Introduction to Abstract Algebra: Sheet 9

For discussion in week 9 skills sessions and week 10 tutorials

The problems marked with (SS) will be discussed in the Skills Session for Week 9. Please attempt these *before* your Skills Session. Some of the other problems will be discussed in your tutorial for Week 10. Please attempt all core problems *before* your tutorial.

Participation mark problems

Exercise 1. Let G be a group, and let $\sim$ be the binary relation on G given by $x \sim y$ if there exists an isomorphism $\phi : G \to G$ such that $\phi(x) = y$. Is $\sim$ an equivalence relation? Either prove it is or explain why it is not.

Exercise 2. Let G and H be groups, let $\phi : G \to H$ be a group homomorphism, and let $K = \ker \phi$. Let $g_1, g_2 \in G$. Show that $g_1 \in g_2K$ if and only if $\phi(g_1) = \phi(g_2)$.

Core problems

Exercise 3. Let G and H be groups, and let $\phi : G \to H$ be an isomorphism. Prove that G is abelian if and only if H is abelian.

Exercise 4. For each of the following relations $\sim$ on a set S, determine whether or not $\sim$ is reflexive, symmetric, transitive. For each property, either prove $\sim$ has this property or give an example to show that $\sim$ does not have the property.

(i) $S = \mathbb{Z} \times \mathbb{Z}$, and $\sim$ is the relation given by $(a, b) \sim (c, d)$ if $ad = bc$.

(ii) S is the set of all subsets of $\{1, \ldots, n\}$. For subsets $A, B \in S$, we write $A \sim B$ if $A \subset B$.

(iii) (SS) S is the set of all subgroups of $\mathbb{Z}_{20}$, and $\sim$ is given by $H_1 \sim H_2$ if there exists an surjective homomorphism $H_1 \to H_2$.

(iv) $S = S_5$, and $x \sim y$ if $xy = yx$.

Exercise 5. Find all the left cosets of each of the following:

(i) the subgroup $\langle [5]_{15} \rangle = \{[0]_{15}, [5]_{15}, [10]_{15}\}$ of $\mathbb{Z}_{15}$;

(ii) (SS) the subgroup $\langle [8]_9 \rangle$ of $\mathbb{Z}_9^\times$;

(iii) the subgroup $\{e, (2\ 3), (2\ 4), (3\ 4), (2\ 3\ 4), (2\ 4\ 3)\}$ of S_4;

(iv) the subgroup $H = \{f \in D_4 \mid f \text{ is a rotation or } f = e\}$ of D_4.

Exercise 6 (SS)**.** Let G be a group, let H be a subgroup of G, and let $g \in G$. Show that the coset gH is a subgroup of G if and only if $g \in H$.

Exercise 7. Find the remainder of the following numbers after division by 13 without using a calculator. You can use the fact that $2023 = 7(17)^2$.

(i) $(2023)^{240}$

(ii) 5^{170}

(iii) $(14)^{2023}$

Additional practice

Exercise 8. Note that $\mathbb{R}^\times$ is a subgroup of $\mathbb{C}^\times$ (the non-zero complex numbers under multiplication). Describe the left coset of $\mathbb{R}^\times$ in $\mathbb{C}^\times$ containing i.

Exercise 9. Consider the real vector space $\mathbb{R}^2$ as a group with respect to vector addition. Verify that the subset $H \subseteq \mathbb{R}^2$ consisting of those vectors (x, y) satisfying $y = 3x$ is a subgroup, and describe the cosets of H. Draw a picture and observe that any two cosets are either equal or disjoint.

Exercise 10. Consider the nonzero complex numbers $\mathbb{C}^\times$ as a group with respect to multiplication. Verify that the subset $H \subseteq \mathbb{C}^\times$ consisting of those complex numbers z satisfying $|z| = 1$ is a subgroup, and describe the cosets of H. Draw a picture and observe that any two cosets are either equal or disjoint.

Exercise 11. Let $H = \{f \in S_n \mid f(1) = 1\}$ be the subgroup of S_n defined in Sheet 8 Exercise 8. Show that $fH = gH$ if and only if $f(1) = g(1)$.

Exercise 12. Let G be a group. Define a binary relation on G by $x \sim y$ if there exists $g \in G$ such that $y = gxg^{-1}$.

(i) Show $\sim$ is an equivalence relation on G. The equivalence classes for this relation are called *conjugacy classes*.

(ii) Describe the conjugacy classes of S_3.

(iii) Let Z_G be the center of G, as defined in Sheet 8. Show that $g \in Z_G$ if and only if its conjugacy class is $\{g\}$.

(iv) Let $\phi : G \to H$ be a group homomorphism, and assume H is an abelian group. Show that if x and y are in the same conjugacy class, then $\phi(x) = \phi(y)$.

(v) Find all homomorphisms $S_3 \to \mathbb{C}^\times$.