is odd is necessarily even.) This gives the group $A(u)$. The character ψ of $A(u)$ is obtained just as in type C_l above. We have $\psi(a_i) = \pm 1$. The number of generators a_i is equal to the number of intervals in the set S of entries which occur exactly once in the symbol of (ξ, η). An interval in this case is defined as a subset of S of the form

$$(i, i+1, \ldots, j) \qquad 0 \leqslant i \leqslant j$$

where $i-1 \notin S$ and $j+1 \notin S$. We decide whether $\psi(a_i)$ is 1 or -1 by comparing the entries from the corresponding interval in the symbols of (ξ, η) and (ξ', η'), just as in type C_l. The restriction of ψ to the subgroup $\{\sum n_i a_i; \sum n_i \text{ even}\}$ then determines the required character of $A(u)$.

Now suppose that $\xi = \eta$. Then we have two irreducible characters $\phi_{(\xi,\xi)}'$, $\phi_{(\xi,\xi)}''$ of W. (ξ, ξ) lies in the image of the above map $\lambda \to (\xi, \eta)$. It is the image of the partition λ given by

$$\lambda = (2\xi_1, 2\xi_1, 2\xi_2, 2\xi_2, 2\xi_3, 2\xi_3, \ldots).$$

Now we recall that since all the parts of λ are even there are two unipotent classes in $G_{\mathrm{ad}}(\mathbb{C})$ with this set of elementary divisors. We call these $C_{(\xi,\xi)}'$, $C_{(\xi,\xi)}''$. The Springer correspondence then satisfies

$$\phi_{(\xi,\xi)}' \to (C_{(\xi,\xi)}', 1)$$
$$\phi_{(\xi,\xi)}'' \to (C_{(\xi,\xi)}'', 1)$$

with appropriate labelling. We must therefore decide which of the characters $\phi_{(\xi,\xi)}'$, $\phi_{(\xi,\xi)}''$ corresponds to which of the classes $C_{(\xi,\xi)}'$, $C_{(\xi,\xi)}''$. This may be done as follows. The unipotent classes $C_{(\xi,\xi)}'$, $C_{(\xi,\xi)}''$ are Richardson classes corresponding to parabolic subgroups of the type

$$A_{2\xi_1-1} + A_{2\xi_2-1} + \ldots.$$

There are two non-associated parabolic subgroups of this type in a group of type D_l when $2|\xi| = l$. We may therefore use a general result of Lusztig and Spaltenstein [1] which states that if u is a unipotent element in a Richardson class corresponding to a parabolic subgroup P_J of G then the representation ϕ of W corresponding to the pair $(u, 1)$ is the Macdonald representation $j_{W_J}{}^W(\varepsilon_{W_J})$. The two Macdonald representations obtained in this way from parabolic subgroups of type $A_{2\xi_1-1} + A_{2\xi_2-1} + \ldots$ are $\phi_{(\xi,\xi)}'$, $\phi_{(\xi,\xi)}''$. This determines which character of W corresponds to which unipotent class. We choose the notation so that $\phi_{(\xi,\xi)}'$ corresponds to $C_{(\xi,\xi)}'$ and $\phi_{(\xi,\xi)}''$ corresponds to $C_{(\xi,\xi)}''$.

We now give some examples to illustrate the Springer correspondence $\phi_{(\xi,\eta)} \to (C_\lambda, \psi)$. These examples are in types $B_2, C_2, B_3, C_3, D_3, D_4$. In order to describe the character ψ we give the generators a_i for which $\psi(a_i) = -1$. All other generators not given in the tables satisfy $\psi(a_i) = 1$.

Type C_2

(ξ, η)	Symbol	λ	ψ
2, –	$\begin{pmatrix} 2 \\ - \end{pmatrix}$	4	
11, –	$\begin{pmatrix} 1 & & 3 \\ & 1 & \end{pmatrix}$	211	
1, 1	$\begin{pmatrix} 0 & & 3 \\ & 2 & \end{pmatrix}$	22	
–, 2	$\begin{pmatrix} 0 & & 2 \\ & 3 & \end{pmatrix}$	22	$\psi(a_2) = -1$
–, 11	$\begin{pmatrix} 0 & & 2 & & 4 \\ & 2 & & 4 & \end{pmatrix}$	1111	

Type B_2

(ξ, η)	Symbol	λ	ψ
2, –	$\begin{pmatrix} 2 \\ - \end{pmatrix}$	5	
11, –	$\begin{pmatrix} 1 & & 3 \\ & 0 & \end{pmatrix}$	311	$\psi(a_1) = -1$
1, 1	$\begin{pmatrix} 0 & & 3 \\ & 1 & \end{pmatrix}$	311	
–, 2	$\begin{pmatrix} 0 & & 2 \\ & 2 & \end{pmatrix}$	221	
–, 11	$\begin{pmatrix} 0 & & 2 & & 4 \\ & 1 & & 3 & \end{pmatrix}$	11111	

Type C_3

(ξ, η)	Symbol	λ	ψ
3, –	$\begin{pmatrix} 3 \\ - \end{pmatrix}$	6	
21, –	$\begin{pmatrix} 1 & & 4 \\ & 1 & \end{pmatrix}$	411	

Type C_3—continued

(ξ, η)	Symbol	λ	ψ
111, –	$\begin{pmatrix} 1 & & 3 & & 5 \\ & 1 & & 3 & \end{pmatrix}$	21111	
2, 1	$\begin{pmatrix} 0 & & 4 \\ & 2 & \end{pmatrix}$	42	
11, 1	$\begin{pmatrix} 1 & & 3 \\ & 2 & \end{pmatrix}$	222	
1, 2	$\begin{pmatrix} 0 & & 3 \\ & 3 & \end{pmatrix}$	33	
1, 11	$\begin{pmatrix} 0 & & 2 & & 5 \\ & 2 & & 4 & \end{pmatrix}$	2211	
–, 3	$\begin{pmatrix} 0 & & 2 \\ & 4 & \end{pmatrix}$	42	$\psi(a_2) = -1$ $\psi(a_4) = -1$
–, 21	$\begin{pmatrix} 0 & & 2 & & 4 \\ & 2 & & 5 & \end{pmatrix}$	2211	$\psi(a_2) = -1$
–, 111	$\begin{pmatrix} 0 & & 2 & & 4 & & 6 \\ & 2 & & 4 & & 6 & \end{pmatrix}$	111111	

Type B_3

(ξ, η)	Symbol	λ	ψ
3, –	$\begin{pmatrix} 3 \\ - \end{pmatrix}$	7	
21, –	$\begin{pmatrix} 1 & & 4 \\ & 0 & \end{pmatrix}$	511	$\psi(a_1) = -1$
111, –	$\begin{pmatrix} 1 & & 3 & & 5 \\ & 0 & & 2 & \end{pmatrix}$	31111	$\psi(a_1) = -1$
2, 1	$\begin{pmatrix} 0 & & 4 \\ & 1 & \end{pmatrix}$	511	
11, 1	$\begin{pmatrix} 1 & & 3 \\ & 1 & \end{pmatrix}$	322	
1, 2	$\begin{pmatrix} 0 & & 3 \\ & 2 & \end{pmatrix}$	331	
1, 11	$\begin{pmatrix} 0 & & 2 & & 5 \\ & 1 & & 3 & \end{pmatrix}$	31111	
–, 3	$\begin{pmatrix} 0 & & 2 \\ & 3 & \end{pmatrix}$	331	$\psi(a_3) = -1$

Type B_3—continued

(ξ, η)	Symbol	λ	ψ
$-, 21$	$\begin{pmatrix} 0 & & 2 & & 4 \\ & 1 & & 4 & \end{pmatrix}$	22111	
$-, 111$	$\begin{pmatrix} 0 & & 2 & & 4 & & 6 \\ & 1 & & 3 & & 5 & \end{pmatrix}$	1111111	

Type D_3

(ξ, η)	Symbol	λ	ψ
$(3, -)$	$\begin{pmatrix} 0 \\ 3 \end{pmatrix}$	51	
$(21, -)$	$\begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$	3111	
$(111, -)$	$\begin{pmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{pmatrix}$	111111	
$(11, 1)$	$\begin{pmatrix} 0 & 3 \\ 1 & 3 \end{pmatrix}$	2211	
$(2, 1)$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	33	

Type D_4

(ξ, η)	Symbol	λ	ψ
$(4, -)$	$\begin{pmatrix} 0 \\ 4 \end{pmatrix}$	71	
$(31, -)$	$\begin{pmatrix} 0 & 2 \\ 1 & 5 \end{pmatrix}$	5111	
$(22, -)$	$\begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}$	3221	
$(211, -)$	$\begin{pmatrix} 0 & 2 & 4 \\ 1 & 3 & 6 \end{pmatrix}$	311111	
$(1111, -)$	$\begin{pmatrix} 0 & 2 & 4 & 6 \\ 1 & 3 & 5 & 7 \end{pmatrix}$	11111111	
$(3, 1)$	$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$	53	

Type D_4—continued

(ξ, η)	Symbol	λ	ψ
(21, 1)	$\begin{pmatrix} 0 & 3 \\ 1 & 4 \end{pmatrix}$	3311	
(111, 1)	$\begin{pmatrix} 0 & 2 & 5 \\ 1 & 3 & 5 \end{pmatrix}$	221111	
$(2, 2)'$	$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$	$(44)'$	
$(2, 2)''$	$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$	$(44)''$	
(11. 2)	$\begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}$	3311	$\psi(a_3) = -1$
$(11, 11)'$	$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$	$(2222)'$	
$(11, 11)''$	$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$	$(2222)''$	

We now consider the groups of exceptional type. The map from irreducible characters of W to pairs (C, ψ) has been determined by Springer [10] for type G_2, by Shoji [3] for type F_4 and by Alvis, Lusztig and Spaltenstein for types E_6, E_7, E_8 (cf. Alvis and Lusztig [1], appendix). For the exceptional groups $A(u)$ is always isomorphic to a symmetric group S_n for $n \in \{1, 2, 3, 4, 5\}$ for each unipotent element u. The irreducible characters of S_n have the form ψ_λ where λ is a partition of n, where $\psi_{(n)}$ is the unit character and $\psi_{(1^n)}$ is the sign character. We shall label irreducible characters ψ of $A(u)$ by partitions in this way.

Type G_2

Unipotent class	$A(u) = C(u)/C(u)^\circ$	Character of $A(u)$	dim $\mathfrak{B}_u$	Character of W
1	1		6	$\phi_{1,6}$
A_1	1		3	$\phi_{1,3}''$
$\tilde{A}_1$	1		2	$\phi_{2,2}$
$G_2(a_1)$	S_3	$\psi_3 = 1$	1	$\phi_{2,1}$
		ψ_{21}		$\phi_{1,3}'$
		$(\psi_{111} = \varepsilon)$		
G_2	1		0	$\phi_{1,0}$

The only pair (C, ψ) which does not correspond to an irreducible character of W is the class C of type $G_2(a_1)$ and the sign character ψ of $A(u) \cong S_3$.

Type F_4

Unipotent class	$A(u) = C(u)/C(u)^0$	Character of $A(u)$	dim $\mathfrak{B}_u$	Character of W
1	1		24	$\phi_{1,24}$
A_1	1		16	$\phi_{2,16}''$
$\tilde{A}_1$	S_2	$\psi_2 = 1$	13	$\phi_{4,13}$
		$\psi_{11} = \varepsilon$		$\phi_{2,16}'$
$A_1 + \tilde{A}_1$	1		10	$\phi_{9,10}$
A_2	S_2	$\psi_2 = 1$	9	$\phi_{8,9}''$
		$\psi_{11} = \varepsilon$		$\phi_{1,12}''$
$\tilde{A}_2$	1		9	$\phi_{8,9}'$
$A_2 + \tilde{A}_1$	1		7	$\phi_{4,7}''$
B_2	S_2	$\psi_2 = 1$	6	$\phi_{9,6}''$
		$\psi_{11} = \varepsilon$		$\phi_{4,8}$
$\tilde{A}_2 + A_1$	1		6	$\phi_{6,6}'$
$C_3(a_1)$	S_2	$\psi_2 = 1$	5	$\phi_{16,5}$
		$\psi_{11} = \varepsilon$		$\phi_{4,7}'$
$F_4(a_3)$	S_4	$\psi_4 = 1$	4	$\phi_{12,4}$
		ψ_{31}		$\phi_{9,6}'$
		ψ_{22}		$\phi_{6,6}''$
		ψ_{211}		$\phi_{1,12}'$
		$(\psi_{1111} = \varepsilon)$		
B_3	1		3	$\phi_{8,3}''$
C_3	1		3	$\phi_{8,3}'$
$F_4(a_2)$	S_2	$\psi_2 = 1$	2	$\phi_{9,2}$
		$\psi_{11} = \varepsilon$		$\phi_{2,4}''$
$F_4(a_1)$	S_2	$\psi_2 = 1$	1	$\phi_{4,1}$
		$\psi_{11} = \varepsilon$		$\phi_{2,4}'$
F_4	1		0	$\phi_{1,0}$

The only pair (C, ψ) which does not correspond to an irreducible character of W is the class C of type $F_4(a_3)$ and the sign character ψ of $A(u) \cong S_4$.

Type E_6

Unipotent class	$A(u) = C(u)/C(u)^0$	Character of $A(u)$	dim $\mathfrak{B}_u$	Character of W
1	1		36	$\phi_{1,36}$
A_1	1		25	$\phi_{6,25}$
$2A_1$	1		20	$\phi_{20,20}$
$3A_1$	1		16	$\phi_{15,16}$
A_2	S_2	$\psi_2 = 1$	15	$\phi_{30,15}$
		$\psi_{11} = \varepsilon$		$\phi_{15,17}$
$A_2 + A_1$	1		13	$\phi_{64,13}$
$2A_2$	1		12	$\phi_{24,12}$
$A_2 + 2A_1$	1		11	$\phi_{60,11}$
A_3	1		10	$\phi_{81,10}$
$2A_2 + A_1$	1		9	$\phi_{10,9}$
$A_3 + A_1$	1		8	$\phi_{60,8}$

Type E_6—continued

Unipotent class	$A(u) = C(u)/C(u)^0$	Character of $A(u)$	dim $\mathfrak{B}_u$	Character of W
$D_4(a_1)$	S_3	$\psi_3 = 1$	7	$\phi_{80,7}$
		ψ_{21}		$\phi_{90,8}$
		$\psi_{111} = \varepsilon$		$\phi_{20,10}$
A_4	1		6	$\phi_{81,6}$
D_4	1		6	$\phi_{24,6}$
$A_4 + A_1$	1		5	$\phi_{60,5}$
A_5	1		4	$\phi_{15,4}$
$D_5(a_1)$	1		4	$\phi_{64,4}$
$E_6(a_3)$	S_2	$\psi_2 = 1$	3	$\phi_{30,3}$
		$\psi_{11} = \varepsilon$		$\phi_{15,5}$
D_5	1		2	$\phi_{20,2}$
$E_6(a_1)$	1		1	$\phi_{6,1}$
E_6	1		0	$\phi_{1,0}$

In this case each pair (C, ψ) corresponds to an irreducible character of W.

Type E_7

Unipotent class	$A(u) = C(u)/C(u)^0$	Character of $A(u)$	dim $\mathfrak{B}_u$	Character of W
1	1		63	$\phi_{1,63}$
A_1	1		46	$\phi_{7,46}$
$2A_1$	1		37	$\phi_{27,37}$
$(3A_1)''$	1		36	$\phi_{21,36}$
$(3A_1)'$	1		31	$\phi_{35,31}$
A_2	S_2	$\psi_2 = 1$	30	$\phi_{56,30}$
		$\psi_{11} = \varepsilon$		$\phi_{21,33}$
$4A_1$	1		28	$\phi_{15,28}$
$A_2 + A_1$	S_2	$\psi_2 = 1$	25	$\phi_{120,25}$
		$\psi_{11} = \varepsilon$		$\phi_{105,26}$
$A_2 + 2A_1$	1		22	$\phi_{189,22}$
A_3	1		21	$\phi_{210,21}$
$2A_2$	1		21	$\phi_{168,21}$
$A_2 + 3A_1$	1		21	$\phi_{105,21}$
$(A_3 + A_1)''$	1		20	$\phi_{189,20}$
$2A_2 + A_1$	1		18	$\phi_{70,18}$
$(A_3 + A_1)'$	1		17	$\phi_{280,17}$
$D_4(a_1)$	S_3	$\psi_3 = 1$	16	$\phi_{315,16}$
		ψ_{21}		$\phi_{280,18}$
		$\psi_{111} = \varepsilon$		$\phi_{35,22}$
$A_3 + 2A_1$	1		16	$\phi_{216,16}$
D_4	1		15	$\phi_{105,15}$
$D_4(a_1) + A_1$	S_2	$\psi_2 = 1$	15	$\phi_{405,15}$
		$\psi_{11} = \varepsilon$		$\phi_{189,17}$
$A_3 + A_2$	S_2	$\psi_2 = 1$	14	$\phi_{378,14}$
		$\psi_{11} = \varepsilon$		$\phi_{84,15}$

Type E_7—continued

Unipotent class	$A(u) = C(u)/C(u)^0$	Character of $A(u)$	dim $\mathfrak{B}_u$	Character of W
A_4	S_2	$\psi_2 = 1$	13	$\phi_{420,13}$
		$\psi_{11} = \varepsilon$		$\phi_{336,14}$
$A_3 + A_2 + A_1$	1		13	$\phi_{210,13}$
$(A_5)''$	1		12	$\phi_{105,12}$
$D_4 + A_1$	1		12	$\phi_{84,12}$
$A_4 + A_1$	S_2	$\psi_2 = 1$	11	$\phi_{512,11}$
		$\psi_{11} = \varepsilon$		$\phi_{512,12}$
$D_5(a_1)$	S_2	$\psi_2 = 1$	10	$\phi_{420,10}$
		$\psi_{11} = \varepsilon$		$\phi_{336,11}$
$A_4 + A_2$	1		10	$\phi_{210,10}$
$(A_5)'$	1		9	$\phi_{216,9}$
$A_5 + A_1$	1		9	$\phi_{70,9}$
$D_5(a_1) + A_1$	1		9	$\phi_{378,9}$
$D_6(a_2)$	1		8	$\phi_{280,8}$
$E_6(a_3)$	S_2	$\psi_2 = 1$	8	$\phi_{405,8}$
		$\psi_{11} = \varepsilon$		$\phi_{189,10}$
D_5	1		7	$\phi_{189,7}$
$E_7(a_5)$	S_3	$\psi_3 = 1$	7	$\phi_{315,7}$
		ψ_{21}		$\phi_{280,9}$
		$\psi_{111} = \varepsilon$		$\phi_{35,13}$
A_6	1		6	$\phi_{105,6}$
$D_5 + A_1$	1		6	$\phi_{168,6}$
$D_6(a_1)$	1		6	$\phi_{210,6}$
$E_7(a_4)$	S_2	$\psi_2 = 1$	5	$\phi_{189,5}$
		$\psi_{11} = \varepsilon$		$\phi_{15,7}$
D_6	1		4	$\phi_{35,4}$
$E_6(a_1)$	S_2	$\psi_2 = 1$	4	$\phi_{120,4}$
		$\psi_{11} = \varepsilon$		$\phi_{105,5}$
E_6	1		3	$\phi_{21,3}$
$E_7(a_3)$	S_2	$\psi_2 = 1$	3	$\phi_{56,3}$
		$\psi_{11} = \varepsilon$		$\phi_{21,6}$
$E_7(a_2)$	1		2	$\phi_{27,2}$
$E_7(a_1)$	1		1	$\phi_{7,1}$
E_7	1		0	$\phi_{1,0}$

In this case also each pair (C, ψ) corresponds to an irreducible character of W.

Type E_8

Unipotent class	$A(u) = C(u)/C(u)^0$	Character of $A(u)$	dim $\mathfrak{B}_u$	Character of W
1	1		120	$\phi_{1,120}$
A_1	1		91	$\phi_{8,91}$
$2A_1$	1		74	$\phi_{35,74}$
$3A_1$	1		64	$\phi_{84,64}$

Type E_8—continued

Unipotent class	$A(u) = C(u)/C(u)^0$	Character of $A(u)$	dim $\mathfrak{B}_u$	Character of W
A_2	S_2	$\psi_2 = 1$	63	$\phi_{112,63}$
		$\psi_{11} = \varepsilon$		$\phi_{28,68}$
$4A_1$	1		56	$\phi_{50,56}$
$A_2 + A_1$	S_2	$\psi_2 = 1$	52	$\phi_{210,52}$
		$\psi_{11} = \varepsilon$		$\phi_{160,55}$
$A_2 + 2A_1$	1		47	$\phi_{560,47}$
A_3	1		46	$\phi_{567,46}$
$A_2 + 3A_1$	1		43	$\phi_{400,43}$
$2A_2$	S_2	$\psi_2 = 1$	42	$\phi_{700,42}$
		$\psi_{11} = \varepsilon$		$\phi_{300,44}$
$2A_2 + A_1$	1		39	$\phi_{448,39}$
$A_3 + A_1$	1		38	$\phi_{1344,38}$
$D_4(a_1)$	S_3	$\psi_3 = 1$	37	$\phi_{1400,37}$
		ψ_{21}		$\phi_{1008,39}$
		$\psi_{111} = \varepsilon$		$\phi_{56,49}$
D_4	1		36	$\phi_{525,36}$
$2A_2 + 2A_1$	1		36	$\phi_{175,36}$
$A_3 + 2A_1$	1		34	$\phi_{1050,34}$
$D_4(a_1) + A_1$	S_3	$\psi_3 = 1$	32	$\phi_{1400,32}$
		ψ_{21}		$\phi_{1575,34}$
		$\psi_{111} = \varepsilon$		$\phi_{350,38}$
$A_3 + A_2$	S_2	$\psi_2 = 1$	31	$\phi_{3240,31}$
		$\psi_{11} = \varepsilon$		$\phi_{972,32}$
A_4	S_2	$\psi_2 = 1$	30	$\phi_{2268,30}$
		$\psi_{11} = \varepsilon$		$\phi_{1296,33}$
$A_3 + A_2 + A_1$	1		29	$\phi_{1400,29}$
$D_4 + A_1$	1		28	$\phi_{700,28}$
$D_4(a_1) + A_2$	S_2	$\psi_2 = 1$	28	$\phi_{2240,28}$
		$\psi_{11} = \varepsilon$		$\phi_{840,31}$
$A_4 + A_1$	S_2	$\psi_2 = 1$	26	$\phi_{4096,26}$
		$\psi_{11} = \varepsilon$		$\phi_{4096,27}$
$2A_3$	1		26	$\phi_{840,26}$
$D_5(a_1)$	S_2	$\psi_2 = 1$	25	$\phi_{2800,25}$
		$\psi_{11} = \varepsilon$		$\phi_{2100,28}$
$A_4 + 2A_1$	S_2	$\psi_2 = 1$	24	$\phi_{4200,24}$
		$\psi_{11} = \varepsilon$		$\phi_{3360,25}$
$A_4 + A_2$	1		23	$\phi_{4536,23}$
A_5	1		22	$\phi_{3200,22}$
$D_5(a_1) + A_1$	1		22	$\phi_{6075,22}$
$A_4 + A_2 + A_1$	1		22	$\phi_{2835,22}$
$D_4 + A_2$	S_2	$\psi_2 = 1$	21	$\phi_{4200,21}$
		$\psi_{11} = \varepsilon$		$\phi_{168,24}$
$E_6(a_3)$	S_2	$\psi_2 = 1$	21	$\phi_{5600,21}$
		$\psi_{11} = \varepsilon$		$\phi_{2400,23}$
D_5	1		20	$\phi_{2100,20}$
$A_4 + A_3$	1		20	$\phi_{420,20}$
$A_5 + A_1$	1		19	$\phi_{2016,19}$
$D_5(a_1) + A_2$	1		19	$\phi_{1344,19}$

Type E_8—continued

Unipotent class	$A(u) = C(u)/C(u)^0$	Character of $A(u)$	dim $\mathfrak{B}_u$	Character of W
$D_6(a_2)$	S_2	$\psi_2 = 1$	18	$\phi_{4200,18}$
		$\psi_{11} = \varepsilon$		$\phi_{2688,20}$
$E_6(a_3) + A_1$	S_2	$\psi_2 = 1$	18	$\phi_{3150,18}$
		$\psi_{11} = \varepsilon$		$\phi_{1134,20}$
$E_7(a_5)$	S_3	$\psi_3 = 1$	17	$\phi_{7168,17}$
		ψ_{21}		$\phi_{5600,19}$
		$\psi_{111} = \varepsilon$		$\phi_{448,25}$
$D_5 + A_1$	1		16	$\phi_{3200,16}$
$E_8(a_7)$	S_5	$\psi_5 = 1$	16	$\phi_{4480,16}$
		ψ_{41}		$\phi_{5670,18}$
		ψ_{32}		$\phi_{4536,18}$
		ψ_{311}		$\phi_{1680,22}$
		ψ_{221}		$\phi_{1400,20}$
		ψ_{2111}		$\phi_{70,32}$
		$(\psi_{11111} = \varepsilon)$		
A_6	1		15	$\phi_{4200,15}$
$D_6(a_1)$	S_2	$\psi_2 = 1$	15	$\phi_{5600,15}$
		$\psi_{11} = \varepsilon$		$\phi_{2400,17}$
$A_6 + A_1$	1		14	$\phi_{2835,14}$
$E_7(a_4)$	S_2	$\psi_2 = 1$	14	$\phi_{6075,14}$
		$\psi_{11} = \varepsilon$		$\phi_{700,16}$
$E_6(a_1)$	S_2	$\psi_2 = 1$	13	$\phi_{2800,13}$
		$\psi_{11} = \varepsilon$		$\phi_{2100,16}$
$D_5 + A_2$	S_2	$\psi_2 = 1$	13	$\phi_{4536,13}$
		$\psi_{11} = \varepsilon$		$\phi_{840,14}$
D_6	1		12	$\phi_{972,12}$
E_6	1		12	$\phi_{525,12}$
$D_7(a_2)$	S_2	$\psi_2 = 1$	12	$\phi_{4200,12}$
		$\psi_{11} = \varepsilon$		$\phi_{3360,13}$
A_7	1		11	$\phi_{1400,11}$
$E_6(a_1) + A_1$	S_2	$\psi_2 = 1$	11	$\phi_{4096,11}$
		$\psi_{11} = \varepsilon$		$\phi_{4096,12}$
$E_7(a_3)$	S_2	$\psi_2 = 1$	10	$\phi_{2268,10}$
		$\psi_{11} = \varepsilon$		$\phi_{1296,13}$
$E_8(b_6)$	S_3	$\psi_3 = 1$	10	$\phi_{2240,10}$
		ψ_{21}		$\phi_{175,12}$
		$\psi_{111} = \varepsilon$		$\phi_{840,13}$
$D_7(a_1)$	S_2	$\psi_2 = 1$	9	$\phi_{3240,9}$
		$\psi_{11} = \varepsilon$		$\phi_{1050,10}$
$E_6 + A_1$	1		9	$\phi_{448,9}$
$E_7(a_2)$	1		8	$\phi_{1344,8}$
$E_8(a_6)$	S_3	$\psi_3 = 1$	8	$\phi_{1400,8}$
		ψ_{21}		$\phi_{1575,10}$
		$\psi_{111} = \varepsilon$		$\phi_{350,14}$
D_7	1		7	$\phi_{400,7}$
$E_8(b_5)$	S_3	$\psi_3 = 1$	7	$\phi_{1400,7}$
		ψ_{21}		$\phi_{1008,9}$
		$\psi_{111} = \varepsilon$		$\phi_{56,19}$

Type E_8—continued

Unipotent class	$A(u) = C(u)/C(u)^0$	Character of $A(u)$	dim $\mathfrak{B}_u$	Character of W
$E_7(a_1)$	1		6	$\phi_{567,6}$
$E_8(a_5)$	S_2	$\psi_2 = 1$	6	$\phi_{700,6}$
		$\psi_{11} = \varepsilon$		$\phi_{300,8}$
$E_8(b_4)$	S_2	$\psi_2 = 1$	5	$\phi_{560,5}$
		$\psi_{11} = \varepsilon$		$\phi_{50,8}$
E_7	1		4	$\phi_{84,4}$
$E_8(a_4)$	S_2	$\psi_2 = 1$	4	$\phi_{210,4}$
		$\psi_{11} = \varepsilon$		$\phi_{160,7}$
$E_8(a_3)$	S_2	$\psi_2 = 1$	3	$\phi_{112,3}$
		$\psi_{11} = \varepsilon$		$\phi_{28,8}$
$E_8(a_2)$	1		2	$\phi_{35,2}$
$E_8(a_1)$	1		1	$\phi_{8,1}$
E_8	1		0	$\phi_{1,0}$

The only pair (C, ψ) which does not correspond to an irreducible character of W is the class C of type $E_8(a_7)$ and the sign character ψ of $A(u) \cong S_5$.

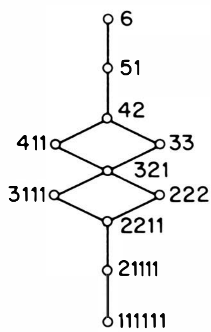
13.4 THE PARTIAL ORDERING ON UNIPOTENT CLASSES

The set of unipotent classes in $G_{ad}(\mathbb{C})$ has a natural partial ordering. If C, C' are unipotent classes we write $C \leqslant C'$ if C lies in the closure of C'. This partial order relation has been investigated in detail by Spaltenstein [3]. We shall describe this partial order relation explicitly, following Spaltenstein, in the individual cases.

Type A_l The unipotent classes of $G_{ad}(\mathbb{C}) = PGL_{l+1}(\mathbb{C})$ are described by partitions α of $l+1$. Then $C_\alpha \geqslant C_\beta$ if and only if $\alpha \geqslant \beta$. Here we are using the natural partial order on partitions defined as follows. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\beta = (\beta_1, \beta_2, \ldots)$ with $\alpha_1 \geqslant \alpha_2 \geqslant \ldots$, $\beta_1 \geqslant \beta_2 \geqslant \ldots$. Then $\alpha \geqslant \beta$ if and only if we have

$$\alpha_1 \geqslant \beta_1$$
$$\alpha_1 + \alpha_2 \geqslant \beta_1 + \beta_2$$
$$\alpha_1 + \alpha_2 + \alpha_3 \geqslant \beta_1 + \beta_2 + \beta_3$$
$$\vdots$$

For example the partial ordering in the group of type A_6 is given by:



The natural duality of partitions gives rise to an order-reversing bijection of the set of unipotent classes into itself.

Type C_l The unipotent classes of $G_{ad}(\mathbb{C}) = PSp_{2l}(\mathbb{C})$ are parametrized by pairs of partitions (α, β) with $|\alpha| + |\beta| = l$ where β has distinct parts. A unipotent element of type (α, β) has elementary divisors in the natural matrix representation of degree $2l$ given by

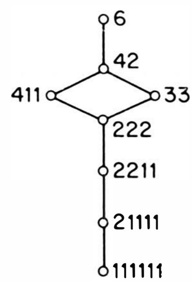
$$(t-1)^{\alpha_1}, (t-1)^{\alpha_1}, \ldots, (t-1)^{\alpha_k}, (t-1)^{\alpha_k}, (t-1)^{2\beta_1}, \ldots, (t-1)^{2\beta_h}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_h)$. These elementary divisors give a partition d of $2l$. Suppose C, C' are unipotent classes giving rise to partitions d, d' of $2l$ in this way. Then we have $C \geqslant C'$ if and only if $d \geqslant d'$ in the natural partial ordering on partitions.

Consider, for example, the group of type C_3. We have the following classes:

(α, β)	d
(3, –)	33
(21, –)	2211
(111, –)	111111
(2, 1)	222
(11, 1)	21111
(1, 2)	411
(–, 3)	6
(–, 21)	42

The partial ordering is shown in the figure:



We see that this partially ordered set does not admit an order-reversing bijection. However if we consider a certain subset of the unipotent classes we recover such an order-reversing bijection.

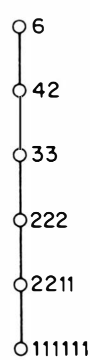
A unipotent class C is defined to be special if the character of the Weyl group associated with the pair $(C, 1)$ as in sections 12.6 and 13.3 is special. There is then a natural bijection between special unipotent classes and special representations of the Weyl group.

In the case being considered of groups of type C_l a unipotent class C with elementary divisors $d = (d_1, d_2, \ldots)$ is special if and only if the following condition is satisfied:

Between any two consecutive odd elementary divisors $2i + 1$ $2j + 1$ with $i < j$ there is an even number of even divisors, and after the largest odd divisor there is an even number of even divisors.

If the partition d of $2l$ satisfies this condition the dual partition does also, and gives the elementary divisors of some special unipotent class in the given group. Thus the partially ordered set of special unipotent classes admits an order-reversing bijection.

In the group of type C_3, for example, the elementary divisors of the special classes are 33, 2211, 111111, 222, 6, 42. Their partial ordering is shown in the figure:



Type B_l The unipotent classes of $G_{\mathrm{ad}}(\mathbb{C}) = SO_{2l+1}(\mathbb{C})$ are parametrized by pairs of partitions (α, β) with $2|\alpha| + |\beta| = 2l + 1$ where β has distinct odd parts. An element of type (α, β) has elementary divisors in the natural matrix representation of degree $2l + 1$ given by

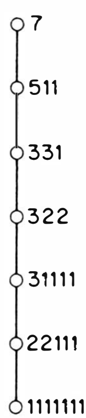
$$(t-1)^{\alpha_1}, (t-1)^{\alpha_1}, \ldots, (t-1)^{\alpha_k}, (t-1)^{\alpha_k}, (t-1)^{\beta_1}, \ldots, (t-1)^{\beta_h}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_h)$. These elementary divisors give a partition d of $2l + 1$. Suppose C, C' are unipotent classes giving rise to partitions d, d' of $2l + 1$ in this way. Then we have $C \geqslant C'$ if and only if $d \geqslant d'$ in the natural partial order on partitions.

Consider, for example, the group of type B_3. We have the following classes:

(α, β)	d
(3, 1)	331
(21, 1)	22111
(111, 1)	1111111
(2, 3)	322
(11, 3)	31111
(1, 5)	511
(–, 7)	7

The partial ordering is as shown:

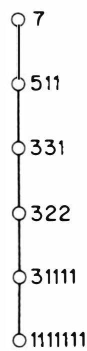


We see that the dual of a partition d representing a unipotent class need not represent a unipotent class. Thus 22111 represents a unipotent class but 52 does not. However let us consider instead the special unipotent classes. A unipotent class C with elementary divisors $d = (d_1, d_2, \ldots)$ is special if and only if the following condition is satisfied:

Between any two consecutive even elementary divisors there is an even number of odd divisors, and after the largest even divisors there is an odd number of odd divisors.

If the partition d of $2l$ represents a unipotent class and satisfies this condition the dual partition also represents a unipotent class and satisfies this condition. Thus the partially-ordered set of special unipotent classes admits an order-reversing bijection.

In the group of type B_3, for example, the elementary divisors of the special classes are 7, 511, 331, 322, 31111, 1111111. The partial ordering on the special unipotent classes is:



Type D_l The unipotent classes of $G_{\mathrm{ad}}(\mathbb{C}) = PSO_{2l}(\mathbb{C})$ are parametrized by pairs of partitions (α, β) with $2|\alpha| + |\beta| = 2l$ where β has distinct odd parts. There are two classes of this kind when β is empty and all parts of α are even. An element of type (α, β) has elementary divisors in the natural matrix representation of degree $2l$ given by

$$(t-1)^{\alpha_1}, (t-1)^{\alpha_1}, \ldots, (t-1)^{\alpha_k}, (t-1)^{\alpha_k}, (t-1)^{\beta_1}, \ldots, (t-1)^{\beta_h}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_h)$. These elementary divisors give a partition d of $2l$. Suppose C, C' are unipotent classes giving rise to partitions d, d' of $2l$ in this way. Suppose also that C, C' do not come from pairs (α, β) where β is empty and all parts of α are even. Then $C \geqslant C'$ if and only if $d \geqslant d'$. The same holds even if C or C' do come from pairs (α, β) where β is empty and all parts of α are even except when C, C' are the two classes coming from the same pair (α, β) of partitions of this special kind, when C, C' are unrelated in the partial ordering.

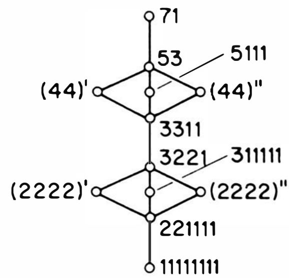
In the groups of type D_4 and D_5, for example, the classes are as shown on the following page.

A unipotent class C with elementary divisors $d = (d_1, d_2, \ldots)$ is special if and only if the following condition is satisfied:

Between any two consecutive even elementary divisors there is an even number of odd divisors, and after the largest even divisor there is an even number of odd divisors.

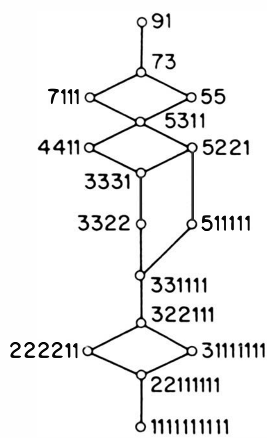
Type D_4

(α, β)	d
$(4, -)'$	$(44)'$
$(4, -)''$	$(44)''$
$(31, 1)$	3311
$(22, -)'$	$(2222)'$
$(22, -)''$	$(2222)''$
$(211, -)$	221111
$(1111, -)$	11111111
$(2, 31)$	3221
$(11, 31)$	311111
$(1, 51)$	5111
$(-, 53)$	53
$(-, 71)$	71



Type D_5

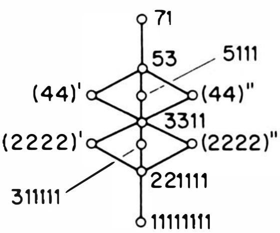
(α, β)	d
$(5, -)$	55
$(41, -)$	4411
$(32, -)$	3322
$(311, -)$	331111
$(221, -)$	222211
$(2111, -)$	22111111
$(11111, -)$	1111111111
$(3, 31)$	3331
$(21, 31)$	322111
$(111, 31)$	31111111
$(2, 51)$	5221
$(11, 51)$	511111
$(1, 53)$	5311
$(1, 71)$	7111
$(-, 73)$	73
$(-, 91)$	91



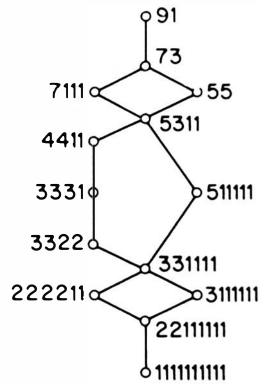
The partially ordered set of special unipotent classes again admits an order-reversing bijection. However this is not given this time by the duality of partitions.

We consider, for example, groups of type D_4 and D_5. In type D_4 all the unipotent classes are special apart from the class with $d = 3221$. In type D_5 all the unipotent classes are special apart from the classes with $d = 322111$ and 5221. The partial ordering on the special unipotent classes is as shown:

Type D_4

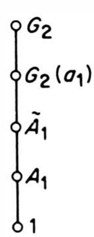


Type D_5

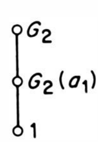


In order to describe explicitly the order-reversing bijection on the special unipotent classes in this case one must consider the corresponding special characters of W and use the involution on the special characters described in 11.3.10.

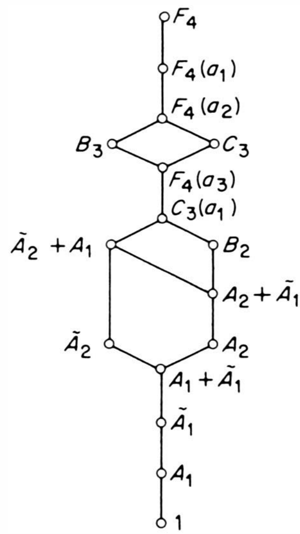
Type G_2 The partial ordering on the unipotent classes of $G_2(\mathbb{C})$, using our earlier notation for these classes, is:



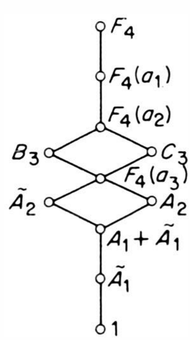
3 of these 5 classes are special. The partial ordering on the set of special unipotent classes is:



Type F_4 The partial ordering on the unipotent classes of $F_4(\mathbb{C})$, using our earlier notation, is as follows:

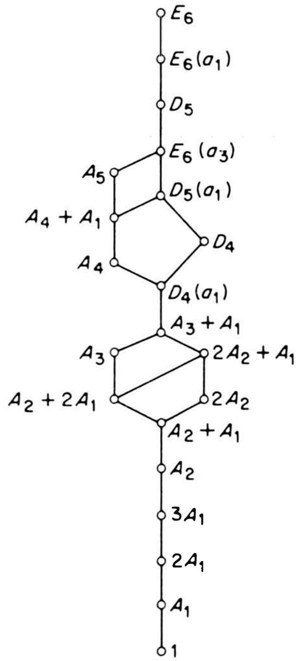


11 out of these 16 classes are special. The partial ordering on the set of special unipotent classes is as follows:

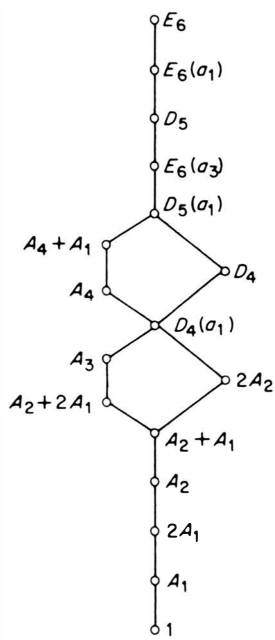


The order-reversing bijection obtained by transferring to the set of special characters of the Weyl group and then applying the involution of 11.3.10 maps C_3 to A_2 and B_3 to $\tilde{A}_2$. Apart from this its effect is clear.

Type E_6 The partial ordering on the unipotent classes of $(E_6)_{\mathrm{ad}}(\mathbb{C})$, using our earlier notation, is as follows:

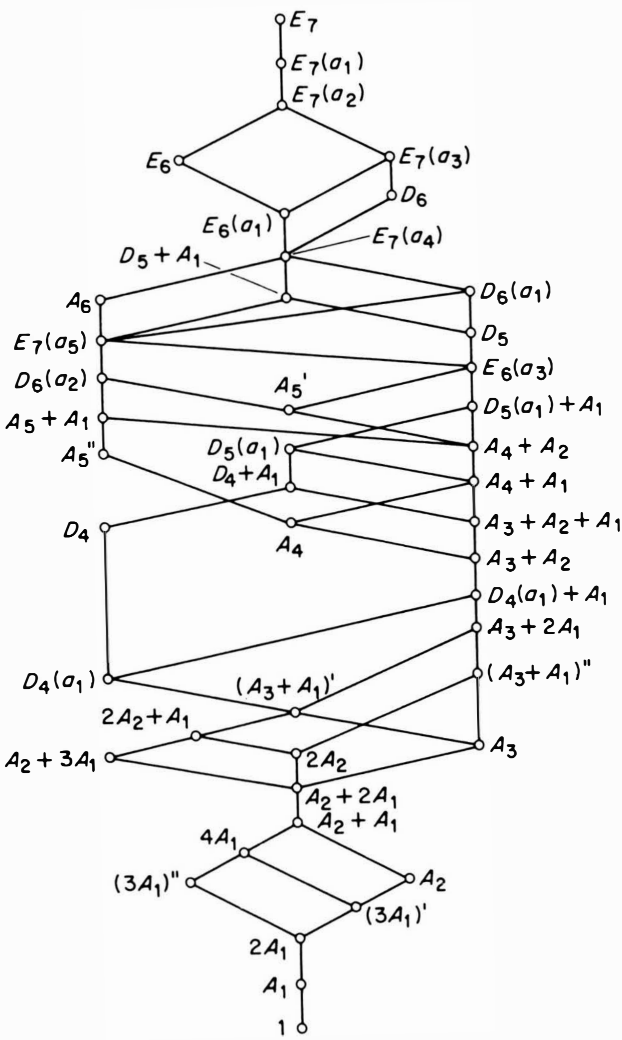


17 of these 21 classes are special. The partial ordering on the set of special unipotent classes is as follows:

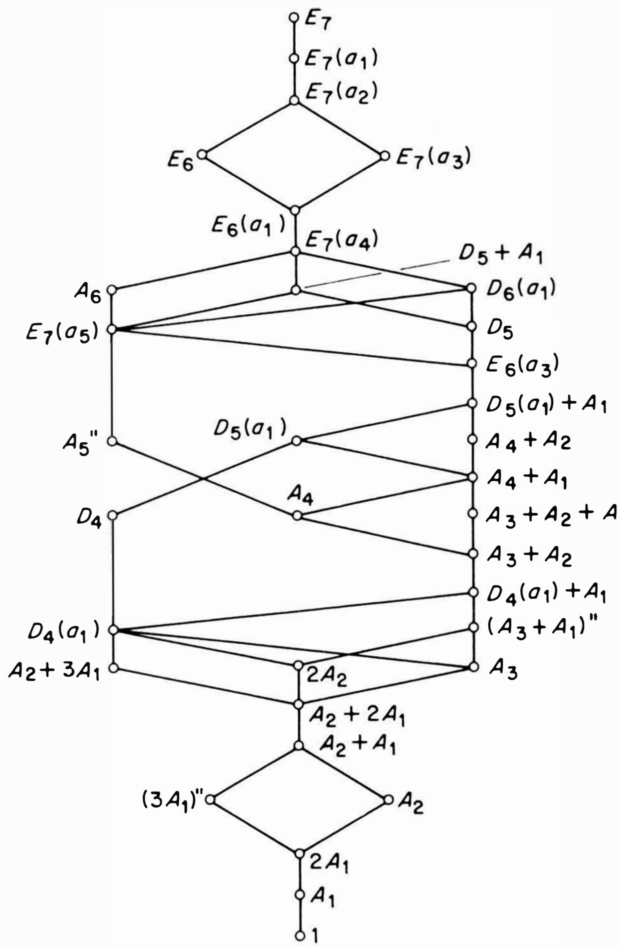


The order-reversing bijection is clear from the figure.

Type E_7 The partial ordering on the unipotent classes of $(E_7)_{ad}(\mathbb{C})$, using our earlier notation, is as follows:

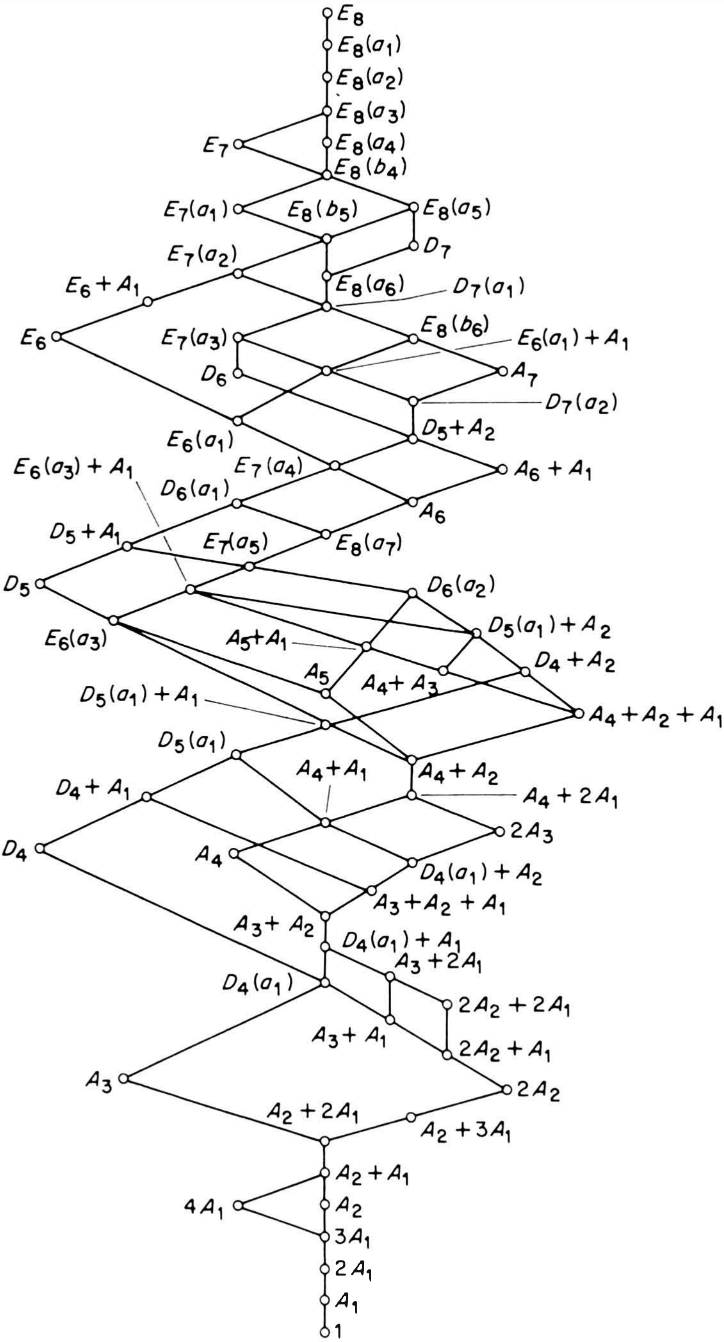


35 out of these 45 classes are special. The partially ordered set of special unipotent classes is shown next:

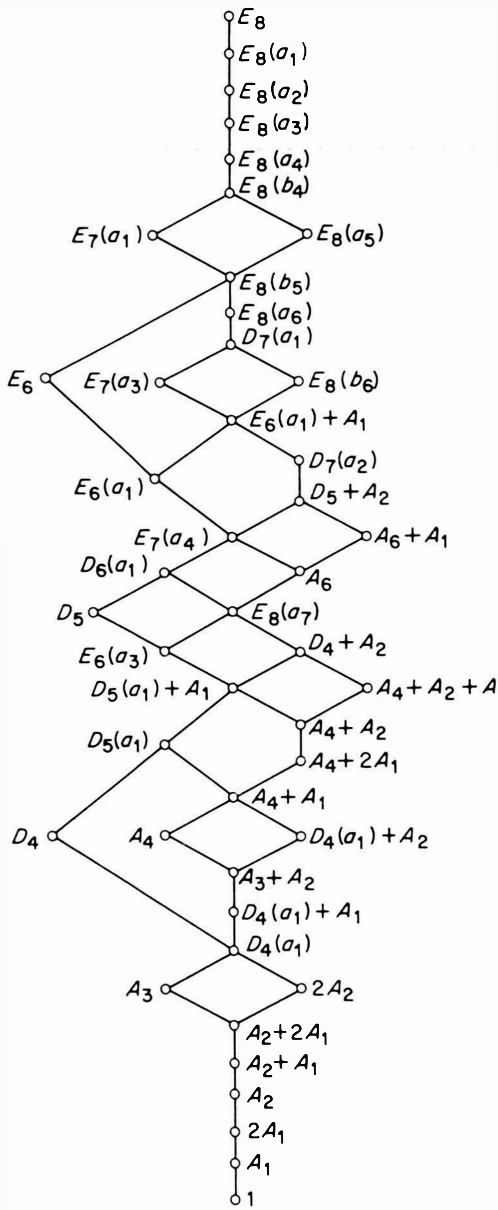


The order-reversing bijection obtained by considering the involution on the special characters of the Weyl group maps E_6 to $(3A_1)''$ and $E_7(a_3)$ to A_2. It is clear from the figure how it acts on the other classes.

Type E_8 The partial ordering on the unipotent classes of $E_8(\mathbb{C})$, using our earlier notation, is as follows:



46 out of these 70 classes are special. The partially ordered set of special unipotent classes is shown next:



The order-reversing bijection obtained from the involution on the special characters of the Weyl group maps $E_7(a_1)$ to A_3, $E_8(a_5)$ to $2A_2$, $E_7(a_3)$ to A_4 and $E_8(b_6)$ to $D_4(a_1) + A_2$. Its effect on the remaining classes is clear from the figure.

Further details on the partial order relation on unipotent classes can be found in Spaltenstein's book [3]. (The preceding eight figures are reproduced from that book by permission of Springer-Verlag.)

13.5 GENERIC DEGREES

Corresponding to each irreducible character of the Weyl group W there is a generic degree, defined as in section 10.11. A knowledge of the generic degrees is necessary to be able to find the degrees of the irreducible components of a character induced from a cuspidal character of a parabolic subgroup. The degrees of these components are obtained by 10.11.5. In the present section we shall give the generic degree corresponding to each irreducible character of W. These will be used to calculate the degrees of the irreducible unipotent characters of G^F, which are given in a later section.

In Weyl groups of type A_l, D_l, E_l where all the roots have the same length the generic degrees are functions of one variable, but in groups of type B_l, G_2, F_4 they are functions of two variables. One of these variables, denoted by u, corresponds to the long roots and the other, v, corresponds to the short roots. The degrees of the unipotent characters in the principal series are obtained from the generic degrees by replacing u and v by q if G^F is split and by replacing u and v by appropriate powers of q if G^F is non-split.

The generic degrees for the groups of classical type were determined by Hoefsmit [1] and can also be found in Lusztig [9]. The generic degrees for the groups of exceptional type were determined by Kilmoyer in type G_2, by Surowski [1] and Lusztig [11] in type F_4, by Surowski [3] in types E_6 and E_7, and by Benson [2] in type E_8.

We shall now describe the situation in the different types individually.

Type A_l Let W have type A_l. Then the irreducible characters of W are parametrized by partitions α of $l+1$, as in 11.4.1. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ where $\alpha_1 \leqslant \alpha_2 \leqslant \ldots$ and let $\lambda_1 = \alpha_1$, $\lambda_2 = \alpha_2 + 1$, $\lambda_3 = \alpha_3 + 2, \ldots$. Then the generic degree corresponding to ϕ_α is

$$\frac{(u-1)(u^2-1)\ldots(u^{l+1}-1)\prod\limits_{\substack{i,i' \\ i'<i}}(u^{\lambda_i}-u^{\lambda_{i'}})}{u^{\binom{m-1}{2}+\binom{m-2}{2}+\ldots+1}\prod\limits_{i}\prod\limits_{k=1}^{\lambda_i}(u^k-1)}$$

where m is the number of parts of α.

For example, let W have type A_2. Then we have generic degrees as follows:

Character of W	Generic degree
ϕ_3	1
ϕ_{21}	$u(u+1)$
ϕ_{111}	u^3

Type B_l, C_l Let W have type B_l. The irreducible characters of W are parametrized by pairs of partitions (α, β) with $|\alpha| + |\beta| = l$, as in 11.4.2. Let

$$\begin{pmatrix} \lambda_1 & & \lambda_2 & & \dots & & \lambda_m & & \lambda_{m+1} \\ & \mu_1 & & \mu_2 & & \dots & & \mu_m & \end{pmatrix}$$

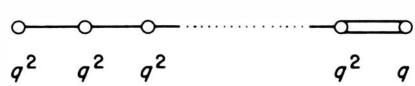
be the symbol associated with (α,β) as in 11.4.2, where λ_1, μ_1 are not both 0. The generic degree corresponding to $\phi_{\alpha,\beta}$ with respect to the labelling

is

$$\frac{u^{m+\binom{m}{2}} \prod_{i=1}^{l} (u^{i-1}v+1) \prod_{i=1}^{l} (u^i - 1) \prod_{\substack{i,i' \\ i'<i}} (u^{\lambda_i} - u^{\lambda_{i'}}) \times \prod_{\substack{j,j' \\ j'<j}} (u^{\mu_j} - u^{\mu_{j'}}) \prod_{i,j} (u^{\lambda_i - 1}v + u^{\mu_j})}{u^{\binom{2m-1}{2}+\binom{2m-3}{2}+\cdots} v^{\binom{m}{2}} \left(\prod_i \prod_{k=1}^{\lambda_i} (u^k - 1)(u^{k-1}v+1) \right) \times \left(\prod_j \prod_{k=1}^{\mu_j} (u^k - 1)(u^{k+1}v^{-1}+1) \right) (u+v)^m}.$$

By making the specialization

we obtain the degrees of the principal series characters of the groups $B_l(q)$ and $C_l(q)$. By making the specialization



we obtain the degrees of the principal series characters of the groups ${}^2A_{2l-1}(q^2)$. By making the specialization

we obtain the degrees of the principal series characters of the groups ${}^2A_{2l}(q^2)$.

For example, suppose W has type B_2. Then we have generic degrees as follows:

Character of W'	Generic degree
$\phi_{(2,\ )}$	1
$\phi_{(11,\ )}$	$\dfrac{u^2(uv+1)}{u+v}$
$\phi_{(1,1)}$	$\dfrac{uv(u+1)(v+1)}{u+v}$
$\phi_{(\ ,2)}$	$\dfrac{v^2(uv+1)}{u+v}$
$\phi_{(\ ,11)}$	u^2v^2

The degrees of the principal series representations in $B_2(q)$, ${}^2A_3(q^2)$, ${}^2A_4(q^2)$ obtained by specializing these generic degrees are as follows:

Generic degree	Degree in $B_2(q)$	Degree in ${}^2A_3(q^2)$	Degree in ${}^2A_4(q^2)$
1	1	1	1
$\dfrac{u^2(uv+1)}{u+v}$	$\frac{1}{2}q(q^2+1)$	$q^3(q^2-q+1)$	$q^2(q^4-q^3+q^2-q+1)$
$\dfrac{uv(u+1)(v+1)}{u+v}$	$\frac{1}{2}q(q+1)^2$	$q^2(q^2+1)$	$q^3(q^2+1)(q^2-q+1)$
$\dfrac{v^2(uv+1)}{u+v}$	$\frac{1}{2}q(q^2+1)$	$q(q^2-q+1)$	$q^4(q^4-q^3+q^2-q+1)$
u^2v^2	q^4	q^6	q^{10}

Type D_l Let W have type D_l. The irreducible characters of W are parametrized by unordered pairs of partitions (α, β) with $|\alpha| + |\beta| = l$, except that when $\alpha = \beta$ there are two corresponding characters $\phi_{\alpha,\alpha}'$, $\phi_{\alpha,\alpha}''$ (see 11.4.4). Let

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_m \\ \mu_1 & \mu_2 & \dots & \mu_m \end{pmatrix}$$

be the symbol associated with (α, β) as in 11.4.4. Then the generic degree corresponding to $\phi_{\alpha,\beta}$ is

$$\frac{(u^2-1)(u^4-1)\dots(u^{2l-2}-1)(u^l-1)\displaystyle\prod_{\substack{i,i'\\ i'<i}}(u^{\lambda_i}-u^{\lambda_{i'}}) \times \prod_{\substack{j,j'\\ j'<j}}(u^{\mu_j}-u^{\mu_{j'}})\prod_{i,j}(u^{\lambda_i}+u^{\mu_j})}{2^c\displaystyle\prod_i\prod_{k=1}^{\lambda_i}(u^{2k}-1)\prod_j\prod_{k=1}^{\mu_j}(u^{2k}-1)u^{\binom{2m-2}{2}+\binom{2m-4}{2}+\dots}}$$

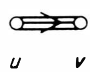
where

$$c = \begin{cases} m-1 & \text{if } \alpha \neq \beta \\ m & \text{if } \alpha = \beta. \end{cases}$$

For example, let W have type D_4. Then we have generic degrees as follows:

Character of W	Generic degree
$\phi_{(4.\ )}$	1
$\phi_{(31.\ )}$	$u^2(u^4+u^2+1)$
$\phi_{(22.\ )}$	$\frac{1}{2}u^3(u^2+1)^2(u^2-u+1)$
$\phi_{(211.\ )}$	$u^6(u^4+u^2+1)$
$\phi_{(1111.\ )}$	u^{12}
$\phi_{(3.1)}$	$u(u^2+1)^2$
$\phi_{(21.1)}$	$\frac{1}{2}u^3(u+1)^3(u^3+1)$
$\phi_{(111.1)}$	$u^7(u^2+1)^2$
$\phi_{(2.2)}'$	$u^2(u^4+u^2+1)$
$\phi_{(2.2)}''$	$u^2(u^4+u^2+1)$
$\phi_{(2.11)}$	$\frac{1}{2}u^3(u^2+1)^2(u^2+u+1)$
$\phi_{(11.11)}'$	$u^6(u^4+u^2+1)$
$\phi_{(11.11)}''$	$u^6(u^4+u^2+1)$

Type G_2 Let W have type G_2. Then the generic degrees with respect to the labelling

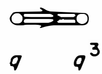


are as follows:

Character of W	Generic degree
$\phi_{1,0}$	1
$\phi_{2,1}$	$\dfrac{uv(u+1)(v+1)(uv+\sqrt{uv}+1)}{2(u+\sqrt{uv}+v)}$
$\phi_{2,2}$	$\dfrac{uv(u+1)(v+1)(uv-\sqrt{uv}+1)}{2(u-\sqrt{uv}+v)}$
$\phi_{1,3}'$	$\dfrac{u^3(u^2v^2+uv+1)}{u^2+uv+v^2}$
$\phi_{1,3}''$	$\dfrac{v^3(u^2v^2+uv+1)}{u^2+uv+v^2}$
$\phi_{1,6}$	u^3v^3

By making the specialization

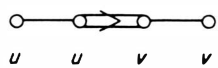
we obtain the degrees of the principal series characters of the group $G_2(q)$. By making the specialization



we obtain the degrees of the principal series characters of the group ${}^3D_4(q^3)$. The degrees of these principal series characters are as follows:

Generic degree	Degree in $G_2(q)$	Degree in ${}^3D_4(q^3)$
1	1	1
$\dfrac{uv(u+1)(v+1)(uv+\sqrt{uv}+1)}{2(u+\sqrt{uv}+v)}$	$\frac{1}{6}q(q+1)^2(q^2+q+1)$	$\frac{1}{2}q^3(q^3+1)^2$
$\dfrac{uv(u+1)(v+1)(uv-\sqrt{uv}+1)}{2(u-\sqrt{uv}+v)}$	$\frac{1}{2}q(q+1)^2(q^2-q+1)$	$\frac{1}{2}q^3(q+1)^2(q^4-q^2+1)$
$\dfrac{u^3(u^2v^2+uv+1)}{u^2+uv+v^2}$	$\frac{1}{3}q(q^4+q^2+1)$	$q(q^4-q^2+1)$
$\dfrac{v^3(u^2v^2+uv+1)}{u^2+uv+v^2}$	$\frac{1}{3}q(q^4+q^2+1)$	$q^7(q^4-q^2+1)$
u^3v^3	q^6	q^{12}

Type F_4 Let W have type F_4. Then the generic degrees with respect to the labelling

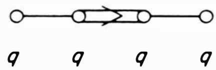


are as follows:

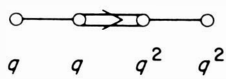
Character of W	Generic degree
$\phi_{1,0}$	1
$\phi_{4,1}$	$\dfrac{uv(uv^2+1)(u^2v+1)(u^2v^2+1)}{u+v}$
$\phi_{9,2}$	$\dfrac{u^2v^2(u^2v+1)(uv^2+1)(u^3v^3+1)(u^2+u+1)(v^2+v+1)}{(uv+1)(u^2+v)(u+v^2)}$
$\phi_{8,3}'$	$\dfrac{u^3v(u+1)(uv+1)(uv^2+1)(u^2v^2+1)(u^3v^3+1)}{(v^3+1)(u^2+v)}$
$\phi_{8,3}''$	$\dfrac{uv^3(v+1)(uv+1)(u^2v+1)(u^2v^2+1)(u^3v^3+1)}{(u^3+1)(u+v^2)}$
$\phi_{2,4}'$	$\dfrac{u^3(u+1)(u^2v+1)(u^2v^2+1)(u^3v^3+1)}{(v^3+1)(u+v)(u+v^2)}$

Character of W'	Generic degree
$\phi_{2,4}''$	$\dfrac{v^3(v+1)(uv^2+1)(u^2v^2+1)(u^3v^3+1)}{(u^3+1)(u+v)(u^2+v)}$
$\phi_{12,4}$	$\dfrac{u^3v^3(u+1)(v+1)(u^2+u+1)(v^2+v+1)(u^2v+1)(uv^2+1)(u^2v^2+1)(u^3v^3+1)}{6(u+v)^2(uv+1)(u^2-u+1)(v^2-v+1)}$
$\phi_{16,5}$	$\dfrac{u^3v^3(u+1)(v+1)(u^2v+1)(uv^2+1)(uv+1)(u^3v^3+1)}{2(u^2+v^2)}$
$\phi_{6,6}'$	$\dfrac{u^3v^3(u^2+u+1)(v^2+v+1)(u^2v+1)(uv^2+1)(u^2v^2+1)(u^3v^3+1)}{3(u^2-uv+v^2)(u+1)(v+1)(uv+1)}$
$\phi_{6,6}''$	$\dfrac{u^3v^3(u^2+u+1)(v^2+v+1)(u^2v+1)(uv^2+1)(uv+1)(u^2v^2+1)(u^3v^3+1)}{3(u^2v^2-uv+1)(u+1)(v+1)(u+v)^2}$
$\phi_{9,6}'$	$\dfrac{u^6v^2(uv+1)(u^2v^2+1)(u^3v^3+1)(u^2+u+1)(v^2+v+1)}{(u+v)^2(u^2+v^2)}$
$\phi_{9,6}''$	$\dfrac{u^2v^6(uv+1)(u^2v^2+1)(u^3v^3+1)(u^2+u+1)(v^2+v+1)}{(u+v)^2(u^2+v^2)}$
$\phi_{4,7}'$	$\dfrac{u^7v(u^2v+1)(uv^2+1)(u^2v^2+1)(u^3v^3+1)}{(u+v)(u^3+v^3)}$
$\phi_{4,7}''$	$\dfrac{uv^7(u^2v+1)(uv^2+1)(u^2v^2+1)(u^3v^3+1)}{(u+v)(u^3+v^3)}$
$\phi_{4,8}$	$\dfrac{u^3v^3(u+1)(v+1)(u^2v+1)(uv^2+1)(u^2v^2+1)(u^2v^2-uv+1)}{2(u+v)^2}$
$\phi_{8,9}'$	$\dfrac{u^7v^3(v+1)(uv+1)(u^2v+1)(u^2v^2+1)(u^3v^3+1)}{(u^3+1)(u+v^2)}$
$\phi_{8,9}''$	$\dfrac{u^3v^7(u+1)(uv+1)(uv^2+1)(u^2v^2+1)(u^3v^3+1)}{(v^3+1)(u^2+v)}$
$\phi_{9,10}$	$\dfrac{u^6v^6(u^2v+1)(uv^2+1)(u^3v^3+1)(u^2+u+1)(v^2+v+1)}{(uv+1)(u^2+v)(u+v^2)}$
$\phi_{1,12}'$	$\dfrac{u^{12}(u^2v+1)(uv^2+1)(uv+1)(u^2v^2+1)(u^3v^3+1)}{(u^2+v)(u+v^2)(u+v)(u^2+v^2)(u^3+v^3)}$
$\phi_{1,12}''$	$\dfrac{v^{12}(u^2v+1)(uv^2+1)(uv+1)(u^2v^2+1)(u^3v^3+1)}{(u^2+v)(u+v^2)(u+v)(u^2+v^2)(u^3+v^3)}$
$\phi_{4,13}$	$\dfrac{u^7v^7(uv^2+1)(u^2v+1)(u^2v^2+1)}{u+v}$
$\phi_{2,16}'$	$\dfrac{u^{12}v^3(v+1)(uv^2+1)(u^2v^2+1)(u^3v^3+1)}{(u^3+1)(u+v)(u^2+v)}$
$\phi_{2,16}''$	$\dfrac{u^3v^{12}(u+1)(u^2v+1)(u^2v^2+1)(u^3v^3+1)}{(v^3+1)(u+v)(u+v^2)}$
$\phi_{1,24}$	$u^{12}v^{12}$

By making the specialization



we obtain the degrees of the principal series characters of the group $F_4(q)$. By making the specialization



we obtain the degrees of the principal series characters of the group ${}^2E_6(q^2)$. The degrees of these principal series characters are as follows:

Character of W	Degree in $F_4(q)$
$\phi_{1,0}$	1
$\phi_{4,1}$	$\frac{1}{2}q(q^3+1)^2(q^4+1)$
$\phi_{9,2}$	$q^2(q^4-q^2+1)(q^4+q^2+1)^2$
$\phi_{8,3}'$	$q^3(q^2+1)(q^4+1)(q^6+1)$
$\phi_{8,3}''$	$q^3(q^2+1)(q^4+1)(q^6+1)$
$\phi_{2,4}'$	$\frac{1}{2}q(q^4+1)(q^6+1)$
$\phi_{2,4}''$	$\frac{1}{2}q(q^4+1)(q^6+1)$
$\phi_{12,4}$	$\frac{1}{24}q^4(q+1)^4(q^2+q+1)^2(q^4+1)(q^4-q^2+1)$
$\phi_{16,5}$	$\frac{1}{4}q^4(q+1)^2(q^2+1)(q^3+1)^2(q^6+1)$
$\phi_{6,6}'$	$\frac{1}{3}q^4(q^4+1)(q^4-q^2+1)(q^4+q^2+1)^2$
$\phi_{6,6}''$	$\frac{1}{12}q^4(q^2+1)^2(q^4+1)(q^4+q^2+1)^2$
$\phi_{9,6}'$	$\frac{1}{8}q^4(q^2+1)(q^4+1)(q^6+1)(q^2+q+1)^2$
$\phi_{9,6}''$	$\frac{1}{8}q^4(q^2+1)(q^4+1)(q^6+1)(q^2+q+1)^2$
$\phi_{4,7}'$	$\frac{1}{4}q^4(q^3+1)^2(q^4+1)(q^6+1)$
$\phi_{4,7}''$	$\frac{1}{4}q^4(q^3+1)^2(q^4+1)(q^6+1)$
$\phi_{4,8}$	$\frac{1}{8}q^4(q+1)^2(q^3+1)^2(q^4+1)(q^4-q^2+1)$
$\phi_{8,9}'$	$q^9(q^2+1)(q^4+1)(q^6+1)$
$\phi_{8,9}''$	$q^9(q^2+1)(q^4+1)(q^6+1)$
$\phi_{9,10}$	$q^{10}(q^4+q^2+1)(q^8+q^4+1)$
$\phi_{1,12}'$	$\frac{1}{8}q^4(q^2-q+1)^2(q^2+1)(q^4+1)(q^6+1)$
$\phi_{1,12}''$	$\frac{1}{8}q^4(q^2-q+1)^2(q^2+1)(q^4+1)(q^6+1)$
$\phi_{4,13}$	$\frac{1}{2}q^{13}(q^3+1)^2(q^4+1)$
$\phi_{2,16}'$	$\frac{1}{2}q^{13}(q^4+1)(q^6+1)$
$\phi_{2,16}''$	$\frac{1}{2}q^{13}(q^4+1)(q^6+1)$
$\phi_{1,24}$	q^{24}

Character of W'	Degree in ${}^2E_6(q^2)$
$\phi_{1,0}$	1
$\phi_{4,1}$	$q^2(q^4 - q^3 + q^2 - q + 1)(q^4 + 1)(q^6 + 1)$
$\phi_{9,2}$	$\frac{1}{2}q^3(q^2 + q + 1)^2(q^4 + 1)(q^4 - q^3 + q^2 - q + 1)(q^6 - q^3 + 1)$
$\phi_{8,3}'$	$\frac{1}{2}q^3(q + 1)(q^3 + 1)(q^5 + 1)(q^9 + 1)$
$\phi_{8,3}''$	$q^6(q^2 + 1)(q^4 + 1)(q^6 + 1)(q^6 - q^3 + 1)$
$\phi_{2,4}'$	$q(q^4 + 1)(q^6 - q^3 + 1)$
$\phi_{2,4}''$	$\frac{1}{2}q^3(q^2 + 1)(q^4 - q^3 + q^2 - q + 1)(q^6 + 1)(q^6 - q^3 + 1)$
$\phi_{12,4}$	$\frac{1}{6}q^7(q^4 + 1)(q^4 - q^3 + q^2 - q + 1)(q^2 + 1)^2(q^6 - q^3 + 1)(q^2 + q + 1)^2$
$\phi_{16,5}$	$\frac{1}{2}q^7(q^4 + 1)(q^4 - q^3 + q^2 - q + 1)(q + 1)^2(q^3 + 1)(q^9 + 1)$
$\phi_{6,6}'$	$\frac{1}{3}q^7(q^4 + 1)(q^4 - q^3 + q^2 - q + 1)(q^2 + q + 1)^2(q^4 - q^2 + 1)(q^6 - q^3 + 1)$
$\phi_{6,6}''$	$\frac{1}{3}q^7(q^4 + 1)(q^4 - q^3 + q^2 - q + 1)(q^2 - q + 1)(q^4 + q^2 + 1)(q^8 + q^4 + 1)$
$\phi_{9,6}'$	$q^6(q^2 - q + 1)(q^4 + q^2 + 1)((q^6 - q^3 + 1)(q^8 + q^4 + 1)$
$\phi_{9,6}''$	$q^{10}(q^2 - q + 1)(q^4 + q^2 + 1)(q^6 - q^3 + 1)(q^8 + q^4 + 1)$
$\phi_{4,7}'$	$q^5(q^4 - q^3 + q^2 - q + 1)(q^4 + 1)(q^6 - q^3 + 1)(q^6 + 1)$
$\phi_{4,7}''$	$q^{11}(q^4 - q^3 + q^2 - q + 1)(q^4 + 1)(q^6 - q^3 + 1)(q^6 + 1)$
$\phi_{4,8}$	$\frac{1}{2}q^7(q^4 + 1)(q^4 - q^3 + q^2 - q + 1)(q^2 + 1)(q^6 + 1)(q^6 - q^3 + 1)$
$\phi_{8,9}'$	$q^{12}(q^2 + 1)(q^4 + 1)(q^6 + 1)(q^6 - q^3 + 1)$
$\phi_{8,9}''$	$\frac{1}{2}q^{15}(q + 1)(q^3 + 1)(q^5 + 1)(q^9 + 1)$
$\phi_{9,10}$	$\frac{1}{2}q^{15}(q^2 + q + 1)^2(q^4 + 1)(q^4 - q^3 + q^2 - q + 1)(q^6 - q^3 + 1)$
$\phi_{1,12}'$	$\frac{1}{2}q^3(q^4 + 1)(q^4 - q^3 + q^2 - q + 1)(q^4 - q^2 + 1)(q^6 - q^3 + 1)$
$\phi_{1,12}''$	$\frac{1}{2}q^{15}(q^4 + 1)(q^4 - q^3 + q^2 - q + 1)(q^4 - q^2 + 1)(q^6 - q^3 + 1)$
$\phi_{4,13}$	$q^{20}(q^4 - q^3 + q^2 - q + 1)(q^4 + 1)(q^6 + 1)$
$\phi_{2,16}'$	$\frac{1}{2}q^{15}(q^2 + 1)(q^4 - q^3 + q^2 - q + 1)(q^6 + 1)(q^6 - q^3 + 1)$
$\phi_{2,16}''$	$q^{25}(q^4 + 1)(q^6 - q^3 + 1)$
$\phi_{1,24}$	q^{36}

Type E_6, E_7, E_8 If W has type E_6, E_7 or E_8 all the roots have the same length and so each generic degree is a polynomial in one variable. These generic degrees can be found as part of the larger tables in section 13.8 giving the degrees of all unipotent characters. We shall not give them here separately.

13.6 FOURIER TRANSFORM MATRICES

We recall from section 12.3 that the degrees of the unipotent characters of a split group G^F can be obtained from the 'fake degrees' by the use of certain square matrices called Fourier transform matrices. The unipotent characters of G^F fall into families in such a way that each family contains a number of characters equal to one of the numbers 1, 2^{2e}, 8, 21, 39. There is a Fourier transform matrix of each such degree which, when multiplied by the vector of fake degrees for the characters in the family, gives the vector of actual degrees. (The fake degree is taken as 0 if the character does not lie in the principal series.)

Each Fourier transform matrix is defined in terms of a certain group Γ isomorphic to one of the following:

$$1,\ C_2 \times \ldots \times C_2\ (e \text{ factors}),\ S_3,\ S_4,\ S_5.$$

The rows and columns of the matrix are labelled by pairs (x, σ) where x lies in Γ and σ is an irreducible character of $C_\Gamma(x)$. One representative x is taken from each conjugacy class of Γ.

We shall describe the Fourier transform matrix explicitly when Γ is isomorphic to C_2, S_3, S_4, S_5. To do this we must have a notation to label the rows and columns. We follow Lusztig [15] in this notation. We are considering the cases when Γ is a symmetric group S_n, $n = 2, 3, 4, 5$. An element of order i in S_n will be selected and denoted by g_i, when $3 \leqslant i \leqslant 6$. g_i is uniquely determined up to conjugacy. For elements of order 2, g_2 will denote a transposition and g_2' an element of cycle type 22. Representatives of the conjugacy classes are then given by

S_2	$1, g_2$
S_3	$1, g_2, g_3$
S_4	$1, g_2, g_2', g_3, g_4$
S_5	$1, g_2, g_2', g_3, g_4, g_5, g_6.$

We now consider the centralizers of these elements. In the case when the centralizer of one of these elements is simply the subgroup generated by this element we denote an irreducible character of this centralizer by the value it takes on the generator. This will be a root of unity. The cases when this does not happen are

$$C_{S_4}(g_2),\ C_{S_4}(g_2'),\ C_{S_5}(g_2),\ C_{S_5}(g_2'),\ C_{S_5}(g_3).$$

$C_{S_4}(g_2)$ is $\langle g_2 \rangle \times \langle \tau \rangle$ where τ is a transposition. The characters of this group will be denoted by $1, \varepsilon', \varepsilon'', \varepsilon$ where $\varepsilon'(g_2) = -1$, $\varepsilon'(\tau) = 1$ and $\varepsilon''(g_2) = 1$, $\varepsilon''(\tau) = -1$.

$C_{S_4}(g_2') = C_{S_5}(g_2')$ is a dihedral group of order 8. It has characters of degrees 1, 1, 1, 1, 2. The character of degree 2 will be denoted by r and the nonprincipal characters of degree 1 by $\varepsilon', \varepsilon'', \varepsilon$. ε' takes value -1 on a transposition and -1 on a 4-cycle. ε'' takes value 1 on a transposition and -1 on a 4-cycle. $\varepsilon = \varepsilon' \otimes \varepsilon''$.

$C_{S_5}(g_2)$ is $\langle g_2 \rangle \times S_3$. The irreducible characters of S_3 will be denoted by $1, r, \varepsilon$ where ε is the sign character and r the reflection character. The irreducible characters of $\langle g_2 \rangle \times S_3$ will be denoted by $1, r, \varepsilon, -1, -r, -\varepsilon$ in the obvious way.

$C_{S_5}(g_3)$ has the form $\langle g_3 \rangle \times \langle g_2 \rangle$. Its characters will be denoted by $1, \theta, \theta^2, \varepsilon, \varepsilon\theta, \varepsilon\theta^2$ where $\theta = e^{2\pi i/3}$. The power of θ shows the restriction of the character to $\langle g_3 \rangle$. The first three characters are trivial on $\langle g_2 \rangle$.

Finally the characters of S_4 and S_5 will be denoted as follows. S_4 has irreducible characters of degree 1, 3, 2, 3, 1. They will be denoted by $1, \lambda^1, \sigma, \lambda^2, \lambda^3$ where λ^1 is the reflection representation and λ^2, λ^3 are its exterior powers. S_5 has irreducible characters of degree 1, 4, 5, 6, 5, 4, 1. They will be denoted by $1, \lambda^1, \nu, \lambda^2, \nu', \lambda^3, \lambda^4$ where λ^1 is the reflection representation and $\lambda^2, \lambda^3, \lambda^4$ are its exterior powers. ν is the character of degree 5 whose value on the reflections is positive.

Using this notation we can write down the set $M(\Gamma)$ of pairs (x, σ) in each of the above cases.

$\Gamma \cong S_2$ $M(\Gamma)$ consists of the 4 pairs

$$(1, 1)\ (1, \varepsilon)\ (g_2, 1)\ (g_2, \varepsilon).$$

$\Gamma \cong S_3$ $M(\Gamma)$ consists of the 8 pairs

$$(1, 1)\ (1, r)\ (1, \varepsilon)\ (g_2, 1)\ (g_2, \varepsilon)\ (g_3, 1)\ (g_3, \theta)\ (g_3, \theta^2)$$

where $\theta = e^{2\pi i/3}$

$\Gamma \cong S_4$ $M(\Gamma)$ consists of the 21 pairs

$$\begin{gathered}(1, 1)\ (1, \lambda^1)\ (1, \lambda^2)\ (1, \lambda^3)\ (1, \sigma)\\ (g_2, 1)\ (g_2, \varepsilon')\ (g_2, \varepsilon'')\ (g_2, \varepsilon)\\ (g_2', 1)\ (g_2', \varepsilon')\ (g_2', \varepsilon'')\ (g_2', \varepsilon)\ (g_2', r)\\ (g_3, 1)\ (g_3, \theta)\ (g_3, \theta^2)\\ (g_4, 1)\ (g_4, i)\ (g_4, -1)\ (g_4, -i).\end{gathered}$$

$\Gamma \cong S_5$ $M(\Gamma)$ consists of the 39 pairs

$$\begin{gathered}(1, 1)\ (1, \lambda^1)\ (1, \lambda^2)\ (1, \lambda^3)\ (1, \lambda^4)\ (1, \nu)\ (1, \nu')\\ (g_2, 1)\ (g_2, r)\ (g_2, \varepsilon)\ (g_2, -1)\ (g_2, -r)\ (g_2, -\varepsilon)\\ (g_2', 1)\ (g_2', \varepsilon')\ (g_2', \varepsilon'')\ (g_2', \varepsilon)\ (g_2', r)\\ (g_3, 1)\ (g_3, \theta)\ (g_3, \theta^2)\ (g_3, \varepsilon)\ (g_3, \varepsilon\theta)\ (g_3, \varepsilon\theta^2)\\ (g_4, 1)\ (g_4, i)\ (g_4, -1)\ (g_4, -i)\\ (g_5, 1)\ (g_5, \zeta)\ (g_5, \zeta^2)\ (g_5, \zeta^3)\ (g_5, \zeta^4)\\ (g_6, 1)\ (g_6, -\theta)\ (g_6, \theta^2)\ (g_6, -1)\ (g_6, \theta)\ (g_6, -\theta^2)\end{gathered}$$

where $\zeta = e^{2\pi i/5}$.

If $\Gamma \cong S_2$ the 4×4 Fourier transform matrix is:

$$\begin{array}{c|cccc} & (1,1) & (1,\varepsilon) & (g_2,1) & (g_2,\varepsilon)\\ \hline (1,1) & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ (1,\varepsilon) & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}\\ (g_2,1) & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\\ (g_2,\varepsilon) & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\end{array}$$

	$(1, 1)$	$(1, \lambda^1)$	$(1, \lambda^2)$	$(1, \lambda^3)$	$(1, \sigma)$	$(g_2, 1)$	(g_2, ε)	(g_2, ε')	(g_2, ε'')	$(g_2', 1)$	(g_2', ε)	(g_2', ε')	(g_2', ε'')	(g_2', r)	$(g_3, 1)$	(g_3, θ)	(g_3, θ^2)	$(g_4, 1)$	$(g_4, -1)$	(g_4, i)	$(g_4, -\mathrm{i})$
$(1, 1)$	$\frac{1}{24}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$(1, \lambda^1)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$	0	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
$(1, \lambda^2)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$(1, \lambda^3)$	$\frac{1}{24}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
$(1, \sigma)$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{6}$	0	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	0	0	0	0
$(g_2, 1)$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0	0	0	0	0
(g_2, ε)	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0	0	0	0	0
(g_2, ε')	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	0	0	0	0	0	0
(g_2, ε'')	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	0	0	0	0	0	0
$(g_2', 1)$	$\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
(g_2', ε)	$\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$-\frac{1}{4}$	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
(g_2', ε')	$\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$	0	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
(g_2', ε'')	$\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$-\frac{1}{4}$	0	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
(g_2', r)	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0	0	0	0	0	0	0
$(g_3, 1)$	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	0	0	0	0	0	0	0	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	0	0	0	0
(g_3, θ)	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	0	0	0	0	0	0	0	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	0
(g_3, θ^2)	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	0	0	0	0	0	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	0	0
$(g_4, 1)$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$(g_4, -1)$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0
(g_4, i)	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	0	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
$(g_4, -\mathrm{i})$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	0	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$

If $\Gamma \cong S_3$ the 8×8 Fourier transform matrix is:

$$\begin{array}{c|c}
 & \begin{array}{cccccccc} (1,1) & (1,r) & (1,\varepsilon) & (g_2,1) & (g_2,\varepsilon) & (g_3,1) & (g_3,\theta) & (g_3,\theta^2) \end{array} \\
\begin{array}{c} (1,1) \\ (1,r) \\ (1,\varepsilon) \\ (g_2,1) \\ (g_2,\varepsilon) \\ (g_3,1) \\ (g_3,\theta) \\ (g_3,\theta^2) \end{array} &
\left[\begin{array}{cccccccc}
\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}
\end{array}\right]
\end{array}$$

If $\Gamma \cong S_4$ we have a 21×21 Fourier transform matrix. There is only one family of unipotent characters which corresponds to such a group Γ, and this family appears in type F_4. The Fourier transform matrix is shown opposite.

If $\Gamma \cong S_5$ we have a 39×39 Fourier transform matrix. There is only one family of unipotent characters which gives rise to such a group Γ, and this family appears in type E_8. Only 17 out of the 39 unipotent characters in this family lie in the principal series. We therefore need to know only 17 of the 39 columns of the matrix to relate the actual degrees to the fake degrees. This 39×17 matrix is shown on pages 458 and 459.

13.7 CUSPIDAL UNIPOTENT CHARACTERS OF G^F

In this section we give the number of irreducible cuspidal unipotent characters of each group G^F, together with their degrees. These results were obtained by Lusztig [9], [10], [11].

Type $A_l(q)$ This group has no cuspidal unipotent characters.

Type ${}^2A_l(q^2)$ This group has no cuspidal unipotent characters except when l has the form $l = \frac{1}{2}(s^2 + s) - 1$ for some s. In this case G^F has one cuspidal unipotent character. It has degree

$$\frac{q^{\binom{s}{2}+\binom{s-1}{2}+\cdots}(q+1)(q^2-1)(q^3+1)\ldots(q^{s(s+1)/2} \pm 1)}{(q+1)^s(q^3+1)^{s-1}(q^5+1)^{s-2}\ldots(q^{2s-3}+1)^2(q^{2s-1}+1)}.$$

Type $B_l(q)$ This group has no cuspidal unipotent characters unless l has the form $l = s^2 + s$ for some s. In this case G^F has one cuspidal unipotent character. It has degree

$$\frac{q^{\binom{2s}{2}+\binom{2s-2}{2}+\cdots}(q^2-1)(q^4-1)\ldots(q^{2(s^2+s)}-1)}{2^s(q+1)^{2s}(q^2+1)^{2s-1}\ldots(q^{2s}+1)}.$$

	$(1, 1)$	$(1, \lambda^1)$	$(1, \lambda^2)$	$(1, \lambda^3)$	$(1, \nu)$	$(1, \nu')$	$(g_5, 1)$	$(g_4, 1)$	$(g_6, 1)$	$(g_3, 1)$	(g_3, ε)	$(g_2', 1)$	(g_2', ε')	(g_2', ε'')	$(g_2, 1)$	(g_2, ε)	(g_2, r)
$(1, 1)$	$\frac{1}{120}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{30}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$
$(1, \lambda^1)$	$\frac{1}{30}$	$\frac{2}{15}$	$\frac{1}{5}$	$\frac{2}{15}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{5}$	0	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
$(1, \lambda^2)$	$\frac{1}{20}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{5}$	0	0	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	0
$(1, \lambda^3)$	$\frac{1}{30}$	$\frac{2}{15}$	$\frac{1}{5}$	$\frac{2}{15}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{5}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{3}$
$(1, \lambda^4)$	$\frac{1}{120}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{30}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{5}$	$-\frac{1}{4}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{1}{6}$
$(1, \nu)$	$\frac{1}{24}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{5}{24}$	$\frac{5}{24}$	0	$-\frac{1}{4}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$
$(1, \nu')$	$\frac{1}{24}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{5}{24}$	$\frac{5}{24}$	0	$\frac{1}{4}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{1}{6}$
$(g_5, 1)$	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$	$-\frac{1}{5}$	0	0	$\frac{4}{5}$	0	0	0	0	0	0	0	0	0	0
(g_5, ζ)	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$	$-\frac{1}{5}$	0	0	$-\frac{1}{5}$	0	0	0	0	0	0	0	0	0	0
(g_5, ζ^2)	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$	$-\frac{1}{5}$	0	0	$-\frac{1}{5}$	0	0	0	0	0	0	0	0	0	0
(g_5, ζ^3)	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$	$-\frac{1}{5}$	0	0	$-\frac{1}{5}$	0	0	0	0	0	0	0	0	0	0
(g_5, ζ^4)	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$	$-\frac{1}{5}$	0	0	$-\frac{1}{5}$	0	0	0	0	0	0	0	0	0	0
$(g_4, 1)$	$\frac{1}{4}$	0	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	0	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	0
$(g_4, -1)$	$\frac{1}{4}$	0	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{2}$	0	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	0
(g_4, i)	$\frac{1}{4}$	0	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0
$(g_4, -\mathrm{i})$	$\frac{1}{4}$	0	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0
$(g_6, 1)$	$\frac{1}{6}$	$-\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$
$(g_6, -1)$	$\frac{1}{6}$	$-\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$

$(g_6, -\theta)$	$\frac{1}{6}$	$-\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	0	0	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	0	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$
$(g_6, -\theta^2)$	$\frac{1}{6}$	$-\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	0	0	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	0	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$
(g_6, θ)	$\frac{1}{6}$	$-\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$
(g_6, θ^2)	$\frac{1}{6}$	$-\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$
$(g_3, 1)$	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$
(g_3, ε)	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$
(g_3, θ)	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$
(g_3, θ^2)	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$
$(g_3, \varepsilon\theta)$	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	0	0	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	0	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$
$(g_3, \varepsilon\theta^2)$	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	0	0	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	0	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$
$(g_2', 1)$	$\frac{1}{8}$	0	$-\frac{1}{4}$	0	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{1}{4}$	0	0	0	$\frac{3}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{1}{4}$	$-\frac{1}{4}$	0
(g_2', ε)	$\frac{1}{8}$	0	$-\frac{1}{4}$	0	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{1}{4}$	0	0	0	$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$-\frac{1}{4}$	$\frac{1}{4}$	0
(g_2', ε')	$\frac{1}{8}$	0	$-\frac{1}{4}$	0	$\frac{1}{8}$	$\frac{1}{8}$	0	$-\frac{1}{4}$	0	0	0	$\frac{3}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$	$\frac{1}{4}$	0
(g_2', ε'')	$\frac{1}{8}$	0	$-\frac{1}{4}$	0	$\frac{1}{8}$	$\frac{1}{8}$	0	$-\frac{1}{4}$	0	0	0	$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{4}$	$-\frac{1}{4}$	0
(g_2', r)	$\frac{1}{4}$	0	$-\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	0
$(g_2, 1)$	$\frac{1}{12}$	$\frac{1}{6}$	0	$-\frac{1}{6}$	$\frac{1}{12}$	$-\frac{1}{12}$	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{6}$
(g_2, ε)	$\frac{1}{12}$	$\frac{1}{6}$	0	$-\frac{1}{6}$	$\frac{1}{12}$	$-\frac{1}{12}$	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$
(g_2, r)	$\frac{1}{6}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
$(g_2, -1)$	$\frac{1}{12}$	$\frac{1}{6}$	0	$-\frac{1}{6}$	$\frac{1}{12}$	$-\frac{1}{12}$	0	0	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{6}$
$(g_2, -\varepsilon)$	$\frac{1}{12}$	$\frac{1}{6}$	0	$-\frac{1}{6}$	$\frac{1}{12}$	$-\frac{1}{12}$	0	0	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$
$(g_2, -r)$	$\frac{1}{6}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	0	0	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	0	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{3}$

Type $C_l(q)$ This group has no cuspidal unipotent characters unless l has the form $l = s^2 + s$ for some s. In this case G^F has one cuspidal unipotent character. It has degree

$$\frac{q^{\binom{2s}{2}+\binom{2s-2}{2}+\cdots}(q^2-1)(q^4-1)\ldots(q^{2(s^2+s)}-1)}{2^s(q+1)^{2s}(q^2+1)^{2s-1}\ldots(q^{2s}+1)}.$$

Type $D_l(q)$ This group has no cuspidal unipotent characters unless l has the form $l = s^2$ for some even integer s. In this case G^F has one cuspidal unipotent character. It has degree

$$\frac{q^{\binom{2s-1}{2}+\binom{2s-3}{2}+\cdots}(q^2-1)(q^4-1)\ldots(q^{2s^2-2}-1)(q^{s^2}-1)}{2^{s-1}(q+1)^{2s-1}(q^2+1)^{2s-2}\ldots(q^{2s-1}+1)}.$$

Type ${}^2D_l(q^2)$ This group has no cuspidal unipotent characters unless l has the form $l = s^2$ for some odd integer s. In this case G^F has one cuspidal unipotent character. It has degree

$$\frac{q^{\binom{2s-1}{2}+\binom{2s-3}{2}+\cdots}(q^2-1)(q^4-1)\ldots(q^{2s^2-2}-1)(q^{s^2}+1)}{2^{s-1}(q+1)^{2s-1}(q^2+1)^{2s-2}\ldots(q^{2s-1}+1)}.$$

Type ${}^3D_4(q^3)$ This group has two cuspidal unipotent characters. They are denoted by ${}^3D_4[1]$, ${}^3D_4[-1]$ and have degrees

$${}^3D_4[-1] \qquad \frac{q^3(q^2-1)(q^6-1)(q^8+q^4+1)}{2(q^3+1)^2(q^4-q^2+1)}$$

$${}^3D_4[1] \qquad \frac{q^3(q^2-1)(q^6-1)(q^8+q^4+1)}{2(q^3+1)^2(q^2+q+1)^2}.$$

Type $G_2(q)$ This group has four cuspidal unipotent characters. They are denoted by $G_2[1]$, $G_2[-1]$, $G_2[\theta]$, $G_2[\theta^2]$ and have degrees

$$G_2[-1] \qquad \frac{q(q^2-1)(q^6-1)}{2(q+1)(q^3+1)}$$

$$G_2[\theta] \qquad \frac{q(q^2-1)(q^6-1)}{3(q^4+q^2+1)}$$

$$G[\theta^2] \qquad \frac{q(q^2-1)(q^6-1)}{3(q^4+q^2+1)}$$

$$G_2[1] \qquad \frac{q(q^2-1)(q^6-1)}{6(q+1)^2(q^2+q+1)}.$$

Type $F_4(q)$ This group has seven cuspidal unipotent characters. They are denoted by $F_4^{\mathrm{I}}[1]$, $F_4^{\mathrm{II}}[1]$, $F_4[-1]$, $F_4[\theta]$, $F_4[\theta^2]$, $F_4[\mathrm{i}]$, $F_4[-\mathrm{i}]$ and have degrees

$$F_4[\mathrm{i}] \quad \frac{q^4(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)}{4(q^2+1)(q^4+1)(q^6+1)}$$

$$F_4[-\mathrm{i}] \quad \frac{q^4(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)}{4(q^2+1)(q^4+1)(q^6+1)}$$

$$F_4[\theta] \quad \frac{q^4(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)}{3(q^4+q^2+1)(q^8+q^4+1)}$$

$$F_4[\theta^2] \quad \frac{q^4(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)}{3(q^4+q^2+1)(q^8+q^4+1)}$$

$$F_4^{\mathrm{I}}[1] \quad \frac{q^4(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)}{8(q+1)^2(q^2+1)^2(q^3+1)^2}$$

$$F_4^{\mathrm{II}}[1] \quad \frac{q^4(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)}{24(q+1)^2(q^2+q+1)^2(q^3+q^2+q+1)^2}$$

$$F_4[-1] \quad \frac{q^4(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)}{4(q+1)^2(q^3+1)^2(q^4+1)}.$$

Type $E_6(q)$ This group has two cuspidal unipotent characters. They are denoted by $E_6[\theta]$, $E_6[\theta^2]$ and have degrees

$$E_6[\theta] \quad \frac{q^7(q^2-1)(q^5-1)(q^6-1)(q^8-1)(q^9-1)(q^{12}-1)}{3(q^2+q+1)(q^4+q^2+1)(q^6+q^3+1)(q^8+q^4+1)}$$

$$E_6[\theta^2] \quad \frac{q^7(q^2-1)(q^5-1)(q^6-1)(q^8-1)(q^9-1)(q^{12}-1)}{3(q^2+q+1)(q^4+q^2+1)(q^6+q^3+1)(q^8+q^4+1)}$$

Type ${}^2E_6(q^2)$ This group has three cuspidal unipotent characters. They are denoted by ${}^2E_6[1]$, ${}^2E_6[\theta]$, ${}^2E_6[\theta^2]$ and have degrees

$${}^2E_6[\theta] \quad \frac{q^7(q^2-1)(q^5+1)(q^6-1)(q^8-1)(q^9+1)(q^{12}-1)}{3(q^2-q+1)(q^4+q^2+1)(q^6-q^3+1)(q^8+q^4+1)}$$

$${}^2E_6[\theta^2] \quad \frac{q^7(q^2-1)(q^5+1)(q^6-1)(q^8-1)(q^9+1)(q^{12}-1)}{3(q^2-q+1)(q^4+q^2+1)(q^6-q^3+1)(q^8+q^4+1)}$$

$${}^2E_6[1] \quad \frac{q^7(q^2-1)(q^5+1)(q^6-1)(q^8-1)(q^9+1)(q^{12}-1)}{6(q+1)^5(q^2+1)^2(q^3+1)(q^4+q^2+1)^2}$$

Type $E_7(q)$ This group has two cuspidal unipotent characters. They are denoted by $E_7[\xi]$, $E_7[-\xi]$ and have degrees

$$E_7[\xi] \quad \frac{q^{11}(q^2-1)(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)}{2(q+1)^2(q^3+1)^2(q^5+1)(q^7+1)(q^9+1)}$$

$$E_7[-\xi] \quad \frac{q^{11}(q^2-1)(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)}{2(q+1)^2(q^3+1)^2(q^5+1)(q^7+1)(q^9+1)}.$$

Type $E_8(q)$ This group has 13 cuspidal unipotent characters. They are denoted by

$$E_8{}^{\mathrm{I}}[1],\ E_8{}^{\mathrm{II}}[1],\ E_8[-1],\ E_8[\theta],\ E_8[\theta^2],\ E_8[\mathrm{i}],\ E_8[-\mathrm{i}],$$
$$E_8[\zeta],\ E_8[\zeta^2],\ E_8[\zeta^3],\ E_8[\zeta^4],\ E_8[-\theta],\ E_8[-\theta^2]$$

where $\theta = \mathrm{e}^{2\pi\mathrm{i}/3}$, $\zeta = \mathrm{e}^{2\pi\mathrm{i}/5}$.

The four characters $E_8[\zeta]$, $E_8[\zeta^2]$, $E_8[\zeta^3]$, $E_8[\zeta^4]$ each have degree

$$\frac{q^{16}(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)}{5(q^{16}+q^{12}+q^8+q^4+1)(q^{24}+q^{18}+q^{12}+q^6+1)}$$

The two characters $E_8[-\theta]$, $E_8[-\theta^2]$ each have degree

$$\frac{q^{16}(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)}{6(q+1)^2(q^2-q+1)(q^4+q^2+1)(q^6-q^3+1)(q^{10}-q^5+1)(q^{16}+q^8+1)}$$

The two characters $E_8[\theta]$, $E_8[\theta^2]$ each have degree

$$\frac{q^{16}(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)}{6(q+1)^2(q^2-q+1)(q^4+q^2+1)(q^6-q^3+1)(q^8+q^4+1)^2(q^{10}+q^5+1)}.$$

The two characters $E_8[\mathrm{i}]$, $E_8[-\mathrm{i}]$ each have degree

$$\frac{q^{16}(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)}{4(q^2+1)(q^4+1)(q^6+1)^2(q^{10}+1)(q^{12}+1)}.$$

The remaining characters have degrees:

$$E_8{}^{\mathrm{I}}[1] \quad \frac{q^{16}(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)}{8(q+1)^2(q^2+1)^2(q^3+1)^4(q^5+1)^2(q^6+1)^2}$$

$$E_8{}^{\mathrm{II}}[1] \quad \frac{q^{16}(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)}{120(q^3+q^2+q+1)^4(q^4+q^3+q^2+q+1)^2(q^5+q^4+q^3+q^2+q+1)^4}$$

$$E_8[-1] \quad \frac{q^{16}(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)}{12(q+1)^4(q^3+1)^2(q^4+1)^2(q^5+1)^2(q^4+q^2+1)(q^8+q^4+1)}.$$

Type ${}^2B_2(q^2)$ We have Suzuki groups of this type where q^2 is an odd power of 2. These groups have two cuspidal unipotent characters. They have degrees

$$\frac{1}{\sqrt{2}}q(q^2-1), \qquad \frac{1}{\sqrt{2}}q(q^2-1).$$

Type ${}^2G_2(q^2)$ We have Ree groups of this type where q^2 is an odd power of 3. These groups have six cuspidal unipotent characters. They have degrees

$$\frac{1}{\sqrt{3}}q(q^4-1)$$

$$\frac{1}{\sqrt{3}}q(q^4-1)$$

$$\frac{1}{2\sqrt{3}}q(q^2-1)(q^2+\sqrt{3}q+1)$$

$$\frac{1}{2\sqrt{3}}q(q^2-1)(q^2+\sqrt{3}q+1)$$

$$\frac{1}{2\sqrt{3}}q(q^2-1)(q^2-\sqrt{3}q+1)$$

$$\frac{1}{2\sqrt{3}}q(q^2-1)(q^2-\sqrt{3}q+1).$$

Type ${}^2F_4(q^2)$ We have Ree groups of this type where q^2 is an odd power of 2. These groups have ten cuspidal unipotent characters. They have degrees

$$\tfrac{1}{12}q^4(q^2-1)^2(q^4-q^2+1)(q^2+\sqrt{2}q+1)^2(q^4+\sqrt{2}q^3+q^2+\sqrt{2}q+1)$$

$$\tfrac{1}{12}q^4(q^2-1)^2(q^4-q^2+1)(q^2-\sqrt{2}q+1)^2(q^4-\sqrt{2}q^3+q^2-\sqrt{2}q+1)$$

$$\tfrac{1}{6}q^4(q^4-1)^2(q^8-q^4+1)$$

$$\tfrac{1}{4}q^4(q^2-1)(q^4-1)(q^6+1)(q^4+\sqrt{2}q^3+q^2+\sqrt{2}q+1)$$

$$\tfrac{1}{4}q^4(q^2-1)(q^4-1)(q^6+1)(q^4+\sqrt{2}q^3+q^2+\sqrt{2}q+1)$$

$$\tfrac{1}{4}q^4(q^2-1)(q^4-1)(q^6+1)(q^4-\sqrt{2}q^3+q^2-\sqrt{2}q+1)$$

$$\tfrac{1}{4}q^4(q^2-1)(q^4-1)(q^6+1)(q^4-\sqrt{2}q^3+q^2-\sqrt{2}q+1)$$

$$\tfrac{1}{3}q^4(q^2-1)^2(q^4-q^2+1)(q^8-q^4+1)$$

$$\tfrac{1}{3}q^4(q^8-1)^2$$

$$\tfrac{1}{3}q^4(q^8-1)^2.$$

In order to obtain the degrees of all the unipotent characters from the degrees of the cuspidal unipotent characters we apply the Howlett–Lehrer theory of

Group G^F	Levi subgroup	Quotient root system	Parameters
Any untwisted group	1	Original root system	All are q
${}^2D_l(q^2)$	1	B_{l-1}	q q q q^2
${}^3D_4(q^3)$	1	G_2	q q^3
${}^2E_6(q^2)$	1	F_4	q q q^2 q^2
${}^2A_{2n+\frac{1}{2}(s^2+s)-1}$ $n \geqslant 1,\ s \geqslant 0$	${}^2A_{\frac{1}{2}(s^2+s)-1}$	B_n	q^2 q^2 q^2 q^{2s+1}
$B_{n+(s^2+s)}$ $n \geqslant 1,\ s \geqslant 1$	B_{s^2+s}	B_n	q q q q^{2s+1}
$C_{n+(s^2+s)}$ $n \geqslant 1,\ s \geqslant 1$	C_{s^2+s}	B_n	q q q q^{2s+1}
D_{n+s^2} $n \geqslant 1,\ s \geqslant 2$, s even	D_{s^2}	B_n	q q q q^{2s}
${}^2D_{n+s^2}$ $n \geqslant 1,\ s \geqslant 3$, s odd	${}^2D_{s^2}$	B_n	q q q q^{2s}
F_4	B_2	B_2	q^3 q^3
E_6	D_4	A_2	q^4 q^4
${}^2E_6$	${}^2A_5$	A_1	q^9
E_7	D_4	C_3	q q^4 q^4
E_7	E_6	A_1	q^9
E_8	D_4	F_4	q q q^4 q^4
E_8	E_6	G_2	q q^9
E_8	E_7	A_1	q^{15}

chapter 10. The required degrees can be calculated using 10.11.5. In order to carry out this calculation we need the generic degrees which have been given in section 13.5. However we also need the parameters p_α which have not yet been given. These parameters were obtained by Lusztig [10] and we list them in the table opposite.

In this table we first give the type of the group G^F. We then give the type of the Levi subgroup of G^F which has a cuspidal unipotent representation. This Levi subgroup gives rise to a quotient root system as in section 10.4, and we describe the type of this quotient root system. For each simple root of the quotient root system there will be a corresponding parameter p_α, and we finally indicate what these parameters are.

13.8 UNIPOTENT CHARACTERS OF GROUPS OF CLASSICAL TYPE

We now turn to a discussion of all the unipotent characters of G^F, not just those which are cuspidal. We begin by considering groups G^F of classical type. The results here are due to Lusztig [9], [18], [20].

Type A_l Suppose $G^F = (A_l)_{\mathrm{ad}}(q) = PGL_{l+1}(q)$. Then the unipotent characters of G^F are all in the principal series. They are parametrized by partitions α of $l+1$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ with $\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_m$ and let $\lambda_1 = \alpha_1$, $\lambda_2 = \alpha_2 + 1$, $\lambda_3 = \alpha_3 + 2, \ldots$. Then the degree of the unipotent character χ^α corresponding to α is given by

$$\chi^\alpha(1) = \frac{(q-1)(q^2-1)\ldots(q^{l+1}-1)\prod\limits_{\substack{i,i' \\ i'<i}}(q^{\lambda_i}-q^{\lambda_{i'}})}{q^{\binom{m-1}{2}+\binom{m-2}{2}+\cdots}\prod\limits_{i}\prod\limits_{k=1}^{\lambda_i}(q^k-1)}$$

where m is the number of parts of α.

Each of these unipotent characters χ^α lies in a family by itself. The group Γ associated to each family is 1. The fake degree of each of these unipotent characters is equal to the actual degree.

Type ${}^2A_l$ Suppose $G^F = ({}^2A_l)_{\mathrm{ad}}(q^2) = PU_{l+1}(q)$. Then the unipotent characters of G^F are again parametrized by partitions α of $l+1$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ with $\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_m$ and let $\lambda_1 = \alpha_1$, $\lambda_2 = \alpha_2 + 1$, $\lambda_3 = \alpha_3 + 2, \ldots$. Then the degree of the unipotent character χ^α corresponding to α is given by

$$\chi^\alpha(1) = \frac{(q+1)(q^2-1)(q^3+1)\ldots(q^{l+1}\pm 1)\prod\limits_{\substack{i,i' \\ i'<i}}(q^{\lambda_i}-(-1)^{\lambda_i+\lambda_{i'}}q^{\lambda_{i'}})}{q^{\binom{m-1}{2}+\binom{m-2}{2}+\cdots}\prod\limits_{i}\prod\limits_{k=1}^{\lambda_i}(q^k-(-1)^k)}.$$

The unipotent characters χ^α do not all lie in the principal series. The division of the χ^α into series can be described as follows. Given a partition α of $n = l + 1$ we take the Young tableau corresponding to α. We consider the hook length of each square in this Young tableau. Let $o(\alpha)$ be the number of squares with odd hook length and $e(\alpha)$ be the number of squares with even hook length. Then $o(\alpha) \geqslant e(\alpha)$ for all partitions α and $o(\alpha) - e(\alpha)$ has the form $\frac{1}{2}s(s+1)$ for some $s \geqslant 0$. Moreover the number of partitions α for which $o(\alpha) - e(\alpha) = \frac{1}{2}s(s+1)$ for a fixed s is equal to the number of pairs of partitions (β, γ) with $|\beta| + |\gamma| = e(\alpha) = \frac{1}{2}(n - \frac{1}{2}s(s+1))$.

Now we recall from section 13.7 that the group ${}^2A_{n-1}$ has a Levi subgroup of type ${}^2A_{\frac{1}{2}s(s+1)-1}$ and that this Levi subgroup has a unique cuspidal unipotent character. Furthermore the irreducible components of the character of ${}^2A_{n-1}$ induced from this cuspidal unipotent character lifted to its parabolic subgroup are in bijective correspondence with the irreducible characters of $W(B_{\frac{1}{2}(n-\frac{1}{2}s(s+1))})$, i.e. with pairs of partitions (β, γ) such that $|\beta| + |\gamma| = \frac{1}{2}(n - \frac{1}{2}s(s+1))$.

This explains how the χ^α fall into series. Two of them lie in the same series if and only if they have the same value of s. The number of χ^α with a given value of s is the number of irreducible characters of the Weyl group of type $B_{\frac{1}{2}(n-\frac{1}{2}s(s+1))}$, and the χ^α with a given value of s can be parametrized by characters of this Weyl group in a natural way. For further details we refer to Lusztig's paper [9].

Types B_l, C_l Suppose $G^F = (B_l)_{\mathrm{ad}}(q) = SO_{2l+1}(q)$ or $(C_l)_{\mathrm{ad}}(q) = PCSp_{2l}(q)$. Then the unipotent characters of G^F are parametrized in terms of symbols of the form

$$\begin{pmatrix} \lambda_1, & & \lambda_2, & & \lambda_3, \ldots, & & \lambda_a \\ & \mu_1, & & \mu_2, & & \ldots, \mu_b & \end{pmatrix}$$

where $0 \leqslant \lambda_1 < \lambda_2 < \ldots < \lambda_a$, $0 \leqslant \mu_1 < \mu_2 < \ldots < \mu_b$, $a - b$ is odd and positive, and λ_1, μ_1 are not both 0. The rank of such a symbol is defined as

$$\sum \lambda_i + \sum \mu_j - \left[\left(\frac{a+b-1}{2}\right)^2\right]$$

($[z]$ denotes the largest integer $\leqslant z$). There is then a bijective correspondence between unipotent characters of G^F and symbols of rank l. The degree of the unipotent character corresponding to the symbol

$$\begin{pmatrix} \lambda_1, & & \lambda_2, & & \ldots, & & \lambda_a \\ & \mu_1, & & \mu_2, \ldots, & \mu_b & & \end{pmatrix}$$

is given by

$$\frac{(q^2-1)(q^4-1)\dots(q^{2l}-1)\prod\limits_{\substack{i,i'\\ i'<i}}(q^{\lambda_i}-q^{\lambda_{i'}})\prod\limits_{\substack{j,j'\\ j'<j}}(q^{\mu_j}-q^{\mu_{j'}})\prod\limits_{i,j}(q^{\lambda_i}+q^{\mu_j})}{2^{[\frac{a+b-1}{2}]}q^{\binom{a+b-2}{2}+\binom{a+b-4}{2}+\cdots}\prod\limits_i\prod\limits_{k=1}^{\lambda_i}(q^{2k}-1)\prod\limits_j\prod\limits_{k=1}^{\mu_j}(q^{2k}-1)}.$$

The division of the unipotent characters into series can be described as follows. The number $d = a - b$ is called the defect of the given symbol. The symbols we are considering at present have odd defect. Let $d = 2s + 1$. Then the number of symbols of rank l and defect d can be shown to be equal to the number of pairs of partitions (α, β) with $|\alpha| + |\beta| = l - (s^2 + s)$. Now the given group of type B_l or C_l has a Levi subgroup of type B_{s^2+s} or C_{s^2+s} respectively, and this has a unique cuspidal unipotent character. If we lift this character to its parabolic subgroup and then induce to G^F, the components of this induced character will be in bijective correspondence with the irreducible characters of the Weyl group of type $B_{l-(s^2+s)}$, as we see from section 13.7. These components thus correspond to pairs of partitions (α, β) with $|\alpha| + |\beta| = l - (s^2 + s)$.

Two unipotent characters of G^F belong to the same series if and only if they have the same defect. The characters with defect $d = 2s + 1$ arise as components of the induced character coming from the cuspidal unipotent character of the Levi subgroup of type B_{s^2+s} or C_{s^2+s}. In particular a unipotent character lies in the principal series if and only if $d = 1$ and in the discrete series if and only if $s^2 + s = l$. The unipotent characters in the principal series are in bijective correspondence with irreducible characters of $W(B_l)$, and we saw in 11.4.2 how to associate with each irreducible character of $W(B_l)$ a symbol with defect 1.

The fake degree of the irreducible character $\phi_{\alpha,\beta}$ of $W(B_l)$, where $|\alpha| + |\beta| = l$, is

$$\frac{\prod\limits_j q^{\mu_j}(q^2-1)(q^4-1)\dots(q^{2l}-1)\prod\limits_{\substack{i,i'\\ i'<i}}(q^{2\lambda_i}-q^{2\lambda_{i'}})\prod\limits_{\substack{j,j'\\ j'<j}}(q^{2\mu_j}-q^{2\mu_{j'}})}{q^{\binom{2m-1}{2}+\binom{2m-3}{2}+\cdots}\prod\limits_i\prod\limits_{k=1}^{\lambda_i}(q^{2k}-1)\prod\limits_j\prod\limits_{k=1}^{\mu_j}(q^{2k}-1)}$$

where

$$\begin{pmatrix}\lambda_1, & \lambda_2, \dots, & \lambda_{m+1}\\ & \mu_1, \quad \dots, \mu_m & \end{pmatrix}$$

is the symbol corresponding to the pair of partitions (α, β).

For example, consider a group of type $B_2 = C_2$. We list the symbols of rank 2 and the degrees of the unipotent characters corresponding to them. We also give the fake degrees of those lying in the principal series.

Symbol	Defect	Degree	Fake degree
$\begin{pmatrix} 2 \\ - \end{pmatrix}$	1	1	1
$\begin{pmatrix} 1 & & 2 \\ & 0 & \end{pmatrix}$	1	$\frac{1}{2}q(q^2+1)$	q^2
$\begin{pmatrix} 0 & & 2 \\ & 1 & \end{pmatrix}$	1	$\frac{1}{2}q(q+1)^2$	$q(q^2+1)$
$\begin{pmatrix} 0 & & 1 \\ & 2 & \end{pmatrix}$	1	$\frac{1}{2}q(q^2+1)$	q^2
$\begin{pmatrix} 0 & & 1 & & 2 \\ & 1 & & 2 & \end{pmatrix}$	1	q^4	q^4
$\begin{pmatrix} 0 & 1 & 2 \\ & - & \end{pmatrix}$	3	$\frac{1}{2}q(q-1)^2$	—

We now consider the way the unipotent characters fall into families, as described in section 12.3. Two unipotent characters lie in the same family if and only if their symbols contain the same entries with the same multiplicities. Thus the set $\{\lambda_1, \lambda_2, \ldots, \lambda_a, \mu_1, \mu_2, \ldots, \mu_b\}$ is constant for symbols in a given family. We shall describe the Fourier transform matrix associated with each family, which relates the fake degrees to the actual degrees.

Now each family of symbols contains a unique special symbol. This is a symbol of the form

$$\begin{pmatrix} \lambda_1, & & \lambda_2, \ldots, & & \lambda_{m+1} \\ & \mu_1, & & \ldots, \mu_m & \end{pmatrix}$$

where $\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \mu_m \leqslant \lambda_{m+1}$. This special symbol contains certain entries which appear in both rows and also a set Z of numbers which appear only once. We have $Z = Z^* \cup Z_*$ where Z^* consists of the elements of Z in the first row and Z_* the elements of Z in the second row. We have $|Z^*| = k + 1$, $|Z_*| = k$ for some k.

Each symbol in the family will have an odd number of elements of Z in one row and an even number in the other row. We pick the row containing a number of elements in Z congruent mod 2 to the number of elements of Z in the lower row of the special symbol. Let M be the set of elements of Z in this row. Then $|M| \equiv k \bmod 2$. In this way we obtain a bijection between symbols in the given family and subsets M of Z with $|M| \equiv k \bmod 2$. The number of such subsets M is 2^{2k}. The special symbol in the family gives $M = Z_*$.

Let $M^{\#} = (M \cup Z_*) - (M \cap Z_*)$. Then $|M^{\#}|$ is even. In this way we obtain a bijection between symbols in the given family and subsets $M^{\#}$ of Z with $|M^{\#}|$ even.

The set of such subsets $M^{\#}$ can be made into a vector space V over the field F_2 of 2 elements with addition being defined as the symmetric difference. The dimension of V is $2d$. We can define a scalar product $V \times V \to F_2$ by

$$\langle M_1^{\#}, M_2^{\#} \rangle \equiv |M_1^{\#} \cap M_2^{\#}| \bmod 2.$$

This makes V into a symplectic space over F_2.

Let $Z = \{z_0, z_1, \ldots, z_{2k}\}$ and let $e_1, e_2, \ldots, e_{2k}$ be the elements of V given by

$$e_1 = \{z_0, z_1\}, e_2 = \{z_1, z_2\}, \ldots, e_{2k} = \{z_{2k-1}, z_{2k}\}.$$

Then $e_1, e_2, \ldots, e_{2k}$ is a basis of V. The symplectic form on V with respect to this basis is given by

$$\begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ & 1 & 0 & & & & \\ & & & \ddots & & & \\ & & & & 0 & 1 & \\ & & & & 1 & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}$$

Now this is not the canonical form for a symplectic scalar product. We can find another basis $f_1, f_2, \ldots, f_{2k}$ of V with respect to which the symplectic form is given by

$$\begin{bmatrix} 0 & 1 & & & & & & & \\ 1 & 0 & & & & & & & \\ & & 0 & 1 & & & & & \\ & & 1 & 0 & & & & & \\ & & & & 0 & 1 & & & \\ & & & & 1 & 0 & & & \\ & & & & & & \ddots & & \\ & & & & & & & 0 & 1 \\ & & & & & & & 1 & 0 \end{bmatrix}$$

Let Γ be the subspace of V spanned by $f_1, f_3, \ldots, f_{2k-1}$ and Γ^* be the subspace spanned by $f_2, f_4, \ldots, f_{2k}$. Thus $V = \Gamma \oplus \Gamma^*$. Now the map $\Gamma \times \Gamma^* \to F_2$ given by $\gamma, \gamma^* \to \langle \gamma, \gamma^* \rangle$ is nondegenerate, and therefore gives rise to an isomorphism between Γ^* and $\mathrm{Hom}(\Gamma, F_2)$. Thus Γ^* may be regarded as the character group of Γ. The elements of V may therefore be regarded as pairs γ, γ^* with $\gamma \in \Gamma$, $\gamma^* \in \Gamma^*$.

In this way each symbol in the given family is associated with a pair (x, σ) where $x \in \Gamma$, $\sigma \in \Gamma^*$ and Γ is a group isomorphic to $C_2 \times \ldots \times C_2$ with k factors.

The Fourier transform matrix for the family is then given as follows. For any two pairs (x, σ), (y, τ) the corresponding entry in the Fourier transform matrix, defined as in section 12.3, will be $\sigma(y)\tau(x)/2^k$.

Take, for example, the group of type $B_2 = C_2$. There are 3 families of symbols for this group which are:

$$\left\{\begin{pmatrix} 2 \\ - \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} 1 & & 2 \\ & 0 & \end{pmatrix}, \begin{pmatrix} 0 & & 2 \\ & 1 & \end{pmatrix}, \begin{pmatrix} 0 & & 1 \\ & 2 & \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ & - & \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & \end{pmatrix}\right\}.$$

The first and third families have Fourier transform matrix (1). The middle family will have a 4×4 Fourier transform matrix. The special symbol in the middle family is $\begin{pmatrix} 0 & & 2 \\ & 1 & \end{pmatrix}$. We thus have $Z = \{0, 1, 2\}$, $Z^* = \{0, 2\}$, $Z_* = \{1\}$. The corresponding sets M and $M^\#$ are given in the following table:

Symbol	M	$M^\#$
$\begin{pmatrix} 0 & & 2 \\ & 1 & \end{pmatrix}$	$\{1\}$	$\varnothing$
$\begin{pmatrix} 1 & & 2 \\ & 0 & \end{pmatrix}$	$\{0\}$	$\{0, 1\}$
$\begin{pmatrix} 0 & & 1 \\ & 2 & \end{pmatrix}$	$\{2\}$	$\{1, 2\}$
$\begin{pmatrix} 0 & 1 & 2 \\ & - & \end{pmatrix}$	$\{0 \quad 1 \quad 2\}$	$\{0, 2\}$

Let $e_1 = \{0, 1\}$ and $e_2 = \{1, 2\}$. The symplectic structure with respect to this basis is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We therefore take $\Gamma = F_2 e_1$, $\Gamma^* = F_2 e_2$ and $e_2(e_1) = -1$. The Fourier transform matrix is therefore

$$\begin{array}{c} \\ \begin{pmatrix} 0 & & 2 \\ & 1 & \end{pmatrix} \\ \begin{pmatrix} 1 & & 2 \\ & 0 & \end{pmatrix} \\ \begin{pmatrix} 0 & & 1 \\ & 2 & \end{pmatrix} \\ \begin{pmatrix} 0 & 1 & 2 \\ & - & \end{pmatrix} \end{array} \begin{array}{c} \begin{pmatrix} 0 & & 2 \\ & 1 & \end{pmatrix} \quad \begin{pmatrix} 1 & & 2 \\ & 0 & \end{pmatrix} \quad \begin{pmatrix} 0 & & 1 \\ & 2 & \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 \\ & - & \end{pmatrix} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}.$$

We illustrate how this Fourier transform matrix relates the fake degrees to the actual degrees of unipotent characters in the family. We have

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} q(q^2+1) \\ q^2 \\ q^2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}q(q+1)^2 \\ \frac{1}{2}q(q^2+1) \\ \frac{1}{2}q(q^2+1) \\ \frac{1}{2}q(q-1)^2 \end{bmatrix}.$$

Thus the Fourier transform matrix multiplied by the vector of fake degrees gives the vector of actual degrees.

Type D_l. Suppose $G^F = (D_l)_{\mathrm{ad}}(q) = P(CO_{2l}(q)^0)$. Then the unipotent characters of G^F are parametrized in terms of symbols of the form

$$\begin{pmatrix} \lambda_1, \lambda_2, \ldots, \lambda_a \\ \mu_1, \mu_2, \ldots, \mu_b \end{pmatrix}$$

where $0 \leqslant \lambda_1 < \lambda_2 < \ldots < \lambda_a$, $0 \leqslant \mu_1 < \mu_2 < \ldots < \mu_b$, $a - b$ is divisible by 4, and λ_1, μ_1 are not both 0. $\begin{pmatrix} \lambda_1, \lambda_2, \ldots, \lambda_a \\ \mu_1, \mu_2, \ldots, \mu_b \end{pmatrix}$ is regarded as the same symbol as $\begin{pmatrix} \mu_1, \mu_2, \ldots, \mu_b \\ \lambda_1, \lambda_2, \ldots, \lambda_a \end{pmatrix}$ in this case. The rank of such a symbol is defined, as before, to be

$$\sum \lambda_i + \sum \mu_j - \left[\left(\frac{a+b-1}{2}\right)^2\right].$$

Each symbol of rank l determines a unipotent character of G^F, except that if $a = b$ and $\{\lambda_1, \ldots, \lambda_a\} = \{\mu_1, \ldots, \mu_a\}$ there are two unipotent characters with the given symbol. The degree of the unipotent character corresponding to the symbol $\begin{pmatrix} \lambda_1, \lambda_2, \ldots, \lambda_a \\ \mu_1, \mu_2, \ldots, \mu_b \end{pmatrix}$ is given by

$$\frac{(q^2-1)(q^4-1)\ldots(q^{2l-2}-1)(q^l-1)\prod\limits_{\substack{i,i' \\ i'<i}}(q^{\lambda_i}-q^{\lambda_{i'}})\prod\limits_{\substack{j,j' \\ j'<j}}(q^{\mu_j}-q^{\mu_{j'}})\prod\limits_{i,j}(q^{\lambda_i}+q^{\mu_j})}{2^c q^{\binom{a+b-2}{2}+\binom{a+b-4}{2}+\cdots}\prod\limits_i\prod\limits_{k=1}^{\lambda_i}(q^{2k}-1)\prod\limits_j\prod\limits_{k=1}^{\mu_j}(q^{2k}-1)}$$

where

$$c = \begin{cases} \left[\dfrac{a+b-1}{2}\right] & \text{if } \lambda \neq \mu \\ a = b & \text{if } \lambda = \mu. \end{cases}$$

The division of the unipotent characters into series can be described as follows. Two unipotent characters of G^F lie in the same series if and only if they have the same defect $d = a - b$. The symbols we are at present considering have defect divisible by 4. The number of symbols of rank l and defect $d = 2s > 0$ is the number of pairs of partitions (α, β) with $|\alpha| + |\beta| = l - s^2$. The unipotent characters of defect $2s$ (s even) are components of the induced character from the cuspidal unipotent character of a Levi subgroup of type D_{s^2} lifted to its parabolic subgroup. Such components are in bijective correspondence with irreducible characters of the Weyl group of type B_{l-s^2}.

The unipotent characters of defect 0 are the ones in the principal series. These are in bijective correspondence with irreducible characters of $W(D_l)$. Each such irreducible character of $W(D_l)$ determines a symbol $\begin{pmatrix} \lambda_1, \lambda_2, \ldots, \lambda_m \\ \mu_1, \mu_2, \ldots, \mu_m \end{pmatrix}$ as in 11.4.4, where $a = b = m$. The fake degree of the irreducible character with this symbol is

$$\frac{(q^2-1)(q^4-1)\ldots(q^{2l-2}-1)(q^l+1)\prod\limits_{\substack{i,i'\\ i'<i}}(q^{2\lambda_i}-q^{2\lambda_{i'}}) \times \prod\limits_{\substack{j,j'\\ j'<j}}(q^{2\mu_j}-q^{2\mu_{j'}})\left(\prod\limits_j q^{\mu_j}+\prod\limits_i q^{\lambda_i}\right)}{q^{\binom{2m-2}{2}+\binom{2m-4}{2}+\cdots}\prod\limits_i\prod\limits_{k=1}^{\lambda_i}(q^{2k}-1)\prod\limits_j\prod\limits_{k=1}^{\mu_j}(q^{2k}-1)}$$

if $\lambda \neq \mu$, and $\frac{1}{2}$ of this expression if $\lambda = \mu$.

We now consider the way the unipotent characters fall into families. Two unipotent characters lie in the same family if and only if their symbols contain the same entries with the same multiplicities, except that if $\lambda = \mu$ the two unipotent characters with this symbol lie in different families.

For example, in the group of type D_4, there is a family containing the symbols $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 \\ & - & & \end{pmatrix}$. We list the degrees of the corresponding unipotent characters, and the fake degrees of those which lie in the principal series.

Symbol	Defect	Degree	Fake degree
$\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$	0	$\frac{1}{2}q^3(q+1)^3(q^3+1)$	$q^3(q^2+1)^3$
$\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$	0	$\frac{1}{2}q^3(q^2+1)^2(q^2-q+1)$	$q^4(q^4+1)$
$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$	0	$\frac{1}{2}q^3(q^2+1)^2(q^2+q+1)$	$2q^4(q^4+q^2+1)$
$\begin{pmatrix} 0 & 1 & 2 & 3 \\ & - & & \end{pmatrix}$	4	$\frac{1}{2}q^3(q-1)^3(q^3-1)$	—

We now describe the Fourier transform matrix of a family, which relates the fake degrees to the actual degrees. Given a family of symbols, just one symbol in the family will be special. Let Z be the set of numbers which appear in just one row and $Z = Z^* \cup Z_*$ where Z^*, Z_* are the elements of Z which appear in the first and second row respectively. Then we have $|Z| = 2k$ and $|Z^*| = |Z_*| = k$ for some k.

Each symbol in the family contains a set M of elements of Z in its first row, and the set $Z - M$ in the second row. We have $|M| \equiv k \bmod 2$. This defines a 2–1 map from subsets M of Z with $|M| \equiv k \bmod 2$ to symbols in the given family. The number of such symbols is 2^{2k-2}. M and $Z - M$ give rise to the same symbol.

Let $M^\# = (M \cup Z_*) - (M \cap Z_*)$. Then $|M^\#|$ is even. In this way we obtain a 2–1 map from subsets $M^\#$ of Z with $|M^\#|$ even to symbols in the family. Complementary subsets of Z give the same symbol.

The set of such subsets $M^\#$ can be made into a vector space V over the field F_2 with addition given by the symmetric difference. The dimension of V is $2k - 1$. We define a scalar product $V \times V \to F_2$ by

$$\langle M_1{}^\#, M_2{}^\# \rangle \equiv |M_1{}^\# \cap M_2{}^\#| \bmod 2.$$

This makes V into a symplectic space over F_2. V is singular since its dimension is odd. The element $M^\# = Z$ is in the radical of V. Let V_0 be the subspace spanned by this element. Then V_0 is the radical of V and V/V_0 is a nonsingular symplectic space. For if $Z = \{z_1, z_2, \ldots, z_{2k}\}$ and $e_1, e_2, \ldots, e_{2k-1}$ are the elements of V given by

$$e_1 = \{z_1, z_2\} \quad e_2 = \{z_2, z_3\} \quad \ldots \quad e_{2k-1} = \{z_{2k-1}, z_{2k}\}$$

then $e_1, e_2, \ldots, e_{2k-1}$ is a basis for V with respect to which the symplectic form is given by

$$\begin{bmatrix} 0 & 1 & & & & & & \\ 1 & 0 & 1 & & & & & \\ & 1 & 0 & 1 & & & & \\ & & & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & 1 & \\ & & & & & 1 & 0 & 1 \\ & & & & & & 1 & 0 \end{bmatrix} \Bigg\updownarrow 2k - 1.$$

Now $V_0 = \langle e_1 + e_3 + \ldots + e_{2k-1}\rangle$ and the elements $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2k-2}$ form a basis for $\bar{V} = V/V_0$ with symplectic form

$$\left.\begin{bmatrix} 0 & 1 & & & & & & \\ 1 & 0 & 1 & & & & & \\ & 1 & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & 1 & \\ & & & & & 1 & 0 & 1 \\ & & & & & & 1 & 0 \end{bmatrix}\right\updownarrow 2k-2.$$

This is nonsingular, although not in canonical form. There exists another basis $f_1, f_2, \ldots, f_{2k-2}$ for $\bar{V}$ with respect to which the form is

$$\left.\begin{bmatrix} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & 0 & 1 & & \\ & & & & 1 & 0 & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{bmatrix}\right\updownarrow 2k-2.$$

Let Γ be the subspace spanned by $f_1, f_3, f_5, \ldots$ and Γ^* be the subspace spanned by $f_2, f_4, f_6, \ldots$. Then Γ^* can be identified with the character group of Γ, just as in the situation already discussed for type B_l.

In this way we get a bijection between the symbols in the given family and the pairs (x, σ) with $x \in \Gamma$, $\sigma \in \Gamma^*$, where Γ is a group isomorphic to $C_2 \times \ldots \times C_2$ with $k - 1$ factors.

The Fourier transform matrix for the family is then given as follows. For any two pairs (x, σ), (y, τ) the corresponding entry in the Fourier transform matrix will be $\sigma(y)\tau(x)/2^{k-1}$.

Take, for example, the family in the group of type D_4 containing the four symbols $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 2 & 3 \\ & - & & \end{pmatrix}$. Then $Z = \{0, 1, 2, 3\}$, $Z^* = \{1, 3\}$, $Z_* = \{0, 2\}$ and $k = 2$. The sets M, M^* corresponding to these symbols are:

Symbol	M	$M^\#$
$\begin{pmatrix}1 & 3\\ 0 & 2\end{pmatrix}$	$\{2, 2\}$ or $\{1, 3\}$	$\varnothing$ or $\{0, 1, 2, 3\}$
$\begin{pmatrix}2 & 3\\ 0 & 1\end{pmatrix}$	$\{0, 1\}$ or $\{2, 3\}$	$\{1, 2\}$ or $\{0, 3\}$
$\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}$	$\{0, 3\}$ or $\{1, 2\}$	$\{2, 3\}$ or $\{0, 1\}$
$\begin{pmatrix}0 \ 1 \ 2 \ 3\\ -\end{pmatrix}$	$\varnothing$ or $\{0, 1, 2, 3\}$	$\{0, 2\}$ or $\{1, 3\}$

This gives rise to the Fourier transform matrix:

$$\begin{array}{c|cccc}
 & \begin{pmatrix}1 & 3\\ 0 & 2\end{pmatrix} & \begin{pmatrix}2 & 3\\ 0 & 1\end{pmatrix} & \begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix} & \begin{pmatrix}0 \ 1 \ 2 \ 3\\ -\end{pmatrix} \\
\hline
\begin{pmatrix}1 & 3\\ 0 & 2\end{pmatrix} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\begin{pmatrix}2 & 3\\ 0 & 1\end{pmatrix} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\begin{pmatrix}0 \ 1 \ 2 \ 3\\ -\end{pmatrix} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}
\end{array}$$

We illustrate how this Fourier transform matrix relates the fake degrees to the actual degrees of unipotent characters in the family. We have

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} q^3(q^2+1)^3 \\ q^4(q^4+1) \\ 2q^4(q^4+q^2+1) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}q^3(q+1)^3(q^3+1) \\ \frac{1}{2}q^3(q^2+1)^2(q^2-q+1) \\ \frac{1}{2}q^3(q^2+1)^2(q^2+q+1) \\ \frac{1}{2}q^3(q-1)^3(q^3-1) \end{bmatrix}$$

Thus the Fourier transform matrix multiplied by the vector of fake degrees gives the vector of actual degrees.

Type ^2D_l. Suppose $G^F = (^2D_l)_{\mathrm{ad}}(q^2) = P(CO_{2l}^-(q)^0)$. Then the unipotent characters of G^F are parametrized in terms of symbols of the form

$$\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_a \\ \mu_1, \mu_2, \ldots, \mu_b\end{pmatrix}$$

where $0 \leqslant \lambda_1 < \lambda_2 < \ldots < \lambda_a$, $0 \leqslant \mu_1 < \mu_2 < \ldots < \mu_b$, $a - b \equiv 2 \bmod 4$. Again in this case

$$\begin{pmatrix}\lambda_1, \lambda_2, \ldots, \lambda_a \\ \mu_1, \mu_2, \ldots, \mu_b\end{pmatrix}, \quad \begin{pmatrix}\mu_1, \mu_2, \ldots, \mu_b \\ \lambda_1, \lambda_2, \ldots, \lambda_a\end{pmatrix}$$

are regarded as the same symbol. There is a bijective correspondence between unipotent characters of G^F and symbols of the above type of rank l. The degree of the unipotent character corresponding to the symbol $\begin{pmatrix} \lambda_1, \lambda_2, \ldots, \lambda_a \\ \mu_1, \mu_2, \ldots, \mu_b \end{pmatrix}$ is

$$\frac{\begin{aligned}(q^2-1)(q^4-1)\ldots(q^{2l-2}-1)(q^l+1)&\prod_{\substack{i,i'\\ i'<i}}(q^{\lambda_i}-q^{\lambda_{i'}})\\ &\times\prod_{\substack{j,j'\\ j'<j}}(q^{\mu_j}-q^{\mu_{j'}})\prod_{i,j}(q^{\lambda_i}+q^{\mu_j})\end{aligned}}{2^{\frac{a+b-2}{2}}q^{\binom{a+b+2}{2}+\binom{a+b-4}{2}+\cdots}\prod_i\prod_{k=1}^{\lambda_i}(q^{2k}-1)\prod_j\prod_{k=1}^{\mu_j}(q^{2k}-1)}.$$

The division of the unipotent characters into series can be described as follows. Two unipotent characters of G^F lie in the same series if and only if they have the same defect $d = a - b$. The symbols we are at present considering have defect congruent to 2 mod 4. Let $d = 2s$ where s is odd. Then the number of symbols of rank l and defect $d = 2s$ is the number of pairs of partitions (α, β) with $|\alpha| + |\beta| = l - s^2$. The unipotent characters of defect $2s$ (s odd), $s \geqslant 3$, are components of the induced character from the cuspidal unipotent character of a Levi subgroup of type ${}^2D_{s^2}$ lifted to its parabolic subgroup. Such components are in bijective correspondence with irreducible characters of the Weyl group of type B_{l-s^2}. The unipotent characters of defect 2 are those in the principal series. These are in bijective correspondence with irreducible characters of $W(B_{l-1})$.

Consider, for example, the group ${}^2D_3(q^2)$. There are 5 possible symbols for this group, all of defect 2. These are

$$\begin{pmatrix} 0 & 3 \\ & - & \end{pmatrix} \begin{pmatrix} 1 & 2 \\ & - & \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 \\ & 1 & \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ & 2 & \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ & 1 & 2 & \end{pmatrix}.$$

Thus all the unipotent characters lie in the principal series, and their degrees are as follows:

Symbol	Degree
$\begin{pmatrix} 0 & 3 \\ & - \end{pmatrix}$	1
$\begin{pmatrix} 1 & 2 \\ & - \end{pmatrix}$	$q(q^2-q+1)$
$\begin{pmatrix} 0 & 1 & 3 \\ & 1 & \end{pmatrix}$	$q^2(q^2+1)$
$\begin{pmatrix} 0 & 1 & 2 \\ & 2 & \end{pmatrix}$	$q^3(q^2-q+1)$
$\begin{pmatrix} 0 & 1 & 2 & 3 \\ & 1 & 2 & \end{pmatrix}$	q^6

13.9 UNIPOTENT CHARACTERS OF GROUPS OF EXCEPTIONAL TYPE

We now turn to the exceptional groups, and shall describe the irreducible unipotent characters in the groups $G_2(q)$, ${}^3D_4(q^3)$, $F_4(q)$, $E_6(q)$, ${}^2E_6(q^2)$, $E_7(q)$, $E_8(q)$, ${}^2B_2(q)$, ${}^2G_2(q)$, ${}^2F_4(q)$. For each irreducible unipotent character we shall give its degree and also how it arises from a cuspidal unipotent character of a Levi subgroup by the Howlett–Lehrer theory. Since the degrees involve polynomial functions of q which can be quite large it is convenient to write these degrees in terms of cyclotomic polynomials. We denote by $\Phi_k = \Phi_k(q)$ the polynomial in q whose roots are the primitive kth roots of unity. The particular cyclotomic polynomials we shall need are the following:

Φ_1	$q - 1$
Φ_2	$q + 1$
Φ_3	$q^2 + q + 1$
Φ_4	$q^2 + 1$
Φ_5	$q^4 + q^3 + q^2 + q + 1$
Φ_6	$q^2 - q + 1$
Φ_7	$q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$
Φ_8	$q^4 + 1$
Φ_9	$q^6 + q^3 + 1$
Φ_{10}	$q^4 - q^3 + q^2 - q + 1$
Φ_{12}	$q^4 - q^2 + 1$
Φ_{14}	$q^6 - q^5 + q^4 - q^3 + q^2 - q + 1$
Φ_{15}	$q^8 - q^7 + q^5 - q^4 + q^3 - q + 1$
Φ_{18}	$q^6 - q^3 + 1$
Φ_{20}	$q^8 - q^6 + q^4 - q^2 + 1$
Φ_{24}	$q^8 - q^4 + 1$
Φ_{30}	$q^8 + q^7 - q^5 - q^4 - q^3 + q + 1.$

In order to describe how a given unipotent character arises in the context of the Howlett–Lehrer theory we give for each such unipotent character a cuspidal unipotent character of a Levi subgroup of G^F and also an irreducible character of the reflection group corresponding to the quotient root system. The possible cuspidal unipotent characters are described in section 13.7 and the quotient root systems are given at the end of section 13.7.

In the case of the split groups G^F we shall also describe how the unipotent characters fall into families, with one family for each special irreducible character of the Weyl group. The characters in a family are described by pairs (x, σ) where x lies in a group Γ and σ is an irreducible character of $C_\Gamma(x)$. x is taken up to conjugacy in Γ. The group Γ depends on the family being considered and, in the case of the exceptional groups, is always isomorphic to a symmetric group S_n for $1 \leqslant n \leqslant 5$. We shall describe the pair (x, σ) for each unipotent character in the family, using the notation of section 13.6.

The results in this section are due to Lusztig [15], [21].

Type $G_2(q)$. The group has 10 unipotent characters. 6 of these lie in the principal series and 4 are cuspidal. There are three families of unipotent characters. These families contain 1, 1, 8 characters respectively. The degrees of the unipotent characters are given below. The characters in the principal series are described by the irreducible characters of $W(G_2)$, given in section 13.2.

Description in terms of cuspidal characters	Degree	Pair (x, σ)
$\phi_{1,0}$	1	
$\phi_{2,1}$	$\frac{1}{6}q\Phi_2{}^2\Phi_3$	$(1, 1)$
$G_2[1]$	$\frac{1}{6}q\Phi_1{}^2\Phi_6$	$(1, \varepsilon)$
$\phi_{2,2}$	$\frac{1}{2}q\Phi_2{}^2\Phi_6$	$(g_2, 1)$
$\phi_{1,3}'$	$\frac{1}{3}q\Phi_3\Phi_6$	$(1, r)$
$\phi_{1,3}''$	$\frac{1}{3}q\Phi_3\Phi_6$	$(g_3, 1)$
$G_2[-1]$	$\frac{1}{2}q\Phi_1{}^2\Phi_3$	(g_2, ε)
$G_2[\theta]$	$\frac{1}{3}q\Phi_1{}^2\Phi_2{}^2$	(g_3, θ)
$G_2[\theta^2]$	$\frac{1}{3}q\Phi_1{}^2\Phi_2{}^2$	(g_3, θ^2)
$\phi_{1,6}$	q^6	

Type ${}^3D_4(q^3)$. This group has 8 unipotent characters. 6 of these lie in the principal series and 2 are cuspidal. The characters in the principal series are described by the irreducible characters of $W(G_2)$. The degrees are as follows:

Description in terms of cuspidal characters	Degree
$\phi_{1,0}$	1
$\phi_{1,3}'$	$q\Phi_{12}$
$\phi_{2,2}$	$\frac{1}{2}q^3\Phi_2{}^2\Phi_{12}$
$\phi_{2,1}$	$\frac{1}{2}q^3\Phi_2{}^2\Phi_6{}^2$
${}^3D_4[-1]$	$\frac{1}{2}q^3\Phi_1{}^2\Phi_3{}^2$
${}^3D_4[1]$	$\frac{1}{2}q^3\Phi_1{}^2\Phi_{12}$
$\phi_{1,3}''$	$q^7\Phi_{12}$
$\phi_{1,6}$	q^{12}

Type $F_4(q)$. This group has 37 unipotent characters. 25 of them lie in the principal series, 5 of them arise from the cuspidal unipotent character of the Levi subgroup $B_2(q)$, and 7 are cuspidal. The quotient root system arising from the Levi subgroup $B_2(q)$ has type B_2 and its Weyl group $W(B_2)$ has 5 irreducible characters, four of degree 1 and 1 of degree 2. These will be denoted by $1, \varepsilon, \varepsilon', \varepsilon'', r$ where ε is the sign character, r the character of degree 2, and $\varepsilon', \varepsilon''$ the characters of degree 1 other than the unit and sign characters.

$F_4(q)$ has 11 families of unipotent characters. 8 of these have 1 element, 2 have 4 elements and one has 21 elements. The degrees of the unipotent characters are given below:

Description in terms of cuspidal characters	Degree	(x, σ)
$\phi_{1,0}$	1	
$\phi_{9,2}$	$q^2\Phi_3{}^2\Phi_6{}^2\Phi_{12}$	
$\phi_{8,3}'$	$q^3\Phi_4{}^2\Phi_8\Phi_{12}$	
$\phi_{8,3}''$	$q^3\Phi_4{}^2\Phi_8\Phi_{12}$	
$\phi_{8,9}'$	$q^9\Phi_4{}^2\Phi_8\Phi_{12}$	
$\Phi_{8,9}''$	$q^9\Phi_4{}^2\Phi_8\Phi_{12}$	
$\phi_{9,10}$	$q^{10}\Phi_3{}^2\Phi_6{}^2\Phi_{12}$	
$\phi_{1,24}$	q^{24}	
$\phi_{4,1}$	$\frac{1}{2}q\Phi_2{}^2\Phi_6{}^2\Phi_8$	$(1, 1)$
$\phi_{2,4}''$	$\frac{1}{2}q\Phi_4\Phi_8\Phi_{12}$	$(g_2, 1)$
$\phi_{2,4}'$	$\frac{1}{2}q\Phi_4\Phi_8\Phi_{12}$	$(1, \varepsilon)$
$B_2, 1$	$\frac{1}{2}q\Phi_1{}^2\Phi_3{}^2\Phi_8$	(g_2, ε)
$\phi_{4,13}$	$\frac{1}{2}q^{13}\Phi_2{}^2\Phi_6{}^2\Phi_8$	$(1, 1)$
$\phi_{2,16}'$	$\frac{1}{2}q^{13}\Phi_4\Phi_8\Phi_{12}$	$(g_2, 1)$
$\phi_{2,16}''$	$\frac{1}{2}q^{13}\Phi_4\Phi_8\Phi_{12}$	$(1, \varepsilon)$
B_2, ε	$\frac{1}{2}q^{13}\Phi_1{}^2\Phi_3{}^2\Phi_8$	(g_2, ε)
$\phi_{12,4}$	$\frac{1}{24}q^4\Phi_2{}^4\Phi_3{}^2\Phi_8\Phi_{12}$	$(1, 1)$
$\phi_{9,6}''$	$\frac{1}{8}q^4\Phi_3{}^2\Phi_4{}^2\Phi_8\Phi_{12}$	$(g_2', 1)$
$\phi_{9,6}'$	$\frac{1}{8}q^4\Phi_3{}^2\Phi_4{}^2\Phi_8\Phi_{12}$	$(1, \lambda^1)$
$\phi_{1,12}''$	$\frac{1}{8}q^4\Phi_4{}^2\Phi_6{}^2\Phi_8\Phi_{12}$	(g_2', ε')
$\phi_{1,12}'$	$\frac{1}{8}q^4\Phi_4{}^2\Phi_6{}^2\Phi_8\Phi_{12}$	$(1, \lambda^2)$
$\phi_{4,7}''$	$\frac{1}{4}q^4\Phi_2{}^2\Phi_4\Phi_6{}^2\Phi_8\Phi_{12}$	$(g_4, 1)$
$\phi_{4,7}'$	$\frac{1}{4}q^4\Phi_2{}^2\Phi_4\Phi_6{}^2\Phi_8\Phi_{12}$	(g_2, ε'')
$\phi_{4,8}$	$\frac{1}{8}q^4\Phi_2{}^4\Phi_6{}^2\Phi_8\Phi_{12}$	(g_2', ε'')
$\phi_{6,6}'$	$\frac{1}{3}q^4\Phi_3{}^2\Phi_6{}^2\Phi_8\Phi_{12}$	$(g_3, 1)$
$\phi_{6,6}''$	$\frac{1}{12}q^4\Phi_3{}^2\Phi_4{}^2\Phi_6{}^2\Phi_8$	$(1, \sigma)$
$\phi_{16,5}$	$\frac{1}{4}q^4\Phi_2{}^4\Phi_4{}^2\Phi_6{}^2\Phi_{12}$	$(g_2, 1)$
B_2, r	$\frac{1}{4}q^4\Phi_1{}^2\Phi_2{}^2\Phi_3{}^2\Phi_6{}^2\Phi_8$	(g_2', r)
B_2, ε'	$\frac{1}{4}q^4\Phi_1{}^2\Phi_3{}^2\Phi_4\Phi_8\Phi_{12}$	$(g_4, -1)$
B_2, ε''	$\frac{1}{4}q^4\Phi_1{}^2\Phi_3{}^2\Phi_4\Phi_8\Phi_{12}$	(g_2, ε')
$F_4[\theta]$	$\frac{1}{3}q^4\Phi_1{}^4\Phi_2{}^4\Phi_4{}^2\Phi_8$	(g_3, θ)
$F_4[\theta^2]$	$\frac{1}{3}q^4\Phi_1{}^4\Phi_2{}^4\Phi_4{}^2\Phi_8$	(g_3, θ^2)
$F_4[\mathrm{i}]$	$\frac{1}{4}q^4\Phi_1{}^4\Phi_2{}^4\Phi_3{}^2\Phi_6{}^2$	(g_4, i)
$F_4[-\mathrm{i}]$	$\frac{1}{4}q^4\Phi_1{}^4\Phi_2{}^4\Phi_3{}^2\Phi_6{}^2$	$(g_4, -\mathrm{i})$
$F_4{}^{\mathrm{I}}[1]$	$\frac{1}{8}q^4\Phi_1{}^4\Phi_3{}^2\Phi_8\Phi_{12}$	(g_2', ε)
$F_4{}^{\mathrm{II}}[1]$	$\frac{1}{24}q^4\Phi_1{}^4\Phi_6{}^2\Phi_8\Phi_{12}$	$(1, \lambda^3)$
$F_4[-1]$	$\frac{1}{4}q^4\Phi_1{}^4\Phi_3{}^2\Phi_4{}^2\Phi_{12}$	(g_2, ε)

***Type* $E_6(q)$.** This group has 30 unipotent characters. 25 of them lie in the principal series, 3 of them arise from the cuspidal unipotent character of the Levi subgroup $D_4(q)$, and 2 are cuspidal. The quotient root system arising from the Levi subgroup $D_4(q)$ has type A_2 and its Weyl group $W(A_2)$ has three irreducible characters $1, \varepsilon, r$.

The unipotent characters fall into 17 families. 14 of these families contain 1 character, 2 contain 4 characters, and one contains 8 characters. The degrees of the unipotent characters are given below:

Description in terms of cuspidal characters	Degree	(x, σ)
$\phi_{1,0}$	1	
$\phi_{6,1}$	$q\Phi_8\Phi_9$	
$\phi_{20,2}$	$q^2\Phi_4\Phi_5\Phi_8\Phi_{12}$	
$\phi_{64,4}$	$q^4\Phi_2{}^3\Phi_4{}^2\Phi_6{}^2\Phi_8\Phi_{12}$	
$\phi_{60,5}$	$q^5\Phi_4\Phi_5\Phi_8\Phi_9\Phi_{12}$	
$\phi_{81,6}$	$q^6\Phi_3{}^3\Phi_6{}^2\Phi_9\Phi_{12}$	
$\phi_{24,6}$	$q^6\Phi_4{}^2\Phi_8\Phi_9\Phi_{12}$	
$\phi_{81,10}$	$q^{10}\Phi_3{}^3\Phi_6{}^2\Phi_9\Phi_{12}$	
$\phi_{60,11}$	$q^{11}\Phi_4\Phi_5\Phi_8\Phi_9\Phi_{12}$	
$\phi_{24,12}$	$q^{12}\Phi_4{}^2\Phi_8\Phi_9\Phi_{12}$	
$\phi_{64,13}$	$q^{13}\Phi_2{}^3\Phi_4{}^2\Phi_6{}^2\Phi_8\Phi_{12}$	
$\phi_{20,20}$	$q^{20}\Phi_4\Phi_5\Phi_8\Phi_{12}$	
$\phi_{6,25}$	$q^{25}\Phi_8\Phi_9$	
$\phi_{1,36}$	q^{36}	
$\phi_{30,3}$	$\frac{1}{2}q^3\Phi_4{}^2\Phi_5\Phi_9\Phi_{12}$	$(1, 1)$
$\phi_{15,5}$	$\frac{1}{2}q^3\Phi_5\Phi_6{}^2\Phi_8\Phi_9$	$(1, \varepsilon)$
$\phi_{15,4}$	$\frac{1}{2}q^3\Phi_5\Phi_8\Phi_9\Phi_{12}$	$(g_2, 1)$
$D_4, 1$	$\frac{1}{2}q^3\Phi_1{}^4\Phi_3{}^2\Phi_5\Phi_9$	(g_2, ε)
$\phi_{30,15}$	$\frac{1}{2}q^{15}\Phi_4{}^2\Phi_5\Phi_9\Phi_{12}$	$(1, 1)$
$\phi_{15,17}$	$\frac{1}{2}q^{15}\Phi_5\Phi_6{}^2\Phi_8\Phi_9$	$(1, \varepsilon)$
$\phi_{15,16}$	$\frac{1}{2}q^{15}\Phi_5\Phi_8\Phi_9\Phi_{12}$	$(g_2, 1)$
D_4, ε	$\frac{1}{2}q^{15}\Phi_1{}^4\Phi_3{}^2\Phi_5\Phi_9$	(g_2, ε)
$\phi_{80,7}$	$\frac{1}{6}q^7\Phi_2{}^4\Phi_5\Phi_8\Phi_9\Phi_{12}$	$(1, 1)$
$\phi_{20,10}$	$\frac{1}{6}q^7\Phi_4{}^2\Phi_5\Phi_6{}^2\Phi_8\Phi_9$	$(1, \varepsilon)$
$\phi_{60,8}$	$\frac{1}{2}q^7\Phi_4{}^2\Phi_5\Phi_8\Phi_9\Phi_{12}$	$(g_2, 1)$
$\phi_{10,9}$	$\frac{1}{3}q^7\Phi_5\Phi_6{}^2\Phi_8\Phi_9\Phi_{12}$	$(g_3, 1)$
$\phi_{90,8}$	$\frac{1}{3}q^7\Phi_3{}^3\Phi_5\Phi_6{}^2\Phi_8\Phi_{12}$	$(1, r)$
D_4, r	$\frac{1}{2}q^7\Phi_1{}^4\Phi_3{}^2\Phi_5\Phi_8\Phi_9$	(g_2, ε)
$E_6[\theta]$	$\frac{1}{3}q^7\Phi_1{}^6\Phi_2{}^4\Phi_4{}^2\Phi_5\Phi_8$	(g_3, θ)
$E_6[\theta^2]$	$\frac{1}{3}q^7\Phi_1{}^6\Phi_2{}^4\Phi_4{}^2\Phi_5\Phi_8$	(g_3, θ^2)

***Type* ${}^2E_6(q^2)$.** This group has 30 unipotent characters. They are in bijective correspondence with the unipotent characters of $E_6(q)$. The degrees of the unipotent characters of ${}^2E_6(q^2)$ are obtained from those of $E_6(q)$ by replacing q by $-q$ and then changing the sign if necessary to make the degree positive. There are 25 characters in the principal series, and these are labelled by the irreducible characters of $W(F_4)$. There are 2 characters which arise from the cuspidal unipotent character of the Levi subgroup ${}^2A_5(q^2)$. The quotient root system has type A_1. Finally there are 3 cuspidal unipotent characters. The degrees of the unipotent characters are as follows:

Description in terms in cuspidal characters	Degree
$\phi_{1,0}$	1
$\phi_{2,4}'$	$q\Phi_8\Phi_{18}$
$\phi_{4,1}$	$q^2\Phi_4\Phi_8\Phi_{10}\Phi_{12}$
${}^2A_5, 1$	$q^4\Phi_1{}^3\Phi_3{}^2\Phi_4{}^2\Phi_8\Phi_{12}$
$\phi_{4,7}'$	$q^5\Phi_4\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$
$\phi_{9,6}'$	$q^6\Phi_3{}^2\Phi_6{}^3\Phi_{12}\Phi_{18}$
$\phi_{8,3}''$	$q^6\Phi_4{}^2\Phi_8\Phi_{12}\Phi_{18}$
$\phi_{9,6}''$	$q^{10}\Phi_3{}^2\Phi_6{}^3\Phi_{12}\Phi_{18}$
$\phi_{4,7}''$	$q^{11}\Phi_4\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$
$\phi_{8,9}'$	$q^{12}\Phi_4{}^2\Phi_8\Phi_{12}\Phi_{18}$
${}^2A_5, \varepsilon$	$q^{13}\Phi_1{}^3\Phi_3{}^2\Phi_4{}^2\Phi_8\Phi_{12}$
$\phi_{4,13}$	$q^{20}\Phi_4\Phi_8\Phi_{10}\Phi_{12}$
$\phi_{2,16}''$	$q^{25}\Phi_8\Phi_{18}$
$\phi_{1,24}$	q^{36}
$\phi_{2,4}''$	$\frac{1}{2}q^3\Phi_4{}^2\Phi_{10}\Phi_{12}\Phi_{18}$
$\phi_{9,2}$	$\frac{1}{2}q^3\Phi_3{}^2\Phi_8\Phi_{10}\Phi_{18}$
$\phi_{1,12}'$	$\frac{1}{2}q^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$
$\phi_{8,3}'$	$\frac{1}{2}q^3\Phi_2{}^4\Phi_6{}^2\Phi_{10}\Phi_{18}$
$\phi_{2,16}'$	$\frac{1}{2}q^{15}\Phi_4{}^2\Phi_{10}\Phi_{12}\Phi_{18}$
$\phi_{9,10}$	$\frac{1}{2}q^{15}\Phi_3{}^2\Phi_8\Phi_{10}\Phi_{18}$
$\phi_{1,12}''$	$\frac{1}{2}q^{15}\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$
$\phi_{8,9}''$	$\frac{1}{2}q^{15}\Phi_2{}^4\Phi_6{}^2\Phi_{10}\Phi_{18}$
${}^2E_6[1]$	$\frac{1}{6}q^7\Phi_1{}^4\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$
$\phi_{12,4}$	$\frac{1}{6}q^7\Phi_3{}^2\Phi_4{}^2\Phi_8\Phi_{10}\Phi_{18}$
$\phi_{4,8}$	$\frac{1}{2}q^7\Phi_4{}^2\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$
$\phi_{6,6}'$	$\frac{1}{3}q^7\Phi_3{}^2\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$
$\phi_{6,6}''$	$\frac{1}{3}q^7\Phi_3{}^2\Phi_6{}^3\Phi_8\Phi_{10}\Phi_{12}$
$\phi_{16,5}$	$\frac{1}{2}q^7\Phi_2{}^4\Phi_6{}^2\Phi_8\Phi_{10}\Phi_{18}$
${}^2E_6[\theta]$	$\frac{1}{3}q^7\Phi_1{}^4\Phi_2{}^6\Phi_4{}^2\Phi_8\Phi_{10}$
${}^2E_6[\theta^2]$	$\frac{1}{3}q^7\Phi_1{}^4\Phi_2{}^6\Phi_4{}^2\Phi_8\Phi_{10}$

Type $E_7(q)$. This group has 76 unipotent characters. 60 of them lie in the principal series, 10 of them arise from the cuspidal unipotent character of the Levi subgroup $D_4(q)$, 4 of them arise from the two cuspidal unipotent characters of the Levi subgroup $E_6(q)$, and 2 of them are cuspidal. The quotient root system arising from the Levi subgroup $D_4(q)$ has type C_3 and its Weyl group $W(C_3)$ has 10 irreducible characters. 4 of these have degree 1, 2 have degree 2, and 4 have degree 3.

They will be denoted by

$$1, \varepsilon_1, \varepsilon_2, \varepsilon, \sigma_2, \sigma_2', r, r\varepsilon_1, r\varepsilon_2, r\varepsilon$$

where ε is the sign character, $\varepsilon_1, \varepsilon_2$ are the characters of degree 1 not equal to the unit or sign character, σ_2, σ_2' are the characters of degree 2, and r is the reflection character of degree 3. σ_2, σ_2' are distinguished by the fact that σ_2 contains the unit character on restriction to $W(C_2)$ whereas σ_2' does not. $\varepsilon_1, \varepsilon_2$ are distinguished by the fact that ε_1 is equal to the unit character on restriction to $W(A_2)$ whereas ε_2 is not.

The quotient root system arising from the Levi subgroup $E_6(q)$ has type A_1.

The unipotent characters fall into 35 families, one for each special character of $W(E_7)$. 24 of these families contain 1 character, 9 contain 4 characters, and 2 contain 8 characters. The degrees of the unipotent characters are given below:

Description in terms of cuspidal characters	Degree	(x, σ)
$\phi_{1,0}$	1	
$\phi_{7,1}$	$q\Phi_7\Phi_{12}\Phi_{14}$	
$\phi_{27,2}$	$q^2\Phi_3{}^2\Phi_6{}^2\Phi_9\Phi_{12}\Phi_{18}$	
$\phi_{21,3}$	$q^3\Phi_7\Phi_9\Phi_{14}\Phi_{18}$	
$\phi_{189,5}$	$q^5\Phi_3{}^2\Phi_6{}^2\Phi_7\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$	
$\phi_{210,6}$	$q^6\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{18}$	
$\phi_{105,6}$	$q^6\Phi_5\Phi_7\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	
$\phi_{168,6}$	$q^6\Phi_4{}^2\Phi_7\Phi_8\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$	
$\phi_{189,7}$	$q^7\Phi_3{}^2\Phi_6{}^2\Phi_7\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$	
$\phi_{378,9}$	$q^9\Phi_3{}^2\Phi_6{}^2\Phi_7\Phi_8\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$	
$\phi_{210,10}$	$q^{10}\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	
$\phi_{105,12}$	$q^{12}\Phi_5\Phi_7\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	
$\phi_{210,13}$	$q^{13}\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	
$\phi_{378,14}$	$q^{14}\Phi_3{}^2\Phi_6{}^2\Phi_7\Phi_8\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$	
$\phi_{105,15}$	$q^{15}\Phi_5\Phi_7\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	
$\phi_{189,20}$	$q^{20}\Phi_3{}^2\Phi_6{}^2\Phi_7\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$	
$\phi_{210,21}$	$q^{21}\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{18}$	
$\phi_{105,21}$	$q^{21}\Phi_5\Phi_7\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	
$\phi_{168,21}$	$q^{21}\Phi_4{}^2\Phi_7\Phi_8\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$	
$\phi_{189,22}$	$q^{22}\Phi_3{}^2\Phi_6{}^2\Phi_7\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$	
$\phi_{21,36}$	$q^{36}\Phi_7\Phi_9\Phi_{14}\Phi_{18}$	
$\phi_{27,37}$	$q^{37}\Phi_3{}^2\Phi_6{}^2\Phi_9\Phi_{12}\Phi_{18}$	
$\phi_{7,46}$	$q^{46}\Phi_7\Phi_{12}\Phi_{14}$	
$\phi_{1,63}$	q^{63}	
$\phi_{56,3}$	$\frac{1}{2}q^3\Phi_2{}^4\Phi_6{}^2\Phi_7\Phi_{10}\Phi_{14}\Phi_{18}$	$(1, 1)$
$\phi_{35,4}$	$\frac{1}{2}q^3\Phi_5\Phi_7\Phi_8\Phi_{12}\Phi_{14}\Phi_{18}$	$(g_2, 1)$
$\phi_{21,6}$	$\frac{1}{2}q^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}$	$(1, \varepsilon)$
$D_4, 1$	$\frac{1}{2}q^3\Phi_1{}^4\Phi_3{}^2\Phi_5\Phi_7\Phi_9\Phi_{14}$	(g_2, ε)
$\phi_{120,4}$	$\frac{1}{2}q^4\Phi_2{}^4\Phi_5\Phi_6{}^2\Phi_9\Phi_{10}\Phi_{14}\Phi_{18}$	$(1, 1)$
$\phi_{15,7}$	$\frac{1}{2}q^4\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$(g_2, 1)$
$\phi_{105,5}$	$\frac{1}{2}q^4\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$	$(1, \varepsilon)$
D_4, ε_1	$\frac{1}{2}q^4\Phi_1{}^4\Phi_3{}^2\Phi_5\Phi_7\Phi_9\Phi_{10}\Phi_{18}$	(g_2, ε)

Description in terms of cuspidal characters	Degree	(x, σ)
$\phi_{405,8}$	$\frac{1}{2}q^8\Phi_3{}^3\Phi_5\Phi_6{}^2\Phi_8\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$	$(1, 1)$
$\phi_{216,9}$	$\frac{1}{2}q^8\Phi_2{}^4\Phi_3{}^2\Phi_6{}^3\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$(g_2, 1)$
$\phi_{189,10}$	$\frac{1}{2}q^8\Phi_3{}^2\Phi_6{}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$	$(1, \varepsilon)$
$D_4, r\varepsilon_1$	$\frac{1}{2}q^8\Phi_1{}^4\Phi_3{}^3\Phi_5\Phi_6{}^2\Phi_7\Phi_9\Phi_{12}\Phi_{18}$	(g_2, ε)
$\phi_{420,10}$	$\frac{1}{2}q^{10}\Phi_4{}^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$	$(1, 1)$
$\phi_{84,12}$	$\frac{1}{2}q^{10}\Phi_4{}^2\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$(g_2, 1)$
$\phi_{336,11}$	$\frac{1}{2}q^{10}\Phi_2{}^4\Phi_6{}^2\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{18}$	$(1, \varepsilon)$
D_4, σ_2	$\frac{1}{2}q^{10}\Phi_1{}^4\Phi_3{}^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{14}\Phi_{18}$	(g_2, ε)
$\phi_{512,11}$	$\frac{1}{2}q^{11}\Phi_2{}^7\Phi_4{}^2\Phi_6{}^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$(1, 1)$
$\phi_{512,12}$	$\frac{1}{2}q^{11}\Phi_2{}^7\Phi_4{}^2\Phi_6{}^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$(1, \varepsilon)$
$E_7[\xi]$	$\frac{1}{2}q^{11}\Phi_1{}^7\Phi_3{}^3\Phi_4{}^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{12}$	$(g_2, 1)$
$E_7[-\xi]$	$\frac{1}{2}q^{11}\Phi_1{}^7\Phi_3{}^3\Phi_4{}^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{12}$	(g_2, ε)
$\phi_{420,13}$	$\frac{1}{2}q^{13}\Phi_4{}^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$	$(1, 1)$
$\phi_{84,15}$	$\frac{1}{2}q^{13}\Phi_4{}^2\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$(g_2, 1)$
$\phi_{336,14}$	$\frac{1}{2}q^{13}\Phi_2{}^4\Phi_6{}^2\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{18}$	$(1, \varepsilon)$
D_4, σ_2'	$\frac{1}{2}q^{13}\Phi_1{}^4\Phi_3{}^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{14}\Phi_{18}$	(g_2, ε)
$\phi_{405,15}$	$\frac{1}{2}q^{15}\Phi_3{}^3\Phi_5\Phi_6{}^2\Phi_8\Phi_9\Phi_{12}\Phi_{14}\Phi_{18}$	$(1, 1)$
$\phi_{216,16}$	$\frac{1}{2}q^{15}\Phi_2{}^4\Phi_3{}^2\Phi_6{}^3\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$(g_2, 1)$
$\phi_{189,17}$	$\frac{1}{2}q^{15}\Phi_3{}^2\Phi_6{}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$	$(1, \varepsilon)$
$D_4, r\varepsilon_2$	$\frac{1}{2}q^{15}\Phi_1{}^4\Phi_3{}^3\Phi_5\Phi_6{}^2\Phi_7\Phi_9\Phi_{12}\Phi_{18}$	(g_2, ε)
$\phi_{120,25}$	$\frac{1}{2}q^{25}\Phi_2{}^4\Phi_5\Phi_6{}^2\Phi_9\Phi_{10}\Phi_{14}\Phi_{18}$	$(1, 1)$
$\phi_{15,28}$	$\frac{1}{2}q^{25}\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$(g_2, 1)$
$\phi_{105,26}$	$\frac{1}{2}q^{25}\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$	$(1, \varepsilon)$
D_4, ε_2	$\frac{1}{2}q^{25}\Phi_1{}^4\Phi_3{}^2\Phi_5\Phi_7\Phi_9\Phi_{10}\Phi_{18}$	(g_2, ε)
$\phi_{56,30}$	$\frac{1}{2}q^{30}\Phi_2{}^4\Phi_6{}^2\Phi_7\Phi_{10}\Phi_{14}\Phi_{18}$	$(1, 1)$
$\phi_{35,31}$	$\frac{1}{2}q^{30}\Phi_5\Phi_7\Phi_8\Phi_{12}\Phi_{14}\Phi_{18}$	$(g_2, 1)$
$\phi_{21,33}$	$\frac{1}{2}q^{30}\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}$	$(1, \varepsilon)$
D_4, ε	$\frac{1}{2}q^{30}\Phi_1{}^4\Phi_3{}^2\Phi_5\Phi_7\Phi_9\Phi_{14}$	(g_2, ε)
$\phi_{315,7}$	$\frac{1}{6}q^7\Phi_3{}^3\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$(1, 1)$
$\phi_{280,8}$	$\frac{1}{2}q^7\Phi_2{}^4\Phi_5\Phi_6{}^3\Phi_7\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$(g_2, 1)$
$\phi_{280,9}$	$\frac{1}{3}q^7\Phi_4{}^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{18}$	$(1, r)$
$\phi_{70,9}$	$\frac{1}{3}q^7\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$(g_3, 1)$
$\phi_{35,13}$	$\frac{1}{6}q^7\Phi_5\Phi_6{}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}$	$(1, \varepsilon)$
D_4, r	$\frac{1}{2}q^7\Phi_1{}^4\Phi_3{}^3\Phi_5\Phi_7\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}$	(g_2, ε)
$E_6[\theta], 1$	$\frac{1}{3}q^7\Phi_1{}^6\Phi_2{}^6\Phi_4{}^2\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{14}$	(g_3, θ)
$E_6[\theta^2], 1$	$\frac{1}{3}q^7\Phi_1{}^6\Phi_2{}^6\Phi_4{}^2\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{14}$	(g_3, θ^2)
$\phi_{315,16}$	$\frac{1}{6}q^{16}\Phi_3{}^3\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$(1, 1)$
$\phi_{280,17}$	$\frac{1}{2}q^{16}\Phi_2{}^4\Phi_5\Phi_6{}^3\Phi_7\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$(g_2, 1)$
$\phi_{280,18}$	$\frac{1}{3}q^{16}\Phi_4{}^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{18}$	$(1, r)$
$\phi_{70,18}$	$\frac{1}{3}q^{16}\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$(g_3, 1)$
$\phi_{35,22}$	$\frac{1}{6}q^{16}\Phi_5\Phi_6{}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}$	$(1, \varepsilon)$
$D_4, r\varepsilon$	$\frac{1}{2}q^{16}\Phi_1{}^4\Phi_3{}^3\Phi_5\Phi_7\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}$	(g_2, ε)
$E_6[\theta], \varepsilon$	$\frac{1}{3}q^{16}\Phi_1{}^6\Phi_2{}^6\Phi_4{}^2\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{14}$	(g_3, θ)
$E_6[\theta^2], \varepsilon$	$\frac{1}{3}q^{16}\Phi_1{}^6\Phi_2{}^6\Phi_4{}^2\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{14}$	(g_3, θ^2)

Type $E_8(q)$. This group has 166 unipotent characters. 112 of them lie in the principal series, 25 of them arise from the cuspidal unipotent character of the Levi subgroup $D_4(q)$, 12 of them arise from the two cuspidal unipotent characters of the Levi subgroup $E_6(q)$, 4 of them arise from the two cuspidal unipotent characters of the Levi subgroup $E_7(q)$, and 13 are cuspidal.

The quotient root system arising from the Levi subgroup $D_4(q)$ has type F_4, so the characters arising from this subgroup are labelled by the irreducible characters of $W(F_4)$. The quotient root system arising from the Levi subgroup $E_6(q)$ has type G_2, so the characters arising from this subgroup are labelled by the irreducible characters of $W(G_2)$. Finally the quotient root system arising from the Levi subgroup $E_7(q)$ has type A_1.

The unipotent characters fall into 46 families, one for each special irreducible character of $W(E_8)$. 23 of these families contain 1 character, 18 contain 4 characters, 4 contain 8 characters, and one family contains 39 characters. The degrees of the unipotent characters are given below:

Description in terms of cuspidal characters	Degree	(x, σ)
$\phi_{1,0}$	1	
$\phi_{8,1}$	$q\Phi_4^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}$	
$\phi_{35,2}$	$q^2\Phi_5\Phi_7\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{30}$	
$\phi_{560,5}$	$q^5\Phi_4^2\Phi_5\Phi_7\Phi_8^2\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	
$\phi_{567,6}$	$q^6\Phi_3^3\Phi_6^3\Phi_7\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	
$\phi_{3240,9}$	$q^9\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	
$\phi_{525,12}$	$q^{12}\Phi_5^2\Phi_7\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	
$\phi_{4536,13}$	$q^{13}\Phi_3^3\Phi_4^2\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	
$\phi_{2835,14}$	$q^{14}\Phi_3^3\Phi_5\Phi_6^3\Phi_7\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	
$\phi_{6075,14}$	$q^{14}\Phi_3^4\Phi_5^2\Phi_6^4\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	
$\phi_{4200,15}$	$q^{15}\Phi_4^2\Phi_5^2\Phi_7\Phi_8\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	
$\phi_{2100,20}$	$q^{20}\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	
$\phi_{4200,21}$	$q^{21}\Phi_4^2\Phi_5^2\Phi_7\Phi_8\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	
$\phi_{2835,22}$	$q^{22}\Phi_3^3\Phi_5\Phi_6^3\Phi_7\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	
$\phi_{6072,22}$	$q^{22}\Phi_3^4\Phi_5^2\Phi_6^4\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	
$\phi_{4536,23}$	$q^{23}\Phi_3^3\Phi_4^2\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	
$\phi_{3240,31}$	$q^{31}\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	
$\phi_{525,36}$	$q^{36}\Phi_5^2\Phi_7\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	
$\phi_{567,46}$	$q^{46}\Phi_3^3\Phi_6^3\Phi_7\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	
$\phi_{560,47}$	$q^{47}\Phi_4^2\Phi_5\Phi_7\Phi_8^2\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	
$\phi_{35,74}$	$q^{74}\Phi_5\Phi_7\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{30}$	
$\phi_{8,91}$	$q^{91}\Phi_4^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}$	
$\phi_{1,120}$	q^{120}	
$\phi_{112,3}$	$\frac{1}{2}q^3\Phi_2^4\Phi_6^2\Phi_7\Phi_8\Phi_{10}^2\Phi_{14}\Phi_{18}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{84,4}$	$\frac{1}{2}q^3\Phi_4^2\Phi_7\Phi_8\Phi_9\Phi_{12}\Phi_{14}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{28,8}$	$\frac{1}{2}q^3\Phi_4^2\Phi_7\Phi_8\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}$	$(1, \varepsilon)$
$D_4, \phi_{1,0}$	$\frac{1}{2}q^3\Phi_1^4\Phi_3^2\Phi_5^2\Phi_7\Phi_8\Phi_9\Phi_{14}\Phi_{15}\Phi_{24}$	(g_2, ε)

Description in terms of cuspidal characters	Degree	(x, σ)
$\phi_{210,4}$	$\frac{1}{2}q^4\Phi_5\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{50,8}$	$\frac{1}{2}q^4\Phi_5^2\Phi_8^2\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{160,7}$	$\frac{1}{2}q^4\Phi_2^4\Phi_4^2\Phi_5\Phi_6^2\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{30}$	$(1, \varepsilon)$
$D_4, \phi_{2,4}'$	$\frac{1}{2}q^4\Phi_1^4\Phi_3^2\Phi_4^2\Phi_5^2\Phi_7\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{20}\Phi_{30}$	(g_2, ε)
$\phi_{700,6}$	$\frac{1}{2}q^6\Phi_4^2\Phi_5^2\Phi_7\Phi_8\Phi_{10}^2\Phi_{12}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{400,7}$	$\frac{1}{2}q^6\Phi_2^4\Phi_5^2\Phi_6^2\Phi_8\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{300,8}$	$\frac{1}{2}q^6\Phi_4^2\Phi_5^2\Phi_8\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, \varepsilon)$
$D_4, \phi_{1,12}'$	$\frac{1}{2}q^6\Phi_1^4\Phi_3^2\Phi_5^2\Phi_7\Phi_8\Phi_9\Phi_{10}^2\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	(g_2, ε)
$\phi_{2268,10}$	$\frac{1}{2}q^{10}\Phi_3^3\Phi_4^2\Phi_6^4\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{972,12}$	$\frac{1}{2}q^{10}\Phi_3^4\Phi_4^2\Phi_6^3\Phi_8\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{1296,13}$	$\frac{1}{2}q^{10}\Phi_2^4\Phi_3^3\Phi_6^4\Phi_8\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	$(1, \varepsilon)$
$D_4, \phi_{9,2}$	$\frac{1}{2}q^{10}\Phi_1^4\Phi_3^4\Phi_5^2\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	(g_2, ε)
$\phi_{2240,10}$	$\frac{1}{2}q^{10}\Phi_2^4\Phi_4^2\Phi_5\Phi_6^2\Phi_7\Phi_8\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{1400,11}$	$\frac{1}{2}q^{10}\Phi_4^2\Phi_5^2\Phi_7\Phi_8^2\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{840,13}$	$\frac{1}{2}q^{10}\Phi_4^2\Phi_5\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, \varepsilon)$
$D_4, \phi_{4,7}'$	$\frac{1}{2}q^{10}\Phi_1^4\Phi_3^2\Phi_4^2\Phi_5^2\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	(g_2, ε)
$\phi_{4096,11}$	$\frac{1}{2}q^{11}\Phi_2^7\Phi_4^4\Phi_6^4\Phi_8^2\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{4096,12}$	$\frac{1}{2}q^{11}\Phi_2^7\Phi_4^4\Phi_6^4\Phi_8^2\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, \varepsilon)$
$E_7[\xi], 1$	$\frac{1}{2}q^{11}\Phi_1^7\Phi_3^4\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{20}\Phi_{24}$	$(g_2\ 1)$
$E_7[-\xi], 1$	$\frac{1}{2}q^{11}\Phi_1^7\Phi_3^4\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{20}\Phi_{24}$	(g_2, ε)
$\phi_{4200,12}$	$\frac{1}{2}q^{12}\Phi_4^2\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{840,14}$	$\frac{1}{2}q^{12}\Phi_4^2\Phi_5\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{3360,13}$	$\frac{1}{2}q^{12}\Phi_2^4\Phi_5\Phi_6^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, \varepsilon)$
$D_4, \phi_{2,4}''$	$\frac{1}{2}q^{12}\Phi_1^4\Phi_3^2\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	(g_2, ε)
$\phi_{2800,13}$	$\frac{1}{2}q^{13}\Phi_2^4\Phi_5^2\Phi_6^4\Phi_7\Phi_8\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{700,16}$	$\frac{1}{2}q^{13}\Phi_4^2\Phi_5^2\Phi_7\Phi_8\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{2100,16}$	$\frac{1}{2}q^{13}\Phi_4^2\Phi_5^2\Phi_7\Phi_8\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, \varepsilon)$
$D_4, \phi_{9,6}'$	$\frac{1}{2}q^{13}\Phi_1^4\Phi_3^4\Phi_5^2\Phi_7\Phi_8\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	(g_2, ε)
$\phi_{5600,15}$	$\frac{1}{2}q^{15}\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_{10}^2\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{3200,16}$	$\frac{1}{2}q^{15}\Phi_2^4\Phi_4^2\Phi_5^2\Phi_6^2\Phi_8^2\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{2400,17}$	$\frac{1}{2}q^{15}\Phi_4^4\Phi_5^2\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, \varepsilon)$
$D_4, \phi_{8,3}''$	$\frac{1}{2}q^{15}\Phi_1^4\Phi_3^2\Phi_4^2\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	(g_2, ε)
$\phi_{5600,21}$	$\frac{1}{2}q^{21}\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_{10}^2\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{3200,22}$	$\frac{1}{2}q^{21}\Phi_2^4\Phi_4^2\Phi_5^2\Phi_6^2\Phi_8^2\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{2400,23}$	$\frac{1}{2}q^{21}\Phi_4^4\Phi_5^2\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, \varepsilon)$
$D_4, \phi_{8,9}'$	$\frac{1}{2}q^{21}\Phi_1^4\Phi_3^2\Phi_4^2\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	(g_2, ε)
$\phi_{4200,24}$	$\frac{1}{2}q^{24}\Phi_4^2\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{840,26}$	$\frac{1}{2}q^{24}\Phi_4^2\Phi_5\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{3360,25}$	$\frac{1}{2}q^{24}\Phi_2^4\Phi_5\Phi_6^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, \varepsilon)$
$D_4, \phi_{2,16}'$	$\frac{1}{2}q^{24}\Phi_1^4\Phi_3^2\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	(g_2, ε)

Description in terms of cuspidal characters	Degree	(x, σ)
$\phi_{2800,25}$	$\frac{1}{2}q^{25}\Phi_2^4\Phi_5^2\Phi_6^4\Phi_7\Phi_8\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{700,28}$	$\frac{1}{2}q^{25}\Phi_4^2\Phi_5^2\Phi_7\Phi_8\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{2100,28}$	$\frac{1}{2}q^{25}\Phi_4^2\Phi_5^2\Phi_7\Phi_8\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, \varepsilon)$
$D_4, \phi_{9,6}''$	$\frac{1}{2}q^{25}\Phi_1^4\Phi_3^4\Phi_5^2\Phi_7\Phi_8\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	(g_2, ε)
$\phi_{4096,26}$	$\frac{1}{2}q^{26}\Phi_2^7\Phi_4^4\Phi_6^4\Phi_8^2\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{4096,27}$	$\frac{1}{2}q^{26}\Phi_2^7\Phi_4^4\Phi_6^4\Phi_8^2\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, \varepsilon)$
$E_7[\xi], \varepsilon$	$\frac{1}{2}q^{26}\Phi_1^7\Phi_3^4\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{20}\Phi_{24}$	$(g_2, 1)$
$E_7[-\xi], \varepsilon$	$\frac{1}{2}q^{26}\Phi_1^7\Phi_3^4\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{20}\Phi_{24}$	(g_2, ε)
$\phi_{2240,28}$	$\frac{1}{2}q^{28}\Phi_2^4\Phi_4^2\Phi_5\Phi_6^2\Phi_7\Phi_8\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{1400,29}$	$\frac{1}{2}q^{28}\Phi_4^2\Phi_5^2\Phi_7\Phi_8^2\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{840,31}$	$\frac{1}{2}q^{28}\Phi_4^2\Phi_5\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, \varepsilon)$
$D_4, \phi_{4,7}''$	$\frac{1}{2}q^{28}\Phi_1^4\Phi_3^2\Phi_4^2\Phi_5^2\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	(g_2, ε)
$\phi_{2268,30}$	$\frac{1}{2}q^{30}\Phi_3^3\Phi_4^2\Phi_6^4\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{972,32}$	$\frac{1}{2}q^{30}\Phi_3^4\Phi_4^2\Phi_6^3\Phi_8\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{1296,33}$	$\frac{1}{2}q^{30}\Phi_2^4\Phi_3^3\Phi_6^4\Phi_8\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	$(1, \varepsilon)$
$D_4, \phi_{9,10}$	$\frac{1}{2}q^{30}\Phi_1^4\Phi_3^4\Phi_5^2\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	(g_2, ε)
$\phi_{700,42}$	$\frac{1}{2}q^{42}\Phi_4^2\Phi_5^2\Phi_7\Phi_8\Phi_{10}^2\Phi_{12}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{400,43}$	$\frac{1}{2}q^{42}\Phi_2^4\Phi_5^2\Phi_6^2\Phi_8\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{300,44}$	$\frac{1}{2}q^{42}\Phi_4^2\Phi_5^2\Phi_8\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, \varepsilon)$
$D_4, \phi_{1,12}''$	$\frac{1}{2}q^{42}\Phi_1^4\Phi_3^2\Phi_5^2\Phi_7\Phi_8\Phi_9\Phi_{10}^2\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	(g_2, ε)
$\phi_{210,52}$	$\frac{1}{2}q^{52}\Phi_5\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{50,56}$	$\frac{1}{2}q^{52}\Phi_5^2\Phi_8^2\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{160,55}$	$\frac{1}{2}q^{52}\Phi_2^4\Phi_4^2\Phi_5\Phi_6^2\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{30}$	$(1, \varepsilon)$
$D_4, \phi_{2,16}''$	$\frac{1}{2}q^{52}\Phi_1^4\Phi_3^2\Phi_4^2\Phi_5^2\Phi_7\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{20}\Phi_{30}$	(g_2, ε)
$\phi_{112,63}$	$\frac{1}{2}q^{63}\Phi_2^4\Phi_6^2\Phi_7\Phi_8\Phi_{10}^2\Phi_{14}\Phi_{18}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{84,64}$	$\frac{1}{2}q^{63}\Phi_4^2\Phi_7\Phi_8\Phi_9\Phi_{12}\Phi_{14}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{28,68}$	$\frac{1}{2}q^{63}\Phi_4^2\Phi_7\Phi_8\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}$	$(1, \varepsilon)$
$D_4, \phi_{1,24}$	$\frac{1}{2}q^{63}\Phi_1^4\Phi_3^2\Phi_5^2\Phi_7\Phi_8\Phi_9\Phi_{14}\Phi_{15}\Phi_{24}$	(g_2, ε)
$\phi_{1400,7}$	$\frac{1}{6}q^{7}\Phi_4^2\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{1344,8}$	$\frac{1}{2}q^{7}\Phi_2^4\Phi_4^2\Phi_6^2\Phi_7\Phi_8\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{1008,9}$	$\frac{1}{3}q^{7}\Phi_3^3\Phi_4^2\Phi_6^3\Phi_7\Phi_8^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, r)$
$\phi_{448,9}$	$\frac{1}{3}q^{7}\Phi_4^4\Phi_7\Phi_8^2\Phi_9\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_3, 1)$
$\phi_{56,19}$	$\frac{1}{6}q^{7}\Phi_4^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}$	$(1, \varepsilon)$
$D_4, \phi_{4,1}$	$\frac{1}{2}q^{7}\Phi_1^4\Phi_3^2\Phi_4^2\Phi_5^2\Phi_7\Phi_8\Phi_9\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}$	(g_2, ε)
$E_6[\theta], \phi_{1,0}$	$\frac{1}{3}q^{7}\Phi_1^6\Phi_2^6\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{20}\Phi_{24}$	(g_3, θ)
$E_6[\theta^2], \phi_{1,0}$	$\frac{1}{3}q^{7}\Phi_1^6\Phi_2^6\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{20}\Phi_{24}$	(g_3, θ^2)

Description in terms of cuspidal characters	Degree	(x, σ)
$\phi_{1400,8}$	$\frac{1}{6}q^8\Phi_4^4\Phi_5^2\Phi_7\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{1050,10}$	$\frac{1}{2}q^8\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{1575,10}$	$\frac{1}{3}q^8\Phi_3^3\Phi_5^2\Phi_6^3\Phi_7\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, r)$
$\phi_{175,12}$	$\frac{1}{3}q^8\Phi_5^2\Phi_7\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_3, 1)$
$\phi_{350,14}$	$\frac{1}{6}q^8\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{30}$	$(1, \varepsilon)$
$D_4, \phi_{8,3}'$	$\frac{1}{2}q^8\Phi_1^4\Phi_2^4\Phi_3^2\Phi_5^2\Phi_6^2\Phi_7\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{30}$	(g_2, ε)
$E_6[\theta], \phi_{1,3}'$	$\frac{1}{3}q^8\Phi_1^6\Phi_2^6\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{30}$	(g_3, θ)
$E_6[\theta^2], \phi_{1,3}'$	$\frac{1}{3}q^8\Phi_1^6\Phi_2^6\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{30}$	(g_3, θ^2)
$\phi_{1400,32}$	$\frac{1}{6}q^{32}\Phi_4^4\Phi_5^2\Phi_7\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{1050,34}$	$\frac{1}{2}q^{32}\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{1575,34}$	$\frac{1}{3}q^{32}\Phi_3^3\Phi_5^2\Phi_6^3\Phi_7\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, r)$
$\phi_{175,36}$	$\frac{1}{3}q^{32}\Phi_5^2\Phi_7\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_3, 1)$
$\phi_{350,38}$	$\frac{1}{6}q^{32}\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{30}$	$(1, \varepsilon)$
$D_4, \phi_{8,9}''$	$\frac{1}{2}q^{32}\Phi_1^4\Phi_2^4\Phi_3^2\Phi_5^2\Phi_6^2\Phi_7\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{30}$	(g_2, ε)
$E_6[\theta], \phi_{1,3}''$	$\frac{1}{3}q^{32}\Phi_1^6\Phi_2^6\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{30}$	(g_3, θ)
$E_6[\theta^2], \phi_{1,3}''$	$\frac{1}{3}q^{32}\Phi_1^6\Phi_2^6\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{30}$	(g_3, θ^2)
$\phi_{1400,37}$	$\frac{1}{6}q^{37}\Phi_4^2\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{1344,38}$	$\frac{1}{2}q^{37}\Phi_2^4\Phi_4^2\Phi_6^2\Phi_7\Phi_8\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{1008,39}$	$\frac{1}{3}q^{37}\Phi_3^3\Phi_4^2\Phi_6^3\Phi_7\Phi_8^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, r)$
$\phi_{448,39}$	$\frac{1}{3}q^{37}\Phi_4^4\Phi_7\Phi_8^2\Phi_9\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_3, 1)$
$\phi_{56,49}$	$\frac{1}{6}q^{37}\Phi_4^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}$	$(1, \varepsilon)$
$D_4, \phi_{4,13}$	$\frac{1}{2}q^{37}\Phi_1^4\Phi_3^2\Phi_4^2\Phi_5^2\Phi_7\Phi_8\Phi_9\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}$	(g_2, ε)
$E_6[\theta], \phi_{1,6}$	$\frac{1}{3}q^{37}\Phi_1^6\Phi_2^6\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{20}\Phi_{24}$	(g_3, θ)
$E_6[\theta^2], \phi_{1,6}$	$\frac{1}{3}q^{37}\Phi_1^6\Phi_2^6\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{20}\Phi_{24}$	(g_3, θ^2)
$\phi_{4480,16}$	$\frac{1}{120}q^{16}\Phi_2^8\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, 1)$
$\phi_{3150,18}$	$\frac{1}{6}q^{16}\Phi_3^3\Phi_5^2\Phi_6^4\Phi_7\Phi_8^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_3, 1)$
$\phi_{4200,18}$	$\frac{1}{8}q^{16}\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2', 1)$
$\phi_{4536,18}$	$\frac{1}{24}q^{16}\Phi_3^4\Phi_4^4\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, \nu)$
$\phi_{5670,18}$	$\frac{1}{30}q^{16}\Phi_3^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, \lambda^1)$
$\phi_{420,20}$	$\frac{1}{5}q^{16}\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_5, 1)$
$\phi_{1134,20}$	$\frac{1}{6}q^{16}\Phi_3^4\Phi_6^3\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	(g_3, ε)
$\phi_{1400,20}$	$\frac{1}{24}q^{16}\Phi_4^4\Phi_5^2\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, \nu')$
$\phi_{2688,20}$	$\frac{1}{8}q^{16}\Phi_2^8\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	(g_2', ε'')
$\phi_{1680,22}$	$\frac{1}{20}q^{16}\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	$(1, \lambda^2)$
$\phi_{168,24}$	$\frac{1}{8}q^{16}\Phi_4^4\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	(g_2', ε')
$\phi_{70,32}$	$\frac{1}{30}q^{16}\Phi_5^2\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}$	$(1, \lambda^3)$
$\phi_{7168,17}$	$\frac{1}{12}q^{16}\Phi_2^8\Phi_4^4\Phi_6^2\Phi_7\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, 1)$
$\phi_{1344,19}$	$\frac{1}{4}q^{16}\Phi_2^4\Phi_4^2\Phi_6^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_4, 1)$
$\phi_{2016,19}$	$\frac{1}{6}q^{16}\Phi_2^4\Phi_3^3\Phi_4^2\Phi_6^4\Phi_7\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_6, 1)$
$\phi_{5600,19}$	$\frac{1}{6}q^{16}\Phi_2^4\Phi_4^2\Phi_5^2\Phi_6^4\Phi_7\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	(g_2, r)
$\phi_{448,25}$	$\frac{1}{12}q^{16}\Phi_2^4\Phi_4^2\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{30}$	(g_2, ε)
$D_4, \phi_{4,8}$	$\frac{1}{4}q^{16}\Phi_1^4\Phi_3^2\Phi_4^2\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_4, -1)$
$D_4, \phi_{6,6}'$	$\frac{1}{6}q^{16}\Phi_1^4\Phi_3^4\Phi_4^2\Phi_5^2\Phi_6^3\Phi_7\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_6, -1)$
$D_4, \phi_{6,6}''$	$\frac{1}{6}q^{16}\Phi_1^4\Phi_3^4\Phi_4^2\Phi_5^2\Phi_7\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}$	$(g_2, -r)$
$D_4, \phi_{12,4}$	$\frac{1}{12}q^{16}\Phi_1^4\Phi_3^4\Phi_4^2\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{30}$	$(g_2, -1)$
$D_4, \phi_{16,5}$	$\frac{1}{4}q^{16}\Phi_1^4\Phi_2^4\Phi_3^2\Phi_5^2\Phi_6^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	(g_2', r)

Description in terms of cuspidal characters	Degree	(x, σ)
$E_6[\theta], \phi_{2,1}$	$\frac{1}{6}q^{16}\Phi_1^6\Phi_2^8\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}$	(g_3, θ)
$E_6[\theta^2], \phi_{2,1}$	$\frac{1}{6}q^{16}\Phi_1^6\Phi_2^8\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}$	(g_3, θ^2)
$E_6[\theta], \phi_{2,2}$	$\frac{1}{6}q^{16}\Phi_1^6\Phi_2^8\Phi_4^4\Phi_5^2\Phi_6^2\Phi_7\Phi_8^2\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{30}$	(g_6, θ)
$E_6[\theta^2], \phi_{2,2}$	$\frac{1}{6}q^{16}\Phi_1^6\Phi_2^8\Phi_4^4\Phi_5^2\Phi_6^2\Phi_7\Phi_8^2\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{30}$	(g_6, θ^2)
$E_8[\zeta]$	$\frac{1}{5}q^{16}\Phi_1^8\Phi_2^8\Phi_3^4\Phi_4^4\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{24}$	(g_5, ζ)
$E_8[\zeta^2]$	$\frac{1}{5}q^{16}\Phi_1^8\Phi_2^8\Phi_3^4\Phi_4^4\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{24}$	(g_5, ζ^2)
$E_8[\zeta^3]$	$\frac{1}{5}q^{16}\Phi_1^8\Phi_2^8\Phi_3^4\Phi_4^4\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{24}$	(g_5, ζ^3)
$E_8[\zeta^4]$	$\frac{1}{5}q^{16}\Phi_1^8\Phi_2^8\Phi_3^4\Phi_4^4\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{24}$	(g_5, ζ^4)
$E_8[-\theta]$	$\frac{1}{6}q^{16}\Phi_1^8\Phi_2^6\Phi_3^2\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{20}$	$(g_6, -\theta)$
$E_8[-\theta^2]$	$\frac{1}{6}q^{16}\Phi_1^8\Phi_2^6\Phi_3^2\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{20}$	$(g_6, -\theta^2)$
$E_8[\theta]$	$\frac{1}{6}q^{16}\Phi_1^8\Phi_2^6\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_3, \varepsilon\theta)$
$E_8[\theta^2]$	$\frac{1}{6}q^{16}\Phi_1^8\Phi_2^6\Phi_4^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_3, \varepsilon\theta^2)$
$E_8[\mathrm{i}]$	$\frac{1}{4}q^{16}\Phi_1^8\Phi_2^8\Phi_3^4\Phi_5^2\Phi_6^4\Phi_7\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{30}$	(g_4, i)
$E_8[-\mathrm{i}]$	$\frac{1}{4}q^{16}\Phi_1^8\Phi_2^8\Phi_3^4\Phi_5^2\Phi_6^4\Phi_7\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{30}$	$(g_4, -\mathrm{i})$
$E_8^{\mathrm{II}}[1]$	$\frac{1}{120}q^{16}\Phi_1^8\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(1, \lambda^4)$
$E_8[-1]$	$\frac{1}{12}q^{16}\Phi_1^8\Phi_3^2\Phi_4^4\Phi_5^2\Phi_7\Phi_9\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$(g_2, -\varepsilon)$
$E_8^{\mathrm{I}}[1]$	$\frac{1}{8}q^{16}\Phi_1^8\Phi_3^4\Phi_5^2\Phi_7\Phi_8^2\Phi_9\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	(g_2', ε)

Type $^2B_2(q^2)$. The Suzuki groups $^2B_2(q^2)$ where $q^2 = 2^{2m+1}$ have 4 unipotent characters. Two lie in the principal series and the other two are cuspidal. Their degrees are:

1	1
ε	q^4
$^2B_2[a]$	$\frac{1}{\sqrt{2}}q\Phi_1\Phi_2$
$^2B_2[b]$	$\frac{1}{\sqrt{2}}q\Phi_1\Phi_2$

Type $^2G_2(q^2)$. The Ree groups $^2G_2(q^2)$ where $q^2 = 3^{2m+1}$ have 8 unipotent characters. Two lie in the principal series and the other 6 are cuspidal. Their degrees are:

1	1
ε	q^6
cuspidal	$\frac{1}{\sqrt{3}}q\Phi_1\Phi_2\Phi_4$
cuspidal	$\frac{1}{\sqrt{3}}q\Phi_1\Phi_2\Phi_4$
cuspidal	$\frac{1}{2\sqrt{3}}q\Phi_1\Phi_2\Phi_{12}'$
cuspidal	$\frac{1}{2\sqrt{3}}q\Phi_1\Phi_2\Phi_{12}'$

cuspidal	$\frac{1}{2\sqrt{3}} q\Phi_1\Phi_2\Phi_{12}''$
cuspidal	$\frac{1}{2\sqrt{3}} q\Phi_1\Phi_2\Phi_{12}''$

where

$$\Phi_{12} = \Phi_{12}'\Phi_{12}''$$

and

$$\Phi_{12}' = q^2 - \sqrt{3}q + 1, \qquad \Phi_{12}'' = q^2 + \sqrt{3}q + 1.$$

Type ${}^2F_4(q^2)$. The Ree groups ${}^2F_4(q^2)$ where $q^2 = 2^{2m+1}$ have 21 unipotent characters. 7 of them lie in the principal series, 4 of them arise from the two cuspidal unipotent characters of the Levi subgroup ${}^2B_2(q^2)$, and 10 are cuspidal.

The characters in the principal series are labelled by the irreducible characters of the dihedral group of order 16. This has four irreducible characters of degree 1, denoted by $1, \varepsilon, \varepsilon', \varepsilon''$, and three of degree 2, denoted by $\rho_2, \rho_2', \rho_2''$. The degrees of the unipotent characters are as follows:

1	1
ε'	$q^2\Phi_{12}\Phi_{24}$
ε''	$q^{10}\Phi_{12}\Phi_{24}$
ε	q^{24}
${}^2B_2[a], 1$	$\frac{1}{\sqrt{2}} q\Phi_1\Phi_2\Phi_4{}^2\Phi_{12}$
${}^2B_2[b], 1$	$\frac{1}{\sqrt{2}} q\Phi_1\Phi_2\Phi_4{}^2\Phi_{12}$
${}^2B_2[a], \varepsilon$	$\frac{1}{\sqrt{2}} q^{13}\Phi_1\Phi_2\Phi_4{}^2\Phi_{12}$
${}^2B_2[b], \varepsilon$	$\frac{1}{\sqrt{2}} q^{13}\Phi_1\Phi_2\Phi_4{}^2\Phi_{12}$
ρ_2'	$\frac{1}{4}q^4\Phi_4{}^2\Phi_8''^2\Phi_{12}\Phi_{24}'$
ρ_2''	$\frac{1}{4}q^4\Phi_4{}^2\Phi_8'^2\Phi_{12}\Phi_{24}''$
ρ_2	$\frac{1}{2}q^4\Phi_4\Phi_8\Phi_{12}\Phi_{24}$
cuspidal	$\frac{1}{12}q^4\Phi_1{}^2\Phi_2{}^2\Phi_8'^2\Phi_{12}\Phi_{24}'$
cuspidal	$\frac{1}{12}q^4\Phi_1{}^2\Phi_2{}^2\Phi_8''^2\Phi_{12}\Phi_{24}''$
cuspidal	$\frac{1}{6}q^4\Phi_1{}^2\Phi_2{}^2\Phi_4{}^2\Phi_{24}$
cuspidal	$\frac{1}{4}q^4\Phi_1{}^2\Phi_2{}^2\Phi_4{}^2\Phi_{12}\Phi_{24}''$
cuspidal	$\frac{1}{4}q^4\Phi_1{}^2\Phi_2{}^2\Phi_4{}^2\Phi_{12}\Phi_{24}''$
cuspidal	$\frac{1}{4}q^4\Phi_1{}^2\Phi_2{}^2\Phi_4{}^2\Phi_{12}\Phi_{24}'$
cuspidal	$\frac{1}{4}q^4\Phi_1{}^2\Phi_2{}^2\Phi_4{}^2\Phi_{12}\Phi_{24}'$

cuspidal	$\frac{1}{3}q^4\Phi_1{}^2\Phi_2{}^2\Phi_4{}^2\Phi_8{}^2$
cuspidal	$\frac{1}{3}q^4\Phi_1{}^2\Phi_2{}^2\Phi_4{}^2\Phi_8{}^2$
cuspidal	$\frac{1}{3}q^4\Phi_1{}^2\Phi_2{}^2\Phi_{12}\Phi_{24}$

where $\Phi_8 = \Phi_8'\Phi_8''$, $\Phi_{24} = \Phi_{24}'\Phi_{24}''$ and

$$\Phi_8' = q^2 + \sqrt{2}q + 1 \qquad \Phi_8'' = q^2 - \sqrt{2}q + 1$$

$$\Phi_{24}' = q^4 + \sqrt{2}q^3 + q^2 + \sqrt{2}q + 1 \qquad \Phi_{24}'' = q^4 - \sqrt{2}q^3 + q^2 - \sqrt{2}q + 1.$$

Appendix

l-ADIC COHOMOLOGY

(a) THE WEIL CONJECTURES

The introduction of l-adic cohomology by M. Artin and Grothendieck was motivated by the conjectures of A. Weil concerning the number of points of an algebraic variety over a finite field.

Let X be a nonsingular irreducible projective variety defined over the finite field F_q where q is a power of the prime p. Let $X(F_{q^n})$ be the set of rational points of X over F_{q^n}. Weil conjectured that the numbers $|X(F_{q^n})|$ for $n = 1, 2, 3, \ldots$ could be obtained from a certain rational function $Z(X, t) \in \mathbb{Q}(t)$ called the zeta function of X. $Z(X, t)$ is defined as a power series over $\mathbb{Q}$ by the two conditions

$$Z(X, 0) = 1$$

$$\frac{d}{dt}(\log Z(X, t)) = \sum_{n=1}^{\infty} |X(F_{q^n})| t^{n-1}.$$

Weil's conjectures were as follows:

(i) $Z(X, t)$ is a rational function, i.e. $Z(X, t) \in \mathbb{C}(t)$.

(ii) Let $d = \dim X$. Then $Z(X, t)$ has the form

$$Z(X, t) = \frac{P_1(t)P_3(t) \ldots P_{2d-1}(t)}{P_0(t)P_2(t) \ldots P_{2d}(t)}$$

where $P_i(t) \in \mathbb{Z}[t]$ has the form

$$P_i(t) = (1 - \alpha_{i1}t)(1 - \alpha_{i2}t) \ldots (1 - \alpha_{iB_i}t)$$

and the α_{ij} are algebraic integers with modulus given by $|\alpha_{ij}| = q^{i/2}$.

(iii) $Z(X, 1/q^d t) = (-1)^{\chi}(q^{d/2}t)^{\chi} Z(X, t)$ where $\chi = \sum_{i=0}^{2d} (-1)^i B_i$.

(iv) Suppose X is obtained from a variety Y defined over a ring A of algebraic integers by reduction modulo a prime ideal $\mathfrak{P}$ of A containing p. Let $Y_{\mathbb{C}}$ be the complex manifold obtained from Y by extending the scalars from A to $\mathbb{C}$. Then B_i is the ith Betti number of $Y_{\mathbb{C}}$ with respect to the classical topology on the complex manifold $Y_{\mathbb{C}}$ (not the Zariski topology).

We now show how the Weil conjectures would follow from the existence of cohomology groups $H^i(X)$ which should be finite-dimensional vector spaces over a field of characteristic 0 satisfying suitable conditions.

Suppose such cohomology groups can be defined such that $H^i(X) = 0$ unless $0 \leqslant i \leqslant 2d$. Let $B_i = \dim H^i(X)$. Suppose that $B_{2d} = 1$ and that there is a cup product map

$$H^i(X) \times H^{2d-i}(X) \to H^{2d}(X)$$

which is a perfect pairing, for each $i = 0, 1, \ldots, 2d$.

Suppose also that, when X is derived from a variety Y in characteristic 0 as in (iv), we have

$$\dim H^i(X) = \dim H^i(Y_{\mathbb{C}}, \mathbb{C})$$

where $H^i(Y_{\mathbb{C}}, \mathbb{C})$ is the classical cohomology of the complex manifold $Y_{\mathbb{C}}$.

Let $F: X \to X$ be the qth power Frobenius map defined by $(x_0, x_1, \ldots, x_m) \to (x_0^q, x_1^q, \ldots, x_m^q)$. Then we have

$$|X(F_{q^n})| = |X^{F^n}|.$$

Suppose F induces a corresponding map on the cohomology $H^i(X)$ such that, for each n, we have

$$|X^{F^n}| = \sum_{i=0}^{2d} (-1)^i \operatorname{trace}(F^n, H^i(X)).$$

Suppose that the eigenvalue of F on $H^{2d}(X)$ is q^d and that the eigenvalues α_{ij} of F on $H^i(X)$ are algebraic integers of modulus $q^{i/2}$ for each $i = 0, 1, \ldots, 2d$.

It is then possible to derive the Weil conjectures as follows. We have

$$|X(F_{q^n})| = |X^{F^n}| = \sum_i (-1)^i \operatorname{trace}(F^n, H^i(X)) = \sum_i (-1)^i \sum_j \alpha_{ij}^n.$$

Thus

$$\frac{d}{dt}(\log Z(x, t)) = \sum_{n=1}^{\infty} |X(F_{q^n})| t^{n-1} = \sum_{n=1}^{\infty} \sum_i (-1)^i \sum_j \alpha_{ij}^n t^{n-1}$$

$$= \sum_i (-1)^i \sum_j \frac{\alpha_{ij}}{1 - \alpha_{ij} t}.$$

Hence

$$\log Z(X, t) = -\sum_i (-1)^i \sum_j \log(1 - \alpha_{ij} t)$$

since $Z(X, 0) = 1$. It follows that

$$Z(X, t) = \frac{P_1(t)P_3(t) \ldots P_{2d-1}(t)}{P_0(t)P_2(t) \ldots P_{2d}(t)}$$

where $P_i(t) = (1 - \alpha_{i1} t)(1 - \alpha_{i2} t) \ldots (1 - \alpha_{iB_i} t)$. Thus $Z(X, t)$ is a rational function of the required form.

We now consider $Z(X, 1/q^d t)$. Assuming that the pairing $H^i(X) \times H^{2d-i}(X) \to H^{2d}(X)$ is compatible with the action of F on the cohomology groups we see that, since F acts on $H^{2d}(X)$ with eigenvalue q^d, the eigenvalues of F on $H^{2d-i}(X)$ are $q^d/\alpha_{i1}, \ldots, q^d/\alpha_{iB_i}$. Thus $B_{2d-i} = B_i$ and we may take $\alpha_{2d-i,j} = q^d/\alpha_{i,j}$. Thus

$$Z\left(X, \frac{1}{q^d t}\right) = \frac{P_1\left(\frac{1}{q^d t}\right) \ldots P_{2d-1}\left(\frac{1}{q^d t}\right)}{P_0\left(\frac{1}{q^d t}\right) \ldots P_{2d}\left(\frac{1}{q^d t}\right)}$$

and

$$\begin{aligned} P_i\left(\frac{1}{q^d t}\right) &= \left(1 - \frac{\alpha_{i1}}{q^d t}\right) \ldots \left(1 - \frac{\alpha_{iB_i}}{q^d t}\right) \\ &= \frac{1}{(q^d t)^{B_i}} (-1)^{B_i} \alpha_{i1} \ldots \alpha_{iB_i} \left(1 - \frac{q^d}{\alpha_{i1}} t\right) \ldots \left(1 - \frac{q^d}{\alpha_{iB_i}} t\right) \\ &= (-1)^{B_i} \frac{1}{(q^d t)^{B_i}} \det(F, H^i(X))(1 - \alpha_{2d-i,1} t) \ldots (1 - \alpha_{2d-i,B_i} t) \\ &= (-1)^{B_i} \frac{1}{(q^d t)^{B_i}} \det(F, H^i(X)) P_{2d-i}(t). \end{aligned}$$

Hence we have

$$Z\left(X, \frac{1}{q^d t}\right) = \frac{(-1)^{B_1 + B_3 + \cdots} (q^d t)^{B_0 + B_2 + \cdots} \prod_{i \text{ odd}} \det(F, H^i(X))}{(-1)^{B_0 + B_2 + \cdots} (q^d t)^{B_1 + B_3 + \cdots} \prod_{i \text{ even}} \det(F, H^i(X))} Z(X, t).$$

Now $\chi = \sum (-1)^i B_i$ and we have

$$\det(F, H^i(X)) \det(F, H^{2d-i}(X)) = (q^d)^{B_i}.$$

Thus

$$\begin{aligned} Z\left(X, \frac{1}{q^d t}\right) &= \frac{(-1)^\chi (q^d t)^\chi (q^d)^{\frac{1}{2}(B_1 + B_3 + \ldots)}}{(q^d)^{\frac{1}{2}(B_0 + B_2 + \ldots)}} Z(X, t) \\ &= (-1)^\chi (q^{d/2} t)^\chi Z(X, t). \end{aligned}$$

Thus the Weil conjectures follow from the existence of cohomology groups $H^i(X)$ over a field of characteristic 0 with suitable properties. We see that, although X is defined over a field of characteristic p, it is crucial that $H^i(X)$ should be over a field of characteristic 0 so that the formula

$$|X^{F^n}| = \sum_{i=0}^{2d} (-1)^i \operatorname{trace}(F^n, H^i(X))$$

gives the number of points on X^{F^n}, and not merely a number congruent to this modulo p. This trace formula leads to the Weil conjecture (i) and the first half of (ii). The duality map $H^i(X) \times H^{2d-i}(X) \to H^{2d}(X)$ leads to conjecture (iii) and the comparison theorem $\dim H^i(X) = \dim H^i(Y_{\mathbb{C}}, \mathbb{C})$ leads to conjecture (iv). Finally a knowledge of the eigenvalues of the Frobenius map F on $H^i(X)$ leads to the second half of (ii).

In this way the Weil conjectures motivated the search for a cohomology theory over characteristic 0 for varieties over characteristic p. Such a cohomology theory was developed by Grothendieck and M. Artin, in which $H^i(X)$ is interpreted as the l-adic cohomology $H^i(X, \mathbb{Q}_l)$. The formal properties of these l-adic cohomology groups led to the proof of the Weil conjectures (i), (iii), (iv) above. The proof of conjecture (ii) proved much more difficult and was settled relatively recently by Deligne in [2], in which he was able to derive the necessary facts about the eigenvalues of the Frobenius map F on the l-adic cohomology $H^i(X, \mathbb{Q}_l)$.

We shall define these l-adic cohomology groups $H^i(X, \mathbb{Q}_l)$ subsequently, but it is first necessary to recall some ideas regarding sheaf theory and sheaf cohomology.

(b) CLASSICAL SHEAF THEORY

Let X be a topological space. A presheaf of abelian groups on X consists of an abelian group $\mathscr{F}(U)$ for each open subset U of X together with a homomorphism $\rho_{U,V}:\mathscr{F}(U) \to \mathscr{F}(V)$ for each pair of open subsets satisfying $V \subset U$, such that $\rho_{U,U}$ is the identity map and $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$ when $W \subset V \subset U$.

A presheaf $\mathscr{F}$ is called a sheaf if it satisfies in addition the following condition. Suppose we have an open subset U of X and open subsets U_α such that $U = \cup_\alpha U_\alpha$. Suppose $s_\alpha \in \mathscr{F}(U_\alpha)$ satisfy

$$\rho_{U_\alpha, U_\alpha \cap U_\beta}(s_\alpha) = \rho_{U_\beta, U_\alpha \cap U_\beta}(s_\beta) \qquad \text{for all } \alpha, \beta.$$

Then there is a unique element $s \in \mathscr{F}(U)$ such that $\rho_{U,U_\alpha}(s) = s_\alpha$ for each α.

One can in a similar way define a sheaf of rings on X or a sheaf of R-modules on X for a ring R.

Let $\mathscr{F}_1, \mathscr{F}_2$ be presheaves or sheaves. A morphism $\phi:\mathscr{F}_1 \to \mathscr{F}_2$ is defined as a family of homomorphisms $\phi(U):\mathscr{F}_1(U) \to \mathscr{F}_2(U)$ such that, for all $V \subset U$,

$$\phi(V) \circ \rho_{U,V} = \rho_{U,V} \circ \phi(U)$$

on $\mathscr{F}_1(U)$.

Given any presheaf $\mathscr{F}$ on X there is a sheaf $\mathscr{F}^+$ and a morphism $\phi:\mathscr{F} \to \mathscr{F}^+$ such that any morphism from $\mathscr{F}$ into a sheaf factors uniquely through ϕ. $\mathscr{F}^+$ is called the sheafification of $\mathscr{F}$.

Let $\mathscr{F}$ be a sheaf on X and let $x \in X$. Then the stalk $\mathscr{F}_x$ is defined by

$$\mathscr{F}_x = \varinjlim_{x \in U} \mathscr{F}(U).$$

Thus $\mathscr{F}_x$ is the direct limit of all the $\mathscr{F}(U)$ for which U contains x. Each element $s \in \mathscr{F}(U)$ thus gives rise to a corresponding element $s_x \in \mathscr{F}_x$ for all $x \in U$.

If $f: X \to Y$ is continuous and if $\mathscr{F}$ is a sheaf on X we can define a sheaf $f_*\mathscr{F}$ on Y by

$$(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$$

for all open subsets V of Y. $f_*\mathscr{F}$ is called the direct image of $\mathscr{F}$.

If $f: X \to Y$ is continuous and $\mathscr{G}$ is a sheaf on Y we can define a sheaf $f^*\mathscr{G}$ on X called the inverse image of $\mathscr{G}$. This is a little more complicated to define than the direct image. $f^*\mathscr{G}$ is the sheafification of the presheaf on X given by

$$U \to \varinjlim_{V \supseteq f(U)} \mathscr{G}(V)$$

where U is open in X and the limit is taken over all open subsets V of Y containing $f(U)$. The stalks of the sheaf $f^*\mathscr{G}$ on X are given by

$$(f^*\mathscr{G})_x \cong \mathscr{G}_{f(x)} \qquad x \in X.$$

The class of sheaves of abelian groups on a topological space together with the morphisms between them form an abelian category. The map f^* from sheaves on Y to sheaves on X is then left adjoint to the map f_* from sheaves on X to sheaves on Y, in the sense that there is a natural bijection between morphisms from $f^*\mathscr{G}$ to $\mathscr{F}$ on X and morphisms from $\mathscr{G}$ to $f_*\mathscr{F}$ on Y.

A constant sheaf on X is one which is the sheafification of some presheaf $\mathscr{F}$ of the form $\mathscr{F}(U) = A$ for all open subsets U of X, where A is fixed.

A sheaf $\mathscr{F}$ on X is called locally constant if X has an open covering $X = \cup U_\alpha$ such that the inclusion map $i_\alpha: U_\alpha \to X$ induces a constant sheaf $i_\alpha{}^*\mathscr{F}$ on U_α for each α.

A sheaf $\mathscr{F}$ on X is called constructible if X can be expressed as the disjoint union $X = \cup_j X_j$ of a finite number of locally closed subsets X_j such that the inclusion map $i_j: X_j \to X$ induces a locally constant sheaf $i_j{}^*\mathscr{F}$ on X_j for each j.

Now each sheaf on X has an injective resolution. Thus we may construct sheaf cohomology groups in the following manner.

The map Γ from sheaves on X to abelian groups given by $\Gamma(\mathscr{F}) = \mathscr{F}(X)$ is called the global section functor. Let

$$0 \to \mathscr{F} \to \mathscr{C}^0 \to \mathscr{C}^1 \to \mathscr{C}^2 \to \cdots$$

be an injective resolution of the sheaf $\mathscr{F}$ on X. Then we obtain a complex of abelian groups and homomorphisms

$$0 \to \mathscr{F}(X) \to \mathscr{C}^0(X) \xrightarrow{d_0} \mathscr{C}^1(X) \xrightarrow{d_1} \mathscr{C}^2(X) \xrightarrow{d_2} \cdots.$$

We define $H^i(X, \mathscr{F}) = \operatorname{Ker} d_i/\operatorname{Im} d_{i-1}$. $H^i(X, \mathscr{F})$ is called the ith cohomology group of X with coefficients in the sheaf $\mathscr{F}$, and is independent of the choice of injective resolution for $\mathscr{F}$.

In order to define the l-adic cohomology groups of a variety we shall need a nonclassical analogue of this sheaf cohomology due to Grothendieck. In order to discuss this nonclassical sheaf cohomology it is best to use the language of schemes. We shall next introduce the necessary concepts.

(c) SCHEMES

A ringed space is a pair (X, O_X) where X is a topological space and O_X is a sheaf of rings on X. A morphism of ringed spaces $(X, O_X) \to (Y, O_Y)$ is a continuous map $f: X \to Y$ together with, for each open subset U of Y, a ring homomorphism $\phi_U: O_Y(U) \to O_X(f^{-1}(U))$ such that if $V \subset U$ then $\phi_V \circ \rho_{U,V} = \rho_{f^{-1}(U), f^{-1}(V)} \circ \phi_U$ on $O_Y(U)$. Every morphism of ringed spaces induces a homomorphism of stalks $O_{Y,f(x)} \to O_{X,x}$ for each $x \in X$.

A locally ringed space is a ringed space (X, O_X) for which each stalk $O_{X,x}$ is a local ring, i.e. is a commutative ring with identity which has a unique maximal ideal m_x. A morphism of locally ringed spaces $(X, O_X) \to (Y, O_Y)$ is a morphism of ringed spaces which induces a local homomorphism of local rings $O_{Y,f(x)} \to O_{X,x}$ on the stalks, for all $x \in X$. This means that the inverse image of the maximal ideal m_x of $O_{X,x}$ is the maximal ideal $m_{f(x)}$ of $O_{Y,f(x)}$.

Let A be a commutative ring with identity element. Let $X = \operatorname{Spec} A$ be the set of prime ideals of A. For each subset S of A let $V(S)$ be the set of all prime ideals containing S. We can make X into a topological space by defining the closed subsets to be the subsets of the form $V(S)$ for some S. We now make X into a locally ringed space by defining a sheaf of rings O_X on X which has the property that, for each $x \in X$, the stalk $O_{X,x}$ is the localization of A at the prime ideal x. The sheaf O_X on $X = \operatorname{Spec} A$ is defined as follows. For each $a \in A$ let

$$D(a) = \{x \in \operatorname{Spec} A;\ a \notin x\}.$$

$D(a)$ is the principal open subset of Spec A which is the complement of $V(\{a\})$. The subsets of the form $D(a)$ for $a \in A$ form a basis for the topology on Spec A. We define

$$O_X(D(a)) = A_a$$

where A_a is the localization of A with respect to a, i.e. the ring of fractions of A with respect to the multiplicative system $\{1, a, a^2, \ldots\}$. This definition is unambiguous since if $D(a) = D(b)$ then A_a and A_b are canonically isomorphic. More generally, if U is any open subset of X, we define

$$O_X(U) = \varprojlim_{U \supseteq D(a)} O_X(D(a))$$

over all principal open subsets $D(a)$ contained in U. Then O_X is a sheaf of rings on X. Moreover for each $x \in X$ the stalk $O_{X,x}$ is the localization of A at the prime ideal x. Thus (X, O_X) is a locally ringed space.

An affine scheme is a locally ringed space isomorphic to Spec A for some commutative ring A with identity.

A scheme is a locally ringed space (X, O_X) in which, for each point $x \in X$, there is an open subset U of X containing x such that the injection $i: U \to X$ induces a pair (U, i^*O_X) which is an affine scheme.

A morphism of schemes is a morphism of the underlying locally ringed spaces.

We now discuss products of schemes. If X and Y are schemes there is a scheme

$X \times Y$ unique up to isomorphism for which there exist morphisms $p_1: X \times Y \to X$, $p_2: X \times Y \to Y$ such that, given any scheme Z and any morphisms $f: Z \to X$, $g: Z \to Y$, there exists a unique morphism $h: Z \to X \times Y$ such that $f = p_1 \circ h$ and $g = p_2 \circ h$. More generally, suppose we start with schemes X and Y and with morphisms $j: X \to S$, $k: Y \to S$ of X and Y into a third scheme S. There is then a scheme denoted by $X \times_S Y$, unique up to isomorphism, called the fibred product which satisfies the following condition. There exist morphisms $p_1: X \times_S Y \to X$, $p_2: X \times_S Y \to Y$ satisfying $j \circ p_1 = k \circ p_2$ such that, given any scheme Z and any morphisms $f: Z \to X$, $g: Z \to Y$ with $j \circ f = k \circ g$, there exists a unique morphism $h: Z \to X \times_S Y$ such that $f = p_1 \circ h$ and $g = p_2 \circ h$.

In the special case when X, Y and S are all affine, so that $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} C$ where A and B are C-algebras, the fibred product is given by $X \times_S Y = \operatorname{Spec}(A \otimes_C B)$. The general case is built up from this special case by a nontrivial patching process.

We now discuss certain special kinds of schemes. Let A be a commutative ring with identity. A scheme over A is a scheme X together with a morphism $X \to \operatorname{Spec} A$. In particular a scheme over a field k is a scheme X together with a morphism $X \to \operatorname{Spec} k$. A scheme X is reduced if, for all open subsets U of X, $O_X(U)$ has no nilpotent elements. A scheme X is separated if the diagonal morphism $\Delta: X \to X \times X$ has image $\Delta(X)$ which is closed in $X \times X$. A scheme X is of finite type over an algebraically closed field K if there are finitely many open subsets U_j of X such that $X = \cup U_j$ and the injection $i_j: U_j \to X$ induces a scheme $(U_j, i_j^* O_X)$ isomorphic to $\operatorname{Spec} A_j$ where A_j is a finitely generated K-algebra.

The concept of a scheme is a far-reaching generalization of the concept of an algebraic variety. Every algebraic variety over an algebraically closed field K gives rise to a scheme which is reduced, separated, and of finite type over K. Conversely every scheme which is reduced, separated, and of finite type over K arises from an algebraic variety over K which is unique up to isomorphism. For example if V is an affine variety over K with coordinate ring $K[V]$ the corresponding scheme will be $X = \operatorname{Spec} K[V]$. The points of V will be in bijective correspondence with the closed points of X. For the points of X correspond to the prime ideals of $K[V]$, the points of V correspond to the maximal ideals of $K[V]$ and the point of X corresponding to a prime ideal I of $K[V]$ is closed if and only if I is a maximal ideal of $K[V]$.

(d) ÉTALE SHEAF THEORY

We shall now describe a modification of the classical sheaf theory discussed in (b). This modification is due to Grothendieck. We shall define étale sheaves on a scheme X, which will always be assumed to be separated and of finite type over an algebraically closed field K.

In order to give a sheaf on X in the classical sense we must associate with each open subset U of X an abelian group $\mathscr{F}(U)$. However to give an étale sheaf on X

it is not sufficient to consider open subsets of X. Grothendieck considered instead étale morphisms into the scheme X. An étale morphism into X is a morphism of schemes $f: U \to X$ satisfying the following conditions:

(i) $f^{-1}(x)$ is finite for each closed point $x \in X$.

(ii) For each closed point $x \in X$ and each $u \in f^{-1}(x)$ the morphism of stalks $O_{U,u} \to O_{X,x}$ gives rise to an isomorphism of their completions $\hat{O}_{U,u} \to \hat{O}_{X,x}$ with respect to their maximal ideals.

If U is an open subset of X and $i: U \to X$ is the injection then (U, i^*O_X) is a scheme and $i: U \to X$ determines a morphism of schemes which is an étale morphism into X. However not all étale morphisms into X will come from open subsets of X in this way.

Let $X_{\text{ét}}$ be the category whose objects are étale morphisms $U \to X$ where U is a scheme and whose morphisms from $U \to X$ to $V \to X$ are commutative triangles

$$\begin{array}{ccc} U & \to & V \\ & \searrow \; \swarrow & \\ & X & \end{array}$$

A presheaf of abelian groups (or rings etc.) on $X_{\text{ét}}$ consists of an abelian group $\mathscr{F}\begin{pmatrix} U \\ \downarrow \\ X \end{pmatrix}$ for each étale morphism $U \to X$ with a homomorphism

$$\rho_{U,V}: \mathscr{F}\begin{pmatrix} U \\ \downarrow \\ X \end{pmatrix} \to \mathscr{F}\begin{pmatrix} V \\ \downarrow \\ X \end{pmatrix}$$

for each morphism $\begin{array}{ccc} V & \to & U \\ & \searrow \; \swarrow & \\ & X & \end{array}$. $\rho_{U,U}$ is the identity and $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$ when we have morphisms

$$\begin{array}{ccccc} W & \to & V & \to & U \\ & \searrow & \downarrow & \swarrow & \\ & & X & & \end{array}$$

In order to define a sheaf on $X_{\text{ét}}$ we need an analogue of the intersection $U_\alpha \cap U_\beta$ of two open sets in X. This analogue is the fibred product defined in (c).

A presheaf $\mathscr{F}$ on $X_{\text{ét}}$ is called a sheaf if it satisfies in addition the following condition. Suppose we have an étale morphism $U \to X$ and finitely many étale morphisms $U_\alpha \xrightarrow[p_\alpha]{} U$ such that $U = \cup\, p_\alpha(U_\alpha)$. Then we have étale morphisms $U_\alpha \to X$ obtained by composition. Let $s_\alpha \in \mathscr{F}\begin{pmatrix} U_\alpha \\ \downarrow \\ X \end{pmatrix}$ be such that

$$\rho_{U_\alpha, U_\alpha \times_U U_\beta}(s_\alpha) = \rho_{U_\beta, U_\alpha \times_U U_\beta}(s_\beta)$$

for all α, β. Then there is a unique element $s \in \mathscr{F}\begin{pmatrix} U \\ \downarrow \\ X \end{pmatrix}$ such that $\rho_{U,U_\alpha}(s) = s_\alpha$ for each α.

Let $\mathscr{F}_1, \mathscr{F}_2$ be presheaves or sheaves on $X_{\text{ét}}$. A morphism $\phi: \mathscr{F}_1 \to \mathscr{F}_2$ is defined as a family of homomorphisms

$$\phi(U): \mathscr{F}_1\begin{pmatrix} U \\ \downarrow \\ X \end{pmatrix} \to \mathscr{F}_2\begin{pmatrix} U \\ \downarrow \\ X \end{pmatrix}$$

such that, whenever $V \to U$ is an étale morphism,

$$\phi(V) \circ \rho_{U,V} = \rho_{U,V} \circ \phi(U) \qquad \text{on } \mathscr{F}_1\begin{pmatrix} U \\ \downarrow \\ X \end{pmatrix}.$$

Given any presheaf $\mathscr{F}$ on $X_{\text{ét}}$ there is a sheaf $\mathscr{F}^+$ on $X_{\text{ét}}$ and a morphism $\phi: \mathscr{F} \to \mathscr{F}^+$ such that any morphism from $\mathscr{F}$ into a sheaf on $X_{\text{ét}}$ factors uniquely through ϕ. $\mathscr{F}^+$ is called the sheafification of $\mathscr{F}$ on $X_{\text{ét}}$.

We now introduce the analogue of the stalk of a sheaf at a point. A geometric point of a scheme X is defined as a morphism Spec $\tilde{K} \to X$ where $\tilde{K}$ is an algebraically closed field. The scheme Spec $\tilde{K}$ has a single point whose image $x \in X$ is called the centre of the given geometric point. If

$$\bar{x}: \operatorname{Spec} \tilde{K} \to X$$

is a geometric point of X we define an étale neighbourhood of $\bar{x}$ to be a commutative triangle

$$\begin{array}{ccc} \operatorname{Spec} \tilde{K} & \to & U \\ & {}_{\bar{x}}\searrow \;\swarrow & \\ & X & \end{array}$$

where $U \to X$ is an étale morphism. The stalk $\mathscr{F}_{\bar{x}}$ of the étale sheaf $\mathscr{F}$ at the geometric point $\bar{x}$ is then defined by

$$\mathscr{F}_{\bar{x}} = \varinjlim \mathscr{F}\begin{pmatrix} U \\ \downarrow \\ X \end{pmatrix}$$

the direct limit being taken over all étale neighbourhoods of $\bar{x}$.

We now define direct and inverse images of étale sheaves. Let $f: X \to Y$ be a morphism of schemes and let $\mathscr{F}$ be a sheaf on $X_{\text{ét}}$. We can then define a sheaf $f_*\mathscr{F}$ on $Y_{\text{ét}}$ as follows. Given an étale morphism $U \to Y$ we define $(f_*\mathscr{F})\begin{pmatrix} U \\ \downarrow \\ Y \end{pmatrix}$ by

$$(f_*\mathscr{F})\begin{pmatrix} U \\ \downarrow \\ Y \end{pmatrix} = \mathscr{F}\begin{pmatrix} X \times_Y U \\ \downarrow \\ X \end{pmatrix}.$$

$f_*\mathscr{F}$ is a sheaf on $Y_{\text{ét}}$ called the direct image of $\mathscr{F}$.

Now let $f: X \to Y$ be a morphism of schemes and $\mathscr{G}$ be a sheaf on $Y_{\text{ét}}$. We can then define an inverse image sheaf $f^*\mathscr{G}$ on $X_{\text{ét}}$. This is uniquely determined by the property that the map f^* from sheaves on $Y_{\text{ét}}$ to sheaves on $X_{\text{ét}}$ is left adjoint to the map f_* from sheaves on $X_{\text{ét}}$ to sheaves on $Y_{\text{ét}}$. The stalks $(f^*\mathscr{G})_{\bar{x}}$ of the inverse image $f^*\mathscr{G}$ are described as follows. Let $\bar{x}:\operatorname{Spec} \tilde{K} \to X$ be a geometric point of X; thus $f \circ \bar{x}:\operatorname{Spec} \tilde{K} \to Y$ is a geometric point of Y. The stalk $(f^*\mathscr{G})_{\bar{x}}$ is then given by

$$(f^*\mathscr{G})_{\bar{x}} \cong \mathscr{G}_{f \circ \bar{x}}.$$

We now define constant, locally constant and constructible étale sheaves. A constant sheaf on $X_{\text{ét}}$ is one which is the sheafification of some presheaf $\mathscr{F}$ on $X_{\text{ét}}$ of the form $\mathscr{F}\begin{pmatrix} U \\ \downarrow \\ X \end{pmatrix} = A$ for all étale morphisms into X, where A is fixed.

A sheaf $\mathscr{F}$ on $X_{\text{ét}}$ is called locally constant if there exist finitely many étale morphisms $U_\alpha \xrightarrow[p_\alpha]{} X$ such that $X = \cup\, p_\alpha(U_\alpha)$ and $p_\alpha{}^*\mathscr{F}$ is a constant sheaf on $(U_\alpha)_{\text{ét}}$ for each α.

A sheaf $\mathscr{F}$ on $X_{\text{ét}}$ is called constructible if X can be expressed as a disjoint union of locally closed subsets X_j such that the inclusion map $i_j: X_j \to X$ induces a locally constant sheaf $i_j{}^*\mathscr{F}$ on $(X_j)_{\text{ét}}$ for each j.

The category of sheaves of abelian groups on $X_{\text{ét}}$ is an abelian category. Moreover each sheaf on $X_{\text{ét}}$ has an injective resolution. Thus we may construct étale sheaf cohomology groups as follows.

The map Γ from sheaves on $X_{\text{ét}}$ to abelian groups given by $\Gamma(\mathscr{F}) = \mathscr{F}\begin{pmatrix} X \\ \downarrow \\ X \end{pmatrix}$ is called the global section functor. Let

$$0 \to \mathscr{F} \to \mathscr{C}^0 \to \mathscr{C}^1 \to \mathscr{C}^2 \to \dots$$

be an injective resolution of the sheaf $\mathscr{F}$ on $X_{\text{ét}}$. Then we obtain a complex of abelian groups and homomorphisms

$$0 \to \Gamma(\mathscr{F}) \to \Gamma(\mathscr{C}^0) \xrightarrow{d_0} \Gamma(\mathscr{C}^1) \xrightarrow{d_1} \Gamma(\mathscr{C}^2) \xrightarrow{d_2} \dots.$$

We define $H^i(X_{\text{ét}}, \mathscr{F}) = \operatorname{Ker} d_i/\operatorname{Im} d_{i-1}$. $H^i(X_{\text{ét}}, \mathscr{F})$ is called the ith étale cohomology group with coefficients in the sheaf $\mathscr{F}$, and is independent of the choice of injective resolution for $\mathscr{F}$.

(e) ÉTALE COHOMOLOGY WITH COMPACT SUPPORT

A scheme X over a field k is said to be universally closed if, for any scheme X' over k, the corresponding morphism $X \times_k X' \to X'$ is closed (i.e. the image of any closed subset is closed). Suppose, as before, that we have a scheme X which is separated and of finite type over an algebraically closed field K. According to a theorem of Nagata there is a scheme $\bar{X}$ over K which is separated, of finite type over K, and universally closed and an injective morphism $j: X \to \bar{X}$ which induces an isomorphism between X and $j(X)$, which is an open subscheme of $\bar{X}$. A scheme $\bar{X}$ over K satisfying the above conditions is said to be proper over K. $\bar{X}$ may be thought of as being a 'compactification' of X, If $K = \mathbb{C}$ then $\bar{X}$ is proper over $\mathbb{C}$ if and only if it is compact in the usual topology. The scheme associated with any projective variety over K is proper over K.

We now consider torsion abelian groups and torsion sheaves. A torsion abelian group is an abelian group in which each element has finite order. A torsion sheaf of abelian groups is a sheaf for which each abelian group is a torsion group. Given a torsion sheaf $\mathscr{F}$ on $X_{\text{ét}}$ there is a sheaf $j_!\mathscr{F}$ on $\bar{X}_{\text{ét}}$ called the extension by zero of $\mathscr{F}$. This is the sheafification of the presheaf on $\bar{X}_{\text{ét}}$ defined as follows. The abelian group corresponding to an etale morphism $U \to \bar{X}$ is $\mathscr{F}\begin{pmatrix} U \\ \downarrow \\ X \end{pmatrix}$ if there is a factorization
$$\begin{tikzcd} & U \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "j"'] & & \bar{X} \end{tikzcd}$$
and zero otherwise. The homomorphism $\rho_{U,V}$ corresponding to a pair of étale morphisms $U \to \bar{X}$ and $V \to \bar{X}$ is the same as that corresponding to the pair $U \to X$, $V \to X$ if both étale morphisms factor through j as above, and is zero otherwise. This presheaf determines as its sheafification the sheaf $j_!\mathscr{F}$ on $\bar{X}_{\text{ét}}$ whose stalks are given by

$$(j_!\mathscr{F})_{\bar{x}} \cong \begin{cases} \mathscr{F}_{\bar{x}} & \text{if } x \in X \\ 0 & \text{if } x \notin X \end{cases}$$

where $\bar{x}$ is a geometric point of $\bar{X}$ centred at $x \in \bar{X}$.

We then define the étale cohomology groups with compact support with coefficients in the sheaf $\mathscr{F}$ by

$$H^i_c(X_{\text{ét}}, \mathscr{F}) = H^i(\bar{X}_{\text{ét}}, j_!\mathscr{F}).$$

It can be shown that this is independent of the choice of $\bar{X}$. In particular, if X is proper over K, we have

$$H^i_c(X_{\text{ét}}, \mathscr{F}) = H^i(X_{\text{ét}}, \mathscr{F}).$$

(f) *l*-ADIC SHEAVES

Let X be a scheme which is separated and of finite type over an algebraically closed field K. Let l be a prime and let $\mathbb{Z}_l$ be the ring of l-adic integers. Thus $\mathbb{Z}_l$ is the inverse limit of the inverse system

$$\cdots \to \frac{\mathbb{Z}}{l^{n+1}\mathbb{Z}} \to \frac{\mathbb{Z}}{l^n\mathbb{Z}} \to \frac{\mathbb{Z}}{l^{n-1}\mathbb{Z}} \to \cdots \to \frac{\mathbb{Z}}{l\mathbb{Z}} \to \frac{\mathbb{Z}}{\mathbb{Z}}.$$

A $\mathbb{Z}/l^n\mathbb{Z}$-module is an abelian group A in which $l^nA = 0$. $\mathbb{Z}/l^n\mathbb{Z}$ is a $\mathbb{Z}/l^n\mathbb{Z}$-module in a natural way. Any $\mathbb{Z}/l^n\mathbb{Z}$-module also has the structure of a $\mathbb{Z}/l^{n+1}\mathbb{Z}$-module since $l^nA = 0$ implies that $l^{n+1}A = 0$. In particular $\mathbb{Z}/l^n\mathbb{Z}$ can be considered as a $\mathbb{Z}/l^{n+1}\mathbb{Z}$-module. Given any $\mathbb{Z}/l^{n+1}\mathbb{Z}$-module A we can construct a $\mathbb{Z}/l^n\mathbb{Z}$-module A' defined by $A' = A \otimes_{\mathbb{Z}/l^{n+1}\mathbb{Z}} \mathbb{Z}/l^n\mathbb{Z}$. Moreover there is a natural homomorphism $A \to A'$ given by $a \to a \otimes 1$.

In a similar way, given any sheaf $\mathscr{F}$ of $\mathbb{Z}/l^{n+1}\mathbb{Z}$-modules on $X_{\text{ét}}$ we can construct a sheaf $\mathscr{F}'$ of $\mathbb{Z}/l^n\mathbb{Z}$-modules on $X_{\text{ét}}$ by tensoring each abelian group over $\mathbb{Z}/l^{n+1}\mathbb{Z}$ by $\mathbb{Z}/l^n\mathbb{Z}$. $\mathscr{F}'$ will be denoted by $\mathscr{F} \otimes_{\mathbb{Z}/l^{n+1}\mathbb{Z}} \mathbb{Z}/l^n\mathbb{Z}$. There is a natural morphism $\mathscr{F} \to \mathscr{F} \otimes_{\mathbb{Z}/l^{n+1}\mathbb{Z}} \mathbb{Z}/l^n\mathbb{Z}$ obtained by applying the above homomorphism $A \to A'$ to each abelian group in $\mathscr{F}$.

A $\mathbb{Z}_l$-sheaf on the scheme X is an inverse system of sheaves $\mathscr{F}_n$ on $X_{\text{ét}}$

$$\ldots \to \mathscr{F}_n \to \mathscr{F}_{n-1} \to \ldots \to \mathscr{F}_1 \to \mathscr{F}_0$$

where $\mathscr{F}_n$ is a constructible sheaf of $\mathbb{Z}/l^{n+1}\mathbb{Z}$-modules for each $n \geqslant 0$ and $\mathscr{F}_{n-1}$ is isomorphic to $\mathscr{F}_n \otimes_{\mathbb{Z}/l^{n+1}\mathbb{Z}} \mathbb{Z}/l^n\mathbb{Z}$ in a way which gives rise to a commutative triangle

$$\begin{tikzcd} \mathscr{F}_n \arrow[r] \arrow[dr] & \mathscr{F}_n \otimes_{\mathbb{Z}/l^{n+1}\mathbb{Z}} \mathbb{Z}/l^n\mathbb{Z} \arrow[d] \\ & \mathscr{F}_{n-1} \end{tikzcd}$$

Since each $\mathscr{F}_n$ is a torsion sheaf we have étale sheaf cohomology groups $H^i(X_{\text{ét}}, \mathscr{F}_n)$ and $H^i_c(X_{\text{ét}}, \mathscr{F}_n)$ for each n. We also have inverse systems of homomorphisms

$$\ldots \to H^i(X_{\text{ét}}, \mathscr{F}_n) \to H^i(X_{\text{ét}}, \mathscr{F}_{n-1}) \to \ldots \to H^i(X_{\text{ét}}, \mathscr{F}_0)$$
$$\ldots \to H^i_c(X_{\text{ét}}, \mathscr{F}_n) \to H^i_c(X_{\text{ét}}, \mathscr{F}_{n-1}) \to \ldots \to H^i_c(X_{\text{ét}}, \mathscr{F}_0).$$

For a $\mathbb{Z}_l$-sheaf $\mathscr{F}$ we define abelian groups $H^i(X_{\text{ét}}, \mathscr{F})$ and $H^i_c(X_{\text{ét}}, \mathscr{F})$ by

$$H^i(X_{\text{ét}}, \mathscr{F}) = \varprojlim H^i(X_{\text{ét}}, \mathscr{F}_n)$$
$$H^i_c(X_{\text{ét}}, \mathscr{F}) = \varprojlim H^i_c(X_{\text{ét}}, \mathscr{F}_n).$$

We now consider the special case when $\mathscr{F}_n$ is the constant sheaf on $X_{\text{ét}}$ determined by the group $\mathbb{Z}/l^{n+1}\mathbb{Z}$. There is a natural inverse system involving these constant sheaves $\mathscr{F}_n$ which determines a $\mathbb{Z}_l$-sheaf $\mathscr{F}$. We define

$$H^i(X, \mathbb{Z}_l) = H^i(X_{\text{ét}}, \mathscr{F})$$
$$H^i_c(X, \mathbb{Z}_l) = H^i_c(X_{\text{ét}}, \mathscr{F})$$

for this $\mathbb{Z}_l$-sheaf $\mathscr{F}$. Then $H^i(X, \mathbb{Z}_l)$ and $H^i_c(X, \mathbb{Z}_l)$ can be considered as $\mathbb{Z}_l$-modules in a natural way.

Let $\mathbb{Q}_l$ be the field of l-adic numbers. We define

$$H^i(X, \mathbb{Q}_l) = H^i(X, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$
$$H^i_c(X, \mathbb{Q}_l) = H^i_c(X, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

$H^i(X, \mathbb{Q}_l)$ and $H^i_c(X, \mathbb{Q}_l)$ are vector spaces over $\mathbb{Q}_l$. $H^i(X, \mathbb{Q}_l)$ is called the ith l-adic cohomology group of X, and $H^i_c(X, \mathbb{Q}_l)$ is called the ith l-adic cohomology group of X with compact support.

It is also useful to have the corresponding spaces over the algebraic closure $\bar{\mathbb{Q}}_l$ of $\mathbb{Q}_l$. We define

$$H^i(X, \bar{\mathbb{Q}}_l) = H^i(X, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \bar{\mathbb{Q}}_l$$
$$H^i_c(X, \bar{\mathbb{Q}}_l) = H^i_c(X, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \bar{\mathbb{Q}}_l.$$

These are vector spaces over the algebraically closed field $\bar{\mathbb{Q}}_l$.

(g) PROPERTIES OF l-ADIC COHOMOLOGY WITH COMPACT SUPPORT

As before X is a scheme which is separated and of finite type over an algebraically closed field K. Then the l-adic cohomology with compact support $H^i_c(X, \mathbb{Q}_l)$ is a finite-dimensional vector space over $\mathbb{Q}_l$. We have $H^i_c(X, \mathbb{Q}_l) = 0$ unless $0 \leqslant i \leqslant 2 \dim X$.

The spaces $H^i_c(X, \mathbb{Q}_l)$ have a functoriality property for finite morphisms. A morphism $f: X \to Y$ is said to be finite if there is a covering $Y = \cup_i V_i$ of Y by open subsets V_i of the form $V_i = \operatorname{Spec} B_i$ such that $f^{-1}(V_i) = U_i$ has the form $U_i = \operatorname{Spec} A_i$ for all i and the map $f: U_i \to V_i$ induces a map $\theta: B_i \to A_i$ which makes A_i into a finitely generated B_i-module under the operation $b.a = \theta(b)a$ for all $a \in A_i$, $b \in B_i$.

If $f: X \to Y$ is a finite morphism of schemes then there is an induced map

$$f^*: H^i_c(Y, \mathbb{Q}_l) \to H^i_c(X, \mathbb{Q}_l)$$

which is a linear map of vector spaces. In particular, since any automorphism of a scheme X is a finite morphism of X into itself, each automorphism f of X induces a nonsingular linear map f^* on $H^i_c(X, \mathbb{Q}_l)$. We have

$$(fg)^* = g^* f^*$$

for all $f, g \in \operatorname{Aut} X$. Thus the map $f \to (f^*)^{-1}$ is a representation of the group $\operatorname{Aut} X$ of automorphisms of X afforded by the $\operatorname{Aut} X$-module $H^i_c(X, \mathbb{Q}_l)$.

We now describe Grothendieck's trace formula. We assume now that the field K has characteristic p. Let q be a power of p. An F_q-structure on the scheme X is a scheme X_0 over F_q such that X is isomorphic to $X_0 \times_{\operatorname{Spec} F_q} \operatorname{Spec} K$. Each F_q-structure X_0 on X gives rise to a morphism $F: X \to X$ called the Frobenius morphism corresponding to X_0. F is defined in the following way. We first define a morphism $F_0: X_0 \to X_0$. F_0 acts as the identity on the set X_0 and as the qth power map $f \to f^q$ on $O_{X_0}(U)$ for each open subset U of X_0. Then F is the morphism on $X = X_0 \times_{\operatorname{Spec} F_q} \operatorname{Spec} K$ given by $F = F_0 \times 1$. The Frobenius morphism $F: X \to X$ induces a linear map $F: H^i_c(X, \mathbb{Q}_l) \to H^i_c(X, \mathbb{Q}_l)$ where l is a prime different from p. We define X^F to be the set of closed points x of X such that $F(x) = x$.

We can now state Grothendieck's trace formula. Let X be a scheme which is separated and of finite type over an algebraically closed field of characteristic p

and let l be a prime with $l \neq p$. Let F be the Frobenius morphism of an F_q-structure on X for some power q of p. Then we have

$$|X^F| = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{trace}(F, H_c^i(X, \mathbb{Q}_l)).$$

If F is the Frobenius morphism of an F_q-structure on X then F^n is the Frobenius morphism of an F_{q^n}-structure on X, for $n \geqslant 1$. Thus we obtain a formula for $|X^{F^n}|$ for each $n \geqslant 1$. Such a formula was needed to derive the first of the Weil conjectures.

In order to describe the remaining properties of $H_c^i(X, \mathbb{Q}_l)$ needed in the proof of the Weil conjectures we assume in addition that X is the scheme associated with a nonsingular irreducible projective variety. In particular X is proper over K and so we have $H^i(X, \mathbb{Q}_l) = H_c^i(X, \mathbb{Q}_l)$. Under these assumptions it can be shown that $\dim H^{2d}(X, \mathbb{Q}_l) = 1$ where $d = \dim X$ and that there is a cup-product mapping

$$H^i(X, \mathbb{Q}_l) \times H^j(X, \mathbb{Q}_l) \to H^{i+j}(X, \mathbb{Q}_l)$$

for all i, j. In particular we have a cup-product map

$$H^i(X, \mathbb{Q}_l) \times H^{2d-i}(X, \mathbb{Q}_l) \to H^{2d}(X, \mathbb{Q}_l)$$

for each i with $0 \leqslant i \leqslant 2d$, which can be shown to be a perfect pairing.

We also have a comparison theorem which interprets the numbers $B_i = \dim H^i(X, \mathbb{Q}_l)$. Suppose X is the scheme associated with a nonsingular irreducible projective variety defined over F_q. Suppose also that this variety is obtained from a variety V defined over a ring A of algebraic integers by reduction modulo a prime ideal $\mathfrak{P}$ of A containing p. Let $V_{\mathbb{C}}$ be the complex manifold obtained from V by extending the scalars from A to $\mathbb{C}$. Then we have

$$\dim H^i(X, \mathbb{Q}_l) = \dim H^i(V_{\mathbb{C}}, \mathbb{C})$$

where $H^i(V_{\mathbb{C}}, \mathbb{C})$ is the classical cohomology of the complex manifold $V_{\mathbb{C}}$.

Finally it was shown by Deligne that the eigenvalues $\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{iB_i}$ of F on $H^i(X, \mathbb{Q}_l)$ are algebraic integers with $|\alpha_{ij}| = q^{i/2}$ and that the polynomial

$$P_i(t) = (1 - \alpha_{i1}t)(1 - \alpha_{i2}t) \ldots (1 - \alpha_{iB_i}t)$$

has coefficients in $\mathbb{Z}$. The eigenvalues α_{ij} are independent of the choice of the prime $l \neq p$.

Thus the l-adic cohomology modules $H^i(X, \mathbb{Q}_l)$ satisfy all the conditions needed to prove the Weil conjectures when X is the scheme associated to a nonsingular irreducible projective variety defined over F_q.

(h) PROPERTIES OF THE LEFSCHETZ NUMBER

Let X be a separated scheme of finite type over an algebraically closed field K of characteristic p and let l be a prime with $l \neq p$. Let $g: X \to X$ be an automorphism of X of finite order. Then g induces a nonsingular linear map

$$g^*: H_c^i(X, \mathbb{Q}_l) \to H_c^i(X, \mathbb{Q}_l) \qquad \text{for each } i.$$

The Lefschetz number $\mathscr{L}(g, X)$ is defined by

$$\mathscr{L}(g, X) = \sum_i (-1)^i \operatorname{trace}(g^{*-1}; H_c^i(X, \mathbb{Q}_l)).$$

Several of the properties which we state in section 7.1 as axioms involve just the Lefschetz number $\mathscr{L}(g, X)$ rather than the action of g on the l-adic cohomology module $H_c^i(X, \mathbb{Q}_l)$. This is because we have been concerned mainly with character values rather than the representations afforded by the modules $H_c^i(X, \mathbb{Q}_l)$.

Now there is a way of defining the Lefschetz number $\mathscr{L}(g, X)$ which does not refer at all to the l-adic cohomology modules $H_c^i(X, \mathbb{Q}_l)$ and which can be used in practice to calculate $\mathscr{L}(g, X)$ and to prove some of the theoretical properties which it satisfies. We shall now describe this alternative definition, which can be derived from Grothendieck's trace formula.

Let g be an automorphism of X of finite order. Then we can find a Frobenius morphism F of an F_q-structure on X for some power q of p such that F commutes with g. In fact if F is any Frobenius morphism on X some power of F will commute with g. We assume that $F: X \to X$ is a Frobenius morphism satisfying $Fg = gF$. Then Fg is itself a Frobenius morphism of some F_q-structure on X. Also, for any $n \geqslant 1$, F^n is a Frobenius morphism on X which commutes with g and so $F^n g$ will be a Frobenius morphism on X. The same applies to $F^n g^{-1}$. $F^n g^{-1}$ induces a linear map denoted by $F^n g^{*-1}$ on $H_c^i(X, \mathbb{Q}_l)$ for each i and by Grothendieck's trace formula we have

$$|X^{F^n g^{-1}}| = \sum_i (-1)^i \operatorname{trace}(F^n g^{*-1}, H_c^i(X, \mathbb{Q}_l))$$

for $n \geqslant 1$.

We consider the power series given by

$$-\sum_{n=1}^{\infty} |X^{F^n g^{-1}}| t^n.$$

Using Grothendieck's trace formula we have

$$\begin{aligned} -\sum_{n=1}^{\infty} |X^{F^n g^{-1}}| t^n &= \sum_i (-1)^i \operatorname{trace}\left(-\sum_{n=1}^{\infty} t^n F^n g^{*-1}, H_c^i(X, \mathbb{Q}_l)\right) \\ &= \sum_i (-1)^i \operatorname{trace}\left(\frac{-tFg^{*-1}}{1-tF}, H_c^i(X, \mathbb{Q}_l)\right). \end{aligned}$$

Now F has eigenvalues $\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{iB_i}$ on $H_c^i(X, \mathbb{Q}_l)$. Let g^{*-1} have eigenvalues $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{iB_i}$ on this space. Then the eigenvalues of the map $-tFg^{*-1}/(1-tF)$ are the numbers $-t\alpha_{ij}\lambda_{ij}/(1-t\alpha_{ij})$ for $j = 1, \ldots, B_i$. Thus we have

$$-\sum_{n=1}^{\infty} |X^{F^n g^{-1}}| t^n = \sum_{i=0}^{2d} (-1)^i \sum_{j=1}^{B_i} \frac{-t\alpha_{ij}\lambda_{ij}}{1-t\alpha_{ij}}.$$

This is therefore a rational function in t. Let $R(t)$ be the rational function given by

$$R(t) = \sum_{i=0}^{2d} (-1)^i \sum_{j=1}^{B_i} \frac{-t\alpha_{ij}\lambda_{ij}}{1 - t\alpha_{ij}}.$$

Then $R(t)$ is the rational function determined by the power series $-\sum_{n=1}^{\infty} |X^{F^n g^{-1}}| t^n$. $R(t)$ has no pole at ∞. In fact we have

$$\begin{aligned}\lim_{t \to \infty} R(t) &= \sum_{i=0}^{2d} (-1)^i \sum_{j=1}^{B_i} \lambda_{ij} \\ &= \sum_i (-1)^i \operatorname{trace}(g^{*-1}, H_c^i(X, \mathbb{Q}_l)) = \mathscr{L}(g, X).\end{aligned}$$

Thus the Lefschetz number may be described, without reference to l-adic cohomology, by the formula

$$\mathscr{L}(g, X) = \lim_{t \to \infty} R(t)$$

where $R(t)$ is the rational function whose power series expansion is

$$-\sum_{n=1}^{\infty} |X^{F^n g^{-1}}| t^n.$$

This formula may be used to derive several useful properties of the Lefschetz number. In particular $\mathscr{L}(g, X)$ is independent of the choice of $l \neq p$, and $\mathscr{L}(g, X) \in \mathbb{Z}$. We also have $\mathscr{L}(g, X) = \mathscr{L}(g^{-1}, X)$.

The detailed properties of the l-adic cohomology and the Lefschetz number which are needed for the representation theory of finite groups of Lie type are described in section 7.1. We have stated these properties in terms of the groups $H_c^i(X, \bar{\mathbb{Q}}_l)$ where X is an algebraic variety over $K = \bar{F}_p$. These groups are related to the groups $H_c^i(X, \mathbb{Q}_l)$ defined in the present appendix by identifying the variety X with the scheme associated to X, and using the formula

$$H_c^i(X, \bar{\mathbb{Q}}_l) = H_c^i(X, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \bar{\mathbb{Q}}_l.$$

Most of the properties described in section 7.1 are concerned with automorphisms of X, but special care must be taken with the property 7.1.5 for which this is not so. In 7.1.5 we are dealing with a morphism between algebraic varieties which is not in general a finite morphism. Thus we may not be able to pass directly to a corresponding map between l-adic cohomology groups. However by using the description of the Lefschetz number which does not refer to the l-adic cohomology it is possible to prove the invariance of the Lefschetz number which is asserted in 7.1.5. A proof can be found in Lusztig [10].

We have of course mentioned only a few of the most basic properties of the l-adic cohomology groups. The reader who wishes to study these groups in detail is referred to M. Artin *et al.* [1] (S.G.A 4), Deligne [1] (S.G.A $4\frac{1}{2}$), Grothendieck *et al.* [1] (S.G.A 5) and to the book of Milne [1].

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Index of Notation

Symbol	*Meaning*	*Page of definition*
A	The matrix of Cartan integers	23
A_{ij}	A Cartan integer	23
A_l	The type of a simple group, linear of degree $l + 1$	24
$(A_l)_{sc}(q)$	The simply-connected group of type A_l over $\mathbb{F}_q$	39
$(A_l)_{ad}(q)$	The adjoint group of type A_l over $\mathbb{F}_q$	39
$({}^2A_l)_{sc}(q^2)$	The simply-connected twisted group of type A_l over $\mathbb{F}_{q^2}$	39
$({}^2A_l)_{ad}(q^2)$	The adjoint twisted group of type A_l over $\mathbb{F}_{q^2}$	39
Ad	The adjoint representation of an algebraic group	10
ad	The adjoint representation of a Lie algebra	11
$A(u)$	The component group $C(u)/C(u)^\circ$	388
B	A Borel subgroup	15
B^-	The opposite Borel subgroup to B	18
B^F	The subgroup of F-stable elements in B	33
B_w	A basis vector of the endomorphism algebra	308
B_l	The type of a simple group, orthogonal of degree $2l + 1$	24
$(B_l)_{sc}(q)$	The simply-connected group of type B_l over $\mathbb{F}_q$	39
$(B_l)_{ad}(q)$	The adjoint group of type B_l over $\mathbb{F}_q$	39
${}^2B_2(q^2)$	The Suzuki group of type B_2 over $\mathbb{F}_{q^2}$, $q^2 = 2^{2n+1}$	41
$B(x, y)$	The Killing form	11
$\mathfrak{B}$	The variety of Borel subgroups	178
$\mathfrak{B}_u$	The variety of Borel subgroups containing u	178
$\mathfrak{B}_w$	The set of Borel subgroups B with B, $F(B)$ in relative position w	246
C_l	The type of a simple group, symplectic of degree $2l$	24
$(C_l)_{sc}(q)$	The simply-connected group of type C_l over $\mathbb{F}_q$	39
$(C_l)_{ad}(q)$	The adjoint group of type C_l over $\mathbb{F}_q$	39
$CSp_{2l}(K)$	The conformal symplectic group of degree $2l$ over K	25
$CO_n(K)$	The conformal orthogonal group of degree n over K	26
$CSp_{2l}(q)$	The conformal symplectic group of degree $2l$ over $\mathbb{F}_q$	39
$CO_n(q)$	The conformal orthogonal group of degree n over $\mathbb{F}_q$	39
$CO^-_{2l}(q)$	The conformal orthogonal group derived from $O^-_{2l}(q)$	40
$C_{J,\phi}$	A complement to the reflection subgroup $R_{J,\phi}$ of $W^{J,\phi}$	331
$C_G(H)$	The centralizer in G of the subgroup H	27
$C_G(x)$	The centralizer in G of the element x in G	27
$C_{\mathfrak{L}(G)}(x)$	The centralizer in $\mathfrak{L}(G)$ of the element x in G	28
$C_G(a)$	The centralizer in G of the element a in $\mathfrak{L}(G)$	28

Symbol	*Meaning*	*Page of definition*
$C_{\mathfrak{L}(G)}(a)$	The centralizer in $\mathfrak{L}(G)$ of the element a in $\mathfrak{L}(G)$	28
$C_{W,F}(w)$	The F-centralizer of w	87
D_l	The type of a simple group, orthogonal of degree $2l$	24
$(D_l)_{sc}(q)$	The simply-connected group of type D_l over $\mathbb{F}_q$	39
$(D_l)_{ad}(q)$	The adjoint group of type D_l over $\mathbb{F}_q$	39
$({}^2D_l)_{sc}(q^2)$	The simply-connected twisted group of type D_l over $\mathbb{F}_{q^2}$	39
$({}^2D_l)_{ad}(q^2)$	The adjoint twisted group of type D_l over $\mathbb{F}_{q^2}$	40
${}^3D_4(q^3)$	The triality twisted group of type D_4 over $\mathbb{F}_{q^3}$	40
$d_1, \ldots, d_l$	The degrees of the basic polynomial invariants of W	48
D_J	The set of distinguished coset representatives of W_J in W	47
$D_{J,K}$	The set of distinguished double coset representatives of W with respect to W_J, W_K	64
$(\mathrm{d}\phi)_v$	The differential of ϕ at v	9
$\mathrm{d}\phi$	The differential of ϕ at 1	10
d_ψ	The generic degree of the character ψ	362
E_{ij}	The elementary matrix with 1 in the (i,j)-position	20
E_6	The type of a simple group, exceptional of rank 6	24
E_7	The type of a simple group, exceptional of rank 7	24
E_8	The type of a simple group, exceptional of rank 8	24
$(E_6)_{sc}(q)$	The simply-connected group of type E_6 over $\mathbb{F}_q$	40
$(E_6)_{ad}(q)$	The adjoint group of type E_6 over $\mathbb{F}_q$	40
$({}^2E_6)_{sc}(q^2)$	The simply-connected twisted group of type E_6 over $\mathbb{F}_{q^2}$	40
$({}^2E_6)_{ad}(q^2)$	The adjoint twisted group of type E_6 over $\mathbb{F}_{q^2}$	40
$(E_7)_{sc}(q)$	The simply-connected group of type E_7 over $\mathbb{F}_q$	41
$(E_7)_{ad}(q)$	The adjoint group of type E_7 over $\mathbb{F}_q$	41
$E_8(q)$	The group of type E_8 over $\mathbb{F}_q$	41
$\mathfrak{E}$	The endomorphism algebra of an induced module	255, 306
f	The determinant of the Cartan matrix	27
F	A Frobenius map	31
F_0	The map given by $F = qF_0$	35
$\mathbb{F}_q$	The finite field with q elements	39
F_4	The type of a simple group, exceptional of rank 4	24
$F_4(q)$	The group of type F_4 over $\mathbb{F}_q$	50
${}^2F_4(q^2)$	The Ree group of type F_4 over $\mathbb{F}_{q^2}$, $q^2 = 2^{2n+1}$	41
$\mathfrak{I}(J, \rho)$	A module affording the induced representation $\rho_{P_J}{}^G$	305
G	A group, or an algebraic group	6
G^0	The connected component of G containing 1	7
$\mathbf{G}_a$	The additive group	7
$\mathbf{G}_m$	The multiplicative group	7
G/H	The quotient variety of G with respect to the subgroup H	13
$G' = [G, G]$	The commutator subgroup of G	16
G_{ad}	The adjoint group isogenous to G	25
G_{sc}	The simply-connected group isogenous to G	25
$\mathfrak{g}$	The Lie algebra of G	27
G^F	The subgroup of G of elements fixed by F	31
G^*	The dual group of G	112
$GL_n(K)$	The general linear group of degree n over K	6
$\mathfrak{gl}_n(K)$	The general linear Lie algebra of degree n over K	10
$GL_n(q)$	The general linear group of degree n over $\mathbb{F}_q$	39
G_2	The type of a simple group, exceptional of rank 2	24
$G_2(q)$	The group of type G_2 over $\mathbb{F}_q$	40

Symbol	*Meaning*	*Page of definition*
${}^2G_2(q^2)$	The Ree group of type G_2 over $\mathbb{F}_{q^2}$, $q^2 = 3^{2n+1}$	41
h	The Coxeter number	20
H	The subgroup $B \cap N$ of a group with BN-pair	42
H_α	The hyperplane orthogonal to α	45
$HS_{2l}(K)$	The half-spin group of degree $2l$ over K	26
$HS_{2l}(q)$	The half-spin group of degree $2l$ over $\mathbb{F}_q$	39
$H_c^i(X, \bar{\mathbb{Q}}_l)$	The ith l-adic cohomology group of X with compact support	202, 503
I	The index set $\{1, 2, \ldots, l\}$	34
I'	The set of ρ-orbits on the index set I	195
$\sqrt{I}$	The radical of the ideal I	2
$\mathscr{I}(V)$	The ideal of functions vanishing on the affine variety V	1
$\mathscr{I}_p(V)$	The homogeneous ideal of functions vanishing on the projective variety V	3
ind W	The order of U_w	324
J	A subset of the index set I	34
$j_{W'}{}^W$	Truncated induction from W' to W	367
K	An algebraically closed field	1
K^*	The multiplicative group of K	7
K^n	The vector space of n-tuples over K	1
$K[x_1, \ldots, x_n]$	The polynomial rang in n variables over K	1
$K[V]$	The coordinate ring of the affine variety V	2
$K(V)$	The function field of the affine variety V	2
$K(X)$	The function field of the irreducible variety X	6
L	Lang's map	32
$\mathfrak{L}(G)$	The Lie algebra of G	10
l	The semisimple rank of G	73
$l(w)$	The length of the element $w \in W$	43
L_J	The standard Levi subgroup corresponding to the subset J of I	60
$\mathfrak{l}_J$	The Lie algebra of L_J	166
$\mathscr{L}(g, X)$	The Lefschetz number of g on X	202, 505
m_v	The maximal ideal of the local ring $O_{V,v}$	8
$M(\Gamma)$	The set of pairs (x, σ) parametrizing the unipotent characters in a family	383
N	The number of positive roots	74
N	The normalizer of a maximal torus	22
N^F	The subgroup of N of elements fixed by F	34
$N_G(H)$	The normalizer in G of the subgroup H	17
n_i	An element of N mapping to $s_i \in W$	22
n_w	An element of N mapping to $w \in W$	36
n_0	An element of N mapping to $w_0 \in W$	50
n_J	An element of N^F mapping to $s_J \in W^F$	34
$N_{J,K}$	A set of double coset representatives of G with respect to P_J, P_K	68
$\mathfrak{n}$	The Lie algebra $\mathfrak{L}(U)$	27
$\mathfrak{n}^-$	The Lie algebra $\mathfrak{L}(U^-)$	27
$\mathfrak{N}$	The nilpotent variety	29
$N(T, T')$	The set of $g \in G$ with $T^g = T'$.	212
O_U	A K-algebra of functions on U	2, 3
$O_{V,v}$	The localization of the coordinate ring $K[V]$ at v	8
$O_n(K)$	The orthogonal group of degree n over K	25

Symbol	*Meaning*	*Page of definition*
$O_n(q)$	The orthogonal group of degree n over $\mathbb{F}_q$ corresponding to a quadratic form of maximal index	39
$O_{2l}^-(q)$	The orthogonal group of degree $2l$ over $\mathbb{F}_q$ corresponding to a quadratic form not of maximal index	40
$\mathfrak{O}_w$	The set of pairs of Borel subgroups in relative position w	246
P	A parabolic subgroup	43
P_J	The parabolic subgroup BN_JB	43
$P_n(K)$	Projective space of dimension n over K	3
p	The characteristic of the field K	28
$\mathfrak{p}_J$	The Lie algebra of P_J	138
$PGL_n(K)$	The projective general linear group of degree n over K	25
$PCSp_{2l}(K)$	The projective conformal symplectic group of degree $2l$ over K	25
$PGL_n(q)$	The projective general linear group of degree n over $\mathbb{F}_q$	39
$PCSp_{2l}(q)$	The projective conformal symplectic group of degree $2l$ over $\mathbb{F}_q$	39
$PU_n(q)$	The projective unitary group of degree n over $\mathbb{F}_q$	39
$\mathfrak{P}$	The algebra of polynomial functions on a vector space	48
$\tilde{P}_\phi(t)$	The generic degree of ϕ in one variable	369
$P_\phi(t)$	The fake degree of ϕ	370
$P_{y,w}(t)$	The Kazhdan–Lusztig polynomial corresponding to $y, w \in W$	386
q	The absolute value of the eigenvalues of F on X	35
$\mathbb{Q}_{p'}$	The additive group of rationals with denominator prime to p	80
$\mathbb{Q}_l$	The field of l-adic numbers	201
$\bar{\mathbb{Q}}_l$	The algebraic closure of $\mathbb{Q}_l$	202
$Q_T(u)$	The Green function $R_{T,1}(u)$	212
$R(G)$	The radical of G	16
$R_u(G)$	The unipotent radical of G	16
$R_{T,\theta}$	A Deligne–Lusztig generalized character	207
$R_{J,\phi}$	A normal reflection subgroup of $W^{J,\phi}$	331
R_w	$R_{T,\theta}$ when $T = T_w$ and $\theta = 1$	382
R_ϕ	$\dfrac{1}{\lvert W\rvert}\sum_w \phi(w)R_w$	383
s_i	The simple reflection w_{α_i} in W	20
s_J	The element of maximal length in W_J	34
$SL_n(K)$	The special linear group of degree n over K	16
$Sp_{2l}(K)$	The symplectic group of degree $2l$ over K	25
$\mathrm{Spin}_n(K)$	The spin group of degree n over K	25
$SO_n(K)$	The special orthogonal group of degree n over K	25
$SL_n(q)$	The special linear group of degree n over $\mathbb{F}_q$	39
$\mathrm{Sp}_{2l}(q)$	The symplectic group of degree $2l$ over $\mathbb{F}_q$	39
$\mathrm{Spin}_n(q)$	The spin group of degree n over $\mathbb{F}_q$	39
$SO_n(q)$	The special orthogonal group of degree n over $\mathbb{F}_q$	39
$SU_n(q)$	The special unitary group of degree n over $\mathbb{F}_q$	39
$SO_{2l}^-(q)$	The special orthogonal group derived from $O_{2l}^-(q)$	40
$\mathfrak{sl}_n(K)$	The Lie algebra of $n \times n$ matrices over K of trace 0	138
St	The Steinberg character	187
$\mathcal{S}_W$	The set of special irreducible characters of W	374, 388
$\bar{\mathcal{S}}_W$	A set of irreducible characters of W in bijective correspondence with unipotent classes of G	388
Spec A	The set of prime ideals of A	496
$T_v(V)$	The tangent space to V at v	8
T	An algebraic torus	14

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