4CCM113A Linear Algebra and Geometry I

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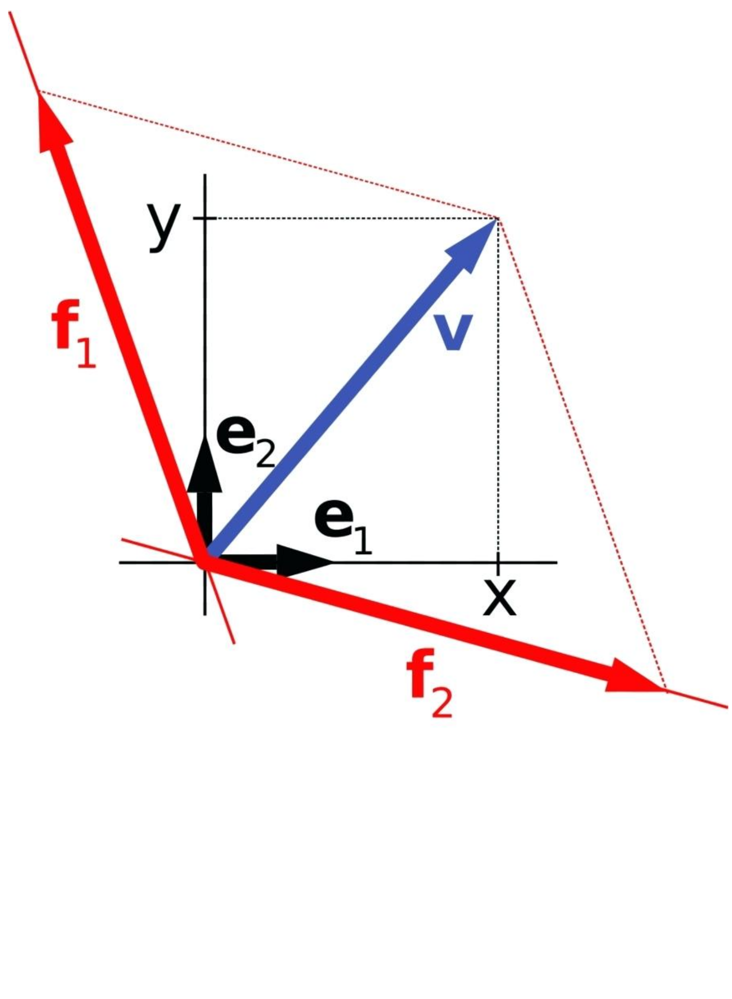
King's College London 2022



Preface

These notes were written in the Autumn term 2018 to accompany the lectures for the module 4CCM113A Linear Algebra and Gemetry I. Much of the material and presentation is based on the excellent book *Linear algebra Done Wrong* by Sergei Treil and I am grateful to him for making this book freely available. These notes were also partly inspired by the lecture notes for the previous iteration of this module *Linear Methods* by George Papadopolous and the book *Introduction to Linear Algebra* by Gilbert Strang. I would strongly recommend these sources to students of this module.

I take this opportunity to thank Kwok-Wing Tsoi for careful reading of these notes and many valuable suggestions throughout their preparation. I also wish to thank the many students, in particular Tamanna Seghal and Senan Sekhon, that took the time to read through earlier drafts of the notes and pointed out typos.

Nazar Miheisi, September 2019

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Chapter 0

Foundational Material

Notation

We will use the following sets of numbers:

$$\begin{aligned} \mathbb{N} = &\text{ The natural numbers } = \{1, 2, 3, 4, \dots\}, \\ \mathbb{Z} = &\text{ The integers } = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}, \\ \mathbb{R} = &\text{ The real numbers.} \end{aligned}$$

0.1 Summation Notation

Throughout this course (and in many other courses) we will perform some standard manipulations of summations. Here we quickly review the sigma notation for summation that is used; it is necessary to be familiar with this.

Let $a_0, a_1, \dots, a_n$ be a collection of numbers. Recall that

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n.$$

Notice that while the summation variable (or summation index) k appears on the left hand side, once the sum is written out in full there is no appearance of k in the sum. Consequently the summation variable k is often called a 'dummy variable' because it does not appear in the full expression. You might think of the summation variable as a label for each of the terms on the right

Remark 0.1.1. It is a good rule of thumb when working with sigma notations: "if in doubt, write it out".

Example 0.1.2. *Evaluate the sum*

$$\sum_{k=2}^{5}(-1)^k(k+1)^2.$$

Solution.

$$\begin{aligned}\sum_{k=2}^{5}(-1)^k(k+1)^2 &= (-1)^2\times 3^2+(-1)^3\times 4^2+(-1)^4\times 5^2+(-1)^5\times 6^2\\ &= 9-16+25-36=-18.\end{aligned}$$

□

Exercise. *Express the sum* $1-3^2+5^4-7^6+9^8-11^{10}$ *using sigma notation.*

0.1.1 Relabelling the indices

Since the summation index (k in the example above) is just used to label the terms in the sum, we can choose any index variable we like, i.e

$$\sum_{k=0}^{n}a_k=\sum_{j=0}^{n}a_j=a_0+a_1+\cdots+a_n.$$

i, j, k, etc. are usually popular choices for index variables, but any symbol can be used.

In addition to renaming indices, you can also shift them, provided you shift the bounds to match. For example, rewriting

$$\sum_{k=1}^{n}(k-1)\quad\text{as}\quad\sum_{k=0}^{n-1}k$$

can make the sum more convenient to work with.

0.1.2 Linearity of sums

Let $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ be collections of numbers. Then

$$\begin{aligned}\sum_{k=0}^{n}(a_k + b_k) &= (a_0 + b_0) + (a_1 + b_1) + \ldots (a_n + b_n) \\ &= (a_0 + a_1 + \ldots a_n) + (b_0 + b_1 + \cdots + b_n) \\ &= \sum_{k=0}^{n} a_k + \sum_{k=0}^{n} b_k.\end{aligned}$$

This means that sums inside sums can be split.

Let C be a constant. Then

$$\begin{aligned}\sum_{k=0}^{n} Ca_k &= Ca_0 + Ca_1 + \ldots Ca_n \\ &= C(a_0 + a_1 + \ldots a_n) \\ &= C\sum_{k=0}^{n} a_k,\end{aligned}$$

so constant factors can be pulled out of the sum.

0.1.3 Some standard sums

- **Sum of ones**:

$$\sum_{k=1}^{n} 1 = \underbrace{1 + 1 + \ldots 1}_{n \text{ terms}} = n.$$

- **Sum of integers up to n**:

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1).$$

Indeed we can check that

$$\begin{aligned}\sum_{k=1}^{n} k &= 1 + 2 + \cdots + n \\ &= \frac{1}{2}\{(1 + 2 + \cdots + n) + (n + (n-1) + \cdots + 1)\} \\ &= \frac{1}{2}\{\underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n \text{ terms}}\} \\ &= \frac{1}{2}n(n+1).\end{aligned}$$

- **Sum of a geometric series**:

$$\sum_{k=1}^{n} r^k = \frac{r - r^{n+1}}{1 - r}.$$

Indeed

$$\sum_{k=1}^{n} r^k = r + r^2 + \cdots + r^n$$

and

$$r\sum_{k=1}^{n} r^k = r^2 + r^3 + \cdots + r^{n+1}.$$

Taking the difference we get

$$(1 - r)\sum_{k=1}^{n} r^k = r - r^{n+1},$$

and so

$$\sum_{k=1}^{n} r^k = \frac{r - r^{n+1}}{1 - r}.$$

Example 0.1.3. *Use the properties above to evaluate the sum*

$$\sum_{k=1}^{5}(2^k - 4k + 3).$$

Solution.

$$\begin{aligned}\sum_{k=1}^{5}(2^k-4k+3) &= \sum_{k=1}^{5}2^k+\sum_{k=1}^{5}k+4\sum_{k=1}^{5}1\\ &= \frac{2-2^6}{1-2}-(4\times\frac{1}{2}\times 5\times 6)+(3\times 5)\\ &= 62-60+15=17.\end{aligned}$$

□

Exercise. *Derive a formula for the sum of the first n odd numbers.*

0.1.4 Double sums

Sometimes the collection of numbers that we are summing is indexed by two variables, i.e we want to sum up $a_{00}, a_{01}\ldots, a_{0n}, a_{10}, a_{11}, \ldots a_{1n}\ldots a_{m0}, a_{m_1}, \ldots a_{mn}$. Then

$$\begin{aligned}\sum_{j=0}^{m}\sum_{k=0}^{n}a_{jk} &= \sum_{j=0}^{m}(a_{j0}+a_{j1}+\cdots+a_{jn})\\ &= a_{00}+a_{01}+\cdots+a_{0n}\\ &+ a_{10}+a_{11}+\cdots+a_{1n}\\ &\ \ \vdots \qquad\qquad \vdots\\ &+ a_{m0}+a_{m1}+\cdots+a_{mn}.\end{aligned}$$

Here, as well as changing summation index, we can also swap the order of summation without changing the sum, i.e

$$\sum_{j=0}^{m}\sum_{k=0}^{n}a_{jk}=\sum_{k=0}^{n}\sum_{j=0}^{m}a_{jk}.$$

A common way of getting a double sum in a calculation occurs when we have to multiply two sums:

$$\begin{aligned}
\left(\sum_{k=0}^{n} a_k\right)\left(\sum_{k=0}^{m} b_k\right) &= (a_0 + a_1 + \ldots a_n)(b_0 + b_1 + \cdots + b_m) \\
&= a_0(b_0 + b_1 + \cdots + b_m) + a_1(b_0 + b_1 + \cdots + b_m) \\
&\qquad\qquad + \cdots + a_n(b_0 + b_1 + \cdots + b_m) \\
&= a_0b_0 + a_0b_1 + \cdots + a_0b_m \\
&+ a_1b_0 + a_1b_1 + \cdots + a_1b_m \\
&\ \ \vdots \qquad\quad \vdots \\
&+ a_nb_0 + a_nb_1 + \cdots + a_nb_m \\
&= \sum_{j=0}^{n}\sum_{k=0}^{m} a_jb_k.
\end{aligned}$$

Observe that while the summation indices on the left hand side are the same, they are different on the right. This is necessary because when we multiply out, we have to sum over all possible pairs of a_k's and b_k's, not just the when the indices are the same.

In summary we have the identities [1]

$$\left(\sum_{k=0}^{n} a_k\right)\left(\sum_{k=0}^{m} b_k\right) = \sum_{j=0}^{n} a_j \sum_{k=0}^{m} b_k = \sum_{k=0}^{m} b_k \sum_{j=0}^{n} a_j = \sum_{j=0}^{n}\sum_{k=0}^{m} a_jb_k.$$

Remark 0.1.4. We can also have triple sums, quadruple sums etc.

Exercise. *(i) Evaluate the sum*

$$\sum_{j=1}^{n}\sum_{k=1}^{m} 1.$$

(ii) Show that

$$\sum_{j=1}^{n}\sum_{k=1}^{m}(a_j + b_k) = m\sum_{j=1}^{n} a_j + n\sum_{k=1}^{m} b_k.$$

[1] These identities do not generally hold for infinite sums. You will do more on infinite sums in *Sequences and Series*.

0.2 Complex Numbers

The real numbers $\mathbb{R}$ have a striking shortcoming: there are many quadratic equations that have no solutions. In particular, the equation

$$x^2 + 1 = 0$$

has no solution in $\mathbb{R}$. This can be remedied by 'inventing' a new number which is a solution to this equation, and so we introduce [2]

$$i = \sqrt{-1}.$$

Remarkably, with the addition of this one number, we can solve all quadratic equations, and indeed all polynomials of any degree!

Definition 0.2.1. The set of all numbers of the form $a + ib$, where $a, b \in \mathbb{R}$ are call the *complex numbers* and denoted $\mathbb{C}$, i.e.

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}.$$

Remark 0.2.2. 1. Note that $z = -i$ is also a solution of $z^2 + 1 = 0$.

2. $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. In particular, $i^3 + i = 0$ (i^3 is the *additive inverse* of i) and $i^3 \times i = 1$ (i^3 is the *multiplicative inverse* of i).

3. $\mathbb{C}$ contains all real numbers since every real number is of the form $a + i0$.

4. A complex number of the form $0 + ib$ is called an *imaginary* number.

5. There is no natural way to order the complex numbers. This sometimes makes them more difficult to work with than the real numbers.

Definition 0.2.3. Let $z = a + ib$. We define

$$\begin{aligned} \mathrm{Re}\,(z) &= a; \quad \text{this is called the } \textit{real part} \text{ of } z \\ \mathrm{Im}\,(z) &= b; \quad \text{this is called the } \textit{imaginary part} \text{ of } z \end{aligned}$$

Note that both $\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z)$ are real numbers.

[2] Engineers usually use j rather than i to denote $\sqrt{-1}$.

0.2.1 Fundamental theorem of algebra

As we mentioned earlier, every non-constant polynomial equation has a solution in $\mathbb{C}$. In fact we can say more. Let P be a polynomial of degree n, i.e

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots a_1 z + a_0,$$

where $a_0, \dots, a_n \in \mathbb{C}$. Then the equation $P(z) = 0$ has n solutions $w_1, w_2, \dots, w_n$ (some of the solutions may be repeated). This means that $P(z)$ can be factorised as

$$P(z) = a_n (z - w_1)(z - w_2) \dots (z - w_n).$$

This fact is known as the *fundamental theorem of algebra*. Unfortunately the proof is beyond the scope of this course, but you are encouraged to look it up. A number system (or *field*) that satisfies the above property is called *algebraically closed*. Thus $\mathbb{C}$ is algebraically closed whereas $\mathbb{R}$ is not.

0.2.2 Arithmetic of complex numbers

When manipulating complex numbers, we simply follow the rules for real numbers together with the additional rule '$i^2 = -1$'. Let $z = a + ib$ and $w = c + id$, where $a, b, c, d \in \mathbb{R}$. Then we have

Addition/Substraction : $z \pm w = (a + ib) \pm (c + id) = (a \pm c) + i(b \pm d)$;
Multiplication : $zw = (a + ib)(c + id) = ac + ibc + iad + i^2 bd = (ac - bd) + i(bc + ad)$.

Example 0.2.4. *Write each of the following complex numbers in the form $a + ib$.*

(i) $(9 + i) + (11 - i)$

(ii) $(-1 - i)(-8 + i)$

(iii) $\dfrac{2 + i}{3} \times \dfrac{1 + 2i}{5}$

Solution. (i) $(9 + i) + (11 - i) = 20 \quad (= 20 + 0i)$.

(ii) $(-1 - i)(-8 + i) = 8 - i + 8i - i^2 = 8 + 7i + 1 = 9 + 7i$.

(iii) $\dfrac{2 + i}{3} \times \dfrac{1 + 2i}{5} = \dfrac{(2 + i)(1 + 2i)}{15} = \dfrac{2 + 5i + 2i^2}{15} = \dfrac{5i}{15} = \dfrac{1}{3} i$.

$\square$

Division is a little trickier. If we want to express

$$\frac{z}{w} = \frac{a+ib}{c+id}$$

in terms of its real and imaginary parts, we need to make the denominator real. We do this by multiplying the numerator and denominator by $c-id$ (because $(c+id)(c-id) = c^2+d^2$ which is a real number). Hence we have

$$\textbf{Division}: \quad \frac{z}{w} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-da}{c^2+d^2}.$$

This looks complicated but it is easy to use, as the next example shows.

Example 0.2.5. *Write each of the following complex numbers in the form $a+ib$.*

(i) $\dfrac{5+2i}{4-3i}$

(ii) $\dfrac{i-4}{3i-7}$

Solution. (i) $\dfrac{5+2i}{4-3i} = \dfrac{(5+2i)(4+3i)}{(4-3i)(4+3i)} = \dfrac{14+23i}{25} = \dfrac{14}{25} + \dfrac{23}{25}i.$

(ii) $\dfrac{i-4}{3i-7} = \dfrac{(i-4)(-3i-7)}{(3i-7)(-3i-7)} = \dfrac{31+5i}{58} = \dfrac{31}{58} + \dfrac{5}{58}i.$

□

Exercise. *Write $1/i$ in the form $a+ib$.*

0.2.3 Complex conjugate and modulus

When dividing by $c+id$, it was useful to invoke the number $c-id$. This operation of changing the sign of the imaginary part of a complex number will prove to be useful more generally; we call this *complex conjugation.*

Definition 0.2.6. Let $z = a+ib$, where $a, b \in \mathbb{R}$. The *complex conjugate* of z is denoted $\overline{z}$ (or sometimes z^*), and is defined by

$$\overline{z} = a - ib.$$

Remark 0.2.7. If $x \in \mathbb{R}$, then $\overline{x} = x$.

Properties of complex conjugation:

Let $z = a + ib$ and $w = c + id$, where $a, b, c, d \in \mathbb{R}$.

1. $\overline{z+w} = \overline{(a+ib)+(c+id)} = \overline{(a+c)+i(b+d)} = (a+c) - i(b+d) = \overline{z} + \overline{w}$.
2. $\overline{zw} = \overline{(a+ib)(c+id)} = (ac-bd) - i(bc+ad) = (a-ib)(c-id) = \overline{z}\ \overline{w}$.
3. $\overline{\overline{z}} = \overline{a-ib} = a + ib = z$.
4. $z + \overline{z} = (a+ib) + (a-ib) = 2a = 2\mathrm{Re}\,(z)$. This gives the formula
$$\mathrm{Re}\,(z) = \frac{1}{2}(z + \overline{z}).$$
5. $z - \overline{z} = (a+ib) - (a-ib) = 2ib = 2i\mathrm{Im}\,(z)$. This gives the formula
$$\mathrm{Im}\,(z) = \frac{1}{2i}(z - \overline{z}).$$

Definition 0.2.8. Let $z = a + ib$. The *modulus* or *absolute value* of z is denoted $|z|$ and defined by
$$|z| = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}}.$$

Properties of the modulus: Let $z = a + ib$ and $w = c + id$, where $a, b, c, d \in \mathbb{R}$.

1. $|z|$ is always a non-negative real number.
2. $|z| = |\overline{z}|$ since $a^2 + b^2 = a^2 + (-b)^2$.
3. $|\mathrm{Re}\,(z)| \leq |z|$ and $|\mathrm{Im}\,(z)| \leq |z|$ since $a^2 \leq a^2 + b^2$ and $b^2 \leq a^2 + b^2$.
4. $|zw| = |z||w|$ since $|zw| = \sqrt{zw.\overline{zw}} = \sqrt{zw\overline{z}\overline{w}} = \sqrt{z\overline{z}w\overline{w}} = \sqrt{z\overline{z}}\sqrt{w\overline{w}} = |z||w|$.

Exercise. *Compute the modulus of*
$$\left(\frac{1+i}{2}\right)^{10}.$$

.

Proposition 0.2.9. *For every $z, w \in \mathbb{C}$,*

$$|z+w| \leq |z| + |w|.$$

This is known as the *triangle inequality.* When we discuss the complex plane, we will see that this has a geometric interpretation which makes this proposition 'obvious', but for now we give an algebraic proof.

Proof. Take $z, w \in \mathbb{C}$. Then

$$\begin{aligned}
|z+w|^2 &= (z+w)(\overline{z+w}) \\
&= (z+w)(\overline{z}+\overline{w}) \\
&= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\
&= |z|^2 + z\overline{w} + w\overline{z} + |w|^2.
\end{aligned}$$

However, $w\overline{z} = \overline{z\overline{w}}$ and so $z\overline{w} + w\overline{z} = 2\mathrm{Re}\,(z\overline{w})$. Substituting back we get

$$\begin{aligned}
|z+w|^2 &= |z|^2 + 2\mathrm{Re}\,(z\overline{w}) + |w|^2 \\
&\leq |z|^2 + 2|z\overline{w}| + |w|^2 \qquad (\text{since } \mathrm{Re}\,(z\overline{w}) \leq |z\overline{w}|) \\
&= |z|^2 + 2|z||w| + |w|^2 \qquad (\text{since } |z\overline{w}| = |z||w|) \\
&= (|z| + |w|)^2.
\end{aligned}$$

Hence $|z+w| \leq |z| + |w|$ as required. □

Using the conplex conjugate and modulus, we can write a more succinct formula for the quotient of two complex numbers:

$$\frac{z}{w} = \frac{z}{w}\frac{\overline{w}}{\overline{w}} = \frac{z\overline{w}}{|w|^2}.$$

In particular, note that

$$\frac{1}{w} = \frac{\overline{w}}{|w|^2}.$$

Exercise. *Which of the following statements are true **for all** complex numbers z and w? In each case either explain why it is true or give a counterexample to show that it is false.*

1. $|z+w| = |z| + |w|$
2. $\mathrm{Re}\,(z-w) = \mathrm{Re}\,(z) - \mathrm{Re}\,(w)$
3. $\mathrm{Im}\,(zw) = \mathrm{Im}\,(z)\mathrm{Im}\,(w)$

0.2.4 The complex plane or Argand diagram

Recall that the real numbers $\mathbb{R}$ can be represented by points on a line. Analogously, the complex numbers $\mathbb{C}$ can be represented by points on a two dimensional plane. Specifically, we can represent each $z = a + ib \in \mathbb{C}$ by the point (a, b) in the Cartesian plane. When we use the Cartesian plane to represent complex numbers we refer to it as the *complex plane* or *Argand diagram.* [3][4]

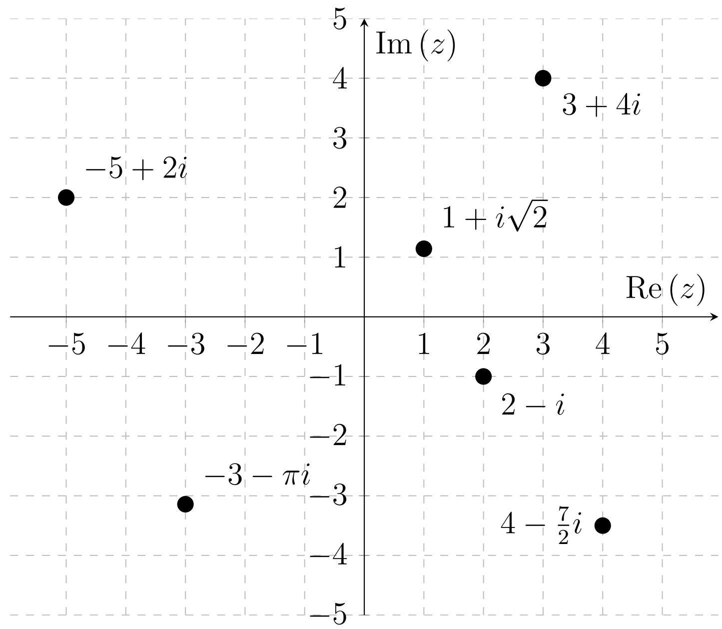

Figure 1: The complex numbers $3 + 4i$, $-5 + 2i$, $2 - i$, $4 - \frac{7}{2}i$, $1 + i\sqrt{2}$, $-3 - \pi i$ plotted on the complex plane.

Remark 0.2.10. The x-axis is called the *real axis* or *real line* and the y-axis is called the *imaginary axis* or *imaginary line.*

The correspondence

$$a + ib \leftrightarrow (a, b)$$

[3] This is similar to the fact that when we use a line to represent real numbers, we usually call it 'the number line'.

[4] The idea of visualising complex numbers on the complex plane was first describe by the Norwegian-Danish mathematician and cartographer Caspar Wessel in 1797 and then independently rediscovered by the amateur mathematician Jean-Robert Argand in 1806.

lets us visualise many algebraic operations with complex numbers as geometric operations of the plane.

- Addition of complex numbers can be visualised by the *parallelogram rule*: the sum $z+w$ is the fourth vertex on the parallelogram with vertices z, w and 0 (see Figure 5.1). Thus if $w \in \mathbb{C}$, we can understand the operation of adding w as translating the complex plane by the "vector" w.

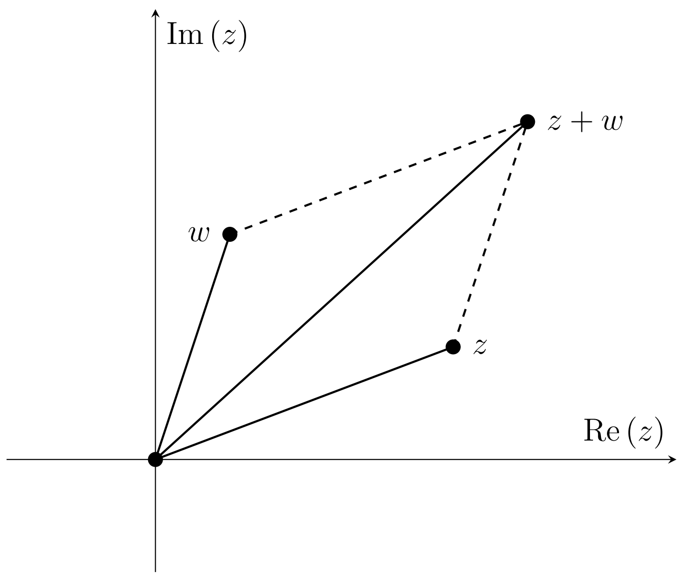

Figure 2: The points 0, z, w and $z+w$ for the vertices of a parallelogram.

- Complex conjugation maps $z = a + ib$ to $\overline{z} = a - ib$. In the plane, this maps (a, b) to $(a, -b)$. Geometrically this is reflection in the real axis (see Figure 3).

- By Pythagoras' theorem, the modulus of $z = a+ib$ is the the distance from z to 0. Also, for $z, w \in \mathbb{C}$, $|z - w|$ is the distance between z and w in the complex plane.

Example 0.2.11. *Find solutions $z \in \mathbb{C}$ to the equation $|z + 1| = |z - 1|$.*

Solution. This can be done using two different ways. We present them both here.

Method 1: We approach this by taking real and imaginary parts and then solving the resulting equation.

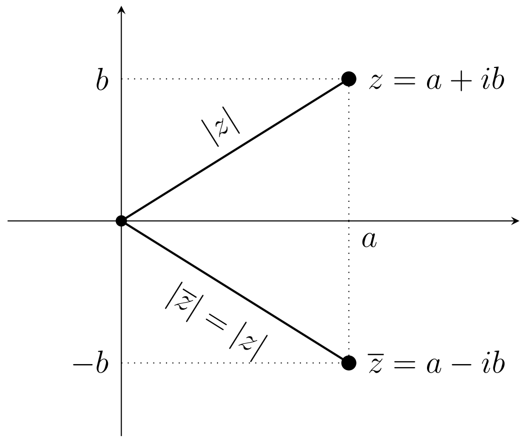

Figure 3: Complex conjugation corresponds to reflection in the real axis and thus it preserves the modulus.

Remember that $|w| = \sqrt{w\overline{w}} = \sqrt{c^2 + d^2}$ if $w = c + id$. So it might help if we solve $|z+1|^2 = |z-1|^2$ instead. Let $z = a + ib$ so that

$$|z+1|^2 = (a+1)^2 + b^2 \quad \text{and} \quad |z-1|^2 = (a-1)^2 + b^2.$$

Then equating these we get that $(a+1)^2 + b^2 = (a-1)^2 + b^2$ and so we must have $(a+1)^2 = (a-1)^2$ Multiplying out both sides we see that $a^2 + 2a + 1 = a^2 - 2a + 1$, which is equivalent to $a = -a$. Therefore $a = 0$. A quick check verifies that any $z \in \mathbb{C}$ with $\operatorname{Re}(z) = 0$ is indeed a solution. Thus the set of solutions is the imaginary axis $\operatorname{Re}(z) = 0$.

Method 2: We think about this geometrically rather than algebraically.

Recall that for $z, w \in \mathbb{C}$, $|z - w|$ is the distance between z and w in the complex plane. Thus $|z+1| = |z-(-1)|$ is the distance between z and -1, and $|z-1|$ is the distance between z and 1. So we need to find all points z that are equidistant from 1 and -1. This is the perpendicular bisector of the straight line joining 1 and -1. Hence the set of solutions is the imaginary axis $\operatorname{Re}(z) = 0$. (If you do not see this straight away, draw a sketch and think about it!)

$\square$

Remark 0.2.12. Although the second approach above is more elegant, it is useful to be able to think about complex numbers both algebraically and geometrically. If you haven't seen complex numbers before, it is worth trying so solve these kinds of problems both ways.

Exercise. *Find all solutions* $z \in \mathbb{C}$ *to the equation* $|z+1| = 2$. *Draw a sketch of this set.*

0.2.5 Argument of a complex number

A point in the plane can also be specified by giving its distance from the origin and the angle it makes with the positive x-axis (these are known as its *polar coordinates*). By the correspondence $a + ib \leftrightarrow (a, b)$, we can do the same with complex numbers.

If $z \neq 0$, $|z| = r$ and z makes an angle θ with the positive real axis (see Figure 4), then

$$\mathrm{Re}\,(z) = r\cos\theta \quad \text{and} \quad \mathrm{Im}\,(z) = r\sin\theta.$$

Therefore

$$z = r(\cos\theta + i\sin\theta).$$

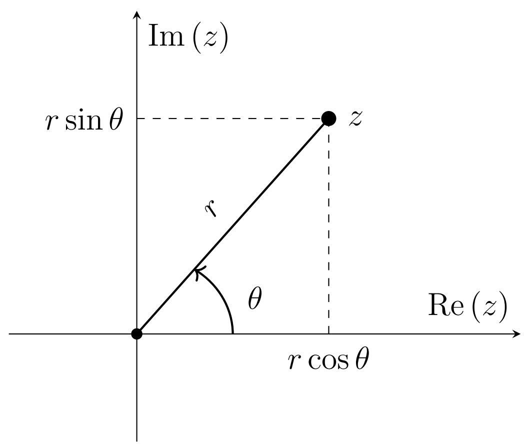

Figure 4: If the complex number z has modulus r and makes an angle θ with the positive real axis, then $z = r(\cos\theta + i\sin\theta)$.

Remark 0.2.13. Every complex number of modulus one is of the form $\cos\theta + i\sin\theta$ for some $-\pi < \theta \leq \pi$.

Definition 0.2.14. Let $z = r(\cos\theta + i\sin\theta)$. The angle θ is called the *argument* of z and is sometimes denoted $\arg z$.

Remark 0.2.15. Although we have defined θ as "the" argument, this is misleading, it would be more accurate to say *an* argument. This is because if θ is an argument of z, then $\theta + 2\pi$ is also an argument of z. However, there is a unique value of the argument θ such that $-\pi < \theta \leq \pi$. This is known as the *principal argument*, and is usually denoted by $\operatorname{Arg} z$ (capital 'A' rather than small 'a').

If $z = a + ib$ with $a, b \in \mathbb{R}$, the argument θ of z satisfies the following equation :

$$\tan(\theta) = \frac{b}{a}.$$

Since $\tan(\theta)$ is periodic with period π, the equation $\tan(\theta) = b/a$ has two solutions in the interval $-\pi < \theta \leq \pi$. To ensure we take the correct one, we have to choose $\tan^{-1}(b/a)$ such that $0 < \theta < \pi$ if $b > 0$, $-\pi < \theta < 0$ if $b < 0$, $\theta = 0$ if $b = 0$ and $a > 0$, and $\theta = \pi$ if $b = 0$ and $a < 0$. It is usually easy to see which value of $\tan^{-1}(b/a)$ to take after drawing a sketch.

Exercise. *Determine the principal argument of the complex number* $z = -3 - i\sqrt{3}$.

Understanding the geometric interpretation of multiplying or dividing complex numbers is most easily done using the modulus and argument of a complex number. However, we will delay this discussion and return to it after introducing one of the most useful formulae in mathematics: *Euler's formula.*

0.2.6 Euler's formula and polar form

Recall from Calculus I that the exponential function $x \mapsto e^x$, $x \in \mathbb{R}$, can be defined as the infinite series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots.$$

This definition can be extended to $z \in \mathbb{C}$ (although we don't consider issues of convergence, this series does converge for all $z \in \mathbb{C}$). Then as is the case for real numbers, we have that for arbitrary $z, w \in \mathbb{C}$

$$e^{z+w} = e^z e^w.$$

We also have the following power series for $\sin(x)$ and $\cos(x)$:

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

Theorem 0.2.16 (Euler's formula)**.** *For every $\theta \in \mathbb{R}$,*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Proof. Substituting $x = i\theta$ into the series for e^x gives

$$e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{x^4}{4!} + \dots.$$

Since $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$, this simplifies to

$$\begin{aligned} e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{3!} + \frac{x^4}{4!} + \dots \\ &= \left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right\} + i\left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right\} \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

□

Remark 0.2.17.

1. When $\theta = 0$, We have $e^0 = \cos(0) + i\sin(0) = 1$ as expected.
2. When $\theta = \pi/2$ we have $e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$.
3. When $\theta = \pi$ we have $e^{i\pi} + 1 = 0$. This is known as *Euler's identity* [5].
4. For every $\theta \in \mathbb{R}$, $e^{i(\theta+2\pi)} = e^{i\theta}$ – that is, the function $\theta \to e^{i\theta}$ is periodic with period 2π. In particular, $e^{2\pi i} = 1$.

We saw in the last section that every complex number $z \neq 0$ can be written in the form

$$z = r(\cos\theta + i\sin\theta).$$

By Euler's theorem it follows that we can write z in the form

$$z = re^{i\theta},$$

where $r = |z|$ and $\theta = \arg z$. This is known as the *polar form* of a complex number.

Complex conjugation is easily carried out in polar form: if $z = re^{i\theta}$, then

$$\overline{z} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = re^{-i\theta}.$$

[5]This is frequently celebrated as one of the most beautiful identities in mathetics – it relates $1, e, i, \pi$ in one elegant equation.

Exercise. *Express the complex number $z = -4 - 4i$ in polar form.*

Recall that for $z \in \mathbb{C}$,

$$\mathrm{Re}\,(z) = \frac{1}{2}(z + \overline{z}) \quad \text{and} \quad \mathrm{Im}\,(z) = \frac{1}{2i}(z - \overline{z}). \tag{0.2.1}$$

If $z = e^{i\theta}$ then Euler's formula states that $\mathrm{Re}\,(z) = \cos\theta$ and $\mathrm{Im}\,(z) = \sin\theta$. We also have that $\overline{z} = e^{-i\theta}$. Substituting these into (0.2.1) gives us the formulae

$$\begin{aligned}\cos\theta &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right);\\ \sin\theta &= \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right).\end{aligned}$$

Compare these with the hyperbolic functions when you meet them in Calculus I.

0.2.7 Multiplication and division using polar form

Let $z = re^{i\theta}$ and $w = \rho e^{i\phi}$. Then

$$zw = re^{i\theta}\rho e^{i\phi} = r\rho e^{i\theta}e^{i\phi} = r\rho e^{i(\theta+\phi)}.$$

Thus

$$|zw| = r\rho = |z||w| \quad \text{and} \quad \arg(zw) = \theta + \phi = \arg z + \arg w.$$

This shows that multiplying two complex numbers involves multiplying their moduli and summing their arguments. This gives a nice geometric interpretation of multiplying by $w = \rho e^{i\phi}$: it corresponds to 'stretching' (or 'squashing' if $|w| < 1$) the complex plane outwards (away from 0) by a factor of ρ and rotating the plane anticlockwise by an angle ϕ.

Similarly, if $z = re^{i\theta}$ and $w = \rho e^{i\phi}$, then

$$\frac{z}{w} = \frac{re^{i\theta}}{\rho e^{i\phi}} = \frac{r}{\rho}e^{i(\theta-\phi)}.$$

Hence

$$\left|\frac{z}{w}\right| = \frac{r}{\rho} = \frac{|z|}{|w|} \quad \text{and} \quad \arg\left(\frac{z}{w}\right) = \theta - \phi = \arg z - \arg w.$$

Remark 0.2.18. As a rule of thumb addition and subtraction of complex numbers is simpler using the $z = a + ib$ notation while multiplication and division are simpler in the polar notation $z = re^{i\theta}$.

0.2.8 De Moivre's theorem

Another consequence of Euler's formula is *De Moivre's theorem*: [6]

Theorem 0.2.19 (De Moivre's theorem). *For every $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$,*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Proof. By Euler's formula,

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

□

De Moivre's theorem and Euler's formula can be used to derive many trigomometric identities. For example, taking $n = 2$ in De Moivre's theorem gives

$$(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta).$$

Then multiplying out the left hand side we get

$$\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta = \cos(2\theta) + i\sin(2\theta).$$

The real part of the left side must equal the real part of the right side; the same is true for the imaginary parts. Equating the real and imaginary parts of each side we find

$$\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta; \\ \sin(2\theta) &= 2\cos\theta\sin\theta. \end{aligned}$$

Thus we have derived the familiar *double angle formulae*. Similar identities can be derived for larger n.

Exercise. *Use Euler's theorem to derive expressions for* $\cos(\theta + \phi)$ *and* $\sin(\theta + \phi)$ *in terms of* $\cos\theta$, $\sin\theta$, $\cos\phi$ *and* $\sin\phi$.

---

[6] Although we present De moivre's theorem follows easily from Euler's formula, it was actually proved about 30 years earlier

0.2.9 Roots of unity

A *root of unity* is a solution in $\mathbb{C}$ of the equation

$$z^n = 1, \tag{0.2.2}$$

for some $n \in \mathbb{N}$. For each particular n these are known as n^{th} *roots of unity.* This equation has n solutions which possess a delightful symmetry in the complex plane. Let us try to find the solutions of (0.2.2). First one notes that

$$1 = |z^n| = |z|^n.$$

Therefore $|z| = 1$. This means that each solution is of the form $z = e^{i\theta}$ for some $-\pi < \theta \leq \pi$. Plugging this back into the (0.2.2) gives

$$z^n = (e^{i\theta})^n = \cos(n\theta) + i\sin(n\theta) = 1.$$

Equating real and imaginary parts we get

$$\cos(n\theta) = 1 \quad \text{and} \quad \sin(n\theta) = 0. \tag{0.2.3}$$

Thus θ is a solution of (0.2.3) if and only if $n\theta$ is a multiple of 2π, i.e $\theta = \frac{2\pi k}{n}$ for some $k \in \mathbb{Z}$. In summary, we have showed that

$$z = e^{i2\pi k/n}$$

is a solution of (0.2.2) for every $k \in \mathbb{Z}$. However, many of these solutions are actually the same. When k is a multiple of n, say $k = mn$, then

$$z = e^{i2\pi mn/n} = e^{i2\pi m} = 1,$$

and so all solutions with $k = \ldots, -3n, -2n, -n, 0, n, 2n, 3n, \ldots$ are the same. Similarly, any two integers that differ by a multiple of n give the same solution, so the only distinct solutions are

$$z = e^{i2\pi k/n}, \quad k = 0, 1, \ldots, n-1.$$

Geometrically, the n-th roots of unity form the vertices of a regular n-polygon, symmetrically arranged around the origin in the complex plane, with one root at $z = 1$ (see Figure 5).

To summarise, we have proved the following:

Proposition 0.2.20. *Let n be a positive integer. The equation $z^n = 1$ has exactly n distinct roots in the complex numbers and they are*

$$z = e^{i2\pi k/n}, \quad k = 0, 1, 2, ..., n-1.$$

Remark 0.2.21. More generally, any $z \in \mathbb{C}$, $z \neq 0$, has n distinct n-th roots in $\mathbb{C}$ and they are all evenly spaced on the circle of radius $|z|^{1/n}$ in the complex plane.

Exercise. *Find all solutions of the equation $z^3 = 1 + i$.*

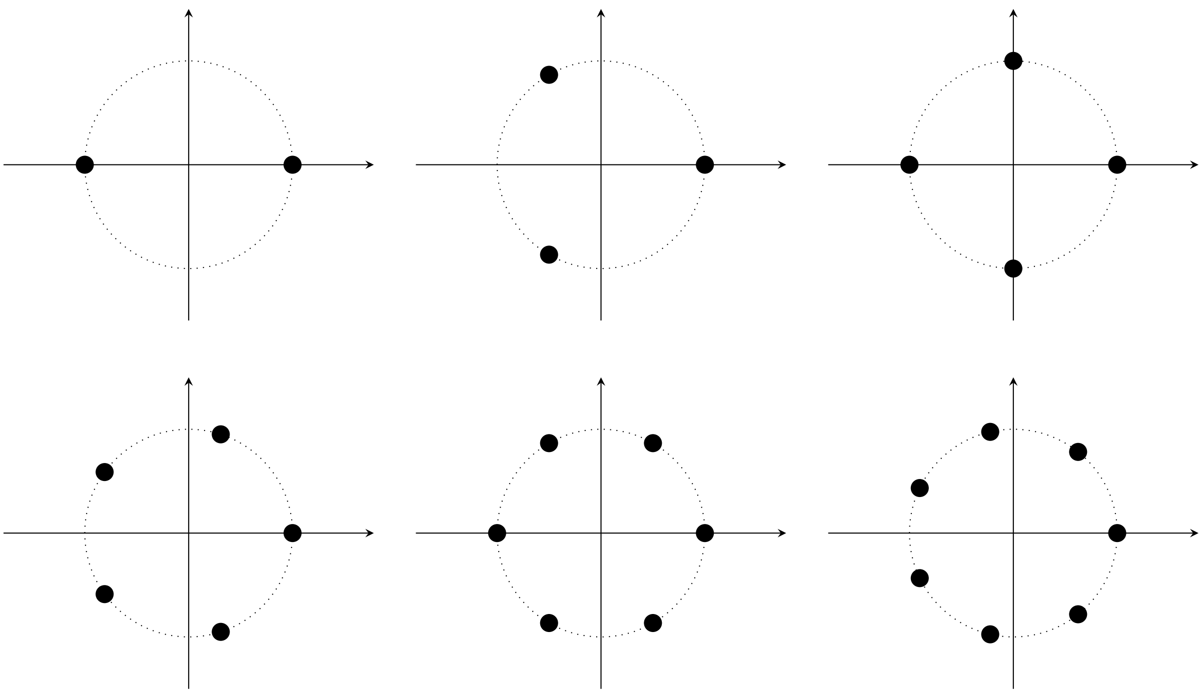

Figure 5: Plot of the n-th roots of unity for $n = 2, 3, 4, 5, 6, 7$. The unit circle (the circle of radius one) is shown as a dotted line for reference. Note that they are evenly spaced around the unit circle.

Recall that the square roots of a real number x are $\sqrt{x}$ and $-\sqrt{x}$, so that the sum of the square roots of x is zero. The fact that the sum of the square roots is zero remains true for all complex numbers z. In fact, for every integer $n \geq 2$ and every complex number z, the sum of the n-th roots of z is equal to zero. The symmetry of the set of n-th roots in the complex plane allows us to give a particularly elegant proof of this. Although, we prove this only for n-th roots of unity, the same proof can be adapted to prove the general case.

Proposition 0.2.22. *Let n be an integer with $n \geq 2$. The sum of the n-th roots of unity is equal to zero, i.e.*

$$\sum_{k=0}^{n-1} e^{2\pi ik/n} = 0.$$

Proof. Let $\omega = e^{2\pi i/n}$, so that the n-th roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1}$. Let

$$S = \sum_{k=0}^{n-1} e^{2\pi ik/n} = \sum_{k=0}^{n-1} \omega^k.$$

Observe that since $\omega^n = \omega^0 = 1$, we have that

$$\omega S = \omega(1 + \omega + \cdots + \omega^{n-1}) = (\omega + \omega^2 + \cdots + \omega^n) = S.$$

Since $\omega \neq 1$ whenever $n \geq 2$, we must have $S = 0$. □

0.2.10 Logarithms and complex powers

We know that if $x \in \mathbb{R}$ and $x > 0$, then the equation $e^y = x$ has a unique solution which we call $\log x$ (or sometimes $\ln x$ but we won't use this notation). Similarly, for $z \in \mathbb{C}$, $z \neq 0$ we can define $\log z$ to be a solution w of the equation

$$e^w = z.$$

To understand this better, let $w = u + iv$ for some $u, v \in \mathbb{R}$ and $z = re^{i\theta}$ for some $r > 0, \theta \in \mathbb{R}$. Then if $e^w = z$ we have

$$e^w = e^{u+iv} = e^u e^{iv} = re^{i\theta}.$$

Since $|e^{iv}| = 1$, we must have $r = e^u$. Dividing both sides by r (recall that $r > 0$), we see that we must also have $e^{iv} = e^{i\theta}$. This happens if and only if $v = \theta + 2\pi k$ for some $k \in \mathbb{Z}$. combining these we see that $e^w = z$ if and only if

$$w = \log r + i\theta + i2\pi k \quad \text{for some} \quad k \in \mathbb{Z}.$$

This shows that $\log z$ has many values. Indeed if w is a solution of $e^w = z$, then $w + 2\pi i$ is also a solution. To make $\log z$ single valued, we insist that θ is the principal argument i.e. $-\pi < \theta \leq \pi$.

Definition 0.2.23. Let $z \in \mathbb{C}$, $z \neq 0$. Then the *principal logarithm* of z is denoted $\operatorname{Log} z$ (capital 'L') and defined by

$$\operatorname{Log} z = \log |z| + i \operatorname{Arg} z.$$

Remark 0.2.24. The principal logarithm is the unique value of $\log z$ such that $-\pi < \operatorname{Im}(\log z) \leq \pi$.

We also wish to make sense of expresions of the form z^w with $z, w \in \mathbb{C}$. We can use the logarithm to do this. For $z, w \in \mathbb{C}$, $z \neq 0$ we define

$$z^w = e^{w \log z}.$$

Remark 0.2.25. Since $\log z$ is multi-valued so is z^w. In particular, if $v \in \mathbb{C}$ is one value of z^w, $e^{2\pi i w} v$ is another value. For example, in Section 0.2.9 we saw that for any $z \neq 0$, $z^{1/n}$ has n possible values.

Exercise. *Find the values of i^i.*

Chapter 1

Vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$

1.1 The space $\mathbb{R}^n$

The set of real numbers $\mathbb{R}$ can be visualised as points on a line. Similarly, pairs of real numbers (x, y) can be visualised as points on a plane (once an origin and unit of length have been fixed). The set of all such pairs is denoted $\mathbb{R}^2$. An element (v_1, v_2) of $\mathbb{R}^2$ can be written as a *column* or *row*:

$$\textbf{Column vector} : \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \textbf{Row vector} : \mathbf{v} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \text{ or } (v_1, v_2)$$

We will predominantly use column vectors in this course, but sometimes write them as rows to save space. [1]

The set of all *triples* of real numbers is denoted $\mathbb{R}^3$. As before we will consider these as columns of length 3:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Elements of $\mathbb{R}^3$ can be visualised as points or arrows in '3-d space' (see Figure 1.2).

Although we can no longer draw a picture, there is nothing to prevent us from considering quadruples of real numbers, quintuples etc.

[1]Although there is no mathematical reason to prefer column vectors over row vectors, columns are slightly more natural from the point of view of linear equations (which will be covered later in the course).

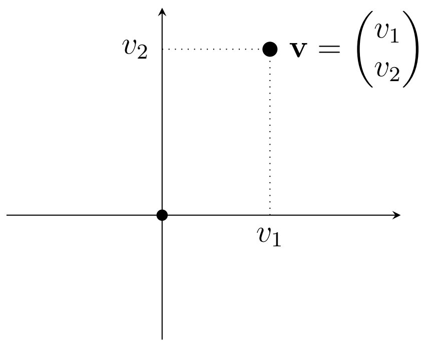

Figure 1.1: An elelment$\mathbf{v} \in \mathbb{R}^2$ can be visualised as a point in the plane (left) or as an arrow to the point with coordinates (v_1, v_2) (right).

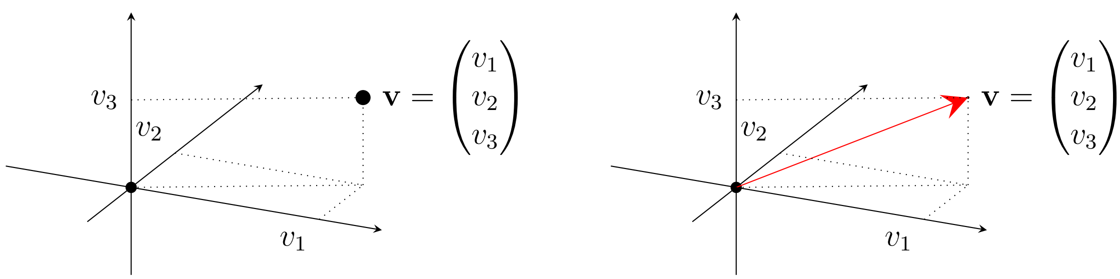

Figure 1.2: Visualisation of $\mathbf{v} \in \mathbb{R}^3$ as a point in space (left) and as an arrow (right).

Definition 1.1.1. Let n be a positive integer. The space $\mathbb{R}^n$ is the set of all columns of size n,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix},$$

whose entries $v_1, v_2, \ldots, v_n$ are real numbers.

We will see later in the course that $\mathbb{R}^n$ is an example of a *vector space* and so we will call elements of $\mathbb{R}^n$ *vectors.* [2]

Remark 1.1.2. 1. The numbers $v_1, \ldots, v_n$ are called *components*, *coordinates* or *entries.*

[2] We will use boldface latin letters to denote elements of $\mathbb{R}^n$ in the notes, and during lectures we will use underlined latin letters. We will continue with this convention when we consider general vector spaces later in the course.

2. We think of elements of $\mathbb{R}^n$ as coordinates of points in "n-dimensional space".

We can add elements of $\mathbb{R}^n$ together and we can multiply them by constants. More precisely, we can perform the following operations:

$$\textbf{Vector addition} \qquad \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix};$$

$$\textbf{Scalar multiplication} \qquad \alpha \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{pmatrix} \qquad (\alpha \in \mathbb{R}).$$

In this context real numbers are called *scalars* (because when a real number multiplies a vector, it 'scales' the vector). [3]

Remark 1.1.3. 1. For $\mathbf{v} \in \mathbb{R}^n$, we write $-\mathbf{v}$ for the element $(-1)\mathbf{v}$.

2. We will use $\mathbf{0}$ to denote the origin, i.e.

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

1.2 Linear combinations and span

We can combine the operations of addition and scalar multiplication to form *linear combinations*.

Definition 1.2.1. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in $\mathbb{R}^n$. A *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is an expression of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_m \mathbf{v}_m = \sum_{k=1}^{m} \alpha_k \mathbf{v}_k,$$

where $\alpha_1, \ldots, \alpha_n$ are scalars (i.e. $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$).

[3] We will usually use small Greek letters, such as $\alpha, \beta, \ldots$ to denote scalars.

Example 1.2.2. *Let* $\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *and* $\mathbf{w} = \begin{pmatrix} -5/2 \\ 4 \end{pmatrix}$.

Evaluate the linear combination $\frac{1}{6}(2\mathbf{u} - 5\mathbf{v} + 2\mathbf{w})$

Solution.

$$\frac{1}{6}(2\mathbf{u} - 5\mathbf{v} + 2\mathbf{w}) = \frac{1}{6}\left\{\begin{pmatrix} 6 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ 5 \end{pmatrix} + \begin{pmatrix} -5 \\ 8 \end{pmatrix}\right\} = \frac{1}{6}\begin{pmatrix} -4 \\ -1 \end{pmatrix} = \begin{pmatrix} -2/3 \\ -1/6 \end{pmatrix}.$$

□

Definition 1.2.3. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be vectors in $\mathbb{R}^n$. The *linear span* (or sometimes just *span*) of $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is the set of all vectors which are linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_m$. We denote the linear span of $\mathbf{v}_1, \ldots, \mathbf{v}_m$ by $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, i.e.

$$\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} = \left\{\sum_{k=1}^{m} \alpha_k \mathbf{v}_k : \alpha_1, \ldots, \alpha_m \in \mathbb{R}\right\}.$$

Remark 1.2.4. We always have $\mathbf{0} \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, since we can take $\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0$.

Example 1.2.5. *Determine whether* $(3, 5, -8)$ *is in the linear span of the vectors* $(1, 3, 0)$ *and* $(0, -2, 4)$.

Solution. $(3, 5, -8)$ is in the linear span of $(1, 3, 0)$ and $(0, -2, 4)$ if and only if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\begin{pmatrix} 3 \\ 5 \\ -8 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}. \tag{1.2.1}$$

Equating each component, we see that this holds if and only if

$$3 = \alpha, \tag{1.2.2}$$
$$5 = 3\alpha - 2\beta, \tag{1.2.3}$$
$$-8 = 4\beta. \tag{1.2.4}$$

From (1.2.2) and (1.2.4) we must have $\alpha = 1$ and $\beta = -2$. Substituting these into the right hand side of (1.2.3) we get $3\alpha - 2\beta = 13 \neq 5$. Hence (1.2.3) does not hold. We have shown that there are no $\alpha, \beta \in \mathbb{R}$ such that (1.2.1) holds and so $(3, 5, -8)$ is not in the linear span of $(1, 3, 0)$ and $(0, -2, 4)$. □

The following examples will be particularly important.

- Let $\mathbf{v} \in \mathbb{R}^n$ be a non-zero vector. Then

  $$\operatorname{span}\{\mathbf{v}\} = \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}.$$

  Geometrically, this is a line. In fact it is the line through the origin in the direction of the vector $\mathbf{v}$. Moreover, every line through the origin is of this form.

- Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ be non-zero vectors. Then

  $$\operatorname{span}\{\mathbf{v}, \mathbf{w}\} = \{\alpha\mathbf{v} + \beta\mathbf{w} : \alpha, \beta \in \mathbb{R}\}.$$

  If $\mathbf{w}$ is a multiple of $\mathbf{v}$, then $\operatorname{span}\{\mathbf{v}, \mathbf{w}\} = \operatorname{span}\{\mathbf{v}\}$ and so we get a line again. If $\mathbf{w}$ is not a multiple of $\mathbf{v}$ (so that $\mathbf{w}$ and $\mathbf{v}$ are not in the "same direction") then $\operatorname{span}\{\mathbf{v}, \mathbf{w}\}$ is a plane. In particular it is the plane containing $\mathbf{v}, \mathbf{w}$ and the origin. Moreover, every plane through the origin is of this form.

1.3 Geometry of $\mathbb{R}^n$

Lengths and Dot products

In two and three dimensions we can compute the length of a vector $\mathbf{v}$ - that is, the distance between the origin and the point $\mathbf{v}$ - using Pythagoras' theorem. More precisely, the length of $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ is

$$\sqrt{v_1^2 + v_2^2 + v_3^2}.$$

We can generalize this to define the length of a vector in $\mathbb{R}^n$.

Definition 1.3.1. Let $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$. The *length* or *norm* of $\mathbf{v}$ is denoted $\|\mathbf{v}\|$ and defined by

$$\|\mathbf{v}\| = \left(\sum_{k=1}^{n} v_k^2\right)^{\frac{1}{2}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

We can define the distance between two points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ to be $\|\mathbf{v} - \mathbf{u}\|$.

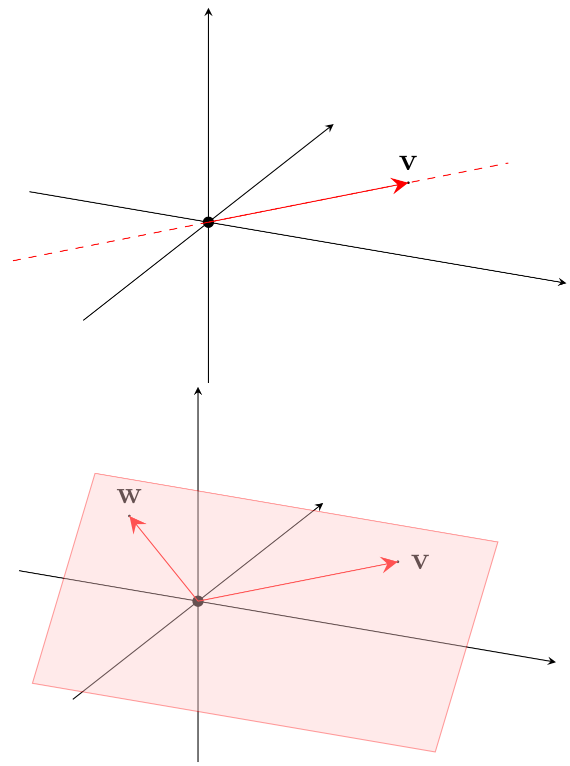

Figure 1.3: The linear span of a single vector is a line (left) and the linear span of two vectors is a plane (right).

Proposition 1.3.2 (Properties of the norm)**.** *For every* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *and every scalar* α *we have*

(i) $\|\mathbf{v}\| \geq 0$ *and* $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$*,*

(ii) $\|\alpha\mathbf{v}\| = |\alpha|\|\mathbf{v}\|$*,*

(iii) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$*.*

Proof. (i) Since $v_1^2 \geq 0$, $v_2^2 \geq 0, \ldots, v_n^2 \geq 0$, it follows that $\|\mathbf{v}\| = \sqrt{v_1^2 + \cdots + v_n^2} \geq 0$. Moreover, if $\|\mathbf{v}\| = 0$ then we must have $v_1^2 = v_2^2 = \cdots = v_n^2 = 0$ and so $\mathbf{v} = \mathbf{0}$.

(ii) This is a direct computaion:

$$\|\alpha\mathbf{v}\| = \sqrt{(\alpha v_1)^2 + \cdots + (\alpha v_n)^2} = \sqrt{\alpha^2(v_1^2 + \cdots + v_n^2)} = \sqrt{\alpha^2}\sqrt{v_1^2 + \cdots + v_n^2} = |\alpha|\|\mathbf{v}\|.$$

(iii) This is the *triangle inequality* that we saw for complex numbers. The proof of this will be left as an assignment question.

□

Example 1.3.3. *Let* $\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ *and* $\mathbf{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$*. Compute the distance between the points* $\mathbf{u}$ *and* $\mathbf{v}$*.*

Solution. The distance between $\mathbf{u}$ and $\mathbf{v}$ is

$$\|\mathbf{u} - \mathbf{v}\| = \left\| \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ 5 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 4 \\ -7 \end{pmatrix} \right\| = \sqrt{65}.$$

□

Definition 1.3.4. Let $\mathbf{u} = (u_1, \ldots, u_n), \mathbf{v} = (v_1, \ldots v_n) \in \mathbb{R}^n$. Then the *dot product* (or *scalar product* or *inner product*) of $\mathbf{u}$ and $\mathbf{v}$ is denoted $\mathbf{u} \cdot \mathbf{v}$ and defined by

$$\mathbf{u} \cdot \mathbf{v} = \sum_{k=1}^{n} u_k v_k = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Remark 1.3.5. For $\mathbf{v} = (v_1, \ldots, v_n)$ we have $\mathbf{v} \cdot \mathbf{v} = v_1^2 + \cdots + v_n^2 = \|\mathbf{v}\|^2$. Thus $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.

Proposition 1.3.6 (Properties of the dot product)**.** *For every* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and every scalar* α *we have*

(i) $\mathbf{v} \cdot \mathbf{v} \geq 0$ *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$,

(ii) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$,

(iii) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$,

(iv) $(\alpha \mathbf{u}) \cdot \mathbf{v} = \alpha(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (\alpha \mathbf{v})$.

Proof. (i) Since $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$, the claim follows from Proposition 1.3.2(i).

(ii) This is immediate since $u_1 v_1 + \cdots + u_n v_n = v_1 u_1 + \cdots + v_n u_n$.

(iii) We compute that

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= u_1(v_1 + w_1) + u_2(v_2 + w_2) + \cdots + u_n(v_n + w_n) \\ &= u_1 v_1 + u_2 v_2 + \ldots u_n v_n + u_1 w_1 + u_2 w_2 + \ldots u_n w_n \\ &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}. \end{aligned}$$

(iv) We have that

$$(\alpha\mathbf{u}) \cdot \mathbf{v} = \alpha u_1 v_1 + \cdots + \alpha u_n v_n = \alpha(u_1 v_1 + \cdots + u_n v_n) = \alpha(\mathbf{u} \cdot \mathbf{v}).$$

The same steps also prove that $\mathbf{u} \cdot (\alpha\mathbf{v}) = \alpha(\mathbf{u} \cdot \mathbf{v})$.

□

When $n = 2$ or 3, the dot product allows us to determine the angle between two vectors.

Proposition 1.3.7. *Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with $n = 2, 3$ and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then*

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta.$$

Proof. The triangle spanned by $\mathbf{u}$ and $\mathbf{v}$ (i.e the triangle with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{0}$) has side lengths $\|\mathbf{u}\|$, $\|\mathbf{v}\|$ and $\|\mathbf{u} - \mathbf{v}\|$ (see Figure 1.4). By the cosine rule, we have that

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta.$$

However, $\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$. We can simplify this using the properties of the dot product above to get

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}. \end{aligned}$$

Substituting this into the above equation we get

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta,$$

and therefore

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta.$$

□

Remark 1.3.8. 1. In particular, we have that two vectors non-zero $\mathbf{u}$ and $\mathbf{v}$ are perpendicular (or *orthogonal*) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

2. Since $|\cos\theta| \leq 1$, we have that

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|.$$

This is known as the *Cauchy-Schwartz* inequality and will be proved in much more generality in Linear Algebra and Geometry II.

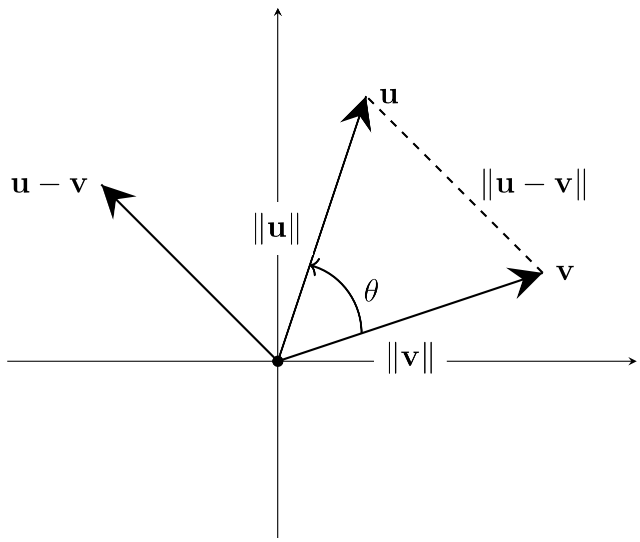

Figure 1.4: The triange spanned by **u** and **v**.

Example 1.3.9. *Find the angle between* $\mathbf{u} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$ *and* $\mathbf{v} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

Solution. Let ϕ denote the angle **u** and **v**. Then by Proposition 1.3.7,

$$\cos\phi = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}.$$

We compute that

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= -3-10 = -13; \\ \|\mathbf{u}\| &= \sqrt{1^2+5^2} = \sqrt{26}; \\ \|\mathbf{v}\| &= \sqrt{(-3)^2+2^2} = \sqrt{13}. \end{aligned}$$

This gives

$$\cos\phi = \frac{-13}{\sqrt{26}\sqrt{13}} = \frac{-1}{\sqrt{2}},$$

and hence $\phi = 3\pi/4$. □

1.3.1 Lines and planes

We saw earlier that the line through the origin in the direction of a vector **p** is of the form $\{\alpha\mathbf{p} : \alpha \in \mathbb{R}\}$. Let L be a line through the point $\mathbf{v}_0$ and parallel to **p**. Then every point $\mathbf{v}$ on L is of the form

$$\mathbf{v} = \mathbf{v}_0 + t\mathbf{p}. \tag{1.3.1}$$

for some $t \in \mathbb{R}$. [4]As the real parameter t varies, we get every point on the line. (1.3.1) is known as the *parametric* equation of the line.

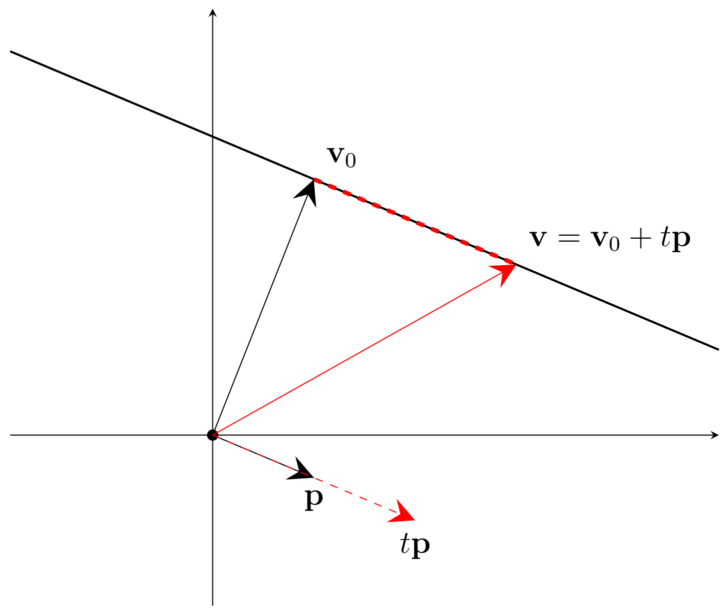

Figure 1.5: Any point $\mathbf{v}$ on the line can be expressed in the form $\mathbf{v} = \mathbf{v}_0 + t\mathbf{p}$ for some choice of $t \in \mathbb{R}$.

Remark 1.3.10. The parametric equation of a line is **not unique**. In particular,

- replacing $\mathbf{p}$ with any non-zero multiple of $\mathbf{p}$ gives the same line;
- replacing $\mathbf{v}_0$ with any other point on the line gives the same line.

Example 1.3.11. *Find a parametric equation of the line in $\mathbb{R}^3$ passing through the points $(1, -3, 1)$ and $(-2, 4, 5)$.*

Solution. For a parametric equation, we need a point $\mathbf{v}_0$ on the line; we will take

$$\mathbf{v}_0 = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}.$$

[4]We use t here rather than α because we think of this parameter as 'time'.

We also need a vector $\mathbf{p}$ parallel to the line. Since the difference of any two vectors on the line will be parallel to it, we can take

$$\mathbf{p} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ -4 \end{pmatrix}$$

Then the parametric equation of the line is

$$\mathbf{v} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ -7 \\ -4 \end{pmatrix}.$$

□

Exercise. *Let* L_1 *be the line in* $\mathbb{R}^2$ *with parametric equation*

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

Let L_2 *be the line in* $\mathbb{R}^2$ *with parametric equation*

$$\mathbf{v} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Determine whether or not L_1 *and* L_2 *are parallel or equal.*

Let $\mathbf{p}$ and $\mathbf{q}$ be two non-zero vectors in $\mathbb{R}^n$ and suppose $\mathbf{p}$ is not a multiple of $\mathbf{q}$. Then the plane spanned by $\mathbf{p}$ and $\mathbf{q}$ (i.e. the plane passing through $\mathbf{p}, \mathbf{q}$ and $\mathbf{0}$) is the set $\operatorname{span}\{\mathbf{p}, \mathbf{q}\} = \{\alpha\mathbf{p} + \beta\mathbf{q} : \alpha, \beta \in \mathbb{R}\}$. Fix a point $\mathbf{v}_0 \in \mathbb{R}^n$ and let Π be the plane parallel to $\operatorname{span}\{\mathbf{p}, \mathbf{q}\}$ and pasing through $\mathbf{v}_0$. Then any point $\mathbf{v}$ on Π is of the form

$$\mathbf{v} = \mathbf{v}_0 + t\mathbf{p} + s\mathbf{q}$$

for some $t, s \in \mathbb{R}$. As the real parameters t, s vary, we get every point on the plane. This is called the *parametric* equation of the plane.

Remark 1.3.12. As with the parametric equation of a line, the parametric equation of a plane is also not unique. In particular,

- replacing $\mathbf{p}$ and $\mathbf{q}$ with any non-zero vectors in $\operatorname{span}\{\mathbf{p}, \mathbf{q}\}$ (provided they are not multiples of each other) gives the same plane;

- replacing $\mathbf{v}_0$ with any other point on the plane gives the same plane.

Proposition 1.3.13. *A plane in $\mathbb{R}^3$ is the set of points $\mathbf{v} = (v_1, v_2, v_3)$ satisfying an equation*

$$av_1 + bv_2 + cv_3 = d, \tag{1.3.2}$$

where $a, b, c, d \in \mathbb{R}$ are constants and a, b, c not all zero.

(1.3.2) is the Cartesian equation of the plane.

Proof. Consider the plane in $\mathbb{R}^3$ with parametric equation $\mathbf{v} = \mathbf{v}_0 + t\mathbf{p} + s\mathbf{q}$. We will show that there exist $a, b, c, d \in \mathbb{R}$ with a, b, c not all zero such that (1.3.2) holds. To do this we take any non-zero vector $\mathbf{u}$ which is perpendicular to both $\mathbf{p}$ and $\mathbf{q}$ (the existence of such a point is geometrically obvious and can be proved rigorously once we study linear equations later in the course). Set a, b, c to be the components of $\mathbf{u}$, so that $av_1 + bv_2 + cv_3 = \mathbf{u} \cdot \mathbf{v}$. Then if $\mathbf{v}$ lies on the plane, so that $\mathbf{v} = \mathbf{v}_0 + t\mathbf{p} + s\mathbf{q}$ for some $t, s \in \mathbb{R}$, we have

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{v}_0 + t\mathbf{p} + s\mathbf{q}) = \mathbf{u} \cdot \mathbf{v}_0 + t\mathbf{u} \cdot \mathbf{p} + s\mathbf{u} \cdot \mathbf{q} = \mathbf{u} \cdot \mathbf{v}_0.$$

Setting $d = \mathbf{u} \cdot \mathbf{v}_0$ we see that if $\mathbf{v}$ lies on the plane then

$$av_1 + bv_2 + cv_3 = d.$$

We also need to show that the set of all vectors satisfying (1.3.2) with a, b, c not all zero is a plane. Suppose without loss of generality that $a \neq 0$. Then if v_1, v_2, v_3 satisfy (1.3.2), we can rearrange to get

$$av_1 = d - bv_2 - cv_3$$

and hence

$$v_1 = d/a - v_2 b/a - v_3 c/a.$$

We conclude that

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} d/a - v_2 b/a - v_3 c/a \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} d/a \\ 0 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} -b/a \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} -c/a \\ 0 \\ 1 \end{pmatrix},$$

which is the equation of a plane. To make it look more familiar we can set $v_2 = t$ and $v_3 = s$ to get

$$\mathbf{v} = \begin{pmatrix} d/a \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -b/a \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -c/a \\ 0 \\ 1 \end{pmatrix}.$$

□

More generally, an equation of the form

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = b$$

where $a_1, \ldots, a_n, b$ are given numbers is called a *linear equation* in the *unknowns* $v_1, \ldots, v_n$. The set of solutions in $\mathbb{R}^n$ of such an equation is called a *hyperplane* in $\mathbb{R}^n$. It is a flat "$n-1$ dimensional surface" in $\mathbb{R}^n$.

1.4 Complex vectors

Much of the theory that we have developed extends to vectors with complex entries.

Definition 1.4.1. Let n be a positive integer. The space $\mathbb{C}^n$ is the set of all columns of size n,

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix},$$

whose entries $z_1, z_2, \ldots, z_n$ are complex numbers.

As in the real case, we can add vectors and multiply by scalars; the only difference being that now we are allowed to multiply by complex numbers and so the complex numbers are called scalars.

Example 1.4.2. *Express* $(-7-3i, i, -5i)$ *as a linear combination of the vectors* $(i, 1, 0)$ *and* $(-3, 0, 1+2i)$.

Solution. We need to find $\alpha, \beta \in \mathbb{C}$ such that

$$\begin{pmatrix} -7-3i \\ i \\ 5i \end{pmatrix} = \alpha \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 0 \\ 1+2i \end{pmatrix}.$$

Equating each component, we see that we must have

$$5 + 3i = i\alpha - 3\beta, \tag{1.4.1}$$
$$i = \alpha, \tag{1.4.2}$$
$$5i = (1+2i)\beta. \tag{1.4.3}$$

It follows from (1.4.2) that $\alpha = i$. Substituting this into (1.4.1) we get $-7-3i = -1-3\beta$, and so $\beta = 2+i$. We can check that with this choice of α and β (1.4.3) is also satisfied:

$$(1+2i)\beta = (1+2i)(2+i) = 5i.$$

We have shown that

$$\begin{pmatrix} -7-3i \\ i \\ 5i \end{pmatrix} = i \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} + (2+i) \begin{pmatrix} -3 \\ 0 \\ 1+2i \end{pmatrix}.$$

□

1.4.1 Length and dot product in $\mathbb{C}^n$

The definitions of the length and the dot product in $\mathbb{R}^n$ cannot be extended directly to $\mathbb{C}^n$. For example, by that definition the length of the vector $(1, i)$ would be $\sqrt{1^2 + i^2} = \sqrt{1-1} = 0$. So a non-zero vector would have zero length. In order to retain the desirable geometric properties we have to modify the definitions of the length and dot product.

Definition 1.4.3. Let $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$. The *length* or *norm* of $\mathbf{z}$ is denoted $\|\mathbf{z}\|$ and defined by

$$\|\mathbf{z}\| = \left(\sum_{k=1}^{n} |z_k|^2 \right)^{\frac{1}{2}} = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}.$$

We can the define distance between two points $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{C}^n$ to be $\|\mathbf{z} - \mathbf{w}\|$.

With this definition the norm on $\mathbb{C}^n$ satisfies the properties in Proposition 1.3.2.

Exercise. *Explain why the triangle inequality in $\mathbb{C}^n$ is a consequence of the triangle inequality in $\mathbb{R}^n$.*

Definition 1.4.4. Let $\mathbf{z} = (z_1, \ldots, z_n), \mathbf{w} = (w_1, \ldots w_n) \in \mathbb{C}^n$. Then the *dot product* (or *scalar product* or *inner product*) of $\mathbf{z}$ and $\mathbf{w}$ is denoted $\mathbf{z} \cdot \mathbf{w}$ and defined by

$$\mathbf{z} \cdot \mathbf{w} = \sum_{k=1}^{n} z_k \overline{w_k} = z_1 \overline{w_1} + z_2 \overline{w_2} + \cdots + z_n \overline{w_n}.$$

Remark 1.4.5. If the entries of $\mathbf{z}$ and $\mathbf{w}$ are real, the definition of the dot product (or length) in $\mathbb{C}^n$ and $\mathbb{R}^n$ coincide. This is because in this case $\overline{w_k} = w_k$ and so

$$\sum_{k=1}^{n} z_k \overline{w_k} = \sum_{k=1}^{n} z_k w_k.$$

Chapter 2

Vector Spaces

In order to define a vector space, we first need to choose a set of scalars, which for us will always be either $\mathbb{R}$ or $\mathbb{C}$. Most of the theory we will develop works equally well for both so we will usually write $\mathbb{F}$ to mean either $\mathbb{R}$ or $\mathbb{C}$ (or neglect to mention the set of scalars at all) when it doesn't matter. For example, we will write $\mathbb{F}^n$ to mean either $\mathbb{R}^n$ or $\mathbb{C}^n$.

Informally, a vector space is a set of objects which can be added together and multiplied by scalars. In addition, we want these operations to obey the usual rules of algebra that we are used to from $\mathbb{R}^n$.

Definition 2.0.1. A *vector space* is any collection of objects V (called *vectors*) for which two operations can be performed:

- *Vector addition*, which takes two vectors $\mathbf{v}, \mathbf{w} \in V$ and returns another vector $\mathbf{v} + \mathbf{w} \in V$ (in this case we say that V is *closed under addition*).
- *Scalar multiplication*, which takes a vector $\mathbf{v} \in V$ and scalar $\alpha \in \mathbb{F}$ and returns a vector $\alpha\mathbf{v} \in V$ (in this case we say that V is *closed under scalar multiplication*).

Furthermore, the following properties (or *axioms*) must be satisfied.

The first four properties relate to addition:

1. Commutativity: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ for all $\mathbf{v}, \mathbf{w} \in V$;
2. Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$;
3. Zero vector: there exists a special vector, denoted $\mathbf{0}$, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$;
4. Additive inverse: For every vector $\mathbf{v} \in V$, there is a vector $-\mathbf{v} \in V$ such that $-\mathbf{v} + \mathbf{v} = \mathbf{0}$;

The next two properties relate to scalar multiplication:

5. Multiplicative identity: $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$;
6. Multiplicative associativity: $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$ for all $\mathbf{v} \in V$ and all $\alpha, \beta \in \mathbb{F}$.

The last two distributive properties connect vector addition and scalar multiplication:

7. $\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$ fora all $\mathbf{v}, \mathbf{w} \in V$ and all $\alpha \in \mathbb{F}$;
8. $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$ for all $\mathbf{v} \in V$ and all $\alpha, \beta \in \mathbb{F}$.

The list of properties above seems daunting but it can be summarised as saying that *the usual laws of algebra work*. Verifying that all of the properties hold can be tedious, but in many cases we don't have to verify all of them.

Remark 2.0.2. If the scalars are the real numbers we say that V is a *real* vector space or V is a vector space *over* $\mathbb{R}$. Similarly, if the scalars are the complex numbers we say that V is a *complex* vector space or that V is vector space over $\mathbb{C}$. Note that every complex vector space is also a real vector space because if we can multiply by complex numbers then we can certainly multiply by real numbers.

2.1 Examples

Column vectors. For each $n \geq 1$, the spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ (with the standard addition and scalar multiplication) are vector spaces. $\mathbb{R}^n$ is a real vector space but not a complex vector space, whereas $\mathbb{C}^n$ can be either a real or complex vector space. In particular, the set of scalars $\mathbb{R} = \mathbb{R}^1$ or $\mathbb{C} = \mathbb{C}^1$ can itself be thought of as a vector space (although this is not a very interesting vector space).

Zero vector space. An even less interesting space is the space $\{\mathbf{0}\}$ consisting of only the zero vector.

Polynomials. For $n \geq 0$, let $\mathbb{P}_n$ denote the space of all polynomials in one variable with degree at most n, i.e. $\mathbb{P}_n$ consists of all polynomials p of the form

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n,$$

where t is the independent variable and $a_0, a_1, \ldots, a_n \in \mathbb{F}$. Note that some, or all of the coefficients a_k can be 0. Addition and scalar multiplication are defined in the standard manner, e.g.

$$(4t^3 - 2t^2 + 5t + 1) + (-6t^3 + t + 4) = -2t^3 - 2t^2 + 6t + 5$$

and

$$3(4t^3 - 2t^2 + 5t + 1) = 12t^3 - 6t^2 + 15t + 3.$$

Then $\mathbb{P}_n$ is a vector space. In particular, if we insist the coefficients are real we have a real vector space, if the coefficients are complex we have a complex vector space.

Functions. Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ denote the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. As before, we define addition and scalar multiplication in the usual way. Then $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is a real vector space. Similarly, the set of all functions from $\mathbb{R}$ to $\mathbb{C}$, $\mathcal{F}(\mathbb{R}, \mathbb{C})$, is a complex vector space.

Solutions of differential equations. An *ordinary differential equation*, or *ODE*, is an equation involving some unknown function and its derivatives. The order of the ODE is the highest derivative to appear in the equation. A *linear* ODE of order n of the unknown function u is an ODE of the form

$$a_n(x)\frac{d^n u}{dx^n} + a_{n-1}(x)\frac{d^{n-1}u}{dx^{n-1}} + \cdots a_1(x)\frac{du}{dx} + a_0(x)u = b(x), \tag{2.1.1}$$

where $a_0(x), a_1(x), \cdots, a_n(x)$ and $b(x)$ are known functions. (2.1.1) is called *homogeneous* if $b(x) = 0$. For example

- $\dfrac{du}{dx} + 4u = 2 + e^{-x}$ is a first order linear ODE of the unknown which is not homogeneous;
- $3\dfrac{d^2u}{dx^2} + x\frac{du}{dx} + u\sin x = 0$ is a second order linear homogeneous ODE.

The set of all solutions of linear homogeneous ODE is a vector space. Let us illustrate this with an example. Consider the first order ODE

$$x\frac{du}{dx} + \sin(x)u = 0. \tag{2.1.2}$$

Suppose that u and v are both solutions of (2.1.2) so that

$$x\frac{du}{dx} + \sin(x)u = 0 \quad \text{and} \quad x\frac{dv}{dx} + \sin(x)v = 0.$$

Then

$$x\frac{d(u+v)}{dx} + \sin(x)(u+v) = x\left(\frac{du}{dx} + \frac{dv}{dx}\right) + \sin(x)(u+v) \tag{2.1.3}$$

$$= x\frac{du}{dx} + \sin(x)u + x\frac{dv}{dx} + \sin(x)v = 0. \tag{2.1.4}$$

So $u+v$ is also a solution. If α is a scalar, we also have

$$x\frac{d(\alpha u)}{dx} + \sin(x)(\alpha u) = \alpha\left(x\frac{du}{dx} + \sin(x)u\right) = 0.$$

Hence αu is also a solution. We have shown that the set of solutions of (2.1.2) is closed under addition and scalar multiplication. One can easily verify that the remaining properties of a vector space hold.

2.1.1 Examples that are not vector spaces

Integers. The set of integers $\mathbb{Z}$ with the usual addition and scalar multiplication is not a vector space since it is not closed under scalar multiplication. For example $\frac{1}{2}3 \notin \mathbb{Z}$. More generally, the set of columns of length n with integer entries, denoted $\mathbb{Z}^n$, is not a vector space for the same reason.

Unit vectors. A vector $\mathbf{v} \in \mathbb{F}^n$ is a *unit vector* if $\|\mathbf{v}\| = 1$. The space of all unit vectors in $\mathbb{F}^n$ with the usual addition and scalar multiplication is not a vector as it is neither closed under addition nor scalar multiplication. For example the sum of the unit vectors $(1,0)$ and $(0,1)$ in $\mathbb{R}^2$ is not a unit vector.

Positive real axis. The space $\mathbb{R}_+ = (0,\infty)$ of all positive real numbers, with the usual addition and scalar multiplication from $\mathbb{R}$, is closed under addition and satisfies most of the properties of vector spaces, but is not a vector space because we can't multiply by negative scalars. In addition, it does not have a zero vector since it does not contain 0.

Exercise. *Does the set of all functions $f : [0,1] \to [0,1]$, with the usual operations, form a vector space?*

2.2 Digression. Matrices

A *matrix* is a rectangular array of numbers. For example

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 6 & 9 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 5 \\ 7 & 9 \end{pmatrix} \qquad \begin{pmatrix} 1+i & 2 \\ \sqrt{3} & -2i \\ 1-i & i\pi/2 \end{pmatrix}.$$

The individual numbers are called the *entries*, *elements* or *components* of the matrix. If the matrix has m rows and n columns we say that it is an $m \times n$ matrix (pronounced 'm by n'). The above examples have respective sizes 2×3, 4×1, 5×5, 2×2 and 3×2. If $m = n$ we say that it is a *square* matrix.

The set of all $m \times n$ matrices with entries in $\mathbb{F}$ will be denoted $M_{m,n}(\mathbb{F})$. If $m = n$ then we will simply write $M_n(\mathbb{F})$.

Remark 2.2.1. Elements of $\mathbb{R}^n$ or $\mathbb{C}^n$ can be thought of as $n \times 1$ matrices (or $1 \times n$ matrices if you are working with row vectors).

2.2.1 Matrix notation

It is often convenient to use subscripts to label the entries of a matrix. Given a matrix A, we typically write a_{ij} to denote the entry in the ith row and jth column. [1] In this notation a typical 3×4 matrix is

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}.$$

At times we may also write A_{ij} to denote the the entry in the ith row and jth column (i.e. we may use the same letter rather than a the lowercase version). Moreover we may write

$$A = (a_{ij})$$

without specifying its size. The columns (or rows) of an $m \times n$ matrix can be regarded as vectors in $\mathbb{F}^m$ (respectively $\mathbb{F}^n$). Let the columns of A be $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbb{F}^m$. We may informally write

$$A = \begin{pmatrix} \uparrow & \uparrow & \ldots & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \\ \downarrow & \downarrow & \ldots & \downarrow \end{pmatrix}$$

when we wish to regard A as an ordered list of its columns. We may occasionally use analogous notation when regarding a matrix as a list of its rows.

Definition 2.2.2. Let A be a matrix. Then the *transpose* of A, denoted A^T, is obtained by interchanging the rows and columns of A, i.e. if $A = (a_{ij})$ then $A^T = (a_{ji})$.

[1] We typically use an uppercase Latin letter to denote a matrix and the same lowercase letter (with subscripts) to denote its entries.

For example

$$\begin{pmatrix}1 & 2 & 3\\ 0 & 6 & 9\end{pmatrix}^T = \begin{pmatrix}1 & 0\\ 2 & 6\\ 3 & 9\end{pmatrix}, \qquad \begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix}^T = \begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}, \qquad \begin{pmatrix}1 & 3 & 5\\ 7 & 9 & 11\\ 2 & 4 & 6\end{pmatrix}^T = \begin{pmatrix}1 & 7 & 2\\ 3 & 9 & 4\\ 5 & 11 & 6\end{pmatrix}.$$

Remark 2.2.3. Observe that $(A^T)^T = A$.

2.2.2 Matrices as vectors

Addition and scalar multiplication are defined entry-wise, i.e if $A = (a_{ij})$ and $B = (b_{ij})$ are two $m \times n$ matrices, and α is a scalar, then

$$A + B = (a_{ij} + b_{ij}) = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n}\\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn}\end{pmatrix};$$

$$\alpha A = (\alpha a_{ij}) = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \dots & \alpha a_{1n}\\ \alpha a_{21} & \alpha a_{22} & \dots & \alpha a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \alpha a_{m1} & \alpha a_{m2} & \dots & \alpha a_{mn}\end{pmatrix}.$$

With these operations $M_{m,n}(\mathbb{F})$ is a vector space. To be precise, the set of $m \times n$ matrices with real entries is a real vector space and the set of $m \times n$ matrices with complex entries is a complex vector space.

Exercise. *Is the set of all 2×2 matrices with integer entries, denoted $M_{2,2}(\mathbb{Z})$, a real vector space?*

2.3 Subspaces

Definition 2.3.1. Let V be a vector space. A non-empty set $W \subseteq V$ is a *subspace* of V if

(1) for every $\mathbf{u}, \mathbf{v} \in W$, $\mathbf{u} + \mathbf{v} \in W$ (W is closed under addition);

(2) for every $\alpha \in \mathbb{F}$ and $\mathbf{v} \in W$, $\alpha\mathbf{v} \in W$ (W is closed under scalar multiplication).

In other words, a subspace of V is a subset of V which is also a vector space with the same operations of addition and scalar multiplication inherited from V.

Remark 2.3.2. The conditions in Definition 2.3.1 can be replaced by the condition

(1') for every $\alpha, \beta \in \mathbb{F}$ and every $\mathbf{u}, \mathbf{v} \in W$, $\alpha\mathbf{u} + \beta\mathbf{v} \in W$.

A consequence of the conditions of the conditions (1) and (2) above (or condition (1')) is the following key point:

> If W is a subspace, it must contain all linear combinations of vectors in W. In particular, W must contain $\mathbf{0}$.

2.3.1 Examples (and non-exmaples) of subspaces

(1) Let V be a vector space. Then $\{\mathbf{0}\}$ and V itself are subspaces of V. These are usually referred to as trivial subspaces.

(2) $\mathbb{P}_2$, the space of polynomials of degree at most 2, is a subspace of $\mathbb{P}_3$.

(3) Let W denote the subset of $\mathbb{R}^2$ consisting of all $(x, y) \in \mathbb{R}^2$ such that $x = 1$.

Let us check if W is closed under addition: Take $\mathbf{v}_1, \mathbf{v}_2 \in W$ with $\mathbf{v}_1 = (x_1, y_1)$ and $\mathbf{v}_2 = (x_2, y_2)$, then

$$\mathbf{v}_1 + \mathbf{v}_2 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}.$$

However $x_1 + x_2 = 1 + 1 = 2$. So $\mathbf{v}_1 + \mathbf{v}_2 \notin W$. Thus W is not closed under addition, and hence not a subspace of $\mathbb{R}^2$. Similarly, one can show that W is not closed under scalar multiplication. However, the easiest way to see that W is not a subspace is to observe that it does not contain $\mathbf{0}$.

(4) Let U be the set of all $(x, y, z) \in \mathbb{R}^3$ such that $x = 2y + z$.

Let us check if U is closed under addition: Take $\mathbf{v}_1, \mathbf{v}_2 \in U$ with $\mathbf{v}_1 = (x_1, y_1, z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2)$, then

$$\mathbf{v}_1 + \mathbf{v}_2 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}.$$

We have that

$$x_1 + x_2 = (2y_1 + z_1) + (2y_2 + z_2) = 2(y_1 + y_2) + (z_1 + z_2).$$

Hence $\mathbf{v}_1 + \mathbf{v}_2 \in U$.

Let us check if U is closed under scalar multiplication: Take $\mathbf{v} \in U$ with $\mathbf{v} = (x, y, z)$ and $\alpha \in \mathbb{R}$, then

$$\alpha\mathbf{v} = \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix}.$$

We have that

$$\alpha x = \alpha(2y + z) = 2\alpha y + \alpha z.$$

Hence $\alpha\mathbf{v} \in U$.

We have shown that U is a subspace of $\mathbb{R}^3$.

(5) Let E be the subset of $\mathbb{P}_n$ consisting of all *even* polynomials, i.e. all $p \in \mathbb{P}_n$ satisfying $p(t) = p(-t)$.

Let us check if E is closed under addition: Take $p, q \in \mathbb{P}_n$. Then $p(t) + q(t) = p(-t) + q(-t)$ and so $p + q \in E$. Let us check if E is closed under scalar multiplication: Take $p \in E$ and $\alpha \in \mathbb{F}$. Then $\alpha p(t) = \alpha p(-t)$ and so $\alpha p \in E$. Hence E is a subspace of $\mathbb{P}_n$.

(6) Let S be the subset of $\mathbb{P}_n$ consisting of all polynomials $p \in \mathbb{P}_n$ such that $p(0) = 1$. Then S doesn not contain the zero function so cannot be a subspace. One can also show that S is not closed under addition or scalar multiplication.

Exercise. *For each of the following conditions, determine if the set of all vectors* $(x, y) \in \mathbb{R}^2$ *which satisfies it is a subspace of* $\mathbb{R}^2$.

(a) $x + y = 1$ *(b)* $x + 2y = 0$ *(c)* $x = y$ *(d)* $x^2 = y^2$

Proposition 2.3.3. *Let* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *be vectors in* V. *Then* $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is a subspace of* V. *Moreover, if* W *is any subspace of* V *containing* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, *then* W *must contain* $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

Proposition 2.3.3 says that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is the *smallest* subspace of V which contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Proof. First let us show that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a subspace. Take $\mathbf{u}, \mathbf{w} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ so that

$$\mathbf{u} = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k \quad \text{and} \quad \mathbf{w} = \sum_{k=1}^{n} \beta_k \mathbf{v}_k.$$

Then

$$\mathbf{u} + \mathbf{w} = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k + \sum_{k=1}^{n} \beta_k \mathbf{v}_k = \sum_{k=1}^{n} (\alpha_k + \beta_k) \mathbf{v}_k \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}.$$

Take $\beta \in \mathbb{F}$. Then

$$\beta \sum_{k=1}^{n} \alpha_k \mathbf{v}_k = \sum_{k=1}^{n} (\beta \alpha_k) \mathbf{v}_k \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}.$$

Hence $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a subspace of V.

Now suppose that W is a subspace of V containing $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Then since W is a subspace, it must also contain all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, so it contains $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. □

Example 2.3.4. The only subspaces of $\mathbb{R}^3$ are $\{\mathbf{0}\}$, lines through the origin (the linear span of a single vector), planes through the origin (the linear span of two vectors) and all of $\mathbb{R}^3$. The only subspaces of $\mathbb{R}^2$ are $\{\mathbf{0}\}$, lines through the origin and all of $\mathbb{R}^2$.

2.4 Bases

Definition 2.4.1. A collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ is a *basis* for V if every vector $\mathbf{v} \in V$ admits a *unique* representation as a linear combination

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k.$$

The coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called *coordinates* of $\mathbf{v}$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Example 2.4.2. Let $V = \mathbb{F}^n$ and consider the vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

The vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ form a basis in $\mathbb{F}^n$. Indeed if

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix},$$

then

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n = \sum_{k=1}^{n} v_k\mathbf{e}_k,$$

and this representation is unique. The basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is called the *standard basis* in $\mathbb{F}^n$. There are many other bases in $\mathbb{F}^n$.

Example 2.4.3. Let $V = \mathbb{P}_n$ be the space of polynomials in one variable with degree at most n and consider the vectors (polynomials)

$$\mathbf{e}_0(t) = 1, \quad \mathbf{e}_1(t) = t, \quad \mathbf{e}_2(t) = t^2, \dots, \mathbf{e}_n(t) = t^n.$$

Then any polynomial $p(t) = a_0 + a_1t + \cdots + a_nt^n$ admits a unique representation

$$p = a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n = \sum_{k=1}^{n} a_k\mathbf{e}_k.$$

Hence $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ form a basis of $\mathbb{P}_n$. We call this the *monomial basis* of $\mathbb{P}_n$.

Remark 2.4.4. Every vector space has a (possibly infinite) basis. The proof of this fact is beyond the scope of this course.

Exercise. *Find a basis for the real vector space $M_{2,2}(\mathbb{R})$ (the space of all 2×2 matrices with real entries).*

2.4.1 Coordinate vector

Let V be a vector space with basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$. Then any $\mathbf{v}$ is uniquely determined its coordinates with respect to $\mathcal{E}$, i.e. the coefficients in the expansion $\mathbf{v} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \cdots + \alpha_n\mathbf{e}_n$. It is convenient to write these coordinates in a column vector.

Definition 2.4.5. Let V be a vector space with basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ ane let $\mathbf{v} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \cdots + \alpha_n\mathbf{e}_n \in V$. Then the *coordinate vector* of $\mathbf{v}$ with respect to $\mathcal{E}$ is the column vector

$$[\mathbf{v}]_{\mathcal{E}} := \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n.$$

Thus we have a one-to-one correspondence between V and $\mathbb{F}^n$:

$$\mathbf{v} \longleftrightarrow [\mathbf{v}]_{\mathcal{E}}.$$

Moreover if $\mathbf{w} = \beta_1\mathbf{e}_1 + \cdots + \beta_n\mathbf{e}_n$ then

$$\mathbf{v} + \mathbf{w} = (\alpha_1 + \beta_1)\mathbf{e}_1 + (\alpha_2 + \beta_2) + \cdots + (\alpha_n + \beta_n)\mathbf{e}_n = \sum_{k=1}^{n} (\alpha_k + \beta_k)\mathbf{e}_k.$$

and if $\beta \in \mathbb{F}$,

$$\beta\mathbf{v} = \beta\alpha_1\mathbf{e}_1 + \beta\alpha_2\mathbf{e}_2 + \cdots + \beta\alpha_n\mathbf{e}_n = \sum_{k=1}^{n} \beta\alpha_k\mathbf{e}_k.$$

Hence

$$[\mathbf{v} + \mathbf{w}]_{\mathcal{E}} = [\mathbf{v}]_{\mathcal{E}} + [\mathbf{w}]_{\mathcal{E}} \quad \text{and} \quad [\beta\mathbf{v}]_{\mathcal{E}} = \beta[\mathbf{v}]_{\mathcal{E}}.$$

In other words, addition and scalar multiplication in V corresponds to addition and scalar multiplication of the coordinate vectors in $\mathbb{F}^n$.

Remark 2.4.6. The coordinate vector of $\mathbf{v}$ depends on the choice of basis. If we choose a different basis, the coordinates of $\mathbf{v}$ will change.

2.4.2 Spanning and linearly independent sets

The definition of a basis states that each vector $\mathbf{v} \in V$ admits a unique representation as a linear combination of vectors in our basis. Thus in order to determine if a set of vectors is a basis we need to determine two distinct facts:

(1) that such a representation exists for each vector;
(2) that it is unique.

First we consider only the existence part.

Definition 2.4.7. We say that a collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ *spans* V, or that it is a *spanning* set (or *generating* set), if every vector $\mathbf{v} \in V$ admits a representation as a linear combination

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k.$$

Equivalently, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span V if

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = V.$$

The difference between the definitions of a basis and a spanning set is that for a basis, the representation of each vector as a linear combination must be unique whereas for a spanning set, there may be many different representations for the same vector.

Example 2.4.8. Consider the following vectors in $\mathbb{R}^2$:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

Then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ spans $\mathbb{R}^2$. Indeed if $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ then

$$\mathbf{v} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = x\mathbf{v}_1 + y\mathbf{v}_2 + 0\mathbf{v}_3 + 0\mathbf{v}_4.$$

However, we also have

$$\mathbf{v} = 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - x \begin{pmatrix} -1 \\ 0 \end{pmatrix} - y \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0\mathbf{v}_1 + 0\mathbf{v}_2 - x\mathbf{v}_3 - y\mathbf{v}_4,$$

and so $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ is not a basis.

Remark 2.4.9. As in the example above, one can always add additional vectors to a spanning set and still get a spanning set. However removing a vector from a spanning set may cause it to stop spanning. For example, the vectors $(1, 0)$ and $(0, 1)$ span $\mathbb{R}^2$ but either one on its own does not.

Example 2.4.10. *Show that the vectors* $(1, 0, 0)$, $(1, 1, 0)$ *and* $(1, 1, 1)$ *span* $\mathbb{R}^3$.

Solution. Take an arbitrary vector $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$. We need to show that it can be expressed as a linear combination of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Equating components we get

$$x = \alpha + \beta + \gamma, \tag{2.4.1}$$
$$y = \beta + \gamma, \tag{2.4.2}$$
$$z = \gamma. \tag{2.4.3}$$

Substituting (2.4.3) into (2.4.2) we get $\beta = y - z$, and then substituting these into (2.4.1) we get $\alpha = x - y$. Hence

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x-y)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (y-z)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Since $\mathbf{v}$ was arbitrary, this shows that every $\mathbf{v} \in \mathbb{R}^3$ is in the linear span of $(1,0,0)$, $(1,1,0)$ and $(1,1,1)$. □

Exercise. *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$. *Suppose* $\operatorname{span}\{\mathbf{u}, \mathbf{v}\} = \mathbb{R}^2$. *Which of the following statements are true?*

(a) $\operatorname{span}\{\mathbf{u}\} = \mathbb{R}^2$ *or* $\operatorname{span}\{\mathbf{v}\} = \mathbb{R}^2$.
(b) $\operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathbb{R}^2$.
(c) $\operatorname{span}\{\mathbf{u}, \mathbf{u}+\mathbf{v}\} = \mathbb{R}^2$

Example 2.4.11. *Determine if the vectors* $(1,1,1)$, $(1,-2,2)$ *and* $(2,-1,3)$ *span* $\mathbb{R}^3$.

Solution. As before we start by taking an arbitrary vector $\mathbf{v} = (x,y,z) \in \mathbb{R}^3$ and trying to solve the equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \gamma\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

Equating components we must have

$$x = \alpha + \beta + 2\gamma, \tag{2.4.4}$$
$$y = \alpha - 2\beta - \gamma, \tag{2.4.5}$$
$$z = \alpha + 2\beta + 3\gamma. \tag{2.4.6}$$

Subtracting (2.4.5) from (2.4.4) and (2.4.6) from (2.4.4) gives

$$x - y = 3\beta + 3\gamma,$$
$$x - z = -\beta - \gamma.$$

Hence $x - y = -3(x - z)$, and so

$$4x - y + 3z = 0 \tag{2.4.7}$$

The set of all $\mathbf{v} = (x, y, z)$ that satisfy (2.4.7) is a plane. Thus $(1, 1, 1)$, $(1, -2, 2)$ and $(2, -1, 3)$ cannot span all of $\mathbb{R}^3$. □

Next we consider only the uniqueness of of a representation of a vector as a linear combination.

Definition 2.4.12. A collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ are *linearly dependent* if there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$, not all zero, such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}.$$

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ are *linearly independent* if they are not linearly dependent.

Of course by taking all the coefficients α_k to be 0, we get one representation of $\mathbf{0}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. So the definition above is saying that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are

- linearly independent if there is precisely one representation of $\mathbf{0}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ (i.e. it is unique for $\mathbf{0}$);
- linearly dependent if there is more than one such representation.

Remark 2.4.13. Clearly any basis is linearly independent; since every vector has a unique representation as a linear combination of its elements, the zero vector certainly has a unique representation.

Proposition 2.4.14. *The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ are linearly dependent if and only if one of the vectors can be represented as a linear combination of the others.*

Proof. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent. This means that there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$, which are not all zero, such that

$$\mathbf{0} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n.$$

Assume without loss of generality that $\alpha_1 \neq 0$ (we can relabel the terms if we wish so that $\alpha_1 \neq 0$). Then

$$\alpha_1\mathbf{v}_1 = -\alpha_2\mathbf{v}_2 - \cdots - \alpha_n\mathbf{v}_2,$$

and so

$$\mathbf{v}_1 = \beta_2\mathbf{v}_2 + \cdots + \beta_n\mathbf{v}_n, \tag{2.4.8}$$

where $\beta_k = -\alpha_k/\alpha_1$.

Conversely, suppose that (2.4.8) holds. Then we can write

$$\mathbf{0} = -\mathbf{v}_1 + \mathbf{v}_1 = -\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_n\mathbf{v}_n,$$

which is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ where the coefficient of $\mathbf{v}_1$ is -1. Thus $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent. □

Remark 2.4.15. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent. Then by Proposition 2.4.14 one of the vectors, say $\mathbf{v}_1$, is a linear combination of the others. Then any vector that can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ can also be written as a linear combination of $\mathbf{v}_2, \ldots, \mathbf{v}_n$, i.e.

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \operatorname{span}\{\mathbf{v}_2, \ldots, \mathbf{v}_n\}.$$

In particular, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are spanning, so are $\mathbf{v}_2, \ldots, \mathbf{v}_n$. To summarise:

> If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent, then one of the $\mathbf{v}_k$ is a linear combination of the others and we can remove this vector without changing their linear span.

Example 2.4.16. *Show that the vectors*

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \textit{and} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

are linearly independent.

Solution. Suppose that $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}$. We need to show that $\alpha = \beta = 0$. Equating components gives

$$\alpha + 2\beta = 0, \tag{2.4.9}$$
$$2\alpha + 3\beta = 0. \tag{2.4.10}$$

From (2.4.9) we get that $\alpha = -2\beta$. Substituting this into (2.4.10) gives $-4\beta + 3\beta = 0$ and so $\beta = 0$ and $\alpha = -2\beta = 0$. □

Proposition 2.4.17. *A collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ is a basis if and only if it is linearly independent and spans V.*

Proof. We have already seen that every basis of V is linearly independent and spans V so it only remains to show the converse.

Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent and span V. Then, since they are spanning, any $\mathbf{v} \in V$ admits a representation

$$\mathbf{v} = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k \tag{2.4.11}$$

We only need to show that this representation is unique. Suppose that we have another representation of $\mathbf{v}$:

$$\mathbf{v} = \sum_{k=1}^{n} \beta_k \mathbf{v}_k.$$

Then

$$\mathbf{0} = \mathbf{v} - \mathbf{v} = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k - \sum_{k=1}^{n} \beta_k \mathbf{v}_k = \sum_{k=1}^{n} (\alpha_k - \beta_k) \mathbf{v}_k.$$

Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent we must have $\alpha_k - \beta_k = 0$ for each $k = 1, \ldots, n$. Hence the representation (2.4.11) is unique. $\square$

Exercise. *Let V be a vector space. Which of the following statements are true?*

(a) Removing a vector from a basis of V produces another basis of V.
(b) Adjoining another vector to a basis of V produces another basis of V.
(c) Adjoining the zero vector to a basis of V produces another basis of V.

Example 2.4.18. *Find a basis for the subspace S of $\mathbb{R}^3$ consisting of all $(x, y, z) \in \mathbb{R}^3$ such that $x + y = 2z$.*

Solution. Suppose $\mathbf{v} = (x, y, z) \in S$. Then $x = 2z - y$ and so

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z - y \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 2z \\ 0 \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

Here y, z can be arbitrary real numbers. Therefore S is the linear span of $(-1, 1, 0)$ and $(2, 0, 1)$. Since we also have that $(-1, 1, 0)$ and $(2, 0, 1)$ are linearly independent, the set $\{(-1, 1, 0), (2, 0, 1)\}$ is a basis of S. $\square$

Finally we show that we can always find a basis from a spanning set.

Proposition 2.4.19. *Any finite spanning set contains a basis.*

Proof. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ be a spanning set in V. If they are linearly independent then we are done, so we can suppose that they are linearly dependent. In this case, by Proposition 2.4.14 and Remark 2.4.15, we can remove one of the vectors and the new set will still span V. If this new set is linearly independent then we are done. Otherwise we repeat this process. Continuing in this way, we must eventually come to a linearly independent spanning set, because otherwise we would remove all the vectors. $\square$

2.4.3 Dimension

Theorem 2.4.20 (Dimension theorem). *Let V be a vector space. Then every basis of V has the same number of elements. Moreover, if V has a basis of n elements then*

(a) any set of vectors in V with less than n elements doesn't span V;

(b) any set of vectors in V with more than n elements is linearly dependent.

We will prove this later in the course.

Definition 2.4.21. The dimension of a vector space V is denoted $\dim V$ and defined to be the number of elements in a basis for V. If V consists of only the zero vector we set $\dim V = 0$ and if V does not have a finite basis we set $\dim V = \infty$.

Exercise. *What is the dimesion of $\mathbb{P}_n$, the vector space of all polynomials of degree at most n?*

In a finite dimensional space, any linearly independent collection of vectors can be completed to a basis.

Proposition 2.4.22. *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in V$ be linearly independent vectors in a finite dimensional vector space V. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ do not span V, there exist vectors $\mathbf{v}_{m+1}, \mathbf{v}_{m+2}, \ldots, \mathbf{v}_n$ such that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis of V.*

Proof. Suppose that $\dim V = n$. Then since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent but do not span V, Theorem 2.4.20 implies that $m < n$. Take any vector $\mathbf{v}_{m+1}$ which is not in the linear span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m, \mathbf{v}_{m+1}$ are still linearly independent (see Question 6 on Assignment sheet 5). We can repeat the process to get $\mathbf{v}_{m+2}$, and so on, until we have a spanning set. Observe that, by Theorem 2.4.20, this process must terminate when we have n vectors since $\dim V = n$. Thus the collection $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ will be a basis for V. $\square$

Theorem 2.4.20 and Proposition 2.4.22 have the following important consequence:

Proposition 2.4.23. *A collection of n vectors in an n dimensional vector space is spanning if and only if it is linearly independent.*

Proof. Let V be an n dimensional vector space and let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$. Assume that they span V. If they are not linearly independent, we can remove one of the $\mathbf{v}_k$ without affecting the span (see Remark 2.4.15). This would leave a collection of $n-1$ vectors which span V. However, by Theorem 2.4.20, any collection with less than n vectors cannot span V. This is a contradiction, and so $\mathbf{v}_1, \ldots, \mathbf{v}_n$ must be linearly independent.

Conversely, suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ are linearly independent. If they do not span V, by Proposition 2.4.22 we can find a vector $\mathbf{v}_{n+1} \in V$ such that $\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}_{n+1}$ are linearly independent. However, by Theorem 2.4.20, any collection with more than n vectors cannot be linearly independent. This is a contradiction, and so $\mathbf{v}_1, \ldots, \mathbf{v}_n$ must span V. $\square$

A useful consequence of Proposition 2.4.23 is the following:

> If we want to check if a collection of n vectors in $\mathbb{F}^n$ is a basis, we only need to check that it is linearly independent (or that it is spanning).

In general, linear independence is easier to check, because you are solving a system of equations where one side is zero.

Example 2.4.24. *Determine if the vectors* $(1, -3, 2)$, $(2, 1, -3)$ *and* $(-3, 2, 1)$ *form a basis of* $\mathbb{R}^3$.

Solution. Since there are 3 vectors in $\mathbb{R}^3$, we only need to check whether they are linearly independent.

Suppose that

$$\alpha \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \gamma \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = 0. \tag{2.4.12}$$

This is equivalent to the linear equations

$$\alpha + 2\beta - 3\gamma = 0, \tag{2.4.13}$$
$$-3\alpha + \beta + 2\gamma = 0, \tag{2.4.14}$$
$$2\alpha - 3\beta + \gamma = 0. \tag{2.4.15}$$

Adding 3×(2.4.13) to (2.4.14) and substracting 2×(2.4.13) from (2.4.15) to eliminate α gives

$$7\beta - 7\gamma = 0, \tag{2.4.16}$$
$$-7\beta + 7\gamma = 0. \tag{2.4.17}$$

Thus $\beta = \gamma$. Substituting this into (2.4.13) gives $\alpha - \beta = 0$. Thus any choice of α, β, γ such that $\alpha = \beta = \gamma$ will satisfy (2.4.12). This shows that $(1, 3, 2)$, $(2, 1, 3)$ and $(3, 2, 1)$ are not linearly independent and hence not a basis. (Observe that this also shows that they do not span $\mathbb{R}^3$). □

Exercise. *What about the vectors* $(1, 3, 2)$, $(2, 1, 3)$ *and* $(3, 2, 1)$*? Do they form a basis of* $\mathbb{R}^3$*?*

Chapter 3

Linear Maps

Definition 3.0.1. Let V and W be vector spaces over $\mathbb{F}$ (either both real or both complex). Then a map (or transformation) $T : V \to W$ is *linear* if it satisfies the following two conditions:

(1) for every $\mathbf{u}, \mathbf{v} \in V$, $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$;

(2) for every $\mathbf{v} \in V$ and every $\alpha \in \mathbb{F}$, $T(\alpha\mathbf{v}) = \alpha T(\mathbf{v})$.

Once again, the two conditions in Definition 3.0.1 can be combined and replaced by the condition

(1') for every $\mathbf{u}, \mathbf{v} \in V$ and every $\alpha, \beta \in \mathbb{F}$, $T(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v})$.

Remark 3.0.2. Observe that we have two different addition rules in the above definition: $\mathbf{u}$ and $\mathbf{v}$ are vectors in V so $\mathbf{u}+\mathbf{v}$ uses the addition in V, but $T(\mathbf{u})$ and $T(\mathbf{v})$ are vectors in W so $T(\mathbf{u})+T(\mathbf{v})$ uses the addition in W. One should keep in mind that an expression of the form $\mathbf{v} + T(\mathbf{v})$ would not necessarily make sense, unless V and W were equal.

3.0.1 Examples (and non-examples) of linear maps

(1) **Identity map.** Let V be a vector space and let $I : V \to V$ be the map given by $I(\mathbf{v}) = \mathbf{v}$. This is a linear map and is called the *identity* map.

(2) **Zero map.** Let V be a vector space and let $T : V \to V$ be the map given by $T(\mathbf{v}) = \mathbf{0}$. This is a linear map and is called the *zero* map.

(3) **Reflection.** Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the map given by reflecting in the x-axis, i.e.

$$T:\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}x\\-y\end{pmatrix}.$$

Then this is a linear map. Indeed for any $\mathbf{v}_1=(x_1,y_1)$, $\mathbf{v}_2=(x_2,y_2)$ we have

$$T\left(\mathbf{v}_1+\mathbf{v}_2\right)=\begin{pmatrix}x_1+x_2\\-(y_1+y_2)\end{pmatrix}=\begin{pmatrix}x_1\\-y_1\end{pmatrix}+\begin{pmatrix}x_2\\-y_2\end{pmatrix}=T(\mathbf{v}_1)+T(\mathbf{v}_2),$$

and for and $\mathbf{v}=(x,y)$ and $\alpha\in\mathbb{R}$ we have

$$T(\alpha\mathbf{v})=\begin{pmatrix}\alpha x\\-\alpha y\end{pmatrix}=\alpha\begin{pmatrix}x\\-y\end{pmatrix}=\alpha T(\mathbf{v}).$$

It can also be shown geometrically that this transformation is linear.

(4) **Rotation.** Let $T_\varphi:\mathbb{R}^2\to\mathbb{R}^2$ be the map that takes a vector in $\mathbb{R}^2$ and rotates it anti-clockwise (about the origin) by an angle φ. Since T_φ rotates the whole plane, it rotates the parallelogram used to define the sum of two vectors (parallelogram law). Therefore T_φ preserves addition. It is also easy to see that it preserves scalar multiplication. Hence T_φ is linear.

(5) **Scaling.** Let V be a vector space and let α be a scalar. Let $T:V\to V$ be the map given by $T(\mathbf{v})=\alpha\mathbf{v}$. Then T is linear. Indeed for $\mathbf{u},\mathbf{v}\in V$ we have

$$T(\mathbf{u}+\mathbf{v})=\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}=T(\mathbf{u})+T(\mathbf{v}),$$

and for every scalar β we have

$$T(\beta\mathbf{v})=\alpha(\beta\mathbf{v})=\beta(\alpha\mathbf{v})=\beta T(\mathbf{v}).$$

The identity map and the zero map are both special cases of this (scaling by 1 and 0 respectively).

(6) **Differentiation.** Let $\mathbb{P}$ denote the vector space of all polynomials in one variable t. Let $D:\mathbb{P}\to\mathbb{P}$ be the map given by

$$D(p)=\frac{dp}{dt}.$$

This is a linear map. Indeed for any $p,q\in\mathbb{P}$ we have

$$D(p+q)=\frac{d(p+q)}{dt}=\frac{dp}{dt}+\frac{dq}{dt}=D(p)+D(q),$$

and for every scalar α we have

$$D(\alpha p) = \frac{d(\alpha p)}{dt} = \alpha \frac{dp}{dt} = \alpha D(p).$$

(7) **Translation.** Let V be a vector space. Fix a non-zero vector $\mathbf{v}_0 \in V$. $T : V \to V$ be the map given by $T(\mathbf{v}) = \mathbf{v}_0 + \mathbf{v}$ (this is usually called *translation* by $\mathbf{v}_0$). Take $\mathbf{u}, \mathbf{v} \in V$. Then

$$T(\mathbf{u} + \mathbf{v}) = \mathbf{v}_0 + \mathbf{u} + \mathbf{v}.$$

But

$$T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{v}_0 + \mathbf{u} + \mathbf{v}_0 + \mathbf{v} = 2\mathbf{v}_0 + \mathbf{u} + \mathbf{v}.$$

So $T(\mathbf{u} + \mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$, and hence T is **not linear**.

Example 3.0.3. *Determine if the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by*

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$$

is linear.

Solution. Let $\mathbf{v} = (x, y)$ and $\alpha \in \mathbb{R}$. Then

$$T(\alpha \mathbf{v}) = \begin{pmatrix} (\alpha x)^2 \\ (\alpha y)^2 \end{pmatrix} = \alpha^2 \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \quad \text{but} \quad \alpha T(\mathbf{v}) = \alpha \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}.$$

Hence $T(\alpha \mathbf{v}) \neq \alpha T(\mathbf{v})$ unless $\alpha = 1$. Thus T is not linear. We could have also shown that T does not preserve addition. □

Example 3.0.4. *Determine if the map $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by*

$$T : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + y \\ z - y \end{pmatrix}$$

is linear.

Solution. Let $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$. Then

$$T(\alpha \mathbf{v}) = \begin{pmatrix} \alpha x + \alpha y \\ \alpha z - \alpha y \end{pmatrix} = \alpha \begin{pmatrix} x + y \\ z - y \end{pmatrix} = \alpha T \mathbf{v}.$$

So T preserves scalar multiplication.

Let $\mathbf{v}_1 = (x_1, y_1, z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2)$. Then

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T\left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}\right) = \begin{pmatrix} (x_1 + x_2) + (y_1 + y_2) \\ (z_1 + z_2) - (y_1 + y_2) \end{pmatrix}$$
$$= \begin{pmatrix} x_1 + y_1 \\ z_1 - y_1 \end{pmatrix} + \begin{pmatrix} x_2 + y_2 \\ z_2 - y_2 \end{pmatrix} = T(\mathbf{v}_1) + T(\mathbf{v}_2).$$

So T also preserves addition. This shows that T is linear. □

Remark 3.0.5. Let $\mathbf{0}_U$ and $\mathbf{0}_V$ denote the zero vectors in the vectors spaces U and V respectively, and let $T : U \to V$ be a linear map. Then $T(\mathbf{0}_U) = \mathbf{0}_V$.

Exercise. *Which of the following definitions would make $f : \mathbb{R}^2 \to \mathbb{R}$ linear?*

(a) $f(x, y) = x$ *(b)* $f(x, y) = xy$ *(c)* $f(x, y) = x + y$ *(d)* $f(x, y) = e^{x+y}$

3.0.2 Linear maps and bases

Let $T : V \to W$ be a linear map and let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ be a basis for V. Suppose we know the vectors $T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_n)$. Then if $\mathbf{v} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \dots + \alpha_n\mathbf{e}_n$, we have

$$T(\mathbf{v}) = T(\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \dots + \alpha_n\mathbf{e}_n) = \alpha_1 T(\mathbf{e}_1) + \alpha_2 T(\mathbf{e}_2) + \dots + \alpha_n T(\mathbf{e}_n).$$

Therefore, if we know the vectors $T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_n)$, we can compute $T(\mathbf{v})$ for every $\mathbf{v} \in V$. In particular, if $T : V \to W$ and $S : V \to W$ are linear maps such that

$$T(\mathbf{e}_1) = S(\mathbf{e}_1), \quad T(\mathbf{e}_2) = S(\mathbf{e}_2), \dots, \quad T(\mathbf{e}_n) = S(\mathbf{e}_n),$$

then $T = S$. We can summarize this as follows:

> A linear map is completely determined by how it acts on a basis.

3.1 Operations on linear maps

Addition and scalar multiplication

Let V and W be vectors spaces (over $\mathbb{F}$) and let $S : V \to W$ and $T : V \to W$ be linear maps (note that they have the same domain and range). We can add S and T by the

rule
$$(S+T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v}) \quad \text{for all} \quad \mathbf{v} \in V.$$
Then $S+T$ is also a linear map from V to W. It's clear that $S+T$ maps V to W so we just need to check that it is linear. To this end, we calculate that if $\mathbf{u}, \mathbf{v} \in V$ then

$$\begin{aligned}
(S+T)(\mathbf{u}+\mathbf{v}) &= S(\mathbf{u}+\mathbf{v}) + T(\mathbf{u}+\mathbf{v}) \qquad \text{(by the definition of } S+T\text{)} \\
&= S(\mathbf{u}) + S(\mathbf{v}) + T(\mathbf{u}) + T(\mathbf{v}) \qquad \text{(because } S \text{ and } T \text{ are both linear)} \\
&= (S+T)(\mathbf{u}) + (S+T)(\mathbf{v}) \qquad \text{(by the definition of } S+T \text{ again).}
\end{aligned}$$

Using the same reasoning we check that if $\mathbf{v} \in V$ and $\alpha \in \mathbb{F}$ then

$$(S+T)(\alpha\mathbf{v}) = S(\alpha\mathbf{v}) + T(\alpha\mathbf{v}) = \alpha S(\mathbf{v}) + \alpha T(\mathbf{v}) = \alpha(S+T)(\mathbf{v}).$$

This shows that $S+T$ is linear.

We can also multiply a linear map $T : V \to W$ by a scalar $\alpha \in \mathbb{F}$ according to the rule

$$(\alpha T)(\mathbf{v}) = \alpha T(\mathbf{v}).$$

One can similarly check that the map αT is also linear. Indeed if $\mathbf{u}, \mathbf{v} \in V$ and $\beta \in \mathbb{F}$ then
$$(\alpha T)(\mathbf{u}+\mathbf{v}) = \alpha T(\mathbf{u}+\mathbf{v}) = \alpha T(\mathbf{u}) + \alpha T(\mathbf{v}) = (\alpha T)(\mathbf{u}) + (\alpha T)(\mathbf{v}),$$
and
$$(\alpha T)(\beta\mathbf{v}) = \alpha T(\beta\mathbf{v}) = \alpha\beta T(\mathbf{v}) = \beta(\alpha T)(\mathbf{v}).$$
So αT is a linear map.

We have shown that the sum of two linear maps is linear and any scalar multiple of a linear map is linear. So with these operations the set of linear maps from V to W is closed under addition and scalar multiplication. Observe that these operations are defined pointwise - that is, they are defined at each point of the domain V separately using the addition and scalar multiplication in W (this is just the usual definition of addition and scalar multiplication of functions). It follows that they must satisfy the axioms of a vector space (because addition and scalar multiplication in W satisfy the axioms). Hence we have the following:

> If we fix vector spaces V and W, then the set of all linear maps from V to W is itself a vector space.

Composition

Let U, V and W be vector spaces and let $S : U \to V$ and $T : V \to W$ be linear maps (note that the range of S is the domain of T). We can consider the map we obtain by first applying S and then applying T, i.e. the map $T \circ S : U \to W$ defined by

$$(T \circ S)(\mathbf{u}) = T(S(\mathbf{u})) \quad \text{for all} \quad \mathbf{u} \in U.$$

Note that even though it is written $T \circ S$, with T first, S is applied first then T. We can visualize how these maps relate to each other using the following diagram:

$$\begin{tikzcd} U \arrow[rr, "T\circ S"] \arrow[dr, "S"'] & & W \\ & V \arrow[ur, "T"'] & \end{tikzcd}$$

It turns out that the map $T \circ S$ is also linear. The proof of this is left as an assignment question.

Notation: For linear maps, we usually write TS instead of $T \circ S$ for the composition of T with S.

3.2 Matrix of a linear map

Example 3.2.1. What are all the linear maps from $\mathbb{R}$ to $\mathbb{R}$?

Let $T : \mathbb{R} \to \mathbb{R}$ be a linear map. Then for any $x \in \mathbb{R}$ we have

$$T(x) = T(x \times 1) = xT(1).$$

Setting $\alpha = T(1)$ we see that $T(x) = \alpha x$ for each $x \in \mathbb{R}$. We have show that every linear map from $\mathbb{R}$ to $\mathbb{R}$ is just multiplication by a constant.

It will turn out that every linear map can be represented as a multiplication, not by a scalar, but by a matrix.

3.2.1 Linear maps $\mathbb{F}^n \to \mathbb{F}^m$. Matrix-vector multiplication

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis in $\mathbb{F}^n$, i.e

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear map. Recall that if $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{F}^n$ then

$$\sum_{k=1}^{n} v_k T(\mathbf{e}_k).$$

Let

$$T(\mathbf{e}_1) = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \quad T(\mathbf{e}_2) = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, \quad T(\mathbf{e}_n) = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

and let A be the matrix with $T(\mathbf{e}_k)$ as its k^{th} column:

$$A = \begin{pmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \\ \downarrow & \downarrow & \cdots & \downarrow \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in M_{m,n}(\mathbb{F}).$$

Then A contains all the information necessary to evaluate T and so we say that A is the matrix of the linear map T. We wish to define a multiplication rule between matrices and vectors so that we can represent $T(\mathbf{v})$ as the product $A\mathbf{v}$. If $T(\mathbf{v}) = A\mathbf{v}$ then we have to define

$$\begin{aligned} A\mathbf{v} = \sum_{k=1}^{n} v_k T(\mathbf{e}_k) &= v_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + v_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + v_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \\ &= \begin{pmatrix} v_1 a_{11} + v_2 a_{12} + \cdots + v_n a_{1n} \\ v_1 a_{21} + v_2 a_{22} + \cdots + v_n a_{2n} \\ \vdots \\ v_1 a_{m1} + v_2 a_{m2} + \cdots + v_n a_{mn} \end{pmatrix}. \end{aligned}$$

Hence the product $A\mathbf{v}$ must be computed according to the following rule:

The k^{th} entry of $A\mathbf{v}$ is the dot product of the k^{th} row of A with $\mathbf{v}$ (without complex conjugation if the vectors are complex):

$$(A\mathbf{v})_k = \sum_{j=1}^{n} a_{kj}v_j = a_{k1}v_1 + a_{k2}v_2 + \cdots + a_{kn}v_n.$$

Remark 3.2.2. Observe that the product of an $m \times n$ matrix and a column vector of length n is a column vector of length m.

Example 3.2.3.

$$\begin{pmatrix} 1 & 0 \\ 2 & 6 \\ 3 & 9 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \cdot 4 + 0 \cdot 5 \\ 2 \cdot 4 + 6 \cdot 5 \\ 3 \cdot 4 + 9 \cdot 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 38 \\ 57 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 3 \cdot 2 + 5 \cdot (-1) \\ 7 \cdot 1 + 9 \cdot 2 + 11 \cdot (-1) \\ 2 \cdot 1 + 4 \cdot 2 + 6 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 2 \\ 14 \\ 4 \end{pmatrix}.$$

Example 3.2.4. *Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map given by*

$$T : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 3x + y \\ x - y - 2z \end{pmatrix}.$$

Find the matrix of T.

Solution. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the standard basis of $\mathbb{R}^3$. To find the matrix for T we need to compute $T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)$ and put them as the columns of the matrix. We have

$$T(\mathbf{e}_1) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad T(\mathbf{e}_2) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad T(\mathbf{e}_1) = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

Hence the matrix for T is

$$\begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & -2 \end{pmatrix}.$$

We can check that

$$\begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x + y \\ x - y - 2z \end{pmatrix}$$

as expected. $\square$

3.2.2 Matrix of linear map between general vector spaces

Let V and W be vector spaces with $\dim V = n$ and $\dim W = m$. Suppose that we fix bases $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ of V and W respectively. Then we can repeat the arguments above to get a matrix for a linear map $T : V \to W$ with respect to the bases $\mathcal{E}$ and $\mathcal{F}$.

Recall that if $\mathbf{v} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \cdots + \alpha_n\mathbf{e}_n$, the coordinate vector of $\mathbf{v}$ with respect to $\mathcal{E}$ is the column vector

$$[\mathbf{v}] = [\mathbf{v}]_{\mathcal{E}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n.$$

Since $\mathcal{F}$ is a basis of W, we can write each of the vectors $T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_n)$ as a linear combination of $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m$. Let

$$\begin{aligned} T(\mathbf{e}_1) &= a_{11}\mathbf{f}_1 + a_{21}\mathbf{f}_2 + \cdots + a_{m1}\mathbf{f}_m \\ T(\mathbf{e}_2) &= a_{12}\mathbf{f}_1 + a_{22}\mathbf{f}_2 + \cdots + a_{m2}\mathbf{f}_m \\ \vdots \quad & \quad\; \vdots \qquad\qquad\qquad\quad \vdots \\ T(\mathbf{e}_n) &= a_{1n}\mathbf{f}_1 + a_{2n}\mathbf{f}_2 + \cdots + a_{mn}\mathbf{f}_m. \end{aligned}$$

As before the coefficients a_{jk} determine T completely. We can put them into a matrix A, so that the k^{th} column of A is the coordinate vector of $T(\mathbf{e}_k)$ with respect to the basis $\mathcal{F}$:

$$A = \begin{pmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ [T(\mathbf{e}_1)]_{\mathcal{F}} & [T(\mathbf{e}_2)]_{\mathcal{F}} & \cdots & [T(\mathbf{e}_n)]_{\mathcal{F}} \\ \downarrow & \downarrow & \cdots & \downarrow \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in M_{m,n}(\mathbb{F}).$$

Then with $\mathbf{v} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \cdots + \alpha_n\mathbf{e}_n$, we have that

$$[T(\mathbf{v})]_{\mathcal{F}} = \alpha_1[T(\mathbf{e}_1)]_{\mathcal{F}} + \alpha_2[T(\mathbf{e}_2)]_{\mathcal{F}} + \cdots + \alpha_n[T(\mathbf{e}_n)]_{\mathcal{F}}$$

and thus

$$[T(\mathbf{v})]_{\mathcal{F}} = A[\mathbf{v}]_{\mathcal{E}}.$$

We say that A is the matrix of T with respect to the bases $\mathcal{E}$ and $\mathcal{F}$.

Let us summarise:

To get the matrix for a linear map T with respect to the bases $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$, we have to

1. express each of $T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_n)$ as a linear combination of $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m$;
2. write the coefficients in a matrix so that the k^{th} column consists of the coefficients of $T(\mathbf{e}_k)$.

We will sometimes denote this matrix by $[T]_{\mathcal{F}}^{\mathcal{E}}$. When the bases are fixed we may just write $[T]$. More often, when it does not lead to confusion, we will not distinguish between a linear map and its matrix and use the same symbol for both. This will particularly be the case for linear maps $\mathbb{F}^n$ to $\mathbb{F}^m$ considered with the standard basis.

Remark 3.2.5. Since a linear map is essentially a multiplication, we often write $T\mathbf{v}$ instead of $T(\mathbf{v})$. Similarly, we often write TS for the composition $T \circ S$ of the linear maps T and S. Note that the expression $T\mathbf{v} + \mathbf{u}$ means $T(\mathbf{v}) + \mathbf{u}$ and not $T(\mathbf{v} + \mathbf{u})$.

Example 3.2.6. *Let $D : \mathbb{P}_3 \to \mathbb{P}_3$ denote the derivative map*

$$Df = \frac{df}{dt}.$$

Determine the matrix of D with respect to the monomial basis $1, t, t^2, t^3$.

Solution. We have that

$$\begin{aligned} D(1) &= 0 = 0 + 0t + 0t^2 + 0t^3 \\ D(t) &= 1 = 1 + 0t + 0t^2 + 0t^3 \\ D(t^2) &= 2t = 0 + 2t + 0t^2 + 0t^3 \\ D(t^3) &= 3t^2 = 0 + 0t + 3t^2 + 0t^3 \end{aligned}$$

Hence the matrix for D is

$$[D] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us verify that this is correct. If $f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$, its coordinate vector is $[f] = (a_0, a_1, a_2, a_3)$. Then

$$[D][f] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{pmatrix}.$$

The right hand side is the coordinate vector of the function $a_1 + 2a_2 t + 3a_3 t^2$. This is indeed the the derivative of f as expected. $\square$

Keep in mind the following key point:

The matrix a linear map $T : V \to W$ depends on the choice of bases for V and W. If one of the bases is changed, then the matrix will be different.

Example 3.2.7. Recall that the identity map $I : \mathbb{F}^n \to \mathbb{F}^n$ is given by $I\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{F}^n$. Clearly the matrix for this map with respect to the standard basis is

$$I = I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Unsurprisingly, this is know as the *identity matrix*.

Let $n = 3$ and consider the following basis of $\mathbb{F}^3$:

$$\mathcal{F} = \left\{ \mathbf{f}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{f}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{f}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

What is the matrix of $I : \mathbb{F}^3 \to \mathbb{F}^3$ when we consider the domain with the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ but the range with the basis $\mathcal{F}$? We have that

$$I(\mathbf{e}_1) = \mathbf{e}_1 = \mathbf{f}_1, \quad I(\mathbf{e}_2) = \mathbf{e}_2 = \mathbf{f}_2 - \mathbf{f}_1, \quad I(\mathbf{e}_3) = \mathbf{e}_3 = \mathbf{f}_3 - \mathbf{f}_2.$$

We conclude that

$$[I]^{\mathcal{E}}_{\mathcal{F}} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

This highlights the fact that the same map can have different matrices when the spaces are considered with different bases.

3.2.3 Composition and matrix multiplication

Let $S : \mathbb{F}^m \to \mathbb{F}^k$ and $T : \mathbb{F}^n \to \mathbb{F}^m$ be linear maps. Then the composition $S \circ T : \mathbb{F}^n \to \mathbb{F}^k$ is well-defined (since the range of T is the domain of S) and linear. Let A be the $k \times m$ matrix for S and B be the $m \times n$ matrix for T. Observe that for $\mathbf{v} \in \mathbb{F}^n$

$$S \circ T(\mathbf{v}) = S(T(\mathbf{v})) = S(B\mathbf{v}) = A(B\mathbf{v}).$$

So to enable us to do algebra in an easy way, we want the matrix for $S \circ T$ to be the "product" of A and B; we just need to define this product appropriately.

Let $\mathbf{b}_j = T(\mathbf{e}_j)$ so that $\mathbf{b}_j$ is the j^{th} column of B:

$$B = \begin{pmatrix} \uparrow & \uparrow & \dots & \uparrow \\ \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_n \\ \downarrow & \downarrow & \dots & \downarrow \end{pmatrix}.$$

Then the j^{th} column in the matrix of $S \circ T$ is

$$S \circ T(\mathbf{e}_j) = S(T(\mathbf{e}_j)) = S(\mathbf{b}_j) = A\mathbf{b}_j.$$

If we want the matrix of $S \circ T$ to be the product AB we have to define matrix multiplication in the following way:

The product AB is obtained by multiplying each column in B by A and adjoining the results into a matrix.

More formally, if A is a $k \times m$ and B is an $m \times n$ matrix,

$$\begin{aligned}(AB)_{ij} &= (i^{\text{th}} \text{ row of } A) \cdot (j^{\text{th}} \text{ column of } B) \\ &= \sum_{l=1}^{m} a_{il} b_{lj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{im} b_{mj}.\end{aligned}$$

Remark 3.2.8. In order for the product AB to be defined we must have that the number of columns in A is the same as the number of rows in B.

We have the following properties of matrix multiplication:

1. $A(BC) = (AB)C$ for all matrices A, B, C provided the products AB and BC are defined.

2. $A(B+C) = AB + AC$ for all matrices A, B, C provided the products AB and AC are defined.
3. $A(\alpha B) = (\alpha A)B = \alpha(AB)$ for all matrices A, B and all scalars α provided the product AB is defined.

These follow from the corresponding properties for linear maps (or alternatively by manipulating the sums in the definition of each product).

Exercise. *Prove each of the properties above.*

Warning: Matrix multiplication is not commutative! It is possible that AB is defined whereas BA is not. Even when they are both defined we may have $AB \neq BA$ (in fact it is very rare that two matrices commute).

Careful analysis of the rule matrix multiplication also gives the following property relating to the transpose:

4. $(AB)^T = B^T A^T$ for all matrices A, B provided the product AB is defined.

Note that the order of multiplication changes when we transpose.

Example 3.2.9. Let

$$A = \begin{pmatrix} -1 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 3 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}.$$

Then

$$A^T(B+C) = \begin{pmatrix} -1 & 1 \\ 2 & 0 \\ 0 & 3 \end{pmatrix} \left(\begin{pmatrix} -1 & 0 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \right) = \begin{pmatrix} -1 & 1 \\ 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 0 & 2 \\ 12 & 6 \end{pmatrix}.$$

Exercise. *Let A be an $m \times n$ matrix. Is the product AA^T well defined? If so, what are the dimensions of AA^T?*

3.3 Invertible linear maps

Let V be a vector space. Recall that the identity map $I : V \to V$ is given by $I\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$ (we will write I_V when we want to emphasize the space on which it acts). Then for linear maps $A : U \to V$ and $B : V \to U$ we have

$$IA = A \quad \text{and} \quad BI = B.$$

Definition 3.3.1. Let $A : V \to W$ be a linear map. Then A is *invertible* or *non-singular* if there exist a linear map $A^{-1} : W \to V$ such that

$$AA^{-1} = I_W \quad \text{and} \quad A^{-1}A = I_V.$$

In this case, we call A^{-1} the *inverse* of A.

The inverse of an invertible linear map is unique. Indeed if B is another inverse of A then

$$B = BI = B(AA^{-1}) = (BA)A^{-1} = IA^{-1} = A^{-1}.$$

Suppose that $\dim V = n$ and $\dim W = m$. Recall that I_n is the $n \times n$ identity matrix. Then if A is the $m \times n$ matrix of an invertible linear map $V \to W$ and A^{-1} is the $n \times m$ matrix corresponding to its inverse (with respect to the same bases), then

$$AA^{-1} = I_m \quad \text{and} \quad A^{-1}A = I_n.$$

In this case we say that that matrix A is invertible and that A^{-1} is the inverse of A. Observe that a linear map is invertible if and only if its matrix (with respect to any bases) is invertible.

Warning: It is possible that there exists $B : W \to V$ such that $AB = I$ but $BA \neq I$ or vice-versa. For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{but} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

However, if there exists linear maps $B, C : W \to V$ such that $AB = I$ and $CA = I$ then $B = IB = CAB = CI = C$, so A is invertible and $B = C = A^{-1}$.

Proposition 3.3.2 (Properties of the inverse)**.** *Let A and B be linear maps (matrices) such that the product AB is defined. Then the following properties hold:*

(i) if A is invertible then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$;

(ii) if A and B are invertible then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof. (i) Using the fact that $(AB)^T = B^T A^T$ with $B = A^{-1}$ we get that

$$A^T(A^{-1})^T = (A^{-1}A)^T = I \quad \text{and} \quad (A^{-1})^T A^T = (AA^{-1})^T = I.$$

(ii) This is a direct computation:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I.$$

$\square$

Exercise. *Let $A : \mathbb{F}^n \to \mathbb{F}^n$ be an invertible linear map. If A is symmetric, i.e $A = A^T$, is A^{-1} also symmetric?*

An invertible linear map (and its inverse) can be used to convert properties of its domain to properties of its range and vice versa. An example of this is the following theorem.

Theorem 3.3.3. *Let $A : V \to W$ be an invertible linear map and let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be a basis of V. Then $A\mathbf{e}_1, A\mathbf{e}_2, \ldots, A\mathbf{e}_n$ is a basis of W.*

Proof. First we will show that $A\mathbf{e}_1, A\mathbf{e}_2, \ldots, A\mathbf{e}_n$ are linearly independent. To this end, we take $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$ such that

$$\sum_{k=1}^{n} \alpha_k A\mathbf{e}_k = \mathbf{0}.$$

Then applying A^{-1} gives

$$A^{-1}\left(\sum_{k=1}^{n} \alpha_k A\mathbf{e}_k\right) = \sum_{k=1}^{n} \alpha_k A^{-1}A\mathbf{e}_k = \sum_{k=1}^{n} \alpha_k \mathbf{e}_k = \mathbf{0}.$$

Since $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are linearly independent we have that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Hence $A\mathbf{e}_1, A\mathbf{e}_2, \ldots, A\mathbf{e}_n$ are linearly independent.

Next, we will show that $A\mathbf{e}_1, A\mathbf{e}_2, \ldots, A\mathbf{e}_n$ span W. For any $\mathbf{w} \in W$, $A^{-1}\mathbf{w} \in V$, and since $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ span V, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$ such that

$$A^{-1}\mathbf{w} = \sum_{k=1}^{n} \alpha_k \mathbf{e}_k.$$

Therefore

$$\mathbf{w} = AA^{-1}\mathbf{w} = A\left(\sum_{k=1}^{n} \alpha_k \mathbf{e}_k\right) = \sum_{k=1}^{n} \alpha_k A\mathbf{e}_k.$$

This shows that $A\mathbf{e}_1, A\mathbf{e}_2, \ldots, A\mathbf{e}_n$ span W. $\square$

Remark 3.3.4. (1) Applying the previous theorem to A^{-1} we see that in fact $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is a basis of V *if and only if* $A\mathbf{e}_1, A\mathbf{e}_2, \dots, A\mathbf{e}_n$ is a basis of W.

(2) Inspecting the proof shows that we can also replace "is a basis of" with "is linearly independent in" or "spans" in the statement and the result will still hold.

If there exists an invertible linear map between vector spaces V and W we say that V and W are *isomorphic*. Isomorphic spaces have almost identical vector space properties (even though they may be very different in other respects).

For example, consider the map from $\mathbb{P}_n$ to $\mathbb{F}^{n+1}$ given by

$$a_0 + a_1 t + \cdots + a_n t^n \mapsto \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

Then this is invertible and linear. This identification reveals that addition and scalar multiplication in $\mathbb{P}_n$ corresponds to addition and scalar multiplication in $\mathbb{F}^{n+1}$ (since we just perform the operations on the coefficients). Therefore, even though these are different vector spaces, we can think of them informally as two ways of representing the same space.

More generally, let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be a basis of V (so that V is n-dimensional). Recall that for $\mathbf{v} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \cdots + \alpha_n\mathbf{e}_n \in V$, the coordinate vector of $\mathbf{v}$ with respect to $\mathcal{E}$ is the column vector

$$[\mathbf{v}]_{\mathcal{E}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n.$$

Then the map $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{E}}$ is an invertible linear map from V to $\mathbb{F}^n$. As a result, any statements that can be proved that can be proved for $\mathbb{F}^n$ can be converted into statements about V.

Chapter 4

Systems of Linear Equations

The main motivation behind the development of linear algebra and matrices was the study and solution of systems of linear equations. Recall that a *linear equation* in the *unknowns* $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

and a *system* of linear equations (or sometimes just *linear system*) is a finite collection of linear equations.

4.1 Geometry of linear equations

In this section, all the equations are real and so we consider only real solutions (the discussion below extends to $\mathbb{C}^n$ but it is difficult to visualise and so we avoid it here).

Consider two linear equations in two unknowns:

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2. \end{aligned}$$

Suppose that the coefficients a_k and b_k on each row are not both zero. Then the set of solutions of each row is a line in $\mathbb{R}^2$. The set of solutions of this linear system is the intersection of these two lines. There are three possible arrangements of three lines in $\mathbb{R}^2$:

(1) **Lines intersect at a point.** The point of intersection is the unique solution.

(2) **Lines are parallel.** Since the lines do not intersect, the system has no solutions.

(3) **Lines are equal.** Every point on the line is a solution and so there are infinitely many solutions (sometimes caled a *1-parameter family* of solutions).

Example 4.1.1. Consider the linear system

$$\begin{aligned} x - 2y &= 3 \\ -3x + 6y &= 7. \end{aligned}$$

This system belongs to case (2) above and hence it has no solutions (it is inconsistent).

Next, let us consider a system with two equations in three unknowns:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2. \end{aligned}$$

In this case, assuming the coefficients on each row are not all zero, each equation determines a plane in $\mathbb{R}^3$. As before, we can determine the number of possible solutions by considering the possible arrangements of two planes in $\mathbb{R}^3$. Now we can have the following arrangements:

(1) **Planes intersect at a line.** There are infinitely many solutions (1-parameter family of solutions).

(2) **Planes are parallel.** The system has no solutions.

(3) **Planes are equal.** There are infinitely many solutions (a *2-parameter* family of solutions).

In particular, this shows that two equations in three unknowns can never have a unique solution. With three equations however, we can have a unique solution, but there is also more variety of things that could go wrong. To be precise, the possible arrangements of three planes in $\mathbb{R}^3$ are:

(1) **Planes intersect at a point.** There is a unique solution.

(2) **Two parallel planes.** The system has no solutions. (This includes the case where exactly two planes are equal.)

(3) **No parallel planes but no intersection.** The intersection of any two planes will be a line but the intersection of all three will be empty so there are no solutions. (This is the most common case when we don't have a unique solution.)

(4) **All three planes intersect at a line.** There are infinitely many solutions (1-parameter family).

(5) **All three planes are parallel.** The system has no solutions.

(6) **All three planes are equal.** There are infinitely many solutions (2-parameter family).

We can extend this reasoning to higher dimensions but we will have difficulty imagining 7-dimensional hyperplanes in $\mathbb{R}^8$! To analyse higher dimensional analogues we need the techniques of linear algebra, but it is helpful to keep these pictures in mind.

4.2 Elimination

4.2.1 Notation. Matrix Form

Consider a linear system of m equations with n unknowns $x_1, x_2, \ldots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad \vdots \qquad\quad &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

To solve this system is to find *all* collections $x_1, x_2, \ldots, x_n \in \mathbb{F}$ such that these equations hold simultaneously. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Then the above system of linear equations can be written in *matrix form*:

$$A\mathbf{x} = \mathbf{b}.$$

To solve this is to find all $\mathbf{x} \in \mathbb{F}^n$ satisfying $A\mathbf{x} = \mathbf{b}$. The set of solutions is generally referred to as *the general solution* when it is given in parametric form. We call A the *coefficient matrix* and $\mathbf{b}$ the *vector of constants*.

Observe that it does not matter what we call the unknowns; we could call them x_k, y_k or anything else. So all the information necessary to solve the system is contained in the the matrix A and the vector $\mathbf{b}$. As a result, we often combine the coefficient matrix and the constant vector to form the *augmented matrix* of the system:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right).$$

It is customary to put a vertical line separating A and $\mathbf{b}$ to distinguish between the coefficient matrix and the augmented matrix.

4.2.2 The idea of elimination

Linear systems are solved by *elimination* (or *row reduction*). This is just a systematic form of the technique that you already know for solving simultaneous linear equations.

Example 4.2.1. Let us solve the system

$$x + 2y = -2 \tag{4.2.1}$$
$$2x + y = 7. \tag{4.2.2}$$

We start by subtracting $2 \times (4.2.1)$ from $(4.2.2)$:

$$x + 2y = -2 \tag{4.2.3}$$
$$-3y = 11. \tag{4.2.4}$$

This instantly gives $y = -11/3$. Then substituting back we get

$$x = -2 - 2y = -2 + 22/3 = 16/3.$$

Alternatively, instead of back substitution, we could eliminate y from the first equation. In particular, we multiply $(4.2.4)$ by $-1/3$:

$$x + 2y = -2 \tag{4.2.5}$$
$$y = -11/3; \tag{4.2.6}$$

then subtract $2 \times (4.2.6)$ from $(4.2.5)$:

$$x = 16/3$$
$$y = -11/3.$$

Observe that we could have carried out this sequence of operations directly on the augmented matrix for the system. The first step would be

$$\left(\begin{array}{cc|c} 1 & 2 & -2 \\ 2 & 1 & 7 \end{array}\right) \xrightarrow{R2 \to R2 - 2R1} \left(\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & -3 & 11 \end{array}\right). \tag{4.2.7}$$

and the final two steps would be

$$\left(\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & -3 & 11 \end{array}\right) \xrightarrow{R2 \to -\frac{1}{3} \times R2} \left(\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -11/3 \end{array}\right) \xrightarrow{R1 \to R1 - 2R2} \left(\begin{array}{cc|c} 1 & 0 & 16/3 \\ 0 & 1 & -11/3 \end{array}\right). \tag{4.2.8}$$

This is the idea of elimination, to perform operations on the rows of the augmented matrix (or equivalently on the equations themselves) until we reduce it to a simple form where one can write down the solutions.

In general, we simplify our system by performing *elementary row operations* of the following types:

I. interchange two rows of the matrix;

II. multiply a row by a non-zero scalar;

III. add a multiple of one row to a different row.

These operations do not change the set of solutions - that is, the solutions of the linear system obtained after applying an elementary row operation are the same as the solutions of the original system.

4.2.3 Echelon and reduced echelon form

Recall that the idea of elimination is to perform row operations to an augmented matrix of a linear system to reduce it to a simple form where we can easily write down the solutions. Here we make precise what is meant by ‘simple form’. In particular we will introduce matrices of two forms:

- *Echelon form* - for augmented matrices in this form we can use back substitution to determine the general solution. For example the matrix on the right of (4.2.7) is in echelon form.
- *Reduced echelon form* - for augmented matrices in this form we can read off the general solution directly without the need for back substitution. For example, the matrix on the right of (4.2.8)

Definition 4.2.2. A matrix is in *echelon form* if

1. all zero rows, if there are any, are below all non-zero rows;
2. for each non-zero row, its left-most non-zero entry is strictly to the right of the left-most non-zero entry of the row above.

The left-most non-zero entry in each row in echelon form is called a *pivot entry* or simply *pivot*.

A matrix is in *reduced echelon form* (or *REF* for short) if it is in echelon form and

3. all pivot entries are equal to 1;
4. all entries above the pivots are zero.

Example 4.2.3. Consider the following matrices:

$$A = \begin{pmatrix} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 5 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 3 & 1 & 1 & 2 \\ 0 & 4 & 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have the following:

- A is in REF (and hence in echelon form).
- B is not in echelon form because the pivot in row 2 is to left of the pivot in row 1. This can be put in echelon form by swapping the first two rows, and then into REF by multiplying the second row by 5.
- C is not in echelon form because the pivot in row 2 is directly below the pivot in row 1 (it must be *strictly* to the right). Making this entry 0, by subtracting the first row, would put it in REF.
- D is not in echelon form because the pivot in row 3 is directly below the pivot in row 2. Subtracting 1/2 times row 2 from row 3 will put it in echelon form. To put it in REF we would then have to multiply each row by a constant to make the pivots equal to 1 and then make the entries above the pivots zero.

- E is in echelon form, but not in REF because the pivots are not equal to 1 and the entry above the pivot in row 2 is not equal to zero.
- F is in REF.

Remark 4.2.4. Every matrix can be put into REF by a sequence of elementary operations and the reduced echelon form of a matrix is unique.

We can easily determine the solutions from an augmented matrix in REF. For example, suppose the REF of the augmented matrix is

$$\left(\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & \boxed{1} & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{array}\right),$$

where we have boxed the pivots. The variables corresponding to columns without pivots are called *free variables*, so in this example the free variables are x_2 and x_4. They are called free because the other variables can be written in terms of them, therefore they can be defined freely and then the remaining variables are fixed.

The aim is to move the free variables to the right side. So the solution of this system is

$$\begin{aligned} &x_1 = 2 - 2x_2 + x_4, \\ &x_2 \text{ is free}, \\ &x_3 = 1 + 3x_4, \\ &x_4 \text{ is free}, \\ &x_5 = 4. \end{aligned}$$

Since x_2 and x_4 are free, let us set $x_2 = t$ and $x_4 = s$ (this is only notational, not mathematical). Then writing the solution in vector form we get that any solution $\mathbf{x}$ is of the form

$$\mathbf{x} = \begin{pmatrix} 2 - 2t + s \\ t \\ 1 + 3s \\ s \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 4 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}.$$

Hence the set of solutions is a plane in $\mathbb{R}^5$ (2-parameter family of solutions).

We can put a matrix into echelelon (or reduced echelon) form using the following algorithm:

Gauss-Jordan elimination algorithm:

1. If the matrix consists entirely of zeros, stop, it is in echelon form.

2. Find the first column from the left containing a non-zero entry and use a type I row operation to put that entry into the top row - this is the pivot entry for the row.

3. Add an appropriate multiple of the first row to each row below (type III operations) to make all entries below the pivot zero.

4. Apply Steps 1-3 to the rows below the current row. Repeat this until no rows remain or the remaining rows are zero.

The above algorithm will put the matrix into echelon form. In order to put it in REF we also carry out the following steps:

5. Multiply each non-zero row by an appropriate constant (a type II operation) to make the pivot entry 1.

6. Working from right to left, use type III operations to make all entries directly above each pivot zero.

Example 4.2.5. *Put the following matrix into REF:*

$$\begin{pmatrix} 0 & 0 & 3 & 1 \\ 2 & 0 & -4 & 0 \\ 4 & 1 & 2 & 1 \end{pmatrix}.$$

Solution. Applying the Gauss-Jordan elimination algorithm we get the following:

$$\begin{pmatrix} 0 & 0 & 3 & 1 \\ 2 & 0 & -4 & 0 \\ 4 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{pmatrix} 2 & 0 & -4 & 0 \\ 0 & 0 & 3 & 1 \\ 4 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{R3 \to R3 - 2R1} \begin{pmatrix} 2 & 0 & -4 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & 10 & 1 \end{pmatrix}$$

$$\xrightarrow{R2 \leftrightarrow R3} \begin{pmatrix} 2 & 0 & -4 & 0 \\ 0 & 1 & 10 & 1 \\ 0 & 0 & 3 & 1 \end{pmatrix} \xrightarrow[R1 \to \frac{1}{2}R1]{R3 \to \frac{1}{3}R3} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 10 & 1 \\ 0 & 0 & 1 & 1/3 \end{pmatrix} \xrightarrow[R1 \to R1 + 2R3]{R2 \to R2 - 10R3} \begin{pmatrix} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -7/3 \\ 0 & 0 & 1 & 1/3 \end{pmatrix}.$$

This is now in REF. $\square$

4.2.4 Elementary matrices

Definition 4.2.6. A square matrix is *elementary* if it differs from the identity matrix by one elementary row operation.

Every elementary row operation is equivalent to multiplication on the left by an elementary matrix. Suppose we want to apply a row operation R to a matrix A. Let E be the matrix obtained by applying R to the identity matrix. Then the matrix obtained by applying R to A is EA. For example, if A has 4 rows we have the following:

(1) Interchanging row 2 and row 4 is equivalent to left multiplication by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(2) Multiplying row 3 by α is equivalent to left multiplication by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(3) Adding α times row 1 to row 2 is equivalent to left multiplication by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark 4.2.7. Since row operations are reversible, elementary matrices are invertible.

Let A be a matrix. Suppose it can be put into echelon form by a sequence of row operations $R_1, R_2, \ldots, R_k$. Then each row operation R_j corresponds to left multiplication by an elementary matrix E_j. Set $E = E_k \ldots E_2 E_1$. Then the matrix

$$EA = E_k \ldots E_2 E_1 A$$

is in echelon form. Moreover, since each E_j is invertible, E is also invertibe. 1We conclude the following:

Given a matrix A, we can find an invertible matrix E such that EA is in echelon form (or reduced echelon form).

Exercise. *Let A be the matrix in Example 4.2.5. Find an invertible matrix E such that EA is in echelon form. Find a matrix F such that FA is in REF.*

4.2.5 Analysing the pivots

All questions regarding existence and uniqueness of solutions of a linear system can be answered by analysing the pivots in the echelon (or reduced echelon) form of the augmented matrix.

First we consider when a linear system has no solutions. We call a system with no sultions *inconsistent*, otherwise we say it is *consistent*.

Proposition 4.2.8. *A linear system is inconsistent (i.e. has no solutions) if and only if the echelon form of the* augmented *matrix has a pivot in the last column.*

Proof. The echelon form of the augmented matrix has a pivot in the last column if and only if it has a row of the form

$$\left(\begin{array}{cccc|c} 0 & 0 & \dots & 0 & b \end{array}\right),$$

for some $b \neq 0$. This row corresponds to the equation

$$0x_1 + 0x_2 + \cdots + 0x_n = b \neq 0,$$

which clearly has no solutions. If we do not have such a row we can put the matrix into REF and read off a solution. □

We can also determine uniqueness of solutions by analysing pivots; here one only needs to consider the *coefficient* matrix.

Proposition 4.2.9. *Let $A : \mathbb{F}^n \to \mathbb{F}^m$ be a linear map (matrix).*

(i) The linear system $A\mathbf{x} = \mathbf{b}$ is consistent for all right sides $\mathbf{b} \in \mathbb{F}^m$ if and only if the echelon form of the coefficient matrix has a pivot in every row.

(ii) The linear system $A\mathbf{x} = \mathbf{b}$ has a unique *solution for every right side $\mathbf{b} \in F^m$ if and only if the echelon form of the coefficient matrix has a pivot in every column and every row.*

Proof. (i) Let E be an invertible matrix such that EA is in echelon form. If EA has a pivot in every row, we cannot have a pivot in the last column of the augmented matrix $\left(EA \mid E\mathbf{b}\right)$. Hence the system is always consistent, regardless of $\mathbf{b}$.

Suppose that EA does not have a pivot in the last row, i.e. it has a zero row. Then we can choose $\mathbf{b} = E^{-1}\mathbf{e}_m$, where $\mathbf{e}_m = (0, \ldots, 0, 1)$, so that

$$\left(EA \mid E\mathbf{b}\right) = \left(EA \mid EE^{-1}\mathbf{e}_m\right) = \left(EA \mid \mathbf{e}_m\right),$$

which is inconsistent.

(ii) It's clear that a solution (if it exists) is unique if and only if there are no free variables (these are the source of the non-uniqueness). This happens precisely when the echelon form of the coefficient matrix has a pivot in every column. Combining this observation with (i) proves (ii). □

Proposition 4.2.9 has the following immediate consequences.

Corollary 4.2.10. *A matrix A is invertible if and only if its echelon form has a pivot in every column and every row.*

Proof. By Question 6 on Assignment 7, A is invertible if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for every right side $\mathbf{b}$. The claim now follows from Proposition 4.2.9(ii). □

Remark 4.2.11. Any row or column of a matrix in echelon form can have at most one pivot in it. Therefore if the echelon form of a matrix has a pivot in every row and every column, it must have the same number of rows and columns. In particular, this shows us that an invertible matrix must be square.

Questions about whether a collection of vectors in $\mathbb{F}^n$ is linearly independent or spanning can also be answered with row reduction.

Proposition 4.2.12. *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in $\mathbb{F}^n$. Let A be the $n \times m$ matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$, i.e.*

$$A = \begin{pmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m \\ \downarrow & \downarrow & \cdots & \downarrow \end{pmatrix}.$$

Then the following hold:

(i) the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ *are linearly independent if and only if the echelon form of* A *has a pivot in every column;*

(ii) the vector $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ *span* $\mathbb{F}^n$ *if and only if the echelon form of* A *has a pivot in every row;*

(iii) the vector $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ *form a basis of* $\mathbb{F}^n$ *if and only if the echelon form of* A *has a pivot in every column and every row.*

Proof. (i) The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent if and only if the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_m\mathbf{v}_m = \mathbf{0}$$

has the unique solution $x_1 = x_2 = \cdots = x_m = 0$. Equivalently, the equation $A\mathbf{x} = \mathbf{0}$ has unique solution $\mathbf{x} = \mathbf{0}$. This happens precisely when there are no free variables, i.e. when the echelon form of A has a pivot in every column.

(ii) The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ span $\mathbb{F}^n$ if and only if the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_m\mathbf{v}_m = \mathbf{b}$$

has a solution $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ for any right side $\mathbf{b}$. By Proposition 4.2.9(i), this happens precisely when the echelon form of A has a pivot in every row.

(iii) Combine (i) and (ii). □

4.2.6 Consequences for square matrices

Recall that it is possible for a matrix A to have have a *left inverse*, i.e. a matrix B such that $BA = I$, but not a *right inverse* so that $AB \neq I$. Similarly, a matrix may have a right inverse but not a left inverse. This is not possible when A is *square*.

Proposition 4.2.13. *Let* A *be a (square)* $n \times n$ *matrix. Then* A *has a left inverse if and only if it has a right inverse.*

In general, if you have a linear map A and you want to check that another linear map B is the inverse of A, you have to check that both $AB = I$ and $BA = I$ hold. This proposition shows that if you know that the matrix of A is square (i.e. that the domain and range have the same dimension), then it is sufficient to check just one of these and you get the other one automatically.

Proof. Suppose that B is a left inverse of A so that $BA = I$. Then if the equation $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$, we must have that $\mathbf{x} = BA\mathbf{x} = B\mathbf{b}$ and so $\mathbf{x} = B\mathbf{b}$ is the unique solution. Since every equation $A\mathbf{x} = \mathbf{b}$ that is consistent has a unique solution, by Proposition 4.2.9(ii) the echelon form of A must have a pivot in every column. Since A is square this means it also has a pivot in every column. Then by Corollary 4.2.10 A must be invertible.

Conversely, suppose that C is a right inverse of A so that $AC = I$. Then for any $\mathbf{b} \in \mathbb{F}^n$ we have $\mathbf{b} = I\mathbf{b} = AC\mathbf{b}$, so that $C\mathbf{b}$ is a solution of the equation $A\mathbf{x} = \mathbf{b}$. Since the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{F}^n$, by Proposition 4.2.9(i) the echelon form of A must have a pivot in every row. As before, this means it also has a pivot in every column and so it is invertible by Corollary 4.2.10. $\square$

The key point in the proof was that for square matrices, having a pivot in every row is equivalent to having a pivot in every column. We can use this to show that all properties of a matrix that are determined by whether or not the its echelon form has pivot in every row or every column are equivalent for square matrices. For example, we have the following:

Proposition 4.2.14. *Let A be a (square) $n \times n$ matrix. Then A is invertible if and only if the equation $A\mathbf{x} = \mathbf{0}$ has a unique solution.*

Proof. We have already seen that if A is invertible then the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{F}^n$, so one direction is immediate.

Suppose that the system $A\mathbf{x} = \mathbf{0}$ has a unique solution. This means that there are no free variables and hence that the echelon form of A has a pivot in every column. Since A is square it must also have a pivot in every row, so by Corollary 4.2.10 it is invertible. $\square$

4.2.7 Computing the inverse by elimination

Let A be an invertible matrix. As discussed above, A must be square and its echelon form must have a pivot in every row and column. Therefore its reduced echelon form must be the identity matrix I. From the discussion in Section 4.2.4, there exists a matrix E such that

$$EA = I.$$

Hence E must be the inverse of A, i.e. $E = A^{-1}$. Recall that E is the product of the elementary matrices corresponding to the row operations required to put A into REF. Thus E can be computed by applying the same row operations to the identity matrix. This gives us the following algorithm for computing A^{-1} for an $n \times n$ matrix A:

1. Form and *augmented* $n \times 2n$ matrix $\big(A \mid I\big)$.
2. Perform row operations on the augmented matrix to transform A to the identity matrix I.
3. The matrix I that was added will be transformed to A^{-1}.

Remark 4.2.15. If it is not possible to transform A to the identity matrix by row operations then A is not invertible.

After applying the algorithm above, the augmented matrix $\big(A \mid I\big)$ will be transformed to the matrix $\big(EA \mid EI\big) = \big(I \mid E\big)$.

Example 4.2.16. *Compute the inverse of the matrix*

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}.$$

Solution. First we form the augmented matrix

$$\big(A \mid I\big) = \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right).$$

Then we apply row reduction to transform A into I:

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R2 \to R2 - R1]{R3 \to R3 - R1} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R2 \to R2 + 2R3} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R1 \to R1 + R2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & -3 & 1 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right).$$

We conclude that A is invertible and

$$A^{-1} = \begin{pmatrix} -2 & 1 & 2 \\ -3 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix}.$$

We can check that

$$\begin{pmatrix} -2 & 1 & 2 \\ -3 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 & 2 \\ -3 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

□

4.3 Principle of linearity

Definition 4.3.1. A linear equation (linear system) $A\mathbf{x} = \mathbf{b}$ is *homogeneous* if $\mathbf{b} = \mathbf{0}$ - that is, a homogeneous linear equation is an equation of the form $A\mathbf{x} = \mathbf{0}$. Otherwise we say it is *inhomogeneous*.

Given a linear system $A\mathbf{x} = \mathbf{b}$, we call the system

$$A\mathbf{x} = \mathbf{0}$$

the *associated homogeneous* system.

For example the associated homogeneous system of the linear system

$$\begin{pmatrix} 1 & -5 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

is the system

$$\begin{pmatrix} 1 & -5 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consider a linear system $A\mathbf{x} = \mathbf{b}$ and its associated homogeneous system $A\mathbf{x} = \mathbf{0}$. Suppose $\mathbf{x}_0$ is solution of the original system, i.,e $A\mathbf{x}_0 = \mathbf{b}$, and suppose that $\mathbf{x}_h$ is a solution of the homogeneous system, i.e. $A\mathbf{x}_h = \mathbf{0}$. Then

$$A(\mathbf{x}_0 + \mathbf{x}_h) = A\mathbf{x}_0 + A\mathbf{x}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

Hence $\mathbf{x}_0 + \mathbf{x}_h$ is also a solution of the original system.

Now suppose that $\mathbf{x}_1$ is another solution of the original system. Then

$$A(\mathbf{x}_1 - \mathbf{x}_0) = A\mathbf{x}_1 - A\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Hence $\mathbf{x}_1 - \mathbf{x}_0$ is a solution of the homogeneous system. Moreover, we have

$$\mathbf{x}_1 = \mathbf{x}_0 + (\mathbf{x}_1 - \mathbf{x}_0),$$

and thus every solution of the original system is the sum of $\mathbf{x}_0$ and a solution of the homogeneous system.

We have proved the following:

Theorem 4.3.2 (Principle of linearity). *Let $A : V \to W$ be a linear map and let $\mathbf{b} \in W$. Let $\mathbf{x}_0 \in V$ satisfy the equation $A\mathbf{x}_0 = \mathbf{b}$ and let H denote the set of solutions of the associated homogeneous equation $A\mathbf{x} = \mathbf{0}$. Then the set*

$$\{\mathbf{x}_0 + \mathbf{x}_h : \mathbf{x}_h \in H\}$$

is the set of all solutions of the equation $A\mathbf{x} = \mathbf{b}$.

This can be restated as

General solution of $A\mathbf{x} = \mathbf{b}$	$=$	*A particular solution of* $A\mathbf{x} = \mathbf{b}$	$+$	*General solution of* $A\mathbf{x} = \mathbf{0}$

Any particular solution will do in the above statement.

Remark 4.3.3. Observe that Theorem 5.2 is valid for all linear maps. In particular, it is valid for linear maps between infinite dimensional spaces where one cannot simply use elimination to determine the general solution.

Exercise. *Let $A : V \to W$ be a linear map. Which, if any, of the following statements are true?*

(a) The equations $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ have the same solutions for all $\mathbf{b} \in W$.

(b) The equations $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ have the same solutions for some $\mathbf{b} \in W$.

(c) The equations $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ have the same number of solutions for all $\mathbf{b} \in W$.

(d) The equations $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ have the same number of solutions for some $\mathbf{b} \in W$.

Example 4.3.4. *Consider the linear system*

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 3 & 0 & 1 & -3 \\ 7 & 1 & 1 & -4 \\ 0 & 3 & -4 & 9 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 7 \end{pmatrix}.$$

Show that the general solution of this sytem is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 5 \\ 6 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ -7 \\ -3 \\ 1 \end{pmatrix}, \quad t, s \in \mathbb{F}. \tag{4.3.1}$$

Solution. One way to proceed is to perform elimination. However, this is generally time consuming. Moreover, elimination produces the general solution

$$\mathbf{x} = \begin{pmatrix} -1/3 \\ 7/3 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1/3 \\ 4/3 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix}, \quad t, s \in \mathbb{F},$$

which looks different, and so we would still be left with the task of verifying that the two sets of solutions are indeed the same.

An easier approach is to use the principle of linearity. We first check that

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 3 & 0 & 1 & -3 \\ 7 & 1 & 1 & -4 \\ 0 & 3 & -4 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 7 \end{pmatrix},$$

and so $(0, 1, -1, 0)$ is a particular solution. We then check that

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 3 & 0 & 1 & -3 \\ 7 & 1 & 1 & -4 \\ 0 & 3 & -4 & 9 \end{pmatrix} \begin{pmatrix} -1 \\ 5 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 3 & 0 & 1 & -3 \\ 7 & 1 & 1 & -4 \\ 0 & 3 & -4 & 9 \end{pmatrix} \begin{pmatrix} 2 \\ -7 \\ -3 \\ 1 \end{pmatrix} = \mathbf{0},$$

and so

$$t \begin{pmatrix} -1 \\ 5 \\ 6 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ -7 \\ -3 \\ 1 \end{pmatrix}$$

is a solution of the associated homogeneous system for every $t, s \in \mathbb{F}$. It follows that (4.3.1) is a solution of the original system for any $t, s \in \mathbb{F}$. This does not tell us that we have *all* of the solutions (the general solution). To see this we note that the first two columns of the coefficient matrix are linearly independent and so there cannot be more than two free variables in the general solution. This shows that (4.3.1) is indeed the general solution. □

4.3.1 Application to differential equations

Fix $\lambda \in \mathbb{R}$ and consider the first order linear ODE

$$\frac{du}{dx} = \lambda u(x).$$

The general solution to this equation is $u(x) = Ce^{\lambda x}$, where C is an arbitrary constant.

This can also be phrased in terms of linear maps. Let V be the (real) vector space of all smooth (infinitely differentiable) functions $u : \mathbb{R} \to \mathbb{R}$. Let $D : V \to V$ be the linear map defined by

$$Du(x) = \frac{du}{dx}, \quad u \in V.$$

Then the statement above can be rephrased as saying that the general solution of the homogeneous linear equation

$$(D - \lambda I)u = 0$$

is $u(x) = Ce^{\lambda x}$, $C \in \mathbb{R}$.

Next, consider the second order equation

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu(x) = 0, \tag{4.3.2}$$

where $a, b, c \in \mathbb{R}$ are constants. If we let $T : V \to V$ be the linear map $T = aD^2 + bD + cI$ (recall that $D^2 = D \circ D$) is the second derivative) then (4.3.2) can be written as

$$Tu = 0.$$

It is still reasonable to look for a solution of the form $u(x) = e^{\lambda x}$. Substituting this into (4.3.2) we get

$$(a\lambda^2 + b\lambda + c)e^{\lambda x} = 0.$$

Since $e^{\lambda x} > 0$ for all $x \in \mathbb{R}$ we must have $a\lambda^2 + b\lambda + c = 0$. This equation has two roots, λ_1 and λ_2 say. Therefore (4.3.2) is satisfied whenever $u(x) = e^{\lambda_1 x}$ or $u(x) = e^{\lambda_2 x}$. Moreover, if $\lambda_1 \neq \lambda_2$ then the set of linear combinations of the form

$$u(x) = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}, \quad C_1, C_2 \in \mathbb{R},$$

give the general solution of (4.3.2) (a proof this last claim is beyond the scope of this course).

Example 4.3.5. *Determine the general solution of the linear ODE*

$$\frac{d^2u}{dx^2} - \frac{du}{dx} - 6u(x) = 6\cos 3x. \tag{4.3.3}$$

Before proceeding to the solution let us observe that this is a linear equation in the vector space V os smooth functions. Indeed setting $T = D^2 - D - 6I$ and $b(x) = 6\cos 3x$, we see that we are looking for the general solution in V of the linear equation

$$Tu = b.$$

Since V is infinite dimensional we cannot use elimination. However, we can use the principle of linearity.

Solution. First we try to "guess" a particular solution. Since the right hand side is $6\cos 3x$, we look for a solution of the form

$$u(x) = \alpha \cos 3x + \beta \sin 3x.$$

In this case, we will have

$$\frac{du}{dx} = -\alpha \sin 3x + \beta \cos 3x \quad \text{and} \quad \frac{d^2u}{dx^2} = -\alpha \cos 3x + -\beta \sin 3x$$

Substituting these into (4.3.3) we get

$$-9\alpha \cos 3x - 9\beta \sin 3x + 3\alpha \sin 3x - 3\beta \cos 3x - 6\alpha \cos 3x - 6\beta \sin 3x = 6\cos 3x$$

and so

$$(-15\alpha - 3\beta)\cos 3x + (3\alpha - 15\beta)\sin 3x = 6\cos 3x.$$

Equating coefficients of $\cos 3x$ and $\sin 3x$ gives

$$\begin{aligned} -15\alpha - 3\beta &= 6, \\ 3\alpha - 15\beta &= 0. \end{aligned}$$

This has solution $\alpha = -5, \beta = -1$. Therefore

$$u(x) = -5\cos 3x - \sin 3x$$

is a particular solution of (4.3.3).

Next we need to find the general solution of the associated homogeneous equation

$$\frac{d^2u}{dx^2} - \frac{du}{dx} - 6u(x) = 0. \tag{4.3.4}$$

Substituting $u(x) = e^{\lambda x}$ into (4.3.4) we get

$$(\lambda^2 - \lambda - 6)e^{\lambda x} = 0.$$

Dividing by $e^{\lambda x}$ we see that $\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0$. Thus the general solution of (4.3.4) is

$$u(x) = C_1 e^{3x} + C_2 e^{-2x}, \quad C_1, C_2 \in \mathbb{R},$$

Combining these we find that the general solution of (4.3.3) is

$$u(x) = -5\cos 3x - \sin 3x + C_1 e^{3x} + C_2 e^{-2x}, \quad C_1, C_2 \in \mathbb{R}.$$

□

4.4 Image and kernel of a linear map

We have seen that many properties of linear maps are related to the existence and uniqueness of solutions of the form $A\mathbf{x} = \mathbf{b}$. Thus we are naturally lead to consider the following subspaces.

Definition 4.4.1. Let $A : V \to W$ be a linear map. The *image* (or *column space*) of A, denoted $\operatorname{Im} A$, is the set of all vectors $\mathbf{w} \in W$ such that $\mathbf{w} = A\mathbf{v}$ for some $\mathbf{v} \in V$, i.e

$$\operatorname{Im} A := \{A\mathbf{v} \in W : \mathbf{v} \in V\}.$$

The *kernel* (or *null space*) of A, denoted $\operatorname{Ker} A$ is the set of all vectors $\mathbf{v} \in V$ such that $A\mathbf{v} = \mathbf{0}$, i.e.

$$\operatorname{Ker} A = \{\mathbf{v} \in V : A\mathbf{v} = \mathbf{0}\}.$$

Equivalently, $\operatorname{Im} A$ is the set of right sides $\mathbf{b}$ such that the equation $A\mathbf{x} = \mathbf{b}$ has solutions and $\ker A$ is the set of solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Let $A : \mathbb{F}^n \to \mathbb{F}^m$ be given by an $m \times n$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in F^m$, i.e

$$A = \begin{pmatrix} \uparrow & \uparrow & \dots & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \\ \downarrow & \downarrow & \dots & \downarrow \end{pmatrix}.$$

Then for $\mathbf{x} = (x_1, x_2, \dots, x_n)$ we have that

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots x_2\mathbf{a}_n.$$

Therefore $\operatorname{Im} A = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$. This is the reason for the term "column space".

Remark 4.4.2. If A is an $m \times n$ matrix, we can also consider the range and kernel of A^T. $\operatorname{Im} A^T$ and $\operatorname{Ker} A^T$ sometimes called the *row space* and *left null space* respectively. The four spaces $\operatorname{Im} A$, $\operatorname{Ker} A$, $\operatorname{Im} A^T$ and $\operatorname{Ker} A^T$ are called the *fundamental subspaces* of A.

Example 4.4.3. Recall that $\mathbb{P}_n$ is the set of all polynomials in one variable t of degree at most n. Let $D : \mathbb{P}_n \to \mathbb{P}_n$ be the differentiation map

$$Dp(t) = \frac{dp}{dt}.$$

Since the derivative of every polynomial in $\mathbb{P}_n$ is a polynomial in $\mathbb{P}_{n-1}$ and every polynomial in $\mathbb{P}_{n-1}$ is the derivative of some polynomial in $\mathbb{P}_n$, we have that $\operatorname{Im} D = \mathbb{P}_{n-1}$.

We also have that the derivative of a function $p(t)$ is zero if and only if $p(t)$ is constant. Hence

$$\operatorname{Ker} D = \{p \in \mathbb{P}_n : p(t) \equiv c \ \text{ for some constant } c \in \mathbb{F}\}.$$

Proposition 4.4.4. *Let $A : V \to W$ be a linear map. Then* $\operatorname{Im} A$ *is a subspace of of* W *and* $\operatorname{Ker} A$ *is a subsapce of* V.

Proof. This is left as an assignment question. $\square$

Remark 4.4.5. Observe that a linear map $A : V \to W$ is surjective if and only if $\operatorname{Im} A = W$ (this is just the definition of being surjective).

Proposition 4.4.6. *Let $A : V \to W$ be a linear map. A is injective if and only if* $\operatorname{Ker} A = \{\mathbf{0}\}$.

Proof. First, let us assume that $\operatorname{Ker} A = \{\mathbf{0}\}$. Take $\mathbf{v}_1, \mathbf{v}_2 \in V$ with $A\mathbf{v}_1 = A\mathbf{v}_2$. We need to show that $\mathbf{v}_1 = \mathbf{v}_2$. We have that $A(\mathbf{v}_1 - \mathbf{v}_2) = A\mathbf{v}_1 - A\mathbf{v}_2 = \mathbf{0}$, and so $\mathbf{v}_1 - \mathbf{v}_2 \in \operatorname{Ker} A$. Since $\operatorname{Ker} A = \{\mathbf{0}\}$ we have that $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$, i.e $\mathbf{v}_1 = \mathbf{v}_2$.

Now let us assume that there exists a non-zero vector $\mathbf{v} \in \operatorname{Ker} A$. Then $A\mathbf{0} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$ with $\mathbf{v} \neq \mathbf{0}$. Hence A is not injective. $\square$

Exercise. *Let $A : V \to V$ be a linear map. Which, if any, of the following statements are true?*

(a) If A is injective then A^2 is injective.

(b) If A is sujective then A^2 is surjective.

(c) If A^2 is injective then A is injective.

(d) If A^2 is sujective then A is surjective.

The dimensions of the kernel and image of a linear map contain important information about it, and are related to each other.

Definition 4.4.7. Let $A : V \to W$ be a linear map.

The *rank* of A, denoted $r(A)$, is the dimension of the image of A, i.e. $r(A) := \dim \operatorname{Im} A$.

The *nullity* of A, denoted $n(T)$, is the dimension of the kernel of A, i.e. $n(A) = \dim \operatorname{Ker} A$.

Let A be an $m \times n$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and let E be an invertible matrix such that EA is in echelon form. Then the columns of EA are $E\mathbf{a}_1, E\mathbf{a}_2, \ldots E\mathbf{a}_n$. Let us suppose that the pivot columns of EA are $E\mathbf{a}_1, \ldots E\mathbf{a}_k$, $k \leq n$ (we can relabel the columns if necessary). Then $E\mathbf{a}_1, \ldots E\mathbf{a}_k$ form a basis for $\operatorname{Im} EA$. Multiplying each of these by the invertible matrix E^{-1} we see that $\mathbf{a}_1, \ldots \mathbf{a}_k$ are linearly independent. We claim that they also span $\operatorname{Im} A$. Indeed for any $\mathbf{v} \in \operatorname{Im} A$, $E\mathbf{v} \in \operatorname{Im} EA$ and so there are scalars $\beta_1, \dots \beta_k$ such that

$$E\mathbf{v} = \beta_1 E\mathbf{a}_1 + \beta_2 E\mathbf{a}_2 + \cdots + \beta_k E\mathbf{a}_k.$$

Then multiplying both sides by E^{-1} gives

$$\mathbf{v} = \beta_1 \mathbf{a}_1 + \beta_2 \mathbf{a}_2 + \cdots + \beta_k \mathbf{a}_k.$$

We conclude that $\mathbf{a}_1, \dots \mathbf{a}_k$ are a basis for $\operatorname{Im} A$.

To compute a basis for the kernel of A, we have to solve the homogeneous equation $A\mathbf{x} = \mathbf{0}$. The dimension of $\operatorname{Ker} A$ will be number of parameters in the general solution, i.e. the number of free variables.

To summarise, we have shown the following:

1. The pivot columns of A, i.e.columns of A (not its echelon form) corresponding to pivot variables, form a basis for $\operatorname{Im} A$.
2. The rank of A is the number of pivots in the reduced echelon form of A.
3. The nullity of A is the number *free* columns (i.e columns without pivots) in the echelon form of A

The following theorem now follows trivially, but it extremely important in linear algebra.

Theorem 4.4.8 (Rank-nullity theorem)**.** *Let $A : V \to W$ be a linear map, where V is finite dimensional. Then*

$$r(A) + n(A) = \dim V.$$

Example 4.4.9. *Consider the matrix*

$$A = \begin{pmatrix} 1 & 1 & 1 & 3 & -2 \\ 3 & 3 & 1 & 2 & -1 \\ 2 & 2 & 0 & -1 & 1 \\ -1 & -1 & 1 & 4 & -3 \end{pmatrix}.$$

(a) Determine the rank of A and find a basis for $\operatorname{Im} A$.

(b) Determine the nullity of A and finds a basis for $\operatorname{Ker} A$.

Solution. After row reducing A we find that its echelon form is

$$\begin{pmatrix} \boxed{1} & 1 & 1 & 3 & -2 \\ 0 & 0 & \boxed{\text{-2}} & -7 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) This matrix has two pivots and so $r(A) = 2$. Moreover, the pivots are in the first and third columns and so the first first and third columns of A,

$$\begin{pmatrix} 1 \\ 3 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

form a basis for $\operatorname{Im} A$.

(b) This matrix has 3 columns corresponding to free variables and so $n(A) = 3$. To find a basis for $\ker A$ we solve $A\mathbf{x} = \mathbf{0}$. We see from the echelon form that

$$\begin{aligned} &x_5 \text{ is free,} \\ &x_4 \text{ is free,} \\ &x_3 = -\frac{7}{2}x_4 + \frac{5}{2}x_5 \\ &x_2 \text{ is free,} \\ &x_1 = -x_2 + \frac{1}{2}x_4 + \frac{1}{2}x_5. \end{aligned}$$

Thus the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1/2 \\ 0 \\ -7/2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1/2 \\ 0 \\ 5/2 \\ 0 \\ 1 \end{pmatrix}.$$

We conclude that the vectors

$$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1/2 \\ 0 \\ -7/2 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1/2 \\ 0 \\ 5/2 \\ 0 \\ 1 \end{pmatrix}$$

form a basis for $\operatorname{Ker} A$. $\square$

Exercise. *Let $D : \mathbb{P}_n \to \mathbb{P}_n$ be the derivative map given by $Df(t) = f'(t)$. What is the rank and nullity of D?*

Corollary 4.4.10. *Let V and W be vector spaces of dimension n and let $A : V \to W$ be a linear map. Then the following properties of V are equivalent:*

(i) *A is surjective.*

(ii) *$r(A) = n$.*

(iii) *$n(A) = 0$.*

(iv) *A is injective.*

Proof. We will show that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$. This will show that they are all equivalent.

$(i) \Leftrightarrow (ii)$. A is surjective precisely means that $\operatorname{Im} A = W$. Since $\operatorname{Im} A$ is a subspace of W, this holds if and only if $r(A) = \dim \operatorname{Im} A = \dim W = n$.

$(ii) \Leftrightarrow (iii)$. By the rank-nullity theorem, $r(A) + n(A) = n$. Hence $r(A) = n$ if and only if $n(A) = 0$.

$(iii) \Leftrightarrow (iv)$. We have that $n(A) = 0$ and only if $\operatorname{Ker} A = \{\mathbf{0}\}$. By Proposition 4.4.6, this holds if and only if A is injective. ☐

Remark. Observe that if a linear map A satisfies the conditions of Corollary 4.4.10, and any (and hence all) of the conclusions (i)–(iv) in the Corollary hold, then A must be invertible.

Proposition 4.4.11. *For any matrix A we have that $r(A) = r(A^T)$.*

Proof. Let E be an invertible matrix such that EA is in echelon form. Then $(EA)^T = A^T E^T$. If $\mathbf{v} = A^T \mathbf{x}$ so that $\mathbf{v} \in \operatorname{Im} A^T$, then $\mathbf{v} = A^T E^T (E^{-1})^T \mathbf{x}$ so $\mathbf{v} \in \operatorname{Im} (EA)^T$. Hence applying row operations to A does not change row space. It follows that the pivot columns of EA form a basis for $\operatorname{Im} A^T$ and so $r(A^T)$ is the number of pivots in EA, which is also the rank of A. ☐

Remark 4.4.12. The above proof also gives an easy algorithm to compute a basis for $\operatorname{Im} A^T$ from the echelon form of A.

As a consequence of Proposition 4.4.11 we have that for $A : V \to W$

$$r(A) + n(A^T) = \dim W.$$

Exercise. *Let $A : \mathbb{F}^n \to \mathbb{F}^m$ be a linear map. Which, if any, of the following statements are true?*

(a) If A is injective then A^T is injective.

(b) If A is surjective then A^T is surjective.

(c) If A is injective then A^T is surjective.

(d) If A is surjective then A^T is injective.

4.5 Change of bases. Similarity

Let V be a vector space and let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis of V. Recall that for $\mathbf{v} \in V$,

$$[\mathbf{v}]_{\mathcal{E}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \Leftrightarrow \mathbf{v} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}2 + \cdots + \alpha_n \mathbf{e}_n.$$

The map $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{E}}$ is an invertible linear map between V and $\mathbb{F}^n$.

Let $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ be a basis of a vector space W and let $T : V \to W$ be a linear map. Then

$$[T\mathbf{v}]_{\mathcal{F}} = [T]^{\mathcal{E}}_{\mathcal{F}} [\mathbf{v}]_{\mathcal{E}}, \quad \mathbf{v} \in V, \tag{4.5.1}$$

where $[T]^{\mathcal{E}}_{\mathcal{F}}$ is the matrix of T with respect to the bases $\mathcal{E}$ and $\mathcal{F}$ given by

$$[T]^{\mathcal{E}}_{\mathcal{F}} = \begin{pmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ [T(\mathbf{e}_1)]_{\mathcal{F}} & [T(\mathbf{e}_2)]_{\mathcal{F}} & \cdots & [T(\mathbf{e}_n)]_{\mathcal{F}} \\ \downarrow & \downarrow & \cdots & \downarrow \end{pmatrix}.$$

Suppose $V = W$, so that $\mathcal{E}$ and $\mathcal{F}$ are both bases of V. Let $I : V \to V$ be the identity map given by $I\mathbf{v} = \mathbf{v}$, $\mathbf{v} \in V$. Then by (4.5.1) we have that for each $\mathbf{v} \in V$,

$$[\mathbf{v}]_{\mathcal{F}} = [I]^{\mathcal{E}}_{\mathcal{F}} [\mathbf{v}]_{\mathcal{E}}.$$

> Multiplying by $[I]^{\mathcal{E}}_{\mathcal{F}}$ transforms coordinates in the $\mathcal{E}$ basis to coordinates in the $\mathcal{F}$ basis.

The matrix $[I]_{\mathcal{F}}^{\mathcal{E}}$ is known as the *transition* matrix (or *change of coordinate* matrix) from $\mathcal{E}$ to $\mathcal{F}$. Observe that we necessarily have

$$[I]_{\mathcal{E}}^{\mathcal{F}} = \left([I]_{\mathcal{F}}^{\mathcal{E}}\right)^{-1}.$$

Example 4.5.1. Let $\mathcal{E}$ and $\mathcal{F}$ be the bases of $\mathbb{R}^2$ given by

$$\mathcal{E} = \left\{\mathbf{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}\right\}, \quad \mathcal{F} = \left\{\mathbf{f}_1 = \begin{pmatrix}1\\1\end{pmatrix}, \mathbf{f}_2 = \begin{pmatrix}-1\\1\end{pmatrix}\right\}.$$

We can relate the two bases as follows:

$$\begin{aligned}\mathbf{f}_1 &= \mathbf{e}_1 + \mathbf{e}_2,\\ \mathbf{f}_2 &= -\mathbf{e}_1 + \mathbf{e}_2,\end{aligned}$$

and

$$\begin{aligned}\mathbf{e}_1 &= \frac{1}{2}\mathbf{f}_1 - \frac{1}{2}\mathbf{f}_2,\\ \mathbf{e}_2 &= \frac{1}{2}\mathbf{f}_1 + \frac{1}{2}\mathbf{f}_2.\end{aligned}$$

Therefore

$$[I]_{\mathcal{F}}^{\mathcal{E}} = \begin{pmatrix}1/2 & 1/2\\ -1/2 & 1/2\end{pmatrix} \quad \text{and} \quad [I]_{\mathcal{E}}^{\mathcal{F}} = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}.$$

We can verify that

$$[I]_{\mathcal{F}}^{\mathcal{E}}[I]_{\mathcal{E}}^{\mathcal{F}} = \begin{pmatrix}1/2 & 1/2\\ -1/2 & 1/2\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Take, for example, $\mathbf{u} = (3, 1)$. Then we can determine the coordinates of $\mathbf{u}$ with respect to the basis $\mathcal{F}$ as follows:

$$[\mathbf{u}]_{\mathcal{F}} = [I]_{\mathcal{F}}^{\mathcal{E}}[\mathbf{u}]_{\mathcal{E}} = \begin{pmatrix}1/2 & 1/2\\ -1/2 & 1/2\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}.$$

We can verify that the equality $\mathbf{u} = 2\mathbf{f}_1 - \mathbf{f}_2$ holds, i.e that

$$\begin{pmatrix}3\\1\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}-1\\1\end{pmatrix}.$$

holds. $\square$

Exercise. *Let $\mathcal{E}$ and $\mathcal{F}$ be bases of V. Suppose $\mathcal{F}'$ contains the same vectors as $\mathcal{F}$ but in a different order. How are $[I]^{\mathcal{E}}_{\mathcal{F}'}$ and $[I]^{\mathcal{E}}_{\mathcal{F}}$ related? What about $[I]^{\mathcal{F}'}_{\mathcal{E}}$ and $[I]^{\mathcal{F}}_{\mathcal{E}}$?*

Let $\mathcal{G}$ be another basis of V. Then

$$[I]^{\mathcal{E}}_{\mathcal{G}} = [I]^{\mathcal{F}}_{\mathcal{G}}[I]^{\mathcal{E}}_{\mathcal{F}}.$$

This identity is useful because when determining the transition matrix from one basis to another, it is often easier to go through an intermediate basis in which it is quicker to do computations.

Example 4.5.2. *Let $\mathcal{E} = \{1, 1+t\}$ and $\mathcal{F} = \{1-t, 2t+1\}$ be bases of $\mathbb{P}_1$. Determine the transition matrix from $\mathcal{E}$ to $\mathcal{F}$.*

Solution. We will do this using two methods:

Method 1. Let $\mathbf{e}_1(t) = 1$, $\mathbf{e}_2(t) = 1+t$, $\mathbf{f}_1(t) = 1-t$ and $\mathbf{f}_2(t) = 2t+1$ so that $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2\}$. Let us write $\mathbf{e}_1 = \alpha\mathbf{f}_1 + \beta\mathbf{f}_2$ so that

$$1 = \alpha(1-t) + \beta(2t+1).$$

Equating constants and coefficients of t we get

$$\begin{aligned} 1 &= \alpha + \beta, \\ 0 &= -\alpha + 2\beta. \end{aligned}$$

We find that $\beta = 1/3$ and $\alpha = 2/3$, and so

$$\mathbf{e}_1 = \frac{2}{3}\mathbf{f}_1 + \frac{1}{3}\mathbf{f}_2. \tag{4.5.2}$$

We repeat this for $\mathbf{e}_2$. Write $\mathbf{e}_2 = \alpha\mathbf{f}_1 + \beta\mathbf{f}_2$ so that

$$1 + t = \alpha(1-t) + \beta(2t+1).$$

Equating constants and coefficients of t we get

$$\begin{aligned} 1 &= \alpha + \beta, \\ 1 &= -\alpha + 2\beta. \end{aligned}$$

We find that $\beta = 2/3$ and $\alpha = 1/3$, and so

$$\mathbf{e}_2 = \frac{1}{3}\mathbf{f}_1 + \frac{2}{3}\mathbf{f}_2. \tag{4.5.3}$$

It follows from (4.5.2) and (4.5.3) that

$$[I]_{\mathcal{F}}^{\mathcal{E}} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}.$$

Method 2. Let $\mathcal{M} = \{1, t\}$ be the monomial basis of $\mathbb{P}_1$. We will first go from $\mathcal{E}$ to $\mathcal{M}$, then from $\mathcal{M}$ to $\mathcal{F}$.

It is immediate that

$$[I]_{\mathcal{M}}^{\mathcal{E}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad [I]_{\mathcal{M}}^{\mathcal{F}} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}.$$

We can invert $[I]_{\mathcal{M}}^{\mathcal{F}}$ to find that

$$[I]_{\mathcal{F}}^{\mathcal{M}} = \left([I]_{\mathcal{M}}^{\mathcal{F}}\right)^{-1} = \begin{pmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{pmatrix}.$$

Finally, we compute that

$$[I]_{\mathcal{F}}^{\mathcal{E}} = [I]_{\mathcal{F}}^{\mathcal{M}}[I]_{\mathcal{M}}^{\mathcal{E}} = \begin{pmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}.$$

□

Let $\mathcal{E}$ and $\mathcal{F}$ be bases of V and let $T : V \to V$ be a linear map. Let us consider how to relate the matrix $[T]_{\mathcal{E}}^{\mathcal{E}}$ to the matrix $[T]_{\mathcal{F}}^{\mathcal{F}}$. We know that for each $\mathbf{v} \in V$

$$[T\mathbf{v}]_{\mathcal{F}} = [T]_{\mathcal{F}}^{\mathcal{F}}[\mathbf{v}]_{\mathcal{F}}.$$

However, we also have that

$$[T\mathbf{v}]_{\mathcal{F}} = [I]_{\mathcal{F}}^{\mathcal{E}}[T\mathbf{v}]_{\mathcal{E}} = [I]_{\mathcal{F}}^{\mathcal{E}}[T]_{\mathcal{E}}^{\mathcal{E}}[\mathbf{v}]_{\mathcal{E}} = [I]_{\mathcal{F}}^{\mathcal{E}}[T]_{\mathcal{E}}^{\mathcal{E}}[I]_{\mathcal{E}}^{\mathcal{F}}[\mathbf{v}]_{\mathcal{F}}.$$

Hence we have the identity

$$[T]_{\mathcal{F}}^{\mathcal{F}} = [I]_{\mathcal{F}}^{\mathcal{E}}[T]_{\mathcal{E}}^{\mathcal{E}}[I]_{\mathcal{E}}^{\mathcal{F}}.$$

Combining this with the fact that $[I]_{\mathcal{F}}^{\mathcal{E}} = \left([I]_{\mathcal{E}}^{\mathcal{F}}\right)^{-1}$ we get the following:

> If P is the transition matrix from $\mathcal{F}$ to $\mathcal{E}$ then
>
> $$[T]_{\mathcal{F}}^{\mathcal{F}} = P^{-1}[T]_{\mathcal{E}}^{\mathcal{E}}P.$$

Example 4.5.3. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by*

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x+3y \\ 3x+y \end{pmatrix}.$$

Let $\mathcal{E}$ be the standard basis in $\mathbb{R}^2$ and let $\mathcal{F}$ be the basis

$$\mathcal{F} = \left\{ \mathbf{f}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{f}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

Determine the matrix of T with respect to the basis $\mathcal{F}$ for both the domain and codomain.

Solution. First Let us observe that

$$[T]_{\mathcal{E}}^{\mathcal{E}} = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}.$$

In Example 4.5.1 we showed that

$$[I]_{\mathcal{F}}^{\mathcal{E}} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \quad \text{and} \quad [I]_{\mathcal{E}}^{\mathcal{F}} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$[T]_{\mathcal{F}}^{\mathcal{F}} = [I]_{\mathcal{F}}^{\mathcal{E}}[T]_{\mathcal{E}}^{\mathcal{E}}[I]_{\mathcal{E}}^{\mathcal{F}} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

From this we can see that T acts by 'scaling' by a factor of 2 in the $\mathbf{f}_1$ direction and reflecting in the line $t\mathbf{f}_1$. □

Example 4.5.4. Let $T, \mathcal{E}, \mathcal{F}$ be as in Example 4.5.3 and let $\mathbf{u} = (3, 1)$. Then

$$T\mathbf{u} = [T\mathbf{u}]_{\mathcal{E}} = [T]_{\mathcal{E}}^{\mathcal{E}}[\mathbf{u}]_{\mathcal{E}} = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

We can also compute this using $[T]_{\mathcal{F}}^{\mathcal{F}}$. In Example 4.5.1 we showed that

$$[\mathbf{u}]_{\mathcal{F}} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Then

$$[T\mathbf{u}]_{\mathcal{F}} = [T]_{\mathcal{F}}^{\mathcal{F}}[\mathbf{u}]_{\mathcal{F}} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

Hence

$$T\mathbf{u} = 4\mathbf{f}_1 - \mathbf{f}_2 = 4(\mathbf{e}_1 + \mathbf{e}_2) + (-\mathbf{e}_1 + \mathbf{e}_2) = 3\mathbf{e}_1 + 5\mathbf{e}_2,$$

which as expected gives

$$T\mathbf{u} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

Definition 4.5.5. Let A and B be square matrices. Then A is *similar* to B if there exists an invertible matrix P such that

$$A = P^{-1}BP.$$

Observe that if A is similar to B then B is also similar to A. Indeed one easily checks that $A = P^{-1}BP$ if and only if $B = PAP^{-1}$. Setting $Q = P^{-1}$ we get that $B = Q^{-1}AQ$.

Remark 4.5.6. Similarity is an equivalence relation on the set of $n \times n$ matrices.

We have seen that if two $n \times n$ matrices represent the same linear map with respect to (possibly) different bases, then they are similar.

Let V be a vector space with basis $\mathcal{E}$ and let $T : V \to V$ be a linear map. Suppose that $B = [T]_{\mathcal{E}}^{\mathcal{E}}$ and $A = P^{-1}BP$ with

$$P = \begin{pmatrix} \uparrow & \uparrow & \dots & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ \downarrow & \downarrow & \dots & \downarrow \end{pmatrix}.$$

For $j = 1, 2, \dots, n$ let $\mathbf{f}_j$ be the unique vector in V such that $[\mathbf{f}_j]_{\mathcal{E}} = \mathbf{v}_j$. Note that since P is invertible, $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis of $\mathbb{F}^n$. Therefore, since the map $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{E}}$ is invertible and linear, Theorem 3.3.3 implies that $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n\}$ is a basis of V. Then

$$P = \begin{pmatrix} \uparrow & \uparrow & \dots & \uparrow \\ [\mathbf{f}_1]_{\mathcal{E}} & [\mathbf{f}_2]_{\mathcal{E}} & \dots & [\mathbf{f}_n]_{\mathcal{E}} \\ \downarrow & \downarrow & \dots & \downarrow \end{pmatrix} = [I]_{\mathcal{E}}^{\mathcal{F}}.$$

Hence

$$A = P^{-1}BP = [I]_{\mathcal{F}}^{\mathcal{E}}[T]_{\mathcal{E}}^{\mathcal{E}}[I]_{\mathcal{E}}^{\mathcal{F}} = [T]_{\mathcal{F}}^{\mathcal{F}}.$$

We have shown the following:

Two $n \times n$ matrices are similar if and only if they represent the same linear map with respect to (possibly) different bases.

Chapter 5

Determinants

The *determinant* is a function that associates a single number to every square matrix. This number contains a surprising amount of information about the matrix. In particular, we will se that a square matrix is invertible if and only if its determinant is non-zero.

The determinant of a square matrix A is usually denoted $\det(A)$ or $|A|$.

Determinant of a 1×1 matrix

Let $A = \begin{pmatrix} a \end{pmatrix}$ be a 1×1 matrix. Then A is invertible if and only if $a \neq 0$. We define

$$\det(A) := a.$$

Determinant of a 2×2 matrix

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If A is invertible then at least one of a or c must be non-zero (otherwise A would have a zero column). We suppose that $a \neq 0$. Then we can row reduce A as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ 0 & d - \frac{cb}{a} \end{pmatrix}.$$

This is invertible if and only if it has a pivot in each row. Hence A is invertible if and only if

$$a\left(d - \frac{cb}{a}\right) = ad - bc \neq 0.$$

We define

$$\det(A) := ad - bc.$$

Example 5.0.1.

$$\det\begin{pmatrix} 3 & 1 \\ -2 & 5 \end{pmatrix} = 15 - (-2) = 17.$$

5.1 Geometric interpretation

Let $\mathbf{e}_1, \mathbf{e}_2$ be the standard basis vectors in $\mathbb{R}^2$ and let A be the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}.$$

Then the matrix A transforms the standard basis to the columns of A and so the square determined by $\mathbf{e}_1$ and $\mathbf{e}_2$, i.e the square with vertices $(0,0), (0,1), (1,0)$ and $(1,1)$, is transformed to the parallelogram determined by the columns of A, i.e the parallelogram with vertices $(0,0), (a,c), (b,d)$ and $(a+b, c+d)$. Suppose that $A\mathbf{e}_1$ and $A\mathbf{e}_2$ have polar coordinates (r, θ) and (s, ϕ) respectively (see Figure 5.1) so that

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} s\cos\phi \\ s\sin\phi \end{pmatrix}.$$

Then the area of the parallelogram determined by the columns of A is

$$|rs\sin(\phi - \theta)| = |rs(\sin\phi\cos\theta - \sin\theta\cos\phi)| = |ad - bc| = |\det(A)|$$

Since any shape (or more precisely, any open set) in the plane is a disjoint[1] union of squares, we conclude the following:

[1]Strictly speaking, the boundaries of the squares may intersect but these have zero area so we can ignore them.

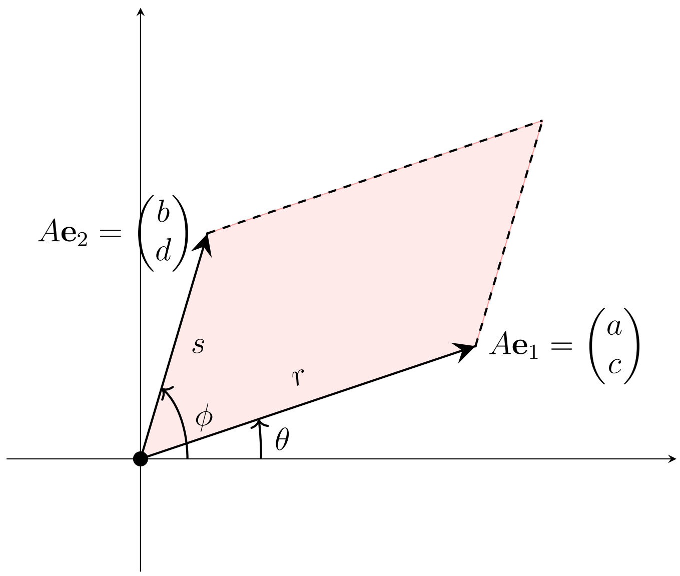

Figure 5.1: $|\det(A)|$ is the area of the parallelogram determined by the columns of A.

> Applying the transformation A scales areas by a factor of $|\det(A)|$.

Observe that $\det(A)$ is negative when the acute angle measured from $A\mathbf{e}_1$ to $A\mathbf{e}_2$ is negative, i.e. when $\phi - \theta$ is negative in Figure 5.1. Thus the sign of $\det(A)$ tells us whether or not the relative orientation of $\mathbf{e}_1$ and $\mathbf{e}_2$ has been preserved or reversed by A.

> The determinant of A is the signed area of the parallelogram determined by $A\mathbf{e}_1$ and $A\mathbf{e}_2$.

Exercise. *Let A be a real 2×2 matrix with positive determinant. Does it follow that the determinant of $-A$ will be negative? Justify your answer.*

5.2 Generalisation to $n \times n$ matrices

Let A be a real $n \times n$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, i.e.

$$A = \begin{pmatrix} \uparrow & \uparrow & \ldots & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \\ \downarrow & \downarrow & \ldots & \downarrow \end{pmatrix}.$$

We want to construct the determinant of A, $\det(A)$ to be the signed n-dimensional volume of the parallelepiped determined by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. We will use the notation

$$D(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) = \det(A)$$

when we want to emphasise the dependence on the columns.

5.2.1 Properties the determinant should have

Linearity in each argument. If we multiply a column, say $\mathbf{a}_k$, by a positive constant β, then the volume $D(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ should also be multiplied by β (see Figure 5.2 for the 2×2 case). Allowing negative volumes, this property should hold for all scalars β. Therefore, for all $\beta \in \mathbb{R}$, the determinant should satisfy

$$D(\mathbf{a}_1, \ldots, \beta\mathbf{a}_k, \ldots, \mathbf{a}_n) = \beta D(\mathbf{a}_1, \ldots, \mathbf{a}_k, \ldots, \mathbf{a}_n) \tag{5.2.1}$$

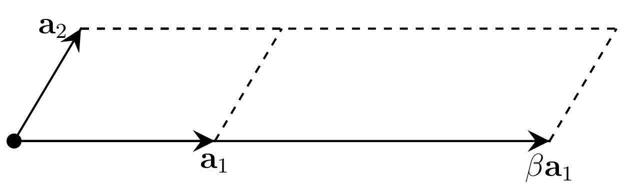

Figure 5.2: The area of the parallelogram determined by $\mathbf{a}_1$ and $\beta\mathbf{a}_2$ is β times the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2$.

Similarly, if we add two vectors, the length of the result in any particular direction is the sum of the lengths of each of the vectors in that direction and so the determinant should satisfy

$$D(\mathbf{a}_1, \ldots, \underbrace{\mathbf{a}_k + \mathbf{b}}_{k}, \ldots, \mathbf{a}_n) = D(\mathbf{a}_1, \ldots, \underbrace{\mathbf{a}_k}_{k}, \ldots, \mathbf{a}_n) + D(\mathbf{a}_1, \ldots, \underbrace{\mathbf{b}}_{k}, \ldots, \mathbf{a}_n). \tag{5.2.2}$$

Combining (5.2.1) and (5.2.2) we see that if we fix $n-1$ columns, then D is a linear function of the remaining column. This can rephrased by saying that D is linear in each argument separately.

Warning: The determinant is not a linear function from $M_n(\mathbb{R})$ to $\mathbb{R}$.

Remark 5.2.1. A function of several variables that is linear in each variable separately is called *multilinear*. So we require that D is multilinear.

Antisymmetry. If two of the $\mathbf{a}_k$ are equal, the volume of the parallelepiped determined by $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ should be zero (the parallelepiped will be "flat"). Hence if $\mathbf{a}_j = \mathbf{a}_k$ with $j \neq k$, we require that

$$D(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n) = 0.$$

Suppose we swap two columns, $\mathbf{a}_1$ and $\mathbf{a}_2$ say. Then by the preceding observation and the multilinearity property, the resulting volume must satisfy

$$\begin{aligned}
&D(\mathbf{a}_2, \mathbf{a}_1, \mathbf{a}_3 \dots, \mathbf{a}_n) + D(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_n) \\
&= D(\mathbf{a}_2, \mathbf{a}_1, \mathbf{a}_3 \dots, \mathbf{a}_n) + D(\mathbf{a}_1, \mathbf{a}_1, \mathbf{a}_3, \dots, \underline{a}_n) \\
&+ D(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \dots, \mathbf{a}_n) + D(\mathbf{a}_2, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_n) \\
&= D(\mathbf{a}_2 + \mathbf{a}_1, \mathbf{a}_1, \mathbf{a}_3, \dots, \mathbf{a}_n) + D(\mathbf{a}_2 + \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_n) \\
&= D(\mathbf{a}_2 + \mathbf{a}_1, \mathbf{a}_2 + \mathbf{a}_1, \mathbf{a}_3, \dots, \mathbf{a}_n) = 0.
\end{aligned}$$

Hence

$$D(\mathbf{a}_2, \mathbf{a}_1, \dots, \mathbf{a}_n) = -D(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n).$$

The same reasoning shows that this must hold for all pairs of columns. Therefore, when we interchange two columns, the determinant is multiplied by -1.

Remark 5.2.2. A function of several variables that that is multiplied by -1 whenever two of the variables are interchanged is called *antisymmetric* (or *alternating*). So we require that D is antisymmetric.

Normalization. Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ be the standard basis is $\mathbb{R}^n$. Then the parallelepiped determined by $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is the unit (n-dimensional) cube in $\mathbb{R}^n$ and the volume of this should be 1, i.e

$$D(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n) = \det(I) = 1.$$

To summarise we have shown that the determinant must satisfy the three basic properties:

(1) *Multilinearity*: The determinant is linear as a function of each column separatedly.

(2) *Antisymmetry*: The determinant is multiplied by -1 whenever we interchange two columns.

(3) *Normalization*: The derminant of the identity matrix is 1.

Remark 5.2.3. Although the motivation for these properties comes from considering real matrices, we can also use them to define the determinant for complex matrices.

Theorem 5.2.4. *For each $n \in \mathbb{N}$, there exists a unique function $\det : M_n(\mathbb{C}) \to \mathbb{C}$ satisfying the basic properties above. In other words, the determinant is well-defined and uniquely determined by the basic properties.*

We will not prove this yet.

Exercise. *Let A be a 3×3 matrix with $\det(A) = 2$. What is $\det(3A)$?*

We can deduce some additional properties from the basic ones.

Proposition 5.2.5. *For a square matrix A, the following statements hold:*

(i) If A has a zero column, the determinant of A is zero.

(ii) If the columns of A are linearly dependent, the determinant of A is zero.

(iii) Adding a multiple of one column to another leaves the determinant unchanged.

Proof. The proofs of (i) and (ii) are left as an assignment question.

Let us prove (iii). Suppose that

$$A = \begin{pmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ \downarrow & \downarrow & \cdots & \downarrow \end{pmatrix}.$$

Then adding β times column k to column 1 and using the multilinearity property we find that the resulting determinant is

$$D(\mathbf{a}_1 + \beta\mathbf{a}_k, \ldots, \mathbf{a}_k, \ldots \mathbf{a}_n) = D(\mathbf{a}_1, \ldots, \mathbf{a}_k, \ldots \mathbf{a}_n) + \beta D(\mathbf{a}_k, \ldots, \mathbf{a}_k, \ldots \mathbf{a}_n).$$

The second determinant on the right hand side has two equal arguments and so must be zero. Hence

$$D(\mathbf{a}_1 + \beta \mathbf{a}_k, \ldots, \mathbf{a}_k, \ldots \mathbf{a}_n) = D(\mathbf{a}_1, \ldots, \mathbf{a}_k, \ldots \mathbf{a}_n).$$

$\square$

Corollary 5.2.6. *If A is not invertible, the determinant of A is zero.*

5.2.2 Determinants of diagonal and triangular matrices

Definition 5.2.7. A square matrix $A = (a_{jk})$ is *diagonal* if $a_{jk} = 0$ whenever $j \neq k$, that is, if A is of the form

$$A = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n \end{pmatrix}.$$

Remark 5.2.8. The entries a_{jk} of A with $j = k$ are called the *main diagonal.* So we say that a diagonal matrix is zero away from the main diagonal.

If A is the diagonal matrix above, the multilinearity and normalisation properties imply that

$$\det(A) = \det \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n \end{pmatrix} = a_1 a_2 \ldots a_n \det(I) = a_1 a_2 \ldots a_n.$$

The determinant of a diagonal matrix is the product of its diagonal entries.

Definition 5.2.9. A square matrix $A = (a_{jk})$ is called *upper triangular* if $a_{jk} = 0$ whenever $k < j$ and *lower triangular* if $a_{jk} = 0$ whenever $k > j$. A matrix is *triangular* if it is either upper or lower triangular.

Remark 5.2.10. Equivalently, a matrix is upper (resp. lower) triangular if its entries are zero below (resp. above) the main diagonal.

For example,

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \quad \text{is upper triangular,} \quad \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} \quad \text{is lower triangular.}$$

First observe that if A is a triangular matrix and one of its diagonal entries is zero then the columns are linearly dependent (since A does not have a pivot in each row). So in this case the determinant of A is zero. If each diagonal entry of A is non-zero, then by repeatedly adding a multiple of one column to another, A can be transformed into a diagonal matrix with the same diagonal entries as A. Since these properties leave the determinant unchanged we get the following:

The determinant of a triangular matrix is the product of its diagonal entries.

Exercise. *Determine whether each of the following statements is true or false.*

(a) The inverse of an elementary matrix is elementary.

(b) The transpose of an elementary matrix is elementary

(c) Every product of elementary matrices is invertible.

(d) Every invertible matrix is a product of elementary matrices.

5.2.3 Determinant of a transpose and product

Consider the following elementary *column* operations:

(I) Interchange column j with column k, where $j \neq k$.

(II) Multiply column j by a scalar α.

(III) Add α times column j to column k.

We know the effect of each of these on the determinant of square matrix: operations of type (I) multiply the determinant by -1, type (II) multiply the determinant by α and type (III) leave the determinant unchanged (multiply by 1). Therefore, for each elementary column operation, there is a scalar $\beta \in \mathbb{F}$ such that applying the column operation to a square matrix, results in multiplying the determinant by β.

Let E be an elementary matrix that corresponds to a particular row operation. Applying the same operation to columns of a matrix A (i.e. to the rows of A^T) produces the matrix

$$(EA^T)^T = AE^T.$$

Observe that E^T is also an elementary matrix of the same type as E. Hence every elementary column operation can be realized as multiplication on the *right* by an elementary matrix.

Lemma 5.2.11. *Let E be an elementary matrix. Then the following statements hold:*

(i) $\det(E) = \det(E^T)$.

(ii) For every square matrix A, $\det(AE) = \det(A)\det(E)$.

Proof. (i) Observe that E^T corresponds to a column operation of the same type as E. Indeed if E corresponds to interchanging columns, then E^T also corresponds to interchanging (possibly different) columns; if E corresponds to multiplying a column by α, E^T corresponds to multiplying a column by α; if E corresponds to adding α times one column to another, E^T corresponds to adding α times one column to another. Hence multiplication on the right E and by E^T will have the same effect on the determinant of a square matrix. In particular,

$$\det(E) = \det(IE) = \det(IE^T) = \det(E^T).$$

(ii) Since right multiplication by E is the same as applying an elementary column operation, there is a non-zero scalar $\beta \in \mathbb{F}$ such that for every square matrix A,

$$\det(AE) = \beta \det(A).$$

It follows that

$$\det(E) = \det(IE) = \beta \det(I) = \beta,$$

and so

$$\det(AE) = \det(A)\det(E).$$

□

Theorem 5.2.12. *Let A be a square matrix. Then*

$$\det(A) = \det(A^T).$$

Proof. If A is not invertible, A^T is not invertible and so $\det(A) = \det(A^T) = 0$.

Let A be an invertible matrix. Then there exist elementary matrices $E_1, E_2, \dots, E_k$ such that $A = E_1 E_2 \dots E_k$. Then by Lemma 5.2.11,

$$\begin{aligned}
\det(A) &= \det(E_1 E_2 \dots E_k) \\
&= \det(E_1)\det(E_2)\dots\det(E_k) \\
&= \det(E_k^T)\dots\det(E_2^T)\det(E_1^T) \\
&= \det(E_k^T \dots E_2^T E_1^T) \\
&= \det(A^T).
\end{aligned}$$

□

Corollary 5.2.13. *The following properties hold for the determinant:*

(i) The determinant is linear in each row separately (i.e. it is a multilinear function of the rows).

(ii) Elementary row operations have the same effect on the determinant as the corresponding column operations.

Theorem 5.2.14. *Let A and B be square matrices. Then*

$$\det(AB) = \det(A)\det(B).$$

Proof. If either A or B is not invertible, the product AB will not be invertible and so $\det(AB) = \det(A)\det(B) = 0$.

Suppose that both A and B are invertible. Then there exist elementary matrices $E_1, E_2, \dots, E_k$ such that $B = E_1 E_2 \dots E_k$. Then by Lemma 5.2.11,

$$\begin{aligned}
\det(AB) &= \det(AE_1 E_2 \dots E_k) \\
&= \det(A)\det(E_1)\det(E_2)\dots\det(E_k) \\
&= \det(A)\det(E_1 E_2 \dots E_k) \\
&= \det(A)\det(B).
\end{aligned}$$

□

Warning: In general, $\det(A + B) \neq \det(A) + \det(B)$.

Corollary 5.2.15. *Let A be a square matrix. Then A is invertible if and only if* $\det(A) \neq 0$.

Proof. We know that if A is not invertible then the columns of A are linearly dependent, and so $\det(A) = 0$. Conversely, if A is invertible, then $A = E_1 E_2 \dots E_k$ for some elementary matrices $E_1, E_2, \dots, E_k$. Since the determinant of an elementary matrix is non-zero, we have that

$$\det(A) = \det(E_1)\det(E_2)\dots\det(E_k) \neq 0.$$

$\square$

Determine which of the following statements are true for all 2×2 matrices A.

Exercise. *Determine which of the following statements are true for all invertible 2×2 matrices A.*

(a) $\det(A + A^T) = 2\det(A)$

(b) $\det(A - A^T) = 0$.

(c) $\det(A^T A^{-1}) = 1$.

What about for 3×3 matrices?

5.2.4 Formal definition. Cofactor formula

Definition 5.2.16. Let A be an $n \times n$ matrix. For $1 \leq i, j \leq n$, the *minor* $M_{ij}(A)$ is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the row i and column j from A.

The *cofactor* $C_{ij}(A)$ is $(-1)^{i+j} M_{ij}(A)$.

Example 5.2.17. Let

$$A = \begin{pmatrix} 3 & 1 & 5 \\ 1 & -1 & 0 \\ 2 & 1 & 2 \end{pmatrix}.$$

Then

$$M_{12}(A) = \det\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} = 2, \quad C_{12}(A) = (-1)^3 \times 2 = -2,$$
$$M_{33}(A) = \det\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = -4, \quad C_{33}(A) = (-1)^6 \times (-4) = -4.$$

Let $A = (a_{ij})$ be an $n \times n$ matrix. We consider several cases in order to define the determinant for general matrices.

Case 1: a_{11} is the only non-zero entry in the first row.

We perform column operations on columns 2 to n to put A into lower triangular form. Let E be an invertible matrix such that AE is in lower triangular form. Then

$$\det(A) = a_{11}(AE)_{22}(AE)_{33} \ldots (AE)_{nn} \times \frac{1}{\det(E)}.$$

Observe that the product

$$(AE)_{22}(AE)_{33} \ldots (AE)_{nn} \times \frac{1}{\det(E)},$$

i.e. the term on the right without a_{11}, is precisely the determinant $M_{11}(A)$. Hence

$$\det(A) = a_{11}M_{11}(A) = a_{11}C_{11}(A).$$

Case 2: a_{1k} is the only non-zero entry in the first row, $1 \leq k \leq n$.

We interchange column k with column $k-1$, then column $k-1$ with column $k-2$, and so on until column k is the left most column. This requires $k-1$ column exchanges and so

$$\det(A) = (-1)^{k-1}a_{1k}M_{1k} = a_{1k}C_{1k}(A).$$

Case 3: General $n \times n$ matrix.

We use linearity in the first row. Let A_k be the matrix obtained from A by making all entries in the first row except a_{1k} zero, and leaving all other entries unchanged. Then since rows 2 to n are the same for $A_1, A_2, \ldots, A_n$ we have that

$$\det(A) = \det(A_1) + \det(A_2) + \ldots \det(A_n).$$

Hence

$$\det(A) = \sum_{j=1}^{n} a_{1j}C_{1j}(A) = a_{11}C_{11}(A) + a_{12}C_{12}(A) + \cdots + a_{1n}C_{1n}(A).$$

As a result of this we introduce the following definition of the determinant.

Definition 5.2.18. Let $A = (a_{ij})$ be an $n \times n$ matrix. Then the determinant of A, $\det(A)$, is defined inductively as follows:

- if $n = 1$, $\det(A) = a$;
- if $n \geq 2$, $\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j}(A) = a_{11}C_{11}(A) + a_{12}C_{12}(A) + \cdots + a_{1n}C_{1n}(A)$.

Proposition 5.2.19. *For each $n \in \mathbb{N}$, the determinant* $\det : M_n(\mathbb{C}) \to \mathbb{C}$ *satisfies the basic properties given in Section 5.2.1 (linearity in each column, antisymmetry and normalization).*

Proof. We prove each property in turn.

Linearity in each column. We prove this by induction on n. It is clear that a 1×1 determinant is linear. Assume that the determinant of $(n-1) \times (n-1)$ matrices is linear in each argument.

Let $A = (a_{ij})$ be an $n \times n$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. Let $B = (b_{ij})$ be the $n \times n$ matrix obtained from A by replacing column k with $\mathbf{b}$ and let $\widetilde{A} = (\widetilde{a}_{ij})$ be the $n \times n$ matrix obtained from A by replacing column k with $\alpha \mathbf{a}_k + \beta \mathbf{b}$. We need to show that

$$\det(\widetilde{A}) = \alpha \det(A) + \beta \det(B).$$

We compute that

$$\det(\widetilde{A}) = \sum_{j=1}^{n} \widetilde{a}_{1j} C_{1j}(\widetilde{A}) = \sum_{j \neq k} \widetilde{a}_{1j} C_{1j}(\widetilde{A}) + (\alpha a_{1k} + \beta b_{1k}) C_{1k}(A).$$

Since $(n-1) \times (n-1)$ determinants are linear in each argument, we have that for $j \neq k$,

$$C_{1j}(\widetilde{A}) = \alpha C_{1j}(A) + \beta C_{1j}(B).$$

Hence

$$\begin{aligned}
\det(\widetilde{A}) &= \sum_{j \neq k} \widetilde{a}_{1j} (\alpha C_{1j}(A) + \beta C_{1j}(B)) + (\alpha a_{1k} + \beta b_{1k}) C_{1k}(A) \\
&= \alpha \sum_{j=1}^{n} a_{1j} C_{1j}(A) + \beta \left(\sum_{j \neq k} a_{1j} C_{1j}(B) + b_{1k} C_{1k}(A) \right).
\end{aligned}$$

Observe that $C_{1k}(A) = C_{1k}(B)$ and $a_{1j} = b_{1j}$ for $j \neq k$. Therefore

$$\det(\widetilde{A}) = \alpha \sum_{j=1}^{n} a_{1j} C_{1j}(A) + \beta \sum_{j=1}^{n} b_{1j} C_{1j}(B) = \alpha \det(A) + \beta \det(B).$$

We have shown that $n \times n$ determinants are linear in each argument. It now follows by induction that this holds for all $n \in \mathbb{N}$.

Antisymmetry. We again prove this by induction on n. Suppose that the $(n-1)\times(n-1)$ determinant is antisymmetric.

Let $A = (a_{ij})$ be an $n \times n$ matrix. Suppose we interchange two adjacent columns of A, column k and column $k+1$ say, to produce a matrix $\widetilde{A} = (\widetilde{a}_{ij})$. Observe that $M_{1k}(A) = M_{1(k+1)}(\widetilde{A})$, and so $C_{1k}(A) = -C_{1(k+1)}(\widetilde{A})$. Similarly, $C_{1(k+1)}(A) = -C_{1k}(\widetilde{A})$. For each $j \neq k, k+1$, $M_{1j}(\widetilde{A})$ and $M_{1j}(A)$ are determinants of $(n-1) \times (n-1)$ matrices that differ by one column exchange. Hence $M_{1j}(\widetilde{A}) = -M_{1j}(A)$ by assumption. We conclude that

$$\det(\widetilde{A}) = -\sum_{j=1}^{n} a_{1j} C_{1j}(A) = -\det(A).$$

Now suppose that $1 \leq j < k \leq n$ and let A have columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. Then exchanging column j and column k can be carried out as follows: exchange column j with column $j+1$, then column $j+1$ with column $j+2$, and so on until $\mathbf{a}_j$ is in column k. At this point, $\mathbf{a}_k$ will be in column $k-1$ and so we exchange column $k-1$ with column $k-2$ and so on until $\mathbf{a}_k$ is in column j. One easily checks that this requires $2(k-j)-1$ exchanges of adjacent columns. Hence the determinant is multiplied by $(-1)^{2(k-j)-1} = -1$.

Since a 1×1 determinant is trivially antisymmetric, it follows by induction that $n \times n$ determinants are antisymmetric for all $n \in \mathbb{N}$.

Normalization. Let I_n denote the $n \times n$ identity matrix. One easily checks that $\det(I_n) = \det(I_{n-1})$. Since $\det(I_1) = 1$, it follows that $\det(I_n) = 1$ for all $n \in \mathbb{N}$. $\square$

Remark 5.2.20. Proposition 5.2.19 combined with the discussion preceding it provide a proof of Theorem 5.2.4.

By exchanging rows, we see that we can expand in any row (rather than just the first row). Moreover, since $\det(A) = \det(A^T)$ we can expand in any column.

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j}(A) \tag{5.2.3}$$

$$= \sum_{j=1}^{n} a_{kj} C_{kj}(A) \quad \text{(expand in row } k\text{)} \tag{5.2.4}$$

$$= \sum_{j=1}^{n} a_{jk} C_{jk}(A) \quad \text{(expand in column } k\text{)} \tag{5.2.5}$$

$$\tag{5.2.6}$$

Example 5.2.21. Let

$$A = \begin{pmatrix} 5 & 1 & 2 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 6 & 0 & -1 \\ -6 & 3 & 1 & 0 \end{pmatrix}.$$

Then expanding in the second row we get that

$$\det(A) = -3 \det \begin{pmatrix} 1 & 2 & 0 \\ 6 & 0 & -1 \\ 3 & 1 & 0 \end{pmatrix}.$$

We compute that

$$\det \begin{pmatrix} 1 & 2 & 0 \\ 6 & 0 & -1 \\ 3 & 1 & 0 \end{pmatrix} = \det \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - 2 \det \begin{pmatrix} 6 & -1 \\ 3 & 0 \end{pmatrix} = 1 - 6 = -5.$$

Therefore $\det(A) = 15$.

5.3 Cofactor formula for A^{-1}

For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

one easily checks that if $\det(A) = ad - bc \neq 0$ then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

There is a similar formula for $n \times n$ matrices.

Proposition 5.3.1. *Let A be an $n \times n$ matrix with $\det(A) \neq 0$. Let C be the $n \times n$ matrix whose i, j-entry is the cofactor $C_{ij}(A)$, i.e*

$$C = \begin{pmatrix} C_{11}(A) & C_{12}(A) & \dots & C_{1n}(A) \\ C_{21}(A) & C_{22}(A) & \dots & C_{2n}(A) \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1}(A) & C_{n2}(A) & \dots & C_{nn}(A) \end{pmatrix}.$$

Then

$$A^{-1} = \frac{1}{\det(A)} C^T.$$

Proof. Suppose $A = (a_{ij})$. We will show that $AC^T \det(A) I$. Then since A is square, it will follow that A is invertible and

$$A^{-1} = \frac{1}{\det(A)} C^T.$$

The i, j-entry of AC^T is the 'dot product' of row i of A and column j of C^T (the latter is just row j of C), i.e.

$$(AC^T)_{ij} = \sum_{j=1}^{n} a_{ik} C_{jk}(A).$$

If $i = j$, this is just the cofactor expansion of A along row j of A, so this equals $\det(A)$. If $i \neq j$, this is the cofactor expansion along row j of the matrix obtained from A by replacing row j with row i (and leaving all other rows unchanged). Since this matrix has two equal rows, it will have determinant equal to zero. Therefore $AC^T = \det(A) I$. $\square$

Chapter 6

Introduction to eigenvalues and canonical forms

This chapter is intended to serve as a brief introduction to some of the concepts that will be covered in more detail in Linear Algebra and Geometry II. The techniques introduced here will also be useful in other courses next semester; in particular, **knowledge of the content in this chapter will be assumed in 4CCM131A Introduction to Dynamical Systems**.

6.1 Eigenvalues and eigenvectors

Definition 6.1.1. Let $A : \mathbb{F}^n \to \mathbb{F}^n$ be a linear map. Then $\lambda \in \mathbb{C}$ is an *eigenvalue* of A if there exists a **non-zero** vector $\mathbf{x} \in \mathbb{F}^n$ such that

$$A\mathbf{x} = \lambda\mathbf{x}. \tag{6.1.1}$$

A non-zero vector $\mathbf{x}$ is an *eigenvector* of A corresponding to the eigenvalue λ if it satisfies (6.1.1).

Observe that (6.1.1) is satisfied if and only if $(A - \lambda I)\mathbf{x} = \mathbf{0}$, so that $\mathbf{x} \in \operatorname{Ker}(A - \lambda I)$. Thus λ is an eigenvalue of A if and only $\operatorname{Ker}(A - \lambda I) \neq \{\mathbf{0}\}$ and the eigenvalues corresponding to λ are the non-zero elements of $\operatorname{Ker}(A - \lambda I)$. Recall that if the kernel of an $n \times n$ matrix contains a non-zero vector, it is not invertible and so its determinant is zero. This gives us the following characterisation of eigenvalues:

$\lambda \in \mathbb{C}$ is an *eigenvalue* of A if and only if $\det(A - \lambda I) = 0$.

When A is an $n \times n$ matrix, the function $p(\lambda) = \det(A - \lambda I)$ is a polynomial in λ of degree n. By the Fundamental Theorem of Algebra, it has precisely n complex roots, and so A has precisely n (possibly repeated) eigenvalues.

Remark 6.1.2. The polynomial $p(\lambda) = \det(A - \lambda I)$ is called the *characteristic polynomial* of A.

Example 6.1.3. *Determine all eigenvalues and eigenvectors of the following matrices:*

(a) $A = \begin{pmatrix} 1 & -10 \\ -5 & -4 \end{pmatrix}$ *(b)* $B = \begin{pmatrix} 3 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

Solution. (a) We first find the eigenvalues by solving the equation $\det(A = \lambda I) = 0$.

We have that

$$A - \lambda I = \begin{pmatrix} 1 & -10 \\ -5 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & -10 \\ -5 & -4-\lambda \end{pmatrix},$$

and so

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)(-4-\lambda) - 50 \\ &= \lambda^2 + 3\lambda - 54 \\ &= (\lambda - 6)(\lambda + 9). \end{aligned}$$

Thus the eigenvalues of A are $\lambda_1 = 6$ and $\lambda_2 = -9$.

To find the eigenvectors of A, we have to solve the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for $\lambda = 6, -9$.

Solving the system

$$(A - 6I)\mathbf{x} = \begin{pmatrix} -5 & -10 \\ -5 & -10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we get $x_1 = -2x_2$. It follows that every eigenvector of A with eigenvalue 6 is a non-zero multiple of

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Similarly, solving the system

$$(A-9I)\mathbf{x} = \begin{pmatrix} 10 & -10 \\ -5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we get $x_1 = x_2$. It follows that every eigenvector of A with eigenvalue -9 is a non-zero multiple of

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(b) As before, we first find the eigenvalues.

$$B - \lambda I = \begin{pmatrix} 3-\lambda & 2 & 2 \\ 0 & 1-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{pmatrix},$$

so $\det(B-\lambda I) = (3-\lambda)(1-\lambda)(2-\lambda)$. Thus the eigenvalues of B are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$.

We now find the eigenvectors corresponding to each of the eigenvalues in turn.

$\lambda = 1$: Solving the equation

$$(B-I)\mathbf{x} = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we get $x_3 = 0$ and $x_1 = -x_2$, and so every eigenvector of B with eigenvalue 1 is a non-zero multiple of

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

$\lambda = 2$: Solving the equation

$$(B-I)\mathbf{x} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we get $x_1 = 0$ and $x_2 = -x_3$, and so every eigenvector of B with eigenvalue 2 is a non-zero multiple of

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

$\lambda = 3$: Solving the equation

$$(B - I)\mathbf{x} = \begin{pmatrix} 0 & 2 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we get $x_2 = x_3 = 0$, and so every eigenvector of B with eigenvalue 3 is a non-zero multiple of

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

□

Exercise. *Let A and B be $n \times n$ matrices and let λ be an eigenvalue of A and μ be an eigenvalue fo B. Which of the following statements must be true?*

(a) 2λ *is an eigenvalue of* $2A$*.*

(b) λ^2 *is an eigenvalue of* A^2*.*

(c) $\lambda + \mu$ *is an eigenvalue of* $A + B$*.*

(d) $\lambda\mu$ *is an eigenvalue of* AB*.*

6.2 Canonical forms of 2×2 matrices

Let A and B be $n \times n$ matrices. Recall that A is *similar* to B if there exists an invertible matrix P such that

$$A = P^{-1}BP.$$

We say that a matrix is *diagonalizable* if it is similar to a diagonal matrix. More precisely, we say that it is diagonalizable over $\mathbb{F}$ if it is similar to a diagonal matrix with entries in $\mathbb{F}$. Note that it is possible that a real matrix is diagonalizable over $\mathbb{C}$ but not over $\mathbb{R}$.

Theorem 6.2.1. *Let A be a 2×2 matrix. Then exactly one of the following hold:*

(i) A is diagonalisable over $\mathbb{C}$ so that there exists an invertible matrix P and $\lambda_1, \lambda_2 \in \mathbb{C}$ such that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

(ii) *A is not diagonalisable over $\mathbb{C}$ but there exists an invertible matrix P and $\lambda_0 \in \mathbb{C}$ such that*

$$P^{-1}AP = \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix}.$$

To determine which class a particular matrix A falls into, we need to determine its eigenvalues. Let A be a 2×2 matrix. Since the function $\det(A - \lambda)$ is a quadratic polynomial in λ we have one of two possibilities for its roots:

(1) $\det(A - \lambda I)$ has two distinct roots $\lambda_1, \lambda_2 \in \mathbb{C}$ (A has two distinct eigenvalues). In this case we have that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

for some invertible matrix P. Moreover, we can construct P by taking its columns to be the eigenvectors of A, i.e the columns of P will be non-zero solutions of the homogeneous linear equations

$$(A - \lambda_1 I)\mathbf{x} = \mathbf{0}, \tag{6.2.1}$$

$$(A - \lambda_2 I)\mathbf{x} = \mathbf{0}. \tag{6.2.2}$$

In particular, we take the first column of P to be any solution of (6.2.1) and the second column of P to be any solution of (6.2.2).

(2) $\det(A - \lambda I)$ has a repeated root $\lambda_0 \in \mathbb{C}$ (A has a repeated eigenvalue). In this case we have to we have to compute the eigenvectors of A, i.e we have to determine $\operatorname{Ker}(A - \lambda_0 I)$. Here we again have two separate cases to consider:

(a) If $\operatorname{Ker}(A - \lambda_0 I)$ is two dimensional then

$$P^{-1}AP = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix},$$

where we can take the columns of P to be any basis of $\operatorname{Ker}(A - \lambda_0 I)$. (Observe that this is case (i) above with $\lambda_1 = \lambda_2 = \lambda_0$.)

(b) If $\operatorname{Ker}(A - \lambda_0 I)$ is one dimensional then

$$P^{-1}AP = \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix},$$

for some invertible matrix P. Let $\mathbf{x}$ be an eigenvector of A (all others are multiples of $\mathbf{x}$) and let $\mathbf{y}$ be any vector which is linearly independent to $\mathbf{x}$. Then there is a non-zero scalar $\alpha \in \mathbb{C}$ such that

$$A\mathbf{v} = \lambda_0 \mathbf{x} + \alpha \mathbf{y}.$$

We can take the first column of P to be $\mathbf{x}$ and the second column to be $\alpha^{-1}\mathbf{y}$.

If A is a real matrix then we can also say that following.

Theorem 6.2.2. *Let A be a real 2×2 matrix. Then exactly one of the following hold:*

(i) A is diagonalisable over $\mathbb{R}$ so that there exists an invertible matrix P and $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

(ii) A is not diagonalisable over $\mathbb{R}$ but there exists an invertible matrix P and $\lambda \in \mathbb{R}$ such that

$$P^{-1}AP = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

(iii) A is not diagonalizable over $\mathbb{R}$ but there exists an invertible matrix P and $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$, such that

$$P^{-1}AP = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

As before, we determine which of these cases applies to a particular matrix A by considering its eigenvalues, i.e. the roots of the quadratic function $\det(A - \lambda I)$. Since we are considering real roots we now have three possibilities:

(1) $\det(A - \lambda I)$ has two real roots $\lambda_1, \lambda_2 \in \mathbb{R}$. In this case the previous discussion applies and we see that A is similar to

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

(2) $\det(A - \lambda I)$ has one real root $\lambda_0 \in \mathbb{R}$. As before we have that if $\dim \mathrm{Ker}\,(A - \lambda I) = 2$ then A is similar to

$$\begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix},$$

otherwise A is similar to

$$\begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix}.$$

(3) $\det(A - \lambda I)$ has two complex roots $\alpha \pm i\beta$, $\beta \neq 0$. In this case A i similar to

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

Observe that A is also similar to

$$\begin{pmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{pmatrix}.$$

Exercise. *Let A be an $n \times n$ matrix. Suppose A is diagonalizable. Is A^T necessarily diagonalizable?*

Example 6.2.3. Consider the matrix

$$A = \begin{pmatrix} 7 & -5 \\ 10 & -8 \end{pmatrix}.$$

To determine the canonical form of A (given by Theorem 6.2.2), we need to find the eigenvalues of A, i.e. we have to find the solutions $\lambda \in \mathbb{C}$ of $\det(A - \lambda I) = 0$.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 7 - \lambda & -5 \\ 10 & -8 - \lambda \end{pmatrix} \\ &= (7 - \lambda)(-8 - \lambda) + 50 \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda + 3)(\lambda - 2). \end{aligned}$$

So $\det(A - \lambda I) = 0$ if and only if $\lambda = -3$ or $\lambda = 2$. Hence there exists an invertible matrix P such that

$$P^{-1}AP = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}.$$

Next, we solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for $\lambda = 2, -3$. If $\lambda = 2$,

$$(A - \lambda I)\mathbf{x} = \begin{pmatrix} 5 & -5 \\ 10 & -10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so $x_1 = x_2$. Hence all of the solutions (eigenvectors with eigenvalue 2) are of the form

$$t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

If $\lambda = -3$,

$$(A - \lambda I)\mathbf{x} = \begin{pmatrix} 10 & -5 \\ 10 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

so $10x_1 = 5x_2$ which gives $2x_1 = x_2$. Hence all of the solutions (eigenvectors with eigenvalue -3) are of the form

$$t \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Therefore we can take

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then

$$P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix},$$

and we check that

$$P^{-1}AP = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 7 & -5 \\ 10 & -8 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}.$$

Example 6.2.4. Let

$$B = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}.$$

Then

$$\begin{aligned} \det(B - \lambda I) &= \det \begin{pmatrix} 3 - \lambda & -1 \\ 1 & 2 - \lambda \end{pmatrix} \\ &= (3 - \lambda)(2 - \lambda) + 1 \\ &= \lambda^2 - 5\lambda + 7 \\ &= (\lambda - 5/2)^2 + 3/4. \end{aligned}$$

So $\det(B - \lambda I) = 0$ if and only if $\lambda = 5/2 \pm i\sqrt{3}/2$. Hence there is an invertible matrix P such that

$$P^{-1}BP = \begin{pmatrix} 5/2 + i\sqrt{3}/2 & 0 \\ 0 & 5/2 - i\sqrt{3}/2 \end{pmatrix}.$$

There is also a *real* invertible matrix Q such that

$$Q^{-1}BQ = \begin{pmatrix} 5/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 5/2 \end{pmatrix}.$$

Example 6.2.5. Let

$$C = \begin{pmatrix} 4 & 9 \\ -1 & -2 \end{pmatrix}.$$

Then

$$\begin{aligned} \det(C - \lambda I) &= \det \begin{pmatrix} 4-\lambda & 9 \\ -1 & -2-\lambda \end{pmatrix} \\ &= (4-\lambda)(-2-\lambda) + 9 \\ &= \lambda^2 - 2\lambda + 1 \\ &= (\lambda - 1)^2. \end{aligned}$$

So the only eigenvalue of C is $\lambda = 1$.

Let us solve $(C - \lambda I)\mathbf{x} = \mathbf{0}$ for $\lambda = 1$:

$$(C - I)\mathbf{x} = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so $x_1 = -3x_2$. Hence each eigenvector is a multiple of

$$\begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Since there is only one linearly independent eigenvector, there exists an invertible matrix P such that

$$P^{-1}CP = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Exercise. *Let λ_1 and λ_2 be distinct real numbers and let A be the matrix*

$$\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}.$$

What is the canonical form of A given by Theorem 6.2.2?

6.3 Application to systems of ODEs

For simplicity, we will only consider first order homogeneous constant coefficient equations.

One equation. For $\lambda \in \mathbb{F}$, consider the ODE

$$\frac{du}{dt} = \lambda u.$$

This has solutions $u(t) = Ce^{\lambda t}$, $C \in \mathbb{F}$. Note that $C = u(0)$.

n **equations.** Now we have n unknown functions $u_1(t), \ldots, u_n(t)$ and a system of ODEs:

$$\begin{aligned}
\frac{du_1}{dt} &= a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n \\
\frac{du_2}{dt} &= a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n \\
\vdots \qquad & \qquad\qquad \vdots \\
\frac{du_n}{dt} &= a_{m1}u_1 + a_{m2}u_2 + \cdots + a_{mn}u_n.
\end{aligned}$$

In general, these equations are *coupled*, i.e each equation cannot be solved on its own. As with ordinary linear equations, we can write this as a single vector ODE:

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}, \tag{6.3.1}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}, \quad \mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}, \quad \frac{d\mathbf{u}}{dt} = \begin{pmatrix} du_1/dt \\ du_2/dt \\ \vdots \\ du_n/dt \end{pmatrix}.$$

The equation (6.3.1) is linear and homogeneous, i.e if $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are solutions, so is $\alpha\mathbf{u}(t) + \beta\mathbf{v}(t)$ for every $\alpha, \beta \in \mathbb{F}$.

Suppose λ is an eigenvalue of A with eigenvector $\mathbf{x}$, so that $A\mathbf{x} = \lambda\mathbf{x}$. Then if

$$\mathbf{u}(t) = e^{\lambda t}\mathbf{x} = \begin{pmatrix} x_1 e^{\lambda t} \\ x_2 e^{\lambda t} \\ \vdots \\ x_2 e^{\lambda t} \end{pmatrix},$$

we have that
$$\frac{d\mathbf{u}}{dt} = \lambda e^{\lambda t}\mathbf{x}$$
and
$$A\mathbf{u} = A(e^{\lambda t}\mathbf{x}) = e^{\lambda t}A\mathbf{x} = \lambda e^{\lambda t}\mathbf{x}.$$
So $\mathbf{u}(t) = e^{\lambda t}\mathbf{x}$ is a solution. Hence if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A with respective eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, then
$$\mathbf{u}(t) = C_1 e^{\lambda_1 t}\mathbf{x}_1 + C_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + C_n e^{\lambda_n t}\mathbf{x}_n$$
is a solution. In fact, we have the following:

If A is diagonalizable and $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent eigenvectors corresponding to $\lambda_1, \ldots, \lambda_n$ respectively, then the general solution of (6.3.1) is
$$\mathbf{u}(t) = C_1 e^{\lambda_1 t}\mathbf{x}_1 + C_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + C_n e^{\lambda_n t}\mathbf{x}_n, \quad C_1, \ldots, C_n \in \mathbb{F}.$$

Remark 6.3.1. If A is not diagonalizable, there are other solutions. In particular, if λ is an eigenvalue which is repeated k times but has only one linearly independent eigenvector $\mathbf{x}$, then $e^{\lambda t}\mathbf{x}, te^{\lambda t}\mathbf{x}, \ldots, t^{k-1}e^{\lambda t}\mathbf{x}$ will be linearly independent solutions.

Example 6.3.2. *Find the general solution of the following system of ODEs:*
$$\begin{aligned} \frac{du_1}{dt} &= 7u_1 - 5u_2, \\ \frac{du_2}{dt} &= 10u_1 - 8u_2. \end{aligned}$$

Solution. This can be written in vector form as
$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} 7 & -5 \\ 10 & -8 \end{pmatrix}\mathbf{u}.$$
We know from Example 6.2.3 that A is diagonalizable and that it has eigenvalues $\lambda_1 = 2, \lambda_2 = -3$ with corresponding eigenvectors
$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Therefore the general solution is

$$\mathbf{u}(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad C_1, C_2 \in \mathbb{F}.$$

or in scalar form,

$$\begin{aligned} u_1(t) &= C_1 e^{2t} + C_2 e^{-3t}, \\ u_2(t) &= C_1 e^{2t} + 2C_2 e^{-3t}. \end{aligned}$$

□

Example 6.3.3. *Find the particular solution of the system of ODEs*

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} 3 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{u}$$

satisfying $\mathbf{u}(0) = (1, 1, 1)$. *(You may use the fact that A is diagonalisable.)*

Solution. We know from Example 6.1.3(b) that the matrix

$$\begin{pmatrix} 3 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

has eigenvalues $1, 2, 3$ with respective eigenvectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore the general solution of the ODE is

$$\mathbf{u}(t) = C_1 e^{t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + C_3 e^{3t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

If $\mathbf{u}(0) = (1, 1, 1)$ we must have

$$C_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

This gives $C_1 = -2$, $C_2 = 0$ and $C_3 = 3$. Thus the solution we seek is

$$\mathbf{u}(t) = -2e^t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 3e^{3t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

$\square$