Linear Algebra and Geometry II - Sheet 5

Please attempt all of the problems on this sheet. You are welcome to make use of the Lecturers' office hours and the tutorial for asking questions about any of the problems. Solutions will be made available in Week 7.

Participation Mark: Please submit your solution to Problems 2, 3 and 4, marked with a blue triangle (△), by using the submission link on Keats. They are due on Monday 26th February 2024 at 4pm GMT.

All matrices are square.

1. Compute the minimal polynomial of
$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
Is the matrix A diagonalizable?

2. △ Find the minimal polynomial for the following matrices:
   (a) $A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}$
   (b) $B = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{pmatrix}$

3. △ Find examples of a matrix A such that its characteristic polynomial is $p_A(x) = (2-x)^2(-1-x)^2$ and its minimal polynomial is given by:
   (a) $m_A(x) = (x-2)(x+1)$;
   (b) $m_A(x) = (x-2)(x+1)^2$.

4. △ Find upper-triangular matrices $T_1, T_2 \in M_3(\mathbb{R})$ with diagonal entries $0, 2, 0$ such that
   (a) T_1 has rank 1;
   (b) T_2 has rank 2.

   Check that T_1 has minimal polynomial $m_{T_1}(x) = x(x-2)$ while T_2 has minimal polynomial $m_{T_2}(x) = x^2(x-2)$.

   [Optional bonus problem:] Explain why the minimal polynomials do not depend on the choice of T_1 and T_2.

5. Compute the characteristic and the minimal polynomials of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then determine whether the matrix A is diagonalizable.

6. Let $A = \begin{pmatrix} 1 & -9 & 4 \\ 1 & -4 & 1 \\ 1 & -7 & 3 \end{pmatrix}$. Compute A^3 using the Cayley-Hamilton theorem.

7. [Optional bonus problem] Let $V = M_n(\mathbb{C})$ be the complex vector space of $n \times n$-matrices viewed as a vector space over $\mathbb{C}$. Let $A \in V$ and let $S = \{I, A, A^2, \ldots, A^n\}$. In this problem we work out the dimension of the subspace of V spanned by S.

   (a) Let $\mathrm{Span}(S)$ denote the subspace of V spanned by the vectors in S. Assuming $n > 1$, show that $\mathrm{Span}(S) \neq V$. [Hint: Consider the dimensions of both V and of $\mathrm{Span}(S)$.]

   (b) Prove that S is linearly *dependent* (i.e. there exists a non-trivial linear relation between the elements of S) and thus $\dim(\mathrm{Span}(S)) \leq n$.

   [Hint: Write down for yourself the statement of the Cayley-Hamilton Theorem applied to the matrix A.]

   (c) Let d be the degree of the minimal polynomial m_A of A and set

   $$S' := \{I, A, A^2, \ldots, A^{d-1}\}.$$

   Prove that S' is a linearly *independent* subset of S, and deduce that therefore $\dim(\mathrm{Span}(S)) \geq d$.

   (d) Show that for *any* polynomial p, the evaluation $p(A)$ is a linear combination of elements of S'. Deduce that $\mathrm{Span}(S) = \mathrm{Span}(S')$ and thus show that $\mathrm{Span}(S)$ has dimension d.

   [Hint: Use the division algorithm for polynomials to divide $p(x)$ by $m_A(x)$: i.e. $p(x) = m_A(x)q(x) + r(x)$, where degree $r(x)$ is less that degree $m(x)$.]