Exercises to Section 5

Exercises in red are from the list of the typical exercises for the exam.
Exercises marked with a star * are for submission to your tutor.

Monotonicity and extrema

1. For the following functions, determine the intervals of monotonicity and local extrema in the natural domain:

   (a) $f(x) = 2 + x - x^2$

   (b) $f(x) = \dfrac{2x}{1+x^2}$

   (c) $f(x) = x^2 2^{-x}$

   (d) $f(x) = x^m(1-x)^n$ (n and m are positive even integers)

Convexity

2. Prove that:

   (a) the functions x^α (with $\alpha > 1$), e^x, $x \log x$ are convex on $(0, \infty)$

   (b) the functions x^α (with $0 < \alpha < 1$), $\log x$ are concave on $(0, \infty)$

3. Using the previous exercise, prove the inequalities

   (a) $\frac{1}{2}(x^\alpha + y^\alpha) \geqslant \left(\frac{x+y}{2}\right)^\alpha$, where $\alpha > 1$, $x > 0$, $y > 0$

   (b) $x \log x + y \log y > (x+y) \log \frac{x+y}{2}$, where $x > 0$ and $y > 0$

4. Let f be a convex function on an interval Δ, let $x_1, \dots, x_n \in \Delta$ and let $\theta_1, \dots, \theta_n$ be positive numbers such that $\theta_1 + \dots + \theta_n = 1$.

   (a) Prove that $\theta_1 x_1 + \dots + \theta_n x_n \in \Delta$.

   (b) Prove that
   $$f(\theta_1 x_1 + \dots + \theta_n x_n) \leqslant \theta_1 f(x_1) + \dots + \theta_n f(x_n).$$
   Hint: use induction in n

Global extrema

5. Find the maximal and minimal values of the function on the given interval:

   (a)* $f(x) = x^2 - 4x + 6$ on $[-3, 10]$

   (b)* $f(x) = |x^2 - 3x + 2|$ on $[-10, 10]$

   (c) $f(x) = x + \frac{1}{x}$ on $[0.01, 100]$

Graph sketching

6. Following the plan given in the notes, sketch the graphs of the following functions on their natural domain:

   (a) $f(x) = (x+1)(x-2)^2$

(b)* $f(x) = \dfrac{x-2}{\sqrt{x^2+1}}$

(c) $f(x) = x^p e^{-x}$, $x \geqslant 0$, $0 < p < 1$

(d) $f(x) = \dfrac{e^x}{1+x}$

Challenging exercises

7. Using the Mean Value Theorem, prove that if f is differentiable and unbounded on a *bounded* interval (a, b), then the derivative f' must also be unbounded on (a, b).

8. Give an example of a differentiable function $f : (0, \infty) \to \mathbb{R}$ such that $\lim_{x\to\infty} f(x)$ exists, but $\lim_{x\to\infty} f'(x)$ does not exist.

9. Let $x_1, \ldots, x_n$ be positive numbers. Using convexity, prove the "AM-GM inequality", i.e. the inequality between the arithmetic mean and the geometric mean:

$$\frac{1}{n}\sum_{k=1}^{n} x_k \geqslant \left(\prod_{k=1}^{n} x_k\right)^{1/n}.$$

10. Let f be a bounded convex function on (a, b). Prove that f is continuous on (a, b).