Introduction to Abstract Algebra: Sheet 3

For discussion in week 3 skills sessions and week 4 tutorials

You should write up your solutions for the participation mark problems and submit them using gradescope on the KEATS page for the course. The problems marked with (SS) will be discussed in the Skills Session for Week 3. Please attempt these *before* your Skills Session. Some of the other problems will be discussed in your tutorial for Week 4. Please attempt these problems *before* your tutorial.

Participation mark problems

Exercise 1. Let A, B, C be subsets of a set X. Prove that $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

Exercise 2. Use the Euclidean algorithm to find $\gcd(50, 2022)$ and to express it in the form $50x + 2022y$ with $x, y \in \mathbb{Z}$.

Exercise 3. Let $a, p \in \mathbb{Z}$, and assume p is prime. Suppose p is not a divisor of a. Show $\gcd(a, p) = 1$ (Note: We use this in the proof of the fundamental theorem of arithmetic, so please don't use the theorem or Corollary 3.26).

Core problems

Exercise 4 (SS)**.** Use the Euclidean algorithm to find $\gcd(98, 40)$ and to express it in the form $98x + 40y$ with $x, y \in \mathbb{Z}$.

Exercise 5. Let $a, p \in \mathbb{Z}$ be primes, and assume $p|a$. Show $p = a$. (Note: We use this in the proof of the fundamental theorem of arithmetic, so please don't use the theorem or Corollary 3.26).

Exercise 6 (SS)**.** Let $a_1, a_2, \dots, a_k \in \mathbb{Z}$, and let $p \in \mathbb{Z}$ be prime. If $p|a_1 a_2 \dots a_k$, then $p|a_i$ for some $i \in \{1, \dots, k\}$. (Hint: Try induction on k.)

Exercise 7. Determine whether the binary operation $*$ on the set X is associative and/or commutative in each of the following cases. If it is associative/commutative, prove it is. If it is not, give an example to show why it is not.

(i) (SS) The binary operation $*$ on $X = \mathbb{R}$ defined by $x * y = xy + 1$;

(ii) The binary operation $*$ on $\mathbb{R}$ defined by $x * y = x + y - 1$;

(iii) The binary operation $*$ on $\mathbb{Z}$ given by $x * y = x$;

(iv) Addition $* = +$ of functions on the X of all functions $\mathbb{Z} \to \mathbb{Z}$. Given functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$, then $f + g$ is the function from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $(f + g)(n) = f(n) + g(n)$ for $n \in \mathbb{Z}$;

(v) Vector addition $* = +$ on $X = \mathbb{R}^3$. Recall that this is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \\ z + z' \end{pmatrix}.$$

(vi) The cross product $* = \times$ on $X = \mathbb{R}^3$. Recall $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} yz' - zy' \\ zx' - xz' \\ xy' - yx' \end{pmatrix}$.

Additional practice

Exercise 8. Let $p \in \mathbb{Z}$ with $p > 1$. Suppose p has the property that for any $a, b \in \mathbb{Z}$, if $p|ab$, then $p|a$ or $p|b$. Show p is prime.

Exercise 9. Suppose that $*$ is a binary operation on a set S. We say $e \in S$ is a *left identity* for $*$ if $e * a = a$ for all $a \in S$. Similarly, we say $e' \in S$ is a *right identity* for $*$ if $a * e' = a$ for all $a \in S$. An *identity element* for $*$ is an element that is both a left identity and a right identity.

(i) Prove that if e is a left identity for $*$ and e' is a right identity for $*$, then $e = e'$ (and so e is in fact an identity element).

(ii) Give an example of a set S with a binary operation $*$ for which there is no left identity or right identity.

(iii) Give an example of an S and $*$ with more than one right identity.

Exercise 10. Suppose that $*$ is a binary operation on a set S with an identity element e. If $a, b \in S$ are such that $a * b = e$, then we say a is a *left inverse* of b and that b is a right inverse of a.

Let S denote the set of functions from $\mathbb{Z}$ to $\mathbb{Z}$ with the operation of $\circ$ (composition).

(i) Show that the function f defined by $f(x) = x$ is an identity element for $\circ$.

(ii) Find a function in S that has a left inverse, but no right inverse.

(iii) Is the left inverse in your example unique?