Linear Algebra and Geometry I 2023 **Problem Sheet 6**

You are encouraged to work with other students on the module. If you are having difficulty with any of the questions or want feedback on specific answers you should ask your tutor in your next tutorial or attend the lecturer's office hours.

1. Determine which of the following transformations are linear. In each case, you should justify your answer. (Recall that $\mathbb{P}$ is the space of all polynomials in one variable).

   (a) $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_3 - x_4, x_1x_2)$.

   (b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x, 2y, 2x - 3y)$.

   (c) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (|x|, -y)$.

   (d) $T : \mathbb{P} \to \mathbb{P}$ defined by $Tf(t) = 3f''(t) + f(t)$.

   (e) $T : \mathbb{P} \to \mathbb{P}$ defined by $Tf(t) = tf'(t) + t^2 f''(t)$.

   (f) $T : \mathbb{P} \to \mathbb{P}$ defined by $Tf(t) = 2f'(t) + t^3$.

2. (a) For each of the following linear maps, determine its matrix (with respect to the standard basis).

   (i) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x + 3y - 4z, 2y - z, x + y + z)$.

   (ii) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x, x + y, 2x - y)$.

   (iii) $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3, x_4) = (x_2 + x_1, x_3 + x_2, x_4 + x_3)$.

   (b) Let $\mathbf{f}_1 = (1, 0, 0)$, $\mathbf{f}_2 = (1, 1, 0)$ and $\mathbf{f}_3 = (1, 1, 1)$. Then $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is a basis for $\mathbb{R}^3$. For the linear map in Part (i) above, determine its matrix with respect to the basis $\mathcal{F}$ (both the domain and range are considered with this basis).

   (c) Recall that the monomial basis in $\mathbb{P}_n$ is $1, t, t^2, \ldots, t^n$. For each of the following linear maps, determine its matrix with respect to the monomial basis.

   (i) $T : \mathbb{P}_4 \to \mathbb{P}_4$ defined by $Tf(t) = f''(t)$.

   (ii) $T : \mathbb{P}_3 \to \mathbb{P}_3$ defined by $Tf(t) = tf'(t)$.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with $T(2, 3) = (4, 0)$ and $T(1, -1) = (-3, 1)$.

   Determine the matrix of T (with respect to the standard basis).

4. Recall that $M_n(\mathbb{R})$ is the space of $n \times n$ matrices with real entries.

(a) Show that the map $\mathcal{T} : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $\mathcal{T}(A) = A^T$ is linear.

(b) Determine the matrix of $\mathcal{T}$ with respect to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

5. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & -2 \end{pmatrix} \; C = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 1 & -1 \end{pmatrix} \; D = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}.$$

(a) State which of the following products are defined and in each case where it is, give the dimensions of the result.

(i) AB	(iv) ABD	(vii) B^TC
(ii) BA	(v) BC	(viii) DC
(iii) ABC	(vi) BC^T	(ix) D^TC^T

(b) Compute the following:

(i) AC (ii) $A(3B + C)$ (iii) B^TA (iv) $(CD)^TA$

6. Let $S : U \to V$ and $T : V \to W$ be linear maps. Prove that $T \circ S : U \to W$ is also linear. (Recall that $T \circ S(\mathbf{u}) = T(S(\mathbf{u}))$.)

7. (a) Show that each of the following maps is linear.

(i) $D : \mathbb{P}_4 \to \mathbb{P}_3$ defined by $Df(t) = f'(t)$.

(ii) $J : \mathbb{P}_3 \to \mathbb{P}_4$ defined by

$$Jf(t) = \int_0^t f(s)\,ds.$$

(b) Explain why the map $F : \mathbb{P} \to \mathbb{P}$ defined by $Ff(t) = (f'(t))^2$ is **not** linear.

(c) For each of the transformations in Part (a), determine its matrix (with respect to the monomial basis in $\mathbb{P}_n$).

(d) Determine the matrices of DJ and JD. What do these matrices tell you about what the linear maps DJ and JD do to functions? Why was this expected?