Problem 0. Prove Proposition 1.1

Problem 1. This problem asks you to think about the relation between homogeneous polynomials and general polynomials.

(1) Given a polynomial of degree d, $f(x_1, \ldots, x_n)$ we define the saturation (or homogenisation) of f to be

$$F(X_0, \ldots, X_n) := X_0^d f(X_1/X_0, \ldots, X_n/X_0).$$

Show that F is a homogeneous polynomial of degree $= d$.

(2) Show that X_0 does not divide F.

(3) Conversely, given a homogeneous polynomial of degree d, $F(X_0, \ldots, X_n)$ we define the affinisation (in the chart $\{X_0 \neq 0\}$) of F to be

$$X_0^{-d} F(X_0, \ldots, X_n).$$

Show that

$$X_0^{-d} F(X_0, \ldots, X_n) = f(X_1/X_0, \ldots, X_n/X_0)$$

where f is a polynomial of degree $\leq d$ in n variables.

(4) Show that the homogenisation/affinisation constructions are inverses to each other.

In standard terminology $[X_0 : \cdots : X_n]$ are called the homogeneous coordinates and the coordinates $x_1 = X_1/X_0, \ldots, x_n = X_n/X_0$ are called the affine coordinates, because they give coordinate functions on $\{X_0 \neq 0\} \cong \mathbb{K}^n$.

Problem 2. Show that we can write $\mathbb{P}_k^n = k^n \sqcup k^{n-1} \sqcup \cdots \sqcup k \sqcup \{0\}$ in a natural way.

Problem 3. Show that if two non-degenerate conics intersect at 5 points, then the two conics must be equal.

Problem 4. let $k = \mathbb{F}_2$ (the field with two elements). Show that $\mathbb{P}_k^2$ has 7 points and 7 lines. Moreover, any line has exactly 3 points on it and through any point there are 3 distinct lines. $\mathbb{P}_k^2$ is often called the Fano plane. Try to draw a picture which represents all the points and lines.

Problem 5. Show that the set of all plane conics can be naturally identified with $\mathbb{P}^5$. (Hint: The set of all plane conics is almost the same as the set of homogeneous degree two polynomials in variables X, Y, Z, $\{aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ\}$. Now, think of (a, b, c, d, e, f) as a point in some other space.)

Problem 6. Show that the set of plane conics passing through a point is naturally identified with $\mathbb{P}^4$. Deduce that if there are 5 points in $\mathbb{P}^2$, such that no three are on a line, then there is a unique non-degenerate conic passing through these 5 points.