CALCULUS 1 TUTORIAL EXERCISES III

In this tutorial, you will practise working with the inverse trigonometric functions, the hyperbolic functions, trigonometric functions with complex arguments, double-angle formulae, and composition of functions.

Questions 19 ans 23 should be submitted to GradeScope via the link under week 3 of the KEATs page for marking and feedback. The deadline is **4pm on Friday 20th October 2023**.

Tutor's Example[1]

Prove that

$$a\cos\theta + b\sin\theta = c\sin(\theta+\alpha)$$

for some $c, \alpha \in \mathbb{R}$ and $c \geq 0$.

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15. Express the following combinations of functions in the form $c\sin(\theta+\alpha)$, where c and α are real constants, which you have to find (assume $|\beta| < \pi/2$):

    (a) $\sin(\theta)+\cos(\theta)$, (b) $\sqrt{3}\sin(\theta)-\cos(\theta)$, (c) $\sin(\theta)-\tan(\beta)\cos(\theta)$.

16. Use the explicit formula for the function $\operatorname{arcsinh}: \mathbb{R} \to \mathbb{R}$, namely $\operatorname{arcsinh}(y) = \ln(y+\sqrt{y^2+1})$, to show that

    $$\operatorname{arcsinh}(\sinh(x)) = x \quad \text{for all } x \in \mathbb{R}, \quad \text{and} \quad \sinh(\operatorname{arcsinh}(y)) = y \quad \text{for all } y \in \mathbb{R}.$$

17. Use the exponential forms for the sinh and cosh functions to derive an analytic expression for arctanh.

18. Show that $\cos(\pi/12) = \frac{1}{2}\sqrt{2+\sqrt{3}}$. [Hint: use the formula for $\cos(2\theta)$ in terms of $\cos(\theta)$]

19. **To be submitted:** Find formulae that express the hyperbolic functions $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$ as ratios of polynomials of $t = \tanh(x/2)$.

20. Find all solutions $\theta \in \mathbb{R}$ (if any) of the equation $\cot(\theta)+\tan(\theta) = \alpha$, for the following three cases:

    (a) $\alpha = 1$, (b) $\alpha = 2$, (c) $\alpha = 4$.

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[1] To be shown by the tutor at the start of the tutorial.

21. Let $\alpha, \beta \in \mathbb{R}$. Rewrite each of the following expressions as a constant times a product of trigonometric functions:

(a) $\sin(\alpha - \beta) + \sin(\alpha + \beta)$,

(b) $\cos(\alpha - \beta) - \cos(\alpha + \beta)$.

22. Find the values of

(a) $\arcsin(\sin(5\pi/6))$,

(b) $\arcsin(\sin(7\pi/6))$,

(c) $\arccos(\cos(7\pi/6))$,

(d) $\arccos(\cos(11\pi/6))$.

23. **To be submitted:** Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) = x^3$, and $g : \mathbb{R} \to \mathbb{R}$ be such that $g(x) = x + 2$.

(i) What is the formula for $f \circ g$?

(ii) What is the formula for $g \circ f$?

(iii) For which values of $x \in \mathbb{R}$ are $f \circ g$ and $g \circ f$ equal? [Hint: you will need to solve a quadratic equation.]

(iv) Are $f \circ g$ and $g \circ f$ equal for all $x \in \mathbb{R}$?

(v) Prove that function composition is <u>not commutative</u> in general, i.e. $f \circ g \neq g \circ f$. [Hint: find a counter-example f and g such that there is some x such that $(f \circ g)(x) \neq (g \circ f)(x)$.]

24. **Exam-style question**[2] The Mercator map (or projection) is perhaps the most well-known way to represent the curved surface of the Earth on a flat piece of paper. Any point on its surface is determined by its *latitude* ϕ, the vertical angle from the Earth's centre to the point measured from the equator (points in the southern hemisphere have negative latitude); and its *longitude* λ, the horizontal angle from the Earth's centre to the point measured from the Greenwich meridian (a line running from the north to south pole through Greenwich park, here in London). The Mercator map is able to represent every point except the two poles.

If the Mercator map is represented in the (x, y)-plane, then the point (ϕ, λ) on the Earth's surface (for $\phi \in (-\pi/2, \pi/2)$ and $\lambda \in (-\pi, \pi]$) has x-coordinate λ, and y-coordinate

$$y(\phi) = \ln\left(\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\right).$$

(i) **[2 marks]** What is the image of $y(\phi)$? Describe how the parts of the Earth near the north and south poles would appear on the map, and whether it would be useful for navigation in these regions?

[2] Some questions on the final exam will be longer with several parts which delve into a topic and build on each other. To help you prepare, each sheet will include such a question.

(ii) **[7 marks]** By using the double angle formula for tan, show that

$$\tan\left(\frac{\phi}{2}+\frac{\pi}{4}\right)=\frac{1+\sin\phi}{\cos\phi},$$

and hence show that

$$y(\phi)=\frac{1}{2}\ln\left(\frac{1+\sin\phi}{1-\sin\phi}\right). \tag{1}$$

(iii) **[4 marks]** By computing $\sinh(y)$ using (1), write down an expression for the inverse of $y(\phi)$ in terms of the *Gudermannian function* $\mathrm{gd}(x) := \arctan(\sinh(x))$.

25. **Advanced Python**[3] *Without* using Python's in-built `sin` and `cos` functions (e.g. these functions are provided by the `math` package), define a function `my_sin` which uses the Taylor expansion of sin up to x^{100} to compute the value of $\sin\theta$. Also write a function `my_cos` to compute $\cos(\theta)$.

   [Hint: you may find it useful to use Python's `%` function, which gives the remainder when one number is divided by another, i.e. `m % n` will return the remainder when `m` is divided by `n`.]

26. **A challenging problem**[4] Assuming that

$$2x(y^2-1)+2y(x^2-1)=(1+x^2)(1+y^2)$$

and

$$4z(1-y^2)+4y(1-z^2)=(1+z^2)(1+y^2)$$

where $x, y, z \in \mathbb{R}$, find the value of the following expression:

$$\left(\frac{2x}{1+x^2}-\frac{2z}{1+z^2}\right)^2+\left(\frac{1-z^2}{1+z^2}-\frac{1-x^2}{1+x^2}\right)^2.$$

[Hint: Use the substitutions $x=\tan\alpha$, $y=\tan\beta$ and $z=\tan\gamma$ to rewrite the expressions.]

[3] Each sheet will contain a Python problem which should complement the programming aspect of the course. In some cases, like this one, you may have to look up unfamiliar functions and use your knowledge of Python to apply them.

[4] All challenging problems in the tutorial problem sets are beyond the scope of the course, and are not examinable, but they are, hopefully interesting and inspiring.