TOPOLOGY

VOLUME II

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VOLUME II

New edition, revised and augmented

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To the memory of my wife

PREFACE TO THE SECOND VOLUME

This volume combined with the first volume forms a whole. According to the plan mentioned in the Preface to the first volume, Chapter IV is on compact spaces, Chapter V on connected spaces, Chapter VI on locally connected spaces, and Chapter VII on retracts, neighbourhood retracts and related topics (homotopy in particular). The last two chapters are more specialized; Chapter IX is on some cutting problems of the sphere $\mathcal{S}_n$ connected with the concept of cohomotopy, and Chapter X on the topology of the plane. Chapter VIII, concerned with group theory, is of rather auxiliary character; it contains however some results which are of importance from the topological point of view, such as the study of spaces contractible relative to the circle and the group of integer-valued measures defined on closed-open subsets of a given space.

There are paragraphs which can be omitted without affecting the understanding of what follows. Such is, for instance, § 46 on dimension theory (which is a continuation of §§ 25–29 of Volume I). However, the author thought that the specific beauty of this theory and of the methods employed in it was a convincing motivation for inserting the theory of dimension in this monograph. Such is also § 51 on the theory of curves.

Also § 48 on the theory of irreducible spaces and of indecomposable spaces is isolated, to some extent, in this volume. However, in recent years, it has acquired considerable interest and importance owing to the results of Bing, Moise and others (a few of these results were mentioned in the Appendix to the second volume of the French edition).

The English edition as compared with the French edition differs essentially in the fact that metric separable spaces are no longer the main object of this treatise. They have been replaced (where it was possible and desirable) by more general topological spaces. Such is the case, in particular, of §§ 41–44 (on compact spaces), which were written anew, §§ 46, 47 (on connected spaces) §§ 49, 50 (on locally connected spaces). Also quite a few new statements have been added in various parts of the book, as well as additional references; the content of the Appendix of the French edition has been inserted in the particular paragraphs of this work.

In addition to the Bibliography given in the Preface to the first volume, I wish to mention the following books, to which I referred in preparing this book.

C. Berge, *Théorie des graphes et ses applications*, Dunod 1958.

K. Borsuk, *Theory of retracts*, Monogr. Mat. 44, Warszawa 1967.

D. C. J. Burgess, *Analytical topology*, Van Nostrand 1966.

R. Busacker and T. Saaty, *Finite graphs and networks*, Mc Graw-Hill 1965.

E. Čech, *Topological spaces*, Czechoslovak Acad. Sci. 1966.

J. Dugundji, *Topology*, Allyn and Bacon 1966.

R. Engelking, *General topology*, North-Holland and PWN 1968.

S. A. Gaal, *Point set topology*, Academic Press 1964.

F. Harary, R. Norman, D. Cartwright, *Structural models*, J. Wiley 1965.

She-Tsen Hu, *Theory of retracts*, Wayne Univ. Press 1965.

— *Elements of general topology*, Holden-Day 1964.

— *Homotopy theory*, Academic Press 1959.

J. R. Isbell, *Uniform spaces*, Amer. Math. Soc. 1964.

J. Nagata, *Modern dimension theory*, North-Holland 1965.

W. Pervin, *Foundations of general topology*, Academic Press 1964.

H. Schubert, *Topologie*, Teubner 1964.

W. J. Thron, *Topological structures*, Holt, Rinehart and Winston 1966.

G. T. Whyburn, *Topological analysis*, Princeton Univ. Press 1964.

My thanks go to numerous colleagues who helped me in preparing this monograph. Among those who I mentioned in the Preface to the first volume I wish to emphasize the support given to me by Professor Engelking and Dr Karłowicz. I also owe thanks to Professors Hilton, Bednarek and Lelek and Dr Kirkor, who suggested many improvements.

Finally, my sincere thanks go to the Academic Press and to Polish Scientific Publishers and personally to Mr Muszyński for their careful work.

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§ 41. Compactness

I. Definitions. Conditions of Borel, Lebesgue, Riesz, Cantor and Bolzano–Weierstrass. Alexander Lemma.

In § 5, VII of Volume I the following definition was given.

DEFINITION 1(¹). A topological space $\mathscr{X}$ is called *compact* if it satisfies the following condition (called the *Borel–Lebesgue condition*)(²):

$$\textit{Every open cover contains a finite subcover.} \tag{1}$$

In other terms, *if* $\mathscr{X} = \bigcup_t G_t$, *where* G_t *is open for each* $t \in T$ (T *arbitrary*), *then there is a finite system* $t_1, \ldots, t_n$ *such that* $\mathscr{X} = G_{t_1} \cup \cup \ldots \cup G_{t_n}$.

DEFINITION 2. A topological space $\mathscr{X}$ is called *countably compact* if it satisfies the condition (called the *Borel condition*)(³):

$$\textit{Every countable open cover contains a finite subcover.} \tag{1'}$$

Obviously a compact space is countably compact, while the converse is not true (as seen on the example of the space of ordinals $a < \Omega$, see § 20, V (ii)). If, however, the space is metric, then compactness and countable compactness are equivalent (see § 21, IX, Theorem 2).

(¹) This definition is essentially due to P. Alexandrov and P. Urysohn. These authors introduced the term *bicompactness* for what is at present called compactness (the term compact space being used at that time in the sense of countably or sequentially compact).

See P. Alexandrov and P. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. K. Akad. Amsterdam 14 (1929), pp. 1–96. See also P. Alexandrov and H. Hopf, *Topologie* I, chap. II, § 1, and L. Vietoris, *Stetige Mengen*, Monatsh. f. Math. u. Phys. 31 (1921), pp. 49–62.

(²) See H. Lebesgue, *Leçons sur l'intégration*, Paris 1905, p. 105. Compare W. H. Young, Proc. London Math. Soc. (1) 35 (1902/3), p. 384.

(³) This condition is also called the *Heine–Borel condition*. See E. Borel, Ann. Ecole Norm. Sup. (3) 12 (1895) (*Thèse*), p. 51. For numerous bibliographical references, see T. H. Hildebrandt, *The Borel theorem and its applications*, Bull. Amer. Math. Soc. 32 (1926), p. 423.

Classical examples of compact spaces are: the interval $\mathscr{I}$, the cube $\mathscr{I}^n$ and, more generally, any closed and bounded subset of the euclidean space $\mathscr{E}^n$ (see Section VI). On the other hand, the space of integers and the space $\mathscr{E}$ of real numbers are obviously non-compact spaces.

One easily sees that the following condition (called the *F. Riesz condition*)(¹) is equivalent (and dual) to the Borel–Lebesgue condition:

If F_t is closed for each $t \in T$, and $\bigcap_t F_t = 0$, then there is a finite system $t_1, \ldots, t_n$ such that $F_{t_1} \cap \ldots \cap F_{t_n} = 0$. (2)

One easily proves that the following two conditions are equivalent to (1), hence to (2):

If the family $\{G_t\}$ is an open cover of $\mathscr{X}$ and is directed relative to the inclusion $\subset$ (i.e. for each pair t_1 and t_2 there is t_3 such that $G_{t_1} \subset G_{t_3}$ and $G_{t_2} \subset G_{t_3}$), then there is t such that $G_t = \mathscr{X}$. (3)

If $F_t = \overline{F}_t \neq 0$ and the family $\{F_t\}$ is directed relative to the inclusion $\supset$ (i.e. for each pair t_1 and t_2 there is t_3 such that $F_{t_3} \subset F_{t_1}$ and $F_{t_3} \subset F_{t_2}$), then $\bigcap_t F_t \neq 0$. (4)

The Borel condition (1′) is equivalent to the following *Cantor condition*(²):

If $\overline{F}_n = F_n \neq 0$ for $n = 1, 2, \ldots$ and $F_1 \supset F_2 \supset \ldots$, then $F_1 \cap F_2 \cap \ldots \neq 0$. (2′)

The implication (2′) → (1′) was given in § 20, V (Remark to Theorem 3). In order to show that (1′) → (2′), let us assume that the hypotheses of (2′) are fulfilled. Let $k_1, \ldots, k_m$ be any finite system of positive integers and let j be the greatest among them. Then $F_{k_1} \cap \ldots \cap F_{k_m} = F_j \neq 0$. Let $G_n = \mathscr{X} - F_n$. Then $G_{k_1} \cup \ldots \cup G_{k_m} \neq \mathscr{X}$ and it follows from (1′) that $G_1 \cup G_2 \cup \ldots \neq \mathscr{X}$, i.e. $F_1 \cap F_2 \cap \ldots \neq 0$.

Remark 1. One easily sees that condition (4) can be expressed using logical symbols in the following way.

(¹) See F. Riesz, Atti del IV Congresso Int. d. Matemat., Roma 1908, vol. 2, p. 21.

(²) See G. Cantor, Math. Ann. 17 (1880).

Given a family $\{\varphi_t(x)\}$ *of propositional functions such that the sets* $F_t = \underset{x}{E}\,\varphi_t(x)$ *are closed and for each pair* t_1, t_2, $\varphi_{t_3}(x) \to \varphi_{t_1}(x) \wedge \wedge \varphi_{t_2}(x)$ *for an appropriate* t_3, *then*

$$\bigwedge_t \bigvee_x \varphi_t(x) \equiv \bigvee_x \bigwedge_t \varphi_t(x). \tag{i}$$

Similarly the Cantor condition can be expressed as follows: *Given an infinite sequence of propositional functions* $\varphi_1(x), \varphi_2(x), \ldots$ *such that* $\varphi_n(x) \Rightarrow \varphi_{n-1}(x)$ *and that the sets* $\underset{x}{E}\,\varphi_n(x)$ *are closed, then*

$$\bigwedge_n \bigvee_x \varphi_n(x) \equiv \bigvee_x \bigwedge_n \varphi_n(x). \tag{ii}$$

Equivalently: *if* $\varphi_n(x) \Rightarrow \varphi_{n+1}(x)$ *and* $\underset{x}{E}\,\varphi_n(x)$ *is open, then*

$$\bigwedge_x \bigvee_n \varphi_n(x) \equiv \bigvee_n \bigwedge_x \varphi_n(x). \tag{iii}$$

Remark 2. Another statement equivalent to the Cantor condition is the following:

$$\textit{if } A_n \neq 0 \textit{ for } n = 1, 2, \ldots, \textit{ then } \mathrm{Ls}\, A_n \neq 0. \tag{iv}$$

For $\mathrm{Ls}\, A_n = \bigcap_{n=1}^{\infty} \overline{A_n \cup A_{n+1} \cup \ldots}$ (see § 29, IV, 8).

Remark 3. In metric spaces (and more generally, in $\mathscr{L}^*$-spaces) the Cantor condition is equivalent to the following *Bolzano–Weierstrass condition* (called also *sequential compactness*, see § 20, V, p. 195, Remark):

Every infinite sequence of points $p_1, p_2, \ldots$ *contains a convergent subsequence*: $\lim\limits_{n=\infty} p_{k_n} = p$, *where* $k_1 < k_2 < \ldots$; (5)

that is

The set A^d, *of accumulation points of* A, *is non-empty, provided that* A *is infinite.* (5′)

Remark 4. Let us add (without proof) that the following conditions are equivalent to compactness(1):

(1) See the paper by W. Sierpiński and myself, *Le théorème de Borel-Lebesgue dans la théorie des ensembles abstraits*, Fund. Math. 2 (1921), p. 173; L. Pontrjagin, *Continuous groups* (Russian), Moscow 1954, p. 80 (theorem 4); P. Alexandrov and P. Urysohn, *op. cit.*, Chap. I, § 2.7.

If $\{F_\alpha\}$ is a transfinite sequence of decreasing closed non-empty sets, then $\bigcap_\alpha F_\alpha \neq 0$. (6)

For every infinite subset E of $\mathscr{X}$ there is a point p of order $\overline{\overline{E}}$ (i.e. such that for each neighbourhood G of p we have $\overline{\overline{G \cap E}} = \overline{\overline{E}}$). (7)

We shall show now that in Definition 1 the open sets can be restricted to members of a given open subbase.

It will be useful to call a cover *essentially infinite* if it does not contain any finite cover of the space (thus compact spaces are spaces which have no essentially infinite open covers).

ALEXANDER LEMMA.(1) *Let $\boldsymbol{A}$ be an open subbase of the topological space $\mathscr{X}$. Suppose that there exists an essentially infinite open cover of $\mathscr{X}$. Then there exists such a cover contained in $\boldsymbol{A}$.*

Proof. Denote by $\mathfrak{M}$ the totality of all essentially infinite open covers of $\mathscr{X}$. By assumption $\mathfrak{M} \neq 0$. We shall show first that $\mathfrak{M}$ has the following property: whenever $\{\boldsymbol{P}_\alpha\}$ is a transfinite monotone sequence (i.e. $\alpha < \beta \Rightarrow \boldsymbol{P}_\alpha \subset \boldsymbol{P}_\beta$) of members of $\mathfrak{M}$, then $(\bigcup_\alpha \boldsymbol{P}_\alpha) \in \mathfrak{M}$.

$\bigcup_\alpha \boldsymbol{P}_\alpha$ is obviously a cover of $\mathscr{X}$. It is essentially infinite. For, suppose not. Then it contains a finite cover $G_1, \ldots, G_n$ and consequently there is a finite system $\alpha_1, \ldots, \alpha_n$ such that $G_i \in \boldsymbol{P}_{\alpha_i}$. Denote by β the greatest α_i $(1 \leqslant i \leqslant n)$. Hence $G_i \in \boldsymbol{P}_\beta$ for each $i = 1, \ldots, n$ and it follows that $\boldsymbol{P}_\beta$ is not essentially infinite.

From the property of $\mathfrak{M}$ just shown, it follows (see Vol. I, Introduction § 3, XI) that $\mathfrak{M}$ contains a maximal element. Denote it by $\boldsymbol{P}$. Thus, if H is open and does not belong to $\boldsymbol{P}$, then $\boldsymbol{P} \cup (H)$ is not essentially infinite, which means that there is a finite system $G_1, \ldots, G_n$ such that

$$H \cup G_1 \cup G_2 \cup \ldots \cup G_n = \mathscr{X} \quad \text{and} \quad G_i \in \boldsymbol{P} \quad \text{for} \quad i = 1, \ldots, n. \tag{8}$$

We shall show that the family of open sets which do not belong to $\boldsymbol{P}$ is a filter, i.e. that (H and G being open)

$$H_1 \notin \boldsymbol{P} \text{ and } H_2 \notin \boldsymbol{P} \quad \text{imply} \quad (H_1 \cap H_2) \notin \boldsymbol{P} \tag{9}$$

and

(1) See, e.g., J. L. Kelley, *General Topology*, p. 139.

$$H \notin \boldsymbol{P} \text{ and } H \subset G \quad \text{imply} \quad G \notin \boldsymbol{P}. \tag{10}$$

The condition $H_j \notin \boldsymbol{P}$, $j = 1, 2$, implies (see (8)) the existence of sets $G_{j,1}, \ldots, G_{j,n_j}$ such that

$$H_j \cup G_{j,1} \cup G_{j,2} \cup \ldots \cup G_{j,n_j} = \mathscr{X} \quad \text{and} \quad G_{j,i} \in \boldsymbol{P}. \tag{11}$$

It follows that

$$(H_1 \cap H_2) \cup \bigcup_{j,i} G_{j,i} = \mathscr{X}, \tag{12}$$

hence $(H_1 \cap H_2) \notin \boldsymbol{P}$, since $\boldsymbol{P}$ is an essentially infinite cover.

Thus (9) has been established.

Now let $H \notin \boldsymbol{P}$. We may suppose that (8) is fulfilled. Therefore, if $H \subset G$, we have

$$G \cup G_1 \cup G_2 \cup \ldots \cup G_n = \mathscr{X}$$

which yields $G \notin \boldsymbol{P}$.

We shall show that (9) and (10) imply that $\boldsymbol{A} \cap \boldsymbol{P}$ is a cover of $\mathscr{X}$.

Let $x_0 \in \mathscr{X}$. Since $\boldsymbol{P}$ is a cover of $\mathscr{X}$, there is a $G \in \boldsymbol{P}$ such that $x_0 \in G$, and since $\boldsymbol{A}$ is a subbase of $\mathscr{X}$, there is a finite system $H_1, \ldots, H_n$ of elements of $\boldsymbol{A}$ such that

$$x_0 \in (H_1 \cap \ldots \cap H_n) \subset G.$$

It follows by (9) and (10) that there is an i such that $H_i \in \boldsymbol{P}$. Hence $x_0 \in H_i \in \boldsymbol{A} \cap \boldsymbol{P}$. Thus $\boldsymbol{A} \cap \boldsymbol{P}$ is a cover of $\mathscr{X}$.

Finally, since $\boldsymbol{P}$ is essentially infinite, so is $\boldsymbol{A} \cap \boldsymbol{P}$.

II. Normality and related properties of compact spaces. First let us prove two elementary properties of compact spaces.

THEOREM 1. *Each compact subset of a $\mathscr{T}_2$-space (i.e. of a Hausdorff space) is closed.*

Proof. Let $A \subset \mathscr{X}$ be compact. We have to show that $\mathscr{X} - A$ is open, i.e. that given a point $b \in \mathscr{X} - A$, there is an open G such that $b \in G \subset \mathscr{X} - A$.

Since $\mathscr{X}$ is a $\mathscr{T}_2$-space, there is for each $x \in A$ a pair of open sets U_x and V_x such that

$$b \in U_x, \quad x \in V_x \quad \text{and} \quad U_x \cap V_x = 0.$$

Consequently, the family of sets $A \cap V_x$, where $x \in A$, is an open cover of A (considered as a space). Since A is compact, there is a finite system $x_1, \ldots, x_n$ such that

$$A = (A \cap V_{x_1}) \cup \ldots \cup (A \cap V_{x_n}), \quad \text{i.e.} \quad A \subset V_{x_1} \cup \ldots \cup V_{x_n}.$$

Put $G = U_{x_1} \cap \ldots \cap U_{x_n}$. Then G is open and $b \in G \subset \mathscr{X} - A$.

THEOREM 2. *Each closed subset of a compact space is compact.*

Proof. Let $F = \overline{F} \subset \mathscr{X}$. Let $\{G_t\}$, $t \in T$, be a cover of F, where G_t is open relative to F. Hence there is an H_t open (relative to $\mathscr{X}$) such that $G_t = F \cap H_t$. Consequently, the family of sets H_t, where $t \in T$, augmented by the set $H = \mathscr{X} - F$ is an open cover of $\mathscr{X}$. Since $\mathscr{X}$ is compact, there is a finite system of indices $t_1, \ldots, t_n$ such that $\mathscr{X} = H \cup H_{t_1} \cup \ldots \cup H_{t_n}$. Hence $F = G_{t_1} \cup \ldots \cup G_{t_n}$.

THEOREM 3. *Each compact $\mathscr{T}_2$-space is normal.*

This theorem follows immediately (applying Theorem 2) from the next lemma:

LEMMA. *Let A and B be two compact disjoint subsets of a space $\mathscr{X}$, and let $\boldsymbol{R}$ be a family of open sets such that for each pair $a \in A$ and $b \in B$, there is a $G \in \boldsymbol{R}$ such that $a \in G$ and $b \notin \overline{G}$. Then there exists in $\boldsymbol{R}$ a finite system $\{G_j^i\}$, where $i = 1, \ldots, k$ and $j = 1, \ldots, m_k$, such that*

$$A \subset (G_1^1 \cap \ldots \cap G_{m_1}^1) \cup \ldots \cup (G_1^k \cap \ldots \cap G_{m_k}^k) \tag{1}$$

and

$$B \cap [(\overline{G_1^1} \cap \ldots \cap \overline{G_{m_1}^1}) \cup \ldots \cup (\overline{G_1^k} \cap \ldots \cap \overline{G_{m_k}^k})] = 0. \tag{2}$$

Proof. Let $a \in A$ be a given point. For each b denote by $G(b)$ a member of $\boldsymbol{R}$ such that $a \in G(b)$ and $b \in \mathscr{X} - \overline{G(b)}$. Since the family of sets $\{\mathscr{X} - \overline{G(b)}\}$, where $b \in B$, is an open cover of the compact set B, there exists a finite set $(b_1, \ldots, b_r)$, where $r = r(a)$ depends on a, such that

$$B \subset [\mathscr{X} - \overline{G(b_1)}] \cup \ldots \cup [\mathscr{X} - \overline{G(b_r)}] = \mathscr{X} - [\overline{G(b_1)} \cap \ldots \cap \overline{G(b_r)}].$$

In other terms, if we put

$$H_1(a) = G(b_1), \ \ldots, \ H_r(a) = G(b_r) \quad \text{and} \quad H(a) = H_1(a) \cap \ldots \cap H_r(a),$$

we have $a \in H(a)$ and $B \cap \overline{H_1(a)} \cap \ldots \cap \overline{H_r(a)} = 0$.

Thus $\{H(a)\}$, for variable $a \in A$, is an open cover of the compact set A. Hence there is a finite cover $H(a_1), \ldots, H(a_k)$ of A. Put $G_j^i = H_j(a_i)$ and $m_i = r(a_i)$. Formulas (1) and (2) follow.

Remark 1. A compact space needs be neither *hereditarily normal* nor *perfectly normal.*

This is a consequence of the following statements:

(i) there exist completely regular $\mathcal{T}_1$-spaces which are not normal (see § 14, II, p. 122),

(ii) every completely regular $\mathcal{T}_1$-space is homeomorphic to a subset of a compact space (see § 16, V, Theorem 5),

(iii) every perfectly normal space is hereditarily normal (see § 14, VI, p. 133).

COROLLARY 1. *Let $A_1, \ldots, A_m$ be a finite system of compact subsets of a $\mathcal{T}_2$-space $\mathcal{X}$. Let $\boldsymbol{R}$ be—according to our previous assumption—a family of open sets having the following separation property:*

$$\textit{if} \qquad A_r \cap A_s = 0 \quad \textit{and} \quad x \in A_r \quad \textit{and} \quad y \in A_s, \tag{3}$$

then there exists $G \in \boldsymbol{R}$ such that

$$x \in G \quad \textit{and} \quad y \notin \overline{G}. \tag{4}$$

Then $\boldsymbol{R}$ contains a finite family $\boldsymbol{R}^*$ having the above separation property.

Proof. If $m = 2$, the corollary follows directly from the lemma assuming that $\boldsymbol{R}^*$ is the family of all sets G_j^i.

Therefore, if $A_r \cap A_s = 0$, there is a finite family $\boldsymbol{R}_{r,s} \subset \boldsymbol{R}$ such that for each $x \in A_r$ and $y \in A_s$ there is a $G \in \boldsymbol{R}_{r,s}$ satisfying (4).

Let $\boldsymbol{R}^*$ be the union of all $R_{r,s}$ such that $A_r \cap A_s = 0$. $\boldsymbol{R}^*$ is the required family.

COROLLARY 2. *Let $\mathcal{X}$ be a compact $\mathcal{T}_2$-space with a countable open base. Then $\mathcal{X}$ is metrizable.*

By Theorem 3, $\mathcal{X}$ is normal and by the Urysohn Metrization Theorem (see § 22, II, Theorem 1) each normal $\mathcal{T}_2$-space with a countable base is metrizable.

Remark 2. By the definition of normality, if A and B are closed and disjoint, then there are G and H open and disjoint such that

$$A \subset G \qquad \text{and} \qquad B \subset H. \tag{5}$$

If the space is compact, this statement can be strengthened as follows.

COROLLARY 3. *Let $\mathscr{X}$ be a compact $\mathscr{T}_2$-space and let $\boldsymbol{B}$ be its open base. Let $\boldsymbol{B}_s$ denote the family of finite unions of members of $\boldsymbol{B}$. If A and B are closed and disjoint, then there are $G \in \boldsymbol{B}_s$ and $H \in \boldsymbol{B}_s$ disjoint and satisfying* (5).

Proof. By Theorem 3, $\mathscr{X}$ is normal. Hence there are open and disjoint G_0 and H_0 satisfying (5). Since $\boldsymbol{B}$ is an open base of $\mathscr{X}$, there is a cover of A composed of members of $\boldsymbol{B}$ contained in G_0, and—as A is compact—this cover may be supposed finite. We denote by G the union of members of this cover.

H is defined similarly.

COROLLARY 4. *Let $\mathscr{X}$ be a compact perfectly normal $\mathscr{T}_1$-space. Let G be open and F closed. Then there are two infinite sequences $G_1, G_2, \ldots$ and $H_1, H_2, \ldots$ such that*

$$G = \bigcup_n G_n, \qquad \overline{G}_n \subset G_{n+1}, \qquad G_n \in \boldsymbol{B}_s, \tag{6}$$

and

$$F = \bigcap_n H_n, \qquad H_n \supset \overline{H}_{n+1}, \qquad H_n \in \boldsymbol{B}_s \tag{7}$$

(*where $\boldsymbol{B}_s$ has the same meaning as in Corollary* 3).

Proof. Since $\mathscr{X}$ is perfectly normal, we have $G = F_1 \cup F_2 \cup \ldots$ where F_n is closed. We define G_n by induction as follows. Since F_1 is compact, there is $G_1 \in \boldsymbol{B}_s$ such that $F_1 \subset G_1$ and $\overline{G}_1 \subset G$. Since $F_n \cup \overline{G}_{n-1}$ is compact $\subset G$, there is $G_n \in \boldsymbol{B}_s$ such that $(F_n \cup \overline{G}_{n-1}) \subset G_n$ and $\overline{G}_n \subset G$. Thus (6) is fulfilled.

In order to prove (7), let us represent F in the form $F = Q_1 \cap \cap Q_2 \cap \ldots$, where Q_n is open. Let $H_1 \in \boldsymbol{B}_s$ be such that $F \subset H_1 \subset Q_1$, and generally, let $H_n \in \boldsymbol{B}_s$ and $F \subset H_n \subset Q_n$ and $\overline{H}_n \subset H_{n-1}$.

THEOREM 4. *If a compact space has a countable open base, then the family of sets which are simultaneously closed and open is countable.*

Proof. Let $R_1, R_2, \ldots$ be the open base of the space. Let, for a given open G, $G = R_{k_1} \cup R_{k_2} \cup \ldots$. If G is closed, then we have $G = R_{k_1} \cup \ldots \cup R_{k_n}$ for an appropriate n. Put $\sigma(G) = (k_1, \ldots, k_n)$. Since $\sigma(G) \neq \sigma(G_1)$ for $G \neq G_1$, and the set of all finite systems of positive integers is countable, the proof is complete.

THEOREM 5. *If $\mathscr{X} = A \cup B$ where A and B are compact, then $\mathscr{X}$ is compact.*

Proof. Let $\{G_t\}$ be a cover of $\mathscr{X}$. Since A and B are compact, there are two finite systems $u_1, \ldots, u_n$ and $v_1, \ldots, v_m$ such that

$$A \subset G_{u_1} \cup \ldots \cup G_{u_n} \quad \text{and} \quad B \subset G_{v_1} \cup \ldots \cup G_{v_m}.$$

Hence $\mathscr{X} \subset G_{u_1} \cup \ldots \cup G_{u_n} \cup G_{v_1} \cup \ldots \cup G_{v_m}$.

THEOREM 6 (Generalized Baire theorem)(1). *Let $\mathscr{X}$ be a compact $\mathscr{T}_2$-space (or, more generally, a countably compact $\mathscr{T}_3$-space) and let $E = N_1 \cup N_2 \cup \ldots$ where N_n is nowhere dense. Then E is a boundary set.*

Proof. Let G be an arbitrary open non-empty set. We have to show that

$$G - E \neq 0. \tag{8}$$

We shall define an infinite sequence of open sets $G_0, G_1, \ldots$ such that

$$G_0 = G \quad \text{and} \quad 0 \neq \overline{G}_n \subset G_{n-1} - N_n \quad \text{for} \quad n = 1, 2, \ldots. \tag{9}$$

Let us proceed by induction. Let $n \geqslant 1$. Since N_n is nowhere dense and $G_{n-1} \neq 0$ (by assumption), there is an open H_n such that $0 \neq H_n \subset G_{n-1} - N_n$. Since $\mathscr{X}$ is regular (by Theorem 3), there is an open G_n such that $0 \neq \overline{G}_n \subset H_n$. Formula (9) follows. Therefore by the Cantor property (I(2')) we get $\left(\bigcap_{n=1}^{\infty} \overline{G}_n\right) \neq 0$. This yields (8), since

$$\bigcap_{n=1}^{\infty} \overline{G}_n \subset \bigcap_{n=1}^{\infty} (G - N_n) = G - \bigcup_{n=1}^{\infty} N_n = G - E. \tag{10}$$

The following statement will be used later.

COROLLARY 5. *Let $\mathscr{X}$ be a compact perfectly normal $\mathscr{T}_1$-space. Let $\boldsymbol{B}$ be an open base of $\mathscr{X}$ and A an $\boldsymbol{F}_\sigma$-set.*

Then there is a sequence $G_1, G_2, \ldots$ in $\boldsymbol{B}$ such that

$$A \subset \bigcup_n G_n, \tag{11}$$

$$\overline{\bigcup_n G_n} \subset \bigcup_n \overline{G}_n \cup \overline{A}. \tag{12}$$

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(1) See E. Čech, *On bicompact spaces*, Annals of Math. 38 (1937), p. 838. Compare R. Sikorski, *On the representation of Boolean algebras as fields of sets*, Fund. Math. 35 (1948), p. 256, footnote. Compare also § 34, IV (for complete metric spaces).

Proof. Put

$$A = F_1 \cup F_2 \cup \ldots, \quad \text{where} \quad F_k = \overline{F}_k. \tag{13}$$

According to (7) let

$$\overline{A} = H_1 \cap H_2 \cap \ldots \quad \text{where } H_k \text{ is open and } \overline{H_{k+1}} \subset H_k. \tag{14}$$

Hence $F_k \subset H_k$, and, since F_k is compact, there is a finite system of sets $G_1^k, \ldots, G_{m_k}^k$ in $\boldsymbol{B}$ such that

$$F_k \subset G_1^k \cup \ldots \cup G_{m_k}^k \quad \text{and} \quad G_i^k \subset H_k \quad \text{for} \quad 1 \leqslant i \leqslant m_k. \tag{15}$$

Let us arrange the sets G_i^k, where $k = 1, 2, \ldots$ and $1 \leqslant i \leqslant m_k$, in an infinite sequence $G_1, G_2, \ldots$ We shall show that conditions (11) and (12) are fulfilled.

(11) is a direct consequence of (13) and (15).

In order to show (12), let us consider for a given k an arbitrary $j > k$. It follows by (15) and (14) that

$$G_i^j \subset H_j \subset H_k, \quad \text{hence} \quad G_n \subset H_k$$

for sufficiently large n. Therefore $G_n \cup G_{n+1} \cup \ldots \subset H_k$ and consequently

$$\bigcap_{n=1}^{\infty} \overline{G_n \cup G_{n+1} \cup \ldots} \subset \overline{H}_k, \quad \text{hence} \quad \bigcap_{n=1}^{\infty} \overline{G_n \cup G_{n+1} \cup \ldots} \subset \bigcap_{k=1}^{\infty} \overline{H}_k = \overline{A},$$

and (12) follows by virtue of the formula (see § 4, III, 9):

$$\overline{\bigcup_n G_n} = \bigcup_n \overline{G}_n \cup \bigcap_n \overline{G_n \cup G_{n+1} \cup \ldots}\,.$$

Remark 3. The conditions which appear in the definitions of $\mathcal{T}_2$-spaces, regular and completely regular spaces can be strengthened as follows with the help of the concept of compactness.

(i) *If $\mathcal{X}$ is a $\mathcal{T}_2$-space and A and B are two compact disjoint subsets of $\mathcal{X}$, then there are two disjoint open sets G and H such that $A \subset G$ and $B \subset H$.*

This follows from the Lemma to Theorem 3.

(ii) *If $\mathcal{X}$ is regular and A compact, F closed and $A \cap F = 0$, then there is G open such that $A \subset G$ and $\overline{G} \cap F = 0$.*

Because for each $x \in A$ there is G_x open and such that $x \in G_x$ and $\bar{G}_x \cap F = 0$. Since A is compact, there are $x_1, \ldots, x_n$ such that $A \subset G_{x_1} \cup \ldots \cup G_{x_n}$ and $\overline{G_{x_1} \cup \ldots \cup G_{x_n}} \cap F = 0$.

(iii) *If $\mathscr{X}$ is a completely regular $\mathscr{T}_1$-space, C compact and F closed such that $F \cap C = 0$, then there is a continuous mapping $f\colon \mathscr{X} \to \mathscr{I}$ such that $f(x) = 0$ for $x \in C$ and $f(x) = 1$ for $x \in F$.*

Proof. Since $\mathscr{X}$ is completely regular, there is, for each $p \in C$, a continuous mapping $f_p\colon \mathscr{X} \to \mathscr{I}$ such that $f_p(p) = 0$ and $f_p(x) = 1$ for $x \in F$. Denote by J the interval $0 \leqslant t < 1/2$. Clearly the family $\{f_p^{-1}(J)\}$ is an open cover of C. Since C is compact, there are $p_1, \ldots, p_n$ such that

$$C \subset f_{p_1}^{-1}(J) \cup \ldots \cup f_{p_n}^{-1}(J).$$

Put $h(x) = \min\big(f_{p_1}(x), \ldots, f_{p_n}(x)\big)$. It is easily seen that the function f defined by the condition: $\frac{1}{2}f(x)$ is the greater of the numbers $h(x) - \frac{1}{2}$ and 0, is the required mapping of $\mathscr{X}$ into $\mathscr{I}$.

III. Continuous mappings.

THEOREM 1. *The image under a continuous mapping of a compact space is compact.*

Proof. Let $\mathscr{X}$ be compact and $f\colon \mathscr{X} \to \mathscr{Y}$ continuous and onto. Let $\{G_t\}$ be an open cover of $\mathscr{Y}$. Hence $\{f^{-1}(G_t)\}$ is an open cover of $\mathscr{X}$. Since $\mathscr{X}$ is compact, there are $t_1, \ldots, t_n$ such that

$$\mathscr{X} = f^{-1}(G_{t_1}) \cup \ldots \cup f^{-1}(G_{t_n}), \qquad \text{hence} \qquad \mathscr{Y} = G_{t_1} \cup \ldots \cup G_{t_n}.$$

THEOREM 2. *Each continuous mapping of a compact space into a $\mathscr{T}_2$-space is a closed mapping.*

Proof. Let $\mathscr{X}$ be compact and $f\colon \mathscr{X} \to \mathscr{Y}$ continuous. Let $F \subset \mathscr{X}$ be closed. By Theorem II, 2, F is compact. Therefore by Theorem 1, $f(F)$ is compact and hence closed by Theorem II, 1 (since $\mathscr{Y}$ is a $\mathscr{T}_2$-space).

THEOREM 3. *Each one-to-one continuous mapping of a compact space into a $\mathscr{T}_2$-space is a homeomorphism.*

This theorem follows from Theorem 2, since each one-to-one continuous closed mapping is a homeomorphism (see § 13, XIII).

Theorem 3 can be generalized as follows.

Given a mapping $f\colon \mathscr{X} \to \mathscr{Y}$, denote by B the set of values of f which are assumed at just one point of $\mathscr{X}$, i.e.

$$(y \in B) \equiv \{f^{-1}(y) \text{ reduces to a single point}\}.$$

THEOREM 3′. *Let $\mathscr{X}$ be compact, $\mathscr{Y}$ a $\mathscr{T}_2$-space and $f\colon \mathscr{X} \to \mathscr{Y}$ continuous. Then the partial mapping $f|f^{-1}(B)$ is a homeomorphism.*

This follows from the fact (see § 13, XIII, Theorem 1) that if f is continuous and closed, then for each $C \subset \mathscr{Y}$ the partial mapping $g = f|f^{-1}(C)$ is closed (and if $C = B$, g is obviously one-to-one).

Remark 1. The assumption of compactness on $\mathscr{X}$ is essential. So, for instance, $z = e^{ix}$ is a continuous and one-to-one transformation of the space $0 \leqslant x < 2\pi$ onto the circumference $|z| = 1$ without being a homeomorphism. (See also Corollary 4a below.)

Moreover, one can define *two non-homeomorphic spaces $\mathscr{X}$ and $\mathscr{Y}$ such that each of them can be obtained from the other by means of a continuous and one-to-one mapping.*(¹)

Namely, $\mathscr{X}$ is the union of open intervals $(3n, 3n+1)$ and individual points of the form $3n+2$, where $n \geqslant 0$. $\mathscr{Y}$ is obtained from $\mathscr{X}$ by replacing the point 2 by 1. The mapping $f\colon \mathscr{X} \to \mathscr{Y}$ is defined by the conditions: $f(x) = x$ for $x \neq 2$ and $f(2) = 1$.

The mapping $g\colon \mathscr{Y} \to \mathscr{X}$ is defined as follows:

$$g(x) = \begin{cases} \frac{1}{2}x & \text{for} \quad x \leqslant 1, \\ \frac{1}{2}x - 1 & \text{for} \quad 3 < x < 4, \\ x - 3 & \text{for} \quad x \geqslant 5. \end{cases}$$

THEOREM 4.(²) *Let $\mathscr{X}$ be a compact $\mathscr{T}_2$-space and F a non-empty closed subset of $\mathscr{X}$. Then there are a compact $\mathscr{T}_2$-space $\mathscr{Y}$, a point $y_0 \in \mathscr{Y}$, and a continuous map $f\colon \mathscr{X} \to \mathscr{Y}$ such that $f(F) = (y_0)$ and f is a homeomorphism of $\mathscr{X} - F$ onto $\mathscr{Y} - (y_0)$.*

(¹) See my paper *Solution d'un problème concernant les images continues d'ensembles de points*, Fund. Math. 2 (1921), p. 158.

In the same direction, see also J. de Groot, *Subcompactness and the Baire category theorem*, Indag. Math. 25 (1963), pp. 761–767.

(²) For $\mathscr{X}$ metric, compare Theorem 1 of § 22, IV, p. 245.

Namely, $\mathscr{Y}$ is the decomposition space (with its quotient topology, see § 19, I, p. 183) of $\mathscr{X}$ into F and single points of $\mathscr{X}-F$.

(Thus, $\mathscr{Y}$ is obtained from $\mathscr{X}$ by "identifying" the points belonging to F.)

Proof. Clearly, the decomposition under consideration is upper semi-continuous (whatever the $\mathscr{T}_1$-space $\mathscr{X}$ is); i.e. for each closed $A \subset \mathscr{X}$, the union of all members of the decomposition which intersect A is closed. It follows (by Theorem 5, p. 185) that the normality of $\mathscr{X}$ implies the normality of $\mathscr{Y}$. Since $\mathscr{Y}$ is a continuous image of $\mathscr{X}$ (by Theorem 1 of § 19), hence a compact space (by Theorem 1), the proof is complete.

COROLLARY 4a. *Let $\mathscr{X}$ be a compact $\mathscr{T}_2$-space and $G \subset \mathscr{X}$ open. Then there is a (compact) $\mathscr{T}_2$-space $\mathscr{Y}$ and a continuous mapping $f\colon \mathscr{X} \to \mathscr{Y}$ such that $f|G$ is a one-to-one mapping of G onto $\mathscr{Y}$.*

Proof. Let $a \in G$ be fixed, and put in Theorem 4:

$$F = (a) \cup (\mathscr{X}-G).$$

Remark 2. Let us draw attention to the following properties of continuous mappings of compact spaces (or—more generally—of continuous closed mappings of topological spaces).

Let $\mathscr{X}$ be compact, $\mathscr{Y}$ a $\mathscr{T}_2$-space and $f\colon \mathscr{X} \to \mathscr{Y}$ continuous and onto. The following statements hold:

(i) *if $V \subset \mathscr{Y}$, then $\overline{V} \subset f[\overline{f^{-1}(V)}]$,*

(ii) *if $V_1 \cup V_2 \subset \mathscr{Y}$ and $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are separated, then so are V_1 and V_2.*

Proof of (i). We have

$$V = ff^{-1}(V) \subset f[\overline{f^{-1}(V)}], \qquad \text{hence} \qquad \overline{V} \subset \overline{f[\overline{f^{-1}(V)}]}.$$

But by Theorem 2, $\overline{f[\overline{f^{-1}(V)}]} = f[\overline{f^{-1}(V)}]$.

Proof of (ii). By (i) and by § 3, III (13), we have

$$\overline{V}_1 \cap V_2 \subset f[\overline{f^{-1}(V_1)}] \cap V_2 = f[\overline{f^{-1}(V_1)} \cap f^{-1}(V_2)].$$

By assumption, $\overline{f^{-1}(V_1)} \cap f^{-1}(V_2) = 0$, hence $\overline{V}_1 \cap V_2 = 0$. Similarly $V_1 \cap \overline{V}_2 = 0$.

IV. Cartesian products.

THEOREM 1[1]. *Let $\mathscr{X}$ be an arbitrary topological space and let $\mathscr{Y}$ be compact. Then the projection of $\mathscr{X}\times\mathscr{Y}$ on the $\mathscr{X}$-axis is a closed mapping of $\mathscr{X}\times\mathscr{Y}$ onto $\mathscr{X}$.*

Equivalently: if G is open in $\mathscr{X}\times\mathscr{Y}$, then the set Q of all points x of $\mathscr{X}$ such that $(x)\times\mathscr{Y}\subset G$ is open in $\mathscr{X}$.

Proof. We shall prove the theorem in the second formulation. By the definition of the product topology, G is of the form $G=\bigcup\limits_t G_t\times H_t$ where G_t is open in $\mathscr{X}$ and H_t in $\mathscr{Y}$. Let x_0 be a fixed point of Q, i.e. $(x_0)\times\mathscr{Y}\subset G$. Hence for each $y\in\mathscr{Y}$ there is an index $t(y)$ such that $\langle x_0, y\rangle\in G_{t(y)}\times H_{t(y)}\subset G$. Thus

$$x_0\in G_{t(y)},\qquad y\in H_{t(y)}\qquad\text{and}\qquad G_{t(y)}\times H_{t(y)}\subset G. \tag{1}$$

Since the family $\{H_{t(y)}\}$, where y ranges over $\mathscr{Y}$, is an open cover of $\mathscr{Y}$, there is a finite system $y_1, \ldots, y_n$ such that

$$\mathscr{Y}=H_{t(y_1)}\cup\ldots\cup H_{t(y_n)}. \tag{2}$$

Put $R(x_0)=G_{t(y_1)}\cap\ldots\cap G_{t(y_n)}$. Hence $R(x_0)$ is open, $x_0\in R(x_0)$ by (1), and by virtue of (2) and (1):

$$[R(x_0)\times\mathscr{Y}]\subset[(G_{t(y_1)}\times H_{t(y_1)})\cup\ldots\cup(G_{t(y_n)}\times H_{t(y_n)})]\subset G.$$

Hence $R(x_0)\subset Q$. Since $R(x_0)$ is open and contains x_0, it follows that Q is open.

COROLLARY 1a. *Under the same assumptions, the projection on the $\mathscr{X}$-axis of an $\boldsymbol{F}_\sigma$-set in $\mathscr{X}\times\mathscr{Y}$ is an $\boldsymbol{F}_\sigma$-set in $\mathscr{X}$.*

COROLLARY 1b. *Under the same assumptions, let $\varphi(x, y)$ be a propositional function defined on $\mathscr{X}\times\mathscr{Y}$. Then, if the set $\underset{xy}{E}\,\varphi(x, y)$ is closed (resp. $\boldsymbol{F}_\sigma$), so is $\underset{x}{E}\,\underset{y}{\bigvee}\varphi(x, y)$; if $\underset{xy}{E}\,\varphi(x, y)$ is open (resp. $\boldsymbol{G}_\delta$), so is $\underset{x}{E}\,\underset{y}{\bigwedge}\varphi(x, y)$.*

For $\underset{x}{E}\,\underset{y}{\bigvee}\varphi(x, y)$ is the projection of $\underset{xy}{E}\,\varphi(x, y)$ on the $\mathscr{X}$-axis (see § 2, V, Theorem 1).

Remark 1. In the same direction one has the following statement (of A. D. Wallace) [2].

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[1] For $\mathscr{L}^*$-spaces, see § 20, V, Theorem 7.

[2] See J. L. Kelley, *General Topology*, p. 142.

Let $\mathscr{X}$ and $\mathscr{Y}$ be two (arbitrary) topological spaces, A and B two compact subsets of $\mathscr{X}$ and $\mathscr{Y}$ respectively and U an open subset of $\mathscr{X}\times\mathscr{Y}$ containing $A\times B$. Then there are two open subsets G and H of $\mathscr{X}$ and $\mathscr{Y}$ such that $A\subset G$, $B\subset H$ and $G\times H\subset U$.

As an application of Corollary 1b, consider the following statement.

COROLLARY 1c. *Let $\mathscr{X}$ be compact and perfectly normal, $\mathscr{Y}$ a $\mathscr{T}_2$-space and $f\colon \mathscr{X}\to\mathscr{Y}$ continuous. Then the set B_f of points x such that*

$$(x'\neq x)\Rightarrow[f(x')\neq f(x)] \tag{3}$$

is a G_δ-set.

Proof. By (3), $B_f = \mathop{E}_{x}\bigwedge_{x'}\{[f(x)=f(x')]\Rightarrow(x=x')\}$. Since $\mathop{E}_{xx'}[f(x)=f(x')]$ is closed (see § 15, IV, Theorem 3), the set of points (x,x') satisfying the condition in braces $\{\,\}$, is a G_δ-set (in fact an union of an open and a closed set). It follows that B_f is also a G_δ-set.

Remark 2. The assumption made in Corollary 1a, of $\mathscr{Y}$ being compact, can be replaced by the weaker assumption, of $\mathscr{Y}$ being the union of a countable sequence of compact sets: $\mathscr{Y}=Y_1\cup \cup Y_2\cup\ldots$.

For, if $S=F_1\cup F_2\cup\ldots$, where $\overline{F}_i=F_i\subset\mathscr{X}\times\mathscr{Y}$, we have

$$\bigvee_{y\in Y}\langle x,y\rangle\in S \equiv \bigvee_{n,m}\bigvee_{y}(\langle x,y\rangle\in F_n)(y\in Y_m),$$

hence

$$\mathop{E}_{x}\bigvee_{y\in Y}(\langle x,y\rangle\in S)=\bigcup_{n,m}\Big\{\mathop{E}_{x}\bigvee_{y\in Y_m}(\langle x,y\rangle\in F_n)\Big\},$$

and for given n and m the set in braces $\{\,\}$ is closed by Theorem 1.

Remark 3. Let us recall that the projection of an open set is open (whatever the space $\mathscr{Y}$ is). But the projection of a G_δ-set may fail to be G_δ. In fact the projections of G_δ subsets of $\mathscr{I}\times\mathscr{I}$ are arbitrary analytic sets in $\mathscr{I}$ (see § 38, IV).

THEOREM 2. *Let $f\colon\mathscr{X}\to\mathscr{Y}$, where $\mathscr{Y}$ is a compact $\mathscr{T}_2$-space. Then f is continuous if and only if $\mathop{E}_{xy}\big(y=f(x)\big)$ is closed (in $\mathscr{X}\times\mathscr{Y}$).*

Proof. The necessity of this condition has been shown in § 15, V (Theorem 2) for $\mathscr{Y}$ being a $\mathscr{T}_2$-space.

In order to show its sufficiency, let us suppose that $F = \overline{F} \subset \mathcal{Y}$. We have to prove that $f^{-1}(F)$ is closed in $\mathcal{X}$.

According to Theorem 1 it suffices to show that the set $f^{-1}(F)$ is the projection of the set $I \cap (\mathcal{X} \times F) = \underset{xy}{E}\big(y = f(x)\big)(y \in F)$ on the $\mathcal{X}$-axis. But this follows from the obvious formula:

$$[x \in f^{-1}(F)] \equiv [f(x) \in F] \equiv \bigvee_{y} \big(y = f(x)\big)(y \in F).$$

Remark 4. Without assuming $\mathcal{Y}$ to be compact, the theorem fails to be true, as seen in the example: $\mathcal{X} = \mathcal{I}$, $\mathcal{Y} = \mathcal{E}$, $f(x) = 1/x$ for $x \neq 0$ and $f(0) = 0$.

THEOREM 3. *The cartesian product $\mathcal{X} \times \mathcal{Y}$ of two compact spaces is compact.*

Proof. It is easily seen that each refinement of an essentially infinite cover is essentially infinite. For every cover of $\mathcal{X} \times \mathcal{Y}$ there is a refinement composed of sets of the form $G \times H$ where G is open in $\mathcal{X}$ and H in $\mathcal{Y}$; thus it suffices to show that each cover $\boldsymbol{C} = \{G_t \times H_t\}$ of $\mathcal{X} \times \mathcal{Y}$ contains a finite subcover.

Denote by $\boldsymbol{R}$ the family of all open subsets Q of $\mathcal{X}$ such that for an appropriate system of indices $t_1, \ldots, t_n$, we have

$$Q \times \mathcal{Y} \subset (G_{t_1} \times H_{t_1}) \cup \ldots \cup (G_{t_n} \times H_{t_n}). \tag{4}$$

We shall show that $\boldsymbol{R}$ is a cover of $\mathcal{X}$. Let $x_0 \in \mathcal{X}$. As $(x_0) \times \mathcal{Y} \underset{\text{top}}{=} \mathcal{Y}$ and as $\mathcal{Y}$ is compact, $(x_0) \times \mathcal{Y}$ is contained in a finite subcover of $\boldsymbol{C}$:

$$(x_0) \times \mathcal{Y} \subset (G_{t_1} \times H_{t_1}) \cup \ldots \cup (G_{t_n} \times H_{t_n}).$$

By Theorem 1, there is an open Q containing x_0 and satisfying (4). Therefore $\boldsymbol{R}$ is a cover of $\mathcal{X}$. Since $\mathcal{X}$ is compact, we have $\mathcal{X} = Q_1 \cup \ldots \cup Q_k$. By (4)

$$Q_i \times \mathcal{Y} \subset (G_{t_{i,1}} \times H_{t_{i,1}}) \cup \ldots \cup (G_{t_{i,n(i)}} \times H_{t_{i,n(i)}}) \quad \text{for} \quad i \leqslant k.$$

Therefore

$$\mathcal{X} \times \mathcal{Y} = \bigcup_{i=1}^{k} (Q_i \times \mathcal{Y}) \subset \bigcup_{i=1}^{k} \bigcup_{j=1}^{n_i} G_{t_{i,j}} \times H_{t_{i,j}}.$$

Remark 5. *The product of two countably compact regular spaces may fail to be countably compact* [1].

[1] See J. Novák, *On the cartesian product of two compact spaces*, Fund. Math. 40 (1953), pp. 106–112.

However, *the product of a compact space and a countably compact space is countably compact* (¹).

Recall that a countable product of countably compact $\mathscr{L}^*$-spaces is countably compact (§ 20, V, Theorem 4).

Remark 6. Theorem 3 can easily be extended to a product of a finite number of factors. In a less elementary way it can be extended—as we shall prove below—to an arbitrary number of factors.

THEOREM 4. (Tychonov product theorem) (²). *The product* $\mathfrak{Z} = \underset{t \in T}{\boldsymbol{P}} X_t$ *of compact spaces* X_t *is compact. In particular the generalized cube* $\mathscr{I}^{\aleph_a}$ *is compact for each* a.

Proof. Let $\boldsymbol{A}$ be the open subbase of $\mathfrak{Z}$ composed of sets (see § 16, I (1)):

$$\mathfrak{G}_{t,G} = \underset{\mathfrak{z}}{E}(\mathfrak{z}^t \in G) \quad \text{where} \quad t \in T \text{ and } G \subset X_t \text{ is open.} \tag{5}$$

Suppose that $\mathfrak{Z}$ is not compact. Then by the Alexander Lemma (of Section I), there is an essentially infinite cover $\boldsymbol{U} \subset \boldsymbol{A}$.

Denote by $\boldsymbol{V}_t$ the family of sets defined by the condition

$$G \in \boldsymbol{V}_t \equiv G_{t,G} \in \boldsymbol{U}. \tag{6}$$

Suppose that there is a $t \in T$ such that $\boldsymbol{V}_t$ is a cover of X_t. Since X_t is compact, we have

$$X_t = G_1 \cup \ldots \cup G_n, \quad \text{where} \quad G_i \in \boldsymbol{V}_t, \text{ i.e. } G_{t,G_i} \in \boldsymbol{U}, \text{ for } i \leqslant n. \tag{7}$$

It follows by (5) and (7) that

$$\begin{aligned} \bigcup_i G_{t,G_i} &= \bigcup_i \underset{\mathfrak{z}}{E}(\mathfrak{z}^t \in G_i) = \underset{\mathfrak{z}}{E} \bigvee_i (\mathfrak{z}^t \in G_i) \\ &= \underset{\mathfrak{z}}{E}\Big(\mathfrak{z}^t \in \bigcup_i G_i\Big) = \underset{\mathfrak{z}}{E}(\mathfrak{z}^t \in X_t) = \mathfrak{Z}. \end{aligned}$$

Thus $\boldsymbol{U}$ contains a finite cover of $\mathfrak{Z}$.

(¹) Theorem of M. Katětov. See *ibid.* p. 111.

(²) A. Tychonov, *Über einen Funktionenraum*, Math. Ann. 111 (1935), p. 702.

For other proofs of the Tychonov Theorem see: J. W. Alexander, *Ordered sets, complexes, and the problem of compactification*, Proc. Nat. Acad. Sc. USA 25 (1939), pp. 296–298; E. Čech, *On bicompact spaces*, Annals of Math. 38 (1937), p. 830; C. Chevalley and O. Frink, Jr., *Bicompactness of cartesian products*, Bull. Amer. Math. Soc. 47 (1941), pp. 612–614; J. W. Tukey, *Convergence and Uniformity in Topology*, Princeton 1940; N. Bourbaki, *Topologie générale*, Chap. I, § 9, No 5.

Therefore we may assume that, for each $t \in T$, $\boldsymbol{V}_t$ is not a cover of X_t. This means that there is $\mathfrak{z} \in \mathfrak{Z}$ such that

$$G \in \boldsymbol{V}_t \Rightarrow \mathfrak{z}^t \notin G, \quad \text{i.e.} \quad G_{t,G} \in \boldsymbol{U} \Rightarrow \mathfrak{z}^t \notin G. \tag{8}$$

As $\boldsymbol{U}$ is a cover of $\mathfrak{Z}$ and $\boldsymbol{U} \subset \boldsymbol{A}$, there is a pair (t, G) such that $\mathfrak{z} \in G_{t,G} \in \boldsymbol{U}$. But this is a contradiction to (5) and (8) since

$$\mathfrak{z} \in G_{t,G} \Rightarrow \mathfrak{z}^t \in G \quad \text{while} \quad G_{t,G} \in \boldsymbol{U} \Rightarrow \mathfrak{z}^t \notin G.$$

COROLLARY. *The following properties are topologically equivalent*:

(i) *being a completely regular $\mathscr{T}_1$-space,*

(ii) *being a subset of a generalized cube $\mathscr{I}^{\aleph_\alpha}$,*

(iii) *being a subset of a compact $\mathscr{T}_2$-space.*

Proof. (i) $\Rightarrow$ (ii) by § 16, V, Theorem 5; (ii) $\Rightarrow$ (iii) since $\mathscr{I}^{\aleph_\alpha}$ is a compact $\mathscr{T}_2$-space by Theorem 3 and § 16, V, Theorem 4; (iii) $\Rightarrow$ (i) since each compact $\mathscr{T}_2$-space is normal by Theorem 3 of Section II, hence completely regular, which is a hereditary property by § 14, I, Theorem 2.

Theorem 4 implies the following (1)

THEOREM 5. *Let* (T, X, f) *be an inverse system* (see § 3, XIII, p. 28) *and let* X_t *be compact for each* $t \in T$. *Then the inverse limit* $\underleftarrow{\mathrm{Lim}}(T, X, f)$, *denoted also by* $\underset{t,t_0 \leqslant t_1}{\mathrm{Lim}}\{X_t, f_{t_0 t_1}\}$, *is compact.*

Moreover, *if $X_t \neq 0$ for each $t \in T$, then this limit is non-void.*

Remark 7. The following interesting theorem can be shown (2).

Each compact $\mathscr{T}_2$-space $\mathscr{X}$ is an inverse limit $\underset{t,t_0 \leqslant t_1}{\mathrm{Lim}}\{P_t, f_{t_0 t_1}\}$ *where P_t is a simplicial polyhedron* (see § 28, p. 308) *and $f_{t_0 t_1}$ is a simplicial mapping $P_{t_1} \to P_{t_0}$.*

Let us add that each polytope P_t is the nerve of an open finite cover of $\mathscr{X}$ (see § 28, p. 318).

V. Compactification of completely regular $\mathscr{T}_1$-spaces. A compact space $\mathscr{Y}$ is said to be a *compactification* of the space $\mathscr{X}$ if $\mathscr{X}$ is homeomorphic to a dense subset of $\mathscr{Y}$.

For example, $\mathscr{I}$ and $\mathscr{S}_1$ are compactifications of $\mathscr{E}$.

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(1) For a proof, see e.g. Eilenberg–Steenrod, *Foundations of algebraic topology*, p. 217, Theorem 3.6.

(2) See H. Freudenthal, *Über die Entwicklung von Räumen und Gruppen*, Compos. Math. 4 (1937); B. A. Pasynkov, Dokl. Akad. Nauk URSS 121 (1958), p. 45; S. Eilenberg and N. Steenrod, *ibid.* p. 284.

It has been shown in § 16, V, Theorem 5, that for each completely regular $\mathscr{T}_1$-space $\mathscr{X}$ there is a subset of the generalized cube $\mathscr{I}^{\aleph_\alpha}$ (for an appropriate α) which is a compactification of $\mathscr{X}$.

More precisely, $\aleph_\alpha = \overline{\overline{\Phi}}$, where $\Phi = \mathscr{I}^{\mathscr{X}}$, and the required homeomorphism $\mathfrak{z}$: $\mathscr{X} \to \mathfrak{W}$, where $\mathfrak{W} = (\mathscr{I}^{\Phi})_{\text{set}}$, is defined by the condition

$$[\mathfrak{z}(x)](\varphi) = \varphi(x) \quad \text{for each } \varphi \in \Phi \text{ and } x \in X \tag{1}$$

($\mathfrak{z}$ is called the evaluation of Φ).

Put

$$\beta\mathscr{X} = \overline{\mathfrak{z}(\mathscr{X})}. \tag{2}$$

Thus $\beta\mathscr{X}$ is a compactification of $\mathscr{X}$ (called the *Čech–Stone compactification*) (¹). We shall see that it can be considered as a *maximal* compactification (among the $\mathscr{T}_2$-spaces).

Denote by $\mathrm{pr}_\varphi(\mathfrak{w})$, for $\mathfrak{w} \in \mathfrak{W}$, the projection of $\mathfrak{w}$ on the φth axis, i.e. (see § 3, VIII):

$$\mathrm{pr}_\varphi(\mathfrak{w}) = \mathfrak{w}(\varphi). \tag{3}$$

We have

$$\mathrm{pr}_\varphi \circ \mathfrak{z} = \varphi \tag{4}$$

since by (3) and (1)

$$(\mathrm{pr}_\varphi \circ \mathfrak{z})(x) = \mathrm{pr}_\varphi[\mathfrak{z}(x)] = [\mathfrak{z}(x)](\varphi) = \varphi(x).$$

Formula (4) yields

$$\varphi \circ \mathfrak{z}^{-1} \subset \mathrm{pr}_\varphi \quad \text{for each } \varphi \in \Phi, \tag{5}$$

since $\varphi \circ \mathfrak{z}^{-1} = \mathrm{pr}_\varphi \circ \mathfrak{z} \circ \mathfrak{z}^{-1} \subset \mathrm{pr}_\varphi$ by § 3, III, (20).

From here follows

LEMMA 1. *Let f: $\mathscr{X} \to \mathscr{I}$ be continuous. Then the function $f \circ \mathfrak{z}^{-1}$: $\mathfrak{z}(\mathscr{X}) \to \mathscr{I}$ has a continuous extension f^*: $\mathfrak{W} \to \mathscr{I}$. Namely* $f^* = \mathrm{pr}_f$.

(¹) See E. Čech, *On bicompact spaces*, Annals of Math. 38 (1937), pp. 823–844, where many properties of $\beta\mathscr{X}$ are given; M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. 41 (1937). See also: H. Wallman, *Lattices and topological spaces*, Ann. of Math. 42 (1941), pp. 687–697; E. Sklyarenko, *On the embedding of normal spaces in bicompact spaces of the same weight and of the same dimension* (Russian), Dokl. Akad. Nauk SSSR 123 (1958), p. 36; R. Engelking, *Sur la compactification des espaces métriques*, Fund. Math. 48 (1960), p. 231; H. de Vries, *Compact spaces and compactifications; an algebraic approach*, Thesis, Amsterdam 1962; R. Engelking and E. Sklyarenko, *On compactifications allowing extensions of mappings*, Fund. Math. 53 (1963), pp. 65–79.

In other terms, identifying $\mathscr{X}$ with $\mathfrak{z}(\mathscr{X})$, we have

$$f \subset f^*\colon \mathfrak{W} \to \mathscr{I}. \tag{6}$$

Lemma 1 can be generalized as follows.

LEMMA 2. *Let T be an arbitrary set and let $f\colon \mathscr{X} \to (\mathscr{I}^T)_{\text{set}}$ be continuous. Let us identify $\mathscr{X}$ with $\mathfrak{z}(\mathscr{X})$. Then we have*

$$f \subset f^*\colon \mathfrak{W} \to (\mathscr{I}^T)_{\text{set}}, \quad \textit{where } f^* \textit{ is continuous}. \tag{7}$$

For, denoting by f_t the t-th coordinate of f, we have $f_t \subset f_t^*\colon \mathfrak{W} \to \mathscr{I}$, whence the complex mapping f^* having f_t^* as its t-th coordinate (see § 3, VIII (7)) satisfies (7).

THEOREM. *Let $\mathscr{X}$ be a completely regular $\mathscr{T}_1$-space, $\mathscr{Y}$ a compact $\mathscr{T}_2$-space and $f\colon \mathscr{X} \to \mathscr{Y}$ continuous. Then identifying $\mathscr{X}$ with $\mathfrak{z}(\mathscr{X})$, we have*

$$f \subset g\colon \beta\mathscr{X} \to \mathscr{Y} \quad \textit{where } g \textit{ is continuous}. \tag{8}$$

Proof. Since $\mathscr{Y}$ is a completely regular $\mathscr{T}_1$-space (by Theorem 3 of Section II), it may be assumed to be a subset of a cube $(\mathscr{I}^T)_{\text{set}}$ for an appropriate set T. Thus $f\colon \mathscr{X} \to (\mathscr{I}^T)_{\text{set}}$. Applying formula (7) put $g = f^* | \beta\mathscr{X}$. It follows that $f \subset g$. Finally $g\colon \beta\mathscr{X} \to \mathscr{Y}$, since the continuity of g and the compactness of $\mathscr{Y}$ imply:

$$g(\beta\mathscr{X}) = g(\overline{\mathscr{X}}) \subset \overline{g(\mathscr{X})} = \overline{f(\mathscr{X})} \subset \overline{\mathscr{Y}} = \mathscr{Y}.$$

Remark. The compactification defined above is *maximal* in the following sense. Given a compactification $\mathscr{Y}$ of $\mathscr{X}$ (where $\mathscr{Y}$ is a $\mathscr{T}_2$-space), there exists a continuous mapping of $\beta\mathscr{X}$ into $\mathscr{Y}$ which is the identity on $\mathscr{X}$.

This is just another form of the preceding theorem. For, let h be a topological immersion of $\mathscr{X}$ into $\mathscr{Y}$ and $h \subset h^*\colon \beta\mathscr{X} \to \mathscr{Y}$. By identifying x with $h(x)$, one obtains the required mapping of $\beta\mathscr{X}$ into $\mathscr{Y}$.

VI. Relationships to metric spaces.

THEOREM 1. *Every compact metric space is complete, totally bounded and separable.*

Completeness of compact metric spaces has been shown in § 33, II. By Theorem 1 of § 21, IX, every compact metric space is totally bounded, hence separable (by Theorem 3 of § 21, VIII).

THEOREM 2. *A complete space is compact if and only if it is totally bounded.*

Proof. Since a compact metric space is totally bounded (by Theorem 1), we must show that, if $\mathscr{X}$ is complete and totally bounded, then $\mathscr{X}$ is countably compact.

Let A be an arbitrary infinite subset of $\mathscr{X}$. According to I (3′) it suffices to show that $A^d \neq 0$.

Now, since $\mathscr{X}$ is totally bounded, we have

$$\mathscr{X} = F_1^n \cup \ldots \cup F_{m_n}^n \quad \text{where} \quad F_i^n = \overline{F_i^n} \text{ and } \delta(F_i^n) < 1/n.$$

Since A is infinite, there is a sequence $k_1, k_2, \ldots$ such that the sets $A \cap F_{k_1}^1$, $A \cap F_{k_1}^1 \cap F_{k_2}^2, \ldots$ are infinite. By the theorem of Cantor on complete spaces (§ 34, II), there is a point p belonging to each set $F_{k_1}^1 \cap \ldots \cap F_{k_n}^n$ for $n = 1, 2, \ldots$. Since the diameter of this set is $< 1/n$, it follows that the ball with center p and radius $1/n$ has an infinite intersection with A. Hence $p \in A^d$.

Remark 1. One can show that every non-compact metric space is homeomorphic to a non-complete space. Thus, a metric space $\mathscr{X}$ is compact if and only if every metric space homeomorphic to $\mathscr{X}$ is complete [1].

COROLLARY 2a. *A subset of* $\mathscr{E}^n$ $(n < \aleph_0)$ *is compact if and only if it is closed and bounded.*

Proof. If $A \subset \mathscr{E}^n$ is compact, then A is closed by Theorem 1 of Section II and is bounded by Theorem 1 of this section.

On the other hand, if $A \subset \mathscr{E}^n$ is bounded, then A is a subset of a cube $C \subset \mathscr{E}^n$. Since C is compact, then if A is closed, A is compact (by Theorem 2 of Section II).

Remark 2. *Every totally bounded space* $\mathscr{X}$ *is isometric to a subset of a compact metric space.*

By a Theorem of Hausdorff (see § 33, VII, p. 410), $\mathscr{X}$ is isometric to a subset of a complete space $\mathscr{X}^*$. The closure $\overline{\mathscr{X}}$ (relative to $\mathscr{X}^*$) is totally bounded and complete, hence compact (by Theorem 2).

THEOREM 3. *For* $\mathscr{T}_2$*-spaces the property of being compact and metrizable is invariant under continuous mappings.*

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(1) See V. Niemytzki and A. Tychonov, *Beweis des Satzes, dass ein metrischer Raum dann und nur dann kompakt ist, wenn er in jeder Metrik vollständig ist,* Fund. Math. 12 (1928), p. 118.

Proof. Let $\mathscr{X}$ be compact metric, $\mathscr{Y}$ a $\mathscr{T}_2$-space and $f\colon \mathscr{X} \to \mathscr{Y}$ continuous and onto. We have to show that $\mathscr{Y}$ is metrizable (the compactness of $\mathscr{Y}$ is a consequence of Theorem 1 of Section III). As $\mathscr{Y}$ is normal (by Theorem 3 of Section II), it remains to be shown (according to the Urysohn metrization theorem, see § 22, II, p. 241) that $\mathscr{Y}$ has an open countable base. We define this base as follows.

By Theorem 1, $\mathscr{X}$ is separable. Hence it contains an open countable base $R_1, R_2, \ldots$. Let $S_1, S_2, \ldots$ be the sequence of all finite unions of sets R_i. We shall prove that the family of sets

$$Q_n = \mathscr{Y} - f(\mathscr{X} - S_n), \quad n = 1, 2, \ldots \tag{1}$$

is an open base of $\mathscr{Y}$.

Let $G \subset \mathscr{Y}$ be open and $y \in G$. We have to show the existence of an n such that

$$y \in Q_n \subset G. \tag{2}$$

Now, the sets $f^{-1}(y)$ and $f^{-1}(\mathscr{Y} - G)$ are compact and disjoint. Hence there is (by the normality of $\mathscr{X}$) an n such that

$$f^{-1}(y) \subset S_n \quad \text{and} \quad S_n \cap f^{-1}(\mathscr{Y} - G) = 0, \quad \text{i.e.} \quad S_n \subset f^{-1}(G). \tag{3}$$

Hence $\mathscr{X} - f^{-1}(G) \subset \mathscr{X} - S_n \subset \mathscr{X} - f^{-1}(y)$, i.e.

$$f^{-1}(\mathscr{Y} - G) \subset \mathscr{X} - S_n \subset f^{-1}\big(\mathscr{Y} - (y)\big),$$

and it follows that $\mathscr{Y} - G \subset f(\mathscr{X} - S_n) \subset \mathscr{Y} - (y)$, which gives (2).

Remark 3. Theorem 3 can be strengthened as follows: *A compact $\mathscr{T}_2$-space, which is the continuous image of a separable metric space, is metrizable* (1).

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(1) Theorem of H. H. Corson. See E. Michael, *Cuts*, Acta Mathematica 111 (1964), p. 14. See also H. H. Corson and E. Michael, *Metrizability of countable unions*, Illinois Journal Math. 8 (1964), pp. 351–360, and A. Mishchenko, *Spaces with point-countable base*, Doklady 144 (1962), p. 985.

For analogous theorems where compactness is not assumed but the mapping (of the metric space) is supposed to be closed or open, see A. H. Stone, *Metrizability of decomposition spaces*, Proc. Amer. Math. Soc. 7 (1956), pp. 690–700, and V. K. Balanchandran, *A mapping theorem for metric spaces*, Duke Math. Journ. 22 (1955), pp. 461–464.

Compare also V. Proizvolov, *On finite-to-one open mappings*, Doklady Akad. Nauk SSSR 166 (1966), p. 38.

COROLLARY 3a. *Each of the following conditions is necessary and sufficient for a $\mathcal{T}_2$-space $\mathcal{X}$ to be compact and metrizable:*

(i) *$\mathcal{X}$ is homeomorphic to a closed subset of the Hilbert cube $\mathcal{H} = \mathcal{J}^{\aleph_0}$,*

(ii) *$\mathcal{X}$ is a continuous image of the Cantor discontinuum $\mathcal{C}$ (provided $\mathcal{X} \neq 0$).*

Proof. Suppose that the metric space $\mathcal{X}$ is compact. Then it is separable and, by the Urysohn Theorem (§ 22, II, Theorem 1), $\mathcal{X} \underset{\text{top}}{=} Y \subset \mathcal{H}$. Hence Y is compact and, consequently, closed.

On the other hand, if $\mathcal{X} \underset{\text{top}}{=} Y = \overline{Y} \subset \mathcal{H}$, then Y is compact, and so is $\mathcal{X}$.

The proof of the second part of the corollary reduces to proving that if F is a closed subset ($\neq 0$) of $\mathcal{H}$, then F is a continuous image of $\mathcal{C}$. Now, by Corollaries 6a of § 16, II and 2 of § 26, II, there are $g\colon \mathcal{C} \to \mathcal{H}$ and $h\colon \mathcal{C} \to g^{-1}(F)$, both continuous and onto. Therefore $g \circ h\colon \mathcal{C} \to F$ is continuous and onto.

THEOREM 4. (Weierstrass theorem.) *Every continuous real-valued function f defined on a compact space $\mathcal{X}$ is bounded and attains its lower and upper bounds.*

This follows from the fact that $f(\mathcal{X})$ is a closed and bounded subset of $\mathcal{E}$ (by Theorem 1 of Section III, Theorem 1 of Section II and Theorem 1 of this Section).

COROLLARY 4a. *Every compact metric (non-void) space $\mathcal{X}$ contains two points a and b such that*

$$|a-b| = \delta(\mathcal{X}).$$

This follows from Theorem 3 of Section 4 and Theorem 4, since the distance is a continuous mapping of $\mathcal{X} \times \mathcal{X}$ into $\mathcal{E}$.

COROLLARY 4b. *If A is compact, B closed and $A \cap B = 0$, then $\varrho(A, B) > 0$ (we assume that $A \neq 0 \neq B$).*

Proof. Obviously $\varrho(A, B) = \inf\limits_{x \in A} \varrho(x, B)$. Since $\varrho(x, B)$ represents a continuous function defined on a compact set, there is x_0 such that $\inf \varrho(x, B) = \varrho(x_0, B)$, and since $x_0 \notin \overline{B}$, we have $\varrho(x_0, B) > 0$. This completes the proof.

COROLLARY 4c. *Let $\{F_t\}$, where $t \in T$, be a family of closed subsets of a compact metric space such that $\bigcap\limits_t F_t = 0$; then there is*

$\varepsilon > 0$ *(called the Lebesgue coefficient of the system* $\{F_t\}$*) such that every set* A *of diameter* $< \varepsilon$ *is disjoint from at least one of the sets* F_t.

Proof. Note that (since the space is compact) we have $F_{t_1} \cap \cap \ldots \cap F_{t_n} = 0$ for an appropriate system $t_1, \ldots, t_n$ of indices. Put

$$f(x_1, \ldots, x_n) = \max_{i,j \leqslant n} |x_i - x_j| \quad \text{where} \quad x_k \in F_{t_k},$$

and denote by ε its lower bound (observe that $f\colon F_{t_1} \times \ldots \times F_{t_n} \to \mathscr{E}$ is continuous). Since $F_{t_1} \cap \ldots \cap F_{t_n} = 0$, we have $\varepsilon > 0$.

Suppose $A \cap F_t \neq 0$ for each $t \in T$. Let $x_k \in A \cap F_{t_k}$. Then $\delta(A) \geqslant \delta(x_1, \ldots, x_n) \geqslant \varepsilon$.

The following statement is dual to Corollary 3a.

COROLLARY 4d. *Let* C *be an open cover of the compact metric space* $\mathscr{X}$. *Then there is* $\varepsilon > 0$ *such that every cover of* $\mathscr{X}$ *composed of sets of diameter* $< \varepsilon$ *is a refinement of* C.

THEOREM 5 (of Heine) [1]. *If* $\mathscr{X}$ *is compact and* $f\colon \mathscr{X} \to \mathscr{E}$ *is continuous, then* f *is uniformly continuous.*

This means, that for each $\varepsilon > 0$ *there is* $\delta > 0$ *such that*

$$|x - x'| < \delta \quad \textit{implies} \quad |f(x) - f(x')| < \varepsilon. \tag{1}$$

Proof. Suppose that the contrary is true. Then there exist two sequences $x_1, x_2, \ldots$ and $x_1', x_2', \ldots$ such that

$$|x_n - x_n'| < 1/n \quad \text{and} \quad |f(x_n) - f(x_n')| \geqslant \varepsilon. \tag{2}$$

As the space is compact, we may assume that the sequence $x_1, x_2, \ldots$ is convergent: $\lim x_n = x$. Hence

$$\lim x_n' = x \quad \text{and} \quad \lim f(x_n) = f(x) = \lim f(x_n').$$

But this contradicts (2).

Remark 4. Theorem 5 can easily be proved using logical symbols.

Put (for a given $\varepsilon > 0$):

$$\varphi_n(x, x') \equiv \{[|x - x'| \leqslant 1/n] \Rightarrow [|f(x) - f(x')| < \varepsilon]\}.$$

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[1] Proved by Heine for the interval in his paper *Elemente der Functionenlehre*, Journ. f. Math. 74 (1872), p. 172.

We have to show that in the formula $\bigwedge_x \bigvee_n \bigwedge_{x'} \varphi_n(x, x')$ (which is true by assumption) the order of the quantifiers $\bigwedge_x$ and $\bigvee_n$ can be interchanged. Now, as $\mathop{E}_{xx'} \varphi_n(x, x')$ is open, so is $\mathop{E}_x \bigwedge_{x'} \varphi_n(x, x')$ (by IV, Corollary 1b), and as

$$\bigwedge_{x'} \varphi_n(x, x') \Rightarrow \bigwedge_{x'} \varphi_{n+1}(x, x'),$$

then by I(iii)

$$\bigwedge_x \bigvee_n \big(\bigwedge_{x'} \varphi_n(x, x')\big) \equiv \bigvee_n \bigwedge_x \big(\bigwedge_{x'} \varphi_n(x, x')\big).$$

THEOREM 6 (1). *Let $\mathcal{X}$ be a metric separable space. If $\mathcal{X}$ is scattered, then there is a metric scattered compactification of $\mathcal{X}$.*

Proof. Since $\mathcal{X}$ is countable (see § 23, V), $\mathcal{X}$ can be considered as a subset of $\mathcal{N}$ (space of irrational numbers between 0 and 1). Furthermore, since every metric separable scattered set is a $\boldsymbol{G}_\delta$ (§ 24, III, Theorem 1a) and every $\boldsymbol{G}_\delta$ subset of $\mathcal{N}$ is homeomorphic to a closed subset of $\mathcal{N}$ (§ 36, II, Corollary), we may assume that $\mathcal{X}$ is closed relative to $\mathcal{N}$. In other words, denoting by $\overline{\mathcal{X}}$ the closure of $\mathcal{X}$ relative to $\mathcal{I}$, we have $\mathcal{X} = \overline{\mathcal{X}} \cap \mathcal{N}$.

$\overline{\mathcal{X}}$ is the required compactification of $\mathcal{X}$. This follows from the fact that $\overline{\mathcal{X}} - \mathcal{X} \subset \mathcal{R}$ (set of rationals) and consequently $\overline{\mathcal{X}}$ is countable; thus $\overline{\mathcal{X}}$ is compact and countable, hence scattered (by Corollary 4 of § 34, IV).

Remark 5. More generally, for each complete, separable, 0-dimensional space $\mathcal{X}$, there is a metric and 0-dimensional compactification $\mathcal{Y}$ such that $\mathcal{Y} - \mathcal{X}$ is countable.

This statement can be proved by replacing $\mathcal{N}$ in the above argument by the set $\mathcal{C}$ minus the ends of the deleted intervals, which is homeomorphic to $\mathcal{N}$.

If $\mathcal{X}$ is compact metric, the statement I(4) can be formulated more precisely as follows.

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(1) Theorem of Knaster–Urbanik. See the paper by these authors *Sur les espaces complets séparables de dimension* 0, Fund. Math. 40 (1953), pp. 194–202.

For a detailed analysis of the problem under consideration, see Z. Semadeni *Sur les ensembles clairsemés*, Rozprawy Matem. 39 (1959), Chapter 4.

THEOREM 7. *Let $\mathscr{X}$ be compact metric and $\{F_t\}$ a directed family of closed sets relative to the inclusion* $\supset$ (see I(4)).

Then

$$\delta\Big(\bigcap_t F_t\Big) = \inf_t \delta(F_t). \tag{3}$$

Otherwise stated:

$$\Big\{\bigwedge_t \delta(F_t) \geqslant \varepsilon\Big\} \Rightarrow \Big\{\delta\Big(\bigcap_t F_t\Big) \geqslant \varepsilon\Big\}. \tag{4}$$

Proof. Let $\delta(F_t) \geqslant \varepsilon$ for each t. Hence there are x and x' such that

$$x \in F_t, \qquad x' \in F_t, \qquad |x - x'| \geqslant \varepsilon. \tag{5}$$

Put

$$K_t = \mathop{E}_{xx'}(x \in F_t)(x' \in F_t)(|x - x'| \geqslant \varepsilon). \tag{6}$$

Clearly K_t is closed in $\mathscr{X} \times \mathscr{X}$. The family $\{K_t\}$ is directed relative to $\supset$. For, $F_{t_3} \subset F_{t_1} \cap F_{t_2}$ implies $K_{t_3} \subset K_{t_1} \cap K_{t_2}$.

Since $\mathscr{X} \times \mathscr{X}$ is compact, it follows (by I(4)) that $\bigcap_t K_t \neq 0$. Let $(x_0, x_0') \in \bigcap_t K_t$, i.e., for each t, $(x_0, x_0') \in K_t$; hence by (6),

$$x_0 \in \bigcap_t F_t, \qquad x_0' \in \bigcap_t F_t \qquad \text{and} \qquad |x_0 - x_0'| \geqslant \varepsilon.$$

Consequently, $\delta\Big(\bigcap_t F_t\Big) \geqslant \varepsilon$.

Remark 6. The second part of the above argument is an easy application of formula I(i). Namely

$$\begin{aligned}\bigwedge_t \bigvee_{x,x'} &\{(|x - x'| \geqslant \varepsilon)(x \in F_t)(x' \in F_t)\} \\ &\equiv \bigvee_{x,x'} \bigwedge_t \{(|x - x'| \geqslant \varepsilon)(x \in F_t)(x' \in F_t)\} \\ &\equiv \bigvee_{x,x'} (|x - x'| \geqslant \varepsilon)\Big(x, x' \in \bigcap_t F_t\Big).\end{aligned}$$

It is known (see IV, Theorem 2) that, if f: $\mathscr{X} \to \mathscr{Y}$, where $\mathscr{Y}$ is compact, and if $I = \mathop{E}_{xy}\big(y = f(x)\big)$ is closed, then f is continuous. If $\mathscr{Y}$ is supposed to be metric, one has the more precise statement.

THEOREM 8. *Let $A \subset \mathscr{X}$ and f: $A \to \mathscr{Y}$ where $\mathscr{Y}$ is compact metric. Then*

$$\omega(x) = \delta\big[\bar{I} \cap \big((x) \times \mathscr{Y}\big)\big], \tag{7}$$

where $\omega(x)$ *denotes the oscillation of* f *at the point* x (see § 21(6)), *i.e.*

$$\omega(x) = \inf_t \delta[f(G_t)], \tag{8}$$

where $\{G_t\}$ *is the family of all open neighbourhoods of* x.

Proof. Put $F_t = \overline{f(G_t)}$. The family $\{F_t\}$ is directed relative to $\supset$. For let t_1 and t_2 be given and let $G_{t_3} = G_{t_1} \cap G_{t_2}$. Then G_{t_3} is an open neighbourhood of x and

$$\overline{f(G_{t_3})} \subset \overline{f(G_{t_1}) \cap f(G_{t_2})} \subset \overline{f(G_{t_1})} \cap \overline{f(G_{t_2})}.$$

Applying Theorem 7, we have by (3) and (8):

$$\omega(x) = \delta\big[\bigcap_t \overline{f(G_t)}\big]. \tag{9}$$

In order to prove (7) it remains to show the following statement (valid for arbitrary topological spaces $\mathscr{X}$ and $\mathscr{Y}$):

$$(x_0, y_0) \in \bar{I} \equiv y_0 \in \bigcap_t \overline{f(G_t)}, \tag{10}$$

where $\{G_t\}$ is the family of all open neighbourhoods of x_0.

Assume first that $(x_0, y_0) \in \bar{I}$. Then for each t and each open H containing y_0, there are $x_1 \in G_t$ and $y_1 \in H$ such that $(x_1, y_1) \in I$, i.e. $y_1 = f(x_1)$, hence $y_1 \in f(G_t)$ and consequently $y_0 \in \overline{f(G_t)}$. Thus $y_0 \in \bigcap_t \overline{f(G_t)}$.

Next suppose that $(x_0, y_0) \notin \bar{I}$. Then there exist t and an open H containing y_0 such that $(G_t \times H) \cap I = 0$. It follows that $y_0 \notin \overline{f(G_t)}$. For otherwise, there would be $y \in H \cap f(G_t)$, hence $y = f(x)$ for some $x \in G_t$; but then $(x, y) \in (G_t \times H) \cap I$.

The Urysohn imbedding theorem can be strengthened as follows.

THEOREM 9 [1]. *Let* $\mathscr{X} = \overline{\mathscr{X}} \subset \mathscr{I}^{\aleph_0}$ *and let* $f\colon \mathscr{X} \to \mathscr{Y}$ *be continuous and onto. Choose* σ *so that*

$$\delta[f^{-1}(y)] < \sigma \quad \textit{for each } y \in \mathscr{Y}. \tag{11}$$

Then there is an imbedding $h\colon \mathscr{Y} \to \mathscr{I}^{\aleph_0}$ *such that*

$$|hf(x) - x| < \sigma \quad \textit{for each } x \in \mathscr{X}. \tag{12}$$

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(1) See my note in Ann. Soc. Polon. Math. 17 (1938), p. 118.

First, we shall prove the following lemma.

LEMMA. *If* σ *satisfies condition* (11), *there is an* $\eta > 0$ *such that*

$$[|f(x)-f(x')| < \eta] \Rightarrow [|x-x'| < \sigma]. \tag{13}$$

Consequently, *if* $B \subset \mathscr{Y}$ *and* $\delta(B) < \eta$, *then* $\delta[f^{-1}(B)] < \sigma$.

Proof of the lemma. Suppose that η does not exist. Then there are two sequences $x_1, x_2, \ldots$ and $x_1', x_2', \ldots$ such that

$$|f(x_k)-f(x_k')| < 1/k, \tag{14}$$

$$|x_k - x_k'| \geqslant \sigma. \tag{15}$$

Since $\mathscr{X}$ is compact, we may assume that the sequences are convergent:

$$\lim_{k=\infty} x_k = x \quad \text{and} \quad \lim_{k=\infty} x_k' = x'. \tag{16}$$

Since f is continuous, it follows that

$$\lim_{k=\infty} f(x_k) = f(x) \quad \text{and} \quad \lim_{k=\infty} f(x_k') = f(x'),$$

hence by (14), $f(x) = f(x')$.

Put $y = f(x)$. Therefore $x, x' \epsilon f^{-1}(y)$ and $|x-x'| < \sigma$ by (11). But this is inconsistent with (15) and (16).

Proof of Theorem 9. Let η satisfy condition (13), and let $G_0, \ldots, G_m$ be an open cover of $\mathscr{Y}$ such that

$$\mathscr{Y} = G_0 \cup \ldots \cup G_m, \qquad \delta(G_i) < \eta/2, \qquad G_i \neq 0. \tag{17}$$

Let $x_i \epsilon f^{-1}(G_i)$. Consider the $\varkappa$-mapping determined by the systems $\{G_0, \ldots, G_m\}$ and $\{x_0, \ldots, x_m\}$ (see § 28, VI)

$$\varkappa(y) = \lambda_0(y)\cdot x_0 + \ldots + \lambda_m(y)\cdot x_m$$

where

$$\lambda_i(y) = \frac{\varrho(y, F_i)}{\varrho(y, F_0)+\ldots+\varrho(y, F_m)} \quad \text{and} \quad F_i = \mathscr{Y} - G_i.$$

We shall show that

$$|\varkappa f(x) - x| < \sigma. \tag{18}$$

Let x be given. Denote by $i_0, \ldots, i_k$ the system of all indices such that $x \epsilon f^{-1}(G_{i_j})$. Then the point $\varkappa f(x)$ belongs to the simplex

$x_{i_0} \ldots x_{i_k}$. Denote by S the simplex $xx_{i_0} \ldots x_{i_k}$. It follows that $\delta(S) < \sigma$. Because the condition $f(x) \epsilon G_{i_j} \cap G_{i_l}$ implies, by (17), $\delta(G_{i_j} \cup G_{i_l}) < \eta$, it follows that $|f(x_{i_j}) - f(x_{i_l})| < \eta$ and this yields, by (13), $|x_{i_j} - x_{i_l}| < \sigma$. On the other hand, $f(x)$ and $f(x_{i_j})$ belong to G_{i_j} and so $|x - x_{i_j}| < \sigma$. Thus $\delta(S) < \sigma$ and (18) follows since x and $\varkappa f(x)$ belong to S.

Now, let us denote by h a homeomorphism of $\mathscr{Y}$ into $\mathscr{I}^{\aleph_0}$, sufficiently close to $\varkappa$ (see § 22, II, Theorem 2); more precisely, such that

$$|h(y) - \varkappa(y)| < \sigma - |\varkappa f(x) - x| \quad \text{for each } x \epsilon \mathscr{X} \text{ and } y \epsilon \mathscr{Y}. \tag{19}$$

Condition (12) follows since by (19) we have

$$|hf(x) - x| \leqslant |hf(x) - \varkappa f(x)| + |\varkappa f(x) - x| < \sigma.$$

LEMMA. [1] *Let $\mathscr{X}$ be a topological space, $\mathscr{Y}$ a $\mathscr{T}_1$-space and $f\colon \mathscr{X} \to \mathscr{Y}$ continuous. Let $\{U_t\}$, $t \epsilon T$, be a base of neighbourhoods of a set $A\,(\neq 0)$ (which means that, whatever the open set $G \supset A$ is, there is t such that $A \subset U_t \subset G$). Then*

$$\bigcap_{t \epsilon T} f(U_t) = f\Big(\bigcap_{t \epsilon T} U_t\Big) = f(A).$$

Proof. Clearly, it is sufficient to show that $\bigcap_{t \epsilon T} f(U_t) \subset f(A)$, or equivalently, that if $y \notin f(A)$, then there is t such that $y \notin f(U_t)$.

Now, if $y \notin f(A)$, then $A \cap f^{-1}(y) = 0$, i.e. $A \subset \mathscr{X} - f^{-1}(y)$. Since $\mathscr{Y}$ is a $\mathscr{T}_1$-space and f is continuous, the set $\mathscr{X} - f^{-1}(y)$ is open. Hence there is t such that $A \subset U_t \subset \mathscr{X} - f^{-1}(y)$. Consequently $U_t \cap f^{-1}(y) = 0$, and thus $y \notin f(U_t)$.

THEOREM 10. *Let F be a compact subset of a metric space $\mathscr{X}$, $\mathscr{Y}$ a $\mathscr{T}_2$-space and let $f\colon \mathscr{X} \to \mathscr{Y}$ be continuous. Denote by S_k the generalized ball of center F and radius $1/k$, i.e.*

$$S_k = \mathop{E}_{x}\big(\varrho(x, F) < 1/k\big).$$

Then

$$\bigcap_{k=1}^{\infty} \overline{f(S_k)} = f\Big(\bigcap_{k=1}^{\infty} S_k\Big), \quad \text{i.e.} \quad = f(F).$$

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[1] I am indebted to Mrs. Karłowicz for this statement.

Proof. Since the sequence $S_1, S_2, \ldots$ is a base of neighbourhoods of F, it follows that

$$\bigcap_{k=1}^{\infty} f(S_k) = f\Big(\bigcap_{k=1}^{\infty} S_k\Big).$$

On the other hand, since $\overline{S}_k \subset S_{k-1}$ and $f(\overline{S}_k)$ is closed, it follows that

$$\overline{f(S_k)} \subset \overline{f(\overline{S}_k)} = f(\overline{S}_k) \subset f(S_{k-1}),$$

hence

$$\bigcap_{k=1}^{\infty} \overline{f(S_k)} = \bigcap_{k=1}^{\infty} f(S_k).$$

VII. Invariants under mappings with small point inverses. Quasi-homeomorphism. Let $\mathscr{X}$ be a metric space. Given a mapping $f\colon \mathscr{X} \to \mathscr{Y}$, we call *point inverses* of f the inverse images of single points of $\mathscr{Y}$, i.e. the sets $f^{-1}(y)$ (written also $\overset{-1}{f}(y)$). Put

$$\delta_f = \sup \delta[f^{-1}(y)] \quad \text{where} \quad y \in \mathscr{Y}.$$

DEFINITION 1 [1]. A property **P** of a space $\mathscr{X}$ is said to be *invariant under mappings with small point inverses*, if there is $\alpha > 0$ such that $f(\mathscr{X})$ has property **P** whenever f is a continuous mapping with $\delta_f < \alpha$.

Remark. Clearly, every invariant of mappings with small point inverses is a topological invariant. The converse is not true (even in the domain of compact spaces). For example, the inequality $\dim \mathscr{X} < n$ is not an invariant of mappings with small point inverses; on the other hand, the property $\dim \mathscr{X} \geqslant n$ is such an invariant (for $\mathscr{X}$ compact, see § 45, IV, Corollary 9).

Thus, to show that a property is invariant under mappings with small point inverses is, generally speaking, much more significant than to show merely that it is a topological invariant.

DEFINITION 2 [2]. Two compact metric spaces $\mathscr{X}_1$ and $\mathscr{X}_2$ are said to be *quasi-homeomorphic* if there are continuous mappings

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(1) P. Alexandrov, *Gestalt und Lage*, Annals of Math. 30 (1928), Chapter I.

(2) See the paper by S. Ulam and myself *Sur un coefficient lié aux transformations continues d'ensembles*, Fund. Math. 20 (1933), p. 252.

of $\mathscr{X}_1$ onto $\mathscr{X}_2$, and of $\mathscr{X}_2$ onto $\mathscr{X}_1$, with point inverses as small as we wish.

In other terms, if for each $\varepsilon > 0$ there are two continuous mappings onto, $f_1: \mathscr{X}_1 \to \mathscr{X}_2$ and $f_2: \mathscr{X}_2 \to \mathscr{X}_1$, such that $\delta_{f_1} < \varepsilon$ and $\delta_{f_2} < \varepsilon$.

Thus, for example, a ball is quasi-homeomorphic with a union of two balls having a single point in common (¹).

The problem of proving invariance under mappings with small point inverses can be reduced to the consideration of mappings g such that $|g(x) - x| < \varepsilon$ for sufficiently small ε. Namely the following is true.

THEOREM (²). *Let* $\mathscr{X} = \overline{\mathscr{X}} \subset \mathscr{I}^{\aleph_0}$. *Every topological property* **P** *invariant under continuous mappings* $g: \mathscr{X} \to \mathscr{I}^{\aleph_0}$ *such that* $|g(x) - x| < \varepsilon$ *for each* $x \in \mathscr{X}$ *is invariant under continuous mappings* $f: \mathscr{X} \to \mathscr{Y}$ *such that* $\delta_f < \varepsilon$.

Proof. According to Theorem 8 of Section VI, let h be a homeomorphism of $\mathscr{Y}$ into $\mathscr{I}^{\aleph_0}$ such that $|hf(x) - x| < \varepsilon$. Put $g = h \circ f$. Then, by assumption, the set $g(\mathscr{X})$ has property **P**. As $g(\mathscr{X})$ is homeomorphic to $f(\mathscr{X})$, and **P** is topological, it follows that $f(\mathscr{X})$ also has property **P**.

REMARKS AND EXAMPLES. 1. Given two compact metric spaces $\mathscr{X}$ and $\mathscr{Y}$ let us consider the following coefficient:

$$\tau(\mathscr{X}, \mathscr{Y}) = \inf \delta_f, \quad \text{where} \quad f: \mathscr{X} \to \mathscr{Y} \text{ is continuous}.$$

Thus $\tau(\mathscr{X}, \mathscr{Y})$ is the largest number such that for each continuous $f: \mathscr{X} \to \mathscr{Y}$ there are two points x_1 and x_2 satisfying the conditions

$$f(x_1) = f(x_2) \quad \text{and} \quad |x_1 - x_2| \geqslant \tau(\mathscr{X}, \mathscr{Y}).$$

Clearly $\mathscr{X}$ and $\mathscr{Y}$ are quasi-homeomorphic if and only if $\tau(\mathscr{X}, \mathscr{Y}) = 0 = \tau(\mathscr{Y}, \mathscr{X})$.

(¹) As remarked, quasi-homeomorphism does not imply homeomorphism. However, in the domain of closed, orientable 2-manifolds, these two notions are equivalent. See M. K. Fort, Jr., *ε-mappings of a disc onto a torus*, Bull. Polish Acad. Sc. 7 (1959), p. 53.

(²) See S. Eilenberg, *Sur l'invariance par rapport aux petites transformations*, C. R. Paris 200 (1935), p. 1003.

2. Let $\mathcal{Q}_n$ be the n-dimensional ball defined by the equation $|p| \leqslant 1$, and $\mathcal{S}_n$ the n-dimensional sphere of radius 1. Then (1)

$$\tau(\mathcal{Q}_n, \mathcal{S}_n) = \frac{2n+2-\sqrt{2n^2+2n}}{n+2}.$$

3. The well-known theorem of Borsuk–Ulam (2) stating that for each continuous mapping of $\mathcal{S}_n$ into $\mathcal{E}^n$ there are two antipodal points of $\mathcal{S}_n$ which are mapped into the same point of $\mathcal{E}^n$, can be expressed as follows:

$$\tau(\mathcal{S}_n, X) = \delta(\mathcal{S}_n) = 2$$

for each compact $X \subset \mathcal{E}^n$.

4. Let $\mathcal{T}$ be a torus and $f\colon \mathcal{Q}_2 \to \mathcal{T}$ continuous. Then (3) there is a point inverse of diameter $\geqslant 1/6$.

VIII. Relationships to Boolean rings(4). Let us recall that a (non-void) set A in which the operations of addition and of multiplication are defined is called a *ring* if it is an abelian group relative to the addition and if the multiplication is associative and distributive relative to the addition.

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(1) See my paper *Sur les transformations des sphères en des surfaces sphériques*, Fund. Math. 20 (1933), p. 206.

(2) See K. Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. 20 (1933), p. 177.

See also § 57, I and § 59, V.

(3) See Fort l.c. and T. Ganea, *On ε-maps onto manifolds*, Fund. Math. 47 (1959), pp. 35–44. Compare Problem 21 of S. Ulam in the Scottish Book.

See also M. Wojdysławski, *Some applications of a criterion for a continuum to be planar* (Russian), Matem. Sb. 18 (1946), pp. 29–40; this paper contains a number of statements about quasi-homeomorphisms.

(4) I am indebted to Professor Mostowski for this section. The theory presented here is due to M. H. Stone. See *The theory of representations for Boolean algebras*, Trans. Amer. Math. Soc. 40 (1936), pp. 37–111. This theory can be extended to distributive lattices without being essentially modified; see G. Birkhoff, *Lattice theory*, 2nd ed., New York 1948, Chapter IX, §§ 5 and 6, containing an extended bibliography. See also R. Sikorski, *Boolean algebras*, 2nd ed., Springer 1964.

For a generalization connected with the papers of I. Gelfand on normed rings, see W. Słowikowski and W. Zawadowski, *A generalization of maximal ideals method of Stone and Gelfand*, Fund. Math. 42 (1955), pp. 215–231.

The ring A is called a *Boolean ring* if it contains the element 1 such that $1a = a$ and if $aa = a$ for each a.

Such, for example, is the family of all subsets of a given set, the multiplication being understood as the intersection and the addition as the symmetric difference.

Clearly the following is true of every Boolean ring:

$$ab = ba \quad \text{and} \quad a + a = 0, \quad \text{hence} \quad -a = a.$$

A (non-empty) subset I of a Boolean ring A is called an *ideal* (comp. § 1, VII) if

$$(a \epsilon I,\ b \epsilon I) \Rightarrow [(a+b) \epsilon I], \tag{1}$$

$$a \epsilon I \Rightarrow (ab) \epsilon I \quad \text{for each } b. \tag{2}$$

If $B \subset A$, then the set of elements of the form

$$a_1 x_1 + \ldots + a_n x_n, \quad \text{where} \quad a_1, \ldots, a_n \epsilon B,$$

is the smallest ideal containing B (it is said to be *generated* by B).

One can prove (with the help of the axiom of choice) that each ideal ($\neq A$) is contained in a *maximal* ideal $I \neq A$.

Obviously $0 \epsilon I$ and $1 \epsilon A - I$.

One easily sees that if I is a maximal ideal and $a \epsilon A - I$, then each element of A is of the form

$$i + ax \quad \text{where} \quad i \epsilon I \text{ and } x \epsilon A.$$

It follows that if $1 \epsilon A$ then

$$(a \epsilon A - I) \equiv \bigvee_x (1 - ax) \epsilon I. \tag{3}$$

We shall deduce from here the following lemma.

LEMMA. *If I is a maximal ideal of a Boolean ring A, then*

$$(a \epsilon A - I) \equiv (1 + a \epsilon I). \tag{4}$$

Proof. Clearly

$$(1 - ax)(1 + a) = 1 + a + a(1 + a)x = 1 + a + (a + aa)x$$
$$= 1 + a.$$

Hence, if $a \epsilon A - I$, we have by (3), $1 - ax \epsilon I$. Therefore $1 + a \epsilon I$ according to (2).

The converse implication follows directly from (3).

THEOREM (of M. H. Stone). *Let A be a Boolean ring and let $\boldsymbol{M}$ denote the family of its maximal ideals. We introduce a topology in $\boldsymbol{M}$ by assuming that the sets*

$$\boldsymbol{G}_a = \underset{I}{E}(a \in A - I), \quad \textit{where } a \textit{ ranges over } A, \tag{5}$$

form an open base of $\boldsymbol{M}$.

Then, with this definition, $\boldsymbol{M}$ becomes a compact 0-dimensional $\mathscr{T}_2$-space.

Proof. First we shall prove that

$$\boldsymbol{G}_a \cap \boldsymbol{G}_b = \boldsymbol{G}_{ab}. \tag{6}$$

By (5) and (2) we have

$$I \in \boldsymbol{G}_{ab} \Rightarrow ab \in A - I \Rightarrow (a \in A - I)(b \in A - I) \Rightarrow I \in \boldsymbol{G}_a \cap \boldsymbol{G}_b.$$

Conversely, let $I \in \boldsymbol{G}_a \cap \boldsymbol{G}_b$. Then by (5), $a, b \in A - I$ and hence

$$1 = i_1 + ax_1 = i_2 + bx_2 \quad \text{where} \quad i_1, i_2 \in I. \tag{7}$$

Since $1 \in A - I$, it follows from (7) that

$$(i_1 i_2 + i_1 bx_2 + i_2 ax_1 + abx_1 x_2) \in A - I.$$

Since I is an ideal, it follows that $ab \in A - I$, which yields $I \in \boldsymbol{G}_{ab}$ (by (5)).

This completes the proof of (6).

Formula (5) implies that the intersection of two sets of the form (5) is again of the same form. Consequently if the family of such sets is taken as an open base of $\boldsymbol{M}$, $\boldsymbol{M}$ becomes a *topological space.*

We shall show that this space is a $\mathscr{T}_2$*-space.*

Let $I, J \in \boldsymbol{M}$ and $a \in I - J$. Hence by (5), $J \in \boldsymbol{G}_a$ and $I \in \boldsymbol{G}_{1+a}$ since $(1 + a) \in (A - I)$ (by (4)).

Furthermore, by (6)

$$\boldsymbol{G}_a \cap \boldsymbol{G}_{1+a} = \boldsymbol{G}_0 = 0. \tag{8}$$

Next, we shall prove that $\boldsymbol{M}$ is *compact.* Let

$$\boldsymbol{M} = \bigcup_{b \in B} \boldsymbol{G}_b \quad \text{where} \quad B \subset A. \tag{9}$$

Let J be an ideal generated by B.

We have $1 \in J$. For suppose that $1 \notin J$. Then there exists a maximal ideal $I \supset J$ and we have $I \in \boldsymbol{M}$ (by the definition of $\boldsymbol{M}$) and, on the other hand, $b \in I$, hence $I \notin \boldsymbol{G}_b$ for each $b \in B$. But this contradicts formula (9).

Since $1 \in J$, we have

$$1 = b_1 x_1 + \ldots + b_n x_n. \tag{10}$$

We shall show that

$$\boldsymbol{M} = \boldsymbol{G}_{b_1} \cup \ldots \cup \boldsymbol{G}_{b_n}, \tag{11}$$

which will complete the proof of the compactness of $\boldsymbol{M}$.

Now, let $I \in \boldsymbol{M}$. As $1 \notin I$, there is (by (10)) $k \leqslant n$ such that $b_k \notin I$. It follows by (5) that $I \in \boldsymbol{G}_{b_k}$. Thus (11) is proved.

Finally, $\boldsymbol{M}$ is *0-dimensional*, i.e. it contains a base composed of closed-open sets. Such is the family of sets $\boldsymbol{G}_a$ with $a \in A$. We have

$$\boldsymbol{M} - \boldsymbol{G}_a = \boldsymbol{G}_{1+a} \tag{12}$$

because of formula (8) and the identity $\boldsymbol{G}_a \cup \boldsymbol{G}_{1+a} = \boldsymbol{M}$, which follows from (4) and (5); namely

$$I \notin \boldsymbol{G}_a \equiv a \in I \equiv 1 + a \in A - I \equiv I \in \boldsymbol{G}_{1+a}.$$

Remark. *The Boolean ring A is isomorphic to the ring $\mathbf{G}$ of all closed-open subsets of $\boldsymbol{M}$.*

This isomorphism is defined by putting

$$f(a) = \boldsymbol{G}_a \quad \text{where} \quad a \in A;$$

otherwise stated:

$$f(A) = \mathbf{G}, \tag{13}$$

$$f(a+b) = [f(a) - f(b)] + [f(b) - f(a)], \tag{14}$$

$$f(ab) = f(a) \cap f(b), \tag{15}$$

$$a \neq b \Rightarrow f(a) \neq f(b). \tag{16}$$

Proof. In order to prove (13), it suffices—as $\boldsymbol{G}_a$ is closed-open—to show that, for each $\boldsymbol{F} \in \mathbf{G}$, there is a such that $\boldsymbol{F} = \boldsymbol{G}_a$.

Now, as $\boldsymbol{F}$ is compact and closed-open (in $\boldsymbol{M}$), $\boldsymbol{F}$ is a finite union of sets $\boldsymbol{G}_x$. Thus we have to show that $\boldsymbol{G}_a \cup \boldsymbol{G}_b = \boldsymbol{G}_c$ for a certain c. We shall show in fact that

$$\boldsymbol{G}_a \cup \boldsymbol{G}_b = \boldsymbol{G}_c \quad \text{where} \quad c = a + b + ab. \tag{17}$$

Since the formula $I \in G_a$ implies $a \in A-I$, hence $c \in A-I$ (since $a = ca$), it follows that $I \in G_c$. Thus $G_a \subset G_c$, hence $G_a \cup G_b \subset G_c$.

Conversely, if $I \notin (G_a \cup G_b)$, we have $a, b \in I$, hence $c \in I$ (by (1) and (2)) and therefore $I \notin G_c$. This completes the proof of (17).

In order to prove (14), let us replace a in (17) by $a+ab$ and b by $b+ab$. Clearly

$$(a+ab)(b+ab) = 0 \quad \text{and} \quad (a+ab)+(b+ab) = a+b.$$

It follows by (17) that

$$G_{a+ab} \cup G_{b+ab} = G_{a+b}.$$

Applying (6) and (12) we obtain

$$G_{a+b} = (G_a \cap G_{1+b}) \cup (G_b \cap G_{1+a}) = (G_a - G_b) \cup (G_b - G_a). \tag{18}$$

This is equivalent to (14).

(15) is equivalent to (6).

Finally, the formula $G_a = G_b$ implies, by (18), $G_{a+b} = 0$, hence $a+b = 0$ and therefore $a = b$.

IX. Dyadic spaces(¹).

DEFINITION (²). A space is called *dyadic* if it is a continuous image of a generalized Cantor discontinuum $D^{\mathfrak{m}}$ (where D is the two-element set $(0,1)$).

(¹) I am indebted to Professor Engelking for this section.

(²) See P. S. Alexandrov, *Zur Theorie der topologischen Räume*, Dokl. Akad. Nauk SSSR 2 (1936), pp. 51–54.

The above definition is "exterior". An "interior" definition is rather complicated; see: P. Alexandrov and V. Ponomarev, *On dyadic bicompacta*, Fund. Math. 50 (1962), pp. 419–429.

For the theory of dyadic spaces, see, in addition to the papers cited below: B. Jefimov, *On the weight structure of dyadic bicompacta* (Russian), Vestnik Mosc. Univ. 1 (1964), pp. 3–11, and *Dyadic bicompacta* (Russian), Trudy Mosc. Math. Soc. 14 (1965), pp. 211–247; B. Jefimov and R. Engelking, *Remarks on dyadic spaces*, II, Colloq. Math. 13 (1965), pp. 181–197; S. Mardešić and P. Papić, *Continuous images of ordered compacta, the Suslin property and dyadic compacta*, Glasnik 17 (1962), pp. 3–25.

By Corollary 2b of Section VI, every compact metric space is a continuous image of the Cantor discontinuum $\mathscr{C} = D^{\aleph_0}$, hence is dyadic. It can also be easily shown that every compact $\mathscr{T}_2$-space of weight $\mathfrak{m} \geqslant \aleph_0$ (compare § 5, XI, Remark 3, p. 53) is a continuous image of a *closed subset* of $D^{\mathfrak{m}}$. It does not need however to be dyadic (see Example below).

THEOREM 1. *Every family of disjoint open subsets of $D^{\mathfrak{m}}$ is countable.*

Proof. Clearly, we may assume that the members of the family under consideration belong to a base of $D^{\mathfrak{m}}$. Thus, we may assume that they are of the form

$$G = G_{t_1\ldots t_n}^{j_1\ldots j_n} = \bigcap_{i=1}^{n} G_{t_i,\{j_i\}} = \bigcap_{i=1}^{n} \mathop{E}_{z}\,(z^{t_i} = j_i), \tag{1}$$

where $t_i \in T$, $\overline{\overline{T}} = \mathfrak{m}$, $j_i = 0, 1$ for $i = 1, \ldots, n$. Let us call n the *length* of the set G and the system $t_1, \ldots, t_n$ the *system of distinguished indices.* It is easy to see that two disjoint members of the base must have a common distinguished index and different projections on the corresponding axis.

For proving the theorem it suffices to show that, for a given n, every family of disjoint sets of length n is countable. We shall proceed by induction. For $n = 1$, our statement follows from the property just mentioned of disjoint members of the base.

Suppose this is true for $n-1$. Let $\boldsymbol{G}$ be the family of sets of length n. Suppose that $\boldsymbol{G}$ is uncountable. Let $G_0 \in \boldsymbol{G}$. Each member of $\boldsymbol{G}$ is disjoint from G_0, and hence has a distinguished index in common with G_0. Therefore $\boldsymbol{G} = \boldsymbol{G}_1 \cup \ldots \cup \boldsymbol{G}_n$, where the members of $\boldsymbol{G}_i$ have a common distinguished index. As $\boldsymbol{G}$ is uncountable, we may assume that so is $\boldsymbol{G}_1$; let t be the common distinguished index of members of $\boldsymbol{G}_1$. Hence either for $i = 0$ or for $i = 1$, there is an uncountable $\boldsymbol{H}_1 \subset \boldsymbol{G}_1$ such that the projection of members of $\boldsymbol{H}_1$ on the t-axis is (i). It follows that any two members of $\boldsymbol{H}_1$ have still another common distinguished index and their projections on the corresponding axis are disjoint. Let $\boldsymbol{K}_1$ be obtained from $\boldsymbol{H}_1$ by omitting (in formula (1)) the index t (and the corresponding upper index i). Therefore $\boldsymbol{K}_1$ is uncountable. But this is contradictory since the length of its members is $n-1$.

Remark 1. It is easily seen, by the above argument, that Theorem 1 is true for every cartesian product of spaces with a countable base ([1]).

Remark 2. Theorem 1 follows also from the property of $D^{\mathfrak{m}}$ of being a compact topological group and of admitting, consequently, a Haar measure ([2]).

Note that *each compact topological group is dyadic* ([3]).

COROLLARY 1. *In a dyadic space, each family of disjoint open sets is countable.*

EXAMPLE. Let $\mathscr{X}$ be a one-point compactification of an uncountable discrete space (see Theorem 5 of Section X) and let us assume that open sets are defined to be the complements of finite sets. Clearly $\mathscr{X}$ is a compact non-dyadic $\mathscr{T}_2$-space.

Remark 3. It follows from the existence of non-dyadic compact spaces that $D^{\mathfrak{m}}$ for $\mathfrak{m} > \aleph_0$ contains closed subsets which are not retracts of $D^{\mathfrak{m}}$ and not even continuous images of $D^{\mathfrak{m}}$.

Note that a closed subset of $D^{\mathfrak{m}}$ can be a continuous image of $D^{\mathfrak{m}}$ without being its retract ([4]).

Given a cartesian product $Z = \underset{t\in T}{\boldsymbol{P}} \mathscr{X}_t$, we denote by $\pi_{T'}$ for $T' \subset T$, its projection on the partial product $Z' = \underset{t\in T'}{\boldsymbol{P}} \mathscr{X}_t$ (i.e. $\pi_{T'}$ assigns to each mapping $z \in Z$ the partial mapping $(z|T') \in Z'$).

THEOREM 2. ([5]) *Let $\mathscr{X}_t$ be separable for $t \in T$, and let $\mathscr{Y}$ be a $\mathscr{T}_2$-space whose points are G_δ-sets. Put $Z = \underset{t}{\boldsymbol{P}} \mathscr{X}_t$ and let $f\colon Z \to \mathscr{Y}$ be con-*

([1]) This statement was proved by E. Marczewski (Szpilrajn). See *Remarque sur les produits cartésiens d'espaces topologiques*, Dokl. Akad. Nauk SSSR 31 (1941), p. 525. For stronger results, see the paper by the same author *Séparabilité et multiplication cartésienne des espaces topologiques*, Fund. Math. 34 (1947), pp. 127-143; and N. Šanin, *On products of topological spaces* (Russian), Trudy Mat. Inst. Steklova 24 (1948).

([2]) See e.g. P. R. Halmos, *Measure Theory*, Chapter 9, New York 1950.

([3]) See L. Ivanovskiĭ, *On a conjecture of P. S. Alexandrov*, Doklady Akad. Nauk SSSR 123 (1958), p. 785, and V. J. Kuzminov, *On a conjecture of P. S. Alexandrov in the theory of topological groups*, *ibid.* 125 (1959), p. 727.

([4]) This answers a question raised by P. R. Halmos. See R. Engelking, *Cartesian products and dyadic spaces*, Fund. Math. 57 (1965), pp. 287–304.

([5]) Theorem of A. M. Gleason. See J. R. Isbell, *Uniform spaces*, Providence 1964, p. 130. For a similar theorem, see S. Mazur, *On continuous images of cartesian products*, Fund. Math. 39 (1952), pp. 229–238, and R. Engelking, *On functions defined on Cartesian products*, Fund. Math. 59 (1966), pp. 221–231.

tinuous. Then there exist a countable $T' \subset T$ and $f'\colon Z' \to \mathcal{Y}$ continuous and such that $f = f' \circ \pi_{T'}$.

Proof. For each $z \in Z$, the set $f^{-1}[f(z)]$ is G_δ. It contains therefore a countable intersection of members of the base of Z containing z. Consequently, there is a countable set $T(z) \subset T$ such that

$$\text{if } \pi_{T(z)}(x) = \pi_{T(z)}(z) \text{ for } x \in Z, \text{ then } f(x) = f(z). \tag{2}$$

We shall define by induction a sequence $Z_0, Z_1, \ldots$ of countable subsets of Z such that

$$\overline{\pi_{T_i}(Z_{i+1})} = \underset{t \in T_i}{P} \mathcal{X}_t \tag{3}$$

where

$$T_i = \bigcup_{j=0}^{i} \bigcup_{z \in Z_j} T(z). \tag{4}$$

Let $Z_0 = (z_0)$ where z_0 is an arbitrary point of Z (we may suppose Z non-empty). Now, suppose that $Z_0, \ldots, Z_k$ satisfy condition (3). According to (4) the set T_k is countable and consequently the product $\underset{t \in T_k}{P} \mathcal{X}_t$ contains a countable dense subset. Hence there exists a countable Z_{k+1} satisfying (3) for $i = k$.

Put $T' = T_0 \cup T_1 \cup \ldots$. In order to show the existence of f' such that $f = f' \circ \pi_{T'}$, it remains to prove that for any x_1 and x_2 in Z

$$[\pi_{T'}(x_1) = \pi_{T'}(x_2)] \Rightarrow [f(x_1) = f(x_2)] \tag{5}$$

(note that $\pi_{T'}$ is an open mapping and hence f' is continuous).

Suppose $f(x_1) \neq f(x_2)$. Let U_1 and U_2 be open subsets of $\mathcal{Y}$ such that

$$f(x_1) \in U_1, \quad f(x_2) \in U_2 \quad \text{and} \quad U_1 \cap U_2 = 0. \tag{6}$$

Since $x_i \in f^{-1}(U_i)$ for $i = 1, 2$, x_i belongs to a product of the form $\underset{t \in T}{P} U_t^i$ where U_t^i is open in $\mathcal{X}_t$ and $U_t^i = \mathcal{X}_t$, except for a finite number of elements t of T and $\underset{t \in T}{P} U_t^i \subset f^{-1}(U_i)$. If the left-hand side member of (5) is supposed to be true, we may assume that $U_t^1 = U_t^2$ for $t \in T'$. Then there is by (3) a point $z \in (Z_0 \cup Z_1 \cup \ldots)$ such that $z_t \in U_t^1$ for $t \in T'$, and it follows by (2) and (4) that $f(z_1) = f(z) = f(z_2)$ whenever $z_i^t = z^t$ for $t \in T'$ and $z_i^t = x_i^t$ for $t \in T - T'$.

Since $z_i \in \mathop{\boldsymbol{P}}_{t\in T} U^i_t$, we have $U_1 \cap U_2 \neq 0$. But this contradicts (6).

THEOREM 3. *Let f and Z be as above. If $\mathscr{Y}$ is a completely regular $\mathscr{T}_1$-space of weight $\mathfrak{m}$, T' may be supposed to have a power $\leqslant \mathfrak{m}$.*

Proof. By hypothesis, $\mathscr{Y}$ can be considered as a subset of $\mathscr{I}^{\mathfrak{m}}$ (compare Theorem 5 of § 16, V, p. 155). Let us write $\mathscr{I}^{\mathfrak{m}} = \mathop{\boldsymbol{P}}_{s\in S} I_s$ where $I_s = \mathscr{I}$ and $\overline{\overline{S}} = \mathfrak{m}$. By Theorem 2 there are a countable set $T_s \subset T$ and a mapping $f_s\colon \mathop{\boldsymbol{P}}_{t\in T_s} \mathscr{X}_t \to \mathscr{I}$ such that $\pi_s \circ f = f_s \circ \pi_{T_s}$ (where π_s is the projection of $\mathscr{I}^{\mathfrak{m}}$ onto I_s). In order to complete the proof, put $T' = \bigcup_{s\in S} T_s$ and let f' be the mapping which has $f_s \circ \pi_{T_s}$ as its sth coordinate.

THEOREM 4 (1). *Each dyadic space $\mathscr{X}$ of weight $\mathfrak{m}$ is a continuous image of $D^{\mathfrak{m}}$.*

Proof. Since $\mathscr{X}$ is dyadic, there are a cardinal $\mathfrak{n} \geqslant \mathfrak{m}$ and a continuous mapping $f\colon D^{\mathfrak{n}} \to \mathscr{X}$. Put $D^{\mathfrak{n}} = \mathop{\boldsymbol{P}}_{t\in T} D_t$ where $D_t = D$ and $\overline{\overline{T}} = \mathfrak{n}$. By Theorem 3 there are $T' \subset T$ and $f'\colon \mathop{\boldsymbol{P}}_{t\in T'} D_t \to \mathscr{X}$ such that $\overline{\overline{T'}} = \mathfrak{m}$ and $f = f' \circ \pi_{T'}$, where f' is continuous. Clearly f' is the required mapping of $D^{\mathfrak{m}}$ onto $\mathscr{X}$.

COROLLARY 2 (2). *Each dyadic space with a countable local base* (see § 5, XI, p. 54) *is a continuous image of the Cantor discontinuum $\mathscr{C}$.*

This is a corollary of Theorem 2.

Remark 3. It follows from Theorem 3, VI, that, if a $\mathscr{T}_2$-space is a continuous image of $\mathscr{C}$, then it is compact and metrizable and hence contains a countable base. Consequently, if a dyadic space has a local countable base, then it has a countable base.

Remark 4. It is easily seen that the totality of all dyadic spaces is the smallest totality of all $\mathscr{T}_2$-spaces containing finite

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(1) Compare the paper of N. Šanin cited above. For the proof given here, see R. Engelking and A. Pełczyński, *Remarks on dyadic spaces*, Colloq. Math. 11 (1963), pp. 55–63.

(2) See A. Jesenin-Volpin, *On the interdependence between local and integral weight in dyadic bicompacta* (Russian), Dokl. Akad. Nauk SSSR 68 (1940), p. 441. For stronger statements, see B. Jefimov, *On dyadic spaces, ibid.* 151 (1963), p. 1021, and *Metrizability and Σ-products of bicompacta, ibid.* 152 (1963), p. 794.

spaces (or, equivalently, containing metrizable compact spaces) and closed relative to the operations of cartesian multiplication and continuous mappings.

THEOREM 5 [1]. *Each closed G_δ-subset of a dyadic space is dyadic.*

Proof. Clearly, it is sufficient to show that every G_δ-set A closed in $D^{\mathfrak{m}}$ is a retract of $D^{\mathfrak{m}}$. By the Theorem of Vedenisov (§ 14, VI, p. 134) there is a continuous map $f\colon D^{\mathfrak{m}} \to \mathscr{I}$ such that $f^{-1}(0) = A$. Put $D^{\mathfrak{m}} = \underset{t\in T}{\boldsymbol{P}} D_t$. By Theorem 2 there are a countable set $T' \subset T$ and a continuous $f_1\colon \underset{t\in T}{\boldsymbol{P}} D_t \to \mathscr{I}$ such that $f = f_1 \circ \pi_{T'}$. Therefore $A = f_1^{-1}(0) \times \underset{t\notin T'}{\boldsymbol{P}} D_t$. Since $f_1^{-1}(0)$ is a retract of $\underset{t\in T'}{\boldsymbol{P}} D_t$, it follows that A is a retract of $D^{\mathfrak{m}}$.

Remark 5. Theorem 5 cannot be extended to *closed* subsets of dyadic spaces.

Consider namely the subset F of $D^{\aleph_1}$ composed of points having at most one coordinate equal to 1. Then F is closed but it is not dyadic, since it is homeomorphic to a one-point compactification of a discrete space of power $\aleph_1$.

Remark 6. As mentioned above, the space $D^{\aleph_1}$ contains the compactification of a discrete space of cardinality $\aleph_1$. This statement can be generalized as follows.

THEOREM 6 [2]. *Let $\mathscr{X}$ be a dyadic space. If the smallest cardinality of a base at the point $x \in \mathscr{X}$ is $\mathfrak{m}$ ($\geqslant \aleph_0$), then there is a discrete $M \subset \mathscr{X}$ such that $\overline{\overline{M}} = \mathfrak{m}$ and $M \cup (x)$ is compact.*

It follows [3] (compare Remark 3) that *all closed subsets of a dyadic space $\mathscr{X}$ are dyadic if and only if $\mathscr{X}$ is metrizable.*

X. Locally compact spaces.

DEFINITION. A space is called *locally compact at the point p* if there is a compact neighbourhood of p; in other words, if there is an open G such that $x \in G$ and $\overline{G}$ is compact.

A space is called *locally compact* if it is locally compact at each of its points.

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(1) See B. Jefimov, *On dyadic bicompacta*, Dokl. Akad. Nauk URSS 149 (1963), p. 1011. For the proof given here, see R. Engelking and A. Pełczyński, *l. cit.* For stronger results, see R. Engelking, *op. cit.* Fund. Math. 57.

(2) For the proof, see B. Jefimov *op. cit.* and R. Engelking *op. cit.*

(3) B. Jefimov *ibid.*

EXAMPLES. The n-dimensional euclidean space is locally compact. Every discrete space is locally compact.

THEOREM 1. *Each closed subset of a locally compact space is locally compact.*

Proof. Let $F = \overline{F} \subset \mathscr{X}$ and let $x \in F$. By assumption, there is an open G such that $x \in G$ and $\overline{G}$ is compact. Now, $G \cap F$ is open in F and the set $\overline{G \cap F} \cap F$ is the closure of $G \cap F$ relative to F. This set is a closed subset of the compact set $\overline{G}$; hence is itself compact (by Theorem 2 of Section II).

THEOREM 2. *Each locally compact $\mathscr{T}_2$-space is completely regular.*

Moreover, if $C \subset \mathscr{X}$ is compact and $F \subset \mathscr{X}$ is closed and $C \cap F = 0$, then there is a continuous $f\colon \mathscr{X} \to \mathscr{I}$ such that

$$f(x) = 0 \text{ for } x \in C, \quad f(x) = 1 \text{ for } x \in F \tag{1}$$

and $f^{-1}([0, a])$ is compact for each $a < 1$.

Proof. For each $x \in C$, let G_x be open and such that $x \in G_x$ and $\overline{G}_x$ is compact. Let $x_1, \ldots, x_k$ be a finite set of points such that $C \subset G = G_{x_1} \cup \ldots \cup G_{x_k}$. Clearly $\overline{G}$ is compact and $\mathscr{T}_2$ (compare II, Theorem 5), hence normal (by II, Theorem 3). Consequently there is a continuous $g\colon \overline{G} \to \mathscr{I}$ such that

$$g(x) = 0 \text{ for } x \in C \quad \text{and} \quad g(x) = 1 \text{ for } x \in [(\overline{G} \cap F) \cup (\overline{G} - G)].$$

Put $f(x) = g(x)$ for $x \in \overline{G}$ and $f(x) = 1$ for $x \notin \overline{G}$. Clearly f is continuous and satisfies condition (1). Finally, if $f(x) < 1$, then $x \in \overline{G}$ and thus, for $a < 1$, the set $f^{-1}([0, a])$ is a closed subset of the (compact) set $\overline{G}$; hence it is compact.

THEOREM 3. *Each open subset of a locally compact $\mathscr{T}_2$-space is locally compact.*

Proof. Let $H \subset \mathscr{X}$ be open and let $x \in H$. By assumption there is an open G such that $x \in G$ and $\overline{G}$ is compact. By Theorem 2, $\mathscr{X}$ is regular. Hence there is an open U such that $x \in U$ and $\overline{U} \subset G \cap H$. Thus U is open relative to H and $\overline{U}$ is compact since $\overline{U} \subset \overline{G}$.

THEOREM 4. *Let $\mathscr{X}$ be a regular $\mathscr{T}_1$-space and $A \subset \mathscr{X}$. If A is locally compact, then A is locally closed in $\mathscr{X}$.*

Consequently, *A is the difference of two closed sets, and hence the set $\overline{A} - A$ is closed.*

Proof. Let $p \in A$. By assumption there is a compact neighbourhood U of p relative to A. Hence there is a neighbourhood E

of p (relative to $\mathscr{X}$) such that $U = E \cap A$ (one can put $E = (\mathscr{X} - \overline{A - U}) \cup U$). Since U is closed, this means that A is locally closed at p (see § 7, V, Theorem 2, p. 66).

The second part of the theorem is a consequence of the first (see § 7, V, Corollary).

THEOREM 5 (the one-point compactification Theorem of Alexandrov) (1). *Each locally compact $\mathscr{T}_2$-space $\mathscr{X}$ is homeomorphic to a subset X_0 of a compact $\mathscr{T}_2$-space $\mathscr{X}^*$ such that $\mathscr{X}^* - X_0$ consists of a single point.*

If $\mathscr{X}$ is metric separable, so is $\mathscr{X}^*$.

Proof. By Theorem 2, $\mathscr{X}$ is completely regular and hence, by Theorem 5 of § 16, V, p. 155, $\mathscr{X}$ can be regarded as a subset of a compact $\mathscr{T}_2$-space C ($\subset$ a generalized cube). Moreover, one can assume that $\mathscr{X}$ is dense in C, i.e. that $C = \overline{\mathscr{X}}$. By Theorem 4, $C - \mathscr{X}$ is closed. Hence there exist, by Theorem 4 of Section III, a compact $\mathscr{T}_2$-space $\mathscr{X}^*$, a point $p \in \mathscr{X}^*$, and a continuous $f\colon C \to \mathscr{X}^*$ such that $f(C - \mathscr{X}) = (p)$ and f is a homeomorphism of $\mathscr{X}$ onto $\mathscr{X}^* - (p)$.

This completes the proof of the first part of the theorem.

If $\mathscr{X}$ is metric separable, then by the Urysohn Theorem, C can be assumed to be a closed subset of the Hilbert cube. Since $\mathscr{X}^*$ is a continuous image of the compact metric space C, $\mathscr{X}^*$ is metrizable by Theorem 3 of Section VI.

Remark 2. Another proof of the first part of Theorem 6, connected with the well-known procedure of adjoining the point at infinity to the euclidean space $\mathscr{E}^n$, is as follows.

Let p be a point not belonging to $\mathscr{X}$. Put $\mathscr{X}^* = \mathscr{X} \cup (p)$ and define a topology in $\mathscr{X}^*$ by taking as members of the open base of $\mathscr{X}^*$:

(i) the open subsets of $\mathscr{X}$,

(ii) sets of the form $(p) \cup (\mathscr{X} - F)$ where $F \subset \mathscr{X}$ is compact.

We must show that $\mathscr{X}^*$ is compact. Let $\{G_t\}$, $t \in T$, be an open cover of $\mathscr{X}^*$. Let $p \in G_{t_0}$. Hence $\mathscr{X} - G_{t_0}$ is compact and so $\{G_t\}$ contains a finite cover $G_{t_1}, \ldots, G_{t_n}$ of $\mathscr{X} - G_{t_0}$. It follows that $G_{t_0}, G_{t_1} \ldots, G_{t_n}$ is a cover of $\mathscr{X}^*$.

Remark 3. By Theorems 3 and 5, *the concepts of locally compact $\mathscr{T}_2$-space and open subset of a compact $\mathscr{T}_2$-space are topologically equivalent.*

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(1) See Alexandrov–Hopf, *Topologie* I, p. 93.

This equivalence leads immediately to the following statements (as well as to Theorem 3):

THEOREM 6 [1]. *If $\mathcal{X}$ is a locally compact $\mathcal{T}_2$-space, then there is a one-to-one continuous mapping of $\mathcal{X}$ onto a compact $\mathcal{T}_2$-space.*

This follows from Corollary 4a of Section III.

THEOREM 7. *If $\mathcal{X}$ and $\mathcal{Y}$ are locally compact $\mathcal{T}_2$-spaces, so is $\mathcal{X} \times \mathcal{Y}$.*

This follows from the invariance of compactness and openness relative to cartesian multiplication (see IV, Theorem 3 and § 15, VII, Theorem 1).

Remark 4. With regard to infinite cartesian products, the following is true.

$\underset{t}{\boldsymbol{P}}\mathcal{X}_t$ is a locally compact $\mathcal{T}_2$-space if and only if each $\mathcal{X}_t$ is a locally compact $\mathcal{T}_2$-space and all $\mathcal{X}_t$—except a finite number—are compact ($\mathcal{X}_t$ are supposed non-empty).

THEOREM 8. *If $\mathcal{X}$ is a locally compact metric separable space, then there is a sequence of compact sets $F_1, F_2, \dots$ such that*

$$\mathcal{X} = F_1 \cup F_2 \cup \dots \quad \textit{and} \quad F_n \subset \operatorname{Int}(F_{n+1}).$$

Consequently $\mathcal{X} = \bigcup_{n=1}^{\infty} \operatorname{Int}(F_n)$ and therefore the family $\mathcal{C}(\mathcal{X})$ of all compact subsets of $\mathcal{X}$ is cofinal with the sequence $F_1, F_2, \dots$; i.e. each member of $\mathcal{C}(\mathcal{X})$ is contained in a member of that sequence.

This follows from Corollary 4 of Section II, since for each open subset X of a compact metric space $\mathcal{X}^*$ there is a sequence of open sets $G_1, G_2, \dots$ such that

$$X = G_1 \cup G_2 \cup \dots \quad \text{and} \quad \bar{G}_n \subset G_{n+1}.$$

Thus the sets $F_1 = \bar{G}_1$, $F_2 = \bar{G}_2, \dots$ form the required sequence of compact sets.

Remark 5. Let us note the following interesting theorem.

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(1) See A. Parhomenko, *Über eineindeutige stetige Abbildungen auf kompakte Räume*, Izwiestia Akad. Nauk SSSR 5 (1941), pp. 225–232.

If $\mathscr{X}$ is paracompact, $\mathscr{Y}$ locally compact and $f\colon \mathscr{X} \to \mathscr{Y}$ continuous and closed, then $\operatorname{Fr}[\overset{-1}{f}(y)]$ *is compact for every* $y \in \mathscr{Y}$ (¹).

Remark 6. *A locally compact space may fail to be normal.*

An example is the Tychonov plank $\underset{\alpha}{E}(\alpha \leqslant \Omega) \times \underset{\beta}{E}(\beta \leqslant \omega)$ with the point (Ω, ω) removed (comp. § 14, V, Remark 1, and Section II, Remark I).

§ 42. The space $2^{\mathscr{X}}$

I. Compactness of the space $2^{\mathscr{X}}$. Let us recall that the topology of $2^{\mathscr{X}}$ (called the *exponential topology*) has been defined by assuming that the totality of all sets

$$\boldsymbol{B}(G) = \underset{F}{E}(F \subset G) \quad \text{and} \quad \boldsymbol{C}(H) = \underset{F}{E}(F \cap H \neq 0) \tag{1}$$

is an open subbase of $2^{\mathscr{X}}$ provided G and H are open subsets of $\mathscr{X}$ and F ranges over $2^{\mathscr{X}}$ (see § 17, I).

It follows that the totality of sets

$$\boldsymbol{B}(G_0, G_1, \ldots, G_n) = \underset{F}{E}(F \subset G_0)(F \cap G_1 \neq 0)\ldots(F \cap G_n \neq 0), \tag{2}$$

where $G_0, \ldots, G_n$ are open, is a base of $2^{\mathscr{X}}$.

THEOREM 1 (²). *If $\mathscr{X}$ is compact, so is $2^{\mathscr{X}}$.*

Proof. By Alexander Lemma (see § 41, I), we have to show that every cover of $2^{\mathscr{X}}$ whose members belong to an open subbase of $2^{\mathscr{X}}$ contains a finite subcover. Thus, let (according to (1)):

$$2^{\mathscr{X}} = \bigcup_t \boldsymbol{B}(G_t) \cup \bigcup_s \boldsymbol{C}(H_s), \tag{3}$$

where G_t and H_s are open in $\mathscr{X}$.

Put $F_0 = \mathscr{X} - \bigcup_s H_s$. Therefore, for each s, we have

$$F_0 \cap H_s = 0, \quad \text{i.e.} \quad F_0 \notin \boldsymbol{C}(H_s), \quad \text{hence} \quad F_0 \in \bigcup_t \boldsymbol{B}(G_t).$$

(¹) E. Michael, *A note on closed maps and compact sets*, Israel Journ. Math. 2 (1964), pp. 173–176.

For $\mathscr{X}$ metric, see I. A. Vainstein, *On closed mappings of metric spaces*, Dokl. Akad. Nauk SSSR 57 (1947), pp. 319–321.

(²) See L. Vietoris, Monatsh. f. Math. u. Phys. 31 (1921), pp. 173–204, and O. Frink, *Topology in lattices*, Trans. Amer. Math. Soc. 51 (1942), pp. 569–582 (Theorem 15).

Consequently, there is t_0 such that

$$F_0 \in \boldsymbol{B}(G_{t_0}), \quad \text{i.e.} \quad F_0 \subset G_{t_0}, \quad \text{hence} \quad \mathscr{X} - G_{t_0} \subset \mathscr{X} - F_0 = \bigcup_s H_s.$$

Since $\mathscr{X} - G_{t_0}$ is compact, there is a finite system $s_1, \ldots, s_n$ such that

$$\mathscr{X} - G_{t_0} \subset H_{s_1} \cup \ldots \cup H_{s_n}. \tag{4}$$

We shall show that

$$2^{\mathscr{X}} = \boldsymbol{B}(G_{t_0}) \cup \boldsymbol{C}(H_{s_1}) \cup \ldots \cup \boldsymbol{C}(H_{s_n}), \tag{5}$$

which will complete the proof.

Let $F \in 2^{\mathscr{X}}$. There are two cases to be considered:

(i) $F \subset G_{t_0}$. Then $F \in \boldsymbol{B}(G_{t_0})$.

(ii) $F \not\subset G_{t_0}$, i.e. $F \cap (\mathscr{X} - G_{t_0}) \neq 0$. Then by (4), there is j such that $F \cap H_{s_j} \neq 0$, i.e. $F \in \boldsymbol{C}(H_{s_j})$.

Thus in both cases F belongs to the right-hand member of (5).

The converse theorem is also true:

THEOREM 2 ([1]). *If $2^{\mathscr{X}}$ is compact, so is $\mathscr{X}$ ($\mathscr{X}$ is supposed to be $\mathscr{T}_1$).*

Proof. Let $\mathscr{X} = \bigcup G_t$, where G_t is open. It follows that $2^{\mathscr{X}} - (0) = \bigcup \boldsymbol{C}(G_t)$. Since $2^{\mathscr{X}} - (0)$ is compact, so $2^{\mathscr{X}} - (0) = \boldsymbol{C}(G_{t_1}) \cup \cup \ldots \cup \boldsymbol{C}(G_{t_n})$. Let $x_0 \in \mathscr{X}$. Then $(x_0) \in \boldsymbol{C}(G_{t_j})$ for some $j \leqslant n$; this means that $x_0 \in G_{t_j}$. Therefore $\mathscr{X} = G_{t_1} \cup \ldots \cup G_{t_n}$.

Remark 1. It is worth noting that *if $2^{\mathscr{X}}$ is metrizable then $\mathscr{X}$ is compact* ([2]).

Remark 2. If $\mathscr{X}$ is a countably infinite compact subset of $\mathscr{E}$, then the space $2^{\mathscr{X}}$ is homeomorphic to the union of the Cantor discontinuum $\mathscr{C}$ with the set of centres of the deleted intervals ([3]).

Incidentally, this statement shows that the homeomorphism of the spaces $2^{\mathscr{X}}$ and $2^{\mathscr{Y}}$ does not imply the homeomorphism of the spaces $\mathscr{X}$ and $\mathscr{Y}$ ([4]).

([1]) See E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. 71 (1951), p. 161.

([2]) See *ibid.*, p. 164.

([3]) Theorem of A. Pełczyński. See *A remark on spaces 2^X for zero-dimensional $\mathscr{X}$*, Bull. Polish Acad. Sci. 13 (1965), p. 85.

([4]) This problem was raised by V. Ponomarev. See *A new space of closed subsets and multivalued mappings of bicompact spaces* (Russian), Matem. Sb. 48 (1959), p. 195.

Remark 3. Let us denote, for an arbitrary topological space $\mathscr{X}$, by $\mathscr{C}(\mathscr{X})$ the set of all compact subsets of $\mathscr{X}$. Since $\mathscr{C}(\mathscr{X}) \subset 2^{\mathscr{X}}$ the (exponential) topology of $\mathscr{C}(\mathscr{X})$ is well defined.

One can show [1] the following interesting properties of the space $\mathscr{C}(\mathscr{X})$ (which correspond to properties of $2^{\mathscr{X}}$ stated in § 17): *$\mathscr{C}(\mathscr{X})$ is regular, resp. completely regular, resp. compact, if and only if $\mathscr{X}$ is such.*

Remark 4. Let $\mathscr{X}$ be a $\mathscr{T}_1$-space. Then $\mathscr{X}$ *is locally compact if and only if $2^{\mathscr{X}}$ is such* [2].

II. Case of $\mathscr{X}$ compact metric. In § 21, VII, we have denoted, for each bounded metric space $\mathscr{X}$, by $(2^{\mathscr{X}})_m$ the metric space of all closed subsets of $\mathscr{X}$, the (Hausdorff) distance, $\operatorname{dist}(A, B)$, of two sets $A \neq 0$ and $B \neq 0$ being defined as the lower upper bound of the numbers $\varrho(x, B)$ and $\varrho(y, A)$ where $x \in A$ and $y \in B$; the void set has been defined as an isolated point of $(2^{\mathscr{X}})_m$.

THEOREM. *Let $\mathscr{X}$ be compact metric. Then*

$$2^{\mathscr{X}} \underset{\text{top}}{=} (2^{\mathscr{X}})_m.$$

Namely, *the identity mapping: $2^{\mathscr{X}} \to (2^{\mathscr{X}})_m$ is a homeomorphism onto.*

More generally,

$$\mathscr{C}(\mathscr{X}) \underset{\text{top}}{=} [\mathscr{C}(\mathscr{X})]_m$$

where $[\mathscr{C}(\mathscr{X})]_m$ denotes the set $\mathscr{C}(\mathscr{X})$ with the Hausdorff distance.

Proof. 1. First we shall show that every open set $\boldsymbol{H}$ in $[\mathscr{C}(\mathscr{X})]_m$ is open in $\mathscr{C}(\mathscr{X})$ (in its exponential topology). Clearly, it is sufficient to prove that each open ball with compact center A:

$$\boldsymbol{R} = \underset{F}{E}\,[\operatorname{dist}(A, F) < \varepsilon], \quad \text{where} \quad F \neq 0 \text{ and } F \in 2^{\mathscr{X}}, \tag{0}$$

is open in $2^{\mathscr{X}}$.

By virtue of the total boundedness of A (see § 41, VI, Theorem 1), there is for each $k = 1, 2, \ldots$, a finite system $a_1^k, \ldots, a_{n_k}^k$

(1) See E. Michael, *op. cit.*

(2) *Ibid.* p. 162. See also P. D. Watson, *On the limits of sequences of sets,* Quart. J. 4 (1953), pp. 1–3.

of points of A such that for each $x \in A$ and k we have $|a_i^k - x| < 1/k$ for some i (see § 21, VIII, Theorem 1). Let

$$G_i^k = \underset{x}{E}[|x - a_i^k| < \varepsilon - 1/k], \tag{1}$$

$$G = \underset{x}{E}[\varrho(x, A) < \varepsilon]. \tag{2}$$

We shall show that (compare I(2)):

$$\begin{aligned} \boldsymbol{R} &= \bigcup_{k=1}^{\infty} \boldsymbol{B}(G, G_1, \ldots, G_{n_k}^k) \\ &= \bigcup_{k=1}^{\infty} \underset{F}{E}(F \subset G)(F \cap G_1^k \neq 0) \ldots (F \cap G_{n_k}^k \neq 0), \end{aligned} \tag{3}$$

which will complete the proof.

First, let $F \in \boldsymbol{R}$. Then $\operatorname{dist}(A, F) < \varepsilon - 1/k$ for some k. It follows that

$$x \in A \Rightarrow \varrho(x, F) < \varepsilon - 1/k \tag{4}$$

and

$$y \in F \Rightarrow \varrho(y, A) < \varepsilon - 1/k. \tag{5}$$

Conditions (5) and (2) imply that $F \subset G$. On the other hand, for $x = a_i^k$ there is by (4) an $y \in F$ such that $|y - a_i^k| < \varepsilon - 1/k$. This means (compare (1)) that $F \cap G_i^k \neq 0$ for $i = 1, 2, \ldots, n_k$. Thus F belongs to the right member of (3).

Now, suppose conversely that $F \subset G$ and $F \cap G_i^k \neq 0$ for some k and for all $i = 1, 2, \ldots, n_k$. Then we have by (2)

$$y \in F \Rightarrow \varrho(y, A) < \varepsilon. \tag{6}$$

On the other hand, let $x \in A$ and let i be such that $|a_i^k - x| < 1/k$. Since $F \cap G_i^k \neq 0$, there is $y \in F$ such that (compare (1)) $|y - a_i^k| < \varepsilon - 1/k$. It follows that $|y - x| < \varepsilon$ and consequently $\varrho(x, F) < \varepsilon$. Thus

$$x \in A \Rightarrow \varrho(x, F) < \varepsilon. \tag{7}$$

Conditions (6) and (7) yield $F \in \boldsymbol{R}$.

2. Next, we must show that for every open set $\boldsymbol{H}$ in $2^{\mathscr{X}}$ the set $\boldsymbol{H} \cap \mathscr{C}(\mathscr{X})$ is open in $[\mathscr{C}(\mathscr{X})]_m$. We can, of course, restrict ourselves to the case where $\boldsymbol{H}$ belongs to a subbase of $2^{\mathscr{X}}$.

Case 2a. $\boldsymbol{H} = \underset{F}{E}(F \subset G)$ where G is open in $\mathscr{X}$ (and $G \neq \mathscr{X}$). Let A be compact $\subset G$. Let $\varepsilon = \varrho(A, \mathscr{X}-G)$. We have $\varepsilon > 0$ by Corollary 3b of § 41, VI. It remains to show that

$$\boldsymbol{R} \subset \boldsymbol{H}, \quad \text{i.e. that} \quad F \in \boldsymbol{R} \Rightarrow F \subset G$$

(where $\boldsymbol{R}$ is defined by formula (0)).

Suppose $p \in F - G$. Hence $\varrho(p, A) \geqslant \varepsilon$ and consequently $\operatorname{dist}(F, A) \geqslant \varepsilon$ and $F \notin \boldsymbol{R}$.

Case 2b. $\boldsymbol{H} = \underset{F}{E}(F \cap G \neq 0)$. Let A be compact and $A \cap G \neq 0$. Let $a \in A \cap G$. Put $\varepsilon = \varrho(a, \mathscr{X} - G)$. It follows that $\boldsymbol{R} \subset \boldsymbol{H}$.

This completes the proof of the Theorem.

Remark 1. One shows easily [1] that if $\mathscr{X}$ is compact metric, then the conditions

$$\lim_{n=\infty} \operatorname{dist}(F_n, F) = 0 \quad \text{and} \quad \operatorname*{Lim}_{n=\infty} F_n = F$$

are equivalent whatever the closed non-void sets F_n and F are. Consequently

$$2^{\mathscr{X}} \underset{\text{top}}{=} (2^{\mathscr{X}})_L,$$

the topology in $(2^{\mathscr{X}})_L$ being induced by the Lim-operation.

Let us note that the homeomorphism between $(2^{\mathscr{X}})_m$ and $(2^{\mathscr{X}})_L$ does not hold for non-compact metric spaces (see § 29, IX).

Remark 2. Denote for every compact ($\neq 0$) subset F of $\mathscr{E}$, by $\mu(F)$ the *first* (or the *last*) point of F. Then *the mapping* $\mu: 2^{\mathscr{I}} \to \mathscr{I}$ *is continuous.*

More generally, μ: $\mathscr{C}(\mathscr{E}) \to \mathscr{E}$ *is continuous.*

Proof. Let $F_1, F_2 \in \mathscr{C}(\mathscr{E})$ and $\mu(F_1) \leqslant \mu(F_2)$. Then

$$|\mu(F_1) - \mu(F_2)| = \varrho[\mu(F_1), F_2] \leqslant \operatorname{dist}(F_1, F_2).$$

III. Families of subsets of $\mathscr{X}$. Operations on sets. Let us recall that if K is closed and G open, then the set $\underset{F}{E}(F \subset K)$ is closed and $\underset{F}{E}(F \subset G)$ is open in $2^{\mathscr{X}}$ whatever the topological space $\mathscr{X}$ is (see § 17, II, Theorem 1).

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[1] See F. Hausdorff, *Mengenlehre*, p. 150. See also § 29, IX.

If $\mathscr{X}$ is a $\mathscr{T}_1$-space, then the set $\underset{F}{E}(A \subset F)$ is closed in $2^{\mathscr{X}}$ whatever the set $A \subset \mathscr{X}$ is (§ 17, II, Theorem 2).

For compact spaces we have the further statements.

THEOREM 1. *Let $\mathscr{X}$ be a compact $\mathscr{T}_2$-space. Then, if A is a $\boldsymbol{G}_\delta$ set, so is $\underset{F}{E}(F \subset A)$.*

Proof. Let us consider the equivalence:

$$F \not\subset A \equiv \bigvee_x (x \in F)(x \in \mathscr{X} - A). \tag{1}$$

The set $\underset{x,F}{E}(x \in F)$ is closed in $\mathscr{X} \times 2^{\mathscr{X}}$ (this is true of every $\mathscr{T}_2$-space $\mathscr{X}$, see § 17, IV, Theorem 1). Hence $\underset{x,F}{E}(x \in F)(x \in \mathscr{X} - A)$ is an $\boldsymbol{F}_\sigma$-set, and so is its projection on the $2^{\mathscr{X}}$-axis (by Corollary 1a of § 41, IV), which means that $\underset{F}{E}(F \not\subset A)$ is an $\boldsymbol{F}_\sigma$-set.

THEOREM 2. *Let $\mathscr{X}$ be compact metric. Then, if A is of projective class $\boldsymbol{CA}$, so is $\underset{F}{E}(F \subset A)$.*

The proof based on equivalence (1) is completely analogous to the preceding one.

Remark. There is no analogous statement for the case of A being $\boldsymbol{F}_\sigma$. In fact, if $\mathscr{X} = \mathscr{I}$ and $A = \mathscr{R}$ (set of rational numbers), then the set $\underset{F}{E}(F \subset A)$ is not an $\boldsymbol{F}_\sigma$-set; moreover, it is not even an analytic set (see § 43, VII, Corollary 3).

THEOREM 3 (1). *Let $\mathscr{X}$ be compact metric. Then the family $\boldsymbol{P}$ of all perfect subsets of $\mathscr{X}$ is a $\boldsymbol{G}_\sigma$-set in $2^{\mathscr{X}}$.*

First, we shall show the following lemma (true for every regular $\mathscr{T}_1$-space $\mathscr{X}$).

LEMMA. *If $G \subset \mathscr{X}$ is open, then the set $\underset{xF}{E}[(F \cap G) = (x)]$ is an intersection of a closed set with an open set in $\mathscr{X} \times 2^{\mathscr{X}}$.*

Proof of the lemma. Since the set $\underset{xF}{E}(x \in F)$ is closed in $\mathscr{X} \times 2^{\mathscr{X}}$ (see § 17, IV, Corollary 5), the set

$$\underset{xF}{E}[(x) \subset (F \cap G)] = \underset{xF}{E}(x \in F) \cap \underset{xF}{E}(x \in G)$$

is an intersection of a closed set with an open set.

(1) This theorem is due to Banach.

On the other hand, (x) represents a continuous mapping (by § 17, III, Corollary 3a) and so does $(x) \cup (\mathscr{X} - G)$ (since the union of two continuous mappings is continuous, see § 17, III, Theorem 4). The set $\underset{FH}{E}(F \subset H)$ is closed in $2^{\mathscr{X}} \times 2^{\mathscr{X}}$ (since $\mathscr{X}$ is regular see § 17, IV, Theorem 1), and it follows that the set

$$\underset{xF}{E}[(F \cap G) \subset (x)] = \underset{xF}{E}[F \subset (x) \cup (\mathscr{X} - G)]$$

is closed in $\mathscr{X} \times 2^{\mathscr{X}}$.

This completes the proof of the lemma.

Proof of Theorem 3. Let $G_1, G_2, \ldots$ denote an open base of $\mathscr{X}$. The condition $F \epsilon 2^{\mathscr{X}} - \boldsymbol{P}$ is equivalent to the existence of a point x and of an index n such that $F \cap G_n = (x)$, i.e.

$$2^{\mathscr{X}} - \boldsymbol{P} = \underset{F}{E} \bigvee_n \bigvee_x [(F \cap G_n) = (x)] = \bigcup_n \underset{F}{E} \bigvee_x [(F \cap G_n) = (x)].$$

By the lemma, the set $\underset{xF}{E}[(F \cap G_n) = (x)]$ is $\boldsymbol{F}_\sigma$ and so is $\underset{F}{E} \bigvee_x [(F \cap G_n) = (x)]$. Consequently $2^{\mathscr{X}} - \boldsymbol{P}$ is an $\boldsymbol{F}_\sigma$-set.

COROLLARY 3a (1). *Let $\mathscr{X}$ be compact metric. Then the family of all countable closed subsets of $\mathscr{X}$ is of projective class $\boldsymbol{CA}$ in $2^{\mathscr{X}}$.*

Proof. Denote by $\boldsymbol{U}$ the family of all uncountable closed subsets of $\mathscr{X}$. Since each $F \epsilon \boldsymbol{U}$ contains a perfect non-void subset (by the Theorem of Cantor–Bendixson, see § 32, V), we have

$$\boldsymbol{U} = \underset{F}{E} \bigvee_K (0 \neq K \subset F)(K \epsilon \boldsymbol{P}).$$

By Theorem 3, the set $\underset{FK}{E}(0 \neq K \subset F)(K \epsilon \boldsymbol{P})$ is $\boldsymbol{G}_\delta$. Hence $\boldsymbol{U}$, as a projection of a $\boldsymbol{G}_\delta$-set, is an analytic set (see § 38, I).

Remark. The family under consideration does not need to be analytic (see § 43, VII, Corollary 3).

THEOREM 4. *Let $\mathscr{X}$ be compact metric. Then the family $\boldsymbol{N}$ of all nowhere dense closed subsets of $\mathscr{X}$ is a $\boldsymbol{G}_\delta$-set in $2^{\mathscr{X}}$.*

Proof. Let $G_1, G_2, \ldots$ be an open base of $\mathscr{X}$ $(G_n \neq 0)$. Then

$$2^{\mathscr{X}} - \boldsymbol{N} = \underset{F}{E} \bigvee_n (G_n \subset F) = \bigcup_n \underset{F}{E}(G_n \subset F).$$

Since $\underset{F}{E}(G_n \subset F)$ is closed, $2^{\mathscr{X}} - \boldsymbol{N}$ is $\boldsymbol{F}_\sigma$.

(1) Theorem of Hurewicz, Fund. Math. 15 (1930), pp. 4–17.

THEOREM 5. *Let $\mathscr{X}$ be a $\mathscr{T}_1$-space. Let $\boldsymbol{A} \subset 2^{\mathscr{X}}$ and $S = \mathrm{S}(\boldsymbol{A})$ (the union of members of $\boldsymbol{A}$). If $\boldsymbol{A}$ is open, so is S. If $\boldsymbol{A}$ is compact and $\mathscr{X}$ regular, then S is closed.*

Proof. Suppose that $\boldsymbol{A}$ is open in $2^{\mathscr{X}}$. With no loss of generality, we may suppose that $\boldsymbol{A}$ is a member of a base of $2^{\mathscr{X}}$. Thus, let $G_0, \ldots, G_n$ be a system of open sets (in $\mathscr{X}$) such that (compare § 17, I):

$$(F \epsilon \boldsymbol{A}) \equiv (F \subset G_0)(F \cap G_1 \neq 0) \ldots (F \cap G_n \neq 0)$$

and $0 \neq G_i \subset G_0$ for $i = 1, \ldots, n$.

We shall prove that $S = G_0$.

Obviously $S \subset G_0$. Conversely, let $x_0 \epsilon G_0$ and denote by x_i any point of G_i for $i = 1, \ldots, n$. Then the set $F = (x_0, x_1, \ldots, x_n)$ is a member of $\boldsymbol{A}$ and consequently $x_0 \epsilon S$. Therefore $G_0 \subset S$ and $S = G_0$, which completes the proof of the first part of the theorem.

Now let $\boldsymbol{A}$ be compact. By definition

$$(x \epsilon S) \equiv \bigvee_{F \epsilon \boldsymbol{A}} (x \epsilon F)$$

and by § 17, IV, Theorem 1, the set $\underset{x,F}{E}(x \epsilon F)$ is closed in $\mathscr{X} \times \boldsymbol{A}$. Since S is its projection on the $\mathscr{X}$ axis, it follows (by the compactness of $\boldsymbol{A}$) that S is closed (see § 41, IV, Theorem 1).

Remark 1. It follows readily from Theorem 5 that if $\boldsymbol{A}$ is a countable union of compact sets, so is $\mathrm{S}(\boldsymbol{A})$ (for instance if $\boldsymbol{A}$ is $\boldsymbol{F}_\sigma$ and $2^{\mathscr{X}}$ compact).

However, there is no similar theorem for $\boldsymbol{G}_\delta$-sets. This is seen in the following example.

Let $\mathscr{X} = \mathscr{I}$, $F_n = \left(\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\right)$ and let $\boldsymbol{A}$ be the family of all F_n, $n = 1, 2, \ldots$. Clearly $\boldsymbol{A}$ is a discrete subset of $2^{\mathscr{I}}$, hence a $\boldsymbol{G}_\delta$ (see § 24, III, Theorem 1a). Now $\mathrm{S}(\boldsymbol{A})$ is the set of all rationals (in $\mathscr{I}$), hence it is not a $\boldsymbol{G}_\delta$-set.

Remark 2. *The family $\boldsymbol{W}$ of all well ordered* (by the relation $<$) *closed subsets of $\mathscr{I}$ is of projective class $\boldsymbol{CA}$ in $2^{\mathscr{I}}$.*

In order to prove this statement, we make use of the obvious fact that a subset of $\mathscr{I}$ is not well ordered if and only if it contains a decreasing sequence of elements. Therefore, denoting by $\mathfrak{z} = [\mathfrak{z}^{(1)}, \mathfrak{z}^{(2)}, \dots]$ a variable point of $\mathscr{I}^{\aleph_0}$, we have

$$2^{\mathscr{X}} - \boldsymbol{W} = \underset{F}{E} \bigvee_{\mathfrak{z}} \bigwedge_{nk} (\mathfrak{z}^{(n)} \in F)(\mathfrak{z}^{(n)} > \mathfrak{z}^{(n+k)}).$$

It follows easily that $2^{\mathscr{X}} - \boldsymbol{W}$ is analytic.

Remark 3. One easily shows that *in any $\mathscr{T}_2$-space, the family of all sets composed of at most n elements is closed* (*for n fixed*).

It follows:

Remark 4. *The family of all finite sets is an $\boldsymbol{F}_\sigma$-set.*

Moreover, *it is dense in* $2^{\mathscr{X}}$ (see § 17, II, Theorem 4).

THEOREM 6. *Let $\mathscr{X}$ be compact and $F_t = \overline{F}_t \subset \mathscr{X}$ for each $t \in T$. Suppose that the family $\boldsymbol{A} = \{F_t\}$ is finitely multiplicative, i.e. for each finite system $t_1, \dots, t_n$ there is $t \in T$ such that $F_t = F_{t_1} \cap \dots \cap F_{t_n}$. Then $\left(\bigcap_{t \in T} F_t\right) \in \overline{\boldsymbol{A}}$.*

Proof. Put $Z = \bigcap_t F_t$. We have to show that if $\boldsymbol{G}$ is open in $2^{\mathscr{X}}$ and $Z \in \boldsymbol{G}$, then there is t such that $F_t \in \boldsymbol{G}$. Clearly we may assume that $\boldsymbol{G}$ belongs to the base considered in I(2). In other terms, given a system of open sets $G_0, \dots, G_n$ (in $\mathscr{X}$) such that

$$Z \subset G_0 \quad \text{and} \quad Z \cap G_i \neq 0 \quad \text{for} \quad i = 1, \dots, n, \tag{2}$$

we have to show that for some $t \in T$

$$F_t \subset G_0 \quad \text{and} \quad F_t \cap G_i \neq 0 \quad \text{for} \quad i = 1, \dots, n. \tag{3}$$

Suppose that this is not true. Since the formula $F_t \cap G_i \neq 0$ follows from the conditions $Z \cap G_i \neq 0$ and $Z \subset F_t$, our assumption means that for each t we have $F_t \not\subset G_0$, i.e. $F_t \cap H \neq 0$ where $H = \mathscr{X} - G_0$. Therefore, for any finite system of indices $t_1, \dots, t_n$, one has $(F_{t_1} \cap H) \cap \dots \cap (F_{t_n} \cap H) \neq 0$; because by assumption there is t such that

$$F_{t_1} \cap \dots \cap F_{t_n} = F_t, \qquad \text{hence}$$
$$(F_{t_1} \cap H) \cap \dots \cap (F_{t_n} \cap H) = (F_t \cap H) \neq 0.$$

It follows, by the Riesz condition (see § 41, I(2)), that $\bigcap_t (F_t \cap H) \neq 0$, i.e. $Z \not\subset G_0$ contrary to (2).

IV. Irreducible sets. Saturated sets. The set F is said to be an *irreducible* set (resp. a *saturated* set) of the family $\boldsymbol{A}$ of sets if $F \epsilon \boldsymbol{A}$ and the conditions $X \neq F$ and $X \subset F$ (resp. $X \supset F$) imply $X \notin \boldsymbol{A}$ (1).

By Theorem III, 6 we have:

THEOREM 1. *Let $\mathscr{X}$ be compact and $\boldsymbol{A} \subset 2^{\mathscr{X}}$ a well-ordered decreasing family of sets*:

$$F_0 \supset F_1 \supset \ldots \supset F_\alpha \supset \ldots, \qquad \alpha < \gamma. \tag{1}$$

Then $\big(\bigcap_\alpha F_\alpha\big) \epsilon \overline{\boldsymbol{A}}$.

THEOREM 2 (2). *Let $\mathscr{X}$ be compact. Then every closed family $\boldsymbol{C} \subset 2^{\mathscr{X}}$ contains an irreducible element.*

Proof. Let F_0 be an arbitrary element of $\boldsymbol{C}$. If F_0 is not irreducible, then there exists an $F_1 \epsilon \boldsymbol{C}$ such that $F_1 \subset F_0$ and $F_1 \neq F_0$. Proceeding in this way, one defines with the aid of transfinite induction the sequence (1) by assuming that $F_{\alpha+1}$ is such that $F_{\alpha+1} \epsilon \boldsymbol{C}$ and $F_{\alpha+1} \subset F_\alpha \neq F_{\alpha+1}$ (provided that F_α is not irreducible) and by putting $F_\lambda = \bigcap_{\xi<\lambda} F_\xi$ if λ is a limit ordinal. Since $\boldsymbol{C}$ is closed, it follows by virtue of Theorem 1 that $F_\alpha \epsilon \boldsymbol{C}$ for each α. Clearly, starting with a certain α_0, the sequence $\{F_\alpha\}$ cannot be continued. Thus F_{α_0} is an irreducible element of $\boldsymbol{C}$.

Remarks. If $\mathscr{X}$ is compact metric, one does not need to use sequences (1) other than of type ω. (3)

In this connection, it is worthy noting that the following statement can be shown in a similar way. (4)

Let $\mathscr{X}$ be a topological space with a countable open base and let $\boldsymbol{A}$ be a family of closed subsets of $\mathscr{X}$ such that for each sequence (1) of type $\gamma = \omega$ the intersection $F_0 \cap F_1 \cap \ldots$ contains an element of $\boldsymbol{A}$. Then each set belonging to $\boldsymbol{A}$ contains a set irreducible in $\boldsymbol{A}$.

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(1) These notions have been introduced by Janiszewski.

(2) Compare L. E. J. Brouwer, Proc. Akad. Wet. Amsterdam 14 (1911), p. 138.

(3) as shown by Mazurkiewicz. See the French edition of my *Topologie II*, p. 27.

(4) See A. Lelek, Prace Mat. 7 (1962), p. 107.

Clearly, the last conclusion remains true if, without assuming that $\mathscr{X}$ has a countable open base, we suppose that $\left(\bigcap_{\alpha} F_\alpha\right) \epsilon \boldsymbol{A}$ for each sequence (1) of subsets of $\mathscr{X}$ (whatever the ordinal γ is). This has been shown, in fact, in proving Theorem 2.

V. Operations $\delta(F)$ **and** $\varrho(F_1, F_2)$**.** Let $\mathscr{X}$ be a compact metric space. Since $\mathscr{X}$ is bounded, so is the diameter $\delta(F)$ for each $F \subset \mathscr{X}$ (let us recall that $\delta(F)$ is the least upper bound of the distances of points of F, see § 21, III).

THEOREM 1. *The function* $\delta\colon 2^{\mathscr{X}} \to \mathscr{E}$ *is continuous.*

More precisely (comp. Section II, Remark),

$$\delta(\mathrm{Li}\, F_n) \leqslant \liminf \delta(F_n), \tag{1}$$

$$\limsup \delta(F_n) \leqslant \delta(\mathrm{Ls}\, F_n). \tag{2}$$

Proof. Let p and q be such that (compare § 41, VI, Corollary 2d):

$$\delta(\mathrm{Li}\, F_n) = |p-q| \quad \text{and} \quad p \epsilon \mathrm{Li}\, F_n, \quad q \epsilon \mathrm{Li}\, F_n. \tag{3}$$

Consequently, there are two sequences $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ such that

$$p = \lim p_n, \quad q = \lim q_n, \quad p_n \epsilon F_n, \quad q_n \epsilon F_n. \tag{4}$$

Therefore

$$|p_n - q_n| \leqslant \delta(F_n) \quad \text{and} \quad \lim |p_n - q_n| = |p-q|$$

hence

$$|p-q| \leqslant \liminf \delta(F_n).$$

This implies (1) by virtue of (3).

Next, assume that

$$\limsup \delta(F_n) = \lim \delta(F_{k_n}) \quad \text{and} \quad \delta(F_{k_n}) = |p_{k_n} - q_{k_n}|, \tag{5}$$

where $p_{k_n} \epsilon F_{k_n}$ and $q_{k_n} \epsilon F_{k_n}$.

Clearly, we may assume that the sequences $p_{k_1}, p_{k_2}, \ldots$ and $q_{k_1}, q_{k_2}, \ldots$ are convergent (otherwise we would replace the sequence $F_{k_1}, F_{k_2}, \ldots$ by a subsequence with the required property). Thus, let us put

$$\lim p_{k_n} = p \text{ and } \lim q_{k_n} = q, \text{ hence } p \epsilon \mathrm{Ls}\, F_n \text{ and } q \epsilon \mathrm{Ls}\, F_n. \tag{6}$$

It follows that

$$|p-q| \leqslant \delta(\mathrm{Ls}\, F_n) \quad \text{and} \quad \lim |p_{k_n} - q_{k_n}| = |p-q|. \tag{7}$$

This implies (2) by virtue of (5).

Thus the proof of Theorem 1 is complete.

Now, let us recall that $\varrho(F_1, F_2)$ denotes the greatest lower bound of the distances $|x_1 - x_2|$ where $x_1 \in F_1$ and $x_2 \in F_2$ (see § 21, IV). Assuming that F_1 and F_2 are variable closed subsets of the compact metric space $\mathscr{X}$, ϱ is a real valued function defined on the cartesian product $2^{\mathscr{X}} \times 2^{\mathscr{X}}$.

THEOREM 2. *The function* $\varrho\colon 2^{\mathscr{X}} \times 2^{\mathscr{X}} \to \mathscr{E}$ *is continuous.*

More precisely (assuming that $F_n \in 2^{\mathscr{X}}$ and $H_n \in 2^{\mathscr{X}}$) we have

$$\varrho(\mathrm{Ls}\, F_n,\ \mathrm{Ls}\, H_n) \leqslant \liminf \varrho(F_n, H_n), \tag{8}$$

$$\limsup \varrho(F_n, H_n) \leqslant \varrho(\mathrm{Li}\, F_n, \mathrm{Li}\, H_n). \tag{9}$$

Proof. Put

$$\liminf \varrho(F_n, H_n) = \lim \varrho(F_{k_n}, H_{k_n}), \qquad \varrho(F_{k_n}, H_{k_n}) = |p_{k_n} - q_{k_n}|, \tag{10}$$

where $p_{k_n} \in F_{k_n}$ and $q_{k_n} \in H_{k_n}$.

Clearly, we may assume that the sequences $p_{k_1}, p_{k_2}, \ldots$ and $q_{k_1}, q_{k_2}, \ldots$ are convergent, hence that (6) is satisfied. It follows that

$$\varrho(\mathrm{Ls}\, F_n, \mathrm{Ls}\, H_n) \leqslant |p-q| = \lim |p_{k_n} - q_{k_n}|,$$

which yields (8) by virtue of (10).

Next, put

$$\varrho(\mathrm{Li}\, F_n, \mathrm{Li}\, H_n) = |p-q|, \qquad p \in \mathrm{Li}\, F_n,\ q \in \mathrm{Li}\, H_n. \tag{11}$$

Hence

$$p = \lim p_n,\ p_n \in F_n, \qquad q = \lim q_n \text{ and } q_n \in H_n.$$

Therefore

$$|p-q| = \lim |p_n - q_n| \quad \text{and} \quad |p_n - q_n| \geqslant \varrho(F_n, H_n),$$

and consequently

$$|p-q| \geqslant \limsup \varrho(F_n, H_n). \tag{12}$$

Formulas (11) and (12) yield (9).

§ 43. Semi-continuity

I. Semi-continuity and the assumption of compactness of $\mathscr{X}$ (1). Let us recall (see § 18, I) that, given a mapping $F\colon \mathscr{Y} \to 2^{\mathscr{X}}$, where $\mathscr{X}$ and $\mathscr{Y}$ are topological spaces, F is called *upper semi-continuous* (*u.s.c.*) if for each closed set $A \subset \mathscr{X}$, the set $\underset{y}{E}[F(y) \cap \cap A \neq 0]$ is closed in $\mathscr{Y}$.

Replacing "closed" by "open" in the above definition, we obtain the definition of *lower semi-continuity* (l.s.c.).

More precisely: F is called *u.s.c. at* y_0 if

$$y_0 \in F^{-1}(2^G) \Rightarrow y_0 \in \operatorname{Int}[F^{-1}(2^G)] \quad \text{whenever } G \text{ is open;}$$

F is *l.s.c. at* y_0 if

$$y_0 \in \overline{F^{-1}(2^K)} \Rightarrow y_0 \in F^{-1}(2^K) \quad \text{whenever } K \text{ is closed.}$$

In § 18 we have stated a number of properties of semi-continuous mappings. At present, we shall add some more, assuming that $\mathscr{X}$ is a *compact* and $\mathscr{Y}$ an arbitrary $\mathscr{T}_2$*-space.*

First, let us note that if $f\colon \mathscr{X} \to \mathscr{Y}$ is continuous, then f is a closed mapping (see § 41, III, Theorem 2). Hence the following statement follows from Theorems 5 and 5a of § 18, III.

THEOREM 1. *If $f\colon \mathscr{X} \to \mathscr{Y}$ is continuous, then the mapping $f^{-1}\colon 2^{\mathscr{Y}} \to 2^{\mathscr{X}}$ is u.s.c.*

In particular, $\overset{-1}{f}\colon \mathscr{Y} \to 2^{\mathscr{X}}$ *is u.s.c.*

THEOREM 2. *Define the mapping $Q\colon \mathscr{Y} \to 2^{\mathscr{X}\times\mathscr{Y}}$ by the condition $Q(y) = \mathscr{X} \times (y)$. If $\mathscr{X}$ is compact, then Q is u.s.c.*

Q is l.s.c. without any restriction on $\mathscr{X}$.

Proof. Let $A \subset \mathscr{X}\times\mathscr{Y}$ be closed. We have

$$[Q(y) \cap A \neq 0] \equiv \bigvee_x (x, y) \in A, \tag{1}$$

which means that the set $\underset{y}{E}[Q(y) \cap A \neq 0]$ is the projection of the set A on the $\mathscr{Y}$-axis. This set is closed, since $\mathscr{X}$ is compact and A closed (compare § 41, IV, Theorem 1). Thus Q is u.s.c.

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(1) Compare my paper *Les fonctions semi-continues dans l'espace des ensembles fermés*, Fund. Math. 18 (1931), p. 148. See also C. Berge, *Espaces topologiques*, Chapter VI.

If we assume A to be open, then so is $\underset{y}{E}[Q(y) \cap A \neq 0]$. Because formula (1) means that (compare § 2, V (1))

$$\underset{y}{E}[Q(y) \cap A \neq 0] = \bigcup_{x} \underset{y}{E}[(x, y) \in A],$$

and $\underset{y}{E}(x, y) \in A$ is open for each x.

Remark. *For non-compact $\mathscr{X}$ the mapping Q may fail to be u.s.c.* Such is the case of $\mathscr{X} = \mathscr{E}$ and $\mathscr{Y} = \mathscr{I}$; consider the set

$$A = \underset{xy}{E}(x = 1/y)(0 < y \leqslant 1).$$

THEOREM 3. *Let $D = \overline{D} \subset \mathscr{X} \times \mathscr{Y}$ and $F(y) = \underset{x}{E}[(x, y) \in D]$ (the horizontal section). Then F is u.s.c.*

Proof. Let $K = \overline{K} \subset \mathscr{X}$. As the mapping Q of Theorem 2 is u.s.c., hence so is $Q \cap D$ (compare § 18, V, Theorem 2). Since $K \times \mathscr{Y}$ is closed in $\mathscr{X} \times \mathscr{Y}$, the following set is closed

$$\underset{y}{E}\big[\big(Q(y) \cap D\big) \cap (K \times \mathscr{Y}) \neq 0\big].$$

It remains to show that this set is identical with the set $\underset{y}{E}[F(y) \cap K \neq 0]$. Now this follows from the formulas:

$$\big((x, y) \in D\big)(x \in K) \equiv (x, y) \in [D \cap (K \times \mathscr{Y})]$$
$$\equiv (x, y) \in [Q(y) \cap D \cap (K \times \mathscr{Y})],$$

hence

$$[F(y) \cap K \neq 0] \equiv \bigvee_{x} \big[x \in \big(F(y) \cap K\big)\big] \equiv \bigvee_{x} [(x, y) \in Q(y) \cap D \cap (K \times \mathscr{Y})]$$
$$\equiv [Q(y) \cap D \cap (K \times \mathscr{Y}) \neq 0].$$

THEOREM 4. *Let $F\colon \mathscr{Y} \to 2^{\mathscr{X}}$. The mapping F is u.s.c. if and only if the set*

$$D = \underset{xy}{E}[x \in F(y)] \tag{2}$$

is closed.

Proof. If F is u.s.c., then D is closed (by Theorem 1 of § 18, III, which is valid for any regular $\mathscr{X}$).

Conversely, if D is closed, our statement follows from Theorem 3 since $(x, y) \in D \equiv x \in F(y)$.

It has been shown in § 18, p. 180, that the intersection of two u.s.c. mappings is an u.s.c. mapping, provided the space $\mathscr{X}$ is normal. We are going to extend this theorem to the intersection of an arbitrary family of u.s.c. mappings under the assumption of $\mathscr{X}$ being compact.

First we shall prove a lemma which corresponds in the finite case to the lemma of § 18, p. 179.

LEMMA (1). *Let T be an arbitrary set, $\mathscr{X}$ a compact $\mathscr{T}_2$-space and F_t: $\mathscr{Y} \to 2^{\mathscr{X}}$ for each $t \in T$. Put $F(y) = \bigcap_t F_t(y)$.*

Then we have for each open $G \subset \mathscr{X}$:

$$F^{-1}(2^G) = \bigcup_U \bigcap_t F_t^{-1}(2^{U_t}), \tag{3}$$

where the union is extended to all families $\boldsymbol{U}$ composed of open U_t having G as their common intersection and such that, except for a finite number of indices, $U_t = \mathscr{X}$.

Proof. 1. Let $y \in F^{-1}(2^G)$, i.e. $\left(\bigcap_t F_t(y)\right) \subset G$ or, equivalently, $\bigcap_t \left(F_t(y) - G\right) = 0$. Since $\mathscr{X}$ is compact and the sets $F_t(y) - G$ are closed, there exists (compare the Riesz condition, § 41, I (2)) a finite system $t_1, \ldots, t_k$ such that

$$\left(F_{t_1}(y) - G\right) \cap \ldots \cap \left(F_{t_n}(y) - G\right) = 0. \tag{4}$$

Since the sets $F_{t_i}(y) - G$ are closed, there exists by (4) a system of open sets $V_{t_1}, \ldots, V_{t_k}$ such that (see § 14, III, p. 124)

$$V_{t_1} \cap \ldots \cap V_{t_k} = 0 \quad \text{and} \quad F_{t_i}(y) - G \subset V_{t_i} \quad \text{for } i = 1, 2, \ldots, k. \tag{5}$$

Let $\boldsymbol{U}$ be the family of all sets $U_{t_i} = G \cup V_{t_i}$ for $i = 1, 2, \ldots, k$ and $U_t = \mathscr{X}$ for $t \neq t_i$. Clearly

$$\bigcap_t U_t = U_{t_1} \cap \ldots \cap U_{t_k} = (G \cup V_{t_1}) \cap \ldots \cap (G \cup V_{t_k}) = G$$

by the first part of (5). By the second part, $F_t(y) \subset U_t$, i.e. $y \in F_t^{-1}(2^{U_t})$ for each $t \in T$, and thus y belongs to the right-hand member of (3).

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(1) This lemma and the following theorem and corollaries are due to R. Engelking. See *Quelques remarques concernant les opérations sur les fonctions semi-continues dans les espaces topologiques*, Bull. Acad. Polon. Sc. 11 (1963), pp. 719–725.

2. Let us now assume that y belongs to the right member of (3). Hence there is a family $\boldsymbol{U}$ of open sets U_t such that $\bigcap_t U_t = G$ and $y \in F_t^{-1}(2^{U_t})$, i.e. $F_t(y) \subset U_t$ for each $t \in T$. Therefore,

$$F(y) = \bigcap_t F_t(y) \subset \bigcap_t U_t = G, \quad \text{i.e.} \quad y \in F^{-1}(2^G).$$

THEOREM 5. *Let $\mathscr{X}$ and F be as in the Lemma. If each F_t is u.s.c. at y_0, then so is F.*

Proof. Let G be open and $y_0 \in F^{-1}(2^G)$. We must show that $y_0 \in \operatorname{Int}[F^{-1}(2^G)]$. Let us apply the Lemma and consider a family $\boldsymbol{U}$ such that $y_0 \in F_t^{-1}(2^{U_t})$ for each $U_t \in \boldsymbol{U}$ and, moreover, $U_t = \mathscr{X}$ except for t belonging to a finite system $t_1, \ldots, t_k$. As F_{t_i} is u.s.c. at y_0, it follows that $y_0 \in \operatorname{Int}[F_{t_i}^{-1}(2^{U_{t_i}})]$ and consequently

$$y_0 \in \bigcap_{i=1}^{k} \operatorname{Int}[F_{t_i}^{-1}(2^{U_{t_i}})] = \operatorname{Int}\Big[\bigcap_{i=1}^{k} F_{t_i}^{-1}(2^{U_{t_i}})\Big].$$

By (3), $\bigcap_{i=1}^{k} F_{t_i}^{-1}(2^{U_{t_i}}) \subset F^{-1}(2^G)$, hence $y_0 \in \operatorname{Int}[F^{-1}(2^G)]$.

THEOREM 6. *Let $\mathscr{X}_j$ be compact and $F_j\colon \mathscr{Y} \to 2^{\mathscr{X}_j}$ for $j = 0, 1$. Put $F(y) = F_0(y) \times F_1(y)$. Then if F_0 and F_1 are u.s.c. at y_0, so is F.*

Proof. As $F_0(y) \times F_1(y) = [F_0(y) \times \mathscr{X}_1] \cap [\mathscr{X}_0 \times F_1(y)]$ and as the intersection of two u.s.c. mappings at y_0 is again u.s.c. at y_0, the proof can be restricted to the particular case where $F_1(y) = \mathscr{X}_1$.

Let $G \subset \mathscr{X}_0 \times \mathscr{X}_1$ be open and such that $F_0(y_0) \times \mathscr{X}_1 \subset G$. We must define (compare § 18, p. 176, Theorem 3) an open $H \subset \mathscr{Y}$, containing y_0 and such that

$$y \in H \Rightarrow F_0(y) \times \mathscr{X}_1 \subset G. \tag{6}$$

Now, owing to the compactness of $\mathscr{X}_1$, there is an open $U \subset \mathscr{X}_0$ such that (compare § 41, IV, Theorem 1):

$$F_0(y_0) \times \mathscr{X}_1 \subset U \times \mathscr{X}_1 \subset G. \tag{7}$$

Thus $F_0(y_0) \subset U$, and since F_0 is u.s.c. at y_0, there is an open $H \subset \mathscr{Y}$ containing y_0 and such that

$$y \in H \Rightarrow F_0(y) \subset U. \tag{8}$$

Clearly, (8) and (7) imply (6).

THEOREM 7. *Let $\mathscr{X}_j$, F_j and F be as in Theorem 6. If F_0 and F_1 are l.s.c. at y_0, so is F.*

Proof. Let $K \subset \mathscr{X}_0 \times \mathscr{X}_1$ be closed and suppose that

$$y_0 \notin F^{-1}(2^K), \quad \text{i.e.} \quad F_0(y_0) \times F_1(y_0) \not\subset K. \tag{9}$$

We must show that $y_0 \notin \overline{F^{-1}(2^K)}$. In other terms, we have to define an open $U \subset \mathscr{Y}$, containing y_0 such that

$$U \cap F^{-1}(2^K) = 0, \quad \text{i.e.} \quad F_0(y) \times F_1(y) \not\subset K \quad \text{for each } y \in U. \tag{10}$$

Now, by (9), there is, for $j = 0, 1$, an $x_j \in F_j(y_0)$ such that $(x_0, x_1) \notin K$. There is therefore an open $V_j \subset \mathscr{X}_j$ such that

$$x_j \in V_j \quad \text{and} \quad (V_0 \times V_1) \cap K = 0. \tag{11}$$

It follows that

$$x_j \in [V_j \cap F_j(y_0)], \text{ hence } F_j(y_0) \not\subset \mathscr{X}_j - V_j, \text{ i.e. } y_0 \notin F_j^{-1}(2^{\mathscr{X}_j - V_j}). \tag{12}$$

Since F_j is l.s.c. at y_0, it follows that $y_0 \notin \overline{F_j^{-1}(2^{\mathscr{X}_j - V_j})}$. Hence there is an open $U_j \subset \mathscr{Y}$, containing y_0 and such that

$$U_j \cap F_j^{-1}(2^{\mathscr{X}_j - V_j}) = 0, \quad \text{i.e. that} \quad F_j(y) \cap V_j \neq 0 \quad \text{for each } y \in U_j. \tag{13}$$

Put $U = U_1 \times U_2$. It follows from (13) that, for $y \in U$,

$$\big(F_0(y) \times F_1(y)\big) \cap (V_0 \times V_1) \neq 0,$$

which implies (10) by virtue of (11).

COROLLARY 1. *Let $\mathscr{X}_j$, F_j and F be as in Theorem 6. If F_0 and F_1 are continuous at y_0, so is F.*

COROLLARY 2. *If $\mathscr{X}_0$ and $\mathscr{X}_1$ are compact, the product operation $K \times L$ is a continuous mapping of $2^{\mathscr{X}_0} \times 2^{\mathscr{X}_1}$ into $2^{\mathscr{X}_0 \times \mathscr{X}_1}$.*

Remark. Theorems 6 and 7 and their corollaries can be extended to *infinite* cartesian multiplication.

II. Case of $\mathscr{X}$ compact metric. Let $\mathscr{X}$ be compact metric and $F: \mathscr{Y} \to 2^{\mathscr{X}}$.

THEOREM 1 [1]. *The mapping F is u.s.c. if and only if*

$$(\lim y_n = y) \Rightarrow \big(\mathrm{Ls}\, F(y_n) \subset F(y)\big); \tag{0}$$

[1] Compare W. A. Wilson, Amer. Journ. Math. 48 (1926), p. 165.

in other terms, *if and only if the conditions*

$$\lim y_n = y, \quad \lim x_n = x \quad \textit{and} \quad x_n \in F(y_n) \tag{1}$$

imply

$$x \in F(y). \tag{2}$$

The theorem follows from Theorem 4 of Section I, because the implication (1) $\Rightarrow$ (2) means that the set D is closed.

THEOREM 2. *F is l.s.c. if and only if*

$$(\lim y_n = y) \Rightarrow \big(F(y) \subset \operatorname{Li} F(y_n)\big); \tag{3}$$

in other terms, *if and only if the conditions*

$$\lim y_n = y \quad \textit{and} \quad x \in F(y) \tag{4}$$

imply the existence of a sequence $x_1, x_2, \ldots$ such that

$$\lim x_n = x \quad \textit{and} \quad x_n \in F(y). \tag{5}$$

Proof. Let $M = \overline{M} \subset \mathscr{X}$ and $N = \underset{y}{E}[F(y) \subset M]$.

1. Suppose that (3) is true, and let $\lim y_n = y$ and $F(y_n) \subset M$; hence $y_n \in N$ and $y \in \overline{N}$. But by (3) we have

$$F(y) \subset \operatorname{Li} F(y_n) \subset M, \quad \text{hence} \quad y \in N.$$

Consequently N is closed, which means that F is l.s.c.

2. Next, suppose that (3) is not true. Hence we have

$$\lim y_n = y, \quad x \in F(y) \quad \text{and} \quad x \notin \operatorname{Li} F(y_n).$$

Therefore, there are an open G and a sequence $k_1 < k_2 < \ldots$ such that $x \in G$ and $G \cap F(y_{k_n}) = 0$. Put $M = \mathscr{X} - G$. If follows that $y_{k_n} \in N$ and $y \notin N$, hence N is not closed, which means that F is not l.s.c.

Remark 1. Theorems 1 and 2 can be easily localized: F is u.s.c. (resp. l.s.c.) in y if and only if proposition (0) (resp. (3)) is fulfilled.

THEOREM 3. *F is l.s.c. if and only if the set*

$$J_\eta = \underset{xy}{E}\{\varrho[x, F(y)] < \eta\}$$

is open for each $\eta > 0$.

Proof. 1. Suppose that F is l.s.c. Let

$$(x, y) \in J_\eta, \quad \lim x_n = x \quad \text{and} \quad \lim y_n = y. \tag{6}$$

We have to show that, for sufficiently large n, one has $(x_n, y_n) \in J_\eta$, i.e. that

$$\varrho[x_n, F(y_n)] < \eta. \tag{7}$$

Now, it follows from $(x, y) \in J_\eta$ that there is $x' \in F(y)$ such that $|x - x'| < \eta$. Since F is l.s.c., we have

$$x' = \lim x_n' \quad \text{and} \quad x_n' \in F(y_n).$$

Therefore $|x_n' - x_n| < \eta$ for sufficiently large n. This completes the proof of (7).

2. Next suppose that F is not l.s.c. at the point y. Hence we have

$$x \in F(y) - \operatorname{Li} F(y_n) \quad \text{and} \quad y = \lim y_n.$$

There are therefore $\eta > 0$ and a sequence $k_1 < k_2 < \dots$ such that

$$\varrho[x, F(y_{k_n})] > \eta > 0, \quad \text{hence} \quad (x, y_{k_n}) \notin J_\eta.$$

Since $(x, y) \in J_\eta$, it follows that J_η is not open.

THEOREM 4. *If F is u.s.c., so is $\delta \circ F$.*

Proof. Let $\lim y_n = y$. We have to show that

$$\limsup \delta[F(y_n)] \leqslant \delta[F(y)], \tag{8}$$

where $\delta(A)$ denotes the diameter of A (note that, since $\mathscr{X}$ is compact, $F(y_n)$ and $F(y)$ are bounded).

Suppose that (8) is not true. Then we may assume that

$$\lim \delta[F(y_n)] > \delta[F(y)]. \tag{9}$$

Since $F(y_n)$ is compact, there are (see § 41, VI, Corollary 2d) x_n and x_n' such that

$$x_n \in F(y_n), \quad x_n' \in F(y_n) \quad \text{and} \quad \delta[F(y_n)] = |x_n - x_n'|. \tag{10}$$

Choose $k_1 < k_2 < \dots$ so that the sequences x_{k_n} and x_{k_n}' be convergent:

$$\lim x_{k_n} = x \quad \text{and} \quad \lim x_{k_n}' = x'. \tag{11}$$

It follows by (10) and (11) that x and x' belong to $\mathrm{Ls}\, F(y_n)$, hence (by Theorem 1) to $F(y)$.

Therefore $|x-x'| \leqslant \delta[F(y)]$.

On the other hand, we have by (11), (10) and (9):

$$|x-x'| = \lim |x_{k_n} - x'_{k_n}| = \lim \delta[F(y_{k_n})] > \delta[F(y)],$$

which is a contradiction.

III. Decompositions of compact spaces. Let us recall (see § 19, I) that, given a family $\boldsymbol{D}$ (called a *decomposition* of the space $\mathscr{X}$) of closed, non-void and disjoint subsets of $\mathscr{X}$ whose union is $\mathscr{X}$, the topology in $\boldsymbol{D}$ (called the *quotient topology*) is defined as follows:

$\boldsymbol{A} \subset \boldsymbol{D}$ *is open* (*in* $\boldsymbol{D}$) *if and only if* $\mathrm{S}(\boldsymbol{A})$ *is open* (*in* $\mathscr{X}$).

The mapping $\boldsymbol{P}$: $\mathscr{X} \to \boldsymbol{D}$, called *projection*, is defined by means of the condition: $[D = \boldsymbol{P}(x)] \equiv (x \in D \in \boldsymbol{D})$.

It is easily seen that *the mapping* $\boldsymbol{P}$ *is continuous.*

The decomposition $\boldsymbol{D}$ is called *upper* (*lower*) *semi-continuous* if for each open (closed) $B \subset \mathscr{X}$ the union of all $D \in \boldsymbol{D}$ contained in B is open (closed); this is equivalent to the condition of $\boldsymbol{P}$ being a closed (open) mapping.

Let us add that the notion of semi-continuity can be localized (at a $D \in \boldsymbol{D}$; see § 19, p. 185).

THEOREM 1. *If* $\mathscr{X}$ *is compact, so is* $\boldsymbol{D}$ (*in its quotient topology*).

If, moreover, $\boldsymbol{D}$ *is upper semi-continuous and* $\mathscr{X}$ *is a* $\mathscr{T}_2$*-space, so is* $\boldsymbol{D}$ (*hence* $\boldsymbol{D}$ *is a normal space*).

Proof. The compactness of $\boldsymbol{D}$ follows from the continuity of the projection $\boldsymbol{P}$: $\mathscr{X} \to \boldsymbol{D}$ (by Theorem 1 of § 41, III).

If $\mathscr{X}$ is a $\mathscr{T}_2$-space, then $\mathscr{X}$ is normal (by Theorem 3 of § 41, II). Therefore, if $\boldsymbol{D}$ is u.s.c., $\boldsymbol{D}$ is also normal since the normality is invariant under u.s.c. decomposition (by Theorem 5 of § 19, II).

Remark 1. As seen from the above theorem, *if* $\boldsymbol{D}$ *is an u.s.c. decomposition of a compact* $\mathscr{T}_2$*-space, there is a compact* $\mathscr{T}_2$*-space* $\mathscr{Y}$ (*namely* $\boldsymbol{D}$) *and a continuous* f: $\mathscr{X} \to \mathscr{Y}$ (*namely* $\boldsymbol{P}$) *such that the elements of* $\boldsymbol{D}$ *are the inverse images* $\overset{-1}{f}(y)$ *of points* y *of* $\mathscr{Y}$.

There is just one space $\mathscr{Y}$ (up to homeomorphisms) satisfying this condition; this means that if f_1: $\mathscr{X} \to \mathscr{Y}_1$ is continuous and such that the elements of $\boldsymbol{D}$ coincide with the sets $\overset{-1}{f_1}(y)$ where $y \in \mathscr{Y}_1$, then $\mathscr{Y}_1$ is homeomorphic to $\mathscr{Y}$.

Namely the composed mapping $g = f_1 \circ \overset{-1}{f}$ is a homeomorphism of $\mathscr{Y}$ onto $\mathscr{Y}_1$ (because for $A = \bar{A} \subset \mathscr{Y}$, we have $g(A) = \overline{g(A)}$ in $\mathscr{Y}_1$).

THEOREM 2. *Let $\mathscr{X}$ and $\mathscr{Y}$ be compact $\mathscr{T}_2$-spaces and $F\colon \mathscr{Y} \to 2^{\mathscr{X}}$ an upper semi-continuous mapping such that*

$$F(y) \cap F(y') = 0 \quad \textit{for} \quad y \neq y' \tag{i}$$

and

$$\mathscr{X} = \bigcup_{y \in \mathscr{Y}} F(y). \tag{ii}$$

Then formula (ii) *represents an u.s.c. decomposition of $\mathscr{X}$.*

Proof. Let $A \subset \mathscr{X}$ be closed. We have to show that the union U of all $F(y)$ such that $A \cap F(y) \neq 0$ is closed. Now

$$(x \in U) \equiv \bigvee_{y} [x \in F(y)][A \cap F(y) \neq 0]$$
$$\equiv \bigvee_{y,x'} [x \in F(y)][x' \in F(y)][x' \in A].$$

By Theorem 1 of § 18, III, the sets $\underset{xy}{E}[x \in F(y)]$ and $\underset{x'y}{E}[x' \in F(y)]$ are closed. Hence, by virtue of the compactness of $\mathscr{X}$ and $\mathscr{Y}$, U is closed (see § 41, IV, Corollary 1b).

THEOREM 3. *If $\mathscr{X}$ is a compact $\mathscr{T}_2$-space and $f\colon \mathscr{X} \to \mathscr{Y}$ is continuous and onto, then the decomposition*

$$\mathscr{X} = \bigcup_{y \in \mathscr{Y}} \overset{-1}{f}(y)$$

is upper semi-continuous.

This follows from Theorem 2 and Theorem 1 of Section I.

IV. Decompositions of compact metric spaces. We assume in this section that $\mathscr{X}$ is *compact metric.*

THEOREM 1 (of P. Alexandrov) (1). *Each upper semi-continuous*

(1) *Über stetige Abbildungen kompakter Räume*, Math. Ann. 96 (1927), pp. 555–571.

For problems of metrizability and paracompactness of $\boldsymbol{D}$, see also A. Archangielski, *A condition for preserving metrizability by factor mappings* (Russian), Dokl. Acad. Nauk URSS 164 (1965), p. 9, and *On the behaviour of metrizability by factor mappings* (Russian), *ibid.* p. 247.

decomposition $\boldsymbol{D}$ *of* $\mathscr{X}$ *is (in its quotient topology) homeomorphic to a compact metric space.*

Proof. By Theorem 1 of Section III, $\boldsymbol{D}$ is a $\mathscr{T}_2$-space. Moreover, $\boldsymbol{D}$ is a continuous image of the compact metric space $\mathscr{X}$. Hence $\boldsymbol{D}$ is compact and metrizable (by Theorem 3 of § 41, VI).

THEOREM 2. *Each of the following conditions is necessary and sufficient for the decomposition* $\boldsymbol{D}$ *of* $\mathscr{X}$ *to be upper semi-continuous*:

(i) $(D \cap \mathrm{Li} D_n \neq 0) \Rightarrow (\mathrm{Ls} D_n \subset D)$ (*where* D, $D_n \in \boldsymbol{D}$),

(ii) *if* $D_1, D_2, \ldots, D_n, \ldots$ *is convergent, its limit is contained in a single element of* $\boldsymbol{D}$.

Proof. 1. Suppose that the decomposition $\boldsymbol{D}$ is u.s.c. Then by the Remark to Theorem 3 of Section III, there is a compact (metric) space $\mathscr{Y}$ and a continuous mapping $f: \mathscr{X} \to \mathscr{Y}$ such that each $D \in \boldsymbol{D}$ is of the form $D = \overset{-1}{f}(y)$, $y \in \mathscr{Y}$.

Suppose now that $x \in (D \cap \mathrm{Li} D_n)$. Hence

$$x \in [\overset{-1}{f}(y) \cap \mathrm{Li}\, \overset{-1}{f}(y_n)],$$

i.e.

$$f(x) = y \quad \text{and} \quad x = \lim x_n \quad \text{where} \quad f(x_n) = y_n.$$

Put

$$x' \in \mathrm{Ls} D_n, \quad \text{i.e.} \quad x' = \lim x'_{k_n} \quad \text{where} \quad f(x'_{k_n}) = y_{k_n}.$$

We must show that $x' \in D$, i.e. that $f(x') = y$.

Now, since f is continuous, we have

$$f(x') = \lim f(x'_{k_n}) = \lim y_{k_n} = \lim f(x_{k_n}) = f(x) = y.$$

Thus $x' \in \overset{-1}{f}(y)$, i.e. $x' \in D$. This completes the proof.

2. Clearly (i) $\Rightarrow$ (ii).

3. Suppose that (ii) is true. We must show that $\boldsymbol{D}$ is u.s.c., which means that if $B \subset \mathscr{X}$ is closed, then so is the union of all D such that $D \cap B \neq 0$. In other terms, suppose that

$$x_n \in D_n, \quad \lim x_n = x \in D, \quad D_n \cap B \neq 0; \tag{4}$$

we have to show that $D \cap B \neq 0$.

As the space $2^{\mathscr{X}}$ is compact, we may assume that the sequence $D_1, D_2, \ldots$ is convergent: $\mathrm{Lim} D_n = L$. By assumption, L is contained in a single element of $\boldsymbol{D}$. Furthermore $L \cap B \neq 0$ since the family of all elements of $2^{\mathcal{X}}$ which have points in common with B is closed (by Theorem 1 of § 17, p. 162), and hence

$$(D_n \cap B \neq 0) \Rightarrow (L \cap B \neq 0) \Rightarrow (D \cap B \neq 0).$$

THEOREM 3. *The following condition is necessary and sufficient for the decomposition* $\boldsymbol{D}$ *of* $\mathcal{X}$ *to be lower semi-continuous*:

$$(D \cap \operatorname{Li} D_n \neq 0) \Rightarrow (D \subset \operatorname{Li} D_n) \quad (\textit{where } D, D_n \in \boldsymbol{D}). \tag{iii}$$

Proof. 1. *Necessity*. Let $D \cap \operatorname{Li} D_n \neq 0$. Thus we have

$$x \in D, \quad x = \lim x_n \quad \text{and} \quad x_n \in D_n. \tag{5}$$

Let $x' \in D$. We must show that $x' \in \operatorname{Li} D_n$ under the assumption the $\boldsymbol{D}$ is l.s.c.; this means that, whatever the open set G containing x' is, we have $D_n \cap G \neq 0$ for $n > n_0$.

Now denote by U the union of all $D' \in \boldsymbol{D}$ such that $D' \cap G \neq 0$. Since $\boldsymbol{D}$ is l.s.c., U is open. Since $x' \in D \cap G$, we have $D \subset U$ and by (5) there is n_0 such that $D_n \cap U \neq 0$ for $n > n_0$. It follows that $D_n \subset U$ (since the elements of $\boldsymbol{D}$ are disjoint). But this means that $D_n \cap G \neq 0$.

2. *Sufficiency*. Suppose $\boldsymbol{D}$ is not l.s.c. This means that there is an open G such that the union U of all $D \in \boldsymbol{D}$ for which $D \cap G \neq 0$ is not open.

Hence there are x and D such that

$$x \in D \subset U, \quad x = \lim x_n \quad \text{and} \quad x_n \notin U. \tag{6}$$

Denote by D_n the element of $\boldsymbol{D}$ such that $x_n \in D_n$. Hence $D_n \cap G = 0$. Since $D \subset U$, we have $D \cap G \neq 0$; put $x_0 \in D \cap G$. Since $x_0 \in G$ while $D_n \cap G = 0$, we have $x_0 \notin \operatorname{Li} D_n$, and therefore $D \not\subset \operatorname{Li} D_n$.

On the other hand, $x \in D \cap \operatorname{Li} D_n$ (by (6)). Thus the implication (iii) is not fulfilled.

V. Continuous decompositions of compact spaces. Let us recall that a decomposition $\boldsymbol{D}$ of an arbitrary topological space $\mathcal{X}$ is called *continuous* if it is simultaneously upper and lower semi-continuous (see § 19, p. 185).

It follows that $\boldsymbol{D}$ is a continuous decomposition if and only if the projection $\boldsymbol{P}$: $\mathcal{X} \to \boldsymbol{D}$ is simultaneously a closed and open mapping.

Theorem 1. *Let $\mathscr{X}$ be a compact $\mathscr{T}_2$-space and $f\colon \mathscr{X} \to \mathscr{Y}$ continuous and onto. Then the decomposition $\boldsymbol{D}$ of $\mathscr{X}$ into inverse images of single points of $\mathscr{Y}$ is continuous if and only if the mapping f is open.*

Proof. 1. Suppose that $\boldsymbol{D}$ is continuous. Then the mapping $\overset{-1}{f}\colon \mathscr{Y} \to 2^{\mathscr{X}}$ is continuous (here the topology of $\boldsymbol{D}$ and of $2^{\mathscr{X}}$ coincide; see § 19, p. 187); and by Theorem 5 of § 18, III, f is open.

2. Suppose that f is open. Let $G \subset \mathscr{X}$ be an arbitrary open set and denote by U the union of all $\overset{-1}{f}(y)$ such that $\overset{-1}{f}(y) \cap G \neq 0$. Clearly $U = f^{-1}f(G)$ and since $f(G)$ is open and f continuous, it follows that U is open. This means that $\boldsymbol{D}$ is l.s.c. By Theorem 3 of Section III, $\boldsymbol{D}$ is also u.s.c., hence $\boldsymbol{D}$ is continuous.

Remark. We shall show in Section VII that each semi-continuous decomposition of a compact metric space contains members of continuity.

The next theorem will be applied later (see § 46, VI).

Theorem 2. [1] *Let $\mathscr{X}$ be compact metric and $f\colon \mathscr{X} \to \mathscr{Y}$ be an open mapping onto. If all sets $\overset{-1}{f}(y)$ are countable (or, more generally, contain isolated points), then there is a sequence of closed sets $F_1, F_2, \ldots$ such that*

$$\bigcup_{n=1}^{\infty} f(F_n) = \mathscr{Y} \text{ and } f|F_n \text{ is a homeomorphism for } n = 1, 2, \ldots. \tag{1}$$

Proof. We need define only a sequence $A_1, A_2, \ldots$ of $\boldsymbol{F}_\sigma$-sets such that

$$\bigcup_{m=1}^{\infty} f(A_m)' = \mathscr{Y} \tag{2}$$

and that the partial mappings $f|A_m$ are one-to-one. Because if we put

$$A_m = \bigcup_{i=1}^{\infty} A_m^i \quad \text{where } A_m^i \text{ is closed},$$

the required sequence $F_1, F_2, \ldots, F_n, \ldots$ is obtained by arranging the double sequence $\{A_m^i\}$ in a simple one.

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[1] Compare P. Alexandrov, C. R. Acad. U.R.R.S. 1936, p. 283.

Now, let $R_1, R_2, \ldots$ be a countable open base of $\mathcal{X}$ and put

$$A_m = \mathop{E}_{x} \bigvee_{y} [R_m \cap \overset{-1}{f}(y) = (x)]. \tag{3}$$

Since $\mathop{E}_{xF}[R_m \cap F = (x)]$ is by the lemma of § 42, III, an $\boldsymbol{F}_\sigma$-set (in the space $\mathcal{X} \times 2^{\mathcal{X}}$), then so is $\mathop{E}_{xy}[R_m \cap \overset{-1}{f}(y) = (x)]$ (in $\mathcal{X} \times \mathcal{Y}$), since the mapping f is open and hence $\overset{-1}{f}$ continuous (by Corollary 2 of § 13, XIV). Its projection on the $\mathcal{X}$-axis is A_m (by formula (3)), and hence A_m is also an $\boldsymbol{F}_\sigma$-set (by Corollary 1a of § 41, IV).

In order to prove (1), put $y \in \mathcal{Y}$. By assumption, the set $\overset{-1}{f}(y)$ contains an isolated point; call it x. Hence there is an m such that

$$R_m \cap \overset{-1}{f}(y) = (x), \qquad \text{whence} \qquad x \in A_m \qquad \text{and} \qquad y \in f(A_m).$$

We shall prove finally that the mapping $f|A_m$ is one-to-one. Let $x \in A_m$, $x' \in A_m$ and $f(x) = f(x')$. By the definition of A_m, there are y and y' such that

$$R_m \cap \overset{-1}{f}(y) = (x) \qquad \text{and} \qquad R_m \cap \overset{-1}{f}(y') = (x'). \tag{4}$$

Therefore, $y = f(x)$ and $y' = f(x')$, and since $f(x) = f(x')$, so $y = y'$. Replace y' by y in the second part of (4). It follows that $x = x'$. This completes the proof.

VI. Examples. Identification of points.

1. Let $\mathcal{X}$ denote the n-dimensional ball and $\boldsymbol{D}$ its decomposition having as one element the boundary $\mathcal{S}_{n-1}$ of $\mathcal{X}$ and as other elements the one-element sets (x) where $x \in \mathcal{X} - \mathcal{S}_{n-1}$. Then $\boldsymbol{D}$ is u.s.c. and is homeomorphic to the sphere $\mathcal{S}_n$ (in this case we say that the points of $\mathcal{S}_{n-1}$ have been identified).

2. Let $\mathcal{X} = \mathcal{S}_n$ and let us identify pairs of antipodal points. Then $\boldsymbol{D}$ is homeomorphic to the *n-dimensional projective space*. Here $\boldsymbol{D}$ is a continuous decomposition of $\mathcal{S}_n$.

3. Let $\mathcal{X} = \mathcal{I}^2$. Let us identify the antipodal points of the vertical sides of $\mathcal{X}$. The decomposition of $\mathcal{I}^2$ thus obtained is the *Möbius strip*.

If we identify points of the vertical sides having the same ordinate, we obtain a *cylindrical surface.*

If in addition we identify points of horizontal sides having the same ordinate we are led to the surface of the *torus.*

4. *Every compact metric (non-empty) space $\mathscr{Y}$ can be considered as an u.s.c. decomposition of the Cantor discontinuum $\mathscr{C}$.*

For, by Corollary 2b of § 41, VI, $\mathscr{Y}$ is a continuous image of $\mathscr{C}$, and hence is homeomorphic to the decomposition space of $\mathscr{C}$ in inverse images of single points of $\mathscr{Y}$.·

VII. Relationships of semi-continuous mappings to the mappings of class 1.

THEOREM 1. *If $\mathscr{X}$ is compact metric, each semi-continuous mapping F: $\mathscr{Y} \to 2^{\mathscr{X}}$ is (B-measurable) of class 1 (i.e. the inverse images of open sets are $\boldsymbol{F}_\sigma$-sets).*

Proof. Since $\mathscr{X}$ is compact metric, so is $2^{\mathscr{X}}$, and hence $2^{\mathscr{X}}$ has a countable open base. As the totality of the sets either of the form $\underset{K}{E}(K \subset G)$ or $\underset{K}{E}(K \cap G \neq 0)$ (where $G \subset \mathscr{X}$ is open) is a subbase of $2^{\mathscr{X}}$ (see § 42, I), it suffices to prove that the inverse images of the sets belonging to that subbase are $\boldsymbol{F}_\sigma$-sets (compare § 31, II, Theorem 1). In other terms, we have to show that, under the assumption of the semi-continuity of F, each of the sets

$$\underset{y}{E}[F(y) \subset G], \tag{i}$$

$$\underset{y}{E}[F(y) \cap G \neq 0], \tag{ii}$$

is an $\boldsymbol{F}_\sigma$-set.

In our proof we shall make use of the following representation of G (compare § 41, II, Corollary 4):

$$G = K_1 \cup K_2 \cup \ldots, \quad \text{where } K_n \text{ is closed and } K_n \subset \operatorname{Int}(K_{n+1}). \tag{1}$$

First, assume that F is u.s.c. Then, by definition, the set (i) is open, hence an $\boldsymbol{F}_\sigma$-set.

Furthermore, it follows by (1) that

$$\underset{y}{E}[F(y) \cap G \neq 0] = \bigcup_{n=1}^{\infty} \underset{y}{E}[F(y) \cap K_n \neq 0],$$

and since $\underset{y}{E}[F(y) \cap K_n \neq 0]$ is closed, the set (ii) is $\boldsymbol{F}_\sigma$.

Next, assume that F is l.s.c. Then the set (ii) is open by definition. Furthermore, if $F(y) \subset G$ then, by formula (1) and by the compactness of $F(y)$, there is n such that $F(y) \subset K_n$. Therefore

$$\underset{y}{E}\,[F(y) \subset G] = \bigcup_{n=1}^{\infty} \underset{y}{E}\,[F(y) \subset K_n]\,,$$

and since $\underset{y}{E}[F(y) \subset K_n]$ is open, the set (i) is $\boldsymbol{F}_\sigma$.

COROLLARY 1. *Let $\mathcal{Y}$ be metric and $\mathcal{X}$ compact metric. If $F\colon\ \mathcal{Y} \to 2^{\mathcal{X}}$ is semi-continuous, then the set of points of discontinuity of F is of the first category.*

Consequently, *if $\mathcal{Y}$ is complete, then the set of continuity points of F, i.e. of points y such that* (comp. Remark 1 of Section II):

$$(\lim y_n = y) \Rightarrow \big(\mathrm{Lim}\, F(y_n) = F(y)\big),$$

is dense in $\mathcal{Y}$.

Proof. The first part of the corollary follows from Theorem 1 of § 31, X, asserting that the set of points of discontinuity of a function of class 1 is of the first category. The second part follows from the Theorem of Baire asserting that in a complete space the complement of a set of the first category is dense in the space (see § 34, IV).

Remark. Corollary 1 can be shown under more general assumptions, namely assuming that $\mathcal{Y}$ is a topological space, $\mathcal{X}$ a metric space and $F(y)$ compact for each $y \in \mathcal{Y}$ (1).

COROLLARY 2 (2). *Let, as in Theorem 3 of Section* I, *D be a closed subset of $\mathcal{X} \times \mathcal{Y}$ (where $\mathcal{X}$ is a compact and $\mathcal{Y}$ a metric space) and $F(y)$ its horizontal section, i.e.*

$$F(y) = \underset{x}{E}\,[(x, y) \in D]. \tag{2}$$

Then if $\boldsymbol{P}$ is a Borel subset of $2^{\mathcal{X}}$, the set $B = \underset{y}{E}\,[F(y) \in \boldsymbol{P}]$ is a Borel subset of $\mathcal{Y}$.

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(1) See M. K. Fort, Jr., *Points of continuity of semi-continuous functions*, Public. Mathem., Debrecen, 2 (1951), pp. 100–102.

(2) See the paper of E. Marczewski-Szpilrajn and myself *Sur les cribles fermés et leurs applications*, Fund. Math. 18 (1931), p. 160.

Proof. Since F is a u.s.c. mapping (by Theorem 3 of Section I), then F is a mapping of class 1, and consequently the set $F^{-1}(\boldsymbol{P}) = B$ is a Borel set.

COROLLARY 3 [1]. *The families $\boldsymbol{P}_1$ of closed countable sets and $\boldsymbol{P}_2$ of closed sets containing no irrational numbers are non-borelian sets in the space $2^{\mathscr{I}}$.*

Proof. Let A be an analytic non-borelian subset of $\mathscr{I}$ (comp. § 38, VI). We shall define a closed subset D of $\mathscr{I}^2$ such that (2) implies

$$\underset{y}{E}[F(y)\in\boldsymbol{P}_1] = \mathscr{I}-A = \underset{y}{E}[F(y)\in\boldsymbol{P}_2]. \tag{3}$$

This will complete the proof since Corollary 2 will imply that $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are non-borelian.

Let $f\colon \mathscr{N}\to A$ be a continuous function onto assuming each value uncountably many times (compare § 39, VII, Remark 1; let us note that the last assumption about f may be omitted in proving that $\boldsymbol{P}_2$ is not borelian). Put

$$D = \overline{\underset{xy}{E}[y=f(x)]} \quad \text{where} \quad x\in\mathscr{I} \text{ and } y\in\mathscr{I}.$$

Since f is continuous, the set $\underset{xy}{E}[y=f(x)]$ is closed in $\mathscr{N}\times\mathscr{I}$, i.e.

$$\underset{xy}{E}[y=f(x)] = D\cap(\mathscr{N}\times\mathscr{I}).$$

Thus the conditions $x\in\mathscr{N}$ and $(x,y)\in D$ yield $y=f(x)$, hence $y\in A$. In other terms, if $y\in\mathscr{I}-A$, the condition $(x,y)\in D$ implies $x\in\mathscr{I}-\mathscr{N}$, i.e. $F(y)\subset\mathscr{I}-\mathscr{N}$ and hence $F(y)\in\boldsymbol{P}_2$.

Conversely, if $y\in A$, the set $\overset{-1}{f}(y)$ is uncountable and so is $F(y)$ since $\overset{-1}{f}(y)\subset F(y)$. Consequently $F(y)\notin\boldsymbol{P}_1$.

This completes the proof of (3) (since $\boldsymbol{P}_2\subset\boldsymbol{P}_1$).

COROLLARY 4 [2]. *Each upper semi-continuous decomposition $\boldsymbol{D}$ of a compact metric space $\mathscr{X}$ contains members of continuity.*

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[1] See W. Hurewicz, *Zur Theorie der analytischen Mengen*, Fund. Math. 15 (1930), pp. 4–17.

[2] See my paper *Sur les décompositions semi-continues d'espaces métriques compacts*, Fund. Math. 11 (1928), p. 176 (Theorem VII). Compare also L. S. Hill, *Properties of certain aggregate functions*, Amer. Journ. Math. 49 (1927), pp. 419–432.

More precisely, *the family of all members of continuity is a dense G_δ-set in $\boldsymbol{D}$ (in its quotient topology).*

Proof. By the Remark of Section III there is $f\colon \mathscr{X} \to \mathscr{Y}$ continuous and such that the members of the decomposition coincide with the sets $\overset{-1}{f}(y)$ where $y \in \mathscr{Y}$ (here $\mathscr{Y}$ can be identified with $\boldsymbol{D}$). Since the mapping $\overset{-1}{f}\colon \mathscr{Y} \to 2^{\mathscr{X}}$ is u.s.c. (compare § 19, IV, Theorem 2) the rest of the proof follows from Corollary 1.

VIII. Examples of mappings of class 2 which are not of class 1.

1. *The boundary* (i.e. the mapping F defined by the condition $F(K) = K \cap \overline{X - K}$ for $K \in 2^{\mathscr{X}}$) *is a mapping of* $2^{\mathscr{X}}$ *into* $2^{\mathscr{X}}$ *of class* 2 ($\mathscr{X}$ is supposed to be compact metric).

F is a composition of two semi-continuous mappings, namely of the intersection and the closure of the complement (by Theorem 1 of § 18, V and the Corollary of § 18, VI), hence—a composition of two mappings of class 1 (by Theorem 1 of Section VII), and therefore is of class 2 (by Theorem 1 of § 31, III).

Furthermore, the boundary may fail to be of class 1. Such is the case where $\mathscr{X}$ is the Cantor discontinuum $\mathscr{C}$. For, it is easily seen (¹) that F is in this case discontinuous at each point K ($\neq 0$).

2. *The derivative of* K (i.e. the set of all accumulation points of K) *is a mapping of class* 2 (²). Furthermore, if $\mathscr{X} = \mathscr{C}$, it is discontinuous at each point $K \neq 0$, since every finite set has an empty derivative, but it is (if it is non-empty) the limit of infinite sets, hence of sets having a non-empty derivative; similarly every infinite set is the limit of finite sets.

The preceding examples show that in the domain of set-valued mappings (unlike the domain of real valued functions of a real variable), very simple and important mappings are neither continuous nor even of class 1.

Let us add that by virtue of Corollary 3 of Section VII, the characteristic function defined on $2^{\mathscr{I}}$ for the family of countable closed subsets of $\mathscr{I}$ is *not B-measurable.*

(¹) See my paper cited above: *Les fonctions semi-continues...*, p. 156.
(²) *Ibid.* p. 157.

IX. Remarks concerning selectors. Given a set-valued mapping $F\colon \mathscr{Y} \to 2^{\mathscr{X}}$, where $F(y) \neq 0$ for each $y \in \mathscr{Y}$, the mapping $f\colon \mathscr{Y} \to \mathscr{X}$ is called a *selector of* F if $f(y) \in F(y)$ for each $y \in \mathscr{Y}$.

One can show (1) that *if* $\mathscr{Y}$ *is metric,* $\mathscr{X}$ *complete and separable and* F *semi-continuous, then there is a selector of class* 1.

This statement follows from the following general theorem.

Let $\boldsymbol{L}$ *be a field of subsets of* $\mathscr{Y}$ *(i.e. if* A *and* B *are members of* $\boldsymbol{L}$, *then so are* $A \cup B$, $A \cap B$ *and* $\mathscr{Y} - A$*). Denote by* $\boldsymbol{L}_\sigma$ *the countably additive family induced by* $\boldsymbol{L}$ *(i.e. the family of all countable unions of members of* $\boldsymbol{L}$*). Suppose further that* $\mathscr{X}$ *is complete separable and* $F\colon \mathscr{Y} \to 2^{\mathscr{X}}$ *is such that* $F(y) \neq 0$ *for each* $y \in \mathscr{Y}$ *and*

$$\mathop{E}_{y}[F(y) \cap G \neq 0] \in \boldsymbol{L}_\sigma \quad \textit{whenever } G \subset \mathscr{X} \textit{ is open.}$$

Then there is a selector $f\colon \mathscr{Y} \to \mathscr{X}$ *such that*

$$f^{-1}(G) \in \boldsymbol{L}_\sigma \quad \textit{whenever } G \subset \mathscr{X} \textit{ is open.}$$

This theorem can be applied to B-measurable mappings, to (Lebesgue) measurable functions, to mappings with Baire property, etc. (see Remark below).

Among others it implies the following theorem.

For each complete separable space $\mathscr{X}$ *there is a choice-function* $f\colon [2^{\mathscr{X}} - (0)] \to \mathscr{X}$ *of class* 1, *i.e.* $f(A) \in A$ *for each* $0 \neq A = \bar{A} \subset \mathscr{X}$. f *may be assumed to be continuous if* $\dim \mathscr{X} = 0$.

An elementary argument shows that there is no continuous choice-function for $\mathscr{X} = \mathscr{I}^2$ (of course, for $\mathscr{X} = \mathscr{I}$ a continuous choice-function does exist). (2)

Remark. Let us recall that a mapping $f\colon \mathscr{Y} \to \mathscr{X}$ is called *B-measurable of class* $\alpha < \Omega$ if $f^{-1}(G)$ is a Borel set of additive class α for each open $G \subset \mathscr{X}$; equivalently, if $f^{-1}(K)$ is of multiplicative class α for each closed $K \subset \mathscr{X}$.

(1) See the paper of C. Ryll-Nardzewski and myself *A general theorem on selectors,* Bull. Acad. Polon. Sc. 13 (1965), pp. 397–402; R. Engelking, *ibid.* vol. 16.

(2) For various conditions which imply the existence of continuous selectors, see E. Michael, *Continuous selections in Banach space,* Studia Math. 1963, pp. 75–76. Compare (also for Banach spaces) J. Lindenstrauss, *A selection theorem,* Israel Journ. Math. 2 (1964), pp. 201–204.

See also in this connection a paper of J. M. Day and myself *On the non-existence of a continuous selector for arcs lying in the plane,* Indagationes 28 (1966), pp. 131–132, and E. Michael, Annals of Math. 63 (1956), p. 374.

A generalization of the concept of upper and lower semi-continuity leads to the following denomination.

The mapping $F\colon \mathcal{Y} \to 2^{\mathcal{X}}$ will be said to be *of class* α^{-}, respectively *of class* α_{-} if

$$\underset{y}{E}\{F(y) \subset G\} \text{ is of additive class } \alpha \text{ for } G \text{ open},$$

respectively

$$\underset{y}{E}\{F(y) \subset K\} \text{ is of multiplicative class } \alpha \text{ for } K \text{ closed}.$$

The next statement concerning B-measurable selectors follows from the theorem stated above.

Let $\mathcal{Y}$ be metric and $\mathcal{X}$ complete separable. If $F\colon \mathcal{Y} \to 2^{\mathcal{X}}$ is of class α_{-} (where $\alpha > 0$) and $F(y) \neq 0$, then there is a B-measurable selector of class α $f\colon \mathcal{Y} \to \mathcal{X}$, $f(y) \in F(y)$.

Finally, our theorem can be applied to mappings measurable in the following sense. Let $\boldsymbol{L}$ be a countably additive (hence countably multiplicative) field of subsets of $\mathcal{Y}$ (e.g. $\boldsymbol{L}$ is the field of Lebesgue measurable subsets of an interval). A mapping $f\colon \mathcal{Y} \to \mathcal{X}$ (where $\mathcal{X}$ is a topological space) is called ***L**-measurable* if

$$f^{-1}(G) \in \boldsymbol{L} \quad \text{whenever} \quad G \subset \mathcal{X} \text{ is open}.$$

Of course, a mapping $F\colon \mathcal{Y} \to 2^{\mathcal{X}}$ is $\boldsymbol{L}$-measurable if the sets

$$\underset{y}{E}\{F(y) \cap G \neq 0\} \quad \text{and} \quad \underset{y}{E}\{F(y) \cap K \neq 0\}$$

are members of $\boldsymbol{L}$ for G open and K closed in $\mathcal{X}$.

It follows that:

If $\mathcal{X}$ is complete separable and $F\colon \mathcal{Y} \to 2^{\mathcal{X}}$ is $\boldsymbol{L}$-measurable and $F(y) \neq 0$, then there exists an $\boldsymbol{L}$-measurable selector $f\colon \mathcal{Y} \to \mathcal{X}$, $f(y) \in F(y)$.

§ 44. The space $\mathcal{Y}^{\mathcal{X}}$

I. The compact-open topology of $\mathcal{Y}^{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary topological spaces. We introduce topology—called *natural topology* or *compact-open* (more concisely, the *c.o. topology*)—in $\mathcal{Y}^{\mathcal{X}}$ in the following way.

DEFINITION [1]. For $C \subset \mathcal{X}$ and $H \subset \mathcal{Y}$, put

$$\Gamma(C, H) = \mathop{E}_{f}[f(C) \subset H], \quad \text{where} \quad f \in \mathcal{Y}^{\mathcal{X}}. \tag{1}$$

The compact-open topology of $\mathcal{Y}^{\mathcal{X}}$ is defined by assuming that *the totality of sets $\Gamma(C, H)$, where C is compact and H open, is an open subbase of $\mathcal{Y}^{\mathcal{X}}$.*

THEOREM 1. *If $F \subset \mathcal{Y}$ is closed, so is $\Gamma(C, F)$ (in $\mathcal{Y}^{\mathcal{X}}$) for each $C \subset \mathcal{X}$.*

Proof. Clearly

$$f \in \Gamma(C, F) \equiv \{f(x) \in F \text{ for each } x \in C\}$$
$$\equiv \{f \in \Gamma[(x), F] \text{ for each } x \in C\}.$$

Hence $\mathcal{Y}^{\mathcal{X}} - \Gamma(C, F) \equiv \bigcup_{x \in C} \Gamma[(x), \mathcal{Y} - F]$.

Since (x) is compact and $\mathcal{Y} - F$ open, $\Gamma[(x), \mathcal{Y} - F]$ is open (by definition) and so is $\mathcal{Y}^{\mathcal{X}} - \Gamma(C, F)$.

THEOREM 2. *If $\mathcal{Y}$ is regular, so is $\mathcal{Y}^{\mathcal{X}}$* [2].

Proof. Let $f \in \Gamma(C, H)$ where C is compact and H open. We shall define an open G in $\mathcal{Y}$ such that

$$f \in \Gamma(C, G) \quad \text{and} \quad \overline{\Gamma(C, G)} \subset \Gamma(C, H). \tag{2}$$

Since $f(C) \subset H$, it follows by virtue of the regularity of $\mathcal{Y}$ that for each $x \in C$ there is an open G_x such that

$$f(x) \in G_x \quad \text{and} \quad \bar{G}_x \subset H. \tag{3}$$

Since $f(C)$ is compact, its cover $\{G_x\}$ contains a finite subcover $G_{x_1}, \ldots, G_{x_n}$. Put $G = G_{x_1} \cup \ldots \cup G_{x_n}$. Thus by (3)

$$f(C) \subset G \quad \text{and} \quad \bar{G} = \bar{G}_{x_1} \cup \ldots \cup \bar{G}_{x_n} \subset H. \tag{4}$$

The first part of (4) gives the first part of (2), and by the second part $\Gamma(C, \bar{G}) \subset \Gamma(C, H)$; since $\Gamma(C, \bar{G})$ is closed (by Theorem 1), we obtain the second part of (2):

$$\overline{\Gamma(C, G)} \subset \overline{\Gamma(C, \bar{G})} = \Gamma(C, \bar{G}) \subset \Gamma(C, H).$$

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[1] This definition is due to R. H. Fox; see *On topologies for function spaces*, Bull. Amer. Math. Soc. 51 (1945), pp. 429–432. See also R. Arens, *A topology for spaces of transformations*, Ann. of Math. 47 (1946), pp. 480–495, and J. L. Kelley, *General Topology*, p. 221 ff.

[2] Concerning *complete* regularity, see Section III, Theorem 5.

From (2) we deduce easily that $\mathscr{Y}^{\mathscr{X}}$ is regular [1].

Remark 1. One can prove also that:

(α) *if* $\mathscr{Y}$ *is a* $\mathscr{T}_2$*-space, so is* $\mathscr{Y}^{\mathscr{X}}$; [2]

(β) *if* $\mathscr{X}$ *and* $\mathscr{Y}$ *are separable metric spaces, then* $\mathscr{Y}^{\mathscr{X}}$ *is normal (even hereditarily normal).*

In fact one proves, under the same assumptions, that $\mathscr{Y}^{\mathscr{X}}$ is both hereditarily Lindelöf and hereditarily separable [3].

But a regular Lindelöf space (i.e. a space in which each cover contains a countable cover; see § 5, VII, p. 50) is normal (see § 14, II, p. 122, Theorem 1); moreover, since $\mathscr{Y}$ is metric (hence regular), $\mathscr{Y}^{\mathscr{X}}$ is regular (by Theorem 2) and regularity is hereditary (see § 5, X, p. 52).

Remark 2. It is interesting to note in that direction that $\mathscr{Y}^{\mathscr{I}}$ may fail to be normal even if $\mathscr{Y}$ is compact [4].

Remark 3. *Let* $\mathscr{X}$, $\mathscr{Y}$ *and* $\mathscr{Z}$ *be topological spaces and* $g\colon \mathscr{Y} \to \mathscr{Z}$ *continuous. Put* $\gamma(f) = g \circ f$ *for* $f \in \mathscr{Y}^{\mathscr{X}}$. *Then the mapping* $\gamma\colon \mathscr{Y}^{\mathscr{X}} \to \mathscr{Z}^{\mathscr{X}}$ *is continuous.*

Because $\gamma^{-1}\big(\Gamma(C, H)\big) = \Gamma\big(C, g^{-1}(H)\big)$, which follows from the equivalence $[gf(C) \subset H] \equiv [f(C) \subset g^{-1}(H)]$ for each $f \in \mathscr{Y}^{\mathscr{X}}$.

II. Joint continuity and related problems. In Sections II–IV we shall assume that $\mathscr{Y}^{\mathscr{X}}$ *has the compact-open topology.*

THEOREM 1. *Given* $f\colon \mathscr{X} \to \mathscr{Y}$, *put* $w(f, x) = f(x)$. *If* $\mathscr{X}$ *is a compact* $\mathscr{T}_2$*-space and* $\mathscr{Y}$ *is an arbitrary space, the mapping* $w\colon \mathscr{Y}^{\mathscr{X}} \times \mathscr{X} \to \mathscr{Y}$ *is continuous.*

Moreover, the compact-open topology is the coarsest topology of $\mathscr{Y}^{\mathscr{X}}$ *for which* w *is continuous.*

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(1) See, e.g., Engelking, *loc. cit.*, Chapter 1, § 5, Theorem 3.

(2) See Kelley, *loc. cit.*, p. 222.

(3) See E. Michael, *On a theorem of Rudin and Klee*, Proc. Amer. Math. Soc. 12 (1961), p. 921.

Compare M. E. Rudin and V. L. Klee, *A note on certain function spaces*, Archiv. d. Mathem. 7 (1951), pp. 469–470.

(4) See A. H. Stone, *A note on paracompactness and normality of mappings spaces*, Proc. Amer. Math. Soc. 14 (1963), pp. 81–83.

For the case where $\mathscr{Y}$ is paracompact, see E. Fadell, *B paracompact does not imply* $B^{\mathscr{I}}$ *paracompact*, Proc. Amer. Math. Soc. 9 (1959), pp. 839–840.

Proof. Let $H \subset \mathscr{Y}$ be open. We have to show that the set

$$w^{-1}(H) = \underset{f,x}{E}[f(x) \in H]$$

is open in $\mathscr{Y}^{\mathscr{X}} \times \mathscr{X}$; in other terms, that for each $f_0(x_0) \in H$ there is Q open in $\mathscr{Y}^{\mathscr{X}} \times \mathscr{X}$ such that

$$(f_0, x_0) \in Q \tag{1}$$

and

$$Q \subset w^{-1}(H), \quad \text{i.e.} \quad [(f, x) \in Q] \Rightarrow [f(x) \in H]. \tag{2}$$

Since f_0 is continuous and $\mathscr{X}$ regular (compare § 41, II, Theorem 3), there is G open in $\mathscr{X}$ such that

$$x_0 \in G \quad \text{and} \quad f_0(\bar{G}) \subset H, \quad \text{i.e.} \quad f_0 \in \Gamma(\bar{G}, H).$$

Put

$$Q = \Gamma(\bar{G}, H) \times G. \tag{3}$$

It follows that Q is open in $\mathscr{Y}^{\mathscr{X}} \times \mathscr{X}$ and that formula (1) is fulfilled.

Formula (2) is also true because the condition $(f, x) \in Q$ means (by (3)) that $f(\bar{G}) \subset H$ and $x \in G$, hence $f(x) \in H$.

This completes the proof of the first part of the theorem.

In order to prove its second part, let us assume that $\mathscr{T}$ is a topology of $\mathscr{Y}^{\mathscr{X}}$ such that the mapping w is continuous. We have to show that each open set G in the compact-open topology of $\mathscr{Y}^{\mathscr{X}}$ is open in the topology $\mathscr{T}$. Clearly, we may assume that G belongs to the subbase of $\mathscr{Y}^{\mathscr{X}}$ considered in Section I; thus, put $G = \Gamma(C, H)$ where $C \subset \mathscr{X}$ is compact and $H \subset \mathscr{Y}$ is open.

Let $f \in \Gamma(C, H)$, i.e. $f(C) \subset H$. Given an $x \in C$, there are (owing to the continuity of w) two open sets V_x in $\mathscr{X}$ and W_x in $\mathscr{Y}^{\mathscr{X}}$ (relative to the $\mathscr{T}$-topology) such that

$$x \in V_x, \; f \in W_x \text{ and } w(f', x') \in H \text{ for each } x' \in V_x \text{ and } f' \in W_x.$$

Since C is compact, the cover $\{V_x\}$ contains a finite cover $V_{x_1}, \ldots, V_{x_n}$ of C. Put $W = W_{x_1} \cap \ldots \cap W_{x_n}$. Obviously W is open in $\mathscr{Y}^{\mathscr{X}}$ (relative to the $\mathscr{T}$-topology) and $f \in W$.

Furthermore, $W \subset \Gamma(C, H)$. For suppose that $f' \in W$ and $x' \in C$. Then there is $k \leqslant n$ such that $x' \in V_{x_k}$ and, since $f' \in W_{x_k}$, it follows that $w(f', x') \in H$, i.e. $f'(x') \in H$. Thus $f'(C) \subset H$, i.e. $f' \in \Gamma(C, H)$.

Theorem 1 can be generalized as follows.

THEOREM 2 [1]. *Let f and $\mathscr{X}$ be as in Theorem 1 and $\mathscr{Y}$ a $\mathscr{T}_2$-space. Let $F(f, K) = f(K)$. Then the mapping F: $\mathscr{Y}^{\mathscr{X}} \times 2^{\mathscr{X}} \to 2^{\mathscr{Y}}$ is continuous.*

Proof. Let $f_0 \in \mathscr{Y}^{\mathscr{X}}$ and $K_0 \in 2^{\mathscr{X}}$ be given. Let $\boldsymbol{W} \subset 2^{\mathscr{Y}}$ be open and $f_0(K_0) \in \boldsymbol{W}$. We must define two open subsets V of $\mathscr{Y}^{\mathscr{X}}$ and $\boldsymbol{U}$ of $2^{\mathscr{X}}$ such that

$$f_0 \in V,\ K_0 \in \boldsymbol{U} \text{ and } f(K) \in \boldsymbol{W} \text{ for each } f \in V \text{ and } K \in \boldsymbol{U}. \tag{4}$$

Clearly we may assume that $\boldsymbol{W}$ belongs to an open subbase of $2^{\mathscr{Y}}$. Thus we can suppose that either

$$\boldsymbol{W} = \underset{B}{E}(B \subset G) \tag{i}$$

or

$$\boldsymbol{W} = \underset{B}{E}(B \cap G \neq 0), \tag{ii}$$

where $G \subset \mathscr{Y}$ is open.

First, consider case (i). Then $f_0(K_0) \subset G$, i.e. for each $x \in K_0$ we have $f_0(x) \in G$. As w is continuous (by Theorem 1), there is an open $V_x \subset \mathscr{Y}^{\mathscr{X}}$ and an open $H_x \subset \mathscr{X}$ such that

$$f_0 \in V_x,\ x \in H_x \text{ and } f(x') \in G \text{ for each } f \in V_x \text{ and } x' \in H_x. \tag{5}$$

Since $\{H_x\}$ is an open cover of the (compact) set K_0, there is a finite subset $x_1, \ldots, x_k$ of K_0 such that $K_0 \subset H_{x_1} \cup \ldots \cup H_{x_k}$. Put

$$V = V_{x_1} \cap \ldots \cap V_{x_k} \quad \text{and} \quad \boldsymbol{U} = \underset{K}{E}(K \subset H_{x_1} \cup \ldots \cup H_{x_k}). \tag{6}$$

The first and second parts of (4) follow directly from (6) and from the first part of (5). In order to prove the last part of (4), put $f \in V$ and $K \in \boldsymbol{U}$. Hence by (6):

$$f \in V_{x_1} \cap \ldots \cap V_{x_k} \quad \text{and} \quad K \subset H_{x_1} \cup \ldots \cup H_{x_k}. \tag{7}$$

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[1] For metric spaces, Theorems 2–6 have been shown in the French edition of this book (in a simpler way); see Chapter IV, Sections VI and VII (see also the remarks of A. Császár, *ibid.* p. 502). For the present formulation of these theorems I am indebted to Professor Engelking.

Let $x' \in K$. Hence there is i such that $x' \in H_{x_i}$ and since $f \in V_{x_i}$, we have by the last part of (5), $f(x') \in G$. Thus $f(K) \subset G$ and by (i), $f(K) \in \boldsymbol{W}$.

Next consider case (ii). As $f_0(K_0) \in \boldsymbol{W}$, it follows that $f_0(K_0) \cap \cap G \neq 0$, hence there is $x_0 \in K_0$ such that $f_0(x_0) \in G$. As w is continuous (by Theorem 1), there are two open sets $V \subset \mathscr{Y}^{\mathscr{X}}$ and $H \subset \mathscr{X}$ such that

$$f_0 \in V, \ x_0 \in H \text{ and } f(x) \in G \text{ for each } f \in V \text{ and } x \in H. \tag{8}$$

Put $\boldsymbol{U} = \underset{K}{E}(K \cap H \neq 0)$. Clearly $\boldsymbol{U}$ is open in $2^{\mathscr{X}}$ and, as $x_0 \in K_0 \cap H$, we have $K_0 \in \boldsymbol{U}$. Thus the two first parts of formula (4) are true. In order to prove the last one, let us assume that $f \in V$ and $K \in \boldsymbol{U}$, i.e. $K \cap H \neq 0$. Let $x \in K \cap H$. Hence by (8), $f(x) \in G$, and consequently $f(K) \cap G \neq 0$, which means (by (ii)) that $f(K) \in \boldsymbol{W}$.

Remark. In Theorem 2 (as well as in Theorem 1) the assumption of compactness of $\mathscr{X}$ cannot be omitted [1]. It can, however, be replaced by local compactness.

COROLLARY. *Given f: $\mathscr{X} \times \mathscr{Y} \to \mathscr{Z}$, where $\mathscr{Z}$ is a $\mathscr{T}_2$-space, put*

$$\gamma(f, K, L) = f(K, L) = \underset{z}{E} \bigvee_{xy} [z = f(x, y)](x \in K)(y \in L).$$

If $\mathscr{X}$ and $\mathscr{Y}$ are compact $\mathscr{T}_2$-spaces and $\mathscr{Z}$ an arbitrary $\mathscr{T}_2$-space, then the mapping γ: $\mathscr{Z}^{\mathscr{X} \times \mathscr{Y}} \times 2^{\mathscr{X}} \times 2^{\mathscr{Y}} \to 2^{\mathscr{Z}}$ is continuous.

In particular, *the mapping η: $2^{\mathscr{X}} \times \mathscr{Y} \to 2^{\mathscr{Z}}$, defined by:*

$$\eta(K, y) = \underset{z}{E} \bigvee_{x} [z = f(x, y)](x \in K),$$

is continuous (for f fixed).

Proof. Put

$$\psi(f, M) = f(M) \quad \text{for} \quad f \in \mathscr{Z}^{\mathscr{X} \times \mathscr{Y}} \text{ and } M \in 2^{\mathscr{X} \times \mathscr{Y}},$$

and

$$\alpha(K, L) = K \times L \quad \text{for} \quad K \in 2^{\mathscr{X}} \text{ and } L \in 2^{\mathscr{Y}}.$$

Then γ is a composed mapping: $\gamma(f, K, L) = \psi[f, \alpha(K, L)]$. As ψ is continuous (by Theorem 2) and α is continuous (by Corollary 2 of § 43, I), so is γ.

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[1] Compare R. Arens, *loc. cit.*

THEOREM 3. *Let $\mathscr{X}$ be a compact $\mathscr{T}_2$-space and $\mathscr{Y}$ an arbitrary $\mathscr{T}_2$-space. Then the set*

$$\Phi = \mathop{E}_{fxy}[y = f(x)]$$

is closed in $\mathscr{Y}^{\mathscr{X}} \times \mathscr{X} \times \mathscr{Y}$.

Proof. We have to show that the complement of Φ is open. Let (f_0, x_0, y_0) belong to that complement, i.e. $f_0(x_0) \neq y_0$. Since $\mathscr{Y}$ is a $\mathscr{T}_2$-space, there are in $\mathscr{Y}$ open sets U and V such that

$$f_0(x_0) \in U, \quad y_0 \in V \quad \text{and} \quad U \cap V = 0. \tag{9}$$

Since f_0 is continuous and $\mathscr{X}$ regular, there is an open $G \subset \mathscr{X}$ such that $x_0 \in G$ and $f_0(\bar{G}) \subset U$. In order to complete the proof, it suffices to show that:

$$(f_0, x_0, y_0) \in \Gamma(\bar{G}, U) \times G \times V, \quad \text{i.e.} \quad f_0(\bar{G}) \subset U, \quad x_0 \in G, \ y_0 \in V, \tag{10}$$

and

$$[\Gamma(\bar{G}, U) \times G \times V] \cap \Phi = 0, \quad \text{i.e.}$$
$$\{(f(\bar{G}) \subset U)(x \in G)(y \in V)\} \Rightarrow [f(x) \neq y]. \tag{11}$$

It remains to prove only (11). Now, if $x \in G$ and $f(\bar{G}) \subset U$, it follows that $f(x) \in U$, hence, by (9), $f(x) \notin V$, while $y \in V$. Thus $f(x) \neq y$.

Under the same assumptions on $\mathscr{X}$ and $\mathscr{Y}$, the following corollaries hold true.

COROLLARY 4. *The set $\mathop{E}_{f}[f(\mathscr{X}) = \mathscr{Y}]$ is closed in $\mathscr{Y}^{\mathscr{X}}$.*

Because $\mathop{E}_{f}[f(\mathscr{X}) = \mathscr{Y}] = \bigcap_{y} \mathop{E}_{f} \bigvee_{x} [y = f(x)]$.

COROLLARY 5. *If $F = \bar{F}$, $\mathop{E}_{fx}[f(x) \in F]$ is closed in $\mathscr{Y}^{\mathscr{X}} \times \mathscr{X}$.*

Because $\mathop{E}_{fx}[f(x) \in F] = w^{-1}(F)$, where $w(f, x) = f(x)$; since w is continuous (by Theorem 1), this completes the proof.

COROLLARY 6. *Let G be open and F closed (in $\mathscr{X}$). Then the sets*

$$\mathop{E}_{fy}[\overset{-1}{f}(y) \subset G] \quad \text{and} \quad \mathop{E}_{f}[f^{-1}(F) \subset G]$$

are open in $\mathscr{Y}^{\mathscr{X}} \times \mathscr{Y}$ and $\mathscr{Y}^{\mathscr{X}}$ respectively.

The first part of the corollary follows from the equivalence

$$[\overset{-1}{f}(y) \not\subset G] \equiv \bigvee_x \{[y = f(x)](x \in \mathscr{X} - G)\}$$

since the set of points (f, x, y) satisfying the condition in braces $\{\ \}$ is closed (by Theorem 3, cf. also Theorem 1 of § 41, IX).

Similarly, the second part follows (by Corollary 5) from the equivalence:

$$[f^{-1}(F) \not\subset G] \equiv \bigvee_x \{[f(x) \in F](x \in \mathscr{X} - G)\}.$$

Remark. Most of the theorems of this section stated for compact spaces can be extended to locally compact spaces.

III. The restriction operation. Inverse systems. Let $f \in \mathscr{Y}^{\mathscr{X}}$. Denote by ϱ_C the operation of restriction, that is $\varrho_C(f) = f|C$ for a given $C \subset \mathscr{X}$. Thus ϱ_C: $\mathscr{Y}^{\mathscr{X}} \to \mathscr{Y}^C$.

THEOREM 1. *If C is compact, then ϱ_C is continuous.*

Proof. We have to show that, if $\Phi \subset \mathscr{Y}^C$ is open, then $\varrho_C^{-1}(\Phi)$ is open in $\mathscr{Y}^{\mathscr{X}}$. Clearly, we may assume that Φ is of the form $\Phi = \Gamma(C_1, H)$, where

$$\Gamma(C_1, H) = \mathop{E}_g [g(C_1) \subset H], \text{ with } g \in \mathscr{Y}^C \text{ and } C_1 \subset C \text{ compact.} \quad (1)$$

Therefore,

$$f \in \varrho_C^{-1}(\Phi) \equiv (f|C) \in \Phi \equiv (f|C)(C_1) \subset H \equiv f(C_1) \subset H,$$

i.e.

$$\varrho_C^{-1}(\Phi) = \Gamma(C_1, H).$$

This completes the proof.

Put as before $f|C = \varrho_C(f)$ and for $\Phi \subset \mathscr{Y}^{\mathscr{X}}$ let $\Phi|C = \varrho_C(\Phi)$, i.e. *$\Phi|C$ is the set of all elements $g \in \mathscr{Y}^C$ of the form $g = f|C$ where $f \in \Phi$.*

THEOREM 2. *The following equivalence is true for each $\Phi \subset \mathscr{Y}^{\mathscr{X}}$.*

$$(f \in \overline{\Phi}) \equiv [(f|C) \in \overline{\Phi|C} \textit{ for each compact } C \subset \mathscr{X}]. \quad (2)$$

Proof. The implication from left to right is a direct consequence of Theorem 1.

In order to prove the converse implication, put $f_0 \notin \overline{\Phi}$. We have to define a compact $C \subset \mathscr{X}$ such that

$$f_0|C \notin \overline{\Phi|C}. \tag{3}$$

Now, by hypothesis, there are two finite systems $C_1, \ldots, C_n$ and $H_1, \ldots, H_n$ (of compact respectively of open sets) such that

$$f_0 \in \Gamma(C_1, H_1) \cap \ldots \cap \Gamma(C_n, H_n) \subset \mathscr{Y}^{\mathscr{X}} - \Phi. \tag{4}$$

Put $C = C_1 \cup \ldots \cup C_n$ and write briefly $g_0 = f_0|C$. In order to prove (3), it suffices obviously to show that

$$g_0 \in \Gamma(C_1, H_1) \cap \ldots \cap \Gamma(C_n, H_n) \subset \mathscr{Y}^C - (\Phi|C). \tag{5}$$

Since $C_i \subset C$, we have $g_0(C_i) = f_0(C_i)$ and since (by (4)), $f_0 \in \Gamma(C_i, H_i)$, it follows that $g_0(C_i) \subset H_i$, hence $g_0 \in \Gamma(C_i, H_i)$.

On the other hand, if $g \in \Phi|C$, we have $g = f|C$ for some $f \in \Phi$, and it follows by (4) that for some $i \leqslant n$ we have $f \notin \Gamma(C_i, H_i)$, i.e. $f(C_i) \not\subset H_i$. But this means that $g(C_i) \not\subset H$ and hence $g \notin \Gamma(C_i, H_i)$.

COROLLARY 2a. *If Φ_0 and Φ_1 are two closed subsets of $\mathscr{Y}^{\mathscr{X}}$, then*

$$(\Phi_0 = \Phi_1) \equiv [(\Phi_0|C) = (\Phi_1|C) \text{ for each compact } C \subset \mathscr{X}].$$

THEOREM 3. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ where $\mathscr{X}$ is metric (or, more generally, is a $\mathscr{T}_2$-space with k-property, see below). If $f|C$ is continuous whatever the compact set $C \subset \mathscr{X}$ is, then f is continuous, i.e. $f \in \mathscr{Y}^{\mathscr{X}}$.*

In other terms (in view of Theorem 2):

$$(f \in \mathscr{Y}^{\mathscr{X}}) \equiv [(f|C) \in \mathscr{Y}^C \text{ for each compact } C \subset \mathscr{X}]. \tag{6}$$

Proof. The proof will be based on the following "*k-property*" of subsets of a metric space:

if $A \cap C$ is closed for each compact C, then A is closed (1). $\quad (k)$

Suppose that A is not closed; then there is $p_0 \notin A$ such that $p_0 = \lim p_n$ where $p_n \in A$ for $n = 1, 2, \ldots$. Denote by C the set $(p_0, p_1, \ldots, p_n, \ldots)$. Clearly C is compact and $A \cap C$ is not closed.

---

(1) For a detailed study of spaces with k-property, see A. Archangielskii, *Bicompact sets...* (Russian), Works of the Moscow Math. Soc. 13 (1965), pp. 3–54.

It remains to show that, assuming that $f|C$ is continuous for each compact $C \subset \mathscr{X}$, f is continuous (on $\mathscr{X}$), i.e. that $f^{-1}(F)$ is closed for each F closed in $\mathscr{Y}$. Put $A = f^{-1}(F)$. Clearly (compare § 3, III, p. 16 (14)):

$$A \cap C = f^{-1}(F) \cap C = (f|C)^{-1}(F).$$

But the last set is closed (because $f|C$ is continuous) and it follows by property (k) that A is closed.

Remark. This theorem can also be applied to the case where $\mathscr{X}$ and $\mathscr{Y}$ are $\mathscr{L}^*$-spaces.

For each $\mathscr{X}$, $\mathscr{C}(\mathscr{X})$ denotes the *family of all compact subsets of* $\mathscr{X}$ (cf. § 42, I). Clearly $\mathscr{C}(\mathscr{X})$ is ordered by the relation of inclusion and, moreover, is a directed set since for $C_1 \in \mathscr{C}(\mathscr{X})$ and $C_2 \in \mathscr{C}(\mathscr{X})$, $(C_1 \cup C_2) \in \mathscr{C}(\mathscr{X})$ and $C_1 \subset C_1 \cup C_2$ and $C_2 \subset C_1 \cup C_2$.

Define $\pi_{C_0 C_1}$, for $C_0 \subset C_1$, by the condition

$$\pi_{C_0 C_1}(g) = g|C_0 \quad \text{for} \quad g \in \mathscr{Y}^{C_1}.$$

Thus $\pi_{C\mathscr{X}} = \varrho_C$ if $\mathscr{X}$ is compact. Clearly $\pi_{CC}(g) = g$ and

$$\pi_{C_0 C_1} \circ \pi_{C_1 C_2} = \pi_{C_0 C_2} \quad \text{for} \quad C_0 \subset C_1 \subset C_2.$$

Therefore, $[\mathscr{C}(\mathscr{X}), \mathscr{Y}^C, \pi]$ is an *inverse system* (compare § 3, XIII). Denote by L its inverse limit:

$$L = \operatorname*{Lim}_{C, C_0 \subset C_1} \{\mathscr{Y}^C, \pi_{C_0 C_1}\}. \tag{7}$$

THEOREM 4. *If $\mathscr{X}$ is metric (or, more generally, has the k-property), then $\mathscr{Y}^{\mathscr{X}} \underset{\text{top}}{=} L$.*

Namely the mapping ϱ having ϱ_C as its C-th coordinate is a homeomorphism of $\mathscr{Y}^{\mathscr{X}}$ onto L.

Proof. ϱ is a homeomorphism, because we have by Theorem 2 (for $\Phi \subset \mathscr{Y}^{\mathscr{X}}$)

$$(f \in \overline{\Phi}) \equiv \{\varrho_C(f) \in \overline{\varrho_C(\Phi)} \text{ for each } C \in \mathscr{C}(\mathscr{X})\}, \tag{8}$$

and the last condition is equivalent to $\varrho(f) \in \overline{\varrho(\Phi)}$ (by Theorem 3 of § 16, VI, p. 159, where we substitute $\varrho(f)$ for $\mathfrak{z}$, $\varrho(\Phi)$ for $\mathfrak{Z}$ and $\mathscr{C}(\mathscr{X})$ for T).

It remains to show that ϱ is a mapping onto L.

Let $\{g_C\} \in L$. We must define $f \in \mathscr{Y}^{\mathscr{X}}$ so that

$$f|C = g_C \quad \text{for each } C \in \mathscr{C}(\mathscr{X}). \tag{9}$$

The required f is obtained by assuming that

$$f(x) = g_{(x)}(x) \quad \text{for each } x \in \mathscr{X}. \tag{10}$$

(9) is true because we have $g_C(x) = g_{(x)}(x)$ for each $x \in C$. Moreover, by Theorem 3, f is continuous (i.e. $f \in \mathscr{Y}^{\mathscr{X}}$) since by (9) $f|C = g_C \in \mathscr{Y}^C$, which means that f is continuous on every compact subset of $\mathscr{X}$.

THEOREM 5. *If $\mathscr{Y}$ is a completely regular $\mathscr{T}_1$-space, then so is $\mathscr{Y}^{\mathscr{X}}$.*

Proof. Let $f_0 \in \mathscr{Y}^{\mathscr{X}}$ and let Φ_0 be an open set in $\mathscr{Y}^{\mathscr{X}}$ containing f_0. We have to define a continuous mapping

$$\gamma: \mathscr{Y}^{\mathscr{X}} \to \mathscr{I} \quad \text{such that} \quad \gamma(f_0) = 0 \text{ and } \gamma(f) = 1 \text{ for } f \notin \Phi_0.$$

Clearly, we may assume that $\Phi_0 = \Gamma(C, H)$ where $C \subset \mathscr{X}$ is compact and $H \subset \mathscr{Y}$ is open. Thus $f_0(C)$ is compact and contained in H. Consequently (see § 41, II, Remark (iii)) there is a continuous

$$g: \mathscr{Y} \to \mathscr{I} \quad \text{such that} \quad gf_0(C) = 0 \text{ and } g(\mathscr{Y} - H) = 1.$$

Denote by $\gamma(f)$ the last point of the set $gf(C)$. Hence $\gamma: \mathscr{Y}^{\mathscr{X}} \to \mathscr{I}$. We shall show that γ is continuous.

Obviously $gf(C) = (gf|C)(C)$. Put

$$F(f) = gf(C) \quad \text{for } f \in \mathscr{Y}^{\mathscr{X}}, \quad \text{and} \quad U(u) = gu(C) \quad \text{for } u \in \mathscr{Y}^C.$$

Hence

$$F: \mathscr{Y}^{\mathscr{X}} \to 2^{\mathscr{I}}, \quad U: \mathscr{Y}^C \to 2^{\mathscr{I}}, \quad \text{and} \quad F(f) = U\varrho_C(f).$$

By Theorem 2 of Section II, U is continuous, and since by Theorem 1 of Section III, ϱ_C is continuous, so is F. This implies that γ is continuous (see Remark 2 of § 42, II).

Obviously $\gamma(f_0) = 0$. Finally, let $f \notin \Phi_0$. Then there is $x \in C$ such that $f(x) \notin H$, and hence $gf(x) = 1$. Thus $\gamma(f) = 1$.

IV. Relations between the spaces $\mathscr{Y}^{\mathscr{X} \times \mathscr{T}}$ and $(\mathscr{Y}^{\mathscr{X}})^{\mathscr{T}}$. Let $g: \mathscr{X} \times \mathscr{T} \to \mathscr{Y}$ and put

$$f_t(x) = g(x, t). \tag{1}$$

Thus f considered as a function of t associates with each $t \in \mathscr{T}$ a mapping having x as the independent variable (compare § 20, Section VII).

THEOREM 1 (1). *If g is continuous, so is f, i.e.*

$$g \in \mathscr{Y}^{\mathscr{X}\times\mathscr{T}} \Rightarrow f \in (\mathscr{Y}^{\mathscr{X}})^{\mathscr{T}}. \tag{2}$$

Proof. Let Q be open in $\mathscr{Y}^{\mathscr{X}}$. We must show that $f^{-1}(Q)$ is open in $\mathscr{T}$. We can, of course, assume that Q is of the form $Q = \Gamma(C, H)$, where C is compact in $\mathscr{X}$ and H open in $\mathscr{Y}$. Thus, given $t_0 \in f^{-1}(Q)$, we have to define U open in $\mathscr{T}$ such that

$$t_0 \in U \quad \text{and} \quad U \subset f^{-1}(Q). \tag{3}$$

Now, we have

$$[t \in f^{-1}(Q)] \equiv (f_t \in Q) \equiv [f_t(x) \in H \text{ for each } x \in C]. \tag{4}$$

Since $t_0 \in f^{-1}(Q)$, it follows by (1) that

$$g(x, t_0) \in H \quad \text{for each } x \in C, \quad \text{i.e.} \quad C \times (t_0) \subset g^{-1}(H).$$

Since g is continuous, $g^{-1}(H)$ is open in $\mathscr{X}\times\mathscr{T}$, hence is a union of products $G_s \times U_s$ of open sets (in $\mathscr{X}$ and $\mathscr{T}$ respectively):

$$g^{-1}(H) = \bigcup_s G_s \times U_s.$$

Since $C \times (t_0) \subset g^{-1}(H)$ is compact, the cover $\{G_s \times U_s\}$ contains a finite subcover:

$$C \times (t_0) \subset (G_{s_1} \times U_{s_1}) \cup \ldots \cup (G_{s_n} \times U_{s_n}), \text{ where } t_0 \in U_{s_1} \cap \ldots \cap U_{s_n}. \tag{5}$$

Put $U = U_{s_1} \cap \ldots \cap U_{s_n}$. Thus $t_0 \in U$ and U is open in $\mathscr{T}$.

In order to prove the second part of (3), put $t \in U$. Since $C \subset G_{s_1} \cup \ldots \cup G_{s_n}$, we have for each $x \in C$

$$(x, t) \in (G_{s_1} \times U_{s_1}) \cup \ldots \cup (G_{s_n} \times U_{s_n}) \subset g^{-1}(H),$$

i.e. $g(x, t) \in H$, hence $f_t(x) \in H$, and by (4), $t \in f^{-1}(Q)$.

The converse to Theorem 1 is the following

THEOREM 2. *Let $\mathscr{X}$ be regular and locally compact. Then, if f is continuous, so is g.*

This means (in view of Theorem 1) that

$$g \in \mathscr{Y}^{\mathscr{X}\times\mathscr{T}} \equiv f \in (\mathscr{Y}^{\mathscr{X}})^{\mathscr{T}}. \tag{6}$$

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(1) See R. H. Fox, *loc. cit.* p. 430, Lemma 1. Comp. also R. Brown, *Function spaces and product topologies*, Quart. J. 15 (1964), pp. 238–250.

Proof. Let H be open in $\mathscr{Y}$. We must show that $g^{-1}(H)$ is open in $\mathscr{X}\times\mathscr{T}$. Thus, given $g(x_0, t_0)\in H$, we have to define two open sets G (in $\mathscr{X}$) and U (in $\mathscr{T}$) such that

$$x_0\in G, \quad t_0\in U \quad \text{and} \quad G\times U \subset g^{-1}(H). \tag{7}$$

Since $f_{t_0}\in\mathscr{Y}^{\mathscr{X}}$ and $f_{t_0}(x_0) = g(x_0, t_0)\in H$, there is G_0 open in $\mathscr{X}$ such that $x_0\in G_0$ and $f_{t_0}(G_0) \subset H$. Since $\mathscr{X}$ is regular and locally compact, there is G open and such that $x_0\in G$, $\overline{G}$ is compact and $\overline{G} \subset G_0$.

It follows that

$$f_{t_0}(\overline{G}) \subset f_{t_0}(G_0) \subset H, \quad \text{i.e.} \quad f_{t_0}\in\Gamma(\overline{G}, H). \tag{8}$$

Put

$$U = f^{-1}[\Gamma(\overline{G}, H)], \quad \text{i.e.} \quad t\in U \equiv f_t\in\Gamma(\overline{G}, H). \tag{9}$$

Since f is continuous and $\Gamma(\overline{G}, H)$ is open (by the definition of the c.o. topology), U is open. Furthermore, $t_0\in U$ by (8). In order to prove the last part of (7), put $x\in G$ and $t\in U$. We have to show that $g(x, t)\in H$, i.e. that $f_t(x)\in H$. But $t\in U$ means by (9) that $f_t\in\Gamma(\overline{G}, H)$, and since $\Gamma(\overline{G}, H) \subset \Gamma(G, H)$, it follows that $f_t\in\Gamma(G, H)$, i.e. $f_t(x)\in H$ for each $x\in G$.

Remark 1 [1]. In connection with formulas (2) and (6), it is worthy noticing that, if we assign to each $g\in\mathscr{Y}^{\mathscr{X}\times\mathscr{T}}$ the mapping $f\in(\mathscr{Y}^{\mathscr{X}})^{\mathscr{T}}$ (defined by (1)), we define a homeomorphism into. This homeomorphism becomes onto under the assumptions of Theorem 2.

Thus for $\mathscr{X}$ regular and locally compact

$$\mathscr{Y}^{\mathscr{X}\times\mathscr{T}} \underset{\text{top}}{=} (\mathscr{Y}^{\mathscr{X}})^{\mathscr{T}}. \tag{10}$$

Remark 2. Similarly, the following statements, which have been proved for the topology of continuous convergence (see § 20, VII, Theorems 1 and 2, and VIII, Theorem 3), can be shown for the c.o. topology.

$$\underset{t\in T}{\boldsymbol{P}} \mathscr{Y}_t^{\mathscr{X}} \underset{\text{top}}{=} \big(\underset{t\in T}{\boldsymbol{P}} \mathscr{Y}_t\big)^{\mathscr{X}}, \tag{11}$$

$$\mathscr{Y}^A \times \mathscr{Y}^B \underset{\text{top}}{=} \mathscr{Y}^{A\cup B} \tag{12}$$

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[1] See J. R. Jackson, *Spaces of mappings on topological products with applications to homotopy theory*, Proc. Amer. Math. Soc. 3 (1952), pp. 327–333.
Compare § 20, VII, Theorem 3, for the space $\mathscr{Y}^{\mathscr{X}}$ with continuous convergence.

provided A and B are closed in $A \cup B$ and $A \cap B = 0$,

$$\mathscr{Y}^{f(\mathscr{X})} \underset{\text{top}}{\subset} \mathscr{Y}^{\mathscr{X}} \tag{13}$$

provided f is continuous, $\mathscr{X}$ compact and $f(\mathscr{X})$ a $\mathscr{T}_2$-space.

Remark 3. Finally the following formula (compare § 21, X, Theorem 8) can be shown

$$\mathscr{Y}^{\mathscr{X}} \underset{\text{top}}{\subset} 2^{\mathscr{X} \times \mathscr{Y}} \tag{14}$$

provided $\mathscr{X}$ is compact.

Namely, we assign to each $f \in \mathscr{Y}^{\mathscr{X}}$ its graph.

LEMMA. *Let $f\colon \mathscr{T} \to \mathscr{Y}^{\mathscr{X}}$ be continuous and let $C \subset \mathscr{X}$ be compact. Put $v(t) = f_t|C$. Then $v\colon \mathscr{T} \to \mathscr{Y}^{C}$ is continuous.*

Proof. Let $A \subset \mathscr{T}$ and $t \in \bar{A}$. We have to show that $v(t) \in \overline{v(A)}$.

Since f is continuous, so $t \in \bar{A}$ implies $f_t \in \overline{f(A)}$. Put $\Phi = f(A)$ in III (2). It follows that $(f_t|C) \in \overline{\Phi|C}$. This completes the proof, because $\Phi|C = v(A)$.

THEOREM 3. *If $\mathscr{X}$ is metric (or, more generally, a $\mathscr{T}_2$-space with k-property) and $\mathscr{T}$ is metric compact, then* (6) *is true.*

Proof. In view of Theorem 1, it remains to show that, if f is continuous, then so is g; and this reduces to show that $g|C \times \mathscr{T}$ is continuous for each compact $C \subset \mathscr{X}$ (compare III (6)).

Write (as above) $v(t) = f_t|C$. Since f is continuous, so is v (by the Lemma), i.e. $v \in (\mathscr{Y}^C)^{\mathscr{T}}$, and by Theorem 2 (where $\mathscr{X}$ is to be replaced by C and g by $g|C \times \mathscr{T}$), $g|C \times \mathscr{T}$ is continuous.

Remark 4. The assumption of $\mathscr{T}$ being metric can be omitted, because the product of a space with k-property and of a compact space has the k-property [1].

V. The topology of uniform convergence of $\mathscr{Y}^{\mathscr{X}}$. Let us recall that, given an arbitrary set $\mathscr{X}$ and a metric space $\mathscr{Y}$, we denote by $\Phi(\mathscr{X}, \mathscr{Y})$ the *family of all bounded mappings* $f\colon \mathscr{X} \to \mathscr{Y}$. This family can be considered as a metric space by defining the distance as follows (see § 21, X, p. 218)

$$|f_1 - f_2| = \sup_{x \in \mathscr{X}} |f_1(x) - f_2(x)|. \tag{1}$$

[1] See A. Archangielskii, *Bicompact sets...* (Russian) Trudy 13 (1965), p. 18. Compare also R. Brown, *Ten topologies for $X \times Y$*, Quart. J. of Math. 14 (1963), pp. 303–319, and *Function spaces and product topologies, ibid.* 15 (1964), pp. 238–250.

Let us add that the convergence in the space $\Phi(\mathscr{X}, \mathscr{Y})$ coincides with the uniform convergence (see Corollary 1a of § 21, X).

The topology of $\Phi(\mathscr{X}, \mathscr{Y})$ induced by the above defined distance is called *topology of uniform convergence* (more concisely, the *u.c. topology*).

THEOREM 1. *If $\mathscr{X}$ is compact and $\mathscr{Y}$ is metric, then $\mathscr{Y}^{\mathscr{X}} \subset \Phi(\mathscr{X}, \mathscr{Y})$ and hence $\mathscr{Y}^{\mathscr{X}}$ is metric (in the u.c. topology).*

This is a direct consequence of the fact that if $f\colon \mathscr{X} \to \mathscr{Y}$ is continuous, then $f(\mathscr{X})$ is bounded (see § 41, VI, Theorem 1).

Remark 1. If $\mathscr{X}$ is compact metric, the uniform convergence of continuous mappings coincides with the continuous convergence; which means (see § 21, X, Theorem 5) that

$$(\lim x_n = x) \Rightarrow [\lim f_n(x_n) = f(x)]. \tag{2}$$

THEOREM 2. *If $\mathscr{X}$ is compact and $\mathscr{Y}$ metric, then the u.c. topology of $\mathscr{Y}^{\mathscr{X}}$ coincides with its c.o. topology.*

Proof. 1. First we shall show that each open set in the u.c. topology is open in the c.o. topology. Clearly, it suffices to prove that, for each $f_0 \in \mathscr{Y}^{\mathscr{X}}$ and $\varepsilon > 0$, there are two finite systems $C_1, \ldots, C_n$ and $H_1, \ldots, H_n$ (where $C_i \subset \mathscr{X}$ is compact and $H_i \subset \mathscr{Y}$ is open) such that

$$f_0 \in [\Gamma(C_1, H_1) \cap \ldots \cap \Gamma(C_n, H_n)] \subset K(f_0, \varepsilon) \tag{3}$$

where $K(f_0, \varepsilon)$ is the ball of center f_0 and radius ε, i.e.

$$K(f_0, \varepsilon) = \mathop{E}_{f} |f - f_0| < \varepsilon.$$

Since $\mathscr{X}$ is compact and f_0 continuous, there is a finite open cover $\mathscr{X} = G_0 \cup \ldots \cup G_n$ such that $\delta[f_0(G_i)] < \varepsilon/2$ and $G_i \neq 0$ for $i = 1, \ldots, n$. Choose $x_i \in G_i$ and put

$$C_i = \overline{G}_i \text{ and } H_i = K[f_0(x_i), \varepsilon/2] = \mathop{E}_{y} [|y - f_0(x_i)| < \varepsilon/2]. \tag{4}$$

Now since $\delta[f_0(C_i)] < \varepsilon/2$, we have for $x \in C_i$ the inequality $|f_0(x) - f_0(x_i)| < \varepsilon/2$, and hence $f_0(x) \in H_i$, i.e. $f_0 \in \Gamma(C_i, H_i)$. This is true for each $i = 1, \ldots, n$. Thus the proof of the first part of (3) is complete.

To prove the second part of (3), put $f \in \Gamma(C_i, H_i)$. Then for each $x \in C_i$, we have $f(x) \in H_i$ and by (4), $|f(x)-f_0(x_i)| < \varepsilon/2$. Since $\delta[f_0(C_i)] < \varepsilon/2$, it follows that

$$|f(x)-f_0(x)| \leqslant |f(x)-f_0(x_i)|+|f_0(x_i)-f_0(x)| < \varepsilon.$$

Therefore $f \in K(f_0, \varepsilon)$.

2. Let G be open in the c.o. topology of $\mathcal{Y}^{\mathcal{X}}$. We must show that it is open in the u.c. topology. [1] Without loss of generality we may assume that $G = \Gamma(C, H)$.

Let $f_0 \in \Gamma(C, H)$. We have to define $\varepsilon > 0$ so that

$$|f-f_0| < \varepsilon \quad \text{implies} \quad f \in \Gamma(C, H). \tag{5}$$

Put

$$\varepsilon = \inf \varrho[f_0(x), \mathcal{Y}-H] \quad \text{where} \quad x \in C. \tag{6}$$

Since $f_0(x) \in H$ for each $x \in C$, so $\varrho[f_0(x), \mathcal{Y}-H] > 0$, and since C is compact and ϱ a continuous function of x (compare § 21, IV(5), p. 209), ϱ attains its lower bound on C (see § 41, VI, Corollary 2c). Thus $\varepsilon > 0$.

Let $|f-f_0| < \varepsilon$. Suppose, contrary to (5), that $f \notin \Gamma(C, H)$, i.e. that $f(x_0) \in \mathcal{Y}-H$ for some $x_0 \in C$. Hence

$$\varrho[f_0(x_0), \mathcal{Y}-H] \leqslant |f_0(x_0)-f(x_0)| < \varepsilon.$$

But this contradicts (6).

Remark 2. It follows from Theorem 2 that the u.c. topology for $\mathcal{X}$ compact and $\mathcal{Y}$ metric is a topological invariant (it does not depend upon the metric of $\mathcal{Y}$); for $\mathcal{X}$ compact metric this follows also from formula (2).

THEOREM 3. *If $\mathcal{X}$ is compact and $\mathcal{Y}$ complete (metric), then $\mathcal{Y}^{\mathcal{X}}$ is complete in its u.c. topology.*

This is an immediate consequence of Theorem 2 of § 33, V, p. 408.

Remark 3. We recall that *if $\mathcal{X}$ is compact metric and $\mathcal{Y}$ separable metric, then $\mathcal{Y}^{\mathcal{X}}$ is separable* (see § 22, III, p. 244).

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[1] Cf. J. R. Jackson, *Comparison of topologies on function spaces*, Proc. Amer. Math. Soc. 3 (1952), pp. 156–158.

THEOREM 4. *Let $\mathscr{Y}$ be compact metric and $\mathscr{X}$ arbitrary. Let $A \subset \mathscr{X}$ and put $H(f) = \overline{f(A)}$. The mapping $H: Y^{\mathscr{X}} \to 2^{\mathscr{Y}}$ is continuous (in the u.c. topology of $Y^{\mathscr{X}}$).*

More precisely, $\operatorname{dist}[H(f), H(g)] \leqslant |f-g|$, *where* $f, g \in Y^{\mathscr{X}}$.

Proof. We have

$$\operatorname{dist}[H(f), H(g)] = \operatorname{dist}[f(A), g(A)] \leqslant |f-g|,$$

because (compare § 21 (2), p. 215) for each $y \in f(A)$ there is $y' \in g(A)$ such that $|y-y'| \leqslant |f-g|$; namely $y' = g(x)$ where $x \in A$ is such that $y = f(x)$.

Remark 4. One can show that Theorem 4 is not true for the c.o. topology of $\mathscr{Y}^{\mathscr{X}}$.

COROLLARY 4a. *Let $\mathscr{Y}$ be compact metric and $\mathscr{X}$ arbitrary. Put $A \subset \mathscr{X}$ and $B \subset \mathscr{X}$. Then the set*

$$\underset{f}{E}\,[\overline{f(A)} \cap \overline{f(B)} = 0] \tag{7}$$

is open in $\mathscr{Y}^{\mathscr{X}}$.

Because the set $\underset{KL}{E}\,(K \cap L = 0)$ is open in $2^{\mathscr{Y}} \times 2^{\mathscr{Y}}$ for $\mathscr{Y}$ normal (compare § 17, V, Theorem 1, p. 169).

Remark 5. Theorem 4 and Corollary 4a can be stated in the following more general form.

Let $\mathscr{X}$ be an arbitrary set and $\mathscr{Y}$ a metric space. For $A \subset \mathscr{X}$, denote by Ψ_A the set of members f of $\Phi(\mathscr{X}, \mathscr{Y})$ such that $\overline{f(A)}$ is compact, and put $H(f) = f(A)$. Then the mapping $H\colon \Psi_A \to \mathscr{C}(\mathscr{Y})$ is continuous (compare the theorem of § 42, II) and the set (7) is open in $\Psi_A \cap \Psi_B$.

To prove this, note that the exponential topology of $\mathscr{C}(\mathscr{Y})$ is identical with its Hausdorff distance topology (by § 42, II) and the set $\underset{KL}{E}\{(K \cap L = 0)\ (K, L \in \mathscr{C}(\mathscr{Y}))\}$ is open in $\mathscr{C}(\mathscr{Y}) \times \mathscr{C}(\mathscr{Y})$.

VI. The homeomorphisms.

THEOREM 1 [1]. *If $\mathscr{X}$ is a perfectly normal compact $\mathscr{T}_2$-space and $\mathscr{Y}$ an arbitrary $\mathscr{T}_2$-space, the set Φ of all homeomorphisms $f\colon \mathscr{X} \to \mathscr{Y}$ is a G_δ-set in $\mathscr{Y}^{\mathscr{X}}$.*

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[1] See my paper *Evaluation de la classe borélienne ou projective...*, Fund. Math. 17 (1931), p. 270.

Proof. Clearly

$$(f \notin \Phi) \equiv \bigvee_{xx'} [f(x) = f(x')](x \neq x').$$

Now the set $\underset{fxx'}{E}[f(x) = f(x')]$ is closed (by Theorem 3 of Section II) and the set $\underset{fxx'}{E}(x \neq x')$ is open (by Theorem 2 of § 15, IV), hence an $\boldsymbol{F}_\sigma$, and so is Φ as a projection parallel to a compact axis (by Corollary 1a of § 41, IV).

Remark. The compactness assumption is essential (1).

LEMMA. *Let $\mathscr{X}$ be compact, $\mathscr{Y} = \mathscr{I}^{\aleph_0}$ and A and B two closed disjoint subsets of $\mathscr{X}$. Let $f \in \mathscr{Y}^{\mathscr{X}}$ and $\varepsilon > 0$. Then there is $g \in \mathscr{Y}^{\mathscr{X}}$ such that*

$$\overline{g(A)} \cap \overline{g(B)} = 0 \quad \text{and} \quad |g-f| < \varepsilon. \tag{1}$$

In other terms,

$$\overline{\underset{g}{E}[\overline{g(A)} \cap \overline{g(B)} = 0]} = \mathscr{Y}^{\mathscr{X}}. \tag{2}$$

Proof. Put $f(x) = [f^1(x), f^2(x), \ldots]$ where $f^n(x) \in \mathscr{I}$. Let n be such that $2^{-n} < \varepsilon$. We define g as follows:

1) $g^i(x) = f^i(x)$ for $i \leqslant n$,

2) $g^{n+1}(x) = \dfrac{\varrho(x, A)}{\varrho(x, A) + \varrho(x, B)}$,

3) $g^i(x) = 0$ for $i > n+1$.

Consequently, $g^{n+1}(x) = 0$ for $x \in A$, and hence $y^{n+1} = 0$ for $y \in g(A)$. It follows that $y^{n+1} = 0$ for $y \in \overline{g(A)}$.

On the other hand, for $y \in \overline{g(B)}$, we have $y^{n+1} = 1$. This completes the proof of the first part of (1). Its second part is proved as follows:

$$|g(x) - f(x)| = \sum_{i=1}^{\infty} 2^{-i} |g^i(x) - f^i(x)| = \sum_{i=n+1}^{\infty} 2^{-i} |g^i(x) - f^i(x)| \leqslant 2^{-n} < \varepsilon.$$

THEOREM 2. *Let $\mathscr{X}$ be metric separable and $\mathscr{Y}$ compact metric and suppose that (2) is true for each pair of closed disjoint subsets A*

(1) Compare J. H. Roberts, Amer. Journ. Math. 70 (1948), p. 126.

and B *of* $\mathcal{X}$ *(such is the case of* $\mathcal{Y} = \mathcal{I}^{\aleph_0}$*). Then the set of elements of* $\mathcal{Y}^{\mathcal{X}}$ *which are not homeomorphisms is of the first category in* $\mathcal{Y}^{\mathcal{X}}$.

More precisely, *if* F *is a closed subset of* $\mathcal{X}$ *and condition* (2) *is true for each pair of closed disjoint subsets* A *and* B *of* F, *then the set of all* $f \in \mathcal{Y}^{\mathcal{X}}$ *such that* $f|F$ *is not a homeomorphism is of the first category in* $\mathcal{Y}^{\mathcal{X}}$.

Proof. Let $R_1, R_2, \ldots$ be a countable open base relative to F. Put

$$\Phi_{kl} = \mathop{E}_{g}[\overline{g(\bar{R}_k)} \cap \overline{g(F-R_l)} = 0]$$

for each pair k, l such that $\bar{R}_k \subset R_l$.

As the space $\mathcal{Y}^{\mathcal{X}}$ is complete (by Theorem 2 of § 33, V, p. 408) and as the sets Φ_{kl} are dense (by (2)) and open (by Corollary 4a of Section IV), their intersection Φ is a G_δ-set whose complement is a first category set (by the Theorem of Baire). Finally, if $f \in \Phi$, then $f(R_k) \cap \overline{f(F-R_l)} = 0$ whenever $\bar{R}_k \subset R_l$. But this implies that $f|F$ is a homeomorphism (by Theorem 3 of § 22, I, p. 239).

VII. Case of $\mathcal{X}$ locally compact. Let us assume in this section that $\mathcal{X}$ and $\mathcal{Y}$ are metric and that $\mathcal{X}$ is locally compact.

By Theorem 8 of § 41, X, we have

$$\mathcal{X} = F_1 \cup F_2 \cup \ldots \quad \text{where} \quad F_i \subset \mathrm{Int}(F_{i+1}) \tag{1}$$

and F_i is compact.

It follows that for each compact $C \subset \mathcal{X}$ there is i such that $C \subset F_i$. In other terms, the family $\mathcal{C}(\mathcal{X})$ of all compact subsets of $\mathcal{X}$ is *cofinal* with $\{F_1, F_2, \ldots, F_i, \ldots\}$. This implies by Theorem 2 of Section III that

$$(f \in \bar{\Phi}) \equiv [(f|F_i) \in \overline{\Phi|F_i}] \tag{2}$$

for each subset Φ of the space $\mathcal{Y}^{\mathcal{X}}$ (endowed with the compact-open topology).

THEOREM 1. *The space* $\mathcal{Y}^{\mathcal{X}}$ *(with its c.o. topology) is metrizable.*

Its metric can be given by the formula

$$|f-g| = \sum_{i=1}^{\infty} 2^{-i} \frac{|(f|F_i)-(g|F_i)|}{1+|(f|F_i)-(g|F_i)|}. \tag{3}$$

Proof. First we shall show that

$$\mathcal{Y}^{\mathcal{X}} \underset{\text{top}}{\subset} \mathcal{Y}^{F_1} \times \mathcal{Y}^{F_2} \times \ldots. \tag{4}$$

The proof is similar to that of Theorem 4 of Section III: we put $\varrho_i(f) = f|F_i$; hence ϱ_i: $\mathcal{Y}^{\mathcal{X}} \to \mathcal{Y}^{F_i}$. Put $\varrho = (\varrho_1, \varrho_2, \ldots)$; then ϱ: $\mathcal{Y}^{\mathcal{X}} \to \mathcal{Y}^{F_1} \times \mathcal{Y}^{F_2} \times \ldots$ and by (1)

$$(f \in \overline{\Phi}) \equiv \bigwedge_i [\varrho_i(f) \in \overline{\varrho_i(\Phi)}] \equiv [\varrho(f) \in \overline{\varrho(\Phi)}]. \tag{5}$$

This means that ϱ is a homeomorphism and thus (4) is proved.

Now, since $\mathcal{Y}^{F_i}$ is metrizable (compare V (1)), the distance between two elements $f = (f_1, f_2, \ldots)$ and $g = (g_1, g_2, \ldots)$ of $\mathcal{Y}^{F_1} \times \mathcal{Y}^{F_2} \times \ldots$ can be assumed to be (compare § 21, VI, p. 213):

$$|f - g| = \sum_{i=1}^{\infty} 2^{-i} \frac{|f_i - g_i|}{1 + |f_i - g_i|}. \tag{6}$$

This completes the proof of Theorem 1.

Moreover, the same argument yields, as in the case of Theorem 4 of Section III, the following theorem

THEOREM 2. $\mathcal{Y}^{\mathcal{X}} \underset{\text{top}}{=} \operatorname{Lim}_{i, i<k} \{\mathcal{Y}^{F_i}, \pi_{ik}\}$ *where* $\pi_{ik}(g) = g|F_i$ *for* $g \in \mathcal{Y}^{F_k}$.

The above defined mapping ϱ is the required homeomorphism.

It follows easily from (3) that

$$\begin{aligned}(f = \lim_{k=\infty} f_k) &\equiv [(f|F_i) = \lim_{k=\infty} (f_k|F_i) \text{ for } i = 1, 2, \ldots] \\ &\equiv [(f|C) = \lim_{k=\infty} (f_k|C) \text{ for each compact } C], \end{aligned} \tag{7}$$

$$(\lim_{k=\infty} x_k = x) \Rightarrow [\lim_{k=\infty} f_k(x_k) = f(x)], \tag{8}$$

which means that the convergence in $\mathcal{Y}^{\mathcal{X}}$ is the continuous convergence.

THEOREM 3. *If* $\mathcal{Y}$ *is separable, so is* $\mathcal{Y}^{\mathcal{X}}$. *If* $\mathcal{Y}$ *is complete, so is* $\mathcal{Y}^{\mathcal{X}}$.

The first statement follows from (4), since $\mathcal{Y}^{F_i}$ is separable (see Remark 3 of Section V).

To prove the second statement we use formula (4) and the fact that the product $\mathcal{Y}^{F_1} \times \mathcal{Y}^{F_2} \times \ldots$ of complete spaces (see Theorem 3 of Section V) metrized by formula (6) is complete (see Theorem 2 of § 33, III, p. 406). Furthermore, ϱ is an isometric mapping of $\mathscr{Y}^{\mathscr{X}}$ onto a closed subset of $\mathscr{Y}^{F_1}\times\mathscr{Y}^{F_2}\times\ldots$ (by Theorem 2).

THEOREM 4. *Given a sequence of sets* $0\neq\Phi_k\subset\mathscr{Y}^{\mathscr{X}}$, *where* $k=1,2,\ldots$, *and a sequence of integers* $m_1, m_2, \ldots$ *such that*

$$\Phi_k|F_i=\Phi_{m_i}|F_i \quad \textit{for} \quad k>m_i \textit{ and } i=1,2,\ldots, \tag{9}$$

there is a convergent sequence $f_1, f_2, \ldots$ *where* $f_k\in\Phi_k$.

More precisely, *there is* $f\in\mathscr{Y}^{\mathscr{X}}$ *such that*

$$f(x)=f_k(x) \quad \textit{for} \quad x\in F_i \textit{ and } k>m_i, \tag{10}$$

hence $f=\lim\limits_{k=\infty}f_k$ (by (7)).

Proof. Clearly, we may assume that $m_1<m_2<\ldots$. First, we shall define a sequence $f_{m_1}, f_{m_2}, \ldots$ such that

$$f_{m_i}\in\Phi_{m_i}, \tag{11}$$

and

$$f_{m_{i+1}}(x)=f_{m_i}(x) \quad \text{for} \quad x\in F_i. \tag{12}$$

We proceed by induction. Let f_{m_1} be an arbitrary mapping satisfying (11) for $i=1$. Let us assume that condition (11) is fulfilled for a given i ($\geqslant 1$). Put in formula (9), $k=m_{i+1}$. It follows by (11) that $(f_{m_i}|F_i)\in(\Phi_{m_{i+1}}|F_i)$. Hence there exists $f_{m_{i+1}}$ such that (11) is true for $i+1$ (i.e. $f_{m_{i+1}}\in\Phi_{m_{i+1}}$) and that $f_{m_i}|F_i=f_{m_{i+1}}|F_i$, i.e. that (12) is true.

This completes the definition of the sequence $f_{m_1}, f_{m_2}, \ldots$.

Now it follows from (12) that there exists a mapping f such that

$$f(x)=f_{m_i}(x) \quad \text{for} \quad x\in F_i \text{ and } i=1,2,\ldots. \tag{13}$$

Since $F_i\subset\mathrm{Int}(F_{i+1})$, f is continuous, i.e. $f\in\mathscr{Y}^{\mathscr{X}}$.

It remains to define f_k for $m_i<k<m_{i+1}$ (for $k<m_1$, f_k is an arbitrary element of Φ_k). Now, by (9) and (11), we have $(f_{m_i}|F_i)\in(\Phi_k|F_i)$. Hence we may assume that

$$f_k\in\Phi_k \quad \text{and} \quad f_k(x)=f_{m_i}(x) \quad \text{for} \quad x\in F_i, \tag{14}$$

and formula (10) follows from (12)–(14).

Remark. *If $\mathcal{X}$ is not locally compact, the space $\mathcal{Y}^{\mathcal{X}}$ (with the compact open topology) is not metrizable* (1). It is however *almost metric* (compare § 21, XV, p. 233) relative to the family of pseudo-distances ψ_C such that

$$\psi_C(f, g) = \max_{x \in C} |f(x) - g(x)| \quad \text{and} \quad C \in \mathscr{C}(\mathcal{X}).$$

If $\mathcal{X}$ is separable and locally compact, the family $\mathscr{C}(\mathcal{X})$ can be replaced by the family $F_1, F_2, \ldots$; this implies the metrizability of $\mathcal{Y}^{\mathcal{X}}$ (see § 21, XV, Remark 2, p. 234).

VIII. The pointwise topology of $\mathcal{Y}^{\mathcal{X}}$. In § 16 we have defined topology (called also Tychonov topology) in the cartesian product $\underset{x \in \mathcal{X}}{\boldsymbol{P}} \mathcal{Y}_x$ of topological spaces $\mathcal{Y}_x$ (where $\mathcal{X}$ does not need to be a topological space). In the particular case where all $\mathcal{Y}_x$ are identical, $\mathcal{Y}_x = \mathcal{Y}$, the set of all mappings $f\colon \mathcal{X} \to \mathcal{Y}$, denoted by $(\mathcal{Y}^{\mathcal{X}})_{\text{set}}$, becomes a topological space. If the mappings f are restricted to continuous mappings ($\mathcal{X}$ being topological), their space, $\mathcal{Y}^{\mathcal{X}}$, is endowed, in this way, with a topology, called the *pointwise topology*.

This denomination is related to the fact that a net (see § 20, IX) $\{f_t\}$ converges to g if and only if $\{f_t(x)\}$ converges to $g(x)$ for each $x \in \mathcal{X}$ (for the case of a countable product, compare § 20, IV).

One easily shows that the sets $\Gamma(F, H)$ where F is a finite subset of $\mathcal{X}$ and H an open subset of $\mathcal{Y}$ form a base for the pointwise topology of $\mathcal{Y}^{\mathcal{X}}$. It follows that the pointwise topology is coarser than the compact-open topology of $\mathcal{Y}^{\mathcal{X}}$.

§ 45. Topics in dimension theory (continued)

§ 45 is a continuation of §§ 25–28 of Volume I. Here use will be made of the notion of compactness, studied in the preceding paragraphs of this chapter.

In § 45 the space $\mathcal{X}$ is supposed to be *metric separable* (2).

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(1) See S. Mrówka, *On function spaces*, Fund. Math. 45 (1958), p. 274. Compare also R. Arens, *A topology for spaces of transformations*, Ann. of Math. 47 (1946), pp. 480–495; R. Arens and J. Dugundji, *Topologies for function spaces*, Pacific Journ. Math. 1 (1951), pp. 5–32.

(2) For a more general approach, see J. Nagata, *Modern dimension theory*, 1965.

I. Mappings of order k. A point y is said to be a *value of order k* of the mapping f if the set $f^{-1}(y)$ consists of k elements. A mapping is said *to be of order $\leqslant k$* provided that each of its values is of order $\leqslant k$.

LEMMA 1. *Let $A_0, \ldots, A_r$ be a system of disjoint subsets of a space $\mathscr{X}$ and let f be a mapping of $\mathscr{X}$ of order $k \geqslant r$. Suppose*

$$B = f(A_0) \cap \ldots \cap f(A_r). \tag{1}$$

Then the partial mapping $f_i = f|[A_i \cap f^{-1}(B)]$ is of order $\leqslant k-r$.

Moreover, if we let $C_i = f^{-1}f(A_i) - A_i$, the partial mapping $f|C_i$ is of order $\leqslant k-1$.

Proof. Assume that $y \epsilon f[A_i \cap f^{-1}(B)]$. Since (compare § 3, II(13))

$$f[A_i \cap f^{-1}(B)] = B \cap f(A_i) = B, \tag{2}$$

there exists a system of $r+1$ points $x_0, \ldots, x_r$ such that

$$x_0 \epsilon A_0, \ \ldots, \ x_r \epsilon A_r \quad \text{and} \quad y = f(x_0) = \ldots = f(x_r).$$

Since the sets $A_0, \ldots, A_r$ are disjoint, the set $f^{-1}(y) \cap A_i$ contains at most $k-r$ points, and so does the set

$$f^{-1}(y) \cap A_i \cap f^{-1}(B), \quad \text{i.e.} \quad f_i^{-1}(y),$$

by much the same reason.

In a similar way, if $y \epsilon f(C_i) = f(A_i) \cap f(\mathscr{X} - A_i)$, then y is a point of order $\leqslant k-1$ of the mapping $f|\mathscr{X} - A_i$ and hence the same holds for the mapping $f|C_i$.

THEOREM 2 (1) (of Hurewicz). *If a continuous mapping f of order $\leqslant k$ $(k \geqslant 1)$ maps a compact space $\mathscr{X}$, then*

$$\dim f(\mathscr{X}) \leqslant \dim \mathscr{X} + k - 1.$$

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(1) Proc. Acad. Amsterdam 30 (1927), p. 164. Compare K. Nagami, *Finite-to-one closed mappings*, Proc. Japan. Acad. 34 (1958), pp. 503–506 and 35 (1958), pp. 437–439; J. Suzuki, *Note on a theorem for dimension*, *ibid.*, p. 201; I. Wainstein and J. Kashdan, *Continuous mappings of finite order raising dimension* (Russian), Iswiestia Acad. Nauk SSSR 8 (1944), pp. 129–138; I. Wainstein, *On raising dimension mappings*, Dokl. Akad. Nauk SSSR 57 (1947), p. 431 and 67 (1949), p. 19.

And more generally, if $A_0, \ldots, A_r$ $(0 \leqslant r \leqslant k)$ *is a system of disjoint closed sets, then*

$$\dim f(A_0) \cap \ldots \cap f(A_r) \leqslant \dim A_i + k - r - 1.$$

Proof. The first part of Theorem 2 will be proved by induction. Let $\dim \mathscr{X} = n$.

Theorem 2 is obvious in case where either $n = -1$ or $k = 1$. Assume that it holds for $n-1$, whatever the value of k may be, and that it also holds for the pair n and $k_0 - 1$.

Let $y \epsilon f(\mathscr{X})$ and let S be an open ball with the center y. We are going to define a neighbourhood E of y in $f(\mathscr{X})$ such that

$$E \subset S \quad \text{and} \quad \dim \mathrm{Fr}(E) \leqslant n + k_0 - 2.$$

By Theorem 1 of § 27, II, there exists such an open set G in an n-dimensional space $\mathscr{X}$ that

$$f^{-1}(y) \subset G, \quad \bar{G} \subset f^{-1}(S) \quad \text{and} \quad \dim(\bar{G} - G) \leqslant n - 1.$$

Let us define $E = f(\bar{G})$. It follows that

$$\begin{aligned} \mathrm{Fr}(E) = \mathrm{Fr}[f(\bar{G})] &= f(\bar{G}) \cap \overline{f(X) - f(\bar{G})} \subset f(\bar{G}) \cap \overline{f(\mathscr{X}) - f(G)} \\ &\subset f(\bar{G}) \cap \overline{f(\mathscr{X}) - G)} = f(\bar{G}) \cap f(\mathscr{X}) - G), \end{aligned}$$

because $f(\mathscr{X}) - f(G) \subset f(\mathscr{X} - G)$ (compare § 3, III, (3)).

We have $y \epsilon E$, because $f^{-1}(y) \subset G$ and hence $y \epsilon f(G) \subset E$.

On the other hand, $y \notin \mathrm{Fr}(E)$ since

$$\mathscr{X} - G \subset \mathscr{X} - f^{-1}(y) = f^{-1}[f(\mathscr{X}) - y],$$

but then $f(\mathscr{X} - G) \subset f(\mathscr{X}) - y$ and so $y \notin f(\mathscr{X} - G) \supset \mathrm{Fr}(E)$.

So E is a neighbourhood of y. Finally we get (compare § 3, III, (13))

$$\begin{aligned} f(\bar{G}) \cap f(\mathscr{X} - G) &= f[\bar{G} \cap f^{-1} f(\mathscr{X} - G)] \\ &= f[(\bar{G} - G) \cap f^{-1} f(\mathscr{X} - G) \cup G \cap f^{-1} f(\mathscr{X} - G)] \\ &= f[(\bar{G} - G) \cap f^{-1} f(\mathscr{X} - G)] \cup \bigcup_{m=1}^{\infty} f[F_m \cap f^{-1} f(\mathscr{X} - G)] \end{aligned}$$

presenting G as a union

$$G = F_1 \cup F_2 \cup \ldots, \quad \text{where} \quad F_m = \bar{F}_m.$$

By hypothesis the inequality $\dim(\bar{G}-G) \leqslant n-1$ implies that

$$\dim f[(\bar{G}-G) \cap f^{-1}f(\mathscr{X}-G)] \leqslant n+k_0-2;$$

the fact that f is of order $\leqslant k_0-1$ on $G \cap f^{-1}f(\mathscr{X}-G)$ (by the second part of Lemma 1) for $A_i = \mathscr{X}-G$ proves that

$$\dim f[F_m \cap f^{-1}f(\mathscr{X}-G)] \leqslant n+k_0-2.$$

By the union theorem (§ 27, I, Theorem 1) it follows that

$$\dim[f(\bar{G}) \cap f(\mathscr{X}-G)] \leqslant n+k_0-2,$$

and therefore

$$\dim \mathrm{Fr}(E) \leqslant n+k_0-2.$$

The second part of Theorem 2 is a consequence of the first one, because by Lemma 1, (1) and (2), we obtain

$$\begin{aligned}\dim B = \dim f[A_i \cap f^{-1}(B)] &\leqslant \dim[A_i \cap f^{-1}(B)]+k-r-1\\ &\leqslant \dim A_i+k-r-1.\end{aligned}$$

II. Parametric representation of n-dimensional perfect, compact spaces on the Cantor set $\mathscr{C}$(1). By Corollary 3a of § 41, VI, every compact space has a parametric representation on the set $\mathscr{C}$ (i.e. it is a continuous image of $\mathscr{C}$). This fact can be restated more precisely in the following way.

THEOREM 1. *If a perfect subset P of a compact space $\mathscr{X}$ is n-dimensional (or more generally, P satisfies condition D_n of* § 27, III), *then there exists a continuous mapping of $\mathscr{C}$ onto $\mathscr{X}$ such that every point of P is of order $\leqslant n+1$.*

Moreover, if Φ is the set of functions f such that $f \in \mathscr{X}^{\mathscr{C}}$ and $f(\mathscr{C}) = \mathscr{X}$, then the subset Ψ of Φ, consisting of functions which have points of order $> n+1$ in P, is of the first category in Φ.

Proof. Since Φ is non-empty (§ 41, VI, Corollary 3a) and closed (§ 44, II, Corollary 4) subset of the complete space $\mathscr{X}^{\mathscr{C}}$ (§ 44, VII, Theorem 3), it is sufficient to prove the second part of Theorem 1, because by the Baire Theorem a first category subset cannot fill up a complete space.

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(1) See my paper, *Sur l'application des espaces fonctionnels à la Théorie de la dimension,* Fund. Math. 18 (1932), p. 285.

By the definition of Ψ, a positive integer l and a system of points $x_0, \ldots, x_{n+1}$ correspond to each $f \in \Psi$ so that

$$f(x_i) = f(x_j) \in P \quad \text{and} \quad |x_i - x_j| \geqslant 1/l \qquad \text{for} \quad 0 \leqslant i < j \leqslant n+1. \tag{1}$$

Let Ψ_l be the set of functions f for which there exists a system of points $x_0, \ldots, x_{n+1}$ satisfying condition (1); then

$$\Psi = \bigcup_{l=1}^{\infty} \Psi_l.$$

The idea of the proof is to show that Ψ_l is not dense in Φ. Since Ψ_l is closed, as can easily be seen(¹), our task is to prove that Ψ_l is a boundary set, i.e. that a mapping f^* corresponds to each $f \in \Phi$ and to every $\varepsilon > 0$ such that

$$f^* \in \Phi - \Psi_l \tag{2}$$

and

$$|f - f^*| \leqslant \varepsilon. \tag{3}$$

Since the mapping f is uniformly continuous on the space $\mathscr{C}$, the latter can be split into disjoint closed parts so that

$$\mathscr{C} = C_0 \cup \ldots \cup C_m, \quad \delta[f(C_i)] < \varepsilon/2 \quad \text{and} \quad \delta(C_i) < 1/l.$$

Let G_i be the open ball with the center $f(C_i)$ and radius $\varepsilon/4$. It follows that

$$\mathscr{X} = G_0 \cup \ldots \cup G_m, \quad f(C_i) \subset G_i \neq 0, \quad \delta(G_i) < \varepsilon. \tag{4}$$

Since P is a perfect set, condition D_n implies (compare § 27, III, Theorem 5 (5)) the existence of a system of closed sets H_i such that

$$\mathscr{X} = H_0 \cup \ldots \cup H_m, \quad 0 \neq H_i \subset G_i, \quad P \cap H_{i_0} \cap \ldots \cap H_{i_{n+1}} = 0 \tag{5}$$

for any distinct indices $i_0, \ldots, i_{n+1}$.

Let f^* be a continuous function such that $f^*(C_i) = H_i$. Since

$$f^*(\mathscr{C}) = f^*(C_0) \cup \ldots \cup f^*(C_m) = \mathscr{X},$$

(¹) E.g. defining Ψ_l by means of logical symbols

$$(f \in \Psi_l) \equiv \bigvee_{x_0, \ldots, x_{n+1}} \bigwedge_{ij} \{(0 \leqslant i < j \leqslant n+1) \Rightarrow [|x_i - x_j| \geqslant 1/l][f(x_i) = f(x_j) \in P]\}.$$

f^* belongs to Φ. Let us suppose, contrary to (2), that $f^* \in \Psi_l$. Let $x_0, \ldots, x_{n+1}$ be a system of points satisfying condition (1) (in which f is replaced by f^*). Since $\delta(C_i) < 1/l$, no C_i contains two distinct points belonging to this system; so there exists a system of distinct indices $i_0, \ldots, i_{n+1}$ such that $x_j \in C_{i_j}$, $j = 0, \ldots, n+1$. Therefore $f^*(x_j) \in f^*(C_{i_j}) = H_{i_j}$, and since $f^*(x_0) = f^*(x_j) \in P$ (compare (1)), it follows that

$$f^*(x_0) \in P \cap H_{i_0} \cap \ldots \cap H_{i_{n+1}}$$

contrary to (5).

Condition (2) is proved by this contradiction.

So is condition (3). In fact, by (4) and (5) we get

$$f(x) \in f(C_i) \subset G_i \quad \text{and} \quad f^*(x) \in H_i \subset G_i$$

for any $x \in C_i$, and hence by (4)

$$|f(x) - f^*(x)| \leqslant \delta(G_i) < \varepsilon.$$

Theorem 1 implies the following three corollaries.

THEOREM 2. *A perfect compact space has dimension $\leqslant n$ if and only if it has a parametric representation of order $\leqslant n+1$ on the set $\mathscr{C}$.*

This corollary follows from the preceding theorem (if we let $\mathscr{X} = P$) and Theorem I, 2.

THEOREM 3. *For any compact*[1] *space $\mathscr{X}$ the condition* D_n *is equivalent to the inequality* $\dim \mathscr{X} \leqslant n$.

Proof. Taking into account Theorem 4 of § 27, III, it will be sufficient to show that condition D_n implies the inequality $\dim \mathscr{X} < n$. Let $\mathscr{X}_0$ be a perfect subset of $\mathscr{X}$ such that $\mathscr{X} - \mathscr{X}_0$ is countable (cf. the Cantor–Bendixson Theorem, § 23, V). Since the open set $\mathscr{X} - \mathscr{X}_0$ has dimension $\leqslant 0$, we get (compare § 27, I, Theorem 1) $\dim \mathscr{X} = \dim \mathscr{X}_0$ (with the exception of the case when $\mathscr{X}_0 = 0$ which may be omitted, because in that case $\dim \mathscr{X} \leqslant 0$). Hence it only remains to show that $\dim \mathscr{X}_0 \leqslant n$ or, by Theorems 1 and 2 of Section I, that $\mathscr{X}_0$ satisfies condition D_n.

Let

$$\mathscr{X}_0 = A_1 \cup \ldots \cup A_m \tag{6}$$

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[1] In case of an arbitrary space see VII, Corollary 3.

be a decomposition into sets, which are open in $\mathscr{X}_0$. Let G_i be an open set such that $A_i = G_i \cap \mathscr{X}_0$, where $i = 1, \ldots, m$, and let $G_0 = \mathscr{X} - \mathscr{X}_0$. It follows that

$$\mathscr{X} = G_0 \cup \ldots \cup G_m.$$

Therefore, property D_n of $\mathscr{X}$ implies the existence of a system of open sets $H_0, \ldots, H_m$ such that

$$\mathscr{X} = H_0 \cup \ldots \cup H_m, \quad H_i \subset G_i \quad \text{and} \quad H_{i_0} \cap \ldots \cap H_{i_{n+1}} = 0.$$

Assuming $B_i = H_i \cap \mathscr{X}_0$, we get $B_0 = 0$ and

$$\mathscr{X}_0 = B_1 \cup \ldots \cup B_m, \quad B_i \subset A_i, \quad B_{i_0} \cap \ldots \cap B_{i_{n+1}} = 0.$$

These conditions together with (6) prove that $\mathscr{X}_0$ satisfies condition D_n.

THEOREM 4. *Let $P_1, P_2, \ldots$ be a sequence of perfect sets in a compact space $\mathscr{X}$. There exists a continuous mapping of $\mathscr{C}$ onto $\mathscr{X}$ such that each point of P_i $(i = 1, 2, \ldots)$ is of order $\leqslant \dim P_i + 1$.*

Proof. Let Φ have the same meaning as in Theorem 1, and let Ψ be the set of functions $f \in \Phi$ which do not satisfy conditions of Theorem 4. We have $\Psi = \Psi_1 \cup \Psi_2 \cup \ldots$, where Ψ_i is the set of functions $f \in \Phi$ which have points of order $> \dim P_i + 1$ in the set P_i.

Since Ψ_i is of the first category in Φ by Theorem 1, the same holds for Ψ. Therefore $\Phi - \Psi \neq 0$.

Now we are going to consider parametric representation of closed sets (not necessarily perfect).

LEMMA 5. *Every closed subset F of a perfect compact space $\mathscr{X}$ is contained in a perfect set P such that $\dim P = \dim F$.*

Hence (by substituting the Hilbert cube $\mathscr{I}^{\aleph_0}$ for $\mathscr{X}$) it follows that every compact space is topologically contained in a perfect compact space of the same dimension.

Proof. If $p_1, p_2, \ldots$ is the sequence of isolated points in F and P_i is a perfect set such that $p_i \in P_i$, $\delta(P_i) < 1/i$ and $\dim P_i = 0$, then the set

$$P = F \cup P_1 \cup P_2 \cup \ldots$$

is perfect and $\dim P = \dim F$ by the union theorem (§ 27, I, Theorem 2).

THEOREM 6. *Every n-dimensional compact space has a parametric representation of order $n+1$ on a 0-dimensional compact space* (1).

Generally speaking, *Theorem 4 remains true if the hypothesis of the sets P_i being perfect is deleted (but the assumption of P_i of being closed is kept) and $\mathscr{C}$ is replaced by a 0-dimensional compact space $\mathscr{C}_0$ properly chosen.*

Proof. Let us consider $\mathscr{X}$ as a subset of the space $\mathscr{I}^{\aleph_0}$ and let P_i^* be a perfect set (in $\mathscr{I}^{\aleph_0}$) such that

$$P_i \subset P_i^* \quad \text{and} \quad \dim P_i = \dim P_i^*.$$

According to Theorem 4, let f be a continuous mapping of $\mathscr{C}$ onto $\mathscr{I}^{\aleph_0}$ such that each point of P_i^* has the order $\leqslant \dim P_i^* + 1$. Let us assume $\mathscr{C}_0 = f^{-1}(\mathscr{X})$; then $f|\mathscr{C}_0$ is the required mapping of $\mathscr{C}_0$ onto $\mathscr{X}$.

THEOREM 7. *If G is an n-dimensional open subset of a compact space $\mathscr{X}$, there exists a 0-dimensional compact space $\mathscr{C}_0$ and a continuous mapping f of $\mathscr{C}_0$ onto $\mathscr{X}$ such that each point of G is of order $\leqslant n+1$.*

Proof. If $P_1, P_2, \ldots$ is a sequence of closed sets such that $G = P_1 \cup P_2 \cup \ldots$, then all that is necessary is to apply the second part of Theorem 6.

Remarks. For $n = 0$, Theorem 2 implies that every 0-dimensional perfect space is homeomorphic to the Cantor set $\mathscr{C}$. All these spaces have the same topological type and they possess all the topological properties of $\mathscr{C}$ (as to be homogeneous for example).

In connection with the countable compact spaces $\mathscr{X}$ let us quote the following

MAZURKIEWICZ–SIERPIŃSKI THEOREM (2). *If $\mathscr{X}^{(a)}$ is the last derived* (compare § 24, IV) *of $\mathscr{X}$ and n is the number of elements in $\mathscr{X}^{(a)}$, the space $\mathscr{X}$ is homeomorphic to a well-ordered subset of a segment of the type $\omega^a \cdot n+1$.*

Therefore the topological type of a countable compact space *is characterized* by the pair (a, n). Hence countable compact spaces have $\aleph_1$ topological types.

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(1) See a generalization by J. H. Roberts, *A theorem on dimension,* Duke Math. Journ. 8 (1941), p. 565.

(2) Fund. Math. 1 (1920), p. 21.

However, there are $\mathfrak{c}$ distinct topological types of *scattered* spaces (which are metric separable, but not necessarily compact) (1).

III. Theorems on decomposition. The fact that every n-dimensional space satisfies condition D_n is a very particular case of the following theorem.

THEOREM 1 (2). *Let a sequence of closed sets $P_1, P_2, \ldots$ be given in a compact* (3) *space $\mathscr{X}$. To any decomposition into open sets $\mathscr{X} = G_0 \cup \ldots \cup G_m$ there corresponds a decomposition into closed sets*

$$\mathscr{X} = F_0 \cup \ldots \cup F_m \quad \textit{where} \quad F_i \subset G_i$$

and

$$\dim(P_j \cap F_{i_0} \cap \ldots \cap F_{i_r}) \leqslant \dim P_j - r$$

for any integers $j = 1, 2, \ldots, r \leqslant \dim P_j + 1$ and $i_0 < \ldots < i_r \leqslant m$.

Proof. According to II, Theorem 6, there exists a 0-dimensional compact space $\mathscr{C}_0$ and a continuous mapping f such that $f(\mathscr{C}_0) = \mathscr{X}$ and every point of P_j is of order $\leqslant \dim P_j + 1$. Since the sets $f^{-1}(G_i)$ are open, the decomposition

$$\mathscr{C}_0 = f^{-1}(G_0) \cup \ldots \cup f^{-1}(G_m)$$

implies that there exists a system of disjoint closed-open sets A_i such that

$$\mathscr{C}_0 = A_0 \cup \ldots \cup A_m \quad \text{and} \quad A_i \subset f^{-1}(G_i).$$

Define $F_i = f(A_i)$. Hence

$$\mathscr{X} = F_0 \cup \ldots \cup F_m \quad \text{and} \quad F_i \subset G_i.$$

Finally assume $Q_j = f^{-1}(P_j)$; then the partial function $f|Q_j$ is of order $\leqslant \dim P_j + 1$ and hence one gets, by I, Theorem 2 (for $k = \dim P_j + 1$) and by the condition $\dim A_i = 0$, the following inequality:

$$\dim[f(A_{i_0} \cap Q_j) \cap \ldots \cap f(A_{i_r} \cap Q_j)] \leqslant \dim P_j - r.$$

(1) *Ibidem.*

(2) See my paper quoted above, p. 290. For the particular case (where the sequence $P_1, P_2, \ldots$ is finite) see K. Menger, *Dimensionstheorie,* p. 170.

(3) The hypothesis of compactness may be omitted; see Section VII (Theorem 5).

But this proves our theorem, because (compare § 3, III, (13))

$$f[A_i \cap f^{-1}(P_j)] = f(A_i) \cap P_j = F_i \cap P_j.$$

COROLLARY 2 (¹). *Every compact n-dimensional space, for any $\varepsilon > 0$, can be decomposed into a finite number of closed sets such that the diameter of each set is $< \varepsilon$ and any product of r sets has dimension $\leqslant n-r+1$ $(r = 1, 2, \ldots, n+2)$.*

Proof. It is sufficient to assume that

$$\mathscr{X} = P_1 = P_2 = \ldots \quad \text{and} \quad \delta(G_i) < \varepsilon.$$

Remarks. Corollary 2 follows more directly from the first part of Theorem II, 6. Let $\mathscr{C}_0$ be the 0-dimensional space and f the required function. Let us decompose $\mathscr{C}_0$ into disjoint closed sets $A_0, \ldots, A_m$ sufficiently small in order that $\delta[f(A_i)] < \varepsilon$; then

$$\mathscr{X} = f(A_0) \cup \ldots \cup f(A_m)$$

is the required decomposition.

Moreover, if a sequence $\varepsilon_1, \varepsilon_2, \ldots$ converging to 0 is considered instead of the number ε, then a sequence of decompositions is obtained, which satisfy conditions of Corollary 2 and the $i+1$-st decomposition is a subdivision of the ith one. All that is needed is to subdivide the sets $A_0, \ldots, A_m$.

IV. n-dimensional degree. The n-dimensional degree of a space $\mathscr{X}$ (written $d_n(\mathscr{X})$) is defined (²) as the greatest lower bound of numbers ε such that there exists a system of open sets $G_0, \ldots, G_m$, which satisfy conditions

$$\mathscr{X} = G_0 \cup \ldots \cup G_m, \tag{1}$$

$$\delta(G_i) < \varepsilon, \tag{2}$$

$$G_{i_0} \cap \ldots \cap G_{i_n} = 0 \quad \text{for} \quad i_0 < \ldots < i_n \leqslant m. \tag{3}$$

In other words, *the inequality $d_n(\mathscr{X}) < \varepsilon$ holds if and only if there exists a system of open sets which satisfy conditions* (1) *to* (3).

(¹) See K. Menger, *Dimensionstheorie*, p. 156.

(²) Compare Urysohn's flattening coefficient, Fund. Math. 8 (1926), p. 353.

In particular, the inequality $d_1(\mathcal{X}) < \varepsilon$ means that $\mathcal{X}$ can be decomposed into a finite number of disjoint closed-open sets with diameters $< \varepsilon$.

Condition (3) can be replaced by the following one

$$\bar{G}_{i_0} \cap \ldots \cap \bar{G}_{i_n} = 0 \quad \text{for} \quad i_0 < \ldots < i_n \leqslant m. \tag{4}$$

It can also be assumed that the sets G_i are closed instead of being open (compare § 14, III).

THEOREM 1. *If $\mathcal{X}$ is compact, then the condition $d_{n+1}(\mathcal{X}) = 0$ is equivalent to the condition* D_n, *and hence it is equivalent to the condition* $\dim \mathcal{X} \leqslant n$ (compare II, Theorem 3).

Proof. The application of the condition D_n to a covering of $\mathcal{X}$ with open sets of diameter $< \varepsilon$ gives $d_{n+1}(\mathcal{X}) = 0$.

On the other hand, let $\mathcal{X} = A_0 \cup \ldots \cup A_m$ be a decomposition into open sets. By § 41, VI, Corollary 4d, there exists a number $\varepsilon > 0$ so that each set of diameter $< \varepsilon$ is contained in at least one of the sets A_i. Let us assume $d_{n+1}(\mathcal{X}) = 0$ and consider the sets $G_0, \ldots, G_m$ which satisfy conditions (1) to (3), where n is replaced by $n+1$. Let I_i be the set of indices j such that $G_j \subset A_i$, let H_0 be the union of all G_j with $j \in I_0$, and generally, let H_{i+1} be the union of all G_j with $j \in I_{i+1} - (I_0 \cup \ldots \cup I_i)$. It follows that

$$\mathcal{X} = H_0 \cup \ldots \cup H_m, \quad H_i \subset A_i \quad \text{and} \quad H_{i_0} \cap \ldots \cap H_{i_{n+1}} = 0,$$

and this shows that $\mathcal{X}$ satisfies condition D_n.

If E is a subset of $\mathcal{X}$, then the condition $d_n(E) < \varepsilon$ means that there exists a system of sets $A_0, \ldots, A_m$ which are open in E and satisfy conditions

$$E = A_0 \cup \ldots \cup A_m, \quad \delta(A_i) < \varepsilon \quad \text{and} \quad A_{i_0} \cap \ldots \cap A_{i_n} = 0.$$

Let $G_0, \ldots, G_m$ be a system of open sets which is similar to the system $A_0, \ldots, A_m$ and such that $A_i = E \cap G_i$ and $\delta(G_i) < \varepsilon$ (compare § 21, XI, Theorem 2), it follows that *the condition $d_n(E) < \varepsilon$ is equivalent to the existence of a system of open sets $G_0, \ldots, G_m$ which satisfy conditions* (2) *and* (3) *and the inclusion $E \subset G_0 \cup \ldots \cup G_m$ as well.* Besides, condition (3) can be replaced by (4).

Remark. The assumption of compactness of $\mathcal{X}$ cannot be omitted. In fact *there exists a set $A \subset \mathcal{E}^3$ of dimension 2 such that, for each $\varepsilon > 0$, there is a continuous mapping of A with point inverses* *of diameter less than ε into a polygonal line.* Thus $d_2(A) = 0$. Moreover, A is a G_δ-set [1].

THEOREM 2. *Suppose that $\mathscr{X}$ is compact. Then the family of all closed sets F such that $d_n(F) < \varepsilon$, is open in the space $2^{\mathscr{X}}$ for each $\varepsilon > 0$.*

In other words, *the function $d_n(F)$ is upper-semicontinuous in the space $2^{\mathscr{X}}$.*

Proof. If Γ is a system of open sets $G_0, \ldots, G_m$ satisfying conditions (2) and (3), then the family

$$\Phi_\Gamma = \mathop{E}_{F}(F \subset G_0 \cup \ldots \cup G_m)$$

is open in $2^{\mathscr{X}}$ (compare § 17, II, Theorem 1). The same holds for the family

$$\mathop{E}_{F}[d_n(F) < \varepsilon] = \bigcup_{\Gamma} \Phi_\Gamma.$$

Theorem 2 can be generalized in the following way.

THEOREM 2′. *If $\mathscr{X}$ is compact, the set*

$$\mathop{E}_{F_1,\ldots,F_k}[d_n(F_1 \cap \ldots \cap F_k) < \varepsilon]$$

is open in the space $(2^{\mathscr{X}})^k$.

Proof. In order to show this, it is sufficient to define

$$\Phi_\Gamma = \mathop{E}_{F_1,\ldots,F_k}(F_1 \cap \ldots \cap F_k \subset G_0 \cup \ldots \cup G_m)$$

and to apply Theorem 1 of § 17, V, instead of Theorem 1 of § 17, II.

It follows that the set Φ_Γ is open in $(2^{\mathscr{X}})^k$.

THEOREM 3. *If $\mathscr{X}$ is compact, there exists a set $F \in 2^{\mathscr{X}}$ such that $d_n(F) = d_n(\mathscr{X})$ and that the conditions $X \in 2^{\mathscr{X}}$ and $X \subset F \neq X$ imply $d_n(X) < d_n(F)$; that means that the set F is irreducible with respect to its n-dimensional degree.*

Proof. Since the set $\mathop{E}_{F}[d_n(F) \geqslant d_n(\mathscr{X})]$ is closed, it contains an irreducible element (by § 42, IV, Theorem 2).

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(1) See K. Sitnikov, *Example of a two-dimensional set in $\mathscr{E}^3$ which can be deformed into a polygonal one-dimensional line by means of a deformation as small as we wish; new characterization of dimension in euclidean spaces*, Dokl. Acad. Nauk URSS 88 (1953), p. 21. See also, of the same author, *Combinatorial topology of non-closed sets*, Matem. Sb. 37 (1955), pp. 385–434 (Russian).

THEOREM 4. *If $\mathscr{X}$ is compact, the set* $\underset{F}{E}(\dim F \leqslant n)$ *is a* G_δ *in* $2^{\mathscr{X}}$. *Hence the function* $n = \dim F$ *is of the second class on* $2^{\mathscr{X}}$.

Proof. This is because

$$\underset{F}{E}(\dim F \leqslant n) = \underset{F}{E}[d_{n+1}(F) = 0] = \bigcap_{k=1}^{\infty} \underset{F}{E}[d_{n+1}(F) < 1/k]$$

and the set $\underset{F}{E}[d_{n+1}(F) < 1/k]$ is open by Theorem 2.

The set $\underset{F}{E}(m \leqslant \dim F \leqslant n)$ is a difference of two G_δ sets. Hence it is an $F_{\sigma\delta}$ and therefore the dimension of a closed set F considered as the function of F is of the second class of Baire.

The application of Theorem 2′ instead of Theorem 2 gives the following one.

THEOREM 4′. *If $\mathscr{X}$ is compact, the set*

$$\underset{F_1,\dots,F_k}{E}[\dim(F_1 \cap \dots \cap F_k) \leqslant n]$$

is a G_δ *in the space* $(2^{\mathscr{X}})^k$.

Remarks. (i) *The function* $\dim F$ *is a limit of an increasing sequence of upper semi-continuous functions* $\Delta_k(F)$.

Proof. Let $\Delta_k(F)$ denote the least integer $n \geqslant -1$ such that there exists a system of open sets $G_0, \dots, G_m$ satisfying conditions

$$F \subset G_0 \cup \dots \cup G_m, \qquad \delta(G_i) < 1/k \qquad \text{and} \qquad G_{i_0} \cap \dots \cap G_{i_{n+1}} = 0$$

for any set of indices $i_0 < \dots < i_{n+1} \leqslant m$.

Hence the condition $\Delta_k(F) \leqslant n$ is equivalent to $d_{n+1}(F) < 1/k$. Therefore, by Theorem 2, the function $\Delta_k(F)$ is upper semi-continuous. Put $\dim F = n$; it follows that $\Delta_k(F) > n-1$ for sufficiently large values of k, since $d_n(F) \geqslant 1/k$ in that case. On the other hand, condition D_n implies $\Delta_k(F) \leqslant n$ and it follows that

$$n = \dim F = \lim_{k=\infty} \Delta_k(F).$$

(ii) *The set* $\underset{F}{E}(\dim F \leqslant n)$ *does not need to be an* F_σ *and hence the function* $\dim F$ *does not need to be of the first class.*

E.g. if $\mathscr{X} = \mathscr{I}$, the families F_0 and F_1 consisting of 0 or 1-dimensional sets F respectively are dense in $2^{\mathscr{X}}$. Therefore, F_0 cannot be an F_σ by the Baire Theorem (§ 34, IV).

(iii) *If $\mathcal{X}$ is a metric separable space, then the function $f(p) = \dim_p \mathcal{X}$ is of the second class.*

Proof. The reason for this is that by § 25, III, Theorem 2 the set $\underset{p}{E}(\dim_p X \leqslant n)$ is a $\boldsymbol{G}_\delta$ (which does not need to be an $\boldsymbol{F}_\sigma$).

THEOREM 5. *If $\mathcal{X}$ and $\mathcal{Y}$ are compact, the set*

$$\underset{f}{E} \underset{y}{\wedge} \{d_n[f^{-1}(y)] < \varepsilon\}$$

is open in the space $\mathcal{Y}^{\mathcal{X}}$ for any $\varepsilon > 0$.

The set $\underset{f}{E} \underset{y}{\wedge} \{\dim[f^{-1}(y)] \leqslant n\}$ is a $\boldsymbol{G}_\delta$.

Proof. Let $\Phi_y = \underset{f}{E}\{d_n[f^{-1}(y)] < \varepsilon\}$. Condition $f \epsilon \Phi_y$ means that there exists a system of open sets $G_0, \ldots, G_m$ which satisfy conditions (2), (3), and the inclusion $f^{-1}(y) \subset G_0 \cup \ldots \cup G_m$. Since the set of pairs (f, y) which satisfy this inclusion is open (§ 44, II, Corollary 6), then so are $\underset{f,y}{E}[f \epsilon \Phi_y]$ and $\underset{f}{E} \underset{y}{\wedge}(f \epsilon \Phi_y)$ as well (compare § 41, IV, Corollary 1b).

Finally,

$$\begin{aligned}\underset{y}{\wedge}\{\dim[f^{-1}(y) \leqslant n]\} &\equiv \underset{y}{\wedge}\{d_{n+1}[f^{-1}(y)] = 0\}\\ &\equiv \underset{y}{\wedge}\underset{k}{\wedge}\{d_{n+1}[f^{-1}(y)] < 1/k\}\\ &\equiv \underset{k}{\wedge}\underset{y}{\wedge}\{d_{n+1}[f^{-1}(y)] < 1/k\}\end{aligned}$$

which proves the second part of the theorem.

THEOREM 6. *If A and B are compact sets, the inequalities $d_n(A) < \varepsilon$, $d_n(B) < \varepsilon$ and $\dim(A \cap B) \leqslant n-2$ imply that $d_n(A \cup B) < \varepsilon$.*

Proof. By hypothesis there exists two systems of open sets $A_0, \ldots, A_1$ and $B_0, \ldots, B_m$ such that

$$\begin{array}{lll} A \subset A_0 \cup \ldots \cup A_l, & \delta(A_i) < \varepsilon, & A_{i_0} \cap \ldots \cap A_{i_n} = 0,\\ B \subset B_0 \cup \ldots \cup B_m, & \delta(B_j) < \varepsilon, & B_{j_0} \cap \ldots \cap B_{j_n} = 0.\end{array}$$

Using Theorem 1 of Section III in the case where $\mathcal{X} = A$ and $P_1 = P_2 = \ldots = A \cap B$, the existence of a system of closed sets $A_0^*, \ldots, A_l^*$ can be inferred, such that

$$A = A_0^* \cup \ldots \cup A_l^*,\ A_i^* \subset A_i \quad \text{and}$$
$$\dim(B \cap A_{i_0}^* \cap \ldots \cap A_{i_r}^*) \leqslant n-r-2$$

for any $r \leqslant n-1$.

The same theorem used for $\mathscr{X} = B$ implies the existence of a system of closed sets $B_0^*, \ldots, B_m^*$ such that

$$B = B_0^* \cup \ldots \cup B_m^*, \quad B_j^* \subset B_j \quad \text{and}$$
$$A_{i_0}^* \cap \ldots \cap A_{i_r}^* \cap B_{j_0}^* \cap \ldots \cap B_{j_{n-r-1}} = 0$$

(where the sets $B \cap A_{i_0}^* \cap \ldots \cap A_{i_r}^*$ take the role of P_i's and where $0 \leqslant r \leqslant n-1$).

Thus a decomposition is obtained

$$A \cup B = A_0^* \cup \ldots \cup A_l^* \cup B_0^* \cup \ldots \cup B_m^*$$

into closed sets of diameter $< \varepsilon$ and such that no point is common to $n+1$ of these sets. Hence $d_n(A \cup B) < \varepsilon$.

Remark. The condition $\dim(A \cap B) \leqslant n-2$ cannot be replaced by $d_{n-1}(A \cap B) \leqslant \varepsilon$.

THEOREM 8 [1]. *If $\mathscr{X}$ is compact, $d_n(\mathscr{X})$ is the greatest lower bound of numbers ε for which there exists a continuous mapping f of $\mathscr{X}$ such that*

$$\dim f(\mathscr{X}) \leqslant n-1, \tag{i}$$
$$\delta[f^{-1}(y)] < \varepsilon \tag{ii}$$

for any $y \in f(\mathscr{X})$.

More precisely, *if $d_n(\mathscr{X}) < \varepsilon$, there exists a continuous mapping f, satisfying condition* (ii), *of $\mathscr{X}$ into an $(n-1)$-dimensional polyhedron; on the other hand, if f is a continuous mapping satisfying conditions* (i) *and* (ii), *it follows that $d_n(\mathscr{X}) < \varepsilon$ for compact $\mathscr{X}$.*

Proof. Suppose that $d_n(\mathscr{X}) < \varepsilon$. Hence there exists a system of open sets $G_0, \ldots, G_m$ satisfying conditions (1) to (3). Let $p_0 \ldots p_m$ be an m-dimensional simplex. The mapping

$$f(x) = \lambda_0(x) \cdot p_0 + \ldots + \lambda_m(x) \cdot p_m,$$

where

$$\lambda_i(x) = \frac{\varrho(x, \mathscr{X} - G_i)}{\varrho(x, \mathscr{X} - G_0) + \ldots + \varrho(x, \mathscr{X} - G_m)},$$

[1] Compare P. Alexandrov, C. R. Paris 183, p. 640, and the paper of Ulam and myself, Fund. Math. 20 (1933), p. 246.

i.e. the mapping $\varkappa$ associated with the systems $p_0, \ldots, p_m$ and $G_0, \ldots, G_m$, is the required mapping (compare § 28, VI, Theorem 4).

On the other hand, let f be the mapping which satisfies conditions (i) and (ii). Since $f(\mathscr{X})$ is compact, there exists by (ii) (compare § 41, VI, Lemma) an $\eta > 0$ such that conditions $f(\mathscr{X}) = H_0 \cup \cup \ldots \cup H_m$ and $\delta(H_i) < \eta$ imply $\delta[f^{-1}(H_i)] < \varepsilon$.

By condition (i) it can be assumed that H_i are open and that $H_{i_0} \cap \ldots \cap H_{i_n} = 0$. Let $G_i = f^{-1}(H_i)$; then conditions (1) to (3) are satisfied; hence $d_n(\mathscr{X}) < \varepsilon$.

COROLLARIES 9. *If $\mathscr{X}$ is a compact space, then*

(1) *the relation* $\dim \mathscr{X} \geqslant n$ *is an invariant of the mappings with small point-inverses* (1), *namely with point-inverses of diameter* $< d_n(\mathscr{X})$;

(2) *if $\mathscr{X} \subset \mathscr{E}^{\aleph_0}$* (2), *the condition* $\dim \mathscr{X} \leqslant n$ *is equivalent to the existence for each $\varepsilon > 0$ of an ε-displacement of $\mathscr{X}$ into an n-dimensional polyhedron* (3);

(3) *the degree $d_n(\mathscr{X})$ is the greatest lower bound of numbers $\tau(\mathscr{X}, \mathscr{Y})$* (4) *where $\mathscr{Y}$ ranges over all compact spaces of dimension $< n$.*

Using the following Theorem, we can define the coefficient d_n by means of d_1.

THEOREM 10 (5). *For compact $\mathscr{X}$ the degree $d_n(\mathscr{X})$ is the greatest lower bound of numbers ε such that there exists a system of closed sets $A_1, \ldots, A_n$ having the following properties*

$$\mathscr{X} = A_1 \cup \ldots \cup A_n \quad \textit{and} \quad d_1(A_i) < \varepsilon, \quad \textit{for} \quad i = 1, \ldots, n. \qquad (+)$$

More precisely: *the condition $d_n(\mathscr{X}) < \varepsilon$ is equivalent to the existence of a decomposition of this kind.*

Proof. First let $d_n(\mathscr{X}) < \varepsilon$. We use induction. The existence of the required decomposition is obvious if $n = 1$; we are going

(1) For the definition see § 41, VII. Corollary 9 (1) is due to L. E. J. Brouwer.

(2) It is worth noticing that according to a recent result of R. D. Anderson, *the space $\mathscr{E}^{\aleph_0}$ is homeomorphic to the Hilbert space.* See Proceed. Moscow Congress 1966.

(3) Compare § 28, VI, Remark (i).

(4) Compare § 41, VII.

(5) Theorem of Eilenberg, *Sur le théorème de décomposition de la théorie de la dimension,* Fund. Math. 26 (1936), p. 147.

to show that such a decomposition exists for a number n, provided that it does exist for the number $n-1$.

Since $d_n(\mathscr{X}) < \varepsilon$, there exists a system of closed sets $F_0, \ldots, F_m$ such that

$$\mathscr{X} = F_0 \cup \ldots \cup F_m, \quad F_{i_0} \cap \ldots \cap F_{i_n} = 0, \quad \delta(F_i) < \varepsilon.$$

Let F be the union of all intersections of n sets belonging to that system. Since these intersections are pairwise disjoint, it follows that $d_1(F) < \varepsilon$. According to Theorem 2, let S be an open ball with center F and with a sufficiently small radius in order that $d_1(\overline{S}) < \varepsilon$.

The conditions

$$\mathscr{X} - S = (F_0 - S) \cup \ldots \cup (F_m - S), \quad \delta(F_i - S) < \varepsilon$$

and

$$F_{i_1} \cap \ldots \cap F_{i_n} - S \subset F - S = 0$$

imply that

$$d_{n-1}(\mathscr{X} - S) < \varepsilon.$$

So it follows by hypothesis that

$$\mathscr{X} - S = A_2 \cup \ldots \cup A_n, \quad \overline{A}_i = A_i, \quad d_1(A_i) < \varepsilon \quad \text{for} \quad i = 2, \ldots, n.$$

The decomposition $(+)$ is obtained setting $A_1 = \overline{S}$.

Conversely, assume that the system of closed sets $A_1, \ldots, A_n$ satisfies condition $(+)$.

Therefore, for each i, there exists a decomposition of A_i into closed disjoint sets

$$A_i = A_1^i \cup \ldots \cup A_{k_i}^i \quad \text{where} \quad \delta(A_j^i) < \varepsilon \quad \text{for} \quad j = 1, \ldots, k_i.$$

So $\mathscr{X} = \bigcup_{ij} A_j^i$ is a decomposition of $\mathscr{X}$ into closed sets so that no point is common to any $n+1$ of them. Therefore, $d_n(\mathscr{X}) < \varepsilon$.

Theorems 1 and 10 imply the following statement.

THEOREM 11. *If $\mathscr{X}$ is compact, then* $\dim \mathscr{X} < n$ *if and only if for each $\varepsilon > 0$ there exists a system of closed sets $A_1, \ldots, A_n$ which satisfies condition* $(+)$.

V. Dimensional kernel of a compact space. N is said to be the *dimensional kernel* (compare § 27, V) of the n-dimensional space $\mathscr{X}$ if N is the set of all points p such that $\dim_p \mathscr{X} = n$. In general, the dimensional kernel of a metric separable space has dimension

$\geqslant n-1$; but it does not need to be n-dimensional (namely, if the space is weakly n-dimensional, compare § 27, VI). However, if the space is compact, the following theorem (of Menger) holds ([1]).

THEOREM. *The dimensional kernel N of an n-dimensional compact space $\mathscr{X}$ is n-dimensional at each of its points.*

Therefore, $\dim N = n$.

Proof. Suppose, on the contrary, that $p \in N$ and $\dim_p N < n$. Let $\varepsilon > 0$. Hence there exists an open set G such that

$$p \in G, \qquad \delta(G) < \varepsilon \qquad \text{and} \qquad \dim[N \cap \operatorname{Fr}(G)] \leqslant n-2. \tag{1}$$

According to § 27, IV, let P be a G_δ-set such that

$$N \cap \operatorname{Fr}(G) \subset P \qquad \text{and} \qquad \dim P \leqslant n-2. \tag{2}$$

By inclusion (2) it follows that $N \cap [\operatorname{Fr}(G) - P] = 0$. Therefore

$$\dim_x \mathscr{X} \leqslant n-1 \qquad \text{for } x \in [\operatorname{Fr}(G) - P]. \tag{3}$$

Hence there exists a family of open sets such that a set H of this family corresponds to each point $x \in [\operatorname{Fr}(G) - P]$ and to each positive k, and the following conditions are satisfied

$$x \in H, \qquad \delta(H) < \varepsilon/k \qquad \text{and} \qquad \dim \operatorname{Fr}(H) \leqslant n-2. \tag{4}$$

Since P is G_δ, the set $\operatorname{Fr}(G) - P$ is F_σ. Therefore a sequence $H_1, H_2, \ldots$ can be chosen in that family (compare § 41, II, Corollary 5) such that

$$\overline{\bigcup_m H_m} \subset \bigcup_m \bar{H}_m \cup \operatorname{Fr}(G) \qquad \text{and} \qquad \operatorname{Fr}(G) - P \subset \bigcup_m H_m, \tag{5}$$

and hence

$$\operatorname{Fr}(G) - \bigcup_m H_m \subset P, \text{ therefore } \dim\Big[\operatorname{Fr}(G) - \bigcup_m H_m\Big] \leqslant n-2 \tag{6}$$

according to (2).

Let

$$Q = G \cup \bigcup_m H_m. \tag{7}$$

It follows by (1) and (4) that $p \in Q$ and $\delta(Q) \leqslant 2\varepsilon$. We are going to prove that

$$\dim \operatorname{Fr}(Q) \leqslant n-2,$$

([1]) Proc. Acad. Amsterdam 30 (1926), p. 138.

which will imply the required contradiction, because by hypothesis $p \in N$ and hence $\dim_p \mathscr{X} = n$.

By (5) it follows that

$$\begin{aligned}\operatorname{Fr}(Q) = \bar{Q}-Q &= (\bar{G}-Q) \cup \big(\overline{\bigcup_m H_m}-Q\big) \\ &\subset \big(\bar{G}-G-\bigcup_m H_m\big) \cup \big\{\big[\bigcup_m \bar{H}_m \cup \operatorname{Fr}(G)\big]-G-\bigcup_m H_m\big\} \\ &\subset \big[\operatorname{Fr}(G)-\bigcup_m H_m\big] \cup \bigcup_m \operatorname{Fr}(H_m),\end{aligned}$$

because

$$\bigcup_m \bar{H}_m - \bigcup_m H_m \subset \bigcup_m (\bar{H}_m - H_m) = \bigcup_m \operatorname{Fr}(H_m).$$

Finally, by (6) and (4) (compare § 22, I, Theorem 2), it follows that

$$\dim \operatorname{Fr}(Q) \leqslant \dim \big\{\big[\operatorname{Fr}(G)-\bigcup_m H_m\big] \cup \bigcup_m \operatorname{Fr}(H_m)\big\} \leqslant n-2.$$

VI. Transformations with k-dimensional point inverses.

THEOREM 1 (of Hurewicz) [1]. *Let f be a continuous mapping of a compact space $\mathscr{X}$. If the condition $\dim f^{-1}(y) \leqslant k$ is satisfied for all $y \in f(\mathscr{X})$, then*

$$\dim f(\mathscr{X}) \geqslant \dim \mathscr{X} - k.$$

Proof. Let $\dim \mathscr{X} = n$ and $\dim f(\mathscr{X}) = m$. We are going to show that

$$m \geqslant n-k.$$

Obviously the theorem holds for $m = -1$; hence we can assume that it holds for a number $m-1$. Since $d_n(\mathscr{X}) \neq 0$, we can also suppose (compare IV, Theorem 3) that $\mathscr{X}$ is irreducible with respect to its n-dimensional degree. Let A_1 and A_2 be two closed sets such that

$$f(\mathscr{X}) = A_1 \cup A_2, \quad A_1 \neq f(\mathscr{X}) \neq A_2, \quad \text{and} \quad \dim(A_1 \cap A_2) \leqslant m-1.$$

The twofold inequality implies that

$$f^{-1}(A_1) \neq \mathscr{X} \neq f^{-1}(A_2),$$

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(1) Proc. Acad. Amsterdam 30 (1927), p. 164.

and it follows that

$$d_n[f^{-1}(A_1)] < d_n(\mathscr{X}) \quad \text{and} \quad d_n[f^{-1}(A_2)] < d_n(\mathscr{X}).$$

Since $\mathscr{X} = f^{-1}(A_1) \cup f^{-1}(A_2)$, it follows (by IV, Theorem 6) that

$$\dim[f^{-1}(A_1) \cap f^{-1}(A_2)] \geqslant n-1, \quad \text{i.e.} \quad \dim f^{-1}(A_1 \cap A_2) \geqslant n-1.$$

On the other hand, since $\dim(A_1 \cap A_2) \leqslant m-1$, we infer by the hypothesis that

$$\dim(A_1 \cap A_2) \geqslant \dim f^{-1}(A_1 \cap A_2) - k.$$

So, we get finally

$$m-1 \geqslant n-1-k.$$

COROLLARY 2 (¹). *If $\mathscr{X}$ is compact and f is an open mapping of $\mathscr{X}$ such that the sets $f^{-1}(y)$ are countable, then*

$$\dim f(\mathscr{X}) = \dim \mathscr{X}.$$

Proof. Theorem 1 implies that $\dim \mathscr{X} \leqslant \dim f(\mathscr{X})$ and, by § 43, V, Theorem 2, there exists a sequence of closed sets $F_1, F_2, \ldots$ such that

$$f(\mathscr{X}) = f(F_1) \cup f(F_2) \cup \ldots$$

and $f(F_i)$ is homeomorphic with F_i, hence such that

$$\dim f(F_i) = \dim F_i \leqslant \dim \mathscr{X}.$$

Thus it follows, by the union theorem (§ 27, I, Theorem 2), that

$$\dim f(\mathscr{X}) \leqslant \dim \mathscr{X}.$$

Remark. The hypothesis of countability of the sets $f^{-1}(y)$ cannot be replaced by the condition $\dim f^{-1}(y) = 0$ (²) (which means that the *mapping f is 0-dimensional*).

(¹) P. Alexandrov, C. R. Acad. U.R.R.S. 4 (1936), p. 293.

(²) This is shown by an example of Kolmogorov (*ibid.*). A non-compact example was given by J. H. Roberts; see Hurewicz and Wallman, *Dimension Theory*, p. 93.

In fact, *for every compact space* $\mathscr{Y}$ *of positive dimension there is a 1-dimensional compact space* $\mathscr{X}$ *and an open 0-dimensional mapping* f *of* $\mathscr{X}$ *onto* $\mathscr{Y}$ (1).

THEOREM 3. *Given three compact spaces* $\mathscr{X}, \mathscr{Y}_1$ *and* $\mathscr{Y}_2$, *and two functions* $f_i\colon \mathscr{X} \to \mathscr{Y}_i$, $i = 1, 2$, *define a function* $f\colon \mathscr{X} \to \mathscr{Y}_1 \times \mathscr{Y}_2$ *by identity* $f(x) = [f_1(x), f_2(x)]$. *If* $\dim f^{-1}(z) \leqslant k$ *for all* $z \in \mathscr{Y}_1 \times \mathscr{Y}_2$, *then*

$$\dim f_1^{-1}(y_1) \leqslant \dim \mathscr{Y}_2 + k \quad \textit{for each } y_1 \in \mathscr{Y}_1.$$

Proof. Let us fix the point y_1 of $\mathscr{Y}_1$. Define $h = f_2 | [f_1^{-1}(y_1)]$. The equivalence

$$\{h(x) = y_2\} \equiv \{[f_1(x) = y_1][f_2(x) = y_2]\} \equiv \{f(x) = (y_1, y_2)\}$$

implies that $h^{-1}(y_2) \subset f^{-1}(y_1, y_2)$, hence $\dim h^{-1}(y_2) \leqslant k$. It follows by Theorem 1 (with $f_1^{-1}(y_1)$ replacing $\mathscr{X}$ and h replacing f) that

$$\dim f_1^{-1}(y_1) - k \leqslant \dim h[f_1^{-1}(y_1)] \leqslant \dim \mathscr{Y}_2.$$

VII. Space $(\mathscr{I}^r)^{\mathscr{X}}$ for $r \geqslant 2 \cdot \dim \mathscr{X} + 1$.

THEOREM 1 (Imbedding Theorem of Menger and Nöbeling) (2). *Every n-dimensional metric separable space is topologically contained in the cube* $\mathscr{I}^{2n+1}$, *i.e.*

$$\textit{if} \quad \dim \mathscr{X} = n, \quad \textit{then} \quad \mathscr{X} \underset{\text{top}}{\subset} \mathscr{I}^{2n+1}.$$

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(1) See B. Pasynkov, *0-dimensional open dimension-raising mappings*, Uspiehi 18 (1963), pp. 183–190. Earlier L. Kieldyš has shown this theorem for $\mathscr{Y} = \mathscr{I}^2$. See *0-dimensional open mappings*, Izviestia Acad. .Nauk 23 (1959), pp. 165–184. See also, by the same author, *An example of a 1-dimensional continuum which can be mapped onto the square using a 0-dimensional open mapping*, Doklady Acad. Nauk 97 (1954), p. 201, and *0-dimensional dimension-raising mappings*, Matemat. Sbornik 28 (1951), pp. 537–566.

For a number of theorems concerning dimension raising open mappings, see R. D. Anderson, *Open mappings of compact continua*, Proc. Nat. Acad. Sc. 42 (1956), p. 347. The author shows, among others, that, for any $n \geqslant 3$ and $m \geqslant 2$, there is a monotone open mapping of $\mathscr{S}_n$ onto $\mathscr{S}_m$.

(2) K. Menger, *Über umfassendste n-dimensionale Mengen*, Proc. Akad. Amsterdam 29 (1926), p. 1125, and G. Nöbeling, *Über eine n-dimensionale Universalmenge im* R_{2n+1}, Math. Ann. 104 (1930), p. 71.

More precisely (1), *if the closed set* $E \subset \mathscr{X}$ *satisfies condition* D_n *and if* $r \geqslant 2n+1$, *then the set of functions* $f \in (\mathscr{I}^r)^{\mathscr{X}}$ *such that the partial function* $f|E$ *is a homeomorphism is a residual set* (*i.e. a complement of a first category set*) *in the space* $(\mathscr{I}^r)^{\mathscr{X}}$.

Proof. If A and B are two closed disjoint subsets of E, denote by Φ_{AB} the family of functions $g \in (\mathscr{I}^r)^{\mathscr{X}}$ such that

$$\overline{g(A)} \cap \overline{g(B)} = 0.$$

Since the set E satisfies condition D_n and $r \geqslant 2n+1$, Theorem 4 of § 28, VII shows that

$$\overline{\Phi}_{AB} = (\mathscr{I}^r)^{\mathscr{X}}.$$

And this gives immediately the second part of Theorem 1 by Theorem 2 of § 44, VI.

In order to obtain the first part from the second one, let $E = \mathscr{X}$ and use the fact, that since the space $(\mathscr{I}^r)^{\mathscr{X}}$ is complete (compare § 33, V, Theorem 1), every residual subset of this space is non-empty (by the Baire theorem).

Remarks. (i) The exponent $2n+1$ *cannot be lessened* in Theorem 1. In fact, there are n-dimensional spaces which are not homeomorphic with any subset of the cube $\mathscr{I}^{2n}$ (2). Such is the union of all at most n-dimensional faces of an $(2n+2)$-dimensional simplex.

(ii) Theorem of Menger and Nöbeling can be restated more precisely in the following manner (3).

Let N_n *be the subset of the cube* $\mathscr{I}^{2n+1}$, *whose points have at most* n *rational coordinates. The condition* $\dim \mathscr{X} = n$ *implies that* $\mathscr{X} \underset{\text{top}}{\subset} N_n$.

Therefore *the space* N_n *has the highest topological rank among all at most* n*-dimensional metric separable spaces*.

First, we are going to prove the following statement.

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(1) W. Hurewicz, *Über Abbildungen endlichdimensionalen Räumen auf Teilmengen Cartesischer Räume*, Sgb. Preuss. Akad. 1933, p. 754 (case where $\mathscr{X}$ is compact). For the general case, see my paper *Sur les théorèmes du "plongement" dans la théorie de la dimension*, Fund. Math. 28 (1937), p. 336.

(2) A. Flores, *Über n-dimensionale Komplexe, die im* R_{2n+1} *absolut selbstverschlungen sind*, Ergebn. math. Koll. 6 (1933), p. 4.

(3) G. Nöbeling, *loc. cit.*

If $\dim \mathscr{X} \leqslant n$, *the set* Φ *of functions* g *such that*

$$\overline{g(\mathscr{X})} \subset N_n \tag{1}$$

is a residual G_δ *in the space* $(\mathscr{I}^{2n+1})^{\mathscr{X}}$.

Proof. Let $n+1 \leqslant m \leqslant 2n+1$. Given a system of m rational numbers $r_1, \ldots, r_m$ and of m positive integers $i_1 < \ldots < i_m$ ($\leqslant 2n+$ $+1$), the set of points

$$x = (x^1, \ldots, x^{2n+1}) \quad \text{where} \quad x^{i_1} = r_1, \ldots, x^{i_m} = r_m$$

is a $(2n+1-m)$-dimensional linear variety.

For variable m, $r_1, \ldots, r_m$ and $i_1, \ldots, i_m$ a sequence $L_1, L_2, \ldots$ of linear varieties is obtained such that

$$\mathscr{I}^{2n+1} - N_n = \mathscr{I}^{2n+1} \cap (L_1 \cup L_2 \cup \ldots).$$

Since $\dim L_k \leqslant n$, the set $\mathop{E}_{g}[\overline{g(\mathscr{X})} \cap L_k = 0]$ is dense in the space $[\mathscr{I}^{2n+1}]^{\mathscr{X}}$ (according to Remark of § 28, VII, Theorem 3). But the same set is open in that space (compare § 17, II, Theorem 1 and § 44, V, Theorem 4) and hence the set Φ is a residual G_δ by the following formula

$$\Phi = \mathop{E}_{g}[\overline{g(\mathscr{X})} \subset N_n] = \mathop{E}_{g}\Big[\overline{g(\mathscr{X})} \cap \bigcup_{k=1}^{\infty} L_k = 0\Big] = \bigcap_{k=1}^{\infty} \mathop{E}_{g}[\overline{g(\mathscr{X})} \cap L_k = 0].$$

Since the set of homeomorphisms is also residual, it follows that the homeomorphisms which satisfy condition (1) constitute a residual set in the space $(\mathscr{I}^{2n+1})^{\mathscr{X}}$. Therefore $\mathscr{X} \mathrel{\underset{\text{top}}{\subset}} N_n$.

It remains to prove that $\dim N_n \leqslant n$.

Denote by $R_{k,m}$ the subset of $\mathscr{I}^m$, whose points have k rational coordinates and $m-k$ irrational ones; then

$$N_n = R_{0,2n+1} \cup \ldots \cup R_{n,2n+1}.$$

Since the union of $n+1$ 0-dimensional sets has dimension $\leqslant n$ (compare § 27, I, Theorem 1), it remains only to show that

$$\dim R_{k,m} = 0 \quad (0 \leqslant k \leqslant m)\ (^1). \tag{2}$$

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(1) See K. Menger, *Dimensionstheorie*, p. 147.

Let $r_1, r_2, \ldots$ be the sequence of rational numbers contained in the interval 01 and let $Z_{j_1,\ldots,j_k}^{i_1,\ldots,i_k}$ be the set of points $x = (x^{(1)}, \ldots$ $\ldots, x^{(m)})$ for which $x^{(i_1)} = r_{j_1}, \ldots, x^{(i_k)} = r_{j_k}$ and whose all other coordinates are irrational.

It follows that

$$R_{k,m} = \bigcup Z_{j_1,\ldots,j_k}^{i_1,\ldots,i_k},$$

where the summation ranges over all systems $i_1 < \ldots < i_k \leqslant m$ and $j_1, \ldots, j_k$.

$Z_{j_1,\ldots,j_k}^{i_1,\ldots,i_k}$ is homeomorphic to the $(m-k)$th power of the set $\mathscr{N}$ (of all irrational numbers of the interval $\mathscr{I}$), hence to $\mathscr{N}$; therefore it has dimension 0. Besides, it is closed in $R_{k,m}$, because every point x of the set $\overline{Z_{j_1,\ldots,j_k}^{i_1,\ldots,i_k}} - Z_{j_1,\ldots,j_k}^{i_1,\ldots,i_k}$ has in addition to the coordinates $x^{i_1} = r_{j_1}, \ldots, x^{i_k} = r_{j_k}$ at least one rational coordinate, and hence $x \notin R_{k,m}$.

The set $R_{k,m}$ has dimension 0 since it is the union of a sequence of 0-dimensional sets closed in $R_{k,m}$ (cf. § 26, III, Corollary 1).

THEOREM 2 (on compactification) (1). *Every n-dimensional space is topologically contained in an n-dimensional compact space.*

More precisely: *if $\mathscr{X}$ satisfies condition D_n, the set of homeomorphisms h such that* $\dim \overline{h(\mathscr{X})} \leqslant n$ *is a residual set in the space* $(\mathscr{I}^r)^{\mathscr{X}}$.

Proof. Let $H(f) = \overline{f(\mathscr{X})}$; the functional H is continuous in the space $(\mathscr{I}^r)^{\mathscr{X}}$, according to § 44, V, Theorem 4. It follows by IV, Theorem 4 (with $\mathscr{I}^r$ replacing $\mathscr{X}$) that the set

$$\Phi = \underset{f}{E}[\dim \overline{f(\mathscr{X})} \leqslant n]$$

is a G_δ in $(\mathscr{I}^r)^{\mathscr{X}}$.

Since the space $\mathscr{X}$ satisfies condition D_n, the set Φ is dense in the space $(\mathscr{I}^r)^{\mathscr{X}}$ (by Theorem 3 of § 28, VII). And hence it is a residual set in that space (1), and so is the intersection of Φ with the set of all the homeomorphisms, because the last one is residual according to Theorem 1.

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(1) W. Hurewicz, *Über das Verhältniss separabler Räume zu kompakten Räumen,* Proc. Acad. Amsterdam 30 (1927), p. 425.

(2) Compare K. Borsuk, Fund. Math. 28 (1937), p. 97.

Remarks. (i) Remark (ii) to Theorem 1, combined with Theorem 2, leads to the following conclusion.

There exists a compact n-dimensional space which has the highest topological rank among all metric separable spaces of dimension $\leqslant n$.

Such is the compact n-dimensional space, which topologically contains the set N_n.

(ii) Theorem 2 can be generalized in the following way [1].

If $\dim \mathscr{X} \leqslant n$, *the set of homeomorphisms* g *such that* $\dim \overline{g(\mathscr{X})} \leqslant n$ *and such that, for any* $x \in \mathscr{X}$, *the dimension of* $\mathscr{X}$ *at the point* x *is equal to the dimension of* $\overline{g(\mathscr{X})}$ *at the point* $g(x)$, *is a residual set in the space* $(\mathscr{I}^r)^{\mathscr{X}}$.

COROLLARY 3. *The inequality* $\dim \mathscr{X} \leqslant n$ *and the condition* D_n *are equivalent.*

Proof. The inequality $\dim \mathscr{X} \leqslant n$ implies the condition D_n by § 27, III, Theorem 4, and conversely, D_n implies, by Theorem 2, that $\mathscr{X}$ is homeomorphic to a subset of an at most n-dimensional set, and so $\dim \mathscr{X} \leqslant n$.

It is easy to see that the proof of Theorem 1 is essentially based on the following statement (which is a straightforward consequence of Theorems § 28, VII, 4 and § 44, V, 4a).

THEOREM 4. *If* A *and* B *are two closed, disjoint, n-dimensional sets, the set of all functions* g *which satisfy condition*

$$\overline{g(A)} \cap \overline{g(B)} = 0$$

is a dense open set in the space $(\mathscr{I}^r)^{\mathscr{X}}$.

THEOREM 4′. *In a metric separable space* $\mathscr{X}$, *let be given* $l+1$ *closed sets* $A_0, \ldots, A_l$ *of dimension* $\leqslant n$. *Then the set* Φ *of functions* g *satisfying the condition*

$$\dim[\overline{g(A_0)} \cap \ldots \cap \overline{g(A_l)}] \leqslant \dim(A_0 \cap \ldots \cap A_l) \tag{3}$$

is a residual G_δ *in the space* $(\mathscr{I}^r)^{\mathscr{X}}$.

Proof. By Theorem 4′ of Section IV, the set

$$\underset{F_0,\ldots,F_l}{E} [\dim(F_0 \cap \ldots \cap F_l) \leqslant \dim(A_0 \cap \ldots \cap A_l)]$$

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(1) For the proof see my paper, Fund. Math. 30 (1937), p. 13. Compare also W. Hurewicz, *loc. cit.*, p. 430.

is a G_δ in the space $(2^{\mathscr{I}^r})^{l+1}$. Since the operation $\overline{g(A_i)}$ is continuous (if considered as function of g, according to § 44, VI, Theorem 2), the set Φ is a G_δ in the space $(\mathscr{I}^r)^{\mathscr{X}}$. Finally, it is dense in that space by Theorem 5 of § 28, VII.

THEOREM 5. *Let a sequence $A_0, A_1, \ldots$ of closed sets in a space $\mathscr{X}$ be given. From the topological point of view $\mathscr{X}$ may be considered as a dense subset of a compact space $\mathscr{X}^*$ such that if A^* denotes the closure of A in $\mathscr{X}^*$, then*

$$\dim(A^*_{i_0} \cap \ldots \cap A^*_{i_l}) = \dim(A_{i_0} \cap \ldots \cap A_{i_l}), \tag{4}$$

for any indices $i_0, \ldots, i_l$ $(l \geqslant 0)$.

Therefore *the condition of compactness can be omitted in Theorem* III, 1.

Proof. By Theorem 2 of § 44, VI, the set Ψ of homeomorphisms is residual in the space $(\mathscr{I}^{\aleph_0})^{\mathscr{X}}$, and so is the set $\Phi(i_0, \ldots, i_l)$ of functions g such that

$$\dim[\overline{g(A_{i_0})} \cap \ldots \cap \overline{g(A_{i_l})}] \leqslant \dim(A_{i_0} \cap \ldots \cap A_{i_l}), \tag{5}$$

according to Theorem 4′, and hence so is the intersection $\Gamma = \Psi \cap \cap \bigcap \Phi(i_0, \ldots, i_l)$, where $i_0, \ldots, i_l$ range over all systems of integers $\geqslant 0$.

Let $g \epsilon \Gamma$. Assume $\mathscr{X}^* = \overline{g(\mathscr{X})}$ and let us identify $\mathscr{X}$ with $g(\mathscr{X})$. It follows that

$$\dim[g(A_{i_0}) \cap \ldots \cap g(A_{i_l})] = \dim(A_{i_0} \cap \ldots \cap A_{i_l}).$$

Therefore, by (5),

$$\dim[\overline{g(A_{i_0})} \cap \ldots \cap \overline{g(A_{i_l})}] = \dim(A_{i_0} \cap \ldots \cap A_{i_l}),$$

which implies (4).

Now consider the second part of Theorem 5.

Let $P_0, P_1, \ldots$ be a sequence of closed subsets of $\mathscr{X}$ and let $G_0, \ldots, G_m$ be a system of open sets such that $\mathscr{X} = G_0 \cup \ldots \cup G_m$. Let

$$Q_i = \mathscr{X} - G_i, \quad \text{hence} \quad Q_0 \cap \ldots \cap Q_m = 0. \tag{6}$$

As proved, $\mathscr{X}$ may be considered as a dense subset of a compact space $\mathscr{X}^*$ such that

$$\dim P^*_j = \dim P_j \quad \text{and} \quad Q^*_0 \cap \ldots \cap Q^*_m = 0 \tag{7}$$

(we replace the sequence $A_0, A_1, \ldots$ by $Q_0, \ldots, Q_m, P_0, P_1, \ldots$).

Put $U_i = \mathscr{X}^* - Q_i^*$. It follows that

$$U_0 \cup \ldots \cup U_m = \mathscr{X}^* - (Q_0^* \cap \ldots \cap Q_m^*) = \mathscr{X}^*.$$

Since the sets U_i are open, there exists by Theorem 1 of Section III a system of closed sets $W_0, \ldots, W_m$ (in $\mathscr{X}^*$) such that

$$\begin{gathered} \mathscr{X}^* = W_0 \cup \ldots \cup W_m, \qquad W_i \subset U_i, \\ \dim(P_j^* \cap W_{i_0} \cap \ldots \cap W_{i_l}) \leqslant \dim P_j^* - l, \end{gathered} \tag{8}$$

for any $j = 1, 2, \ldots,$ $l \leqslant \dim P_j + 1$ and $i_0 < \ldots < i_l \leqslant m$.

Let $F_i = \mathscr{X} \cap W_i$. It follows that

$$\begin{gathered} F_0 \cup \ldots \cup F_m = \mathscr{X} \cap (W_0 \cup \ldots \cup W_m) = \mathscr{X} \cap \mathscr{X}^* = \mathscr{X}, \\ F_i = \mathscr{X} \cap W_i \subset \mathscr{X} \cap U_i = \mathscr{X} \cap \mathscr{X}^* - Q_i^* = \mathscr{X} - Q_i^*, \end{gathered} \tag{9}$$

and since Q_i is closed in $\mathscr{X}$, it follows by (6) that

$$\mathscr{X} \cap Q_i^* = Q_i, \qquad \text{whence} \qquad \mathscr{X} - Q_i^* = \mathscr{X} - Q_i = G_i,$$

and hence $F_i \subset G_i$ according to (9). Finally, by (8) and (7),

$$\begin{aligned} \dim(P_j \cap F_{i_0} \cap \ldots \cap F_{i_l}) &\leqslant \dim(P_j^* \cap W_{i_0} \cap \ldots \cap W_{i_l}) \\ &\leqslant \dim P_j - l. \end{aligned}$$

THEOREM 6 [1]. *In a metric separable space $\mathscr{X}$ there are given two closed sets A and B of dimension $\leqslant n$ and such that the intersection $A \cap B$ is compact. Then the set of functions g, which satisfy condition*

$$\overline{g(A)} \cap \overline{g(B)} = g(A \cap B), \tag{10}$$

is residual in the space $(\mathscr{I}^r)^{\mathscr{X}}$.

Proof. Let $S_k = \underset{x}{E}[\varrho(x, A \cap B) < 1/k]$ and

$$A_k = A - S_k, \text{ hence } A_k \cap B = 0 \text{ and } A - B = A_1 \cup A_2 \cup \ldots.$$

By Theorem 4, the set $\Phi_k = \underset{g}{E}[\overline{g(A_k)} \cap \overline{g(B)} = 0]$ is open and dense in $(\mathscr{I}^r)^{\mathscr{X}}$. Hence the set $\Phi = \Phi_1 \cap \Phi_2 \cap \ldots$ is residual.

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[1] See my paper, *Quelques théorèmes sur le plongement topologique des espaces*, Fund. Math. 30 (1937), p. 8. See also, K. Morita, *A generalization of a theorem of Kuratowski concerning functional spaces*, Science Reports Tokyo 4 (1949), p. 151.

Let $g \in \Phi$. We are going to prove (10). Now

$$\overline{g(A)} = \bigcup_k \overline{g(A_k)} \cup [\overline{g(A)} - \bigcup_k \overline{g(A_k)}] = \bigcup_k \overline{g(A_k)} \cup \bigcap_k [\overline{g(A)} - \overline{g(A_k)}]$$

$$\subset \bigcup_k \overline{g(A_k)} \cup \bigcap_k \overline{g(A - A_k)} \subset \bigcup_k \overline{g(A_k)} \cup \bigcap_k \overline{g(S_k)}.$$

Since (compare § 41, VI, Theorem 10) $\bigcap_k \overline{g(S_k) = g(\bigcap_k} S_k) = g(A \cap B)$, it follows that

$$\overline{g(A)} \cap \overline{g(B)} \subset \bigcup_k \overline{g(A_k)} \cap \overline{g(B)} \cup g(A \cap B),$$

and so

$$\overline{g(A)} \cap \overline{g(B)} \subset g(A \cap B),$$

because the hypothesis $g \in \Phi_1 \cap \Phi_2 \cap \ldots$ means that

$$\bigcup_k \overline{g(A_k)} \cap \overline{g(B)} = 0.$$

The next theorem is an application of Theorem 4.

THEOREM 7 (¹). *In the cube $\mathcal{I}^r$, let be given a sequence $A_1, A_2, \ldots$ of sets of dimension $\leqslant n$. For each $\varepsilon > 0$ there exists a sequence $h_0, h_1, \ldots$ of homeomorphisms such that*

$$h_i(A_i) \subset \mathcal{I}^r, \quad h_i(A_i) \cap h_j(A_j) = 0 \quad (\textit{if } j \neq i),$$

$$|h_i(x) - x| < \varepsilon.$$

Proof. To every point $x = [x^1, x^2, \ldots]$ of A_i let us attach the point $f_i(x) = [i, x^1, x^2, \ldots]$ of $\mathcal{E}^{r+1}$. Clearly, f_i is a homeomorphism and the sets $B_i = f_i(A_i)$ are pairwise disjoint and closed-open in their union $S = B_0 \cup B_1 \cup \ldots$. Since $\dim B_i \leqslant n$, it follows that $\dim S \leqslant n$.

Let g be the function, which is equal to f_i^{-1} on B_i for $i = 0, 1, \ldots$ Therefore $g \in (\mathcal{I}^r)^S$ and hence there exists, by Theorems 1 and 4, a homeomorphism $h \in (\mathcal{I}^r)^S$ such that

$$|h - g| < \varepsilon \quad \text{and} \quad h(B_i) \cap h(B_j) = 0 \quad \text{for} \quad j \neq i.$$

So it is sufficient to assume $h_i(x) = hf_i(x)$ for $x \in A_i$.

(¹) In that direction, see P. Alexandrov, *Dimensionstheorie*, Math. Ann. 106 (1932), p. 210 (2. Zusatz) and W. Hurewicz, *Über Abbildungen von endlichdimensionalen Räumen auf Teilmengen Cartesischer Räume*, Sgb. Preuss. Akad. 24 (1933), p. 760.

VIII. Space $(\mathscr{I}^r)^{\mathscr{X}}$ for $r > \dim \mathscr{X}$.

THEOREM OF HUREWICZ [1]. *If $\dim \mathscr{X} \leqslant n$, there exists a function $g\colon \mathscr{X} \to \mathscr{I}^r$, no value of which is attained at more than m points, where m is the least integer $\geqslant \dfrac{n+1}{r-n}$.*

More generally, *if V_k is the set of values of a function g of order $> k$, then*

$$\dim V_k \leqslant n - k(r-n) \quad \textit{for} \quad k \leqslant \frac{n+1}{r-n}. \tag{1}$$

Furthermore, *the set Ψ_k of functions g which satisfy condition* (1), *and in consequence the set $\Psi_0 \cap \Psi_1 \cap \ldots$, are residual in the space $(\mathscr{I}^r)^{\mathscr{X}}$.*

Proof. Let $R_1, R_2, \ldots$ be the base of the space $\mathscr{X}$. Let $g\colon \mathscr{X} \to \mathscr{I}^r$ and $y \in V_k$. There exists a system of $k+1$ distinct points $x_0, \ldots, x_k$ such that $y = g(x_0) = \ldots = g(x_k)$. Consequently, there exists a system of indices $S = (i_0, \ldots, i_k)$ such that the sets $\bar{R}_{i_0}, \ldots, \bar{R}_{i_k}$ are disjoint and that

$$y \in g(\overline{R_{i_0}}) \cap \ldots \cap g(\overline{R_{i_k}}) \subset \overline{g(\overline{R_{i_0}})} \cap \ldots \cap \overline{g(\overline{R_{i_k}})},$$

and it follows

$$V_k \subset \bigcup_S \overline{g(\overline{R_{i_0}})} \cap \ldots \cap \overline{g(\overline{R_{i_k}})}. \tag{2}$$

Since the union is countable and its terms are closed sets, then if g does not satisfy condition (1), there exists a system S such that

$$\dim[\overline{g(\overline{R_{i_0}})} \cap \ldots \cap \overline{g(\overline{R_{i_k}})}] > n - k(r-n). \tag{3}$$

In other words, if $g \notin \Psi_k$, there exists a system S satisfying condition (3). If Γ_S is the set of functions g which satisfy condition (3), it follows that

$$(\mathscr{I}^r)^{\mathscr{X}} - \Psi_k \subset \bigcup_S \Gamma_S.$$

Since Γ_S is a boundary set (by § 28, VII, Theorem 6) and an $\boldsymbol{F}_\sigma$-set (by IV, Theorem 4′ and § 44, V, Theorem 4), so Ψ_k is a residual set.

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(1) *Op. cit.*, p. 755. In the same direction see S. Eilenberg, *Remarque sur un théorème de M. Hurewicz*, Fund. Math. 24 (1935), p. 156.

EXAMPLE. If $\dim \mathscr{X} = 1$ (if $\mathscr{X}$ is a curve, for instance), $\mathscr{X}$ can be mapped into the plane in such a manner that no point of the plane will be covered by more than two points of $\mathscr{X}$ and such, in addition, that the set of points of the plane, which are covered by two points of $\mathscr{X}$, is 0-dimensional.

IX. Space $(\mathscr{I}^r)^{\mathscr{X}}$ for $r \leqslant \dim \mathscr{X}$.

THEOREM OF HUREWICZ (¹). *If* $\dim \mathscr{X} \leqslant n$, *there exists a function* $g\colon \mathscr{X} \to \mathscr{I}^r$ *such that*

$$\dim[g^{-1}(y)] \leqslant n-r \quad \textit{for each } y \in g(\mathscr{X}). \tag{1}$$

Furthermore, *if $\mathscr{X}$ is compact and* $\dim \mathscr{X} \leqslant n$, *the set of functions g satisfying condition* (1) *is a residual G_δ in the space* $(\mathscr{I}^r)^{\mathscr{X}}$.

Proof. Since every space is contained in a compact space of the same dimension (VII, Theorem 2), it is sufficient to establish the second part of the theorem.

First, consider the case where $r = n$. Since the set $g^{-1}(y)$ is compact, the condition $\dim[g^{-1}(y)] = 0$ is equivalent (by IV, Theorem 1) to $d_1[g^{-1}(y)] = 0$, which means that

$$d_1[g^{-1}(y)] < 1/k \tag{2}$$

for $k = 1, 2, \ldots$.

Finally, condition (2) is equivalent to the existence of a decomposition of the set $g^{-1}(y)$ into a finite number of closed disjoint sets of diameter $< 1/k$ (compare Section IV). Denote the set of functions g satisfying condition (1) by Ψ and the set of functions g satisfying condition (2) by Ψ_k; it follows that

$$\Psi = \Psi_1 \cap \Psi_2 \cap \ldots.$$

Since the set Ψ_k is dense according to Theorem 7 of § 28, VII and open by Theorem IV, 5, the set Ψ is a residual G_δ.

Now consider the case where $r < n$. Since the set Ψ is a G_δ (by Theorem IV, 5), it remains only to prove that it is dense.

(¹) *Op. cit.* In the case where $\mathscr{X}$ is a polyhedron, the first part of the theorem becomes elementary. See also V. Boltianskii and P. Soltan, *Generalization of a Theorem of Hurewicz on the Dimension of point-inverses* (Russian), Matem. Sbornik 69 (1966), pp. 257–285.

Let $f\colon \mathscr{X} \to \mathscr{I}^r$ and $\varepsilon > 0$. Let $f_*\colon \mathscr{X} \to \mathscr{I}^n$ be the function defined in the following way

$$f_*^i(x) = \begin{cases} f^i(x) & \text{for} \quad i \leqslant r, \\ 0 & \text{for} \quad r < i \leqslant n. \end{cases}$$

As we have just proved, there exists a function $g_*\colon \mathscr{X} \to \mathscr{I}^n$ with 0-dimensional point inverses and such that $|g_* - f_*| < \varepsilon$.
Define $g(x) = [g_*^1(x), \ldots, g_*^r(x)]$. It follows that

$$g\colon \mathscr{X} \to \mathscr{I}^r \quad \text{and} \quad |g - f| < \varepsilon.$$

Finally, $g \in \Psi$, because condition (1) follows directly from Theorem VI, 3, where we set

$$\mathscr{Y}_1 = \mathscr{I}^r, \qquad \mathscr{Y}_2 = \mathscr{I}^{n-r},$$
$$f_1 = g, \quad f_2 = (g_*^{r+1}, \ldots, g_*^n), \quad \text{hence} \quad f = g_*.$$

CHAPTER FIVE

CONNECTED SPACES

§ 46. Connectedness

I. Definition. General properties. Monotone mappings. The space $\mathscr{X}$ is said to be *connected* if it contains no set X such that

$$0 \neq X \neq \mathscr{X} \quad \text{and} \quad \bar{X} \cap \overline{\mathscr{X}-X} = 0\ (^1); \tag{1}$$

in other words, if no set X, which satisfies condition $0 \neq X \neq \mathscr{X}$, has an empty *boundary*.

THEOREM 0. *A space is connected if and only if it is not the union of two disjoint, closed and non-empty sets.*

This is so, for if the space can be split that way into sets X and Y, then conditions (1) are fulfilled; conversely, if X satisfies conditions (1), then $\mathscr{X} = \bar{X} \cup \overline{\mathscr{X}-X}$ is a decomposition into two disjoint, closed and non-empty sets.

It follows that a *set* C (lying in a given space) is connected if and only if every subset X of C such that $0 \neq X \neq C$ satisfies the condition $C \cap \bar{X} \cap \overline{C-X} \neq 0$; in other words, that C cannot be split into two non-empty, disjoint sets X and Y which are closed in C; or, into two non-empty *separated* sets X and Y, i.e. which satisfy the following conditions

$$C = X \cup Y, \quad (\bar{X} \cap Y) \cup (X \cap \bar{Y}) = 0, \quad X \neq 0 \neq Y. \tag{2}$$

An open connected set is called a *region.*

THEOREM 1. *If C is connected and $C \cap A \neq 0 \neq C-A$, then $C \cap \mathrm{Fr}(A) \neq 0$.*

Because

$$0 \neq C \cap \overline{C \cap A} \cap \overline{C-A} \subset C \cap \bar{A} \cap \overline{\mathscr{X}-A} = C \cap \mathrm{Fr}(A).$$

(1) The definition of connectedness adopted here originates from C. Jordan, *Cours d'analyse I,* p. 25, Paris 1893. Compare N. J. Lennes, Amer. Journ. Math. 33 (1911), p. 303. It aims to express in a topological way the intuitive notion of the continuity of a set of points.

THEOREM 2. *If C is not connected, there exists an open G such that*

$$C \cap G \neq 0 \neq C - \bar{G} \quad \text{and} \quad C \cap \operatorname{Fr}(G) = 0.$$

(The space is assumed to be metric; more generally, one could assume that the space is hereditarily normal).

Proof. Suppose that X and Y satisfy conditions (2), and let

$$G = \underset{x}{E}[\varrho(x, X) < \varrho(x, Y)].$$

It follows that

$$X \subset G \quad \text{and} \quad \bar{G} \cap Y = 0, \quad \text{hence} \quad (X \cup Y) \cap (\bar{G} - G) = 0.$$

THEOREM 3. *Connectedness is invariant under continuous mappings.*

Proof. If $f(C) = X \cup Y$ where X and Y are two disjoint, closed and non-empty sets, it follows that

$$C = f^{-1}f(C) = f^{-1}(X) \cup f^{-1}(Y),$$

and the sets $f^{-1}(X)$ and $f^{-1}(Y)$ are also disjoint, closed and non-empty.

EXAMPLES AND REMARKS. *The space $\mathscr{E}$ of real numbers is connected.* This is an easy consequence of the Dedekind axiom.

The set with *only one point* and the *empty set* are connected.

It is easy to see that there are no other connected subsets of $\mathscr{E}$ (containing more than one point) except the following: *closed or open intervals, closed or open rays, and intervals without one end point.*

Since $\mathscr{I}$ is connected, so is the graph $\underset{xy}{E}[y = f(x)]$ where $f: \mathscr{I} \to \mathscr{Y}$ is continuous and $\mathscr{Y}$ arbitrary (since this graph is homeomorphic to $\mathscr{I}$, see Theorem 1 of § 15, V, p. 141).

COROLLARY 3′. *A space $\mathscr{X}$ is connected if and only if every continuous function f: $\mathscr{X} \to \mathscr{E}$ has the Darboux property (i.e. it passes from one value to another through all intermediate values).*

Proof. If $\mathscr{X}$ is connected and f continuous, then $f(\mathscr{X})$ is connected by Theorem 3, and it follows at once from the above remarks that f has the Darboux property.

If $\mathscr{X}$ is not connected, then $\mathscr{X} = A \cup B$ where A and B are disjoint, closed and non-empty sets. Put $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$. Then f is continuous but has not the Darboux property.

THEOREM 4. *The cardinality of a completely regular connected space C containing more than one point is at least* $\mathfrak{c}$.

Proof. Let a and b be two distinct points of C. Then there exists a continuous function $f\colon C \to \mathscr{I}$ such that $f(a) = 0$ and $f(b) = 1$. By Corollary 3′, $\mathscr{I} \subset f(C)$, which completes the proof.

Remark. There are connected $\mathscr{T}_2$-spaces which are (infinitely) countable [1].

THEOREM 5. *If C is connected metric and separable, then* $\dim_p C \neq 0$ *for each $p \in C$*. (Provided that the space contains more than one point).

For otherwise, there would exist a closed and open neighbourhood of p distinct from C.

THEOREM 6. *Let $f\colon \mathscr{I} \to \mathscr{E}^n$ be a function of the* 1*st class of Baire such that $f(x) \in \overline{f(xx')}$ and $f(x') \in \overline{f(xx')}$ for each open interval xx'. Let $\mathscr{Y} = f(\mathscr{I})$. Then $\mathscr{Y}$ is connected.*

Proof. Suppose that F is a closed-open subset of $\mathscr{Y}$ and that $0 \neq F \neq \mathscr{Y}$. Define $A = f^{-1}(F)$. The connectedness of $\mathscr{I}$ implies that $\mathrm{Fr}(A) = \bar{A} \cap \overline{\mathscr{I} - A} \neq 0$. Since the function f is of the 1st class, it follows that the partial function $f|\mathrm{Fr}(A)$ has a point of continuity a (§ 34, VII). Since F and $\mathscr{Y} - F$, and hence A and $\mathscr{I} - A$, satisfy the same assumptions, it can be assumed that $a \in A$ and so $a \in A \cap (\overline{\mathscr{I} - A})$.

The set $A = f^{-1}(F)$ contains a neighbourhood of a relative to $\mathrm{Fr}(A)$, because the function $f|\mathrm{Fr}(A)$ is continuous at a and F is a neighbourhood of $f(a)$; thus there exists a number $\varepsilon > 0$ such that the conditions $x \in \mathrm{Fr}(A)$ and $|x - a| < \varepsilon$ imply $x \in A$. Let b be a point of $\mathscr{I} - A$ such that $|a - b| < \varepsilon$ (in conformity with condition $a \in \overline{\mathscr{I} - A}$). Therefore $b \notin \mathrm{Fr}(A)$ and hence $b \in \mathscr{I} - \bar{A}$. Let cd be an (open) interval which is contiguous to $\bar{A}$ and contains b. One of the end-points, say c, lies between a and b; this gives $|c - a| < \varepsilon$ and since $c \in \mathrm{Fr}(A)$, it follows that $c \in A$ and therefore $f(c) \in F$. But this contradicts the formula

$$f(c) \in \overline{f(cd)} \subset \overline{f(\mathscr{I} - A)} = \overline{ff^{-1}(\mathscr{Y} - F)} = \mathscr{Y} - F.$$

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[1] The first example of such a space was given by P. Urysohn. See *Über die Mächtigkeit der zusammenhängenden Mengen*, Math. Ann. 94 (1925), pp. 262–295. See also I. Rauchvarger, Utch. Zapiski Mosc. Univ. 4 (1945), and N. Bourbaki, *op. cit.* p. 175.

Theorem 6 implies the following

THEOREM 7. *Let $g: \mathscr{I} \to \mathscr{E}$ be a real-valued function. If g is of the 1st class and has the Darboux property, the set $C = \underset{xy}{E}[y = g(x)]$ is connected* (1).

Proof. The Darboux condition implies that to every x correspond two sequences $\{x_n'\}$ and $\{x_n''\}$ converging to x, the one increasing and the other decreasing, and such that

$$\lim_{n=\infty} g(x_n') = g(x) = \lim_{n=\infty} g(x_n'') \text{ (2)}.$$

In consequence, the complex function $f(x) = [x, g(x)]$ satisfies the hypotheses of Theorem 6. Hence the set $C = f(\mathscr{I})$ is connected.

Theorem 7 immediately implies the following one.

THEOREM 8. *If the derivative $dg(x)/dx$ is finite for any x, the set*

$$\underset{xy}{E}\left[y = \frac{dg(x)}{dx}\right]$$

is connected (3).

Moreover, being a G_δ set (compare § 31, VII, Theorem 1), *this set is a topologically complete space* (§ 33, VI).

Remarks. Theorems 7 and 8 lead to definitions of connected sets which possess a variety of remarkable properties (compare VI, (iii)) (4).

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(1) See the paper of Sierpiński and myself, *Les fonctions de classe* 1 *et les ensembles connexes ponctiformes*, Fund. Math. **3** (1922), p. 304. Compare F. Hausdorff, *Mengenlehre*, p. 256.

See also F. B. Jones and E. S. Thomas Jr., *Connected G_δ graphs*, Duke Math. Journ. **33** (1966), pp. 341–345.

(2) In the range of the functions of 1st class, the inverse implication is also true. See W. H. Young, Rend. di Palermo 24 (1907), p. 187.

(3) See the paper of B. Knaster and myself, *Sur quelques propriétés topologiques des fonctions dérivées*, Rend. di Palermo 49 (1925).

(4) For a similar purpose the functions can be used which satisfy the functional equation $f(x+y) = f(x)+f(y)$. Compare F. B. Jones, Bull. Amer. Math. Soc. 48 (1942), p. 115.

Outside the functions of 1st class the Darboux condition remains (as is easily proved) *necessary* for the connectedness of the graph of function, but it is not *sufficient* any more.

In fact, consider the function (of the 2nd class) of Cesàro

$$\omega(x) = \limsup \frac{a_1 + \ldots + a_n}{n}, \qquad \text{where} \qquad 0 < x < 1$$

and $x = (0, a_1, a_2, \ldots)_2$ is the binary expansion of x.

The function ω attains each value between 0 and 1 in every interval. Thus it has the Darboux property, and so does the function g defined as follows: $g(x) = 0$ if $x = \omega(x)$ and $g(x) = \omega(x)$ otherwise. Therefore the straight line $y = x$ does not intersect the set $\underset{xy}{E}[y = g(x)]$; and hence this set is not connected (1).

DEFINITION. A continuous mapping $f\colon \mathcal{X} \to \mathcal{Y}$ is called *monotone* if the inverse image $f^{-1}(C)$ of each connected $C \subset \mathcal{Y}$ is connected (2).

Such are, for instance, the real-valued functions of the real variable (defined on an interval), which are monotone in the usual sense.

THEOREM 9. *If the point inverses of a closed mapping $f\colon \mathcal{X} \to \mathcal{Y}$ are connected, then f is monotone.*

Proof. Let $C \subset \mathcal{Y}$ be connected. Let A and B be two separated sets such that $f^{-1}(C) = A \cup B$. It follows that if $y \in C$ and $f^{-1}(y) \cap A \neq 0$, then $f^{-1}(y) \subset A$. Let M be the set of all points $y \in C$ such that $f^{-1}(y) \subset A$. Hence $A = f^{-1}(M)$.

In a similar way, there exists a set N such that $B = f^{-1}(N)$. Since the sets A and B are separated, then so are M and N (compare § 41, III, Remark 2 (ii)). Hence the hypothesis that the set $C = M \cup N$ is connected implies that either $M = 0$ or $N = 0$, and therefore that either $A = 0$ or $B = 0$.

II. Operations.

THEOREM 1. *If C is a connected subset of the union $M \cup N$ of two separated sets M and N, then either $C \cap M = 0$ or $C \cap N = 0$.*

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(1) *Ibid.* However, the set $\underset{xy}{E}[y = \omega(x)]$ is connected, as proved by L. Vietoris, Mon. Math. u. Ph. 3 (1921), p. 173.

(2) Compare G. T. Whyburn, *Analytic Topology*, p. 127. Compare also A. D. Wallace, *Quasi-monotone transformations*, Duke Math. Journ. 7 (1940), p. 136 (a generalization of the concept of monotone mapping).

Proof. This is because otherwise the set C would be the union of the sets $C \cap M$ and $C \cap N$, which are separated and non-empty.

THEOREM 2 [1]. *Let $\{C_t\}$ be a family of connected sets. The union $\bigcup_t C_t$ is connected, provided that there exists such a set C_0 which is not separated from any set C_t.*

Proof. Let the union $\bigcup_t C_t$ be the whole space $\mathscr{X}$. Suppose that $\mathscr{X} = M \cup N$, where M and N are closed-open and disjoint. We are going to show that either $M = 0$ or $N = 0$. According to Theorem 1 it is admissible to assume that $C_0 \cap M = 0$, which implies $C_0 \subset N$, and therefore N is an (open) neighbourhood of C_0. Since the sets C_0 and C_t are not separated, it follows that $C_t \cap N \neq 0$, and hence $C_t \cap M = 0$ by Theorem 1 for any t. Hence $M = 0$.

Theorem 2 can be derived also from the following

THEOREM 2′. *Let $\{C_t\}$ be a directed family of connected sets (this means that for each pair t_1, t_2 there is t_3 such that $C_{t_1} \subset C_{t_3}$ and $C_{t_2} \subset C_{t_3}$). Then the union $S = \bigcup_t C_t$ is connected.*

Proof. Suppose $S = M \cup N$ where M and N are separated sets. By Theorem 1, we have for each t, either $C_t \subset M$ or $C_t \subset N$. Let $C_{t_0} \neq 0$. Obviously we may assume that $C_{t_0} \subset M$; hence $C_{t_0} \not\subset N$. We shall show that $S \subset M$, which will complete the proof.

Let t be an arbitrary index and let t' be such that $C_{t_0} \subset C_{t'}$ and $C_t \subset C_{t'}$. The first inclusion yields $C_{t'} \not\subset N$. Hence $C_{t'} \subset M$ and therefore $C_t \subset M$. It follows that $S \subset M$.

COROLLARIES 3. (i) *The union of connected sets, which have a common point, is a connected set.*

(ii) *If C is connected and $C \subset X \subset \bar{C}$, X is connected.*

(iii) *If every pair of points of the space is contained in a connected set, the whole space is connected.*

Proof. Corollary (i) follows immediately from Theorem 2. In order to prove (ii), put $X = M \cup N$, where M and N are separated. According to Theorem 1 we may assume that $C \subset M$. It follows that $\bar{C} \subset \bar{M}$ and consequently $\bar{C} \cap N = 0$. As $X \subset \bar{C}$, we obtain $X \cap N = 0$ and hence $N = 0$.

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[1] See a paper of B. Knaster and myself, *Sur les ensembles connexes*, Fund. Math. 2 (1921), p. 210.

THEOREM 4 (of decomposition) [1]. *If C is a connected subset of a connected space $\mathscr{X}$, and if M and N are two separated sets such that $\mathscr{X}-C = M \cup N$, the sets $C \cup M$ and $C \cup N$ are connected.* Furthermore, *if C is closed, then $C \cup M$ and $C \cup N$ are also closed.*

Proof. Assume that $C \cup M = A \cup B$, where A and B are separated. We are going to prove that either $A = 0$ or $B = 0$. According to Theorem 1 it is admissible to assume that $C \cap A = 0$, which gives $A \subset M$, because $A \subset C \cup M$. Since the sets M and N are separated, the set A is separated from N and therefore from $N \cup B$, because A and B are separated (compare § 6, V, Theorem 4). The identity

$$\mathscr{X} = C \cup M \cup N = A \cup B \cup N$$

provides a decomposition of the connected space into two separated sets A and $B \cup N$. And hence one of them is empty, which completes the proof.

Finally, if C is closed, then

$$\overline{C \cup M} = C \cup \bar{M} = (C \cup \bar{M}) \cap (C \cup M \cup N) = C \cup M$$

since $\bar{M} \cap N = 0$.

COROLLARY 5 [2]. *Let A and B be two closed (or two open) sets. If the sets $A \cup B$ and $A \cap B$ are connected, the sets A and B are also connected.*

Proof. Let $A \cup B$ be the whole space $\mathscr{X}$ and assume in Theorem 4 that

$$M = A-B, \quad N = B-A \quad \text{and} \quad C = A \cap B.$$

Since the sets $A-B$ and $B-A$ are separated (compare § 6, V, Theorem 2), the sets

$$A = A \cap B \cup (A-B) \quad \text{and} \quad B = A \cap B \cup (B-A)$$

are connected.

THEOREM 6. *If E is not the union of n connected sets, there exist $n+1$ pairwise separated sets $A_1, \dots, A_{n+1}$ such that*

$$E = A_1 \cup \dots \cup A_{n+1}, \quad A_i \neq 0 \quad \textit{for} \quad 1 \leqslant i \leqslant n+1.$$

(1) *Ibid.*, Theorem VI.

(2) See the paper of S. Janiszewski and myself in Fund. Math. 1 (1920), new edition 1937, p. 211, Theorem I.

Proof. Since this theorem obviously holds for $n = 1$, let us assume that it is true for $n-1$. Thus $E = A_1 \cup \ldots \cup A_n$ and A_j are pairwise separated and non-empty. By hypothesis one of them, say A_n, is not connected; i.e. there exist two separated non-empty sets A_n^* and A_{n+1} such that $A_n = A_n^* \cup A_{n+1}$; this implies the required conclusion.

THEOREM 7 (Generalized Theorem 4) (1). *If $C_1, \ldots, C_n$ are connected subsets of a connected space $\mathscr{X}$, and M and N are two separated sets such that*

$$\mathscr{X} - (C_1 \cup \ldots \cup C_n) = M \cup N,$$

then the set $C_1 \cup \ldots \cup C_n \cup M$ consists of n connected sets (which may be distinct or not).

Proof. Suppose conversely that there exists a decomposition into $n+1$ pairwise separated, non-empty sets (compare Theorem 6):

$$C_1 \cup \ldots \cup C_n \cup M = A_1 \cup \ldots \cup A_{n+1}.$$

No set A_i is separated from N for $i \leqslant n+1$, because otherwise there would exist a decomposition

$$\mathscr{X} = A_i \cup (A_1 \cup \ldots \cup A_{i-1} \cup A_{i+1} \cup \ldots \cup A_{n+1} \cup N)$$

into two separated non-empty sets contradicting the connectedness of $\mathscr{X}$. Since M and N are separated, it follows that $A_i \not\subset M$, and therefore that $A_i \cap C_j \neq 0$ for some $j \leqslant n$. Since C_j is a connected subset of the union of separated sets $A_1, \ldots, A_{n+1}$, it follows that $C_j \subset A_i$.

Finally, we conclude that each of the sets $A_1, \ldots, A_{n+1}$ contains a (non-empty) set belonging to the system $\{C_1, \ldots, C_n\}$, which is obviously impossible.

THEOREM 7′. *Every connected space, which contains more than one point, is the union of two connected sets, which are distinct from the space and contain more than one point* (2).

---

(1) See the paper of B. Knaster and myself in Proc. Nat. Acad. of Sc. 13 (1927), p. 648. Compare also (in the case of $n = 2$) G. T. Whyburn, Fund. Math. 10 (1927), p. 181.

(2) For a more precise result of A. H. Stone and for related questions see P. Erdös, Bull. Amer. Math. Soc. 50 (1944), p. 442.

Proof. If for each x the set $\mathscr{X}-x$ is connected, there is a decomposition $\mathscr{X}=(\mathscr{X}-x_1)\cup(\mathscr{X}-x_2)$ where $x_1\neq x_2$. On the other hand, if there exists an x such that $\mathscr{X}-x$ is not connected, it follows that $\mathscr{X}-x=M\cup N$, where M and N are separated and non-empty; therefore (compare Theorem 4) $\mathscr{X}=(x\cup M)\cup(x\cup N)$.

Remark. However, there exist spaces, called *biconnected*, which admit no decomposition into two connected, disjoint proper subsets containing more than one point.

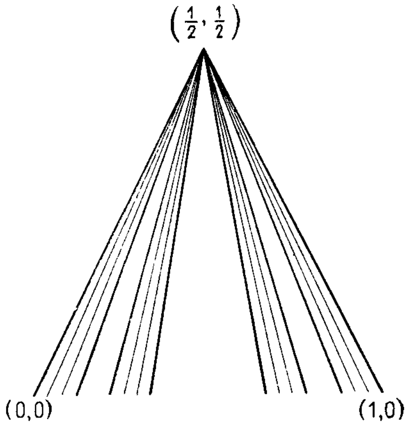

Fig. 1

Consider the following example ([1]). Let us join the point $(\frac{1}{2},\frac{1}{2})$ with each point of the Cantor set $\mathscr{C}$ by a straight line segment. If that segment contains an end-point of an interval which is contiguous to $\mathscr{C}$, let us take its points with rational ordinates; in the opposite case, take its points with irrational ordinates. All these points constitute the required biconnected set.

The above defined biconnected space $\mathscr{X}$ has a point p (called a "dispersed point"), namely the point $(\frac{1}{2},\frac{1}{2})$, such that $\mathscr{X}-p$ *contains no connected set* (having more than one point). It can be proved using the continuum hypothesis that there exist biconnected spaces which do not possess that property ([2]).

([1]) For the proof see the above quoted paper of B. Knaster and myself, Fund. Math. 2 (1921), p. 241. For another example, see the paper of B. Knaster and myself in Rend. di Palermo 49 (1925), p. 3.

See also P. M. Swingle, *Generalization of biconnected sets*, Amer. Journ. Math. 53 (1931), p. 385, P. Erdös, *op. cit.*, R. Duda, *On biconnected sets with dispersion points*, Rozpr. Matem. 37, Warszawa 1964), and M. Rudin, Proceed. Madison Seminar 1955, p. 84.

Compare J. Martin, *A countable Hausdorff space with a dispersion point*, Duke Math. Journ. 33 (1966), pp. 165–167, and P. Roy, *A countable connected Urysohn space with a dispersion point*, *ibid.* pp. 331–333.

([2]) See E. W. Miller, *Concerning biconnected sets*, Fund. Math. 29 (1927), p. 123.

Let us add that the set of all rational points (i.e. points with rational coordinates only) of the Hilbert space becomes biconnected upon the addition of a single point (¹).

THEOREM 8. *If C is a connected space and $\{G_t\}$ an open cover of C, each pair of points can be joined by a chain with links belonging to this cover, i.e. to every pair (x, y) there corresponds a finite set of indices $t_1, \ldots, t_n$ such that*

$$x \in G_{t_1}, \quad G_{t_k} \cap G_{t_{k+1}} \neq 0 \quad \textit{for} \quad 1 \leqslant k \leqslant n, \quad y \in G_{t_n}. \tag{1}$$

Proof. Let x be a fixed point and E the set of all points y, which can be joined to x by a chain. Our task is to show that $E = C$. Since $E \neq 0$ (because $x \in E$) and since E is obviously open, the task is reduced to prove that E is closed.

Let $p \in \bar{E}$ and $p \in G_t$. Since G_t is open, there exists a point $y \in E \cap G_t$. If the chain $G_{t_1}, \ldots, G_{t_n}$ satisfies condition (1), the chain $G_{t_1}, \ldots$ $\ldots, G_{t_n}, G_t$ joins x with p. Hence $p \in E$, so that $\bar{E} = E$.

This implies immediately

THEOREM 9. *Under the same hypotheses about C and $\{G_t\}$, to any pair of non-empty sets (A, B) corresponds an irreducible chain between A and B, i.e. there exists a system of sets $R_1, \ldots, R_n$ $(n > 0)$ belonging to the family $\{G_t\}$ such that if we let $R_0 = A$ and $R_{n+1} = B$, the condition $R_i \cap R_{i'} \neq 0$ is equivalent to $|i - i'| \leqslant 1$.*

THEOREM 10. *If, under the hypotheses of Theorem 8, the indices t range over the set of positive integers, there exists a permutation $k_1, k_2, \ldots$ of this set such that*

$$(G_{k_1} \cup \ldots \cup G_{k_n}) \cap G_{k_{n+1}} \neq 0 \quad \textit{for} \quad n = 1, 2, \ldots, \tag{2}$$

provided that the sets G_k are non-empty.

Proof. Let $k_1 = 1$. For $n > 0$, let k_{n+1} be the least positive integer which satisfies condition (2) and which is different from the numbers $k_1, \ldots, k_n$.

Such an integer exists, because otherwise the set $G_{k_1} \cup \ldots \cup G_{k_n}$ would be closed-open and distinct from the whole space.

(¹) J. H. Roberts, *The rational points in Hilbert space*, Duke Math. Journ. **23** (1956), pp. 489–491. See also R. L. Wilder, *A point set which has no true quasicomponents and which becomes connected upon the addition of a single point*, Bull. Amer. Math. Soc. **33** (1927), pp. 423–427.

We are going to show that the sequence $\{k_n\}$ contains all positive integers.

Suppose that this is not true, and let $l_1, l_2, \ldots$ be the sequence (finite or infinite) of positive integers, which do not belong to the sequence $\{k_n\}$. Since the space is connected, it follows that

$$(G_{k_1} \cup G_{k_2} \cup \ldots) \cap (G_{l_1} \cup G_{l_2} \cup \ldots) \neq 0.$$

Therefore, there exist two indices n and m such that

$$G_{k_n} \cap G_{l_m} \neq 0.$$

By the definition of k_{n+1}, it follows that $k_{n+1} < l_m$ and similarly $k_{n+2} < l_m$ and so on. But this is impossible since the sequence $\{k_n\}$ is not bounded.

THEOREM 11. *The cartesian product* $\mathscr{X} = \underset{t \in T}{\boldsymbol{P}} C_t$ *of connected spaces is connected.*

In particular, *the euclidean space* $\mathscr{E}^n$, *the space* $\mathscr{E}^{\aleph_0}$ *of Fréchet and the Hilbert cube* $\mathscr{I}^{\aleph_0}$ *are connected.*

Proof. If (x_1, y_1) and (x_2, y_2) are two points of $C_1 \times C_2$, the set $(C_1 \times y_1) \cup (x_2 \times C_2)$ is connected as the union of two connected sets having the point (x_2, y_1) in common. And hence $C_1 \times C_2$ is connected by Corollary 3 (iii).

So, the product of two connected spaces (and therefore, of any finite number of them) is connected.

Consider now the general case where T is arbitrary.

We may assume, of course, that $C_t \neq 0$ for each $t \in T$.

Let f_0 be a fixed element of $\mathscr{X}$ (for instance, the point $0, 0, \ldots$ if $\mathscr{X}$ is the Hilbert cube). Let us assign to each finite system $S = (t_1, \ldots, t_n)$ the product K_S of sets C_t, with $t \in S$, and of one-element sets $(f_0(t))$ for $t \notin S$. Otherwise stated, if $f \in \mathscr{X}$, we have

$$(f \in K_S) \equiv [f(t) = f_0(t) \text{ for } t \notin S]. \tag{3}$$

First, note that K_S is connected. This follows from the connectedness (just shown) of the set $L_S = C_{t_1} \times \ldots \times C_{t_n}$ and of the fact that K_S can be obtained from this set by means of a continuous mapping; namely by the mapping $h\colon L_S \to K_S$ where $h^{t_i}(x_{t_1}, \ldots, x_{t_n}) = x_{t_i}$ if $t_i \in S$ and $h^t(x_{t_1}, \ldots, x_{t_n}) = f_0(t)$ if $t \notin S$.

Clearly $(S \subset S') \Rightarrow (K_S \subset K_{S'})$, and it follows that the family of all K_S is directed, which means that for any pair S_1 and S_2

there is S_3 such that $S_1 \subset S_3$ and $S_2 \subset S_3$ (namely $S_3 = S_1 \cup S_2$). Hence, by Theorem 2′, the union $U = \bigcup_S K_S$ is connected, and so is $\overline{U}$ (by Corollary 3 (ii)). It remains to be shown that $\overline{U} = \mathscr{X}$, i.e. that for each $Q \subset \mathscr{X}$ open and non-empty, we have $U \cap Q \neq 0$.

Clearly, we may assume that Q belongs to a base of $\mathscr{X}$. Thus we may assume that there are $S = (t_1, \ldots, t_n)$ and open sets G_{t_i} relative to C_{t_i} (for $t_i \in S$) such that Q is the product of these sets and of the sets C_t with $t \notin S$:

$$Q = \underset{t}{\boldsymbol{P}} G_t \quad \text{where} \quad G_t = C_t \quad \text{for} \quad t \notin S.$$

Let $f \in Q$ be such that $f(t) = f_0(t)$ for $t \notin S$. By (3), we have $f \in K_s$. Hence $f \in U \cap Q$.

LEMMA. *Let $f_i\colon \mathscr{Y} \to \mathscr{X}$ be continuous for $i = 1, \ldots, n$ (where $\mathscr{X}$ is $\mathscr{T}_1$). Put $F(y) = (f_1(y), \ldots, f_n(y))$. Then $F\colon \mathscr{Y} \to 2^{\mathscr{X}}$ is continuous* (note that $F(y)$ is a *set* composed of at most n elements, and is not a finite *sequence*).

Proof. To prove that F is continuous, means to show that the inverse image $F^{-1}(\mathfrak{G})$ of each open set $\mathfrak{G}$ in $2^{\mathscr{X}}$ is open in $\mathscr{Y}$. Clearly, we may limit the range of variability of $\mathfrak{G}$ to an open subbase of $2^{\mathscr{X}}$. Thus we have to prove that the sets

$$\underset{y}{E}[F(y) \subset H] \quad \text{and} \quad \underset{y}{E}[F(y) \cap H \neq 0]$$

are open in $\mathscr{Y}$ whatever is the set H open in $\mathscr{Y}$.

In the first case, we must show that if $F(y_0) \subset H$, then there is Q open in $\mathscr{Y}$ such that $y_0 \in Q$ and $F(Q) \subset H$. Now, the inclusion $F(y_0) \subset H$ means that $f_i(y_0) \in H$ for each $i \leqslant n$. Since f_i is continuous, there is Q_i open in $\mathscr{Y}$ such that $y_0 \in Q$ and $f_i(Q_i) \subset H$. It remains to put $Q = Q_1 \cup \ldots \cup Q_n$.

In the second case, we must show that if $F(y_0) \cap H \neq 0$, then there is an open Q such that $y_0 \in Q$ and $F(y) \cap H \neq 0$ for each $y \in Q$. Now, the formula $F(y_0) \cap H \neq 0$ means that there is $i \leqslant n$ such that $f_i(y_0) \in H$. Let Q_i be as above. Clearly, it remains to put $Q = Q_i$.

THEOREM 12 [1]. *If C is a connected $\mathscr{T}_1$-space, the family $\boldsymbol{F}_n \subset 2^C$ of all sets composed of at most n elements is connected.*

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[1] See E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. 71 (1951), p. 165.

Consequently, *the family* $\boldsymbol{F}$ *of all finite subsets of* C *is connected.*

Proof. Let us put in the Lemma: $\mathscr{X} = C$, $\mathscr{Y} = C^n$, and $f_i(x_1, \ldots, x_n) = x_i$. Therefore $F(x_1, \ldots, x_n)$ is the subset of C composed of the elements $x_1, \ldots, x_n$. Clearly, $F\colon C^n \to \boldsymbol{F}_n$ is a mapping onto and is continuous (since f_i is continuous by Theorem 1 of § 15, II, p. 138). As C^n is connected (by Theorem 11), so is $\boldsymbol{F}_n$ (by Theorem 3 of Section I).

As $\boldsymbol{F} = \boldsymbol{F}_1 \cup \boldsymbol{F}_2 \cup \ldots$ and $\boldsymbol{F}_k \subset \boldsymbol{F}_{k+1}$, $\boldsymbol{F}$ is connected by Theorem 2.

COROLLARY 13. *If* C *is a connected* $\mathscr{T}_1$*-space, so is* 2^C.

Because $2^C = \overline{\boldsymbol{F}}$ (by Theorem 4 of § 17, II, p. 163).

THEOREM 14. *The family* $\boldsymbol{C}$ *of all closed connected subsets of a normal space* $\mathscr{X}$ *is closed in* $2^{\mathscr{X}}$. (1)

Proof. Let $A \subset \mathscr{X}$ be a non-connected closed set. Then $A = M \cup N$ where M and N are closed and $M \cap N = 0$, $M \neq 0 \neq N$. Since $\mathscr{X}$ is normal, there are two open sets G and H such that $M \subset G$, $N \subset H$ and $G \cap H = 0$. Denote by $\boldsymbol{V}$ the family of all closed sets F such that

$$F \subset G \cup H \quad \text{and} \quad F \cap G \neq 0 \neq F \cap H. \tag{4}$$

Clearly $\boldsymbol{V}$ is open in $2^{\mathscr{X}}$. Furthermore, $A \in \boldsymbol{V}$ and no connected set F satisfies conditions (4), i.e. $\boldsymbol{V} \cap \boldsymbol{C} = 0$. Hence $2^{\mathscr{X}} - \boldsymbol{C}$ is open.

III. Components. A set is said to be a *component* of the space provided it is saturated with respect to the property of being connected (2); in other words, C is a component if C is connected and if the inclusion $C \subset C_1$ implies $C = C_1$ for any connected set C_1.

It is easily seen (referring to Theorem 2 and Corollary 3 of Section II) that:

THEOREM 1. *The components are disjoint closed sets.*

THEOREM 2. *Every non-empty connected set is contained in one and only one component of the space.*

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(1) See E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. 71 (1951), p. 166, Theorem 4. 13. 5.

(2) Following F. Hausdorff, *Mengenlehre*, p. 152.

THEOREM 3. *If C is a component of A and if $C \subset B \subset A$, then C is a component of B.*

THEOREM 4. *Every closed-open set F is the union of a family of components of the space. In particular, if F is connected and non-empty, then it is a component.*

Proof. This is so because, by Theorem II, 1, there is no component C such that $C \cap F \neq 0 \neq C - F$.

THEOREM 5. *Let $\mathscr{X}$ be a connected space. If A is a connected set and C is a component of $\mathscr{X} - A$, then $\mathscr{X} - C$ is connected.*

Proof. Assume that

$$\mathscr{X} - C = M \cup N, \qquad (\overline{M} \cap N) \cup (M \cap \overline{N}) = 0.$$

Our task is to show that either $M = 0$ or $N = 0$. Since

$$A \subset \mathscr{X} - C = M \cup N,$$

it can be assumed, according to Theorem II, 1, that $A \cap M = 0$, which implies that

$$A \cap (C \cup M) = 0, \qquad \text{so that} \qquad C \subset (C \cup M) \subset \mathscr{X} - A.$$

Since the set $C \cup M$ is connected (by Theorem II, 4), this double inclusion implies, by the definition of component, that $C = C \cup M$ and hence $M = 0$.

Theorem 5 has the following consequences.

THEOREM 6. *If the space $\mathscr{X}$ is connected, then every finite system $\boldsymbol{S}$ (containing at least two elements) of disjoint connected subsets contains at least two elements, X and Y, which have the following property.*

(**P**) *There exists a connected set disjoint from X (respectively from Y) which contains all the elements of $\boldsymbol{S}$ other than X (respectively other than Y).*

Proof. Let $\boldsymbol{S} = (C_0, C_1, \ldots, C_n)$ and proceed by induction. Since the theorem is obvious for $n = 1$, let us assume that it holds for $n - 1$ $(\geqslant 1)$.

We are going to show that there exists a number $k > 0$ such that the set C_k has the property **P**.

Suppose that the set C_1 does not possess this property. Hence there exist at least two components A and B of the set $\mathscr{X} - C_1$ which contain the sets of the system $\boldsymbol{S}$; let A be that one which does not contain C_0.

Let $m_1, \ldots, m_j$ be the sequence of indices of the sets C_i contained in A. It follows that

$$1 \leqslant j \leqslant n-1, \tag{1}$$

$$0 \neq m_1, \ \ldots, \ 0 \neq m_j, \tag{2}$$

$$\textit{if } r \neq m_1, \ldots, r \neq m_j \textit{ and } r \leqslant n, \textit{ then } C_r \subset \mathscr{X}-A. \tag{3}$$

Since the set $\mathscr{X}-A$ is connected (by Theorem 5) and the system

$$\boldsymbol{S}^* = (\mathscr{X}-A, C_{m_1}, C_{m_2}, \ldots, C_{m_j})$$

contains at most n elements (by (1)), there exists by hypothesis an index $s \leqslant j$ such that the set C_{m_s} has the property $\mathbf{P}$ with respect to the system $\boldsymbol{S}^*$. Therefore there exists a connected set K such that

$$(\mathscr{X}-A) \cup C_{m_1} \cup \ldots \cup C_{m_{s-1}} \cup C_{m_{s+1}} \cup \ldots \cup C_{m_j} \subset K \subset \mathscr{X}-C_{m_s}.$$

It follows by (3) that $C_q \subset K$ for every $q \neq m_s$.

That means that C_{m_s} has the property $\mathbf{P}$ (with respect to the system $\boldsymbol{S}$).

Finally, $m_s > 0$ by (2); hence m_s is the required index k.

THEOREM 7. *In a connected space $\mathscr{X}$, let $\boldsymbol{S}$ be an infinite family of disjoint connected sets. If S_0 and S_1 are two arbitrary elements of $\boldsymbol{S}$, there exists in $\mathscr{X}-S_0$ or in $\mathscr{X}-S_1$ a connected set, which contains infinitely many elements of $\boldsymbol{S}$.*

Proof. Let C_j (for $j = 0, 1$) be the component of $\mathscr{X}-S_j$ which contains S_{1-j}. Condition $S_j \subset C_{1-j} \subset \mathscr{X}-S_{1-j}$ implies the double inclusion

$$S_{1-j} \subset \mathscr{X}-C_{1-j} \subset \mathscr{X}-S_j,$$

which implies in turn that

$$\mathscr{X}-C_{1-j} \subset C_j \tag{4}$$

since the set $\mathscr{X}-C_{1-j}$ is connected (by Theorem 5).

Suppose that C_0 contains only a finite number of elements of $\boldsymbol{S}$. Hence there exist infinitely many elements of $\boldsymbol{S}$ contained in $\mathscr{X}-C_0$ and therefore in C_1 according to (4). So $\mathscr{X}-S_1$ contains a connected set, namely C_1, which contains infinitely many elements of the family $\boldsymbol{S}$.

THEOREM 8. *Let* $\mathscr{X} = \underset{t}{\boldsymbol{P}}\mathscr{X}_t$, $\mathfrak{z} = \{\mathfrak{z}^t\} \in \mathscr{X}$ *and* C_t *be the component of* $\mathfrak{z}^t$ *in* $\mathscr{X}_t$. *Then* $C = \underset{t}{\boldsymbol{P}}C_t$ *is the component of* $\mathfrak{z}$ *in* $\mathscr{X}$.

Proof. Since C_t is connected for each t, so C is connected by Theorem 11 of Section II, and obviously $\mathfrak{z} \in C$. Let D be connected and such that $C \subset D \subset \mathscr{X}$. We have to show that $D = C$.

Let D_t be the projection of D on $\mathscr{X}_t$. Hence D_t is connected and $C_t \subset D_t$. Since C_t is a component of $\mathscr{X}_t$, it follows that $D_t = C_t$. Consequently, $\underset{t}{\boldsymbol{P}}D_t = C$. But $C \subset D \subset \underset{t}{\boldsymbol{P}}D_t$. Thus $D = C$.

THEOREM 9. *Let* $f\colon \mathscr{X} \to \mathscr{Y}$ *be a monotone continuous mapping onto. Then the set* C *is a component of* $D \subset \mathscr{Y}$ *if and only if* $f^{-1}(C)$ *is a component of* $f^{-1}(D)$.

Proof. Obviously,

$$f^{-1}(C) \subset E \subset f^{-1}(D) \quad \text{implies} \quad C \subset f(E) \subset D.$$

Now, if C is supposed to be a component of D and E is supposed to be connected, it follows that

$$C = f(E), \quad \text{hence} \quad f^{-1}(C) = f^{-1}f(E) \supset E, \quad \text{and} \quad f^{-1}(C) = E,$$

i.e. $f^{-1}(C)$ is a component of $f^{-1}(D)$.

Conversely, if $f^{-1}(C)$ is supposed to be a component of $f^{-1}(D)$ and if H is a connected set such that $C \subset H \subset D$, it follows that

$$f^{-1}(C) \subset f^{-1}(H) \subset f^{-1}(D),$$

and since the set $f^{-1}(H)$ is connected, it follows that

$$f^{-1}(C) = f^{-1}(H), \quad \text{which implies} \quad C = H.$$

Hence C is a component of D.

IV. Connectedness between sets. A space is said to be *connected between* A *and* B if there is no closed-open set F such that $A \subset F$ and $F \cap B = 0$ [1].

The connectedness between two sets is a symmetric relation. Because $\mathscr{X} - F$ is closed-open, $B \subset \mathscr{X} - F$ and $(\mathscr{X} - F) \cap A = 0$.

The following statements can easily be shown.

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(1) Compare S. Mazurkiewicz, C. R. Paris 151 (1910), p. 296.

THEOREM 1a. *If $\mathscr{X}$ is connected between A and B, then $A \neq 0 \neq B$, and if $A \subset A_1$ and $B \subset B_1$, then $\mathscr{X}$ is connected between A_1 and B_1.*

THEOREM 1b. *If $\mathscr{X}$ is connected between $\bar{A}$ and $\bar{B}$, it is also connected between A and B.*

If $A \cap B \neq 0$, then $\mathscr{X}$ is connected between A and B.

THEOREM 1c. *A connected space is connected between any pair of its non-empty subsets.*

THEOREM 1d. *If a subset of the space is connected between A and B, then so is the whole space.*

THEOREM 2. *A metric (separable) space has dimension 0 if and only if it is not connected between any pair of disjoint closed sets* (§ 26, II, Theorem 2); *it has a positive dimension at the point a if and only if it is connected between a and some closed set B, which does not contain a.*

THEOREM 3. *If the space is connected neither between A and B_0, nor between A and B_1, then it is not connected between A and $B_0 \cup B_1$.*

Proof. Let F_j, where $j = 0, 1$, be a closed-open set such that $A \subset F_j$ and $F_j \cap B_j = 0$. Define $F = F_0 \cap F_1$. Hence the set F is closed-open and satisfies the following conditions:

$$A \subset F \quad \text{and} \quad F \cap (B_0 \cup B_1) = 0.$$

THEOREM 4. *Let $A_0, \ldots, A_n$ be a system of sets such that the space is not connected between any pair A_i, A_j, where $i \neq j$, there exists a system of closed-open disjoint sets $F_0, \ldots, F_n$ such that*

$$\mathscr{X} = F_0 \cup \ldots \cup F_n \quad \textit{and} \quad A_i \subset F_i \quad \textit{for} \quad i = 0, \ldots, n. \tag{1}$$

Proof. We shall proceed by induction. The theorem is obvious for $n = 1$. It will be sufficient to establish it for $n > 1$ assuming it holds for $n-1$.

Define $A_i^* = A_i$ for $i < n-1$ and $A_{n-1}^* = A_{n-1} \cup A_n$. According to Theorem 3, the space is not connected between any pair A_i^*, A_j^* for $i \neq j$. Thus, by hypothesis, there exists a system of disjoint closed sets $F_0^*, \ldots, F_{n-1}^*$ such that

$$\mathscr{X} = F_0^* \cup \ldots \cup F_{n-1}^* \quad \text{and} \quad A_i^* \subset F_i^* \quad \text{for}$$
$$i = 0, \ldots, n-1. \tag{2}$$

Since the space is not connected between A_{n-1} and A_n, then so is the set F^*_{n-1} (compare Theorem 1d). Hence there exist two disjoint closed sets, F_{n-1} and F_n, such that

$$F^*_{n-1} = F_{n-1} \cup F_n, \quad A_{n-1} \subset F_{n-1} \quad \text{and} \quad A_n \subset F_n. \tag{3}$$

Let $F_i = F^*_i$ for $i < n-1$. Condition (1) follows from (2) and (3).

THEOREM 5. *If the space is not connected between A and B, the function f, which takes the value 1 on A and 0 on B, has a continuous extension to the whole space with the only values 0 and 1.*

Conversely, if f is a continuous integer-valued function, and if the sets $f(A)$ and $f(B)$ are disjoint, the space is not connected between A and B.

Proof. First, the characteristic function of the closed-open set F such that $A \subset F$ and $F \cap B = 0$, is the required extension of f.

Second, if we let $F = f^{-1}[f(A)]$, it follows that $A \subset F$ and $F \cap B = 0$, because

$$A \subset f^{-1}[f(A)] \quad \text{and}$$
$$F \cap B \subset f^{-1}[f(A)] \cap f^{-1}[f(B)] = f^{-1}[f(A) \cap f(B)] = 0.$$

THEOREM 6. *In every space with a countable open base there exists a sequence $F_1, F_2, \dots$ of closed-open sets such that for every pair of points p, q, between which the space is not connected, there is n such that F_n contains p but does not contain q.*

Proof. Let $R_1, R_2, \dots$ be an open base of the space.

If the space is not connected between R_i and R_j, let F_{ij} be a closed-open set such that $R_i \subset F_{ij}$ and $F_{ij} \cap R_j = 0$. Arrange the sets F_{ij} in a simple sequence $F_1, F_2, \dots$. This is the required sequence. Because, if F is a closed-open set such that $p \in F$ and $q \in \mathscr{X} - F$, then there exist sets R_i and R_j such that $p \in R_i \subset F$ and $q \in R_j \subset \mathscr{X} - F$, which proves that $\mathscr{X}$ is not connected between R_i and R_j. Hence there is n such that $p \in R_i \subset F_n$ and $q \in R_j \subset \mathscr{X} - F_n$.

Remark 1. If the space is compact, the family of all closed-open sets can be taken as the family $\{F_n\}$. The mentioned family is countable according to Theorem 4 of § 41, II.

Remark 2. Theorem 6 can be generalized as follows. If X is of (infinite) weight $\mathfrak{m}$, i.e. if it has an open base $\boldsymbol{B}$ of cardinality $\mathfrak{m}$, then there is a family $\boldsymbol{C}$ of cardinality $\leqslant \mathfrak{m}$ composed of closed-open sets such that, for every pair of points p, q, between which the space is not connected, there is a member of C which contains p but does not contain q.

THEOREM 7. *The subset E of the hereditarily normal space $\mathcal{X}$ is connected between two subsets A and B if and only if there exists no open set G (in $\mathcal{X}$) such that*

$$E \cap \mathrm{Fr}(G) = 0, \quad A \subset G, \quad \bar{G} \cap B = 0.$$

Proof. If such a G exists, $E \cap G$ is closed-open in E. Conversely, if F is closed-open in E, the sets F and $E-F$ are separated and there exists (by § 14, V, Theorem 3) a set G such that

$$F \subset G \quad \text{and} \quad \bar{G} \cap E \subset F, \quad \text{which implies} \quad E \cap \bar{G} - G = 0.$$

THEOREM 8. *If the space $\mathcal{X}$ is connected, every proper closed subset A is connected between its boundary (i.e. between the set $A \cap \overline{\mathcal{X}-A}$) and every of its points.*

If the metric separable space $\mathcal{X}$ has a positive dimension at the point a, there exists a number $\varepsilon > 0$ such that every closed set A, containing a and having diameter $< \varepsilon$, is connected between a and $A \cap \overline{\mathcal{X}-A}$.

Proof. Suppose that F is a closed subset of A such that $A-F$ is closed and that $a \in F$ and $F \cap \overline{\mathcal{X}-A} = 0$. Then we have the decomposition

$$\mathcal{X} = F \cup [(A-F) \cup \overline{\mathcal{X}-A}]$$

into two disjoint, closed and non-empty sets (and such that $\delta(F) < \varepsilon$).

THEOREM 9. *If in a metric space the sets $C_1, C_2, \ldots$ are connected and $a \in \operatorname{Li}_{n=\infty} C_n$ and $b \in \operatorname{Li}_{n=\infty} C_n$, the set $E = a \cup b \cup C_1 \cup C_2 \cup \ldots$ is connected between a and b.*

Proof. Otherwise there would exist in E a closed-open set F such that $a \in F$ and $b \in E-F$. Since $a \in F \cap \operatorname{Li}_{n=\infty} C_n$, then $F \cap C_n \neq 0$ for sufficiently large values of n; thus $C_n \subset F$ (since C_n is connected and F is separated from $E-F$). But then $\operatorname{Li}_{n=\infty} C_n \subset \bar{F}$, and $b \in F$.

EXAMPLE. Let

$$C_n = \underset{xy}{E}[(-1 \leqslant x \leqslant 1)(y = 1/n)], \quad a = (-1, 0), \quad b = (+1, 0).$$

E is connected between a and b, but a and b belong to two different components of E.

Moreover, $E-(b)$ is connected between a and the set B of points $(1, 1/n)$, $n = 1, 2, \ldots$, but *is not connected between a and any individual point of B* (compare also § 47, II, Theorem 1).

Let $\mathscr{X}$ be a $\mathscr{T}_2$-space. Write briefly

$$(x_0 \approx x_1) \equiv (\mathscr{X} \textit{ is connected between } x_0 \textit{ and } x_1).$$

Clearly, the relation $x_0 \approx x_1$ is an *equivalence relation*.

THEOREM 10. *The relation $x_0 \approx x_1$ is multiplicative.*

That means that, given a family of spaces $\{\mathscr{X}_t\}$, $t \in T$, we have for each pair of points $\mathfrak{z}$ and $\mathfrak{y}$ of $\mathscr{X} = \underset{t}{\boldsymbol{P}} \mathscr{X}_t$ the following equivalence:

$$(\mathfrak{z} \approx \mathfrak{y}) \equiv \bigwedge_t (\mathfrak{z}^t \approx \mathfrak{y}^t). \tag{1}$$

Proof. First suppose that, for a given $t = t_0$, $\mathfrak{z}^t \approx \mathfrak{y}^t$ does not hold. Hence there are two open sets G_t and H_t such that

$$G_t \cap H_t = 0, \quad G_t \cup H_t = \mathscr{X}_t, \quad \mathfrak{z}^t \in G_t, \quad \mathfrak{y}^t \in H_t.$$

Let $\mathfrak{G}$ be the cartesian product of G_t with all axes $\mathscr{X}_{t'}$ where $t' \neq t$. Define $\mathfrak{H}$ in an analogous way. Hence

$$\mathfrak{G} \cap \mathfrak{H} = 0, \quad \mathfrak{G} \cup \mathfrak{H} = \mathscr{X}, \quad \mathfrak{z} \in \mathfrak{G}, \quad \mathfrak{y} \in \mathfrak{H}. \tag{2}$$

This means that $\mathfrak{z} \approx \mathfrak{y}$ does not hold.

Next, assume that $\mathfrak{z}^t \approx \mathfrak{y}^t$ for each t. We have to show that $\mathfrak{z} \approx \mathfrak{y}$.

Let us consider the case where T is finite: $T = (1, 2, \ldots, n)$ and let us proceed by induction. Clearly, our assertion is true for $n = 1$. Suppose that it is true for $n-1$. Put $\mathfrak{w} = (\mathfrak{z}^1, \ldots, \mathfrak{z}^{n-1}, \mathfrak{y}^n)$.

By assumption,

$$(\mathfrak{z}^1, \ldots, \mathfrak{z}^{n-1}) \approx (\mathfrak{y}^1, \ldots, \mathfrak{y}^{n-1}) \quad \text{in} \quad \mathscr{X}_1 \times \ldots \times \mathscr{X}_{n-1}.$$

Therefore

$$\mathfrak{z} \approx \mathfrak{w} \text{ in } (\mathfrak{z}^1, \ldots, \mathfrak{z}^{n-1}) \times \mathscr{X}_n \quad \text{and} \quad \mathfrak{w} \approx \mathfrak{y} \text{ in } \mathscr{X}_1 \times \ldots \times \mathscr{X}_{n-1} \times (\mathfrak{y}^n).$$

It follows that

$$\mathfrak{z} \approx \mathfrak{w} \text{ in } \big((\mathfrak{z}^1, \ldots, \mathfrak{z}^{n-1}) \times \mathscr{X}_n\big) \cup \big(\mathscr{X}_1 \times \ldots \times \mathscr{X}_{n-1} \times (\mathfrak{y}^n)\big),$$

hence in $\mathscr{X}_1 \times \mathscr{X}_2 \times \ldots \times \mathscr{X}_n$.

This completes the proof for the case of T finite.

Consider the general case of T arbitrary, and suppose that $\mathfrak{z}^t \approx \mathfrak{y}^t$ for each $t \in T$, while $\mathfrak{z} \approx \mathfrak{y}$ is not true. There exist therefore in $\mathcal{X}$ two open sets $\mathfrak{G}$ and $\mathfrak{H}$ such that (2) is satisfied. By the definition of topology in $\mathcal{X}$ (compare § 16, I, p. 147), there is a finite system of indices $t_1, \ldots, t_n$ such that

$$\mathfrak{z} \in \underset{t}{\boldsymbol{P}} G_t \subset \mathfrak{G},$$

where G_t is open in $\mathcal{X}_t$ and $G_{t'} = \mathcal{X}_{t'}$ if $t' \neq t_i$ for each $i \leqslant n$.

Let us define the point $\mathfrak{w} \in \mathcal{X}$ by the condition

$$\mathfrak{w}^t = \begin{cases} \mathfrak{z}^t & \text{for} \quad t = t_i, \quad i \leqslant n, \\ \mathfrak{y}^t & \text{otherwise.} \end{cases}$$

It follows that $\mathfrak{w} \in \underset{t}{\boldsymbol{P}} G_t$, hence $\mathfrak{w} \in \mathfrak{G}$. We shall show that $\mathfrak{w} \approx \mathfrak{y}$ in $\mathcal{X}$, which will imply the expected contradiction (since this relation is contradictory to (2)).

Now, let

$$\mathfrak{K} = \underset{t}{\boldsymbol{P}} K_t \quad \text{where} \quad K_t = \begin{cases} \mathcal{X}_t & \text{for} \quad t = t_i, \ i \leqslant n, \\ (\mathfrak{y}^t) & \text{otherwise.} \end{cases}$$

As shown above

$$(\mathfrak{z}^{t_1}, \ldots, \mathfrak{z}^{t_n}) \approx (\mathfrak{y}^{t_1}, \ldots, \mathfrak{y}^{t_n}) \quad \text{in} \quad \mathcal{X}_{t_1} \times \ldots \times \mathcal{X}_{t_n},$$

and it follows that $\mathfrak{w} \approx \mathfrak{y}$ in $\mathfrak{K}$, hence in $\mathcal{X}$.

THEOREM 11. *The relation $x_0 \approx x_1$ is closed.* Otherwise stated, *the set $\mathfrak{F} = \underset{x_0, x_1}{E} (x_0 \approx x_1)$ is closed in $\mathcal{X} \times \mathcal{X}$.*

Proof. Let $\langle x_0, x_1 \rangle \notin \mathfrak{F}$. There is therefore a closed-open A such that $x_0 \in A$ and $x_1 \notin A$, i.e. $\langle x_0, x_1 \rangle \in [A \times (\mathcal{X} - A)]$. Furthermore, if $x \in A$ and $x' \in \mathcal{X} - A$, $\mathcal{X}$ is not connected between x and x'. Hence

$$[A \times (\mathcal{X} - A)] \cap \mathfrak{F} = 0,$$

and thus

$$\langle x_0, x_1 \rangle \in [A \times (\mathcal{X} - A)] \subset \mathcal{X} \times \mathcal{X} - \mathfrak{F}.$$

Therefore $\mathcal{X} \times \mathcal{X} - \mathfrak{F}$ is open.

THEOREM 12. *Let $\mathcal{X}$ be a hereditarily normal space and $A \subset \mathcal{X}$. Then A is connected between x_1 and x_2 if and only if every open G is connected between these points, provided that $A \subset G$.*

Proof. Clearly, if A is connected between x_1 and x_2, then so is G (this is true for arbitrary $\mathcal{X}$).

Next, suppose that A is not connected between x_1 and x_2. Then there are two separated sets P_1 and P_2 such that $x_1 \in P_1$ and $x_2 \in P_2$. As $\mathcal{X}$ is hereditary normal, so there are open sets G_1 and G_2 such that $P_1 \subset G_1$, $P_2 \subset G_2$ and $G_1 \cap G_2 = 0$ (compare § 14, V, Theorem 1, p. 130). Obviously $G = G_1 \cup G_2$ is not connected between x_1 and x_2.

V. Quasi-components. The *quasi-component* of the point p is the intersection of all closed-open sets containing p (1). In other words, it is the set of all points x such that the space is connected between p and x.

Clearly, the quasi-components are equivalence sets determined by the relation $x_1 \approx x_2$. Their family is the *quotient-family* $X/\approx$ (compare § 2, VII, p. 11).

The following statements can be easily established.

THEOREM 1. *The component of* p *is contained in the quasi-component of* p.

The inverse inclusion is not generally true (as is shown by example of Section IV). However, it is true in compact $\mathcal{T}_2$-spaces (see § 47, II, Theorem 2).

THEOREM 2. *The quasi-components are disjoint closed sets.*

THEOREM 3. *For every* $\mathcal{T}_2$*-space* $\mathcal{X}$ *there are a generalized Cantor discontinuum* $D^{\mathfrak{m}}$ *(where* D *is the two-elements set* $(0,1)$*) and a continuous* $f: \mathcal{X} \to D^{\mathfrak{m}}$ *such that the quasi-components of* $\mathcal{X}$ *coincide with the point inverses of* f.

In particular, *if* $\mathcal{X}$ *has a countable open base,* $D^{\mathfrak{m}}$ *can be replaced by the Cantor discontinuum* $\mathcal{C}$ (2).

Proof. Let $\boldsymbol{A} = \{F_t\}$, $t \in T$, be the family of all closed-open subsets of $\mathcal{X}$ and let f be its characteristic function; this means (see § 3, VII) that

$$f^t(x) = 1 \text{ if } x \in F_t \quad \text{and} \quad f^t(x) = 0 \text{ if } x \in \mathcal{X} - F_t.$$

Let $\mathfrak{m} = \overline{\overline{T}}$ be the cardinality of T. Thus $f: \mathcal{X} \to D^{\mathfrak{m}}$.

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(1) See F. Hausdorff, *Grundzüge*, p. 248.

(2) See my paper in Fund. Math. 30 (1938), p. 245, where it is shown that, if $\mathcal{X}$ is metric separable the set of functions f, which satisfy the conclusion of Theorem 3, is a residual set in the space $\mathcal{C}^{\mathcal{X}}$.

Clearly, f^t is continuous for each $t \in T$, and so is f.

Furthermore, if $x \approx x'$, then $f^t(x) = f^t(x')$ for each $t \in T$; hence $f(x) = f(x')$. Conversely, if the relation $x \approx x'$ does not hold, then for some t we have $x \in F_t$, while $x' \in \mathscr{X} - F_t$, i.e. $f^t(x) = 1$ and $f^t(x') = 0$, hence $f(x) \neq f(x')$.

This completes the proof of the first part of the theorem.

Now, let us note that the family $\boldsymbol{A}$ can be restricted to the subfamily $\{F_t\}$, $t \in T_0$, with the following property: for every pair of points x, x' such that $x \approx x'$ does not hold, there is $t \in T_0$ such that $x \in F_t$ while $x' \in \mathscr{X} - F_t$.

In particular, if $\mathscr{X}$ has a countable base, T_0 can be assumed to be countable (by Theorem 6 of Section IV); this completes the proof of the second part of the theorem.

THEOREM 4. *Every metric separable space $\mathscr{X}$ is topologically contained in a compact space $\mathscr{X}^*$ such that two distinct quasi-components of $\mathscr{X}$ are always contained in two distinct quasi-components of $\mathscr{X}^*$* (1).

Moreover, *in the functional space $(\mathscr{I}^{\aleph_0})^{\mathscr{X}}$ the set of homeomorphisms f such that the set $\mathscr{X}^* = \overline{f(\mathscr{X})}$ satisfies the required condition is residual.*

Proof. Let $F_1, F_2, \ldots$ be the sequence of closed-open sets, which was considered in Theorem IV, 6 (while $\mathscr{X}$ is supposed not to be connected). By § 44, V, Corollary 4a and § 44, VI, Theorem 2, the set Φ_n of homeomorphisms f such that

$$f \in (\mathscr{I}^{\aleph_0})^{\mathscr{X}} \quad \text{and} \quad \overline{f(F_n)} \cap \overline{f(\mathscr{X} - F_n)} = 0 \tag{0}$$

is residual in the space $(\mathscr{I}^{\aleph_0})^{\mathscr{X}}$. And so is the set $\Phi = \Phi_1 \cap \Phi_2 \cap \ldots$.

Let $f \in \Phi$. Let p and q be two points, which belong to two different quasi-components P and Q of $\mathscr{X}$. Thus there exists an index n such that

$p \in F_n$ and $q \in \mathscr{X} - F_n$, and hence $f(p) \in f(F_n)$ and $f(q) \in f(\mathscr{X} - F_n)$.

Since

$$\mathscr{X}^* = \overline{f(\mathscr{X})} = \overline{f(F_n)} \cup \overline{f(\mathscr{X} - F_n)}$$

and $f \in \Phi_n$, it follows by (0) that $\mathscr{X}^*$ is not connected between $f(p)$ and $f(q)$; therefore these points belong to two different quasi-components of $\mathscr{X}^*$.

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(1) *Ibid.*, p. 243. This theorem was suggested by B. Knaster.

Theorem 10 of Section IV can be reformulated as follows.

THEOREM 5. *Let* $\mathfrak{z} = \{\mathfrak{z}^t\}$ *be a point of the product* $Z = \mathop{\boldsymbol{P}}_{t \in T} \mathscr{X}_t$ *and let* Q *be the quasi-component of* $\mathfrak{z}$ *in* Z. *Then* $Q = \mathop{\boldsymbol{P}}_{t} Q_t$ *where* Q_t *is the quasi-component of* $\mathfrak{z}^t$ *in* $\mathscr{X}_t$.

Va. The space of quasi-components. Denote by $\mathfrak{C}(\mathscr{X})$ the quotient-topology $\mathscr{X}/\approx$ where $x_0 \approx x_1$ means that $\mathscr{X}$ is connected between x_0 and x_1 (according to Section IV). In other words, the set $\boldsymbol{G} \subset \mathfrak{C}(\mathscr{X})$ is open in $\mathfrak{C}(\mathscr{X})$ if the union of all quasi-components in $\boldsymbol{G}$ is open in $\mathscr{X}$ (see § 19, I).

Besides $\mathfrak{C}(\mathscr{X})$ another topology of the family of all quasi-components (call it $Q(\mathscr{X})$) is considered.

Namely, $Q(\mathscr{X})$ is defined by taking as its open base all sets of the form

$$\mathop{E}_{A}(A \subset F), \tag{1}$$

where F is closed-open in $\mathscr{X}$ and A is a quasi-component of $\mathscr{X}$ (¹).

The following two statements are obvious.

THEOREM 1. *The topology* $Q(\mathscr{X})$ *is weaker than* $\mathfrak{C}(\mathscr{X})$.

In other terms, *the identity mapping is a one-to-one continuous mapping of* $\mathfrak{C}(\mathscr{X})$ *onto* $Q(\mathscr{X})$.

THEOREM 2. *The sets of the form* $\mathop{E}_{A}(A \subset F)$ *are closed-open in* $Q(\mathscr{X})$. *Consequently,* $Q(\mathscr{X})$ *has dimension* 0 *and is completely regular.*

Remark. Clearly, if each quasi-component of $\mathscr{X}$ is composed of a single point, then $\mathfrak{C}(\mathscr{X})$ is homeomorphic with $\mathscr{X}$. Consequently, if $\mathscr{X}$ is of positive dimension (and such spaces do exist, see Section VI), then the topologies $\mathfrak{C}(\mathscr{X})$ and $Q(\mathscr{X})$ are different.

THEOREM 3. *Denote by* $P(x)$ *the quasi-component containing* x. *Then the (natural) mapping* $P\colon \mathscr{X} \to Q(\mathscr{X})$ *is continuous.*

This follows from the equivalence

$$\big(P(x) \subset F\big) \equiv (x \in F) \quad \text{where } F \text{ is closed-open.}$$

(¹) Compare J. de Groot, *A note on 0-dimensional spaces*, Indag. Math. 9 (1947), pp. 94–98, where the case of $\mathscr{X}$ metric separable is considered. For the general case, see E. Michael, *Cuts*, Acta Math. 111 (1964), p. 15. See also H. de Vries, *Compact spaces and compactifications*, Chapter 3, Thesis, Amsterdam 1962; and A. Lelek, *On the Knaster totally disconnected sets*, Bull. Acad. Polon. Sc. 15 (1967), p. 81.

THEOREM 4. *If $\mathscr{X}$ is a compact $\mathscr{T}_2$-space, then so is $\mathfrak{C}(\mathscr{X})$. Moreover, $\mathfrak{C}(\mathscr{X})$ and $Q(\mathscr{X})$ are homeomorphic.*

(In this case the quasi-components of $\mathscr{X}$ are identical with its components, see § 47, II, Theorem 2.)

Proof. By Theorem 3 of Section IV, there is a continuous mapping $f: \mathscr{X} \to D^{\mathfrak{m}}$ such that $\mathfrak{C}(\mathscr{X})$ coincides with the decomposition of $\mathscr{X}$ into point inverses $\overset{-1}{f}(y)$ where $y \in f(\mathscr{X})$. It follows (by Theorem 2 of § 19, II) that $\mathfrak{C}(\mathscr{X})$ is homeomorphic with $f(\mathscr{X})$.

Since $\mathfrak{C}(\mathscr{X})$ is compact, then so is $Q(\mathscr{X})$ (by Theorem 1).

Theorem 3 of Section V can be strengthened as follows.

THEOREM 5. *For every $\mathscr{T}_2$-space $\mathscr{X}$ there is a one-to-one continuous mapping $g: Q(\mathscr{X}) \to D^{\mathfrak{m}}$ (for an appropriate $\mathfrak{m}$).*

Namely, $\big(g(A) = y\big) \equiv \big(\overset{-1}{f}(y) = A\big)$ *for each* $A \in Q(\mathscr{X})$.

Proof. By Theorem 3, g is one-to-one. It remains to show that g is continuous. Since $D^{\mathfrak{m}}$ has a closed-open base, it remains to show that if G is closed-open in $D^{\mathfrak{m}}$, then $g^{-1}(G)$ is open in $Q(\mathscr{X})$.

Now, since f is continuous, the set $f^{-1}(G)$ is closed-open in $\mathscr{X}$. We shall show that

$$\big(A \in g^{-1}(G)\big) \equiv \big(A \subset f^{-1}(G)\big),$$

which will complete the proof.

Let $A \in g^{-1}(G)$. Hence $g(A) \in G$. Put $g(A) = y$. Hence $A = \overset{-1}{f}(y)$ and $A \subset f^{-1}(G)$.

Conversely, if $A \subset f^{-1}(G)$, then $f(A) \subset G$ and $g(A) \in G$. Hence $A \in g^{-1}(G)$.

VI. Hereditarily disconnected spaces. Totally disconnected spaces.

DEFINITIONS. The space is said to be *hereditarily disconnected* (or *dispersed*) if it contains no connected set consisting of more than one point; in other words—if every component consists of a single point.

The space is said to be *totally disconnected* (or *nowhere connected*) if the space is not connected between any pair of points; in other words, if every quasi-component consists of a single point.

The following statement is obvious.

THEOREM 1. *Every* 0-*dimensional space is totally disconnected and every totally disconnected space is hereditarily disconnected.*

THEOREM 2. *Every weakly* 1-*dimensional space* (*i.e. such that the set* N *of points where it has positive dimension is of dimension* $\leqslant 0$, compare § 22, VI) *is hereditarily disconnected* (¹).

Proof. If C is a connected set containing more than one point, then $\dim_p C > 0$ for every $p \in C$ (compare I, Theorem 5). Therefore $C \subset N$, hence $\dim N > 0$.

The following theorem is obvious.

THEOREM 3. *Any space which admits a one-to-one continuous mapping into a totally disconnected space is itself totally disconnected.*

In particular, the weakly 1-dimensional set, defined in § 22, VI, is a *totally disconnected* G_δ *of positive dimension* (²).

Because this set admits a one-to-one projection onto the Cantor set.

Remarks. (i) *There exist totally disconnected spaces of arbitrary finite or infinite dimension.*

Because the Cantor set $\mathscr{C}$ is a one-to-one continuous image of a metric separable space of arbitrary dimension (finite or infinite (³)).

Moreover, there exist complete, separable and totally disconnected spaces of arbitrary dimension (⁴).

(ii) *There exist* (*complete separable*) *hereditarily disconnected spaces, which are not totally disconnected* (⁵).

Proof. Let E be a weakly 1-dimensional G_δ-set in a compact space (e.g. the set defined in § 27, VI). Let $\dim_a E = 1$. There exists, as can be shown (compare § 47, II, Theorem 8), a point $b \neq a$

(¹) K. Menger, *Dimensionstheorie*, p. 204.

(²) The first example of a totally disconnected space of positive dimension was given by W. Sierpiński in his paper *Sur les ensembles connexes et non connexes*, Fund. Math. 2 (1921), p. 81.

(³) Theorem of Hilgers, *Bemerkung zur Dimensionstheorie*, Fund. Math. 28 (1937), p. 303.

(⁴) Theorem of Mazurkiewicz, *Sur les problèmes* $\varkappa$ *et* λ *de Urysohn*, Fund. Math. 10 (1927), p. 311. For the λ problem of Urysohn, see Fund. Math. 8 (1926), p. 324. See also B. Knaster, *Sur les coupures biconnexes des espaces euclidiens de dimension* $n > 1$ *arbitraire*, Mat. Sb. 19 (1946), pp. 9–18.

(⁵) Theorem of W. Sierpiński, *loc. cit.*, p. 8.

such that $E \cup (b)$ is connected between a and b. Beside that, $E \cup (b)$ is hereditarily disconnected since it is weakly 1-dimensional.

(iii) Pompeiu has defined a function g on $\mathscr{I}$ whose derivative g' is finite at each point and takes zero value in every interval, but is constant in none. Its definition can be derived from the formula

$$x = \sum_{n=1}^{\infty} 2^{-n}(y-a_n)^{1/3}, \quad \text{where} \quad a_n = (2k+1)/2^{m+1},$$

k and m are integers which satisfy conditions

$$n = 2^m + k \quad \text{and} \quad 0 \leqslant k \leqslant 2^m - 1. \text{ (}^1\text{)}$$

By Theorem 8 of Section I and § 27, VII, Theorem 1, the set $C = \underset{xy}{E}[y = g(x)]$ is a connected $\boldsymbol{G}_\delta$.

Let $g'(a) \neq 0$. *The set* $A = C - \mathscr{I} \cup (a, 0)$ *is connected between the points* $p = [a, 0]$ *and* $q = [a, g'(a)]$.

Suppose conversely that M and N are separated and that

$$A = M \cup N, \quad p \in M \quad \text{and} \quad q \in N.$$

Let bc be an interval such that

$$b < a < c, \quad \overline{N} \cap bc = 0 \quad \text{and} \quad g'(b) = 0 = g'(c).$$

Denote by N_1 the part of N whose projection lies in the interval bc; hence N_1 is separated from $N - N_1$ and also from $\mathscr{I}$, and therefore from $(N - N_1 \cup M \cup \mathscr{I})$. Since

$$C \subset N_1 \cup (N - N_1 \cup M \cup \mathscr{I})$$

and $p \in M$ and $q \in N$, this contradicts the connectedness of the set C.

It follows (compare § 26, III, Theorem 3) that the set $C - \mathscr{I}$ has dimension 1 at each of its points.

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(1) See Pompeiu, Math. Ann. 63 (1907), p. 326. Compare also Köpcke, *ibid.* vols. 29, 34 and 35, and the paper of B. Knaster and myself, Rend. di Palermo 49 (1925).

Furthermore, let us notice that the identity $\overline{C \cap \mathscr{I}} = \mathscr{I}$ implies that $C - \mathscr{I}$ is *totally disconnected*. And hence A is hereditarily *disconnected* (without being totally disconnected) (¹).

(iv) A space is called *extremally disconnected* if the closure of each open set is open (²).

Clearly, every extremally disconnected regular space has an open base composed of closed-open sets (namely of closures of open sets), hence is 0-dimensional. The converse is not true.

VII. Separators. By definition (§ 6,V) C *separates the space* $\mathscr{X}$ *between* A *and* B (C is a *separator* between A and B) if $\mathscr{X} - C$ is not connected between A and B; in other words, if there exist two sets M and N such that

$$\mathscr{X} - C = N \cup M, \quad (\bar{M} \cap N) \cup (\bar{N} \cap M) = 0,$$
$$A \subset M, \quad B \subset N. \qquad \text{(i)}$$

The set C is said (concisely) to be a *separator of the space* $\mathscr{X}$ if there exists a pair of closed sets A and B, between which the space $\mathscr{X}$ is connected, but the set $\mathscr{X} - C$ is not.

If the space $\mathscr{X}$ is *connected*, C is a separator if and only if $\mathscr{X} - C$ is not connected (since every connected space is connected between each pair of its points).

The set C is said to be an *irreducible separator between* a *and* b if C is a separator between these points, but any set X such that $X \subset C \neq X$, is not.

The set C is said to be a *complete irreducible separator* if C is an irreducible separator between every pair of points, which are separated by C (and if there exists at least one pair of points of that kind).

The set C is said *to separate locally* the space if C separates an open set, which contains it; in other words, if there exists an open set G and two sets A and B closed in G such that $A \cup B \cup C \subset G$ and G is connected between A and B, but $G - C$ is not.

THEOREM 1. *If* $\bar{D} = \mathscr{X}$, *then each closed separator is a separator between a pair of points belonging to* D.

(¹) Compare my paper in Ann. Soc. Pol. Math. 5 (1927), p. 109.

(²) See M. H. Stone, *Algebraic characterization of special Boolean rings*, Fund. Math. 29 (1937), pp. 223–303.

Proof. This is so because, if M and N are two non-empty open sets such that $\mathscr{X}-C = M \cup N$ and $M \cap N = 0$, then $M \cap D \neq 0 \neq N \cap D$.

THEOREM 2. *A closed set C is separating between A and B if and only if there exist two closed sets P and Q such that*

$$\mathscr{X} = P \cup Q, \quad C = P \cap Q, \quad A \cap Q = 0 = B \cap P. \tag{ii}$$

Moreover, *if the sets A, B and C are disjoint, the double identity can be replaced by the inclusions $A \subset P$ and $B \subset Q$.*

Proof. On the one hand, if condition (i) is satisfied, it can be assumed that

$$P = M \cup C \quad \text{and} \quad Q = N \cup C.$$

On the other hand, if P and Q satisfy conditions (ii), it can be assumed that

$$M = \mathscr{X}-Q \quad \text{and} \quad N = \mathscr{X}-P.$$

THEOREM 3. *Let $\mathscr{X}$ be a hereditarily normal space. If the set E is separating every pair of sets belonging to the system $A_1, \ldots, A_n$, then E contains a closed set F with the same property.*

Proof. By the hypothesis and according to IV, Theorem 4, it follows that

$$\mathscr{X}-E = X_1 \cup \ldots \cup X_n, \quad A_i \subset X_i,$$

where the sets X_i are disjoint and open in $\mathscr{X}-E$. According to Theorem 4 of § 14, V, there exists a system of disjoint open sets $G_1, \ldots, G_n$ such that $X_i \subset G_i$. It is sufficient to set $F = \mathscr{X}-(G_1 \cup \ldots \cup G_n)$.

THEOREM 4. *Every set C, which is the common boundary of two components A and B of its complement, is an irreducible separator between each pair of points (a, b) such that $a \in A$ and $b \in B$.*

Proof. On the one hand, the set $C = \bar{A} \cap \overline{\mathscr{X}-A}$ decomposes the space into two separated sets $\bar{A}-C$ and $\overline{\mathscr{X}-A}-C$, one of which contains a and the other b. On the other hand, if $X \subset C \neq X$, hence if $c \in C-X$, the set $A \cup (c) \cup B$ is connected and joins a

with b outside of X. Therefore, the set X is not a separator between a and b, and thus C is an irreducible separator between a and b.

VIII. Separation of connected spaces(1). In the Sections VIII–XI the *space* $\mathscr{X}$ *is supposed to be connected, metric and separable.*

If C is a closed separator between a and b, there exist according to VII (i) two open sets M and N such that

$$\mathscr{X}-C = M \cup N, \quad M \cap N = 0, \quad a \in M,\ b \in N. \tag{1}$$

There are, in general, many decompositions satisfying conditions (1); it is so, for example, of a triod consisting of three segments ac, bc, and dc, which have only the point c in common.

THEOREM 1. *Let* $\boldsymbol{C}$ *be a family of closed, disjoint and connected separators between* a *and* b. *If we let correspond to every* $C \in \boldsymbol{C}$ *a pair of open sets* $M(C)$ *and* $N(C)$ *satisfying conditions* (1) *and if we define* $A(C) = M(C) \cup C$, *then the family* $\boldsymbol{F}$ *of all sets* $A(C)$, *where* $C \in \boldsymbol{C}$, *is strictly monotone* (2).

Moreover, *if* $C \neq D$, *then either* $A(D) \subset M(C)$ *or* $A(C) \subset M(D)$.

Proof. Let $C \neq D$ be two elements of $\boldsymbol{C}$. Since the sets $M(C)$ and $N(C)$ are separated and D is a connected subset of their union, one of them contains D, while the other is disjoint from it. Assume that $D \subset M(C)$. We are going to show that $M(D) \subset M(C)$.

First, $C \subset N(D)$. Because otherwise we would have $C \subset M(D)$ (for the above mentioned reason). But the conditions

$$\mathscr{X} = M(C) \cup C \cup N(C) \quad \text{and} \quad \mathscr{X} = M(D) \cup D \cup N(D) \tag{2}$$

yield

$$\mathscr{X} = M(C) \cup M(D) \cup C \cup D \cup \big(N(C) \cap N(D)\big),$$

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(1) See G. T. Whyburn, *Non-separated cuttings of connected point sets*, Trans. Amer. Math. Soc. 33 (1931), p. 444, and my paper *Sur les familles monotones d'ensembles fermés et leurs applications à la théorie des espaces connexes*, Fund. Math. 30 (1937), p. 17.

(2) Let us recall that a family of sets is said to be *strictly monotone* if for each pair $X \neq Y$ of its elements either $X \subset \operatorname{Int}(Y)$ or $Y \subset \operatorname{Int}(X)$. Theorem 1 leads to applications of theorems established in § 24, VIII, to the family $\boldsymbol{F}$.

and since $D \subset M(C)$, the inclusion $C \subset M(D)$ implies the decomposition of the space

$$\mathscr{X} = [M(C) \cup M(D)] \cup [N(C) \cap N(D)]$$

into two non-empty, disjoint open sets (containing a and b respectively), which contradicts the connectedness of the space.

Thus $C \subset N(D)$. Now consider the decomposition

$$\mathscr{X} = M(C) \cup N(D) \cup C \cup D \cup \big(N(C) \cap M(D)\big)$$
$$= [M(C) \cup N(D)] \cup [N(C) \cap M(D)],$$

which also follows from conditions (2).

Since the last two terms of the second union are disjoint open sets and the first one is non-empty, it follows that

$$N(C) \cap M(D) = 0, \quad \text{which implies} \quad M(D) \subset M(C) \cup C,$$

and since $C \cap M(D) = 0$ (because $C \subset N(D)$), it follows finally that

$$M(D) \subset M(C).$$

This yields

$$A(D) = D \cup M(D) \subset M(C) \subset \mathrm{Int}[A(C)],$$

because $M(C)$ is an open subset of $A(C)$.

Similarly, if we assume that $D \subset N(C)$, it follows that $C \subset M(D)$ and, by the symmetry, that

$$A(C) \subset M(D) \subset \mathrm{Int}[A(D)],$$

which completes the proof.

Combining Theorem 1 with Theorem 1 of § 24, VII and taking account of the implication $C \neq D$ implies $A(C) \neq A(D)$, the following statement is obtained.

THEOREM 2. *The elements C of the family* **C** *can be indexed with indices y such that $0 < y < 1$ and that the condition $u < y$ implies $M(C_u) \subset A(C_u) \subset M(C_y)$.*

It follows that, *if $u < y < z$, then C_y separates C_u and C_z.*

THEOREM 3. *Except for a countable set of elements of the family* $\mathbf{C}$, *every* $C \in \mathbf{C}$ *satisfies the following conditions*

$$\mathrm{Int}[A(C)] = M(C), \quad \textit{hence} \quad \mathrm{Fr}[A(C)] = C, \tag{3}$$

$$C = \mathrm{Fr}[M(C)] = \mathrm{Fr}[N(C)], \tag{4}$$

there exist in $\mathbf{C}$ *two sequences* $\{D_n\}$ *and* $\{E_n\}$ *such that*

$$C = \bigcap_n M(D_n) \cap N(E_n) = \bigcap_n A(D_n) \cap \overline{[\mathscr{X} - A(E_n)]}, \tag{5}$$

$$M(C) \textit{ and } N(C) \textit{ are connected.} \tag{6}$$

Proof. The following inclusion holds by Theorem 2

$$\bigcup_{u<y} \mathrm{Int}[A(C_n)] \subset M(C_y) \subset \mathrm{Int}[A(C_y)].$$

According to § 24, VII Theorem 4, let

$$\mathrm{Int}[A(C_y)] = \bigcup_{u<y} \mathrm{Int}[A(C_u)].$$

It follows that

$$\mathrm{Int}[A(C_y)] = M(C_y),$$

and hence

$$\mathrm{Fr}[A(C_y)] = A(C_y) - \mathrm{Int}[A(C_y)] = C_y.$$

In virtue of (3), it follows by § 24, VIII (5) that

$$C = \mathrm{Fr}[A(C)] = A(C) - \mathrm{Int}[A(C)]$$
$$= \overline{\mathrm{Int}[A(C)]} - \mathrm{Int}[A(C)] = \mathrm{Fr}\big[\mathrm{Int}\big(A(C)\big)\big] = \mathrm{Fr}[M(C)].$$

Condition (5) follows from (3) and § 24, VIII, Theorem 9:

$$C_y = \bigcap_n \{\mathrm{Int}[A(D_n)] - A(E_n)\} \subset \bigcap_n M(D_{n-1}) \cap N(E_n)$$
$$\subset \bigcap_n A(D_{n-1}) \cap \overline{[\mathscr{X} - A(E_n)]} = C_y,$$

because $A(D_n) \subset M(D_{n-1})$ by Theorem 2.

And finally, as we shall see, (5) implies (6).

Let $M(C) = M_1 \cup M_2$, where M_i are disjoint open sets and $a \in M_1$. We shall show that $M_2 = 0$.

Since the sets $M(C)$ and $N(C)$ are separated, and $\mathscr{X}$ and C are connected, the sets $C \cup M(C)$ and $C \cup N(C)$ are connected (by Theorem 4 of Section II). By the same reason, the sets $C \cup N(C) \cup M_1$ and $C \cup N(C) \cup M_2$ are connected. But since the set $C \cup \cup N(C) \cup M_1$ joins the points a and b, and E_n is a separator between these points, it follows that $E_n \cap [C \cup N(C) \cup M_1] \neq 0$. Since the set $A(E_n)$ precedes $A(C)$, it follows that $E_n \cap N(C) = 0$, but then $E_n \cap M_1 \neq 0$ and hence $E_n \cap M_2 = 0$. Since the sets M_2 and $M_1 \cup N(C)$ are separated, $M_2 \cup C$ is connected and disjoint from E_n according to the last identity. The inclusion $C \subset N(E_n)$, which follows from (5), implies that

$$M_2 \cup C \subset N(E_n), \quad \text{and hence} \quad M_2 \subset N(E_n).$$

On the other hand, it follows by Theorem 2 that

$$M_2 \subset A(C) \subset M(D_n),$$

and so

$$M_2 \subset \bigcap_n M(D_n) \cap N(E_n) = C,$$

and since $C \cap M_2 = 0$, it follows that $M_2 = 0$.

THEOREM 4. *In each family of disjoint, closed and connected separators, every separator, except at most $\aleph_0$, separates the space into two connected sets and is their common boundary; hence it is a complete, irreducible separator.*

Proof. Let $p_1, p_2, \ldots$ be a sequence of points dense in the space. By Theorem 1 of Section VII, the considered family can be decomposed into a (countable) sequence of sub-families $\boldsymbol{C}_{ij}$ of separators between p_i and p_j. Therefore, Theorem 4 follows from statements (4) and (6) since the irreducibility is a consequence of Theorem 4 of Section VII.

THEOREM 5. *In the space $\mathscr{X}$ is given a family $\boldsymbol{C}$ of separators between a and b, which are disjoint, closed, connected and indexed according to Theorem 2. There exists an onto continuous transformation $f\colon \mathscr{X} \to \mathscr{I}$ such that $f(a) = 0, f(b) = 1$ and for each index y*

$$\textit{either} \quad f^{-1}(0y) = A(C_y), \quad \textit{or} \quad f^{-1}(0y) = \bigcap_{z>y} A(C_z),$$

according to whether the index y has an immediate successor or not.

Proof. Let $\boldsymbol{F}^*$ be the family of sets $A_y = A(C_y)$ with $0 < y < 1$, augmented by the sets $A_0 = (a)$ and $A_1 = \mathscr{X}$. Consider the function f

defined in § 24, IX, Theorem 3. Obviously, $f(a) = 0$, and $f(b) = 1$ if there is no index which immediately precedes 1. In the opposite case, where r is the index which immediately precedes 1, all that is needed is to change the definition of f in the set $\mathcal{X} - A_r$, namely

$$f(x) = r + (1-r)\frac{\varrho(x, A_r)}{\varrho(x, A_r) + |x-b|}.$$

Once Theorem 5 has been proved, many properties of the family $\boldsymbol{C}$ can be derived from conditions (13) to (18) of § 24, IX. In particular, the following statement holds (by means of (3) and § 24, IX, (17)).

THEOREM 6. *The identity $C_y = f^{-1}(y)$ holds for all, except at most $\aleph_0$, sets C_y.*

IX. Separating points. Following Section VIII let us assume that the space $\mathcal{X}$ is *connected metric and separable.* Let $S(a, b)$ be the set of all points, which separate a and b. These points can be indexed according to Theorem 2 of Section VIII; hence the condition $u < y < z$ implies that p_y separates the points p_u and p_z.

Theorem 4 of Section VIII implies the following

THEOREM 1 [1]. *The set $\mathcal{X} - (x)$ is connected or is a union of two connected sets for every point x except at most a countable set of them.*

THEOREM 2. *If A is connected, then every point of $A \cap S(a, b)$, except the first and the last one (if they exist), separates the set A.*

This implies, by Theorem VII, 1, the following statement.

THEOREM 3 [3]. *With exception of a countable set, every point of a connected set A, which is a separating point of the space, is also a separating point of A.*

Remark. Let the following statement be added without proof.

If $\boldsymbol{C}$ is a family of non-degenerate, disjoint connected sets, each of which contains a separating point of the space, then $\boldsymbol{C}$ is countable [3].

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(1) See the paper of C. Zarankiewicz and myself, Bull. Amer. Math. Soc. 33 (1927), p. 571.

(2) See C. Zarankiewicz, *Sur les points de division dans les espaces connexes*, Fund. Math. 9 (1927), p. 17, and, under more restrictive hypotheses, R. L. Moore, Proc. Nat. Acad. 9 (1923), p. 102.

(3) Theorem of C. Zarankiewicz, Trans. Amer. Math. Soc. 33, p. 447. For the proof, see my previously quoted paper in Fund. Math. 30, p. 30.

THEOREM 4. *If the set C is the topological limit* (compare § 29, VI) *of a sequence $C_1, C_2, \ldots$ of disjoint connected sets, the set $C \cap S(a, b)$ contains at most two points. Hence the set of points of C which separate the space is countable.*

Proof. Suppose that $p_u, p_y, p_z \in C$ and $u < y < z$. It follows that $p_u \in M(p_y)$ and $p_z \in N(p_y)$ (compare VIII, Theorem 2). Since the sets $M(p_y)$ and $N(p_y)$ are open, the condition $p_y \in \operatorname{Lim}_{n=\infty} C_n$ implies that for sufficiently large n

$$C_n \cap M(p_y) \neq 0 \neq C_n \cap N(p_y),$$

which yields

$$C_n \cap \operatorname{Fr} M(p_y) = 0, \quad \text{i.e.} \quad p_y \in C_n.$$

Hence the sets C_n are not disjoint.

THEOREM 5. *There exists a continuous $f \colon \mathscr{X} \to \mathscr{I}$ such that the set of points at which f is one-to-one, augmented by a countable set properly chosen, coincides with the set $S(a, b) \cup a \cup b$.*

Namely f is the function of Theorem 5 *of Section* VIII, *where C is the family of sets which are reduced to individual points separating a and b.*

Proof. If p is a point different from a and b, at which the function f is one-to-one, there exists a number y such that $p = f^{-1}(y)$ and $0 \neq y \neq 1$. Therefore

$$\mathscr{X} - p = f^{-1}(0y - y) \cup f^{-1}(y1 - y)$$

is a decomposition into two open sets, one of which contains a and the other b. It follows that $p \in S(a, b)$.

On the other hand, at every point of $S(a, b)$, with exception of a countable subset, the function f is one-to-one according to VIII (7).

THEOREM 6 (of Lennes [1]). *If $\mathscr{X} = S(a, b) \cup a \cup b$, i.e. if every point different from a and b separates the space between a and b, there exists a one-to-one, onto continuous transformation $f \colon \mathscr{X} \to \mathscr{I}$.*

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(1) Amer. Journ. of Math. 33 (1911), where this theorem was proved under additional hypotheses. Compare also F. Hausdorff, *Mengenlehre*, p. 220 and § 2 of the paper, *Sur les ensembles connexes*, of B. Knaster and myself, Fund. Math. 2 (1921), where this kind of spaces is studied (they are called there connected spaces, *irreducible between a and b*).

Proof. Assume that $f(p_y) = y$, $f(a) = 0$ and $f(b) = 1$.

Since the function f is obviously one-to-one, we have only to show that it is continuous, i.e. that the set $f^{-1}(G)$ is open (in the space $\mathscr{X}$) provided that G is an open interval in $\mathscr{I}$. But this is a consequence of the conditions (compare VIII, Theorem 2)

$$f^{-1}(0y-y) = M(p_y) \quad \text{and} \quad f^{-1}(y1-y) = N(p_y)$$

since the sets $M(p_y)$ and $N(p_y)$ are open.

Remarks. It is clear that the hypothesis of Theorem 6 is equivalent to the condition that *to every point x there correspond two closed sets A and B such that*

$$\mathscr{X} = A \cup B, \quad a \in A, \quad b \in B, \quad A \cap B = (x).$$

It is also equivalent to the hypothesis that *the points a and b cannot be joined by any proper connected subset of the space.*

Proof. Suppose that the point $p \in (\mathscr{X} - a - b)$ does not separate a from b. So, there exists a connected set C such that $a, b \in C \neq \mathscr{X}$. Because in the case where $\mathscr{X} - p$ is not connected there exist two open sets G and H such that

$$\mathscr{X} - p = G \cup H, \quad G \cap H = 0, \quad H \neq 0 \quad \text{and} \quad a, b \in G$$

since p does not separate a from b.

Thus the set $C = (p) \cup G$ is a connected proper subset (compare II, Theorem 4) of the space joining the points a and b.

X. Unicoherence. Discoherence.

Definition. A topological space $\mathscr{X}$ is said to be *unicoherent* if it is connected and if for each pair A, B of closed connected sets such that $\mathscr{X} = A \cup B$, the intersection $A \cap B$ is connected.

$\mathscr{X}$ is said to be *discoherent*, if for any pair of closed sets A and B such that

$$\mathscr{X} = A \cup B \quad \text{and} \quad A \neq \mathscr{X} \neq B \tag{1}$$

the intersection $A \cap B$ is not connected.

According to Theorem 5 of Section II, if a connected space is not discoherent, there exist two closed *connected* sets A and B which have a connected intersection and satisfy conditions (1).

It is clear that every discoherent space is connected.

Examples. The interval $\mathcal{I}$ is unicoherent, the circle $\mathcal{S}$ is discoherent. As will be later seen, $\mathcal{I}^n$ is unicoherent for every n and $\mathcal{S}_n$ is unicoherent for $n \geqslant 2$.

Theorem 1. *A connected space $\mathcal{X}$ is discoherent if and only if the complement of each closed connected subset C is connected.*

Proof. If

$$\mathcal{X} - C = M \cup N, \quad (\bar{M} \cap N) \cup (\bar{N} \cap M) = 0 \quad \text{and} \quad M \neq 0 \neq N, \tag{1'}$$

the sets $A = C \cup M$ and $B = C \cup N$ are closed, $A \neq \mathcal{X} \neq B$ and $A \cap B = C$. On the other hand, if the closed sets A and B satisfy (1) and if the intersection $A \cap B$ is connected, then (1′) follows provided $M = \mathcal{X} - A$ and $N = \mathcal{X} - B$. And so, the (closed and connected) set C is a separator of the space.

Theorem 2. *If the space $\mathcal{X}$ is discoherent and the sets C and D are closed, connected and such that*

$$\mathcal{X} = C \cup D \quad \textit{and} \quad C \neq \mathcal{X} \neq D, \tag{2}$$

then the sets $A = \overline{\mathcal{X} - C}$ and $B = \overline{\mathcal{X} - A}$ are connected, satisfy condition (1) *and moreover $A = \overline{\mathcal{X} - B}$.*

Proof. The set $\mathcal{X} - C$ is connected by Theorem 1. Hence, so is the set $A = \overline{\mathcal{X} - C}$. This implies that $\mathcal{X} - A$ is connected; and therefore the set $B = \overline{\mathcal{X} - A}$ is also connected.

According to condition (2), $\mathcal{X} - C \subset D \neq \mathcal{X}$, which yields $\overline{\mathcal{X} - C} \subset D \neq \mathcal{X}$, so that $A \neq \mathcal{X}$. Since $0 \neq \mathcal{X} - C \subset \operatorname{Int}(A) = \mathcal{X} - \overline{\mathcal{X} - A}$, it follows that $B \neq \mathcal{X}$.

Then

$$A \cup B = A \cup \overline{\mathcal{X} - A} = \mathcal{X}.$$

Finally, by § 8, VIII,

$$\overline{\mathcal{X} - B} = \overline{[\mathcal{X} - \overline{\mathcal{X} - A}]} = \overline{\{\mathcal{X} - \overline{[\mathcal{X} - \overline{\mathcal{X} - C}]}\}} = \overline{\mathcal{X} - C} = A.$$

Theorem 3. *The properties of being unicoherent and discoherent are invariant under continuous monotone mappings.*

Proof. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be continuous monotone and onto. Let A and B be two closed connected sets such that $f(\mathcal{X}) = A \cup B$. Hence $\mathcal{X} = f^{-1}(A) \cup f^{-1}(B)$ and the sets $f^{-1}(A)$ and $f^{-1}(B)$ are closed and connected. Therefore, if the space $\mathcal{X}$ is unicoherent, the set

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

is connected and so is the set

$$A \cap B = f[f^{-1}(A \cap B)].$$

Hence the space $f(\mathcal{X})$ is unicoherent.

If $\mathcal{X}$ is discoherent, the set $f^{-1}(A \cap B)$ is not connected and neither is the set $A \cap B$. Hence the space $f(\mathcal{X})$ is discoherent.

*__XI.__ __n-dimensional connectedness__ (1). The notion of connectedness can be related to dimension in the following way.

Let $\mathcal{X}$ be a metric separable space (containing more than one point). $\mathcal{X}$ is said to be *at most n-dimensionally connected* if there are two closed sets M and N such that

$$\mathcal{X} = M \cup N, \quad M \neq \mathcal{X} \neq N \quad \text{and} \quad \dim(M \cap N) \leqslant n-1; \tag{1}$$

in other words, if there exists an open set G such that

$$0 \neq G, \quad \bar{G} \neq \mathcal{X} \quad \text{and} \quad \dim\operatorname{Fr}(G) \leqslant n-1 \tag{2}$$

or if there exists a closed, at most $(n-1)$-dimensional set which separates the space $\mathcal{X}$.

The least integer of this kind (finite or infinite) is said to be the *dimension of connectedness of* $\mathcal{X}$ and is denoted by $\operatorname{dc}\mathcal{X}$ (2). Therefore, if $\operatorname{dc}\mathcal{X} < \infty$, there exists a closed separator of dimension $\operatorname{dc}\mathcal{X}-1$, but there is none of dimension $\operatorname{dc}\mathcal{X}-2$.

Furthermore, let us assume that $\operatorname{dc}(p) = 0$ and $\operatorname{dc}0 = -1$.

It can be easily established that $\operatorname{dc}\mathcal{X} \leqslant \dim\mathcal{X}$ and that condition $\operatorname{dc}\mathcal{X} \geqslant 1$ is equivalent to the hypothesis that $\mathcal{X}$ is connected and contains more than one point.

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(1) See the paper of E. Otto and myself, *Sur les espaces à connexité n-dimensionnelle*, Fund. Math. 32 (1939), p. 259.

(2) See my paper *Sur la compactification des espaces à connexité n-dimensionnelle*, Fund. Math. 30 (1938), p. 242 (n is to be replaced there by $n+1$).

The *compact* spaces $\mathscr{X}$ such that $\operatorname{dc}\mathscr{X} = \dim\mathscr{X}$ are said to be *Cantor manifolds.*

Remarks. If the space is the union of two cubes which have only one vertex in common, then dc $= 1$; if they have only one edge in common, then dc $= 2$; finally, if they have only one face in common, then dc $= 3$.

Thus it is seen that the number $\operatorname{dc}\mathscr{X}$ allows to express more exactly the geometric idea, which assigns a more or less "strong" connectedness to a polyhedron consisting of two cubes according as they are joined by a common face, an edge or a vertex.

Theorems 6 and 3 of § 45, IV imply the following two theorems.

THEOREM 1. *If $\mathscr{X}$ is a compact space which is irreducible with respect to its n-dimensional degree, then* $\operatorname{dc}\mathscr{X} \geqslant n$.

THEOREM 2. *Every compact space whose dimension $\geqslant n$ contains a closed set F such that* $\operatorname{dc}F \geqslant n$; *therefore it contains in particular a component of dimension* $\geqslant n$ (1).

Many theorems in the theory of connected sets can be generalized and restated in such a manner as to become theorems on dimension of connectedness. Let us quote some of them without proof.

THEOREM 3. *If C is a set containing more than one point, condition* $\operatorname{dc}C \leqslant n$ *is equivalent to the existence of two sets M and N such that*

$$C = M \cup N, \quad C - M \neq 0 \neq C - N,$$
$$\dim[(\overline{M} \cap N) \cup (\overline{N} \cap M)] \leqslant n-1, \tag{3}$$

and also to the existence of an open set G such that

$$C \cap G \neq 0 \neq C - \overline{G}, \quad \dim[C \cap \operatorname{Fr}(G)] \leqslant n-1. \tag{4}$$

THEOREM 4. *If* $\operatorname{dc}C \geqslant n$, $C \subset M \cup N$ *and* $\dim[(\overline{M} \cap N) \cup (\overline{N} \cap M)] \leqslant n-2$, *then either $C \subset \overline{M}$ or $C \subset \overline{N}$.*

THEOREM 5. *If $\{C_t\}$ is a family of sets such that* $\operatorname{dc}C_t \geqslant n$ *and if it contains a set C_0 such that* $\dim(C_0 \cap C_t) \geqslant n-1$ *for each t, then* $\operatorname{dc}\left(\bigcup_t C_t\right) \geqslant n$.

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(1) In that direction compare the notion of "dimensional component" of P. Alexandrov, Math. Ann. 106 (1932), p. 215. Compare also L. Tumarkin, C. R. Paris 186 (1928), p. 420.

THEOREM 6. *If* $C \subset E \subset \bar{C}$ *and* $\mathrm{dc}\, C \geqslant n$, *then* $\mathrm{dc}\, E \geqslant n$.

THEOREM 7. $\mathrm{dc}(\mathscr{X}-C) \geqslant \mathrm{dc}\,\mathscr{X} - \dim C - 1$.

THEOREM 8. *If* $\mathscr{X}-C = M \cup N$, $(\bar{M} \cap N) \cup (\bar{N} \cap M) = 0$ *and* $\mathrm{dc}\, C \leqslant \mathrm{dc}\,\mathscr{X}$, *then* $\mathrm{dc}\, C \leqslant \mathrm{dc}(C \cup M)$ *and* $\mathrm{dc}\, C \leqslant \mathrm{dc}(C \cup N)$.

THEOREM 9. *If* A *and* B *are two closed sets such that* $\mathrm{dc}(A \cap B) \leqslant \mathrm{dc}(A \cup B)$, *then* $\mathrm{dc}(A \cap B) \leqslant \mathrm{dc}\, A$ *and* $\mathrm{dc}(A \cap B) \leqslant \mathrm{dc}\, B$.

THEOREM 10. *If f is a continuous transformation of a (connected) compact space $\mathscr{X}$, such that $\dim f^{-1}(y) \leqslant k$ for each y, then*

$$\mathrm{dc} f(\mathscr{X}) \geqslant \mathrm{dc}\,\mathscr{X} - k.$$

***XII. n-dimensional connectedness between two sets.** The space $\mathscr{X}$ is said to be *n-dimensionally connected between two subsets A and B* (we write it: $\mathrm{dc}_{A,B}\mathscr{X} = n$) if n is the least integer such that there exist two closed sets M and N which satisfy conditions

$$\mathscr{X} = M \cup N, \quad A \cap N = 0 = B \cap M, \quad \dim(M \cap N) \leqslant n-1;$$

this means that there exists a closed set of dimension $\leqslant n-1$ which separates $\mathscr{X}$ between A and B.

One can prove that (1)

THEOREM 1. *A set C in the space $\mathscr{X}$ is at most n-dimensionally connected between two subsets A and B if and only if there exists an open set G such that*

$$A \subset G, \quad \bar{G} \cap B = 0, \quad \dim[C \cap \mathrm{Fr}(G)] \leqslant n-1.$$

THEOREM 2. *If* $\mathrm{dc}_{A_1,B}\mathscr{X} \leqslant n$ *and* $\mathrm{dc}_{A_2,B}\mathscr{X} \leqslant n$, *then* $\mathrm{dc}_{A_1+A_2,B}\mathscr{X} \leqslant n$.

THEOREM 3. *For every compact space, the n-dimensional connectedness between two closed sets A and B implies the n-dimensional connectedness between a pair of points $a \in A$ and $b \in B$.*

More precisely, *if in a compact space are given two subsets A and B of a set C, then there exist two points $a \in \bar{A}$ and $b \in \bar{B}$ such that*

$$\mathrm{dc}_{A,B} C \leqslant \mathrm{dc}_{a,b}(C \cup a \cup b).$$

THEOREM 4. $\dim_a \mathscr{X} \leqslant n$ *if and only if* $\mathrm{dc}_{a,B}\mathscr{X} \leqslant n$ *for every closed set B such that* $a \in \mathscr{X} - B$.

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(1) See footnote 1 to page 164 and pp. 263–264.

THEOREM 5. *$\dim \mathscr{X} \leqslant n$ if and only if $\mathrm{dc}_{A,B}\mathscr{X} \leqslant n$ for any closed disjoint sets A and B.*

If $\mathscr{X}$ is compact, then $\dim \mathscr{X} \leqslant n$ if and only if $\mathrm{dc}_{a,b}\mathscr{X} \leqslant n$ for any points $a \neq b$.

THEOREM 6. *If C is a subset of a compact space, then to every point $a \in C$ corresponds a point $b \neq a$ such that*

$$\mathrm{dc}_{a,b}(C \cup b) = \dim_a C \quad \textit{provided} \quad \dim_a C < \infty.$$

THEOREM 7 (¹). *Each separable metric space can be compactified without increasing its dimension of connectedness between any pair of points.*

Moreover, *the set of homeomorphisms $f\colon \mathscr{X} \to \mathscr{I}^{\aleph_0}$ such that the dimension of connectedness of $\mathscr{X}$ between p and q is equal to the dimension of connectedness of $\overline{f(\mathscr{X})}$ between $f(p)$ and $f(q)$ for any pair of points $p, q \in \mathscr{X}$, is a residual set in the space $(\mathscr{I}^{\aleph_0})^{\mathscr{X}}$.*

§ 47. Continua

I. Definition. Immediate consequences. A compact connected $\mathscr{T}_2$-space is said to be a *continuum* (²). All spaces considered in § 47 are supposed to be *$\mathscr{T}_2$-spaces.*

THEOREM 0. *A compact metric space $\mathscr{X}$ is a continuum if and only if to every pair of points $a, b \in \mathscr{X}$ and to each $\varepsilon > 0$ corresponds a finite system of points*

$$p_0 = a, p_1, \ldots, p_{n-1}, p_n = b \quad \textit{where} \quad |p_i - p_{i+1}| < \varepsilon. \text{ (³)}$$

Proof. The condition is necessary, because the set $F(a, \varepsilon)$ of points, which can be joined with a by a "chain" with links $< \varepsilon$, is closed and open; it coincides with the space $\mathscr{X}$ if $\mathscr{X}$ is connected. The condition is sufficient, because if $\mathscr{X} = A \cup B$ is a decomposition of the compact space $\mathscr{X}$ into two non-empty, disjoint closed

(¹) See my paper, quoted above in Fund. Math. 30, p. 243. Theorem 7 is a generalization of Theorem 4 of Section V.

(²) The term "continuum" is also used by different authors to denote a closed connected set.

(³) This is the original G. Cantor's definition of continuum. See Math. Ann. 21 (1883), p. 576.

sets, it follows that $\varrho(A, B) > 0$; and hence there is no chain joining a and b with links $< \varrho(A, B)$.

The next statements follow directly from the corresponding theorems on connected sets of § 46.

THEOREM 1. *The union of two continua, which have a point in common, is a continuum* (compare § 46, II, Corollary 3 (i)).

THEOREM 2. *If A and B are two compact sets such that $A \cup B$ and $A \cap B$ are continua, then A and B are continua* (compare § 46, II, Corollary 5).

THEOREM 3. *If C is a subcontinuum of a continuum $\mathscr{X}$ and if M and N are two separated sets such that $\mathscr{X} - C = M \cup N$, the sets $C \cup M$ and $C \cup N$ are continua* (compare § 46, II, Theorem 4).

THEOREM 4. *The cartesian product (finite or infinite) of continua is a continuum* (§ 46, II, Theorem 11).

THEOREM 5. *A continuous image of a continuum is a continuum* (§ 46, I, Theorem 3).

THEOREM 6. *The components of a compact space are continua* (§ 46, III, Theorem 1).

II. Connected subsets of compact spaces.

THEOREM 1. *Every compact space has the following property.*

(M) *If the space is connected between two closed sets A and B, it is connected between a pair of points a and b, where $a \in A$ and $b \in B$* [1].

Proof. Suppose that the space $\mathscr{X}$ is not connected between any pair of points a and b, where $a \in A$ and $b \in B$. Let M_{ab} be a closed open set such that

$$a \in M_{ab} \quad \text{and} \quad b \notin M_{ab}.$$

For a given $b \in B$, the family $\{M_{ab}\}_{a \in A}$ is an open cover of the (compact) set A; hence there is a finite subset A' of A such that $\{M_{ab}\}_{a \in A'}$ is still an open cover of A. Put for $b \in B$

$$M_b = \bigcup_{a \in A'} M_{ab}.$$

The set M_b is a closed-open subset of $\mathscr{X}$ and $A \subset M_b$ while $b \notin M_b$. Then the family $\{\mathscr{X} - M_b\}_{b \in B}$ is an open cover of the (com-

[1] Compare S. Mazurkiewicz, C. R. Paris 151 (1910), p. 296.

pact) set B; hence there is a finite set $B' \subset B$ such that $\{\mathcal{X} - M_b\}_{b \in B'}$ still covers B.

It follows that the closed open set $\bigcap_{b \in B'} M_b$ contains A and is disjoint from B, which implies the required contradiction.

THEOREM 1'. *Every compact hereditarily normal space has the following property: if a subset E is connected between two sets A and B (where $A \cup B \subset E$), there exists a pair of points $a \in \bar{A}$, $b \in \bar{B}$ such that the set $E \cup a \cup b$ is connected between a and b.*

Proof. Let $\boldsymbol{G}$ be the family of all open sets G such that $E \cap \cap \operatorname{Fr}(G) = 0$. Suppose that the set $E \cup a \cup b$ is not connected between a and b whenever $a \in \bar{A}$ and $b \in \bar{B}$; this means (compare § 46, IV, Theorem 7) that to every pair of points $a \in \bar{A}$, $b \in \bar{B}$ corresponds an open set G such that

$$a \in G, \quad b \notin \bar{G} \quad \text{and} \quad (E \cup a \cup b) \cap \operatorname{Fr}(G) = 0,$$

hence such that $G \in \boldsymbol{G}$.

According to the lemma of § 41, II, there exists a set

$$H = (G_1^1 \cap \ldots \cap G_{l_1}^1) \cup \ldots \cup (G_1^k \cap \ldots \cap G_{l_k}^k), \quad \text{where} \quad G_j^i \in \boldsymbol{G},$$

such that $\bar{A} \subset H$ and $\bar{B} \cap \bar{H} = 0$. Since (compare § 6, II (8) and (9))

$$\operatorname{Fr}(H) \subset \bigcup_{i,j} \operatorname{Fr}(G_j^i),$$

it follows that $E \cap \operatorname{Fr}(H) = 0$. Thus, E is not connected between A and B.

THEOREM 2 (¹). *In compact spaces (or in spaces with property* (M)) *the quasi-components are connected and coincide therefore with the components.*

Proof. Suppose that the quasi-component Q of the point p is not connected. So, there exists (compare § 46, I, Theorem 2) an open set G and a point q such that

$$p \in G, \quad q \in Q - \bar{G} \quad \text{and} \quad Q \cap \operatorname{Fr}(G) = 0. \tag{1}$$

(¹) For a more direct proof, see M. Šura-Bura, *Zur Theorie der bikompakten Räume*, Recueil Math. Acad. Sc. URSS, 9 (1941), pp. 385–388; see also my book *Introduction à la théorie des ensembles et à la topologie*, Genève 1966, p. 227.

The latter identity means that the space is not connected between p and any point of the set $\mathrm{Fr}(G)$. Hence property (M) implies that the space is not connected between p and $\mathrm{Fr}(G)$. So there exists a closed-open set F such that

$$p \in F \quad \text{and} \quad F \cap \mathrm{Fr}(G) = 0.$$

It follows that $p \in F \cap G$ and $q \in \mathscr{X} - (F \cap G)$; besides $F \cap G$ is open and closed since

$$\overline{F \cap G} \subset \overline{F} \cap \overline{G} = F \cap \overline{G} = F \cap \mathrm{Fr}(G) \cup (F \cap G) = F \cap G.$$

Therefore the space is not connected between p and q.

THEOREM 3. *If a compact space (or a space with property* (M)) *is connected between two closed sets A and B, there exists a component C such that $C \cap A \neq 0 \neq C \cap B$.*

Proof. Let (a, b) be a pair of points such that $a \in A$, $b \in B$ and that the space is connected between a and b; then C is the component (and hence the quasi-component) of the point a.

THEOREM 4 (1). *In every compact metric space the limit of a convergent sequence of connected sets is a connected set.*

This follows directly from Theorem 14 of § 46, II.

THEOREM 5. *If $C_1, C_2, \ldots$ is a sequence of continua such that*

$$C_1 \supset C_2 \supset \ldots \supset C_n \supset \ldots, \tag{2}$$

the intersection $C_1 \cap C_2 \cap \ldots \cap C_n \cap \ldots$ is a continuum.

Proof. This is so because condition (2) implies by Theorem 1 of § 42, IV that $C_1 \cap C_2 \cap \ldots$ belongs to the closure of the family of all subcontinua of C_1. But this family is closed by Theorem 14 of § 46, II.

Remark. If in a complete space, $C_1 \supset C_2 \supset \ldots$ *is a descending sequence of closed connected sets such that* $\lim\limits_{n=\infty} \alpha(C_n) = 0$, *then the intersection $C_1 \cap C_2 \cap \ldots$ is a continuum* (2).

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(1) Compare L. Zoretti, Journ. de Math. (6) 1 (1905), p. 8.

(2) See my paper in Fund. Math. 15 (1930), p. 304. For the definition of $\alpha(C)$ see § 37, IV, Remark 2.

Proof. Suppose conversely that

$$\bigcap_n C_n = A \cup B, \tag{3}$$

$$A = \bar{A}, \quad B = \bar{B}, \quad A \neq 0 \neq B, \quad A \cap B = 0. \tag{4}$$

Let G be an open set such that

$$A \subset G \quad \text{and} \quad \bar{G} \cap B = 0. \tag{5}$$

Since the set C_n is connected, the condition $C_n \cap G \neq 0 \neq C_n - G$ implies that $C_n \cap \mathrm{Fr}(G) \neq 0$. Let $F_n = C_n \cap \mathrm{Fr}(G)$. Since

$$F_1 \supset F_2 \supset \ldots, \quad 0 \neq F_n = \bar{F}_n$$

and

$$\alpha(F_n) \leqslant \alpha(C_n), \quad \text{which yields} \quad \lim_{n=\infty} \alpha(F_n) = 0,$$

it follows according to § 34, II, that

$$\bigcap_n F_n \neq 0, \quad \text{i.e.} \quad \bigcap_n C_n \cap \mathrm{Fr}(G) \neq 0.$$

But this contradicts conditions (3) and (5).

So, it is established that the set $\bigcap_n C_n$ is connected. Its compactness follows from the condition $\alpha(\bigcap_n C_n) = 0$, which means that $\bigcap_n C_n$ is totally bounded (compare § 41, VI, Theorem 2).

THEOREM 6. *If $C_1, C_2, \ldots$ is a sequence of subcontinua of a compact metric space such that* $\mathop{\mathrm{Li}}_{n=\infty} C_n \neq 0$, *the set* $\mathop{\mathrm{Ls}}_{n=\infty} C_n$ *is a continuum.*

Proof. According to Corollary of § 29, VIII, $\mathop{\mathrm{Ls}}_{n=\infty} C_n$ is the union of the limits of convergent subsequences $\{C_{k_n}\}$. Since these limits are connected (by Theorem 4) and contain the set $\mathop{\mathrm{Li}}_{n=\infty} C_n$ (compare § 29, II, Theorem 5) their union is connected (compare § 46, II, Theorem 2).

THEOREM 7. *If f is a continuous mapping of a continuum $\mathscr{X}$, there exists a subcontinuum C of $\mathscr{X}$ which is irreducible with respect to the property: to be a continuum such that $f(C) = f(\mathscr{X})$.*

Proof. This is a consequence of Theorem 2 of § 42, IV, and of the fact that the families of continua and of the closed sets X such that $f(X) = f(\mathscr{X})$ are closed in $2^{\mathscr{X}}$ (compare § 44, II, Theorem 2).

THEOREM 8. *Let E be a subset of a compact metric space. If* $\dim_a E > 0$ (*where a is a given point of E*), *there exists a point $b \neq a$ such that $E \cup (b)$ is connected between a and b* [1].

Proof. Let B_0 be a closed set in E such that $a \in E - B_0$ and that E is connected between a and B_0 (compare § 46, IV, Theorem 2), and let us put in Theorem 1 $A = a$ and $B = B_0$.

THEOREM 9. *If a compact metric space* (*or a space with property* (M)) *has a positive dimension at the point p, then p lies in a connected set* (*which contains more than one point*).

Proof. According to Theorem 8 the space is connected between p and a point $q \neq p$. The quasi-component of p, which coincides with its component, contains q.

III. Closed subsets of a continuum.

THEOREM 1. *If A is a closed proper subset of the continuum $\mathscr{X}$ and if C is a component of A, then*

$$C \cap \overline{\mathscr{X} - A} \neq 0, \quad i.e. \quad C \cap \mathrm{Fr}(A) \neq 0. \text{ [2]}$$

Proof. This is so because (compare II, Theorem 3) A is connected between each point a of A and the set $A \cap \overline{\mathscr{X} - A}$ (by § 46, IV, Theorem 8).

Theorem 1 can be generalized as follows

THEOREM 2. *If X is an arbitrary proper subset of the continum $\mathscr{X}$ and C is a component of X, then*

$$\bar{C} \cap \overline{\mathscr{X} - X} \neq 0, \quad i.e. \quad \bar{C} \cap \mathrm{Fr}(X) \neq 0.$$

Proof. Let $a \in C$. Clearly, we may assume that $a \notin \overline{\mathscr{X} - X}$. Consider the family $\mathbf{A}$ of all open sets G such that $\overline{\mathscr{X} - X} \subset G$ and $a \notin \bar{G}$. Let C_G denote the component of a in $\mathscr{X} - G$. By Theorem 1, we have $C_G \cap \bar{G} \neq 0$.

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[1] Theorem of K. Menger, *Dimensionstheorie*, p. 207.

[2] Theorem of S. Janiszewski, Bull. Acad. Sc. Cracovie 1912, p. 907. See also L. Vietoris, *Stetige Mengen*, Monatsh. 31, and P. Szymański, *Sur les constituants d'ensembles situés sur des continus arbitraires*, Fund. Math. 10 (1927), p. 363, where bibliographic references are given. For a more modern approach, see M. Šura-Bura, *op. cit.* p. 386.

Since $\mathscr{X}-X \subset G$, we have $\mathscr{X}-G \subset X$. Therefore

$$C_G \subset C, \quad \text{hence} \quad C \cap \bar{G} \neq 0.$$

Since the family $\boldsymbol{A}$ is directed relative to the inclusion $\supset$, i.e. to each pair G_1 and G_2 there is G_3 such that $G_3 \subset G_1$ and $G_3 \subset G_2$ (namely $G_3 = G_1 \cap G_2$), so is the family of all sets $\bar{C} \cap \bar{G}$ where $G \in \boldsymbol{A}$. Since the sets $\bar{C} \cap \bar{G}$ are compact and non-empty, it follows (by § 41, I (4)) that their intersection F is non-empty. Obviously the intersection of all $\bar{G}$ where $G \in \boldsymbol{A}$ is equal to $\overline{\mathscr{X}-X}$, hence $0 \neq F = \bar{C} \cap \overline{\mathscr{X}-X}$.

THEOREM 3. *If A is a closed proper subset of a continuum, every component of A contains at least one component of the boundary of A.*

Therefore, *the power of the family of components of A does not exceed the power of the family of components of* $\mathrm{Fr}(A)$.

Proof. If C is a component of A, it follows by Theorem 1 that $C \cap \mathrm{Fr}(A) \neq 0$. Let D be a component of $\mathrm{Fr}(A)$ such that $C \cap D \neq 0$. Since $D \subset \mathrm{Fr}(A) \subset A$, it follows that $D \subset C$.

THEOREM 4. *Let $\mathscr{X}$ and K be continua and G an open set such that $K \subset G$ and $\bar{G} \neq \mathscr{X}$. Then there is a continuum C such that $K \subset C \subset \bar{G}$ and $K \neq C$.*

In particular, *if $\mathscr{X}$ is metric (and contains more than one point), each point of $\mathscr{X}$ belongs to a continuum (containing more than one point) with an arbitrarily small diameter.*

Proof. Replace A by $\bar{G}$ in Theorem 1 and denote by C the component of $\bar{G}$ containing K.

The next theorem follows immediately from Theorem 4.

THEOREM 5 (1). *If K is a proper subcontinuum of the continuum $\mathscr{X}$, there exists a continuum C such that*

$$K \subset C \quad \text{and} \quad K \neq C \neq \mathscr{X}.$$

THEOREM 6 (of Sierpiński) (2). *No continuum can be decomposed into a countable family of non-empty, disjoint closed sets.*

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(1) See also Section VII and § 48, VI, Theorem 1.

(2) Tôhoku Math. Journ. 13 (1918), p. 300.

Proof. Suppose conversely that the space $\mathscr{X}$ is a continuum such that

$$\mathscr{X} = \bigcup_n A_n,$$

where A_n are closed and disjoint and at least two of them are non-empty. We are going to define a sequence of continua $C_1, C_2, \ldots$ such that

$$C_1 \supset C_2 \supset \ldots, \qquad C_n \cap A_n = 0 \quad \text{and} \quad C_n \neq 0;$$

which will imply a contradiction since

$$\bigcap_n C_n \cap \bigcup_n A_n = 0, \quad \text{which implies} \quad \bigcap_n C_n = 0$$

contrary to the Cantor theorem.

Our task is reduced to show that there exists a continuum C such that $C \cap A_1 = 0$ and such that at least two of the terms $C \cap A_2$, $C \cap A_3, \ldots$ are non-empty (C will be denoted by C_1 and, in order to define C_2, C_1 will be considered as the whole space, etc.).

Now, it can obviously be assumed that $A_1 \neq 0$ (since otherwise it could be assumed that $C = \mathscr{X}$). Let $A_m \neq 0$ with $m \neq 1$. Let F be a closed neighbourhood of A_m (i.e. $A_m \cap \overline{\mathscr{X} - F} = 0$) such that $F \cap A_1 = 0$ and let C be a component of F such that $C \cap A_m \neq 0$. Since $F \cap A_1 = 0$, it follows that $C \cap A_1 = 0$. On the other hand, it follows by Theorem 1 that

$$C \cap \overline{\mathscr{X} - F} \neq 0, \quad \text{hence} \quad C \not\subset A_m, \quad \text{i.e.} \quad C - A_m \neq 0,$$

and since

$$C - A_m \subset (C \cap A_2) \cup \ldots \cup (C \cap A_{m-1}) \cup (C \cap A_{m+1}) \cup \ldots,$$

there exists an index $n \neq m$ such that $C \cap A_n \neq 0$.

Remarks. (i) Theorem 6 implies the following

THEOREM 6a. *If a compact space admits a decomposition in (non-empty) disjoint continua $C_1, C_2, \ldots$, each C_n is a component of the space.*

Proof. Because, otherwise, there would exist a continuum K such that $K \supset C_n$ and $K \neq C_n$, and the series

$$K = (K \cap C_1) \cup (K \cap C_2) \cup \ldots$$

would contain at least two non-empty terms.

(ii) In Theorem 6 *the hypothesis of compactness is essential.* An example will be given of a connected *locally compact* space E which admits a decomposition into a series of disjoint, closed connected sets.

E is the union

1st of segments $x = 2^{-n}$, $0 \leqslant y \leqslant 1$, $n = 0, 1, 2, \ldots$,

2nd of the segment $x = 0$, $0 < y \leqslant 1$, where the points with ordinates $3/2^n$ are deleted,

3rd of the arcs $\varrho = 2^{-n}$, $\pi/2 \leqslant \theta \leqslant 2\pi$ (in polar coordinates). (1)

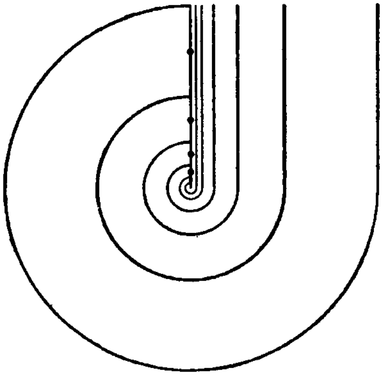

Fig. 2

The set E can easily be mapped by a homeomorphism onto a closed connected set lying in the Euclidean 3-*dimensional* space. However, *in the plane* there is no closed connected set which admits a decomposition into a series of disjoint, closed *connected* sets (2). On the other hand, a closed connected set can be constructed in the plane, which admits a decomposition into a series of disjoint closed (but not connected) sets (3).

(iii) *Applications to accessibility.* A point p is said to be *accessible* from the set A if there exists a continuum C such that

$$p \in C \subset A \cup p \quad \text{and} \quad C \neq p. \tag{1}$$

So, e.g., if E is the curve $y = \sin(1/x)$, $0 < |x| \leqslant 1$, augmented by the segment $|y| \leqslant 1$, $x = 0$, then only the end-points of this segment are accessible from $\mathcal{E}^2 - E$ and all other points of this segment are inaccessible.

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(1) See W. Sierpiński, Fund. Math. 4 (1923), p. 5, and Wiad. Mat. 23 (1919), p. 188. Compare also the paper of B. Knaster and myself, Fund. Math. 5 (1924), p. 58.

(2) S. Mazurkiewicz, *Sur les continus plans non bornés*, Fund. Math. 5 (1924), p. 188, and R. L. Moore, *Concerning the sum of a countable number of mutually exclusive continua in the plane*, *ibid.* 6 (1924), p. 189.

(3) S. Mazurkiewicz, *loc. cit.*

THEOREM 7 (of Urysohn) (¹). *If E is an F_σ-set in a compact metric space $\mathcal{X}$, the set E_a of points which are accessible from $\mathcal{X}-E$ is analytic.*

Proof. Because (compare § 41, IV, Corollary 1b)

$$(x\in E_a)\equiv(x\in E)\bigvee_C\{(C\neq x)\bigwedge_y[(y\in C)(y\neq x)\Rightarrow(y\in\mathcal{X}-E)]\},$$

where C ranges over the compact space (compare II, Theorem 4) of subcontinua of the space $\mathcal{X}$.

Remark. Examples of closed sets lying in $\mathcal{E}^3$ show that E_a can be a non-Borel set (²).

If E is a compact subset of the *plane*, E_a is a Borel set (³).

IV. Separation of compact metric spaces. Theorem II, 3 implies the following

THEOREM 1. *If $\mathcal{X}$ is a compact space, then C is its separator if and only if there exists a component Q of $\mathcal{X}$ containing two points between which $\mathcal{X}-C$ is not connected.*

In this case, $C\cap Q$ is clearly a separator of Q.

However, a separator of a component of $\mathcal{X}$ is not necessarily a separator of $\mathcal{X}$.

Theorem 1 of § 46, VII can be strengthened in the following way.

THEOREM 2. *In every compact space $\mathcal{X}$, there exists a sequence of points $p_1, p_2, \ldots$ such that each closed separator F is a separator between a pair (p_i, p_j), between which $\mathcal{X}$ is connected.*

More precisely, *if $\{Q_n\}$ is a family of components, which is dense in the family of all components of $\mathcal{X}$, and if $P=\{p_n\}$ is a countable set of points such that $\overline{P\cap Q_n}=Q_n$ for $n=1,2,\ldots$, then to every closed separator F correspond three indices i, j, k such that $p_i, p_j\in Q_k$ and F separates the space between p_i and p_j.*

(¹) *Sur les points accessibles des ensembles fermés*, Proc. Akad. Amsterdam 28 (1925), p. 984. For the proof, see my paper in Fund. Math. 17 (1931), p. 263. For an analogous theorem concerning the rectilinear accessibility, compare § 38, VIII.

(²) P. Urysohn, *ibid.* and O. Nikodym, C. R. Soc. Sc. de Varsovie 19 (1926), p. 285.

(³) S. Mazurkiewicz, Fund. Math. 26 (1936), p. 153.

Proof. According to the hypothesis there exist two open sets M and N and a component C of $\mathcal{X}$ such that

$$\mathcal{X} - F = M \cup N, \quad M \cap N = 0, \tag{1}$$

$$C \cap M \neq 0 \neq C \cap N. \tag{2}$$

Since the set C is a limit of a sequence contained in $\{Q_n\}$, there exists an index k such that

$$Q_k \cap M \neq 0 \neq Q_k \cap N.$$

Finally, condition $\overline{P \cap Q_k} = Q_k$ implies the existence of two indices i and j such that $p_i \epsilon Q_k \cap M$ and $p_j \epsilon Q_k \cap N$. By (1), F is a separator of the space between p_i and p_j.

THEOREM 3 (of G. T. Whyburn) (1). *If $\mathcal{X}$ is a continuum, the set $S(a, b)$ of all points which separate $\mathcal{X}$ between a and b is a $\boldsymbol{G}_\delta$ augmented by a countable set.*

Proof. According to § 46, IX, Theorem 5 the set $S(a, b)$, after removal of a countable set, coincides with the set of points, at which a continuous transformation of $\mathcal{X}$ is one-to-one, and this set is a $\boldsymbol{G}_\delta$ (compare Corollary 1c of § 41, III).

THEOREM 4. *The set of all points which separate $\mathcal{X}$ is a $\boldsymbol{G}_{\delta\sigma}$.*

Proof. This is a consequence of Theorems 2 and 3.

Remark. If $\mathcal{X}$ is not compact, $S(a, b)$ can be a non-Borel set. Let $\mathcal{X}$ be a subset of the square $\mathcal{I}^2$ consisting of the base of the square and of vertical segments with abscissae belonging to a set D which is dense in $\mathcal{I}$.

Let $a = (0, 0)$ and $b = (1, 0)$. It follows that $S(a, b) = \mathcal{I} - D$.

THEOREM 5 (of R. L. Moore) (2). *In every continuum $\mathcal{X}$ (which contains more than one point), there exist at least two points which do not separate it.*

Proof. We must prove that for each point p there exists a point $q \neq p$ which does not separate $\mathcal{X}$. Let $p_0, p_1, p_2, \ldots$ be a sequence

(1) Trans. Amer. Math. Soc. 32 (1930), p. 151.

(2) Trans. Amer. Math. Soc. 21 (1920), p. 340, theorem 2 and Proc. Nat. Ac. Sc. 9 (1923), p. 101. For the generalizations see Section VI. Compare also S. Mazurkiewicz, Fund. Math. 2 (1921), p. 119, H. M. Gehman, Proc. Nat. Ac. Sc. 14 (1928), p. 433, R. H. Bing, Amer. Journ. of Math. 70 (1948), p. 501, and my paper in Fund. Math. 5 (1924), p. 113.

dense in $\mathscr{X}$, where $p_0 = p$ and $p_i \neq p_j$ for $i \neq j$. It can be assumed that p_n does not separate $\mathscr{X}$ for $n > 0$.

Let $i_0 = 0, i_1, i_2, \ldots$ be a sequence of indices and let $A_0 = \mathscr{X}$, $A_1, A_2, \ldots$ be a sequence of open sets defined by the following conditions.

(i) i_n is the least index $> i_{n-1}$ such that $p_{i_n} \in A_{n-1}$,

(ii) A_n is an open set such that

$$\bar{A}_n = A_n \cup p_{i_n} \quad \text{and} \quad p_{i_{n-1}} \in \mathscr{X} - A_n$$

(the existence of A_n follows from the fact that $\mathscr{X} - p_{i_n}$ can be decomposed into two non-empty, disjoint open sets; this one which does not contain $p_{i_{n-1}}$ is denoted by A_n).

Since A_n is a continuum (compare § 46, II, Theorem 4), the conditions

$$p_{i_n} \in \bar{A}_n \cap A_{n-1} \quad \text{and} \quad \mathrm{Fr}(A_{n-1}) = p_{i_{n-1}},$$
$$\text{hence} \quad \bar{A}_n \cap \mathrm{Fr}(A_{n-1}) = 0 \tag{1}$$

imply that $\bar{A}_n \subset A_{n-1}$ (compare § 46, I, Theorem 1).

Therefore $A_1 \cap A_2 \cap \ldots \neq 0$ (compare § 41, I, Remark 4). Let

$$q \in (A_1 \cap A_2 \cap \ldots), \text{ which yields } q \neq p_{i_n} \text{ for } n = 0, 1 \ldots. \tag{2}$$

Let M and N be two open sets such that

$$\mathscr{X} - q = M \cup N \quad \text{and} \quad M \cap N = 0. \tag{3}$$

We have to show that one of them is empy. Suppose that there exist infinitely many indices i_n such that $p_{i_n} \in N$, hence that

$$(M \cup q) \cap \mathrm{Fr}(A_n) = 0 \tag{4}$$

according to (1), (2) and (3).

Since the set $M \cup q$ is a continuum (compare § 46, II, Theorem 4), the condition $(M \cup q) \cap A_n \neq 0$ (compare (2) and (4)) implies by § 46, I, Theorem 1

$$M \cup q \subset A_n, \quad \text{hence} \quad M \cup q \subset A_1 \cap A_2 \cap \ldots. \tag{5}$$

If $M \neq 0$, there exists a point $p_k \in M$; but then, if n satisfies conditions $i_n > k \geqslant i_{n-1}$, it follows that $p_k \in M \subset A_n$ (according to (5)) which implies that $k \neq i_{n-1}$, $p_k \in A_{n-1}$ and i_n is not the least index $> i_{n-1}$ satisfying condition $p_{i_n} \in A_{n-1}$. But this contradicts condition (i). Hence $M = 0$.

V. Arcs. Simple closed curves. A space homeomorphic to the interval $\mathcal{I}$ is said to be an *arc*. A space homeomorphic to the circle $x^2+y^2=1$ is said to be a *simple closed curve.*

Every interval contains exactly two points which do not separate it, and so does each arc. These two points are called the *end-points* of the arc (an "arc ab" is an arc with the end-points a and b).

THEOREM 1. *If every point x of a metric continuum $\mathcal{X}$, with exception of two points a and b, is a separator, then $\mathcal{X}$ is an arc* (1).

Proof. Let $a \neq x \neq b$ and $\mathcal{X}-x = M \cup N$, where M and N are two non-empty disjoint open sets. Let $a \in M$. It follows that $b \in N$ since, otherwise, if y denotes (according to Theorem IV, 5) a point of N which does not separate the continuum $N \cup x$, the set

$$\mathcal{X}-y = (N \cup x)-y \cup (M \cup x)$$

would be connected contrary to the hypothesis.

Since each x is a separating point between a and b, the interval $\mathcal{I}$ is a continuous one-to-one image of $\mathcal{X}$ according to Theorem 6 of § 46, IX. And hence, as a compact space, $\mathcal{X}$ is homeomorphic to the interval (compare Theorem 3 of § 41, III).

THEOREM 1′. *If a metric continuum $\mathcal{X}$ contains two points a and b such that to every point x correspond two closed sets A and B satisfying the conditions*

$$\mathcal{X} = A \cup B, \quad a \in A, \quad b \in B \quad \textit{and} \quad A \cap B = x,$$

then $\mathcal{X}$ is an arc (2).

Because each point $x \in \mathcal{X}-a-b$ is a separator.

Remark. The conditions given in Theorems 1 and 1′ are not only sufficient, but also necessary in order that a metric continuum be an arc. In general, since all arcs have the same topological type, it follows that every topological property, which is sufficient

(1) Compare the footnote to the theorem of Lennes (§ 46, IX, Theorem 6). See also R. L. Moore, *Concerning simple continuous curves,* Trans. Amer. Math. Soc. 21 (1920), p. 340.

(2) Compare W. Sierpiński, *L'arc simple comme un ensemble de points dans l'espace à m dimensions,* Ann. di Mat. 26 (1916), p. 131, and *Le continu linéaire comme un ensemble abstrait,* Prace Mat.-Fiz. 27 (1916), p. 203; S. Straszewicz, *Über den Begriff des einfachen Kurvenbogens,* Math. Ann. 78 (1918), p. 369.

for a space to be an arc (and which holds for at least one space) is also a necessary condition.

THEOREM 2 (of R. L. Moore) (1). *If every pair of points separates the metric continuum $\mathscr{X}$, then $\mathscr{X}$ is a simple closed curve.*

Proof. First, we claim that for each point a the set $\mathscr{X}-a$ is connected. For otherwise there would exist two continua P and Q such that

$$\mathscr{X} = P \cup Q \quad \text{and} \quad P \cap Q = a.$$

Therefore there would exist, according to Theorem 5 of Section IV, two points $p \in P$, $q \in Q$, $p \neq a \neq q$, such that the sets $P-p$ and $Q-q$ would be connected. But then the set

$$\mathscr{X}-p-q = (P-p) \cup (Q-q)$$

would be connected (because $a \in (P-p) \cap (Q-q)$), contrary to hypothesis.

By hypothesis every point $x \neq a$ is a separator of the set $\mathscr{X}-a$. According to theorem 4 of § 46, VIII, there exists among these points x a point b which separates the set $X-a$ into two non-empty connected separated sets M and N. Therefore

$$\mathscr{X}-a-b = M \cup N.$$

It follows that $\overline{M} = M \cup a \cup b$, since the assumption that $a \notin \overline{M}$ would imply a decomposition of the (connected) set $\mathscr{X}-b$ into two separated sets M and $N \cup a$. In a similar way, $\overline{N} = N \cup \cup a \cup b$.

We have to prove that $\overline{M}$ and $\overline{N}$ are two arcs ab.

Suppose that, e.g., $\overline{M}$ is not an arc ab. Hence there exists, by Theorem 1, a point $x \in M$ such that $\overline{M}-x$ is connected. There are two cases to be considered according to whether $\overline{N}$ is or is not an arc ab.

If $\overline{N}$ is not ab, there exists a point $y \in N$ such that $\overline{N}-y$ is connected. But in that case the set

$$\mathscr{X}-x-y = (\overline{M}-x) \cup (\overline{N}-y)$$

is connected contrary to the hypothesis.

(1) *Op. cit.*, p. 342. Compare also R. H. Bing, *op. cit.*, p. 505.

If $\overline{N}$ is an arc, every point $y \in N$ determines a decomposition of $\overline{N}$ into two connected sets $ay-y$ and $by-y$, which leads to the same contradiction, since the set

$$\mathcal{X}-x-y = (\overline{M}-x) \cup (ay-y) \cup (by-y)$$

is connected.

The following two statements are immediate consequences of Theorem 2.

THEOREM 2'. *If to every pair of points a, b of a metric continuum $\mathcal{X}$ correspond two closed sets A and B such that*

$$\mathcal{X} = A \cup B, \quad A \cap B = (a, b) \quad \textit{and} \quad A \neq \mathcal{X} \neq B,$$

then $\mathcal{X}$ is a simple closed curve (¹).

THEOREM 2''. *If no connected subset separates the metric continuum $\mathcal{X}$, then $\mathcal{X}$ is a simple closed curve* (²).

THEOREM 3. *In a compact metric space, let C_n, $n = 1, 2, \ldots$, be connected sets containing a and b, and such that to every $x \in \bigcap_n C_n$ corresponds a decomposition $C_n = A_n \cup B_n$ which satisfies the following conditions*

(i) $a \in A_n, \; b \in B_n, \; x \in A_n \cap B_n$,

(ii) $\lim_{n=\infty} \delta(A_n \cap B_n) = 0$,

(iii) $\overline{A}_{n+1} \subset A_n, \; \overline{B}_{n+1} \subset B_n$,

then the intersection $\bigcap_n C_n$ is an arc ab.

In a more general way, *instead of assuming that the space is compact, it can be supposed that it is complete and that* $\lim_{n=\infty} \alpha(C_n) = 0$.

Proof. Since $\overline{C}_{n+1} = \overline{A}_{n+1} \cup \overline{B}_{n+1} \subset A_n \cup B_n = C_n$, the intersection $\bigcap_n \overline{C}_n$ is a continuum (compare II, Remark to Theorem 5) and

$$\bigcap_n \overline{C}_{n+1} \subset \bigcap_n C_n \quad \text{implies that} \quad \bigcap_n C_n = \bigcap_n \overline{C}_n.$$

(¹) A more restrictive condition, which is obtained from Theorem 2' by supposing that A and B are continua, is due to S. Janiszewski, Thesis, Journ. Éc. Polyt. 2s., 16 (1912), p. 137.

(²) Compare J. R. Kline, Fund. Math. 5 (1924), p. 3.

Let (for a fixed x)

$$A = \bigcap_n A_n, \quad B = \bigcap_n B_n.$$

We have to prove that A and B satisfy Theorem 1' (if $\mathcal{X} = \bigcap_n C_n$). But since $\bar{A}_{n+1} \subset A_n \subset \bar{A}_n$, it follows that $A = \bigcap_n \bar{A}_n$, which shows that A is closed. B is closed by an analogous argument. The inclusions $A_{n+1} \subset A_n$ and $B_{n+1} \subset B_n$ imply that

$$\bigcap_n C_n = \bigcap_n (A_n \cup B_n) = \bigcap_n A_n \cup \bigcap_n B_n = A \cup B.$$

Finally, $x \in \bigcap_n (A_n \cap B_n) = A \cap B$, and since $\delta(A \cap B) = 0$, it follows that $A \cap B = x$.

VI. Decompositions of compact spaces into continua.

THEOREM 1 (1). *The decomposition of a compact space $\mathcal{X}$ into components is upper semi-continuous.*

Proof. By Theorem 3 of § 46, V, there is a continuous mapping f of $\mathcal{X}$ into a generalized Cantor discontinuum $D^{\mathfrak{m}}$ such that the quasi-components of $\mathcal{X}$ (hence its components, see II, Theorem 2) coincide with the point inverses of f. Since $\mathcal{X}$ is compact, its decomposition in point inverses of a continuous mapping is upper semi-continuous (by Theorem 3 of § 41, III).

Theorem 1 implies the following two corollaries.

COROLLARY 2. *In a decomposition of a compact space into a countable family $C_1, C_2, \ldots$ of non-empty disjoint continua, there exists at least one C_n which is open* (2).

Proof. According to III, Theorem 6a, the continua C_k are the components of the space and by Theorem 1, the function f considered above maps the space onto a countable closed set; let y be an isolated point of that set, then $f^{-1}(y)$ is the required open continuum.

COROLLARY 3. *Every closed set F, which is the union of a family of components of a compact metric space, is the intersection of a sequence of closed-open sets.*

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(1) See L. E. J. Brouwer, Proc. K. Akad. Wet., Amsterdam 12 (1910), p. 785. See also N. B. Vedenisov, Compos. Math. 7 (1939), p. 194.

(2) R. L. Moore, *An extension of the theorem that no countable point set is perfect*, Proc. Nat. Acad. Sc. 10 (1924), p. 168.

Proof. Let $f\colon \mathcal{X} \to \mathcal{C}$ be the mapping considered above and put $A = f(F)$. It follows that

$$A = \bar{A}, \quad F = f^{-1}(A) \quad \text{and} \quad \dim A = 0.$$

Hence there exists (compare § 26, I, Corollary 1b) a sequence $\{G_n\}$ of closed-open sets in $\mathcal{C}$ such that $A = G_1 \cap G_2 \cap \ldots$ Therefore, $f^{-1}(G_n)$ is closed-open and $F = f^{-1}(A) = f^{-1}(G_1) \cap f^{-1}(G_2) \cap \ldots$

Theorem 5 of Section IV admits the following generalization.

THEOREM 4. *If C is a proper connected subset of a metric continuum $\mathcal{X}$, there exists in $\mathcal{X} - C$ a point which does not separate $\mathcal{X}$* (1).

Proof. Since the theorem is obvious when $\bar{C} = \mathcal{X}$ (compare § 46, II, Corollary 3 (ii)), assume that $\bar{C} \neq \mathcal{X}$. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous mapping of $\mathcal{X}$ onto $\mathcal{Y}$ such that the set $f(\bar{C})$ is reduced to a single point y_0 and that, for $y \neq y_0$, the set $f^{-1}(y)$ consists of one point of $\mathcal{X} - \bar{C}$ (compare § 22, IV, Theorem 1). According to IV, Theorem 5, there exists a point $y_1 \neq y_0$ such that $\mathcal{Y} - y_1$ is connected. And hence the set $f^{-1}(\mathcal{Y} - y) = \mathcal{X} - f^{-1}(y_1)$ is also connected.

Let us quote the following generalization of Theorem 5 of Section IV.

THEOREM 5. *Let the metric continuum $\mathcal{X}$ be the union of (two at least) connected sets C_t; then there exist two indices $t_1 \neq t_2$ such that the sets $\bigcup_{t \neq t_1} C_t$ and $\bigcup_{t \neq t_2} C_t$ are connected* (2).

We shall use the following theorem in the proof of Theorem 7 (3).

THEOREM 6. *If $\boldsymbol{D}$ is a semi-continuous decomposition of a compact metric space $\mathcal{X}$, then the family $\boldsymbol{F}$ of components of members of $\boldsymbol{D}$ is itself a semi-continuous decomposition.*

Proof. Let $F_0, F_1, \ldots$ be a sequence of sets, which are elements of $\boldsymbol{F}$, such that

$$F_0 \cap \mathrm{Li}\, F_n \neq 0. \tag{1}$$

(1) Theorem of H. M. Gehman, *Concerning irreducible continua*, Proc. Nat. Ac. Sc. 14 (1928), p. 435.

(2) S. Eilenberg, Fund. Math. 22 (1934), p. 297. See also A. Lelek, *Remarks on Brouwer reduction theorem*, Prace Mat. 7 (1962), p. 108.

(3) For Theorems 6 and 7, see S. Eilenberg, Fund. Math. 22 (1934), p. 292, G. T. Whyburn, Amer. Journ. Math. 56 (1934), p. 294, and J. Hocking–G. S. Young, p. 137, where these theorems are shown without the assumption of metrizability.

We must show that

$$\operatorname{Ls} F_n \subset F_0. \tag{2}$$

Let D_n $(n \geqslant 0)$ be the member of the decomposition $\boldsymbol{D}$ whose component is F_n. According to (1), $D_0 \cap \operatorname{Li} D_n \neq 0$, which implies $\operatorname{Ls} D_n \subset D_0$ since the decomposition $\boldsymbol{D}$ is semi-continuous.

Therefore $\operatorname{Ls} F_n \subset D_0$. According to (1) and to Theorem 6 of Section II, $\operatorname{Ls} F_n$ is a continuum. But since F_0 is a component of the set D_0 and since $\operatorname{Ls} F_n$ is a subcontinuum of D_0 which has common points with F_0 (by (1)), inclusion (2) follows.

THEOREM 7. *Every continuous mapping f of a compact metric space $\mathscr{X}$ can be represented as a composition of two continuous mappings h and g:*

$$f(x) = gh(x), \qquad x \in \mathscr{X},$$

where h is monotone and g has 0*-dimensional point inverses.*

More precisely, *if h is* (according to Theorem 6) *a continuous mapping, whose point inverses are the components of point inverses of the mapping f, the mapping g defined by the identity*

$$g(y) = f[h^{-1}(y)] \quad \textit{where} \quad y \in h(\mathscr{X}), \tag{3}$$

is continuous and

$$\dim g^{-1}(z) = 0 \quad \textit{for} \quad z \in f(\mathscr{X}). \tag{4}$$

Proof. Notice first that the right term of identity (3) consists of one single element, because the set $h^{-1}(y)$ is contained in one single point inverse of the mapping f.

The mapping g is continuous since (compare § 41, IV, Theorem 2) the set

$$\underset{yz}{E}[z = g(y)] = \underset{yz}{E} \bigvee_x [z = f(x)][x \in h^{-1}(y)]$$
$$= \underset{yz}{E} \bigvee_x [z = f(x)][y = h(x)]$$

s closed as a projection of a closed set.

In order to establish (4) consider a continuum C such that

$$C \subset g^{-1}(z). \tag{5}$$

We have to show that C consists only of one point.

It follows by (3) and (5) that

$$h^{-1}(C) \subset f^{-1}f[h^{-1}(C)] = f^{-1}g(C) \subset f^{-1}g[g^{-1}(z)] = f^{-1}(z).$$

Since the function h is monotone (by Theorem 9 of § 46, I), the set $h^{-1}(C)$ is a continuum and hence is a subcontinuum of a component Q in the set $f^{-1}(z)$. Because the mapping h is constant on Q, and therefore on $h^{-1}(C)$, the set $C = h[h^{-1}(C)]$ consists of only one point.

THEOREM 8. *Let G be an open subset of a continuum $\mathscr{X}$ and p an isolated point of* $\mathrm{Fr}(G)$. *Then p is accessible from G.*

Proof. Denote by C the component of p in $\bar{G}$ and put $F = \mathrm{Fr}(G) - (p)$. Since p is isolated in $\mathrm{Fr}(G)$, so F is closed.

Consider first the case where $C \cap F \neq 0$. According to Theorem 4 of Section III (where we put $\mathscr{X} = C$, $K = (p)$ and $\bar{G} \cap F = 0$), it follows that there exists a continuum C_0 such that $C_0 \neq (p)$ and $p \in C_0 \subset C - F$ and hence $C_0 - (p) \subset G$. Thus, in this case, p is accessible from G.

We have now to consider the case where $C \cap F = 0$, i.e. where $C \cap \mathrm{Fr}(G) = (p)$. It remains to show that $C \cap G \neq 0$ (i.e. that C does not reduce to the point p). Denote by U the union of all components of $\bar{G}$ which have points in common with F. Since F is closed and the decomposition of $\bar{G}$ in components is (by Theorem 1) upper semi-continuous, so U is closed. Since each component of $\bar{G}$ has points in common with $\mathrm{Fr}(\bar{G})$, hence with $\mathrm{Fr}(G)$ (since $\mathrm{Fr}(\bar{G}) \subset \mathrm{Fr}(G)$), we have $C \cup U = \bar{G}$.

It follows that $C \cap G \neq 0$, for otherwise $G \subset U$, hence $\bar{G} \subset U$ and consequently $C \subset U$, which means that $C \cap F \neq 0$, contrary to our assumption.

This completes the proof.

VII. The space $2^{\mathscr{X}}$. According to Theorem 14 of § 46, II, the family $\boldsymbol{C}$ of all subcontinua of a compact space $\mathscr{X}$ is closed in the space $2^{\mathscr{X}}$. A similar result follows.

THEOREM 1. *If A and B are two subsets of a compact space $\mathscr{X}$, the family of the sets $X \in 2^{\mathscr{X}}$ connected between A and B is closed.*

Proof. A set X is not connected between A and B if and only if there exists an open set G such that (compare § 46, IV, Theorem 7)

$$A \subset G, \quad B \cap \bar{G} = 0 \quad \text{and} \quad X \cap \mathrm{Fr}(G) = 0.$$

Since the family of the sets X, which satisfy these conditions for a given G, is open (compare Theorem 1 of § 17, II), the required conclusion follows immediately.

THEOREM 2. *Let $\boldsymbol{F}$ be a monotone family of closed sets in a compact metric space $\mathscr{X}$. If $\boldsymbol{F}$ is a continuum (in $2^{\mathscr{X}}$), then $\boldsymbol{F}$ is an arc.*

Proof. If E is an element of $\boldsymbol{F}$, let $\boldsymbol{M}$ and $\boldsymbol{N}$ be subfamilies of $\boldsymbol{F}$ consisting of sets X such that

$$X \subset E \neq X \quad \text{respectively} \quad E \subset X \neq E.$$

Supposing that E is neither the first, nor the last element of $\boldsymbol{F}$, we have $\boldsymbol{M} \neq 0 \neq \boldsymbol{N}$. Moreover, $\boldsymbol{M}$ and $\boldsymbol{N}$ are separated, because the condition $L = \operatorname*{Lim}_{n=\infty} X_n$ implies that either $L \subset E$ or $E \subset L$ according to whether $X_n \in \boldsymbol{M}$ or $X_n \in \boldsymbol{N}$ for all n.

Thus it is easily seen that with exception of the first and the last element, every element of $\boldsymbol{F}$ is a separating point. And hence $\boldsymbol{F}$ is an arc by Theorem 1 of Section V.

THEOREM 3. *If A and C are two (non-empty) metric continua such that $A \subset C \neq A$, there exists a monotone family of continua which form, in the space 2^C, an arc with end-points A and C* (1).

Proof. We shall show first that if $0 < \varepsilon \leqslant \operatorname{dist}(A, C)$, there exists a continuum A_ε such that

$$A \subset A_\varepsilon \subset C \quad \text{and} \quad \operatorname{dist}(A, A_\varepsilon) = \varepsilon.$$

Let S be the set of points x such that $\varrho(x, A) \leqslant \varepsilon$ and $x \in C$, and let A_ε be the component of S which contains A; it follows that $A_\varepsilon = C$ in case where $\varepsilon = \operatorname{dist}(A, C)$, and that $A_\varepsilon \cap \overline{(X - S)} \neq 0$ otherwise (compare III, Theorem 1); but for $p \in A_\varepsilon \cap \overline{(X - S)}$ it follows that $\varrho(p, A) = \varepsilon$, hence $\operatorname{dist}(A, A_\varepsilon) = \varepsilon$.

In the case where $\operatorname{dist}(A_\varepsilon, C) > \varepsilon$, one proceeds in a similar way: the set $A_{\varepsilon,\varepsilon}$ is defined replacing A by A_ε in the definition of A, next $A_{\varepsilon,\varepsilon,\varepsilon}$ is defined, and so on.

After a finite number of steps a continuum is necessarily obtained, the distance of which from C is $\leqslant \varepsilon$; because, otherwise, there would exist an infinite sequence of points of the space 2^C, the mutual distances of which would be $\geqslant \varepsilon$ (but this is impossible since 2^C is compact).

---

(1) This theorem and the next corollary are due to K. Borsuk and S. Mazurkiewicz; see *Sur l'hyperespace d'un continu*, C. R. Soc. Sc. Varsovie 24 (1931), p. 149.

Thus to every $\varepsilon > 0$ corresponds a finite system of continua

$$B_0 = A \subset B_1 \subset B_2 \subset \ldots \subset B_n = C,$$

where $\operatorname{dist}(B_{i-1}, B_i) \leqslant \varepsilon$ for $i = 1, \ldots, n$. So, there exists a monotone (countable) family $\boldsymbol{D}$ of continua such that $A \epsilon \boldsymbol{D}$, $C \epsilon \boldsymbol{D}$ and for every pair $D_1, D_2 \epsilon \boldsymbol{D}$ and for each $\varepsilon > 0$, there exists in $\boldsymbol{D}$ a system $B_0, B_1, \ldots, B_n$ such that

$$B_0 = D_1, \quad B_n = D_2 \quad \text{and} \quad \operatorname{dist}(B_{i-1}, B_i) \leqslant \varepsilon.$$

Therefore, the family $\overline{\boldsymbol{D}}$ is a continuum (in 2^C, compare Section I) and its elements are continua (in C). Hence it is an arc by Theorem 2 since the family is monotone.

COROLLARY 4. *If C is a metric continuum, every pair A, B of elements belonging to 2^C can be joined by an arc in 2^C.* (¹)

More precisely, *if $A \epsilon 2^C$ and $A \neq C$, there exists a monotone family of closed subsets of C which constitute an arc with end-points A and C in the space 2^C.*

Proof. Let $p \epsilon A$. In the space 2^C there exists an arc $\boldsymbol{A}$ with end-points p and C (by Theorem 3). The family $\boldsymbol{B}$ of all sets $A \cup X$, where $X \epsilon \boldsymbol{A}$, is monotone, contains A and C and, as a continuous image of $\boldsymbol{A}$ (compare § 17, III, Corollary 4a), is a continuum and hence is an arc (by Theorem 2).

In order to derive the first part of the theorem from the second one, two arcs are considered, the arc $\boldsymbol{A}_0$ joining A to C and the arc $\boldsymbol{A}_1$ joining B to C. Let E be the first element of the arc $\boldsymbol{A}_1$ (oriented from B to C) which belongs to $\boldsymbol{A}_0$; then the subarcs $\boldsymbol{B}_0$ of $\boldsymbol{A}_0$ and $\boldsymbol{B}_1$ of $\boldsymbol{A}_1$ with end-points A, E and B, E, respectively, constitute an arc which joins A to B.

Remark. Corollary 4 admits the following generalization. *If C is a metric continuum, the space 2^C is a continuous image of the continuum, obtained by joining the point $(\frac{1}{2}, \frac{1}{2})$ to every point of the Cantor set $\mathscr{C}$* (compare § 46, II, Remark). (²)

Let us mention that $\mathscr{I}^{\aleph_0} \underset{\text{top}}{\subset} 2^C$ (³).

(¹) Compare also R. J. Koch, *Arcs in partially ordered spaces*, Pacific Journ. Math. 9 (1959), pp. 273–728.

(²) S. Mazurkiewicz, Fund. Math. 18 (1932), p. 171.

(³) S. Mazurkiewicz, *Sur le type de dimension de l'hyperspace d'un continu*, C. R. Soc. Sc. Varsovie 24 (1931), p. 191.

VIII. Semi-continua. Cuts of the space.

DEFINITION 1. A space, whose every pair of points can be joined by a continuum, is said to be a *semi-continuum.*

The union of all continua containing a given point p is said to be the *constituant* of that point. Theorefore the constituant of p is the largest semi-continuum containing p. It is clear that two distinct semi-continua are always disjoint.

THEOREM 1. *Every semi-continuum is connected.*

So, the constituant of p is contained in the component of p; therefore the decomposition of the space into constituants is a refinement of the decomposition into components.

THEOREM 2. *Every locally compact (but non-compact) metric semi-continuum $\mathscr{X}$ is the union of an increasing (countable) sequence of continua.*

Proof. Since every locally compact space is homeomorphic to a compact space with one point deleted (see § 41, X, Theorem 5) we have to show that, if C is a continuum and p is a point of C such that $C-p$ is a semi-continuum (homeomorphic to $\mathscr{X}$), there exists a sequence of continua $K_1, K_2, \ldots$ such that

$$C-p = K_1 \cup K_2 \cup \ldots \quad \text{and} \quad K_1 \subset K_2 \subset \ldots. \tag{1}$$

Let $q \in C-p$, let S_n be the open ball with the center p and the diameter $\frac{1}{n}|q-p|$, and K_n the component of q in $C-S_n$. Since the set $C-p$ is a semi-continuum, to every $x \in C-p$ corresponds a continuum Q_x such that $x, q \in Q_x \subset C-p$, and hence $Q_x \subset K_n$ for sufficiently large values of n.

This implies identity (1).

DEFINITION 2. A set is said *to cut* the space (or to be a *cut* or a *cutting* of the space) if its complement is not a semi-continuum. A set E is said *to cut* the space $\mathscr{X}$ between a and b, provided that these two points belong to two different constituants of $\mathscr{X}-E$ (this means that they belong to $\mathscr{X}-E$ but they cannot be joined by a continuum outside of E). *E irreducibly cuts* the space $\mathscr{X}$ between a and b if no closed proper subset cuts $\mathscr{X}$ between these points.

It is clear that *if E separates the space between a and b, it cuts the space between these points.*

The inverse implication is false. So, for instance, the closure of the curve $y = \sin(1/x)$, $0 < x \leqslant 1$, considered as the space, is cut by the point $(0, 0)$ between the points $(0, +1)$ and $(0, -1)$, but is not separated between them.

However, *an open set separates a compact space between a and b, if and only if it cuts the space between these points.*

Because every compact space, which is connected between two given points, contains a continuum which joins them (II, Theorem 3 and I, Theorem 6).

THEOREM 3 [1]. *A metric continuum C is a simple closed curve, provided that no subcontinuum does cut it.*

IX. Hereditarily discontinuous spaces.

DEFINITION. The space is said to be *hereditarily discontinuous* ("ponctiforme") if it contains no continuum consisting of more than one point; in other words, if every constituant of the space reduces to a single point.

Clearly, every hereditarily disconnected space is hereditarily discontinuous. Conversely, for compact (metric) spaces the following notions coincide: of being hereditarily discontinuous, hereditarily disconnected, totally disconnected and 0-dimensional. This is so because every space of positive dimension is connected between two disjoint closed sets (§ 46, IV, Theorem 2), and hence it contains, as a compact space, a continuum joining these sets according to Theorem 3 of Section II.

There exist also *hereditarily discontinuous, connected (complete separable) spaces* [2]. So, for instance, if g is the function of Pompeiu (§ 46, VI, (iii)), the set

$$C = \underset{xy}{E}\left[y = \frac{dg(x)}{dx}\right] \tag{1}$$

is connected by Theorem 8 of § 46, I, and is hereditarily discontinuous, because the derivative $dg(x)/dx$ has points of discontinuity in every interval (and hence C can contain neither an arc nor a continuum having more than one point).

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(1) For the proof, see my paper of Fund. Math. 5 (1924), p. 119.

(2) The first example of this kind was given by Mazurkiewicz; see *Sur un ensemble G_δ ponctiforme qui n'est homéomorphe à aucun ensemble linéaire*, Fund. Math. 1 (1920), p. 61.

A simple example of a (complete separable) space, which is *hereditarily discontinuous and connected*, can be also constructed by condensing the singularity of the function defined in the following way: $\varphi(x) = \sin(1/x)$ for $x \neq 0$ and $\varphi(0) = 0$. So, if

$$\psi(x) = \sum_{n=1}^{\infty} \frac{\varphi(x - r_n)}{2^n} \tag{2}$$

where $\{r_n\}$ is the sequence of rational numbers, the set $\underset{xy}{E}[y = \psi(x)]$ is connected (by Theorem 7 of § 46, I) and is hereditarily discontinuous, since the function ψ is discontinuous at every point r_n (1).

§ 48 (2). Irreducible spaces. Indecomposable spaces

The space $\mathcal{X}$ is supposed to be *metric separable.*

I. Definition. Examples. General properties. A space is said to be *irreducible between the points a and b* provided that it is connected and these two points cannot be joined by any closed connected set which is different from the whole space; in other words, the space is irreducible with respect to the property of being a closed connected set containing a and b (3).

The point a is said to be a *point of irreducibility* of the space.

EXAMPLES. 1. Every interval is irreducible between its end-points.

2. The curve "$\sin 1/x$", defined by means of conditions

$$y = \sin 1/x, \quad 0 < |x| \leqslant 1 \quad \text{and} \quad -1 \leqslant y \leqslant 1, \quad x = 0,$$

is an irreducible continuum between the points $[-1, \sin(-1)]$ and $[1, \sin 1]$.

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(1) See the paper of W. Sierpiński and myself in Fund. Math. 3 (1922), p. 306.

(2) Compare my *Théorie des continus irréductibles entre deux points*, Fund. Math. 3 (1922) and 10 (1927).

(3) This definition is due to L. Zoretti, who introduced it in an attempt to characterize topologically the interval 01. See *La notion de ligne*, Ann. Éc. Norm. Sup. 26 (1909). The concept of a continuum which is irreducible between two points was studied systematically by S. Janiszewski in his Thesis (Paris 1911); compare Journ. Éc. Polyt. II, 16 (1912), or *Oeuvres choisies*, Instyt. Matem. PAN, Warszawa 1962.

Its "half" consisting of points with abscissa $\geqslant 0$ is irreducible between the point $(1, \sin 1)$ and each point $(0, y)$, where $-1 \leqslant y \leqslant 1$.

2a. If ψ is the function defined as in § 47, IX (2), the set

$$\overline{\underset{xy}{E}[y = \psi(x)][0 \leqslant x \leqslant 1]}$$

is an irreducible continuum.

3. The curve $r = 1+1/\theta$, $\theta \geqslant 1$ (r and θ denote the polar coordinates), augmented by the circle $r = 1$, is irreducible between the point $r = 2$, $\theta = 1$ and each point of the circle.

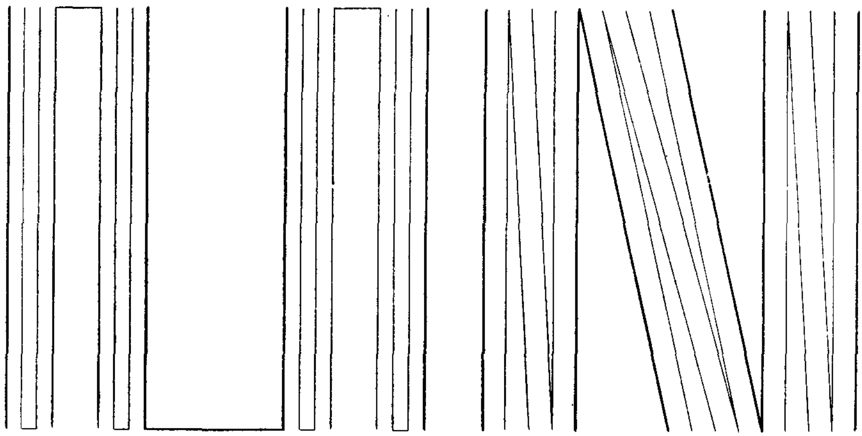

Fig. 3

4. Let C_0 be the Cantor set situated on the x axis, and C_1 be the same set located on the line $y = 1$ (of the plane XY). Let us join each point of C_0 with the corresponding point of C_1 by a vertical segment and add the contiguous intervals to C_0 with lengths $1/3, 1/3^3, \ldots$ and the contiguous intervals to C_1 with lengths $1/3^2, 1/3^4, \ldots$ The continuum thus obtained is irreducible between each point with abscissa 0 and each point with abscissa 1.

5. Let us join by a straight line segment each point x of the interval 01 on the X-axis, with abscissa which can be written without digit 1 in the enumeration system of base 4, to the point (or with two points) of the line $y = 1$, the abscissa of which can be obtained from the one of x by replacing in the repre-

sentation of the latter the digit 2 by 1. (¹) This continuum is irreducible between the same points as the one in Example 4.

If $\mathscr{X}$ is a continuum, the family of all subcontinua, which contain two given points $a, b \in \mathscr{X}$, is closed in the space $2^{\mathscr{X}}$ (compare § 47, II, Theorem 4). Hence we have by Theorem 1 of § 42, IV the following

THEOREM 1 (²). *Every continuum which joins two points a and b contains an irreducible continuum between them.*

The example of "the left half of the curve $\sin 1/x$" (Example 2), diminished by the point $(0, 0)$, shows that a connected and locally compact space may be deprived of subsets, which are irreducible between two given points (the points $0, 1$ and $0, -1$).

THEOREM 2. *If $\mathscr{X}$ is a continuum and f is a continuous mapping of $\mathscr{X}$ such that $f(\mathscr{X})$ is irreducible between two points, then $\mathscr{X}$ contains a continuum C irreducible between two points and such that $f(C) = f(\mathscr{X})$.*

Proof. Let C be a continuum which is irreducible with respect to the property of being a continuum such that $f(C) = f(\mathscr{X})$ (compare § 47, II, Theorem 7); if a and b are two points of C such that $f(\mathscr{X})$ is irreducible between $f(a)$ and $f(b)$, then C is irreducible between a and b.

THEOREM 3. *If $\mathscr{X}$ is an irreducible continuum between a and b, and if f is a continuous monotone mapping of $\mathscr{X}$, then $f(\mathscr{X})$ is an irreducible continuum between $f(a)$ and $f(b)$.*

Proof. Let C be a continuum such that $f(a), f(b) \in C \subset f(\mathscr{X})$. Since f is monotone, the set $f^{-1}(C)$ is a continuum. Since $a, b \in f^{-1}(C)$, so $f^{-1}(C) = \mathscr{X}$, and $C = ff^{-1}(C) = f(\mathscr{X})$.

THEOREM 4 (³). *If $\mathscr{X}$ is a continuum and a is a point of $\mathscr{X}$ such that $\mathscr{X}$ is not a union of two proper subcontinua both of which contain a, then a is a point of irreducibility of $\mathscr{X}$.*

II. Connected subsets of irreducible spaces. Let $\mathscr{X}$ be an irreducible space between the points a and b.

(¹) This example is due to B. Knaster. See also W. A. Wilson, Amer. Journ. Math. 48 (1926).

(²) Theorem 1 is due to S. Janiszewski and S. Mazurkiewicz, C. R. Paris 151 (1910). Compare S. Mazurkiewicz, Bull. Acad. Polon. Sc., 1912, p. 44.

(³) For the proof, see Fund. Math. 10, *op. cit.*, p. 270.

THEOREM 1. *If C is a connected subset of the space $\mathcal{X}$ joining a to b, then C is irreducible between a and b.*

Proof. If F is a connected subset which contains a and b, then $\overline{F} = \mathcal{X}$. Therefore, if F is supposed to be closed in C, i.e. $F = \overline{F} \cap C$, it follows that $F = C$.

THEOREM 2. *The space $\mathcal{X}$ is not a union of two closed connected sets A and B such that $a \in A \cap B$ and $A \neq \mathcal{X} \neq B$.*

Proof. Otherwise, one of these sets would contain the point b and therefore both a and b, but this would contradict the irreducibility of the space $\mathcal{X}$.

THEOREM 3. *Let C be a closed connected set. If $\mathcal{X} - C$ is not connected, then it is the union of two open connected sets, one of which contains a and the other contains b.*

Thus, *if $a \in C$, the set $\mathcal{X} - C$ is connected.*

Proof. Suppose that $\mathcal{X} - C$ is not connected; then there exist two open sets P and Q such that

$$\mathcal{X} - C = P \cup Q, \quad P \cap Q = 0, \quad P \neq 0 \neq Q.$$

According to Theorem 4 of § 46, II, the sets $A = C \cup P$ and $B = C \cup Q$ are connected and closed and it follows that

$$\mathcal{X} = A \cup B, \quad A \cap B = C \quad \text{and} \quad A \neq X \neq B. \tag{i}$$

Hence, by Theorem 2, $a \notin C$. Thus the second part of the theorem is established.

According to (i) neither A nor B can contain both a and b. Let $a \in A$ and $b \in B$. Since the set A is closed and connected, its complement Q is connected, as has just been proved, and $b \in Q$. By the symmetry, P is connected and $a \in P$.

THEOREM 4. *If A and B are two closed connected sets such that $a \in A$ and $b \in B$, the set $\mathcal{X} - (A \cup B)$ is connected.*

Proof. It is admissible to suppose that $A \cap B = 0$, because otherwise $A \cup B = \mathcal{X}$. The set $C = \mathcal{X} - A$ is connected by Theorem 3. Suppose that

$$C - B = U \cup V, \quad U \cap V = 0, \quad U \text{ and } V \text{ open}.$$

We have to show that

$$\text{either} \quad U = 0 \quad \text{or} \quad V = 0.$$

According to Theorem 4 of § 46, II, the sets $B \cup U$ and $B \cup V$ are connected, and so are the sets $B \cup \overline{U}$ and $B \cup \overline{V}$. Since

$$\mathscr{X} = A \cup B \cup \overline{\mathscr{X} - A - B} \quad \text{and} \quad A \cap B = 0,$$

it follows that

$$A \cap \overline{\mathscr{X} - A - B} \neq 0, \quad \text{what means} \quad A \cap \overline{U \cup V} \neq 0.$$

So it follows that either $A \cap \overline{U} \neq 0$ or $A \cap \overline{V} \neq 0$. Assume, for instance, that $A \cap \overline{U} \neq 0$. Therefore, $A \cup \overline{U} \cup B$ is connected. Since $a, b \in (A \cup \overline{U} \cup B)$, it follows that

$$A \cup \overline{U} \cup B = \mathscr{X}, \quad \text{hence} \quad \mathscr{X} - A - B \subset \overline{U}.$$

And since $V \subset C - B = \mathscr{X} - A - B$, then $V \subset \overline{U}$.

Since $U \cap V = 0$ and V is open, it follows that $V = 0$.

THEOREM 5. *If the closed set C is connected, then so is* $\operatorname{Int}(C)$.

Proof. It is admissible to assume that $C \neq \mathscr{X}$, and hence that $a \in \mathscr{X} - C$. There are two cases to be considered.

If $\mathscr{X} - C$ is connected, then so is $\overline{\mathscr{X} - C}$ and, since $a \in \overline{\mathscr{X} - C}$, the set $\operatorname{Int}(C) = \mathscr{X} - \overline{\mathscr{X} - C}$ is connected by Theorem 3.

If $\mathscr{X} - C$ is not connected, then $\mathscr{X} - C = P \cup Q$ is the union of two connected sets and $a \in P$, $b \in Q$. Assuming in Theorem 4 that $A = \overline{P}$ and $B = \overline{Q}$, we infer that the set $\operatorname{Int}(C) = \mathscr{X} - (\overline{P} \cup \overline{Q})$ is connected.

THEOREM 6. *If C is a closed connected set and $a \in \operatorname{Fr}(C)$, then C is nowhere dense.*

Proof. The set $\mathscr{X} - C$ is connected by Theorem 3. Therefore, if we let $A = C$ and $B = \overline{\mathscr{X} - C}$, it follows from Theorem 2 that $\overline{\mathscr{X} - C} = \mathscr{X}$.

THEOREM 7. *If C is a closed connected set and $a \in C$, the set $\overline{\mathscr{X} - C}$ is irreducible between b and every point of* $\operatorname{Fr}(C)$.

In particular, *if C is nowhere dense, the space $\mathscr{X}$ is irreducible between b and every point of C.*

Proof. The set $\overline{\mathscr{X}-C}$ is connected by Theorem 3. On the other hand, if F is a closed connected set and if $b \in F \subset \overline{\mathscr{X}-C}$ and $F \cap \mathrm{Fr}(C) \neq 0$, the set $C \cup F$ is connected, which implies that $C \cup F = \mathscr{X}$, hence $\mathscr{X}-C \subset F$ and therefore $F = \overline{\mathscr{X}-C}$.

III. Closed connected subdomains. As above, let $\mathscr{X}$ be a space irreducible between a and b. Let $\boldsymbol{D}$ be the family of all closed connected domains containing the point a; besides let $0 \in \boldsymbol{D}$. Let us recall that by the definition (compare § 8, VIII) D is a closed domain if

$$D = \overline{\mathscr{X}-\overline{\mathscr{X}-D}}. \tag{0}$$

THEOREM 1. *If* $0 \neq D \in \boldsymbol{D}$, *then* D *is irreducible between* a *and every point of* $\mathrm{Fr}(D)$ (the latter set is never empty, unless $D = \mathscr{X}$).

Proof. If $D \neq \mathscr{X}$, then the set $\overline{\mathscr{X}-D}$ is connected and contains b. Therefore, according to Theorem 7 of Section II, the set $D = \overline{\mathscr{X}-\overline{\mathscr{X}-D}}$ is connected between a and every point of the set $\mathrm{Fr}(\overline{\mathscr{X}-D})$.

But, since D is a closed domain, it follows that $\mathrm{Fr}(\overline{X-D}) = \mathrm{Fr}(D)$ (compare § 8, VIII).

THEOREM 2. $\boldsymbol{D}$ *is a strictly monotone family.*

Proof. Our task is to show that

$$\text{if } D_1, D_2 \in \boldsymbol{D} \text{ and } D_1 \neq D_2 \not\subset \mathrm{Int}(D_1), \quad \text{then} \quad D_1 \subset \mathrm{Int}(D_2).$$

But, the condition $D_2 \not\subset \mathrm{Int}(D_1)$ is equivalent to $D_2 \cap \overline{\mathscr{X}-D_1} \neq 0$ and this one implies that $D_2 \cup \overline{\mathscr{X}-D_1} = \mathscr{X}$ (because $D_1 \neq \mathscr{X}$, so $b \in \overline{\mathscr{X}-D_1}$ and therefore $\overline{\mathscr{X}-D_1}$ is connected by Theorem 3 of Section II). It follows that

$$\overline{\mathscr{X}-\overline{\mathscr{X}-D_1}} \subset D_2, \quad \text{and hence} \quad D_1 \subset D_2$$

according to (0).

Since D_2 is irreducible between a and every point of the set $D_2 \cap \overline{\mathscr{X}-D_2}$ (by Theorem 1), the condition $D_1 \neq D_2$ implies

$$D_1 \cap \overline{X-D_2} = 0, \quad \text{which yields} \quad D_1 \subset \mathrm{Int}(D_2).$$

THEOREM 3. *The family* $\boldsymbol{D}$ *has no gaps* (*if ordered by the relation* $D_1 \subset D_2 \neq D_1$).

In other words, if the family $\boldsymbol{D}$ is decomposed into two families $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ such that each element of $\boldsymbol{D}_1$ is a subset of every element of $\boldsymbol{D}_2$, then either there exists the first element in $\boldsymbol{D}_2$, or the last one in $\boldsymbol{D}_1$. In fact, such an element is the set $\overline{S}$, where S is the union of all elements of $\boldsymbol{D}_1$ ($\overline{S}$ is a closed domain, compare § 8, VIII).

Theorems 2 and 3 combined with Theorem 2 of § 24, VII imply the following one.

THEOREM 4. *The members D of $\boldsymbol{D}$ can be indexed by the elements of a closed subset $J \subset \mathscr{I}$ in such a manner that*

$$[y_1 < y_2] \equiv [D_{y_1} \subset D_{y_2} \neq D_{y_1}].$$

THEOREM 5. *Let $\boldsymbol{E}$ be the family of all closed connected domains which contain the point b (augmented by the empty set). $\boldsymbol{E}$ coincides with the family of sets $\overline{\mathscr{X}-D}$ where $D \in \boldsymbol{D}$.*

In other notation

$$\boldsymbol{E} = \{E_y\}, \quad E_y = \overline{\mathscr{X}-D_y}, \quad \overline{\mathscr{X}-E_y} = D_y.$$

In general, *if c is a point of irreducibility of the space, the family $\boldsymbol{F}$ of closed connected domains containing c (augmented by the empty set) coincides either with $\boldsymbol{D}$ or with $\boldsymbol{E}$.*

The first part of the theorem is obvious and the second one follows from the next statement.

LEMMA [1]. *If the space $\mathscr{X}$ is irreducible between a and b as well as between c and d, then $\mathscr{X}$ is irreducible either between a and c or between b and c.*

Proof. Suppose that $\mathscr{X}$ is irreducible neither between a and c, nor between b and c. So, there exist two closed connected sets K and L such that

$$c \in K \cap L, \quad a \in K, \quad b \in L, \quad K \neq \mathscr{X} \neq L.$$

Since $\mathscr{X}$ is irreducible between c and d, it follows that

$$d \in (\mathscr{X}-K) \cap (\mathscr{X}-L) = \mathscr{X}-(K \cup L), \quad \text{hence} \quad K \cup L \neq \mathscr{X}.$$

---

[1] See K. Yoneyama, Tôhoku Math. Journ. 1917, theorem 3, p. 48.

But this contradicts the hypothesis that the space $\mathscr{X}$ is irreducible between a and b, because $a, b \in (K \cup L)$.

Remark. Let a denote the order type of the family $\boldsymbol{D}$ (and hence of the set J); the order type a^*, inverse to a, is the type of $\boldsymbol{E}$. If $|a|$ is the "absolute value of a", which does not depend on the sense of the order (we stipulate that $|a| = |a^*|$), it follows from Theorem 5 that $|a|$ *is uniquely determined by the irreducible space* (i.e., independently of the choice of its points of irreducibility).

All the spaces in the Examples 1 to 5 of Section I have the same type $|a|$, namely the type of the interval $\mathscr{I}$ [1]. As it will be seen in Section VII, to every $J = \bar{J} \subset \mathscr{I}$ there corresponds a space of the same order type as J.

Let

$$I_y = \text{the set of points } x \text{ such that } D_y \text{ is irreducible between } a \text{ and } x,$$

$$J_y = \text{the set of points } x \text{ such that } E_y \text{ is irreducible between } b \text{ and } x.$$

THEOREM 6. *The following conditions are satisfied by D_y and E_y*

$$\mathrm{Fr}(D_y) = \mathrm{Fr}(E_y) = D_y \cap E_y = I_y \cap J_y, \tag{1}$$

$$\textit{if } y_1 < y_2, \textit{ then } D_{y_1} \cap I_{y_2} = 0 = E_{y_2} \cap J_{y_1} \textit{ and } D_{y_1} \cap E_{y_2} = 0, \tag{2}$$

$$\textit{if } y_1 < y_2, \textit{ then } I_{y_1} \cap I_{y_2} = 0 = J_{y_1} \cap J_{y_2} \textit{ and } I_{y_1} \cap J_{y_2} = 0, \tag{3}$$

$$\textit{if } y_1 < y_2 < y_3, \textit{ then } I_{y_3} \cap J_{y_1} = 0. \tag{4}$$

Proof. Condition (1) is a straightforward consequence of Theorem 1.

Conditions (2) and (3) follow from (1).

According to (2), $E_{y_2} \cap J_{y_1} = 0$, therefore $J_{y_1} \subset \mathscr{X} - E_{y_2} \subset D_{y_2}$; if $y_2 < y_3$, then $D_{y_2} \cap I_{y_3} = 0$ and it follows that $J_{y_1} \cap I_{y_3} = 0$.

THEOREM 7. $\mathscr{X} = \bigcup_{y \in J} (I_y \cup J_y)$.

In other words, *to every point p there corresponds a closed domain irreducible either between a and p, or between b and p.*

Proof. Since the family $\boldsymbol{D}$ has no gaps, there are two cases to be considered according to whether there exists the least $D \in \boldsymbol{D}$ such that $p \in D$, or there exists the largest $D \in \boldsymbol{D}$ such that $p \in \mathscr{X} - D$.

---

(1) The spaces of this type are also said to be of type λ.

In the first case, suppose that D is not irreducible between a and p, hence that there exists a connected set C such that

$$a, p \in C, \quad C = \bar{C} \subset D \neq C.$$

The set $\overline{\mathscr{X} - \overline{\mathscr{X} - C}}$ is a proper subset of D (as a subset of C); moreover, it belongs to $\boldsymbol{D}$ by Theorems 5 and 6 of Section II. Therefore

$$p \in \mathscr{X} - \overline{\mathscr{X} - \overline{\mathscr{X} - C}} \subset \overline{\mathscr{X} - C},$$

and it follows that $p \in \mathrm{Fr}(C)$. Hence the set $\overline{\mathscr{X} - C}$ is a closed domain irreducible between b and p (according to Theorem 7 of Section II).

In the second case, assume that $E = \overline{\mathscr{X} - D}$. Therefore $E \in \boldsymbol{E}$. By the symmetry, we can assume that $\boldsymbol{E}$ does not contain the least set containing p. So let

$$F \in \boldsymbol{E}, \quad p \in F \subset E \neq F.$$

It follows that

$$D = \overline{\mathscr{X} - E} \subset \overline{\mathscr{X} - F} \neq D \quad \text{and hence} \quad p \in \overline{\mathscr{X} - F} \in \boldsymbol{D}$$

by the definition of the sets $\boldsymbol{D}$. Therefore $p \in \mathrm{Fr}(F)$ and F is irreducible between b and p.

COROLLARY [1]. *If there exist two closed sets irreducible between a and p, then there exists only one closed set irreducible between b and p.*

This Corollary follows from Theorem 7 combined with the next statement.

LEMMA. *If D is a closed domain irreducible between a and p, then D is the only closed set irreducible between these points.*

Proof. Suppose, conversely, that

$$a, p \in F = \bar{F} \not\subset D, \quad \text{hence} \quad F \cap \overline{\mathscr{X} - D} \neq 0.$$

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[1] S. Janiszewski, *Démonstration d'une propriété des continus irréductibles entre deux points*, Bull. Acad. Sc. Cracovie 1912, p. 906. Compare this corollary with Example 3 of Section I.

Since F is connected, then so is $F \cup \overline{\mathscr{X}-D}$, and since $D \neq \mathscr{X}$, it follows that $b \in \overline{\mathscr{X}-D}$, hence $F \cup \overline{\mathscr{X}-D} = \mathscr{X}$. So

$$\mathscr{X}-(\overline{\mathscr{X}-D}) \subset F,$$

which yields

$$D = \overline{\mathscr{X}-\overline{\mathscr{X}-D}} \subset F, \quad \text{and hence} \quad F = D.$$

THEOREM 8. *If $y_1 < y_2$, then $D_{y_2} \cap E_{y_1} = \overline{D_{y_2}-D_{y_1}}$ and the set $\overline{D_{y_2}-D_{y_1}}$ is connected.*

Proof. We have

$$D_{y_2} \cap E_{y_1} = D_{y_2} \cap \overline{\mathscr{X}-D_{y_1}} = \overline{D_{y_2}-D_{y_1}},$$

where the last identity is a consequence of inclusion $D_{y_1} \subset \operatorname{Int}(D_{y_2})$.

For, let $A \subset \operatorname{Int}(B)$ and $B = \overline{B}$. Therefore

$$A \cap \overline{\mathscr{X}-A} \subset \overline{\mathscr{X}-A}-\overline{\mathscr{X}-B} \subset \overline{(\mathscr{X}-A)-(\mathscr{X}-B)} = \overline{B-A},$$

which implies that

$$\begin{aligned} B \cap \overline{\mathscr{X}-A} &= (A \cap \overline{\mathscr{X}-A}) \cup \big((B-A) \cap \overline{\mathscr{X}-A}\big) \\ &\subset \overline{B-A} \subset \overline{B} \cap \overline{\mathscr{X}-A} = B \cap \overline{\mathscr{X}-A}. \end{aligned}$$

Hence $B \cap \overline{\mathscr{X}-A} = \overline{B-A}$.

Since a is a point of irreducibility of D_{y_2} (by Theorem 1), it follows from Theorem 3 of Section II that $D_{y_2}-D_{y_1}$ is connected, and so is the set $\overline{D_{y_2}-D_{y_1}}$.

IV. Layers of an irreducible space. Let us substitute $\boldsymbol{D}$ for $\boldsymbol{F}$ in Section X of § 24 and let us consider the function g. The point inverses $g^{-1}(t)$ of this function will be called the *layers* T_t *of the space* $\mathscr{X}$ *irreducible between* a *and* b.

According to § 24, X, Theorem 2, it follows that

$$T_t = g^{-1}(t) = \bigcap_{\Gamma(t)<z} D_z \cap \bigcap_{u<\gamma(t)} E_u, \qquad 0 \leqslant t \leqslant 1. \tag{1}$$

The layers are independent of the choice of the irreducibility points of $\mathscr{X}$.

If $\overline{\overline{\boldsymbol{D}}} \leqslant \aleph_0$, then $\mathscr{X}$ is reduced to one layer ($\mathscr{X}$ is *monostratic*). If $\overline{\overline{\boldsymbol{D}}} > \aleph_0$, the decomposition into layers represents a linear stratification (a partial order) of $\mathscr{X}$ into non-empty closed sets.

THEOREM 1. *Let $h\colon \mathscr{X} \to \mathscr{I}$ be a continuous function; if the sets $h^{-1}(0t)$ are connected and contain a for all except* $\aleph_0$ (at most) *values of t, then each point inverse of the function h is the union of some point inverses of the function g.*

Proof. This is a consequence of Theorem 4 of § 24, X (according to § 24, IX, Theorem 2 and VIII (5), $h^{-1}(0t)$ can be supposed to be a closed domain).

THEOREM 2. *The layers of an irreducible continuum are continua.*

Proof. If $\gamma(t) \neq 0$ and $\Gamma(t) \neq 1$, there exist in J two sequences such that

$$\gamma(t) = \lim_{n=\infty} u_n,\ u_n < u_{n+1} \quad \text{and} \quad \Gamma(t) = \lim_{n=\infty} z_n,\ z_n > z_{n+1}.$$

It follows that (compare § 24, X, Theorem 2)

$$T_t = \bigcap_{n=1}^{\infty} D_{z_n} \cap E_{u_n} \quad \text{and} \quad D_{z_n} \cap E_{u_n} \supset D_{z_{n+1}} \cap E_{u_{n+1}}.$$

Since $D_{z_n} \cap E_{u_n}$ are continua by Theorem 8 of Section III, then so is the set T_t according to Theorem 5 of § 47, II.

If $\gamma(t) = 0$ or $\Gamma(t) = 1$, then

$$\bigcap_{u<\gamma(t)} E_u = \mathscr{X} \quad \text{or} \quad \bigcap_{\Gamma(t)<z} D_z = \mathscr{X},$$

and T_t is a continuum by Theorem 1 of Section III and Theorem 5 of § 47, II.

THEOREM 3. *The decomposition of an irreducible continuum into layers is the finest of all linear semi-continuous decompositions into continua* (1).

In other words, *if $h\colon \mathscr{X} \to \mathscr{I}$ is a continuous mapping onto and each set $h^{-1}(t)$, where $0 \leqslant t \leqslant 1$, is a continuum, this continuum is the union of some layers of the continuum $\mathscr{X}$.*

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(1) *Op. cit.*, Fund. Math. 10, p. 259, Fundamental theorem. Less fine decompositions, or such which require additional hypotheses about the space, were considered by H. Hahn (called “Primteile”), L. Vietoris (“Schichten”) and by W. A. Wilson (“complete oscillatory sets”); compare *ibid.* pp. 226 to 229 and 264.

See also E. S. Thomas, Jr., *Monotone decompositions of irreducible continua*, Rozpr. Matem. 50, Warszawa 1966.

Proof. Let

$$h(a) = t_0, \quad h(b) = t_1.$$

Theorem 4 of § 47, VI can be applied since $\mathscr{X}$ is compact and the sets $h^{-1}(t)$ are connected and hence h is monotone. In other words, if C is connected, then so is $h^{-1}(C)$. Therefore the set $h^{-1}(t_0 t_1)$ (respectively $h^{-1}(t_1 t_0)$ if $t_1 < t_0$) is a continuum. Since a and b belong to it, it coincides with $\mathscr{X}$. Hence it can be assumed that $t_0 = 0$ and $t_1 = 1$.

Let

$$0 < t < 1.$$

The set $h^{-1}(0t)$ is connected, as just has been proved. Moreover, this set contains the point a. Thus applying Theorem 1 we derive the required conclusion.

Remarks. By Theorem 2 of § 24, IX, all the layers T_t, except a countable infinity, satisfy condition

$$T_t = \overline{\bigcup_{u<t} T_u} \cap \overline{\bigcup_{v>t} T_v}.$$

The layers of this kind are called *layers of cohesion*. The layer T_t is a layer of cohesion provided that t is a continuity point of both functions $g^{-1}(0t)$ and $g^{-1}(t1)$. However, a layer of cohesion T_t does not need be a *layer of continuity* (¹).

So, for instance, the vertical segment of the curve $\sin 1/x$ (see Section I, Example 2) is a discontinuity layer, while it is a layer of cohesion.

In Example 4 the vertical segments of the continuum are discontinuity layers; those, which contain end-points of intervals contiguous to the Cantor set, are not cohesion layers.

In Example 5 it is easily seen that there exists a dense set of indices t such that T_t is not a cohesion layer; these are the layers of the shape $\vee$ or $\wedge$. All the other layers are continuity layers.

(¹) The layer T_t is called a *layer of continuity* if it is a continuity member of the decomposition of $\mathscr{X}$ into layers (compare § 19, II, p. 185); in other words, if t is a point of continuity of the mapping $\overset{-1}{g}$.

B. Knaster has proved [1] that *there exists an irreducible continuum such that* $J = \mathscr{I}$ *and that each layer is a continuity layer containing more than one point.*

However, as shown by E. E. Moise [2], *it is not possible that all layers be arcs* (for a stronger result see Section V, Remark 3) [3].

On the other hand, there is a continuum such that *all layers are homeomorphic* (in fact, they are pseudoarcs) [4].

Finally, let us recall that, according to Corollary 1 of § 43, VII, *the set of indices such that* T_t *is a continuity layer is a dense* $\boldsymbol{G}_\delta$*-set in the interval* $\mathscr{I}$.

THEOREM 4. $T_t = \bigcup\limits_{\gamma(t)\leqslant y\leqslant \Gamma(t)} (I_y \cup J_y)$.

Proof. Let $\gamma(t) \leqslant y \leqslant \Gamma(t)$. Therefore,

$$I_y \subset D_y \subset \bigcap_{\Gamma(t)<z} D_z$$

and it follows by III, Theorem 6 (2) for $u < \gamma(t)$ that

$$I_y \subset \mathscr{X} - D_u \subset E_u, \quad \text{which implies} \quad I_y \subset \bigcap_{u<\gamma(t)} E_u.$$

Therefore, $I_y \subset T_t$ (compare (1)) and by the symmetry, $J_y \subset T_t$. This yields

$$\bigcup_{\gamma(t)\leqslant y\leqslant \Gamma(t)} (I_y \cup J_y) \subset T_t.$$

Since the layers are pairwise disjoint, this inclusion combined with Theorem 7 of Section III implies the inverse inclusion.

THEOREM 5. $g^{-1}(0t - t) = D_{\gamma(t)} - I_{\gamma(t)}$, $g^{-1}(t1 - t) = E_{\Gamma(t)} - J_{\Gamma(t)}$. *Therefore, both sets are connected (disjoint and open).*

Proof. Write concisely $\gamma(t) = \gamma$ and $\Gamma(t) = \Gamma$. By Theorem 4 it follows that

$$g^{-1}(0t - t) = \bigcup_{t'<t} T_{t'} = \bigcup_{u<\gamma} (I_u \cup J_u).$$

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(1) Fund. Math. 25 (1935), p. 568.

(2) *A theorem on monotone interior transformations*, Bull. Amer. Math. Soc. 55 (1949), pp. 810–811.

(3) See also M. E. Hamstrom, *Concerning continuous collections of curves*, Proc. Amer. Math. Soc. 4 (1953), p. 240.

(4) R. D. Anderson, *Open mappings of compact continua*, Proc. Nat. Acad. Sci. 42 (1956), p. 347.

We have to show that

$$\bigcup_{u<\gamma}(I_u \cup J_u) = D_\gamma - I_\gamma.$$

Let $u<\gamma$. By the definition of γ there exists u_1 such that $u<u_1<\gamma$. Therefore, according to III, Theorem 6 (2),

$$E_{u_1} \cap J_u = 0, \quad \text{which implies} \quad J_u \subset \mathscr{X} - E_u \subset D_{u_1},$$
$$\text{hence} \quad I_u \cup J_u \subset D_{u_1}.$$

On the other hand, $D_{u_1} \cap I_\gamma = 0$, which yields $D_{u_1} = D_{u_1} - I_\gamma \subset D_\gamma - I_\gamma$. So $I_u \cup J_u \subset D_\gamma - I_\gamma$ and hence

$$\bigcup_{u<\gamma}(I_u \cup J_u) \subset D_\gamma - I_\gamma.$$

In order to prove the inverse inclusion assume that $p \in D_\gamma - I_\gamma$ and $p \in I_u \cup J_u$ (compare Theorem 4). We must show that $u<\gamma$. But, if $p \in I_u$, it follows that $u \neq \gamma$ (because $p \notin I_\gamma$) and D_γ cannot be a proper subset of D_u. Therefore, $u<\gamma$. If $p \in J_u$, the hypothesis that $p \in D_\gamma - I_\gamma$ implies $p \in \mathscr{X} - E_\gamma$ according to III, Theorem 6 (1). Since $p \in E_u$, it follows that $E_\gamma \subset E_u \neq E_\gamma$, hence $u<\gamma$.

THEOREM 6. $T_t = I_{\gamma(t)} \cup (D_{\Gamma(t)} \cap E_{\gamma(t)}) \cup J_{\Gamma(t)}$.

Proof. Assume as above that $\gamma(t) = \gamma$ and $\Gamma(t) = \Gamma$. Taking the intersection of the unions $\mathscr{X} = D_\gamma \cup E_\gamma$ and $\mathscr{X} = D_\Gamma \cup E_\Gamma$ and using the conditions

$$D_\gamma \cap D_\Gamma = D_\gamma, \quad E_\gamma \cap E_\Gamma = E_\Gamma \quad \text{and} \quad D_\gamma \cap E_\Gamma \subset D_\Gamma \cap E_\gamma,$$

which follow from the inequality $\gamma \leqslant \Gamma$, we have

$$\mathscr{X} = D_\gamma \cup (D_\Gamma \cap E_\gamma) \cup E_\Gamma,$$

hence

$$\mathscr{X} = (D_\gamma - I_\gamma) \cup \big(I_\gamma \cup (D_\Gamma \cap E_\gamma) \cup J_\Gamma\big) \cup (E_\Gamma - J_\Gamma).$$

The terms of this union are disjoint since, according to III, Theorem 6 (1),

$$D_\gamma \cap J_\Gamma \subset D_\gamma \cap E_\Gamma \subset D_\gamma \cap E_\gamma \subset I_\gamma$$

and

$$E_\Gamma \cap I_\gamma \subset E_\Gamma \cap D_\gamma \subset E_\Gamma \cap D_\Gamma \subset J_\Gamma.$$

It follows by Theorem 5 that

$$T_t = \mathscr{X} - [g^{-1}(0t-t) \cup g^{-1}(t1-t)]$$
$$= \mathscr{X} - [(D_\gamma - I_\gamma) \cup (E_\Gamma - J_\Gamma)] = I_\gamma \cup (D_\Gamma \cap E_\gamma) \cup J_\Gamma.$$

THEOREM 7. *Let $\mathscr{X}$ be a continuum and $f\colon \mathscr{X} \to \mathscr{I}$ a continuous mapping onto. If all the layers $f^{-1}(y)$, $0 \leqslant y \leqslant 1$, are nowhere dense continua, there exists a connected set C which contains exactly one point from each layer.*

Therefore, *if $a \in C \cap f^{-1}(0)$ and $b \in C \cap f^{-1}(1)$, C is irreducible with respect to the property of being a connected set joining the points a and b* (¹).

V. Indecomposable spaces (²).

DEFINITION. A space $\mathscr{X}$ is *indecomposable* if it is connected and is not the union of two closed connected sets different from $\mathscr{X}$.

EXAMPLES. 1. The simplest example of an indecomposable continuum can be defined in the following way (³).

The continuum consists of

(i) all semi-circles with ordinates $\geqslant 0$, with center $(\frac{1}{2}, 0)$ and passing through every point of the Cantor set $\mathscr{C}$,

(ii) all semi-circles with ordinates $\leqslant 0$, which have for $n \geqslant 1$ the center at $\big(5/(2\cdot 3^n), 0\big)$ and pass through each point of the set $\mathscr{C}$ lying in the interval $2/3^n \leqslant x \leqslant 1/3^{n-1}$.

(¹) Theorem of B. Knaster. For the proof, see *Sur les ensembles connexes irréductibles entre deux points*, Fund. Math. 10 (1927), p. 277, theorem η.

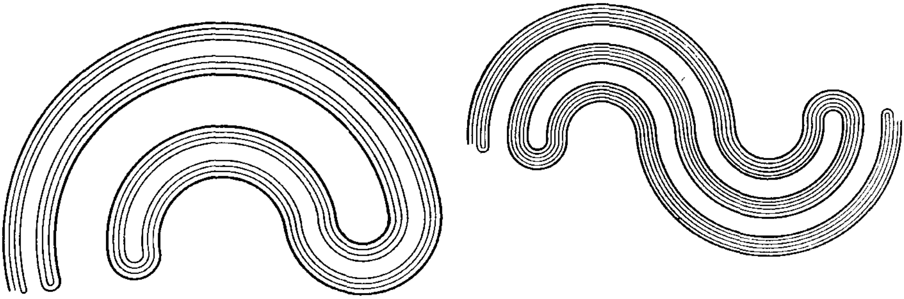
(²) The indecomposable continua were discovered by L. E. J. Brouwer to disprove the conjecture (of Schönflies) that every common boundary of two plane regions is decomposable (Math. Ann. 68 (1910), p. 426). As will be seen later, they are involved in many topological questions. For the applications of indecomposable continua to the theory of topological groups see D. v. Dantzig, Fund. Math. 5 (1930), p. 102, L. Vietoris, Math. Ann. 97 (1927) p. 454, v. Heemert, *Topologische Gruppen und unzerlegbare Kontinua*, Comp. Math. 5, 319–326 (1937).

Besides Brouwer, examples of indecomposable continua were also announced by Denjoy, C. R. Paris 151 (1910) and by Yoneyama, Tôhoku Math. Journ. 1917, p. 60 (example due to Wada).

(³) This definition is due to B. Knaster (see Fund. Math. 3, p. 209). It is obtained by simplifying of Janiszewski's definition (Thesis, p. 36), which in its turn is intimately related to the quoted definition of Brouwer.

The proof that the described set is indecomposable will be given in Remark to Theorem 8 of Section VI.

2. If the point $(\frac{1}{2}, \frac{1}{2})$ is deleted from the continuum of Example 1, an indecomposable space is obtained (compare Theorem 3) (1) which has the following singularity. The arc joining the points $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$, with the last one deleted, is saturated with respect to the property of being a closed connected, proper subset.



Figs. 4–5

3. Let E be the set of numbers of the interval $\mathscr{I}$, which can be written in the enumeration system of base 5 without the digits 1 and 3. Let

$$E_n = \underset{x}{E}(x \in E)(2/5^{n+1} \leqslant x \leqslant 1/5^n) \quad \text{and} \quad F_n = \underset{x}{E}[(1-x) \in E_n].$$

The required continuum is the union of

(i) all the semi-circles with ordinates $\leqslant 0$, with the center at the point $7/10 \cdot 5^n$ and passing through all the points of E_n,

(ii) all the semi-circles with ordinates $\geqslant 0$, with the center at the point $1-7/(10\cdot 5^n)$ and passing through all the points of F_n $(n \geqslant 0)$ (2).

The essential difference between the continua of Examples 1 and 3 is that the first continuum has only one composant (in the sense of Section VI) containing accessible points, while the second

(1) It is clear that this space is homeomorphic to the closed plane set which is obtained from the continuum of Example 1 by means of inversion with the inversion center at $(\frac{1}{2}, \frac{1}{2})$. See the paper of B. Knaster and myself, Fund. Math. 5 (1924), p. 43, Fig. II.

(2) This example is also due to B. Knaster.

continuum has two of them (the one of the point 0 and the other one of the point 1).

Remark 1 (1). (i) The union of composants of an indecomposable plane continuum which contain an accessible point is of the 1st category in that continuum.

(ii) The family of composants of an indecomposable plane continuum which contain more than one accessible point is (at most) countable.

Remark 2. Many examples of indecomposable continua can be derived from the following theorem (2).

Every compact space of dimension $\geqslant 2$ *contains an indecomposable continuum.*

Remark 3. Every irreducible continuum such that $J = \mathcal{I}$ and that each layer is a continuity layer (containing more than one point, compare IV, Remarks) contains indecomposable continua (3).

Remark 4. There exist *hereditarily indecomposable* continua, i.e. continua every subcontinuum of which is indecomposable (4). Moreover, (5) *in the space of all subcontinua of the square* $\mathcal{I}^2$ *the set of all hereditarily indecomposable continua is a dense* $\boldsymbol{G}_\delta$, *hence a residual set.*

This fact seemed to be very surprising: among the subcontinua of a square the most singular are the most frequent.

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(1) S. Mazurkiewicz, Fund. Math. 14 (1929), pp. 107 and 271.

Compare also my paper *Sur une condition qui caractérise les continus indécomposables*, *ibid.* 14 (1929), p. 116.

(2) For the proof, see S. Mazurkiewicz, *Sur l'existence des continus indécomposables*, Fund. Math. 25 (1935), p. 327. This theorem answers a question raised by P. Alexandrov.

(3) For the proof, see Eldon Dyer, *Irreducibility of the sum of the elements of a continuous collection of continua*, Duke Math. Journ. 20 (1953). See also E. L. Bethel, *A note on a reducible continuum*, Proc. Amer. Math. Soc. 16 (1965), pp. 1331–1333.

(4) For the proof, see B. Knaster, Fund. Math. 3 (1922), p. 247. For an interesting application, see E. E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua*, Trans. Amer. Math. Soc. 63 (1948), p. 58.

(5) S. Mazurkiewicz, Fund. Math. 16 (1930), p. 151.

Clearly, a hereditarily indecomposable continuum contains no arc [1].

Remark 5. The fact that in the space of all subcontinua of a compact space, the indecomposable continua or the hereditarily indecomposable continua constitute a G_δ-set, follows immediately from their definitions (compare § 17, IV, Theorem 1, § 17, III, Corollary 4a and § 41, VI):

$$\{C \text{ is decomposable}\} \equiv \bigvee_{K,L} (C = K \cup L)(K \neq C \neq L),$$

$$\{C \text{ is not hereditarily indecomposable}\} \equiv \bigvee_{K} (K \subset C)(K \text{ is decomposable}),$$

where C, K and L are the elements of the space of continua.

THEOREM 1. *No closed connected subset C of an indecomposable space is its separator.*

Proof. If

$$\mathscr{X} - C = M \cup N, \quad M \neq 0 \neq N, \quad M \cap N = 0,$$

where M and N are open, the sets $C \cup M$ and $C \cup N$ are closed and connected and it follows (compare § 46, II, Theorem 4) that

$$\mathscr{X} = (C \cup M) \cup (C \cup N), \quad C \cup M \neq \mathscr{X} \neq C \cup N.$$

THEOREM 2. *A connected space $\mathscr{X}$ is indecomposable if and only if every connected subset is either dense or nowhere dense; or if and only if every closed, connected proper subset is nowhere dense.*

Proof. Suppose that $\mathscr{X} = A \cup B$ where A and B are closed connected sets such that $A \neq \mathscr{X} \neq B$. Hence, neither A nor B is nowhere dense (or dense).

Conversely, let C be a connected set which is neither dense nor nowhere dense, i.e. $\bar{C} \neq \mathscr{X} \neq \overline{\mathscr{X} - \bar{C}}$. Since the set $\mathscr{X} - \bar{C}$ is connected (by Theorem 1), the space $\mathscr{X}$ is decomposable because $\mathscr{X} = \bar{C} \cup \overline{\mathscr{X} - \bar{C}}$.

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[1] The existence of continua containing no arcs has been announced by S. Janiszewski at the Int. Congress of Math. in Cambridge, 1912.

THEOREM 3. *Every dense connected subset of an indecomposable space is indecomposable.*

Proof. Let C be a dense connected set, and let D be a connected subset of C. According to Theorem 2 we must prove that D is either dense or nowhere dense in C. But, since D is either dense or nowhere dense in the space (i.e., in $\bar{C}$) by Theorem 2, the required conclusion follows from Theorems 2 and 4 of § 8, VI.

THEOREM 4. *If $\mathscr{X}$ is a continuum and f is a continuous mapping of $\mathscr{X}$ such that $f(\mathscr{X})$ is indecomposable, then $\mathscr{X}$ contains an indecomposable subcontinuum.*

Such is each continuum irreducible with respect to the condition $f(C) = f(\mathscr{X})$.

Proof. If A and B are two subcontinua of C such that

$$A \neq C \neq B \quad \text{and} \quad C = A \cup B,$$

it follows that

$$f(A) \neq f(C) \neq f(B) \quad \text{and} \quad f(C) = f(A) \cup f(B).$$

VI. Composants.

DEFINITION. The set C of all points of the space $\mathscr{X}$, which can be joined with the point p by a closed, connected proper subset of $\mathscr{X}$, is said to be the *composant of the point* p ([1]).

Examples and remarks.

1. The composants of a non-connected space coincide with its components (compare § 46, III).

If the space $\mathscr{X}$ is connected, the set $\mathscr{X} - C$ coincides with the set of points x such that the space is irreducible between p and x.

If the space is connected, but p is not any of its irreduciblity points, it follows that $C = \mathscr{X}$.

So it is clear that the notion of the composant of a point p is of no interest unless p is an irreducibility point of the space.

2. It is easily proved ([2]) that the composant of the point zero in Example 1 of Section V, consists of an infinite sequence of half-

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([1]) Compare the concept of the nerve by L. E. J. Brouwer.

For further properties of the composants, see also H. Cook, *On subsets of indecomposable continua*, Colloq. Math. 13 (1964), pp. 37–43.

([2]) See Fund. Math. 5 (1924), p. 40.

circles which pass through the end-points of intervals contiguous to $\mathscr{C}$. This composant is a one-to-one continuous image of the half-line $x \geqslant 0$. All the other composants of this continuum are one-to-one continuous images of the whole straight line.

3. The family of composants of a given continuum is *strictly transitive in the sense of category* (1), i.e., if a set with the Baire property is the union of some composants, then either this set or its complement (with respect to the continuum considered as the whole space) is of the first category (2).

THEOREM 1. *If C is the composant of the point p, there exists a sequence of closed connected sets $K_1, K_2, \ldots$ such that*

$$C = K_1 \cup K_2 \cup \ldots, \qquad p \in K_n, \qquad K_n \neq \mathscr{X}. \tag{1}$$

Proof. Let $R_1, R_2, \ldots$ be the members of an open base of the space which do not contain the point p. Let K_n be the composant of the point p in the set $\mathscr{X} - R_n$. It follows that

$$p \in K_n = \overline{K}_n \neq \mathscr{X}, \quad \text{hence} \quad K_n \subset C \text{ and}$$
$$K_1 \cup K_2 \cup \ldots \subset C.$$

Conversely, if $x \in C$ and Q is a closed connected set such that $p, x \in Q \neq \mathscr{X}$, there exists a set R_n disjoint from Q. Hence $Q \subset \mathscr{X} - R_n$, which implies $Q \subset K_n$ and $x \in K_n$; finally, $C \subset K_1 \cup K_2 \cup \ldots$.

THEOREM 2. *If a composant C is not dense, it is closed. And then it is saturated with respect to the property of being a closed, connected proper subset of the space.*

Therefore (compare § 47, III, Theorem 5) *every composant of a continuum is dense.*

Proof. Suppose that $\overline{C} \neq \mathscr{X}$; it follows that $\overline{C} \subset C$, since $\overline{C}$ is a closed, connected proper subset of the space.

Remarks. 1. Example 2 of Section V shows that there exist sets which satisfy the hypothesis of Theorem 2.

2. The existence of a set saturated with respect to the properties considered in Theorem 2 is intimately related with the

(1) This property is a counterpart of the strict transitivity (in the sense of measure) considered in the Statistical Mechanics. For instance, see G. D. Birkhoff, Proc. Nat. Ac. Sc. 17 (1932), p. 650 and 656.

(2) See my paper of Fund. Math. 19 (1932), p. 252.

existence of an indecomposable subset. Actually, the following two theorems hold [1].

THEOREM 2′. *If in a connected space $\mathscr{X}$ the subset S is saturated with respect to the property of being a closed, connected proper subset of $\mathscr{X}$, then $\overline{\mathscr{X}-S}$ is indecomposable.*

THEOREM 2″. *Every connected space which contains two disjoint sets, saturated with respect to the said properties, is indecomposable.*

THEOREM 3. *If C is a composant of a continuum $\mathscr{X}$, the set $\mathscr{X}-C$ is connected.*

Proof. It is admissible to assume that the space $\mathscr{X}$ is irreducible between a and b and that C is the composant of the point a (compare Example and Remark 1). Let

$$\mathscr{X}-C = M \cup N, \quad \overline{M} \cap N = 0 = \overline{N} \cap M, \quad b \in M. \tag{2}$$

We have to show that $N = 0$.

Since the sets M and N are separated, there exists (compare § 14, V, Theorem 1) an open set G such that

$$M \subset G \text{ and } \overline{G} \cap N = 0, \quad \text{which yields}$$
$$(\overline{G}-G) \cap (M \cup N) = 0. \tag{3}$$

Let K be a composant of b in G. Suppose that $N \neq 0$, and hence that $G \neq \mathscr{X}$; it follows $\overline{K}-G \neq 0$ (by Theorem 2 of § 47, III). Let

$$p \in \overline{K}-G, \quad \text{therefore} \quad p \in \overline{G}-G \subset \mathscr{X}-(M \cup N) = C$$

according to (3) and (2). So, there exists a continuum P such that $a, p \in P \subset C$. Since $b, p \in \overline{K}$, the set $\overline{K} \cup P$ is a continuum joining a to b; thus

$$\overline{K} \cup P = \mathscr{X} \quad \text{and therefore} \quad N \subset \overline{K} \cup P.$$

Since $K \subset G$, it follows that $N \cap \overline{K} \subset N \cap \overline{G} = 0$ according to (3), and since $P \subset C$, it follows that $N \cap P \subset N \cap C = 0$ by (2). And hence $N = 0$.

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[1] See the paper of B. Knaster and myself, Fund. Math. 5 (1924), p. 45.

By Theorem 2 the complement of a composant of a continuum is always a boundary set. In that direction we have the following theorem.

THEOREM 4. *Let C be a composant of a continuum $\mathscr{X}$. If $\mathscr{X}-C$ is not a boundary set, it is an indecomposable domain.*

Proof. Assume as before that the space $\mathscr{X}$ is irreducible between a and b and that C is the composant of the point a.

Let $Q = \overline{\mathscr{X}-\overline{\mathscr{X}-C}}$. Assume that the set $\overline{\mathscr{X}-C}$ is not a boundary set, i.e. that $Q \neq \mathscr{X}$.

Since the set $\overline{\mathscr{X}-C}$ is a continuum containing the point b (by Theorem 3), the set Q is also a continuum (by II, Theorem 3). It follows that $Q \subset C$. Because, if we suppose that $Q \neq 0$, hence that $\overline{\mathscr{X}-C} \neq \mathscr{X}$, it follows that $a \in \mathscr{X}-\overline{\mathscr{X}-C} \subset Q$ (since $\mathscr{X}$ is irreducible between a and b) and $Q \subset C$ since $Q \neq \mathscr{X}$. It follows that $\mathscr{X}-C \subset \mathscr{X}-Q$, and

$$\overline{\mathscr{X}-C} \subset \overline{\mathscr{X}-Q} = \overline{\mathscr{X}-\overline{\mathscr{X}-\overline{\mathscr{X}-C}}} \subset \overline{\mathscr{X}-C},$$

and therefore

$$\overline{\mathscr{X}-C} = \overline{\mathscr{X}-Q}, \tag{4}$$

which proves that $\overline{\mathscr{X}-C}$ is a closed domain.

Suppose that $\overline{\mathscr{X}-C}$ is decomposable. Let M and N be two continua such that

$$\overline{\mathscr{X}-C} = M \cup N \quad \text{and} \quad M \neq \overline{\mathscr{X}-C} \neq N. \tag{5}$$

It follows that

$$M-C \neq 0 \neq N-C. \tag{6}$$

Because the condition $N-C = 0$ implies $\mathscr{X}-C \subset \mathscr{X}-N$, hence according to (5),

$$\mathscr{X}-C \subset \overline{\mathscr{X}-C}-N \subset M, \quad \text{and therefore} \quad \overline{\mathscr{X}-C} \subset M,$$

which contradicts inequality (5).

Consider two cases according to whether $Q = 0$ or $Q \neq 0$. In the first case it follows that $\overline{\mathscr{X}-C} = \mathscr{X} = M \cup N$. Let $a \in M$. But then condition (6) implies that $M = \mathscr{X}$, which contradicts inequality (5).

Now consider the second case. In that case, $a \epsilon Q$ as we have already proved. Since the space is connected, the identity

$$\mathscr{X} = \overline{\mathscr{X} - \overline{\mathscr{X} - C}} \cup \overline{\mathscr{X} - C} = Q \cup M \cup N$$

(compare (5)) implies that $Q \cap (M \cup N) \neq 0$. It can be assumed that $Q \cap M \neq 0$. Therefore the set $Q \cup M$ is a continuum which joins the point a with a point of $M - C$ (according to (6)). Consequently

$$Q \cup M = \mathscr{X}, \quad \text{i.e.} \quad \mathscr{X} - Q \subset M, \quad \text{whence} \quad \overline{\mathscr{X} - Q} \subset M,$$

and identities (4) and (5) imply that $\overline{\mathscr{X} - C} = M$, contrary to (5).

THEOREM 5. *The composants of an indecomposable space are disjoint.*

Proof. Let P be the composant of p and Q the composant of q. Suppose that $a \epsilon P \cap Q$ and $b \epsilon P - Q$. So there exist three closed connected sets A, B and C such that

$$p, a \epsilon A \neq \mathscr{X}, \quad p, b \epsilon B \neq \mathscr{X}, \quad q, a \epsilon C \neq \mathscr{X}.$$

Therefore $A \cup B \cup C$ is a closed connected set joining b to q. Hence

$$A \cup B \cup C = \mathscr{X}.$$

It follows that the space is decomposable. This conclusion is obvious if $A \cup B = \mathscr{X}$; on the other hand, if $A \cup B \neq \mathscr{X}$, the formula $\mathscr{X} = (A \cup B) \cup C$ yields the required decomposition.

Theorem 1 combined with Theorem 2 of Section V implies the following one.

THEOREM 6. *Every composant of an indecomposable space is an F_σ-set of the first category.* (1)

The next theorem follows by Baire Theorem.

THEOREM 7. *Every complete indecomposable space contains a countable infinity of composants.*

Every point of this space is a point of irreducibility.

More precisely, *the set of points x such that the space is irreducible between a and x is a dense G_δ.*

(1) Theorem of S. Mazurkiewicz, Fund. Math. 1 (1920), p. 35.

Remarks. (i) The first part of Theorem 7 can be strengthened in the following way.

In every indecomposable continuum there exists a perfect set (of the power $\mathfrak{c}$*) which contains at most one point of each composant.* [1]

(ii) Theorem 7 implies the following statement.

THEOREM 7'. *A complete connected space $\mathcal{X}$ is indecomposable if and only if $\mathcal{X}$ contains three points a, b and c between every pair of which it is irreducible*; *equivalently*: *if every point of $\mathcal{X}$ is a point of irreducibility.*

Proof. The necessity of this condition follows from the fact that the space contains at least 3 composants. To show its sufficiency, suppose that K and L are closed, connected and that

$$\mathcal{X} = K \cup L, \quad K \neq \mathcal{X} \neq L.$$

If a, b and c are any three points of $\mathcal{X}$, then either K or L (or both) contain two of them. So $\mathcal{X}$ is not irreducible between them.

THEOREM 8. *Every connected space containing a composant, which is a boundary set, is indecomposable.*

Consequently (compare Theorem 7), *a complete connected space is indecomposable if and only if it contains a composant, which is a boundary set.*

Proof. Let p be a point whose composant P is a boundary set, i.e. $\overline{\mathcal{X}-P} = \mathcal{X}$. If the space is decomposable, then

$$\mathcal{X} = A \cup B, \quad A \neq \mathcal{X} \neq B,$$

where A and B are closed and connected.

Let $p \in A$. Thus $A \subset P$, hence

$$\mathcal{X}-P \subset \mathcal{X}-A \subset B, \quad \text{and therefore} \quad \mathcal{X} = \overline{\mathcal{X}-P} \subset B,$$

which contradicts the inequality $B \neq \mathcal{X}$.

Remark. Theorem 8 combined with Remark 2, p. 208, implies that Example 1 of Section V is an indecomposable continuum.

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[1] See S. Mazurkiewicz, Fund. Math. 10 (1927), p. 305.

THEOREM 9. *A continuum $\mathscr{X}$ irreducible between a and b is indecomposable if and only if it contains a semi-continuum S, which is a dense boundary set containing a* (1).

Proof. Let S be the composant of the point a. Clearly, the condition is necessary (compare Theorems 2 and 6).

To prove its sufficiency, it remains (by Theorem 8) to show that the composant C of b is a boundary set, or that $C \cap S = 0$ (since S is dense).

But if $C \cap S \neq 0$, there exist two continua K and L such that

$$a \in K \subset S, \quad b \in L \subset C \quad \text{and} \quad K \cap L \neq 0.$$

Since the space is irreducible between a and b, it follows that

$$K \cup L = \mathscr{X}, \text{ thus } \mathscr{X}-S \subset \mathscr{X}-K \subset L, \text{ so that } \overline{\mathscr{X}-S} \subset L.$$

By hypothesis, $\overline{\mathscr{X}-S} = \mathscr{X}$, therefore $L = \mathscr{X}$, and since $L \subset C$, it follows that $a \in C$. But this contradicts the hypothesis that $\mathscr{X}$ is irreducible between a and b.

VII. Indecomposable subsets of irreducible spaces.

THEOREM 1. *If C is an indecomposable closed subset of a space $\mathscr{X}$ irreducible between a and b, then C is either a closed domain or a nowhere dense set. In the first case,* (*if $C \neq 0$*) *there exists in the family* $\boldsymbol{D}$ (*considered in Section* III) *a set D such that D and $D \cup C$ constitute a jump.*

Proof. According to Theorem 5 of Section II, the set $A = \mathscr{X}-\overline{\mathscr{X}-C}$ is connected. So, by Theorem 2 of Section V, A is either dense or nowhere dense in C.

In the first case, $\overline{\mathscr{X}-\overline{\mathscr{X}-C}} = C$, which means that C is a closed domain.

In the second case, A is nowhere dense (in $\mathscr{X}$), and therefore is empty (as an open set), so it follows that $\overline{\mathscr{X}-C} = \mathscr{X}$.

Now consider the second part of the theorem. Let C be an indecomposable, non-empty closed domain. If $C = \mathscr{X}$, it is sufficient to assume $D = 0$. So let $C \neq \mathscr{X}$. It follows that either $a \in \mathscr{X}-C$

(1) Theorem of Urysohn, Fund. Math. 8 (1926), p. 226.

or $b \in \mathcal{X}-C$. By the symmetry, it can be assumed that $a \in \mathcal{X}-C$. According to Theorem 3 of Section II, $\overline{\mathcal{X}-C}$ consists of two closed connected domains one of which, say D, contains a and the other is empty or contains b. Since $D \cap C \neq 0$, the set $D \cup C$ is connected and belongs to $\boldsymbol{D}$ (since it is a closed domain, compare § 8, VIII).

Let
$$D^* \in \boldsymbol{D}, \qquad D \subset D^* \subset D \cup C.$$

We have to prove that either $D^* = D$ or $D^* = D \cup C$.

According to Theorem 1 of Section III, a is an irreducibility point of D^*. Thus, by Theorem 3 of Section II, $\overline{D^*-D}$ is connected. Since $\overline{D^*-D}$ is a closed domain relatively to D^*, which itself is a closed domain, then $\overline{D^*-D}$ is a closed domain (compare § 8, VIII). Therefore $\overline{D^*-D}$, as a subset of C, is its connected, relatively closed subdomain. Since C is indecomposable, it follows by Theorem 2 of Section V that either $\overline{D^*-D} = 0$ or $\overline{D^*-D} = C$. In the first case $D^* = D$, and in the second one

$$D \cup C = D \cup \overline{D^*-D} \subset D \cup D^* = D^*, \quad \text{thus} \quad D^* = D \cup C.$$

Conversely, the following theorem holds.

THEOREM 2. *If the elements D_{y_1} and D_{y_2} of $\boldsymbol{D}$ constitute a jump, the set $\overline{D_{y_2}-D_{y_1}}$ is indecomposable.*

Proof. By Theorem 1 of Section III and Theorem 3 of Section II the set $\overline{D_{y_2}-D_{y_1}}$ is connected. Let A and B be two closed connected sets such that $\overline{D_{y_2}-D_{y_1}} = A \cup B$. We have to prove that one of them coincides with $\overline{D_{y_2}-D_{y_1}}$. Let $A^* = \overline{\mathcal{X}-\overline{\mathcal{X}-A}}$ and $B^* = \overline{\mathcal{X}-\overline{\mathcal{X}-B}}$. According to Theorem 5 of Section II, the sets A^* and B^* are connected. Moreover, the set $A \cup B = \overline{D_{y_2}-D_{y_1}}$ is a closed domain (as a closed domain relatively closed in the closed domain D_{y_2}):

$$A \cup B = \overline{\mathcal{X}-\overline{\mathcal{X}-(A \cup B)}},$$

which implies (compare § 8, VII (i))

$$A^* \cup B^* = A \cup B = \overline{D_{y_2}-D_{y_1}}.$$

If $D_{y_1} \neq 0$, we have either $D_{y_1} \cap A^* \neq 0$ or $D_{y_1} \cap B^* \neq 0$. It can be assumed that the first inequality holds. Consequently,

$(D_{y_1} \cup A^*) \epsilon \boldsymbol{D}$. The same condition is satisfied provided that $D_{y_1} = 0$ and A^* denotes this one of the two sets A^* and B^* which contains a.

It follows by the hypothesis that

$$\text{either} \quad D_{y_1} \cup A^* = D_{y_1} \quad \text{or} \quad D_{y_1} \cup A^* = D_{y_2}.$$

In the first case

$$D_{y_2} = D_{y_1} \cup \overline{D_{y_2} - D_{y_1}} = D_{y_1} \cup A^* \cup B^* = D_{y_1} \cup B^*,$$

hence

$$D_{y_2} - D_{y_1} \subset B^* \subset B, \quad \text{and therefore} \quad \overline{D_{y_2} - D_{y_1}} = B.$$

In the second case

$$D_{y_2} - D_{y_1} \subset A^* \subset A, \quad \text{hence} \quad \overline{D_{y_2} - D_{y_1}} = A.$$

Theorems 1 and 2 imply the following

THEOREM 3. *The family* $\boldsymbol{D}$ *has the order type of the interval if and only if the considered irreducible space contains no closed indecomposable subset which is not a boundary set.*

Remark. *If* F *is a closed subset of* $\mathscr{I}$ *containing the points* 0 *and* 1, *there exists an irreducible continuum between* 0 *and* 1 *for which* $J = F$ (in other words, such that $\boldsymbol{D}$ is similar to F).

In fact, it is only necessary to replace each interval contiguous to F by the continuum of Example 3 of Section V properly diminished and having the same end-points as that interval.

In particular, the pair $(0, 1)$ coincides with the set J for any non-empty indecomposable continuum.

THEOREM 4. *The layers of a continuum* $\mathscr{X}$, *irreducible between* a *and* b, *coincide with the sets saturated with respect to the property of being a continuum, which is the union of a (finite or infinite) sequence of nowhere dense continua and indecomposable continua.*

Proof. Let f be a continuous monotone mapping such that $f(\mathscr{X}) = \mathscr{I}$. Let C be a nowhere dense continuum, $\overline{\mathscr{X} - C} = \mathscr{X}$. We are going to show first that $f(C)$ consists of a single point.

Suppose, conversely, that $f(C) = \alpha\beta$, where $0 \leqslant \alpha < \beta \leqslant 1$. Since the sets $f^{-1}(0\alpha)$ and $f^{-1}(\beta 1)$ are continua (compare § 47, VI, Theorem 4) one of which contains a and the other b and which have points in common with C, it follows that

$$f^{-1}(0a) \cup C \cup f^{-1}(\beta 1) = \mathscr{X}, \quad \text{which implies}$$

$$\mathscr{X} - C \subset f^{-1}(0a) \cup f^{-1}(\beta 1),$$

thus $\overline{\mathscr{X}-C} \subset f^{-1}(0a) \cup f^{-1}(\beta 1)$ and, since $\overline{\mathscr{X}-C} = \mathscr{X}$, it follows finally that $\mathscr{X} = f^{-1}(0a) \cup f^{-1}(\beta 1)$, which contradicts the formula $f(\mathscr{X}) = \mathscr{I}$.

Thus $f(C)$ consists of only one point, and $f(S) = (p)$, provided S is a semi-continuum and the union of a sequence of nowhere dense continua. Since every indecomposable continuum K is, by Theorems 1 and 6 of Section VI, the closure of a semi-continuum S of that kind, it follows that $f(K) = (p)$. It follows, finally, that, if Q is a continuum, which is the union of a sequence of nowhere dense continua and of indecomposable continua, then $f(Q)$ consists also of only one point; in other words, Q is contained in one layer (all that is needed is to substitute the function g of Section IV for f).

Conversely, according to Theorem 4 of Section IV

$$T_t = \bigcup (I_y \cup J_y) = \bigcup (\bar{I}_y \cup \bar{J}_y),$$

where the index y runs over the (countable) set of y's such that $y \in J$ and $\gamma(t) \leqslant y \leqslant \Gamma(t)$. Since each $\bar{I}_y$ and $\bar{J}_y$ is either a nowhere dense continuum or an indecomposable continuum (compare Theorem 4 of Section VI), the layer T_t is the union of a sequence of nowhere dense continua and of indecomposable continua. Since every set of that kind is contained, as we have just proved, in only one layer, and the layers are pairwise disjoint, it follows that each layer is saturated with respect to the said properties.

THEOREM 5. *The layers of an irreducible continuum, which has the order type of the interval, coincide with the nowhere dense saturated continua* [1] *(i.e., with nowhere dense continua, which are not proper subcontinua of any other nowhere dense continuum).*

Proof. Every nowhere dense continuum is contained in only one layer (as we have just shown). On the other hand, since the space does not contain any indecomposable continuum, which is

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[1] There are many properties of nowhere dense, saturated continua in my paper of Fund. Math. 10, § 2.

not nowhere dense, every layer is, by Theorem 4, the union of a sequence of nowhere dense continua; thus it is a nowhere dense continuum.

THEOREM 6. *Every continuum irreducible between two points, which contains no nowhere dense continuum (consisting of more than one point), is an arc.* (1)

Proof. The considered continuum does not contain any indecomposable subcontinuum (consisting of more than one point), because every indecomposable continuum has nowhere dense subcontinua (containing more than one point, compare Theorem 2 of Section V). Therefore, according to Theorems 3 and 5, the layers are nowhere dense continua, so they are individual points. Consequently, the function g (of Section IV) is a homeomorphism.

THEOREM 7. *Every discoherent decomposable space $\mathscr{X}$ which is irreducible between a_0 and a_1 is the union of two indecomposable sets A_0 and A_1 such that*

$$A_0 = \overline{\mathscr{X}-A_1} \quad \textit{and} \quad A_1 = \overline{\mathscr{X}-A_0}\ (^2). \tag{1}$$

Proof. According to § 46, X, Theorem 2, let A_0 and A_1 be two connected sets satisfying condition (1) and such that $A_0 \neq \mathscr{X} \neq A_1$. So let $a_0 \in A_0$ and $a_1 \in A_1$.

Suppose that

$$A_1 = B_0 \cup B_1, \quad B_0 \neq A_1 \neq B_1,$$

where B_0 and B_1 are closed and connected. Let $a_1 \in B_1$. By Theorem 1 of § 46, X the set $\overline{\mathscr{X}-B_0}$ is connected. But

$$\overline{\mathscr{X}-B_0} = \overline{A_0-B_0} \cup \overline{A_1-B_0} = \overline{A_0-B_0} \cup \overline{B_1-B_0}$$

and

$$A_0-B_0 \neq 0 \neq B_1-B_0 \quad \text{thus} \quad 0 \neq \overline{A_0-B_0} \cap \overline{B_1-B_0} \subset A_0 \cap B_1.$$

(1) Theorem of Janiszewski, Thesis. See also § 47, V, Theorem 1. Compare Hallett, Bull. Amer. Math. Soc. 25 (1919).

(2) See Math. Ann. 98 (1927), p. 403. Compare P. Alexandrov, Math. Ann. 96, p. 537.

Therefore, $A_0 \cup B_1$ is connected and, since $a_0 \in A_0$ and $a_1 \in B_1$, it follows that

$$A_0 \cup B_1 = \mathscr{X}, \text{ i.e. } \mathscr{X} - A_0 \subset B_1, \text{ thus } A_1 = \overline{\mathscr{X} - A_0} \subset B_1,$$

contradicting the inequality $A_1 \neq B_1$.

VIII. Spaces irreducibly connected between A and B. Such is a space $\mathscr{X}$ if it is connected between A and B, but no subset of the form $F \cup A \cup B$ is connected between A and B for any $F = \overline{F} \neq \mathscr{X}$; in other words, if the space is irreducible with respect to the property of being a closed set X such that $X \cup A \cup B$ is connected between A and B.

THEOREM 1. *If the space $\mathscr{X}$ is irreducibly connected between A and B, the sets A and B are separated and non-empty and the space $\mathscr{X}$ is connected.*

Consequently, *if $A \neq 0 \neq B$, the connectedness of the space $\mathscr{X}$ between A and B can be replaced in the definition simply by its connectedness.*

Proof. We have to prove that the space is connected since the remainder of the theorem is a straightforward consequence of § 46, IV, 1a.

Suppose that

$$\mathscr{X} = M_0 \cup M_1, \quad M_0 \cap M_1 = 0, \quad M_j = \overline{M}_j \neq \mathscr{X}, \quad \text{where} \quad j = 0, 1.$$

The last inequality implies by the hypothesis that $M_j \cup A \cup B$ is not connected between A and B, and therefore, that M_j is not connected between $A \cap M_j$ and $B \cap M_j$.

Hence it follows that

$$M_j = P_j \cup Q_j, \quad P_j = \overline{P}_j, \quad Q_j = \overline{Q}_j, \quad P_j \cap Q_j = 0, \quad A \cap P_j = 0 = B \cap Q_j.$$

Assume that $P = P_0 \cup P_1$ and $Q = Q_0 \cup Q_1$. It follows that

$$\mathscr{X} = P \cup Q, \quad \overline{P} = P, \quad \overline{Q} = Q, \quad P \cap Q = 0 \quad \text{and} \quad A \cap P = 0 = B \cap Q,$$

which shows that the space is not connected between A and B.

THEOREM 2. *A space irreducibly connected between A and B is irreducible between every pair of points $a \in A$, $b \in B$.*

Proof. If C is a closed connected set such that $a, b \in C$, the set $C \cup A \cup B$ is obviously connected between A and B; thus $C = \mathscr{X}$.

Remark. The converse theorem is not generally true (it holds in compact spaces, compare Theorem 2 of Section IX).

In order to show this consider Example 2 of Section V. The considered space is irreducible between $a = (0, 0)$ and $b = (1, 0)$. However it is not irreducibly connected between these points; because the sequence of half-circles with the center $(\frac{1}{2}, 0)$ passing through the points $(3^{-n}, 0)$, where $n = 0, 1, \ldots$ (for $n = 0$, the point $(\frac{1}{2}, \frac{1}{2})$ of the half-circle is deleted), constitutes a closed set, connected between a and b.

THEOREM 3. *If the space is irreducibly connected between A and B, it is also irreducibly connected between any pair A_1, B_1 where $0 \neq A_1 \subset A$ and $0 \neq B_1 \subset B$.*

Proof. Since $A_1 \neq 0 \neq B_1$ and since the space is connected, it is connected between A_1 and B_1 and, on the other hand, if $F \cup A \cup B$ is not connected between A and B, then $F \cup A_1 \cup B_1$ is not connected between A_1 and B_1 either.

THEOREM 4. *If the space $\mathscr{X}$ is irreducibly connected between A and $B_0 = \overline{B}_0$ and between A and $B_1 = \overline{B}_1$, then $\mathscr{X}$ is irreducibly connected between A and $B_0 \cup B_1$.*

Proof. According to Theorem 1a of § 46, IV, the space $\mathscr{X}$ is connected between A and $B_0 \cup B_1$ (because $B_0 \subset B_0 \cup B_1$). On the other hand, let $F = \overline{F} \neq \mathscr{X}$. Hence the set $F \cup A \cup B_j$, $j = 0, 1$, is not connected between A and B_j and therefore it is different from $\mathscr{X}$; thus $F \cup B_j \neq \mathscr{X}$. It follows that $(F \cup B_0) \cup A \cup B_1$ is connected neither between A and B_1 nor between A and B_0 (by reason of symmetry); therefore, by Theorem 3 of § 46, IV, this set is not connected between A and $B_0 \cup B_1$ either.

THEOREM 5. *If the space $\mathscr{X}$ is irreducibly connected between the closed sets A and B, the set $\mathscr{X} - (A \cup B)$ is connected and dense in $\mathscr{X}$.*

Proof. Suppose that $\mathscr{X} - (A \cup B)$ is not connected. Let G and H be two open sets such that

$$\mathscr{X} - (A \cup B) = G \cup H, \quad G \cap H = 0, \quad G \neq 0 \neq H.$$

Since the set $G \cup A \cup B = \mathscr{X} - H \neq \mathscr{X}$ is closed, it is not connected between A and B. So it follows that

$$G \cup A \cup B = P \cup Q, \quad 0 = P \cap Q = A \cap P = B \cap Q,$$
$$H \cup A \cup B = W \cup Z, \quad 0 = W \cap Z = A \cap W = B \cap Z,$$

where the sets P, Q, W and Z are closed.

Therefore, $\mathscr{X} = (P \cup W) \cup (Q \cup Z)$ is a decomposition of the space into two closed and disjoint sets, since

$$\begin{aligned} P \cap Z &= \big(P \cap (Z \cap H)\big) \cup \big(P \cap (Z \cap A)\big) \\ &= \big((P \cap G) \cap (Z \cap H)\big) \cup \big((P \cap B) \cap (Z \cap H)\big) = 0, \end{aligned}$$

one of which contains B and the other A.

Since the space is connected, the formulas

$$\mathscr{X} = A \cup \overline{\mathscr{X} - (A \cup B)} \cup B \quad \text{and} \quad A \cap B = 0$$

imply that

$$A \cap \overline{\mathscr{X} - (A \cup B)} \neq 0 \neq B \cap \overline{\mathscr{X} - (A \cup B)}.$$

Since the set $\overline{\mathscr{X} - (A \cup B)}$ is connected, it follows that the set $\overline{\mathscr{X} - (A \cup B)} \cup A \cup B$ is connected between A and B. This implies by the hypothesis that $\overline{\mathscr{X} - (A \cup B)} = \mathscr{X}$.

THEOREM 6. *If two closed sets E_j, $j = 0, 1$, are irreducibly connected between $A_j = \bar{A}_j$ and $E_0 \cap E_1$, the union $E_0 \cup E_1$ is irreducibly connected between A_0 and A_1* (1).

Proof. Since the set $E_0 \cup E_1$ is connected, it remains to show that given any set

$$F = \bar{F} \subset E_0 \cup E_1 \neq F,$$

the set $F \cup A_0 \cup A_1$ is not connected between A_0 and A_1.

But, the condition $F \neq E_0 \cup E_1$ implies that either $F \cap E_0 \neq E_0$ or $F \cap E_1 \neq E_1$. Assume that $F \cap E_0 \neq E_0$. Thus $(F \cap E_0) \cup A_0 \cup \cup (E_0 \cap E_1)$ is not connected between A_0 and $E_0 \cap E_1$, i.e.

$$(F \cap E_0) \cup A_0 \cup (E_0 \cap E_1) = M \cup N, \quad M = \bar{M}, \quad N = \bar{N},$$
$$0 = M \cap N = M \cap E_1 = N \cap A_0.$$

(1) Compare J. R. Kline, Fund. Math. 7 (1925), p. 315.

It follows that

$$F \cup A_0 \cup A_1 = (F \cap E_0) \cup A_0 \cup (F \cap E_1) \cup A_1 \subset M \cup (N \cup E_1),$$

where $M \cap (N \cup E_1) = 0$, $A_0 \subset M$ (because $A_0 \cap N = 0$) and $A_1 \subset N \cup E_1$.

Thus $F \cup A_0 \cup A_1$ is not connected between A_0 and A_1.

IX. Irreducibly connected compact spaces.

THEOREM 1. *Every compact space connected between two closed disjoint sets A and B contains a closed set C irreducibly connected between $C \cap A$ and $C \cap B$.*

Proof. According to Theorem 1 of § 47, VII, the family of closed sets F, connected between A and B, is closed. And so is the family $\boldsymbol{F}$ of closed sets X such that $X \cup A \cup B$ is connected between A and B, because $X \cup A \cup B$ represents a continuous function of the variable X (compare § 17, III, Corollary 4a).

Let C be an irreducible element of the family $\boldsymbol{F}$.

C is connected between $C \cap A$ and $C \cap B$, because otherwise we would have

$$C = M \cup N, \quad C \cap A \subset M = \overline{M}, \quad C \cap B \subset N = \overline{N},$$
$$M \cap N = 0,$$

and the decomposition

$$C \cup A \cup B = (A \cup M) \cup (B \cup N)$$

into two closed disjoint sets would be inconsistent with the condition $C \in \boldsymbol{F}$.

Moreover, if $H = \overline{H} \subset C \neq H$, the set $H \cup (C \cap A) \cup (C \cap B)$ is not connected between $C \cap A$ and $C \cap B$ since $H \cup A \cup B$ is not connected between A and B.

THEOREM 2. *A compact space $\mathscr{X}$ is irreducibly connected between two closed disjoint sets A and B if and only if $\mathscr{X}$ is irreducible between every pair of points a, b where $a \in A$ and $b \in B$.*

Proof. According to Theorems 1 and 2 of Section VIII it remains to show that if $\mathscr{X}$ is irreducible between every pair $a \in A$ and $b \in B$, and C is closed and irreducibly connected between $A \cap C$ and $B \cap C$, then $C = \mathscr{X}$.

Let $a \in A \cap C$ and $b \in B \cap C$. Since the set C is closed, connected (according to VIII, Theorem 1) and joins a to b, it follows that $C = \mathscr{X}$.

THEOREM 3. *If an indecomposable continuum $\mathscr{X}$ is irreducibly connected between two closed sets A_0 and A_1, then there exists a composant C such that $C \cap (A_0 \cup A_1) = 0$.*

Proof. Let $R_0, R_1, \ldots$ be the base of $\mathscr{X}$, and let S_{jn}, where $j = 0, 1$, be the union of components of $\mathscr{X} - R_n$ which intersect A_j. The union

$$S_j = S_{j0} \cup S_{j1} \cup \ldots \tag{1}$$

is identical with the union of the composants of the space which intersect the set A_j; because, if K is a continuum such that $K \cap A_j \neq 0$ and $K \neq \mathscr{X}$, there exists a number n such that $K \subset \mathscr{X} - R_n$, and therefore such that $K \subset S_{jn} \subset S_j$.

So it is sufficient to show that S_j is of the 1st category; or—since S_j is an $\boldsymbol{F}_\sigma$ (according to (1))—that S_j is a boundary set, i.e.

$$\overline{\mathscr{X} - S_j} = \mathscr{X}. \tag{2}$$

But, each composant contained in S_{1-j} is dense in the space (compare VI, Theorem 2); hence

$$\overline{S}_{1-j} = \mathscr{X}. \tag{3}$$

Since $\mathscr{X}$ is irreducible between every pair $a_0 \in A_0$, $a_1 \in A_1$, it follows that

$$S_0 \cap S_1 = 0, \text{ thus } S_{1-j} \subset \mathscr{X} - S_j, \text{ and hence } \overline{S}_{1-j} \subset \overline{\mathscr{X} - S_j},$$

which implies (2) by (3).

THEOREM 4. *If E_0 and E_1 are two indecomposable continua and A_0 and A_1 two closed sets such that $E_0 \cap E_1 = A_0 \cup A_1$ and E_j (where $j = 0, 1$) is irreducibly connected between A_0 and A_1, then the union $E_0 \cup E_1$ is irreducible between two points.*

Proof. According to Theorem 3, there exists a composant C_j of E_j, where $j = 0, 1$, such that

$$C_j \cap (A_0 \cup A_1) = 0, \quad \text{therefore such that} \quad C_j \cap E_0 \cap E_1 = 0.$$

Let $a_j \in C_j$. The continuum E_j is irreducible between a_j and every point of $E_0 \cap E_1$ and therefore (by Theorem 2) is irreducibly connected between a_j and $E_0 \cap E_1$.

It follows, according to Theorem 6 of Section VIII, that $E_0 \cup E_1$ is irreducible between a_0 and a_1.

X. Additional remarks. A *chain* is a finite collection of open sets $G_1, \ldots, G_n$ such that $G_i \cap G_j \neq 0$ if and only if $|i-j| \leqslant 1$. If $\delta(G_i) < \varepsilon$ for $i = 1, \ldots, n$, the chain is called an *ε-chain*. A continuum is called *snake-like* if for each positive ε it can be covered by an ε-chain (1).

Let us quote without proof the following interesting statements on snake-like continua and on related problems.

1. A snake-like continuum is irreducible between a pair of its points.

2. Every subcontinuum of a snake-like continuum is snake-like.

3. Every two snake-like hereditarily indecomposable continua (containing more than one point) are homeomorphic (such continua are called *pseudo-arcs* (2); such is, in particular, the Knaster continuum mentioned in Remark 4 of Section V).

4. A pseudo-arc has the fixed point property (3).

5. Each snake-like continuum is contained topologically in the plane.

6. There exist hereditarily indecomposable continua *of all dimensions* (4).

7. The totality of topological types of (plane) hereditarily indecomposable continua is of *power* $\mathfrak{c}$ (5).

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(1) For the definitions and for most of the theorems, see R. H. Bing, *Snake-like continua*, Duke Math. Journ. 18 (1951), pp. 653–663.

Compare also L. K. Barrett, *The structure of decomposable snake-like continua*, ibid. vol. 28 (1961), pp. 515–521; C. E. Burgess, *Chainable continua and indecomposability*, Pacific Journ. Math. 9 (1959), pp. 655–660; J. B. Fugate, *Decomposable chainable continua*, Trans. Amer. Math. Soc. 123 (1966), pp. 460–468, Topological Seminar Wisconsin 1965,; R. M. Schori, *A universal snake-like continuum*, Proc. Amer. Math. Soc. 16 (1965), pp. 1313–1316.

(2) E. E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua*, Trans. Amer. Math. Soc. 63 (1948), p. 583.

(3) O. H. Hamilton, *A fixed point theorem for pseudo-arcs and certain other metric continua*, Proc. Amer. Math. Soc. 2 (1951), pp. 173–174.

(4) R. H. Bing, *Higher-dimensional hereditarily indecomposable continua*, Trans. Amer. Math. Soc. 71 (1951), pp. 267–273. For the infinite dimension, compare J. L. Kelley, *The hyperspaces of a continuum*, ibid. 52 (1942), pp. 22–36.

(5) R. H. Bing, *Concerning hereditarily indecomposable continua*, Pacific Journ. Math. 1 (1951), pp. 43–52.

8. There is a monotone open mapping $f\colon \mathscr{S}_2 \to \mathscr{S}_2$ such that all point-inverses $f^{-1}(y)$ are pseudo-arcs (1).

9. A pseudo-arc is homeomorphic to each of its subcontinua (containing more than one point) (2).

10. Every decomposable continuum which is homeomorphic to each of its subcontinua (containing more than one point) is an arc (3).

11. In the space of all subcontinua of the square $\mathscr{I}^n$ the pseudo-arcs and the homogeneous continua as well form a residual set. A pseudo-arc is homogeneous (4).

12. Each homogeneous snake-like continuum (containing more than one point) is a pseudo-arc (5).

It follows that *there exists in the plane a homogeneous continuum different from the simple closed curve* (question raised by B. Knaster and myself in Fund. Math. 1 (1920), p. 223). Let us recall that, by a theorem of Mazurkiewicz, *every plane homogeneous and locally connected continuum is a simple closed curve* (6).

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(1) See R. D. Anderson, Proc. Nat. Acad. Sc. 42 (1956), pp. 347–349.

(2) Theorem of E. E. Moise, *loc. cit.*, p. 594. This theorem gives a solution to the problem of S. Mazurkiewicz published in Fund. Math. 2 (1921), p. 286 (the question was whether there exists a continuum different from an arc which has the considered property).

(3) G. W. Henderson, Annals of Math. 12 (1960), pp. 421–428.

(4) See R. H. Bing, *A homogeneous indecomposable plane continuum*, Duke Math. Journ. 15 (1948), pp. 729–742; E. E. Moise, *A note on the pseudo-arc*, Trans. Amer. Math. Soc. 64 (1949), pp. 57–58.

See also R. H. Bing and F. B. Jones, *Another homogeneous plane continuum*, Trans. Amer. Math. Soc. 90 (1959), pp. 171–192.

(5) R. H. Bing, Proc. Amer. Math. Soc. 10 (1959), p. 345.

(6) S. Mazurkiewicz, *Sur les continus homogènes*, Fund. Math. 5 (1924), pp. 137–146.

In that direction, see H. J. Cohen, *Some results concerning homogeneous plane continua*, Duke Math. Journ. 18 (1951), pp. 467–474; F. B. Jones, *A note on homogeneous plane continua*, Bull. Amer. Math. Soc. 55 (1949), pp. 113–114 and *On a certain type of homogeneous plane continuum*, Proc. Amer. Math. Soc. 6 (1955), pp. 735–740; C. E. Burgess, *Some theorems on n-homogeneous continua*, *ibid.* 5 (1954), pp. 136–143, *Certain types of homogeneous continua*, *ibid.* 6 (1955), pp. 348–350, *Some condition under which a homogeneous continuum is a simple closed curve*, *ibid.* 10 (1959), pp. 613–615, and *Homogeneous continua*, Summer Institute on Set Theoretic Topology, Madison 1955, pp. 73–76.

13. The cartesian product of n snake-like continua is topologically contained in $\mathscr{E}^{n+1}$ (1).

14. Each snake-like continuum is a continuous image of the pseudo-arc (2).

15. Each snake-like continuum is an inverse limit of arcs with projections being continuous mappings onto (3).

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(1) R. Benneth, *Embedding products of chainable continua*, Proc. Nat. Acad. Sc. 16 (1965), p. 1026.

(2) See J. Mioduszewski, *A functional conception of snake-like continua*, Fund. Math. 51 (1962), pp. 178–189. See also G. R. Lehner, *Extending homeomorphisms on the pseudo-arc*, Trans. Amer. Math. Soc. 98 (1961), pp. 369–394.

(3) See J. R. Isbell, *Embedding of inverse limits*, Ann. of Math. 70 (1959), pp. 73–84. See also G. W. Henderson, *The pseudo-arc as an inverse limit with one binding map*, Duke Math. Journ. 31 (1964), p. 421; B. Pasynkov, *On snake-like compact spaces*, Čechoslov. Math. Journ. 13 (1963), p. 474.

For continuous images of snake-like continua, see A. Lelek, Fund. Math. 51 (1962), pp. 271–283, and L. Fearnley, Trans. Amer. Math. Soc. 111 (1964), pp. 380–392, Proc. London Math. Soc. 15 (1965), pp. 289–300, and Pacific Journ. Math. 23 (1967), pp. 491–513.

CHAPTER SIX

LOCALLY CONNECTED SPACES

§ 49. Local connectedness

I. Points of local connectedness.

DEFINITION. A topological space is said to be *locally connected at the point* p (*l.c. at the point* p) (1) if every open neighbourhood G of p contains a connected neighbourhood of p; in other words, if C denotes the component of p in G, then $p \in \mathrm{Int}(C)$. If the space is metric, this means that for every $\varepsilon > 0$ there exists a connected neighbourhood E of p such that $\delta(E) < \varepsilon$.

Remark. The variability of G in the above Definition can be restricted to an open base $\boldsymbol{B}$ of the space.

For, let $p \in G$ where G is an arbitrary open set. Then there is $R \in \boldsymbol{B}$ such that $p \in R \subset G$. Let C be the component of p in R and D the component of p in G. By assumption, $p \in \mathrm{Int}(C)$. But $C \subset D$. Hence $\mathrm{Int}(C) \subset \mathrm{Int}(D)$ and thus $p \in \mathrm{Int}(D)$. This means that the space is l.c. at p.

The local connectedness is a topological invariant (by the definition). It is a local property, this means:

THEOREM 0. *If G is open and $p \in G$, then G is l.c. at p if and only if the space $\mathscr{X}$ is l.c. at p.*

Proof. Let $\mathscr{X}$ be l.c. at p. Let $p \in H$ where H is open relative G, i.e. H is open and $H \subset G$. Let C be the component of p in H. By assumption, $p \in \mathrm{Int}(C)$, i.e. $p \notin \overline{\mathscr{X}-C}$, hence $p \in G-\overline{G-C}$. But this means that p is an interior point of C relative G. Therefore G is l.c. at p.

Conversely, let G be l.c. at p. Let H be open and $p \in H$. As $G \cap H$ is open, then p is an interior point of its component in $G \cap H$, hence—of its component in H. Thus $\mathscr{X}$ is l.c. at p.

THEOREM 1. *The set of points at which a metric space is locally connected is a $\boldsymbol{G}_\delta$-set.*

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(1) Compare Pia Nalli, Rend. di Palermo 32 (1911), p. 392, S. Mazurkiewicz, C. R. Soc. de Varsovie 6 (1913), H. Hahn, Wiener Ber. 123 (1914), p. 2433.

Proof. This set is an infinite intersection $G_1 \cap G_2 \cap \ldots$ of sets G_n where G_n is the union of all sets $\mathrm{Int}(E)$ such that E is connected and $\delta(E) < 1/n$.

THEOREM 2. *A metric space is l.c. at the point p if and only if for every $\varepsilon > 0$ there exists an $\eta > 0$ such that condition $|x-p| < \eta$ implies the existence of a connected subset C such that $x, p \in C$ and $\delta(C) < \varepsilon$.*

Proof. If the space is l.c. at the point p, there exists a connected neighbourhood E of p such that $\delta(E) < \varepsilon$ and it remains to set $C = E$ and to denote by η the radius of a sphere contained in E with the center p.

On the other hand, if the condition of the theorem is satisfied, let us associate to every x such that $|x-p| < \eta$, a connected set C_x such that $x, p \in C_x$ and $\delta(C_x) < \varepsilon$, and define $E = \bigcup_x C_x$. It follows that $p \in \mathrm{Int}(E)$ and $\delta(E) \leqslant 2\varepsilon$.

THEOREM 3. *If $p \in A_0 \cap A_1$ and each of the sets A_0 and A_1 is l.c. at p, then so is the set $A_0 \cup A_1$.*

Proof. Let E_j be a connected neighbourhood of p in A_j, where $j = 0, 1$. Therefore

$$p \notin \overline{A_j - E_j}, \quad \text{this means} \quad p \notin \overline{(A_0 - E_0) \cup (A_1 - E_1)}.$$

Since

$$\begin{aligned}(A_0 - E_0) \cup (A_1 - E_1) &\supset [A_0 - (E_0 \cup E_1)] \cup [A_1 - (E_0 \cup E_1)] \\ &= (A_0 \cup A_1) - (E_0 \cup E_1),\end{aligned}$$

it follows that $p \notin \overline{(A_0 \cup A_1) - (E_0 \cup E_1)}$, which shows that $E_0 \cup E_1$ is a connected neighbourhood of p in $A_0 \cup A_1$.

THEOREM 4. *If $\mathscr{X}_j$ is l.c. at the point a_j $(j = 0, 1)$, the cartesian product $\mathscr{X}_0 \times \mathscr{X}_1$ is l.c. at the point (a_0, a_1).*

Proof. Let G be an open subset of $\mathscr{X}_0 \times \mathscr{X}_1$ containing the point (a_0, a_1). Let H_j be an open set (in $\mathscr{X}_j$) (compare Theorem 3 of § 15, I) such that

$$a_j \in H_j \subset \mathscr{X}_j \quad \text{and} \quad H_0 \times H_1 \subset G.$$

Let C_j be the component of a_j in H_j. By hypothesis, $a_j \in \mathrm{Int}(C_j)$, hence (compare § 15, III (2))

$$(a_0, a_1) \in \mathrm{Int}(C_0) \times \mathrm{Int}(C_1) = \mathrm{Int}(C_0 \times C_1) \quad \text{and}$$
$$C_0 \times C_1 \subset H_0 \times H_1 \subset G.$$

Thus $C_0 \times C_1$ is a neighbourhood of (a_0, a_1) which is contained in G and is connected (as a cartesian product of two connected sets, compare § 46, II, Theorem 11).

Clearly, Theorem 4 can be extended to an arbitrary *finite* number of factors. If the number of factors is arbitrary, the following is true.

THEOREM 4′. *Let* $\mathfrak{Z} = \underset{t\in T}{\boldsymbol{P}} \mathscr{X}_t$ *where all* $\mathscr{X}_t$, *except a finite number, are connected. Let* $\mathfrak{z} = \{\mathfrak{z}^t\} \in \mathfrak{Z}$. *If* $\mathscr{X}_t$ *is l.c. at* $\mathfrak{z}^t$ *for each* t, *then* $\mathfrak{Z}$ *is l.c. at* $\mathfrak{z}$.

Proof. Let $G \subset \mathfrak{Z}$ be open, $\mathfrak{z} \in G$ and C a component of $\mathfrak{z}$ in G. We have to show that $\mathfrak{z} \in \operatorname{Int}(C)$. According to the Remark to the Definition, we may assume that $G = \underset{t}{\boldsymbol{P}} G_t$ where G_t is open in $\mathscr{X}_t$ and except for a finite system of indices, say $t_1, \ldots, t_n$, $G_t = \mathscr{X}_t$ and $\mathscr{X}_t$ is connected.

For each $t \in T$ denote by C_t the component of $\mathfrak{z}^t$ in G_t. Then by Theorem 8 of § 46, III, $C = \underset{t\in T}{\boldsymbol{P}} C_t$. For $t' \neq t_i$, where $i = 1, \ldots, n$, we have $G_{t'} = \mathscr{X}_{t'}$. Hence $C_{t'} = \mathscr{X}_{t'}$ (for $\mathscr{X}_{t'}$ is connected). It follows (compare § 16, III (3)) that $\operatorname{Int}(C) = \underset{t\in T}{\boldsymbol{P}} \operatorname{Int}(C)$. But $\mathfrak{z}^t \in \operatorname{Int}(C_t)$, and thus $\mathfrak{z} \in \operatorname{Int}(C)$.

Remark. The assumption of $\mathscr{X}_t$ being connected, except for a finite number of indices, is essential. This is seen on the example where $\mathscr{X}_n$ is composed of the elements 0 and 1, for $n = 1, 2, \ldots$ Then $\mathscr{X}_1 \times \mathscr{X}_2 \times \ldots$ is the Cantor discontinuum.

EXAMPLES. (i) The spaces $\mathscr{E}^n$ and $\mathscr{I}^n$ are locally connected at each point for each n (finite or infinite).

(ii) Every isolated point is a point of local connectedness.

(iii) The curve $\sin 1/x$, defined in the following way

$$y = \sin 1/x \quad \text{for} \quad 0 < |x| \leqslant 1 \quad \text{and} \quad -1 \leqslant y \leqslant 1 \quad \text{for} \quad x = 0,$$

is l.c. at no point of the y axis.

(iv) The continuum, which is obtained joining by the segments the point $(0, 1)$ with the points 0 and $1/n$, $n = 1, 2, \ldots$, of the x axis, is not l.c. at the points $0 \leqslant y < 1$ of the y axis.

(v) In a similar way, if the point $(\frac{1}{2}, \frac{1}{2})$ is joined with every point of the Cantor set $\mathscr{C}$ (see § 46, II, Remark), a continuum is obtained which is l.c. at no point except $(\frac{1}{2}, \frac{1}{2})$.

(vi) In Section VII it will be shown that an indecomposable continuum is not l.c. at any point.

(vii) The segments joining the points $1/(n-1)$, $n = 2, 3, \ldots$, of the x axis with the points $(1/nk, 1/n)$, $k = 1, 2, \ldots$, together with the segment 01 of the x axis form a continuum which is l.c. at the origin. However, there is no small neighbourhood of this point which would be connected and open.

(viii) There are connected and locally connected spaces which contain no arc [1] or even are totally imperfect [2].

II. Locally connected spaces.

DEFINITION. A space which is l.c. at every point is said to be l.c. *The space* $\mathscr{X}$ *of Section* II *is supposed to be l.c.*

Theorems 3 and 4 of Section I imply the following ones.

THEOREM 1. *The finite union of closed l.c. sets is l.c.*

THEOREM 2. *The finite cartesian product of l.c. spaces is l.c.*

THEOREM 3. *Every open subset of a l.c. space is l.c.*

Because the local connectedness is a local property (compare Theorem 0, Section I).

THEOREM 4. *Every component of an open subset* G *of the space* $\mathscr{X}$ *is an open set* [3], *and hence it is a region* [4].

In particular, *every component of the space* $\mathscr{X}$ *is a closed-open set.*

This is so because $p \in \operatorname{Int}(C)$ if C is the component of p in G. Conversely, the following is true.

THEOREM 4'. *If every component of each open subset* G *of a given topological space is open, then the space is locally connected.*

Moreover, *the range of variability of* G *can be limited to an open base of the space.*

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(1) See R. L. Moore, Bull. Amer. Math. Soc. 32 (1926), p. 331.

(2) See the paper of B. Knaster and myself, *A connected and connected im kleinen point set which contains no perfect subset*, Bull. Amer. Math. Soc. 33 (1927), p. 106. Compare also § 51, I, 7.

(3) See H. Hahn, *Über die Komponenten offener Mengen*, Fund. Math. 2 (1921), pp. 189–192.

(4) Let us remind that a region is an open connected set.

THEOREM 5. *Let $\mathscr{X}$ be hereditarily normal and let A and B be two separated sets. If A is connected, there exists a region R such that*

$$A \subset R \quad \textit{and} \quad \bar{R} \cap B = 0. \tag{1}$$

If both sets A and B are connected, there exist two regions R and S such that

$$A \subset R, \quad B \subset S \quad \textit{and} \quad R \cap S = 0. \tag{2}$$

Proof. Since $\mathscr{X}$ is hereditarily normal, there exists an open set G such that $A \subset G$ and $\bar{G} \cap B = 0$. Let R be the component of G which contains A and, in case where B is connected, let S be the component of $\mathscr{X}-\bar{G}$ which contains B. According to Theorem 4, R and S are regions.

THEOREM 6. *The family of components of an open set in a l.c. space with a countable base is countable.*

This is so because every family of disjoint open sets is countable (compare § 24, I, Theorem 2).

THEOREM 7. *The family of components of a compact l.c. space is finite.*

Therefore *every compact l.c. $\mathscr{T}_2$-space is the union of a finite number of l.c. continua.*

This follows from Theorem 4.

THEOREM 8. *Every l.c. space $\mathscr{X}$ has a base consisting of regions. If $\mathscr{X}$ is regular, then every open set G is the union of regions:*

$$G = \bigcup_t R_t \quad \textit{where} \quad \bar{R}_t \subset G.$$

If $\mathscr{X}$ is metric separable, then for each $\varepsilon > 0$ there is a sequence $R_1, R_2, \ldots$ of regions such that

(i) $\delta(R_n) < \varepsilon$ *for* $n = 1, 2, \ldots,$

(ii) *every open set G is the union*

$$G = \bigcup_{n=1} R_{k_n} \quad \textit{where} \quad \bar{R}_{k_n} \subset G.$$

Proof. 1. Let $\boldsymbol{B}$ be an open base of $\mathscr{X}$. Then the family $\boldsymbol{C}$ of all components of members of $\boldsymbol{B}$ is the required base consisting of regions.

2. If $\mathscr{X}$ is regular, then G is the union of open sets G_s such that $\bar{G}_s \subset G$. Consequently, G is the union of all components of the sets G_s. We denote by $\{R_t\}$ the family of those components.

3. If $\boldsymbol{B}$ is countable, then so is $\boldsymbol{C}$ (because each member of $\boldsymbol{B}$ has a countable family of components). Furthermore, if $\boldsymbol{B} = (G_1, G_2, \ldots)$ and $\mathscr{X}$ is metric, we may assume that conditions (i) and (ii) are fulfilled, when R_n is replaced by G_n (compare § 22, II). It follows easily that $\boldsymbol{C} = (R_1, R_2, \ldots)$ is the required base satisfying (i) and (ii).

THEOREM 9. *Every region C relative to E is of the form $C = E \cap H$, where H is a region.*

Proof. Let G be an open set such that $C = E \cap G$, and let H be the component of G which contains C.

THEOREM 10. *If A and B are closed sets such that the sets $A \cup B$ and $A \cap B$ are l.c., then A and B are l.c.*

Proof. It can be assumed that $A \cup B = \mathscr{X}$. Since the set $A - B = \mathscr{X} - B$ is open, A is l.c. at every point of the set $A - B$. So, it must be shown that A is l.c. at every point $p \epsilon A \cap B$.

Let G be an open neighbourhood of p. By hypothesis, the component C of the set $A \cap B \cap G$ which contains p is open relative to this set. By Theorem 9 (for $\mathscr{X} = G$ and $E = A \cap B \cap G$), there exists a region H such that $H \subset G$ and $C = A \cap B \cap H$. The union and the intersection of the sets $H \cap A$ and $H \cap B$ (closed in H) are connected because

$$(H \cap A) \cup (H \cap B) = H \cap (A \cup B) = H \quad \text{and}$$
$$(H \cap A) \cap (H \cap B) = H \cap A \cap B = C,$$

and hence $H \cap A$ and $H \cap B$ are connected (§ 46, II, Theorem 5). Since the set $H \cap A$ is a connected neighbourhood of p relative to A and is contained in G, therefore A is l.c. at the point p.

Theorem 10 can be generalized as follows.

THEOREM 10a. *If $\mathscr{X} = F_1 \cup \ldots \cup F_n$, $\bar{F}_j = F_j$ and $F_j \cap F_k$ is l.c. (for $j \neq k$), the sets F_j are l.c.*

Proof. Let

$$A = F_j \quad \text{and} \quad B = F_1 \cup \ldots \cup F_{j-1} \cup F_{j+1} \cup \ldots \cup F_n.$$

Since $F_j \cap F_k$ are l.c., $A \cap B$ is l.c. by Theorem 1. Thus A is l.c. according to Theorem 10.

THEOREM 11. *If F is closed and l.c. and C is a component of $\mathscr{X}-F$, the sets $\mathscr{X}-C$ and $C \cup F$ are l.c.*

Furthermore, *if S is the union of some components of $\mathscr{X}-F$, the sets $\mathscr{X}-S$ and $S \cup F$ are l.c.*

Proof. This is a consequence of Theorem 10, because

$$\mathscr{X} = (\mathscr{X}-S) \cup (S \cup F) \quad \text{and} \quad (\mathscr{X}-S) \cap (S \cup F) = F$$

since S is a closed-open set.

THEOREM 12. *Let F and S be as in Theorem 11. If the space $\mathscr{X}$ is connected, every component C of $\mathscr{X}-S$ contains a component of the set F (supposed to be non-void).*

Consequently, *the set $\mathscr{X}-S$ has not more components than F (if F is connected, then so is $\mathscr{X}-S$).*

Proof. Since $F \subset \mathscr{X}-S$, we have to show that $C \cap F \neq 0$. Suppose that $C \cap F = 0$, i.e. that $C \subset \mathscr{X}-F$. Let R be the component of $\mathscr{X}-F$ containing C. By the definition of S, every component of $\mathscr{X}-F$ which is not contained in S is disjoint from S. Thus

$$R \cap S = 0, \quad \text{i.e.} \quad R \subset \mathscr{X}-S, \quad \text{which implies} \quad R \subset C$$

since $R \cap C \neq 0$ and C is a component of $\mathscr{X}-S$.

So it follows that $C = R$, which shows that C is closed-open (as a component of the closed set $\mathscr{X}-S$ and of the open set $\mathscr{X}-F$). But this is inconsistent with the connectedness of the space $\mathscr{X}$.

THEOREM 13 [1]. *If $\{A_t\}$ is the family of components of A, then*

$$\mathrm{Int}(A) = \bigcup_t \mathrm{Int}(A_t).$$

Proof. Clearly $\mathrm{Int}(A_t) \subset \mathrm{Int}(A)$.

On the other hand, let $p \in \mathrm{Int}(A)$, let A_t be the component of p in A and let G be the component of p in $\mathrm{Int}(A)$. We have $G \subset A_t$ because G is connected. Since the space $\mathscr{X}$ is l.c. and G is open, so $G \subset \mathrm{Int}(A_t)$ and finally $p \in \mathrm{Int}(A_t)$.

THEOREM 14 [2]. *If R is a region in a metric separable space, then there exists a sequence of regions $R_1, R_2, \ldots$ such that*

$$R = R_1 \cup R_2 \cup \ldots \quad \textit{and} \quad \bar{R}_n \subset R_{n+1}. \tag{3}$$

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(1) Compare F. Hausdorff, *Grundzüge der Mengenlehre*, p. 331.

(2) Compare R. L. Wilder, Bull. Amer. Math. Soc. 34 (1928), p. 650.

Proof. By Theorem 8 there exists a sequence $Q_1, Q_2, \ldots$ of regions such that

$$R = Q_1 \cup Q_2 \cup \ldots, \tag{4}$$

$$\bar{Q}_n \subset R. \tag{5}$$

Since R is connected, we can suppose that (compare § 46, II, Theorem 10)

$$(Q_1 \cup \ldots \cup Q_n) \cap Q_{n+1} \neq 0 \quad \text{for} \quad n = 1, 2, \ldots. \tag{6}$$

The sets $\bar{Q}_1$ and $\mathscr{X} - R$ are closed and disjoint (compare (5)), so they are separated and there exists by Theorem 5 a region R_1 such that

$$\bar{Q}_1 \subset R_1 \quad \text{and} \quad \bar{R}_1 \subset R.$$

We continue by induction. Let

$$\overline{Q_1 \cup \ldots \cup Q_n} \subset R \quad \text{and} \quad \bar{R}_{n+1} \subset R; \tag{7}$$

hence the set $\overline{R_n \cup Q_{n+1}}$ is connected (according to (6)) and separated from $\mathscr{X} - R$ (by (5) and (7)). Therefore, by Theorem 5, there exists a region R_{n+1} such that

$$\overline{R_n \cup Q_{n+1}} \subset R_{n+1} \quad \text{and} \quad \bar{R}_{n+1} \subset R.$$

Thus, conditions (7) are satisfied for $n = 1, 2, \ldots$. Combined with (4), they imply condition (3).

THEOREM 15. *Let $\mathscr{X}$ be regular. If F is a compact subset of a region R, there exists a region Q such that $F \subset Q$ and $\bar{Q} \subset R$.*

Proof. Since $\mathscr{X}$ is regular, there is an open cover $A = \{G_t\}$ of R such that G_t is a region and $\bar{G}_t \subset R$ (by Theorem 8). Since F is compact, we have $F \subset G_{t_1} \cup \ldots \cup G_{t_n}$. Join G_{t_1} to each G_{t_j}, where $1 < j \leqslant n$, by a finite chain of members of A (compare Theorem 8 of § 46, II). The union of these chains is the required region Q.

THEOREM 16 [1]. *If F is a compact subset of an open set G, then F intersects only a finite number of components of G.*

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[1] Compare K. Borsuk, *Über eine Bedingung die dem lokalen Zusammenhange äquivalent ist*, Mathematica 7 (1933), p. 144.

Moreover, if the space $\mathcal{X}$ is compact, then all components of $\mathcal{X}-F$, except a finite number, are contained in G.

Proof. Since the components of G are open, there exists a finite set of them $R_1, \ldots, R_n$ such that $F \subset R_1 \cup \ldots \cup R_n$. Therefore, if R is a component of G different from $R_1, \ldots, R_n$, it follows that $F \cap R = 0$.

In order to prove the second part of the theorem, let $\mathcal{X}-F = G^*$ and $\mathcal{X}-G = F^*$. Since $F \subset G$, it follows that $F^* \subset G^*$. Therefore F^* intersects only a finite number of components of G^*, and this means that, except a finite number, all these components are disjoint from F^* and hence contained in G.

THEOREM 17. *If the space $\mathcal{X}$ is connected between M and N, it contains a component C such that $C \cap M \neq 0 \neq C \cap N$.*

Proof. Suppose that the contrary is true. Let S be the union of components intersecting M; it follows that $M \subset S$ and $N \cap S = 0$. Since the set S is closed-open, the space $\mathcal{X}$ is not connected between M and N.

Theorem 17 implies that every l.c. space has property (M) (compare § 47, II, Theorem 1).

THEOREM 18. *The quasi-components of a l.c. space coincide with its components.*

Proof. Let C and Q be the component and the quasi-component of the point p, respectively. Obviously $C \subset Q$. Since C is closed-open, so $Q \subset C$, and hence $Q = C$.

THEOREM 19. *If the l.c. space $\mathcal{X}$ is connected and C is a component of an arbitrary set $X \neq \mathcal{X}$, then $\overline{C} \cap \overline{\mathcal{X}-X} \neq 0$.*

Proof. Let $p \in C$. It can be assumed that $p \in \mathcal{X}-\overline{\mathcal{X}-X} = \mathrm{Int}(X)$. Let C_0 be the component of $\mathrm{Int}(X)$ containing p. Since the set $\mathrm{Int}(X)$ is open, so is the set $\mathrm{Int}(X)-C_0$, since it is the union of components of $\mathrm{Int}(X)$ different from C_0. This implies $\overline{C}_0 \cap \overline{\mathcal{X}-X} \neq 0$, because otherwise the identity

$$\mathcal{X} = \{\overline{\mathcal{X}-X} \cup [\mathrm{Int}(X)-C_0]\} \cup C_0$$

would provide a decomposition of the (connected) space into two non-empty separated sets. Since $C_0 \subset C$, it follows that $\overline{C} \cap \overline{\mathcal{X}-X} \neq 0$.

The set $L(A)$. (1)

Let $\mathcal{X}$ be an arbitrary metric space (l.c. or not) and let A be a subset of $\mathcal{X}$. Let $L(A)$ be the set of all points $x \in \bar{A}$ for which there exist open sets G of arbitrary small diameter, containing x and such that $G \cap A$ is connected.

THEOREM 20. *The set* $L(A)$ *is a* $\boldsymbol{G_\delta}$.

Proof. This is because $L(A) = \bar{A} \cap G_1 \cap G_2 \cap \ldots$, where G_n is the union of open sets G such that $G \cap A$ is connected and $\delta(G) < 1/n$.

THEOREM 21. *If* A *is l.c., then* $A \subset L(A)$.

THEOREM 22. *If* $A \subset B \subset L(A)$, *then* B *is l.c.*

Proof. Because the inclusions $G \cap A \subset G \cap B \subset G \cap \bar{A} \subset \overline{G \cap A}$ imply that $G \cap B$ is connected (provided that $G \cap A$ is connected).

THEOREM 23. *If* $\mathcal{Y}$ *is a complete space,* A *is a l.c. subset of* $\mathcal{X}$ *and* $f\colon A \to \mathcal{Y}$ *continuous, there exists a continuous extension* $g\colon B \to \mathcal{Y}$ *of* f, *where* B *is a l.c.* $\boldsymbol{G_\delta}$.

Moreover, $A \subset B \subset L(A)$.

Proof. By Theorem 1 of § 35, I, there exist a set A^*, which is a $\boldsymbol{G_\delta}$ containing A, and an extension $f^*\colon A^* \to \mathcal{Y}$ of f. It remains to put

$$B = A^* \cap L(A) \quad \text{and} \quad g = f^* | B.$$

III. Properties of the boundary (2).

THEOREM 1. *If* $\{A_t\}$ *is a family of arbitrary sets in a l.c. space* $\mathcal{X}$, *then*

$$\operatorname{Fr}\Big(\bigcup_t A_t\Big) \subset \overline{\bigcup_t \operatorname{Fr}(A_t)}. \tag{1}$$

More precisely, *if* $p \in \operatorname{Fr}\big(\bigcup_t A_t\big) - \overline{\bigcup_t \operatorname{Fr}(A_t)}$, *then* $\mathcal{X}$ *is not l.c. at the point* p.

Proof. Would the space $\mathcal{X}$ be l.c. at the point p, there would exist a connected neighbourhood E of p such that

$$E \subset \mathcal{X} - \overline{\bigcup_t \operatorname{Fr}(A_t)}, \quad \text{so that} \quad p \in \operatorname{Int}(E) \cap \operatorname{Fr}\Big(\bigcup_t A_t\Big). \tag{2}$$

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(1) S. Eilenberg, Fund. Math. 26 (1936), p. 83.

(2) See my paper in Fund. Math. 8 (1926), p. 140.

It follows that (compare § 7, II, Corollary 1a)

$$\operatorname{Int}(E) \cap \bigcup_t A_t \neq 0 \neq \operatorname{Int}(E) - \bigcup_t A_t,$$

thus

$$E \cap \bigcup_t A_t \neq 0 \neq E - \bigcup_t A_t.$$

Hence there exists a subscript t_0 such that

$$E \cap A_{t_0} \neq 0 \neq E - A_{t_0}, \quad \text{which implies} \quad E \cap \operatorname{Fr}(A_{t_0}) \neq 0,$$

since E is connected (compare § 46, I, Theorem 1). But this contradicts condition (2) because

$$E \subset \mathscr{X} - \overline{\bigcup_t \operatorname{Fr}(A_t)} \subset \mathscr{X} - \bigcup_t \operatorname{Fr}(A_t).$$

Remark. *Condition* (1) *characterizes the l.c. spaces.* In fact, every space which is not l.c. contains an infinite sequence of disjoint open sets which do not satisfy condition (1) ([1]).

THEOREM 2. *If $G_1, G_2, \ldots$ is a sequence of disjoint open sets in a l.c. metric space, then*

$$\operatorname*{Ls}_{n=\infty} G_n \subset \operatorname*{Ls}_{n=\infty} \operatorname{Fr}(G_n).$$

Proof. If $p \in \operatorname{Ls} G_n$, then $p \in \mathscr{X} - G_n$ for each n (since the sets G_n are disjoint), so that $p \in \mathscr{X} - \bigcup_n G_n$, and the inclusion $\operatorname*{Ls}_{n=\infty} G_n \subset \overline{\bigcup_n G_n}$ gives

$$p \in \overline{\bigcup_n G_n} - \bigcup_n G_n = \operatorname{Fr}\Big(\bigcup_n G_n\Big) \subset \overline{\bigcup_n \operatorname{Fr}(G_n)}$$

according to Theorem 1. Thus

$$\operatorname*{Ls}_{n=\infty} G_n \subset \overline{\bigcup_n \operatorname{Fr}(G_n)},$$

hence

$$\operatorname*{Ls}_{n=\infty} G_n = \operatorname*{Ls}_{n=\infty} G_{m+n} \subset \overline{\bigcup_{n=m}^{\infty} \operatorname{Fr}(G_n)}$$

([1]) For a proof, see *op. cit.*, p. 142, where this statement has been proved for general topological spaces.

for any m (compare § 29, IV, Theorem 7). Therefore (compare § 29, IV, Theorem 8)

$$\operatorname*{Ls}_{n=\infty} G_n \subset \bigcap_{m=1}^{\infty} \overline{\bigcup_{n=m}^{\infty} \mathrm{Fr}(G_n)} = \operatorname*{Ls}_{n=\infty} \mathrm{Fr}(G_n).$$

THEOREM 3. *If A is a subset of a l.c. space and C is a component of A, then* $\mathrm{Fr}(C) \subset \mathrm{Fr}(A)$.

Moreover, if $p \in \mathrm{Fr}(C) - \mathrm{Fr}(A)$, the space is not l.c. at the point p.

Proof. Let

$$p \in \mathrm{Fr}(C) - \mathrm{Fr}(A) = \bar{C} \cap \overline{\mathscr{X} - C} - (\bar{A} \cap \overline{\mathscr{X} - A}).$$

Therefore

$$p \in \bar{A} - (\bar{A} \cap \overline{\mathscr{X} - A}) \subset \mathscr{X} - \overline{\mathscr{X} - A}.$$

Would $\mathscr{X}$ be l.c. at the point p, there would exist a connected neighbourhood E of p such that

$$E \subset \mathscr{X} - \overline{\mathscr{X} - A}, \quad \text{thus} \quad p \in \mathrm{Int}(E) \cap \mathrm{Fr}(C),$$

which implies (compare § 7, II, Corollary 1a and § 46, I, Theorem 1)

$$\mathrm{Int}(E) \cap C \neq 0 \neq \mathrm{Int}(E) - C \quad \text{and} \quad E \cap C \neq 0 \neq E - C.$$

But this contradicts the hypothesis that C is a component of A, because the conditions $E \subset A$ and $E \cap C \neq 0$ imply that $E \subset C$ since E is connected.

THEOREM 4. *Let A be a subset of a l.c. space. If* $\mathrm{Fr}(A)$ *is l.c., then so is $\bar{A}$.*

Proof. This is a corollary to Theorem 10, Section II, because

$$\bar{A} \cup \overline{\mathscr{X} - A} = \mathscr{X} \quad \text{and} \quad \bar{A} \cap \overline{\mathscr{X} - A} = \mathrm{Fr}(A).$$

IV. Separation of locally connected spaces. *Let $\mathscr{X}$ be a l.c. regular space.*

THEOREM 1. *If F is a closed set which does not separate the sets M and N, disjoint from F (this means that $\mathscr{X} - F$ is connected between M and N), there exists a region R such that*

$$R \cap M \neq 0 \neq R \cap N \quad \textit{and} \quad \bar{R} \cap F = 0. \tag{1}$$

Proof. According to Theorem 17 of Section II, there exists a component C of $\mathscr{X} - F$ such that $C \cap M \neq 0 \neq C \cap N$. Let

$p \in C \cap M$ and $q \in C \cap N$. According to Theorem 15 of Section II there exists a region R such that $p, q \in R$ and $\bar{R} \subset C$, which implies condition (1).

THEOREM 2. *The set G of points of $\mathscr{X}-(M \cup N)$ which do not separate the space between M and N is open in $\mathscr{X}-(M \cup N)$.*

Proof. Let $x \in G$ and let R be a connected set such that $R \cap M \neq 0 \neq R \cap N$ and $x \in \mathscr{X}-\bar{R}$. Therefore no point of $\mathscr{X}-\bar{R}$ separates M from N.

LEMMA. *If Z is an arbitrary set of pairs (m, n) of positive integers, there exists an infinite sequence $k_1 < k_2 < \ldots$ such that for all subscripts i*

$$\textit{either} \quad (k_i, k_{i+1}) \in Z \quad \textit{or} \quad (k_i, k_{i+1}) \notin Z. \tag{2}$$

Proof. Suppose that there is no infinite sequence of the first kind, so there exists a number m (moreover, m can be arbitrarily large) such that condition $n > m$ implies that $(m, n) \notin Z$. The sequence of these numbers m is the required sequence $\{k_i\}$ of the second kind.

Now, let $S(a, b)$ denote the set of points which separate a from b (compare § 46, IX).

THEOREM 3 (of G. T. Whyburn) (1). *The set $S(a, b) \cup a \cup b$ is compact (the space $\mathscr{X}$ is supposed to be connected and metric).*

Proof. By Theorem 2 the set $S(a, b) \cup a \cup b$ is closed. Suppose that it is not compact. So $S(a, b)$ contains an infinite sequence of (distinct) points $p_1, p_2, \ldots$ whose every subsequence constitutes a closed set. By hypothesis, for every n there exist two open sets such that

$$\mathscr{X}-p_n = M_n \cup N_n, \quad M_n \cap N_n = 0, \quad a \in M_n, \quad b \in N_n. \tag{3}$$

Let Z be the set of pairs (m, n) such that $p_m \in M_n$. According to the preceding lemma there exists an infinite sequence $k_1 < k_2 < \ldots$ such that for all subscripts i

$$\text{either} \quad p_{k_i} \in M_{k_{i+1}} \quad \text{or} \quad p_{k_i} \notin M_{k_{i+1}}, \quad \text{i.e.} \quad p_{k_i} \in N_{k_{i+1}}.$$

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(1) *Concerning connected and regular point sets*, Bull. Amer. Math. Soc. **33** (1927), p. 685 and *On the structure of connected and connected im kleinen point sets*, Trans. Amer. Math. Soc. **32** (1930), p. 927.

By the symmetry it can be supposed that the first formula is true. Since $\mathrm{Fr}(M_n) = (p_n)$, it follows by Theorem 1 of Section III that

$$\mathrm{Fr}\Big(\bigcup_{n=1}^{\infty} M_{k_n}\Big) \subset \overline{\bigcup_{n=1}^{\infty} \mathrm{Fr}(M_{k_n})} = \overline{(p_{k_1}, p_{k_2}, \ldots)}$$
$$= (p_{k_1}, p_{k_2}, \ldots) \subset \bigcup_{n=1}^{\infty} M_{k_{n+1}} \subset \bigcup_{n=1}^{\infty} M_{k_n},$$

which shows that the set $M_{k_1} \cup M_{k_2} \cup \ldots$ is closed (compare § 6, II, (7)). This set is open and different from $\mathscr{X}$ (because $b \notin M_n$ for each n), which contradicts the connectedness of $\mathscr{X}$.

THEOREM 4. *If the metric space $\mathscr{X}$ is connected and satisfies the condition $\mathscr{X} = S(a, b) \cup a \cup b$, it is an arc ab.*

Proof. This space is compact by Theorem 3, and hence it is an arc according to Theorem 1 of § 47, V.

Remark. The union of two sets A and B, irreducibly connected between the same pair of points (p, q) and such that $A \cap B = (p, q)$, does not need be a simple closed curve even though it were locally connected [1].

THEOREM 5. *If the metric space $\mathscr{X}$ (containing more than one point) is connected and is separated by each pair of points, while no single point separates it, then the space is a simple closed curve.*

Proof. Let a_0 be a fixed point. By hypothesis, $\mathscr{X} - a_0$ is connected and every point $x \in (\mathscr{X} - a_0)$ separates the set $\mathscr{X} - a_0$. According to Theorem 4 of 46, VIII, all these points x, except perhaps a countable set of them, separate $\mathscr{X} - a_0$ into two regions. Let a_1 be a point x of this kind. So, there exist two regions R_0 and R_1 such that

$$\mathscr{X} - a_0 - a_1 = R_0 \cup R_1, \quad R_0 \cap R_1 = 0 \quad \text{and}$$
$$R_j \neq 0 \quad \text{for} \quad j = 0, 1.$$

It follows that $\mathrm{Fr}(R_j) = (a_0, a_1)$ because, if $\mathrm{Fr}(R_j)$ would consist of a single point, this point would separate the space. So the proof is reduced to show that $\bar{R}_j$ is an arc $a_0 a_1$.

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(1) See R. H. Bing, *Solution of a problem of R. L. Wilder*, Amer. Journ. Math. 70 (1948), p. 95.

Since $\mathrm{Fr}(R_j)$ is l.c., then so is $\bar{R}_j$ (by Theorem 4 of Section III). The hypothesis that $\bar{R}_j$ is not an arc $a_0 a_1$ implies therefore, by Theorem 4, the existence of a point $p \in R_j$ such that $\bar{R}_j - p$ is connected between a_0 and a_1. We are going to show that $\bar{R}_j - p$ is a connected set.

Would we have

$$\bar{R}_j = U \cup V, \quad U = \bar{U}, \quad V = \bar{V}, \quad U \cap V = p,$$
$$U \neq \bar{R}_j \neq V, \quad a_0 \in U, \tag{4}$$

it would follow that $a_1 \in U$ (since $\bar{R}_j - p$ is connected between a_0 and a_1), so that

$$\bar{R}_{1-j} \cap V = 0, \quad \text{and hence} \quad (\bar{R}_{1-j} \cup U) \cap V = U \cap V = p,$$

and the decomposition $\mathscr{X} = (\bar{R}_{1-j} \cup U) \cup V$ would be inconsistent with the hypothesis that p does not separate the space.

Now suppose that $\bar{R}_0$ is not an arc $a_0 a_1$. So there exists a point $p_0 \in R_0$ such that $\bar{R}_0 - p_0$ is connected. Consider two cases according to whether $\bar{R}_1$ is an arc $a_0 a_1$ or not. In the first case, let $p_1 \in R_1$, in the second one let p_1 be such a point of R_1 that $\bar{R}_1 - p$ is connected (the existence of such a point has just been proved). But in both cases the pair p_0, p_1 does not separate the space contrary to the hypothesis.

THEOREM 6 [1]. *If the metric space $\mathscr{X}$ (containing more than one point) is discoherent* (i.e. no closed connected subset separates it), *$\mathscr{X}$ is a simple closed curve.*

Proof. Let Q be a region such that $0 \neq \bar{Q} \neq \mathscr{X}$. Let

$$R_0 = \mathscr{X} - \bar{Q} \quad \text{and} \quad R_1 = \mathscr{X} - \bar{R}_0.$$

The sets R_0 and R_1 are connected by hypothesis and non-empty, and (compare § 8, VIII)

$$R_j = \mathscr{X} - \bar{R}_{1-j} \quad \text{for} \quad j = 0, 1. \tag{5}$$

Let $F = \mathrm{Fr}(R_0)$. Thus

$$F = \bar{R}_0 - R_0 = \bar{R}_0 \cap \bar{R}_1 = \mathrm{Fr}(R_1) = \mathscr{X} - (R_0 \cup R_1). \tag{6}$$

[1] See R. L. Wilder, *Concerning simple closed curves and related point sets*, Amer. Journ. Math. 53 (1931), p. 54.

It follows that F contains more than one point (because no point separates the space). We are going to show that F consists of exactly two points.

Suppose that F contains three distinct points, a_0, a_1 and a_2. Since the space $\mathscr{X}$ is l.c., so let A_0, A_1 and A_2 be three regions such that

$$a_k \epsilon A_k \quad (k = 0, 1, 2) \quad \text{and}$$
$$\bar{A}_0 \cap \bar{A}_1 = \bar{A}_1 \cap \bar{A}_2 = \bar{A}_2 \cap \bar{A}_0 = 0. \tag{7}$$

Since $a_k \epsilon F = \operatorname{Fr}(R_j)$, it follows that

$$A_k \cap \operatorname{Fr}(R_j) \neq 0, \text{ thus } R_j \cap A_k \neq 0 \text{ for } j = 0, 1 \text{ and } k = 0, 1, 2. \tag{8}$$

Therefore the set $R_j^+ = R_j \cup \bar{A}_0 \cup \bar{A}_1$ is connected and it follows that the set

$$R_{1-j}^- = R_{1-j} - (\bar{A}_0 \cup \bar{A}_1) = \mathscr{X} - \bar{R}_j - \bar{A}_0 - \bar{A}_1 = \mathscr{X} - \bar{R}_j^+ \tag{9}$$

is a region. According to (6), (7) and (9), $a_2 \epsilon \bar{R}_j \cap \overline{R_{1-j}^-}$. Therefore $\bar{R}_j \cup \overline{R_{1-j}^-}$ is connected and so is the set (compare (5))

$$\mathscr{X} - (\bar{R}_j \cup \overline{R_{1-j}^-}) = R_{1-j} - \overline{R_{1-j}^-}.$$

Since the sets $\bar{A}_0$ and $\bar{A}_1$ are separated (compare (7)) and

$$R_{1-j} - \overline{R_{1-j}^-} = R_{1-j} - \overline{R_{1-j} - (\bar{A}_0 \cup \bar{A}_1)} \subset \bar{A}_0 \cup \bar{A}_1,$$

one of the sets

$$\text{either} \quad \bar{A}_0 \cap R_{1-j} - \overline{R_{1-j}^-} \quad \text{or} \quad \bar{A}_1 \cap R_{1-j} - \overline{R_{1-j}^-}$$

is empty. Assume that the first one is empty,

$$\bar{A}_0 \cap R_{1-j} - \overline{R_{1-j}^-} = 0. \tag{10}$$

But according to (9), $A_0 \cap R_{1-j}^- = 0$, which yields $A_0 \cap \overline{R_{1-j}^-} = 0$ (because A_0 is open) and if follows by (10) that

$$0 = A_0 \cap R_{1-j} - \overline{R_{1-j}^-} = A_0 \cap R_{1-j},$$

which contradicts (8).

So it has been proved that the set $F = \operatorname{Fr}(R_j)$ consists of two points. Let $F = (a_0, a_1)$. According to Theorem 4 of Section III, $\bar{R}_j$ is l.c. It remains to be proved that $\bar{R}_j$ is an arc $a_0 a_1$, i.e. (compare Theorem 4) that each point of R_j separates $\bar{R}_j$ between a_0 and a_1.

Suppose, conversely, that $p \in R_0$ and that $\bar{R}_0 - p$ is connected between a_0 and a_1. By Theorem 1 there exists a closed connected set C such that

$$a_0, a_1 \in C \subset \bar{R}_0 - p, \quad \text{which implies} \quad F \subset C \subset \mathscr{X} - R_1.$$

It follows according to (6) that

$$\mathscr{X} - C = (F \cup R_0 \cup R_1) - C = (R_0 - C) \cup R_1.$$

This identity yields a decomposition of $\mathscr{X} - C$ into two non-empty, disjoint open sets (because $p \in R_0 - C$), which contradicts the hypothesis that the space is discoherent.

THEOREM 7. *The set E separates the metric space $\mathscr{X}$ into n separated (non-empty) parts if and only if E contains a closed set F such that, if H satisfies conditions $F \subset H = \bar{H} \subset E$, then H separates $\mathscr{X}$ into n (disjoint and non-empty) open sets.*

Proof. The condition of the theorem is necessary according to Theorem 3 of § 46, VII.

In order to show that it is sufficient suppose that the set $\mathscr{X} - E$ does not consist of n non-empty separated parts. Then by Theorem 6 of § 46, II, there exist k connected sets $C_1, \ldots, C_k$ such that

$$\mathscr{X} - E = C_1 \cup \ldots \cup C_k, \quad \text{where} \quad k < n.$$

Let $F = \bar{F} \subset E$ and let $G_1, \ldots, G_j$ $(j \leqslant k)$ be a system of components of $\mathscr{X} - F$ such that

$$\mathscr{X} - E \subset G_1 \cup \ldots \cup G_j.$$

Put $H = \mathscr{X} - (G_1 \cup \ldots \cup G_j)$. It follows that $F \subset H = \bar{H} \subset E$ since $\mathscr{X} - E \subset G_1 \cup \ldots \cup G_j \subset \mathscr{X} - F$. But, H does not separate the space $\mathscr{X}$ into n non-empty open sets.

THEOREM 8. *Let a and b be two points of a compact l.c. $\mathscr{T}_2$-space and $\{F_n\}$ a sequence of closed sets. If no set F_n does cut any region between a and b, then neither does the union $F_1 \cup F_2 \cup \ldots$* [1].

Proof. Let R be a region containing a and b. We are going to define by induction a sequence of regions $R_0, R_1, \ldots$ such that

$$a, b \in R_n, \tag{11}$$

$$\bar{R}_{n+1} \subset R_n - F_n. \tag{12}$$

[1] Compare J. R. Kline, Bull. Amer. Math. Soc. (1917).

Let $R_0 = R$. Assume that R_n satisfies condition (11), and let R_n^* be the component of $R_n - F_n$ which contains the two-element set (a, b). According to Theorem 15 of Section II, there exists a region R_{n+1} such that $a, b \in R_{n+1}$ and $\bar{R}_{n+1} \subset R_n^*$, which yields inclusion (12).

Now, let $K = \bar{R}_1 \cap \bar{R}_2 \cap \ldots$.

Since $\bar{R}_1, \bar{R}_2, \ldots$ is a decreasing sequence of continua, K is a continuum according to Theorem 5 of § 47, II. Besides, it follows by (11) and (12) that

$$a, b \in K \subset R - (F_1 \cup F_2 \cup \ldots).$$

V. Irreducible separators. *Let $\mathscr{X}$ be a l.c. space.*

THEOREM 1. *Let C be a closed set, A and B two distinct components of the set $\mathscr{X} - C$, and let $a \in A$ and $b \in B$. The set C is an irreducible separator between a and b if and only if*

$$\mathrm{Fr}(A) = C = \mathrm{Fr}(B). \tag{1}$$

Proof. Since (1) is a sufficient condition according to Theorem 4 of § 46, VII, we have to show that it is necessary. By Theorem 3 of Section III,

$$\mathrm{Fr}(A) \subset \mathrm{Fr}(\mathscr{X} - C) \subset C.$$

Since $\mathrm{Fr}(A)$ is a separator between a and b (compare Theorem 6 of § 6, V) and C is an irreducible separator between a and b (by the hypothesis), it follows that $C = \mathrm{Fr}(A)$. Similarly, $C = \mathrm{Fr}(B)$.

THEOREM 2. *If A is a region and B is a component of $\mathscr{X} - \bar{A}$, the set $\mathrm{Fr}(B)$ is an irreducible separator between each pair of points $a \in A$ and $b \in B$.*

Proof. The following is true (compare III, Theorem 3)

$$\mathrm{Fr}(B) \subset \mathrm{Fr}(\mathscr{X} - \bar{A}) = \bar{A} \cap (\overline{\mathscr{X} - \bar{A}}) \subset \bar{A} \cap (\overline{\mathscr{X} - A}) = \bar{A} - A. \tag{2}$$

Therefore $a \notin \mathrm{Fr}(B)$, which implies that $a \in \mathscr{X} - \bar{B}$ and hence $\mathrm{Fr}(B)$ separates a from b; on the other hand, (2) implies that $x \in \bar{A} \cap \bar{B}$ for all $x \in \mathrm{Fr}(B)$ and hence $A \cup x \cup B$ is connected, thus no proper subset of $\mathrm{Fr}(B)$ separates a from b.

THEOREM 3 (of Mazurkiewicz) (1). *Every closed separator C between a and b contains a closed irreducible separator F between a and b.*

(1) Fund. Math. 5 (1924), p. 193, Lemma 1.

Proof. Let A be the component of a in $\mathscr{X}-C$. It follows that $b\in\mathscr{X}-\bar{A}$, because $\bar{A}=A\cup\mathrm{Fr}(A)\subset A\cup C$. Let B be the component of b in $\mathscr{X}-\bar{A}$; then (compare (2)) $\mathrm{Fr}(B)\subset\mathrm{Fr}(A)\subset C$. So by Theorem 2 it will be sufficient to set $F=\mathrm{Fr}(B)$.

THEOREM 4. *If in a normal space $a_0\in F_0=\bar{F}_0$, $a_1\in F_1=\bar{F}_1$ and $F_0\cap F_1=0$, then there exists a closed separator irreducible between a_0 and a_1 and disjoint from $F_0\cup F_1$.*

Proof. If G is an open set such that $F_0\subset G$ and $\bar{G}\cap F_1=0$, then $\mathrm{Fr}(G)$ contains the required separator.

VI. The set of points at which a continuum is not l.c. Convergence continua. In Sections VI and VII the space is supposed to be *metric.*

DEFINITION. A continuum K is said to be a *convergence continuum* (¹) of $\mathscr{X}$ if it is the topological limit of a sequence of continua K_n such that

$$K=\operatorname*{Lim}_{n=\infty}K_n \quad\text{and}\quad K\cap K_n=0.$$

If the space $\mathscr{X}$ is compact, the continua $K_1,K_2,\ldots$ can be assumed to be pairwise disjoint. Because they can be replaced by $K_{n_1},K_{n_2},\ldots$ where $n_1=1$ and

$$\mathrm{dist}(K,K_{n_i})<\varrho(K,K_{n_{i-1}})\quad\text{and}\quad n_i>n_{i-1}\quad\text{for}\quad i>1.$$

EXAMPLE. In the curve $\underset{xy}{E}(y=\sin 1/x)$ $(0<|x|\leqslant 1)$ the segment $-1\leqslant y\leqslant 1$, $x=0$ is a convergence continuum.

THEOREM 1. *If the space $\mathscr{X}$ is a continuum and N is the set of points at which $\mathscr{X}$ is not l.c., every point $p\in N$ belongs to a convergence continuum $K\subset N$ (such that $K\neq p$).* (²)

Proof. By the hypothesis there exists a closed neighbourhood E of p such that, if C is the component of p in E, then p does not belong to its interior, $p\in\overline{E-C}$. Let

$$p=\lim_{n=\infty}p_n, \tag{1}$$

$$p_n\in E-C. \tag{2}$$

(¹) Following C. Zarankiewicz, Fund. Math. 9 (1927), p. 127. Compare P. Urysohn, Verh. Akad. Amsterdam 13 (1927), p. 43.

(²) Compare R. L. Moore, Proc. Nat. Acad. Sc. 4 (1918), C. Zarankiewicz *loc. cit.*, p. 132 and P. Urysohn, *loc. cit.* p. 48.

Let C_n be the component of p_n in E. It follows that

$$C \cap C_n = 0. \tag{3}$$

For otherwise, the set $C \cup C_n$ would be a subcontinuum of E, thus $C \cup C_n \subset C$ and hence $p_n \in C$ contrary to (2).

Let F be a closed neighbourhood of p such that

$$F \subset \operatorname{Int}(E). \tag{4}$$

It can be assumed that $p_n \in F$ for $n = 1, 2, \ldots$

Let D_n be the component of p_n in F. Since $F \subset E$, it follows that

$$D_n \subset C_n. \tag{5}$$

From the sequence $\{D_n\}$ we choose a convergent subsequence (compare § 42, I, Theorem 1 and § 42, II) and define

$$K = \operatorname*{Lim}_{n=\infty} D_{m_n}. \tag{6}$$

By (1) it follows that

$$p \in K \subset F. \tag{7}$$

Hence (compare (4)) $K \subset E$, and since K is a continuum (compare § 47, II, Theorem 4), it follows that

$$K \subset C. \tag{8}$$

The condition $K \neq p$ holds, because, according to Theorem 2 of § 47, III,

$$D_n \cap \operatorname{Fr}(F) \neq 0, \quad \text{and hence} \quad K \cap \operatorname{Fr}(F) \neq 0,$$

while $p \notin \operatorname{Fr}(F)$ (since F is a neighbourhood of p).

Next, $K \subset N$. Suppose conversely that $x \in K - N$. According to (7) and (4), E is a neighbourhood of x and, since K is a subcontinuum of E which joins x with p, then C is the component of x in E. Therefore $x \in \operatorname{Int}(C)$ (since we suppose that $x \notin N$). But

$$x \in \operatorname{Ls} D_n \subset \operatorname{Ls} C_n$$

(compare (5) and § 29, IV, Theorem 2), and there exists a subscript n such that $C \cap C_n \neq 0$, contrary to (3).

Finally, K is a convergence continuum, because (8), (5) and (3) imply that $K \cap D_n = 0$.

THEOREM 2. *Every continuum which has no nowhere dense subcontinuum (containing more than one point)* (1) *is l.c.*

Proof. This is so because every convergence continuum is nowhere dense.

Remark. A nowhere dense continuum does not need be a convergence continuum. This shows the following example.

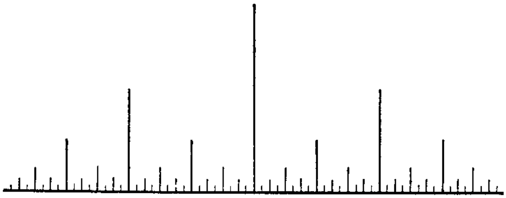

Fig. 6

The space consists of the segment 01 of the x axis and of a sequence of vertical segments

$$0 \leqslant y \leqslant 1/2^n \quad \text{and} \quad x = (2k-1)/2^n, \quad \text{where} \quad 1 \leqslant k \leqslant 2^{n-1}$$

and $n = 1, 2, \ldots$

The level segment is nowhere dense, but it is not a convergence continuum.

THEOREM 3 (of R. L. Moore) (2). *Let $\mathcal{X}$ be a continuum and N the set of points at which $\mathcal{X}$ is not l.c.; then the decomposition space of $\mathcal{X}$ into components of $\overline{N}$ and individual points of $\mathcal{X}-\overline{N}$, is a l.c. continuum.*

In other words, *there exists a continuous mapping $f\colon \mathcal{X} \to \mathcal{Y}$ onto such that $\mathcal{Y}$ is a l.c. continuum and the family of sets $f^{-1}(y)$,*

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(1) A nowhere dense continuum containing more than one point is also called *condensation continuum*.

(2) See *Concerning the prime parts of a continuum*, Math. Zft. 22 (1925), p. 307.

$y \in \mathcal{Y}$, is identical with the family of components of $\overline{N}$ and of individual points of $\mathcal{X}-\overline{N}$.

Proof. The function f induced by the decomposition under consideration is a homeomorphism at every point of $\mathcal{X}-\overline{N}$ (compare Theorem 1 of § 43, IV). Therefore the points of $f(\mathcal{X}-\overline{N})$ are points of local connectedness of $\mathcal{Y}$. If Q is the set of points at which $\mathcal{Y}$ is not l.c., then $Q \subset f(\overline{N})$. Since $\dim f(\overline{N}) \leqslant 0$ (see § 46, V, Theorem 3 and § 47, VI, Theorem 1), it follows that $\dim Q \leqslant 0$, and hence $Q = 0$ by Theorem 1.

Theorem 3 combined with Theorem 6 of Section IV implies the following one.

THEOREM 4. *If $\mathcal{X}$ is a discoherent continuum and $\overline{N} \neq \mathcal{X}$, the space $\mathcal{Y}$ considered in Theorem 3 is a simple closed curve.*

This yields the following statement.

THEOREM 5. *Every discoherent continuum $\mathcal{X}$ is locally an arc at each point of $\mathcal{X}-\overline{N}$.*

Proof. A simple closed curve is locally an arc and the function f is locally a homeomorphism at every point of $\mathcal{X}-\overline{N}$.

Remark. Theorem 3 can easily be generalized in the following way [1]. Let us recall that a property is said to be local provided that the space has it at a point p if and only if each neighbourhood of p has this property at p. Let **P** denote a local topological property of a point such that the set of points, at which a continuum has it, is never 0-dimensional (this means that it is either empty or of positive dimension). Then the following is true.

THEOREM 6. *If N denotes the set of points at which $\mathcal{X}$ has the property* **P**, *then Theorem 3 remains valid.*

Thus the following properties can be substituted for **P**.

(i) $\dim_p \mathcal{X} \geqslant n$ $(n \geqslant 1)$,

(ii) $\operatorname{ord}_p \mathcal{X} \geqslant \aleph_0$ (see § 51, III, Theorem 5),

(iii) $\operatorname{ord}_p \mathcal{X} \geqslant \mathfrak{c}$,

(iv) p is an element of a convergence continuum (or of a nowhere dense continuum) which does not consist only of the point p.

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[1] G. T. Whyburn, Bull. Amer. Math. Soc. 41 (1935), p. 95.

A proof similar to that of Theorem 1 yields the following statement.

THEOREM 7. *Every not l.c. continuum contains*

(i) *a connected set which is not a semi-continuum,*

(ii) *a non-connected set, which is however connected between two points.*

Proof. Reconsider the proof of Theorem 1 retaining the meaning of symbols p, E, F, C and C_n. It can be assumed that the continua C_n are pairwise disjoint and that they are convergent,

$$L = \operatorname*{Lim}_{n=\infty} C_n. \tag{9}$$

By (1) it follows that $p \in L$, which implies $L \subset C$ because L is a continuum (§ 47, II, Theorem 4). According to Theorem 1 of § 47, III, $C_n \cap \mathrm{Fr}(E) \neq 0$, and hence $L \cap \mathrm{Fr}(E) \neq 0$ by (9). The identity $F \cap \mathrm{Fr}(E) = 0$, which follows from (4), implies that $L - [F \cup \mathrm{Fr}(E)] \neq 0$, because L is connected. So there exists a point $q \in L \cap \mathrm{Int}(E) - F$. Let H be a closed neighbourhood of q such that $H \subset E - F$. Let D be the component of p in F and let I be the component of q in H. It follows that $D \cup I \subset C$ and $D \cap I = 0$. Therefore, the sets $D, I, C_1, C_2, \ldots$ are pairwise disjoint and there exist by Theorem 7, § 46, III, an infinite sequence of subscripts $k_1 < k_2 < \ldots$ and a connected set W such that

$$C_{k_1} \cup C_{k_2} \cup \ldots \subset W \quad \text{and either} \quad W \cap D = 0 \quad \text{or} \quad W \cap I = 0.$$

By the symmetry we can consider the first case only.

Since $p \in L$, the set $P = W \cup p$ is connected. However, it is not a semi-continuum. Because otherwise, there would exist a continuum R (compare § 47, III, Theorem 4) such that

$$p \in R \subset P \cap F \quad \text{and} \quad R \neq p$$

(since F is a neighbourhood of p).

But the condition $p \in R \subset F$ implies that $R \subset D$, and hence $R \cap W = 0$ and inasmuch $R \subset P$, it follows that $R = p$.

Finally, the set $p \cup q \cup \bigcup_n C_n$ is connected between p and q (§ 46, IV, Theorem 9) without being a connected set.

VII. Relative distance. Oscillation (1).

DEFINITION 1. The greatest lower bound of the diameters $\delta(E)$, where E denotes a variable connected set joining the points x and y, is said to be the *relative distance between x and y* (2)

$$\varrho_r(x, y) = \inf \delta(E). \tag{1}$$

It follows that the condition $\varrho_r(x, y) < \varepsilon$ holds if and only if there exists a connected set E such that

$$x, y \in E \quad \text{and} \quad \delta(E) < \varepsilon. \tag{2}$$

THEOREM 1. *The relative distance yields a metric to every space such that each pair of its points can be joined by a connected bounded set.*

In other words, *the following conditions hold* (compare § 21, I)

$$[\varrho_r(x, y) = 0] \equiv (x = y) \tag{3}$$

and

$$\varrho_r(y, z) \leqslant \varrho_r(x, y) + \varrho_r(x, z). \tag{4}$$

Proof. Equivalence (3) is obvious. In order to establish condition (4), let Y and Z be two connected bounded sets, which join x with y and x with z respectively. It follows that (compare § 21, III, (4))

$$\varrho_r(y, z) \leqslant \delta(Y \cup Z) \leqslant \delta(Y) + \delta(Z).$$

Assuming further that

$$\delta(Y) < \varrho_r(x, y) + \varepsilon \quad \text{and} \quad \delta(Z) < \varrho_r(x, z) + \varepsilon,$$

we conclude that $\varrho_r(y, z) < \varrho_r(x, y) + \varrho_r(x, z) + 2\varepsilon$, which implies condition (4).

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(1) These concepts are due to S. Mazurkiewicz, C. R. Soc. de Varsovie 6 (1913) and 9 (1916) and Fund. Math. 1 (1920), p. 166. Numerous theorems concerning these concepts are to be found in C. R. du I Congrès des math. des Pays Slaves, Warsaw 1930, p. 66 (of the same author) and by P. Urysohn, Verh. Akad. Amsterdam 13 (1928), p. 38.

For further results, see O. Lokutzievski, *Concerning the topology of continua*, Dokl. Akad. Nauk SSSR 164 (1965), p. 1235. See also M. Shtanko, Utchonyje Zapiski (Russian) 18 (1963).

(2) Generally speaking, if the variable set E runs through a family of sets $\boldsymbol{F}$, a relative distance with respect to $\boldsymbol{F}$ can be defined. Compare N. Aronszajn, Fund. Math. 19 (1932), p. 97.

THEOREM 2. *If the space is a continuum, the set E in* (1) *can be supposed to be a continuum without affecting the number $\varrho_r(x, y)$.*

Proof. This is so because $\bar{E}$ is a continuum if E is connected, and $\delta(\bar{E}) = \delta(E)$.

THEOREM 3. *The relative distance gives a metric to every connected and locally connected space.*

Moreover, *in* (1) *the set E can be supposed to be a region.*

Proof. In order to establish the first part of Theorem 3, it is sufficient, according to Theorem 1, to show that each pair of points of the considered space can be joined by a connected bounded set. But this follows directly from Theorem 8 of § 46, II, provided $\{G_t\}$ is the family of bounded regions.

In order to prove the second part of Theorem 3, it is sufficient to observe that, if C is a connected set, then to each $\varepsilon > 0$ there corresponds a region R such that

$$C \subset R \quad \text{and} \quad \delta(R) \leqslant \delta(C) + \varepsilon;$$

namely R is the component of the set C in the open ball with center C and radius $\varepsilon/2$.

THEOREM 4. *Let $\mathscr{X}$ be the space which satisfies conditions of Theorem* 1. *Let $\mathscr{X}_r$ be the same space with the metric given by the relative distance. The identity transformation $f\colon \mathscr{X}_r \to \mathscr{X}$ is a continuous function; the inverse transformation f^{-1} is continuous at the point p if and only if $\mathscr{X}$ is l.c. at this point.*

Proof. The function f is continuous because

$$|x - y| \leqslant \varrho_r(x, y).$$

Conversely, if the function f^{-1} is continuous at the point x, then to each $\varepsilon > 0$ corresponds an $\eta > 0$ such that the condition $|x - y| < \eta$ implies that $\varrho_r(x, y) < \varepsilon$, which means that there exists a connected set E satisfying condition (2). According to Theorem 2 of Section I, the space $\mathscr{X}$ is l.c. at the point x.

The following is a direct consequence of Theorem 4.

THEOREM 5. *If the space $\mathscr{X}$ is connected and l.c., then the usual distance and the relative distance are topologically equivalent* (*this means that f is a homeomorphism*).

It follows easily that a connected metric space is locally connected if and only if it is homeomorphic with a space in which every open ball is connected [1].

DEFINITION 2. Let the number

$$\delta_r(A) = \sup \varrho_r(x, y) \quad \text{where} \quad x, y \in A$$

be called the *relative diameter of* A.

The oscillation of the function f^{-1} at the point p (which is also called the *oscillation of the space at the point* p) is by the definition (compare § 21, III)

$$\omega(p) = \inf \delta_r(X) \quad \text{where} \quad p \in \operatorname{Int}(X)$$
$$= \limsup_{x,y \to p} \varrho_r(x, y).$$

EXAMPLES. Consider the curve $\sin 1/x$ of Example 3 of Section I; here $\omega(p) = 2$ at every point of this curve with the abscissa 0. In Example 4, $\omega(y) = 1-y$ on the y axis.

THEOREM 6. *On every indecomposable continuum $\mathscr{X}$ the oscillation is constant* $\big(= \delta(\mathscr{X})\big)$.

Because in every neighbourhood of a given point p there exists a point q such that $\mathscr{X}$ is irreducible between p and q (§ 48, VI, Theorem 6); therefore $\varrho_r(p, q) = \delta(\mathscr{X})$.

§ 50. Locally connected metric continua*

I. Arcwise connectedness.

DEFINITIONS. A space is said to be (*integrally*) *arcwise connected* (*i.a.c.*) if every pair of its points can be joined by an arc. It is said to be *locally arcwise connected* (*l.a.c.*) at the point p if in every neighbourhood of p there exists an i.a.c. neighbourhood of p; that means, if to each $\varepsilon > 0$ there corresponds an $\eta > 0$ for which the condition $|x-p| < \eta$ implies the existence of an arc A such that $x, p \in A$ and $\delta(A) < \varepsilon$.

THEOREM 1. *The local arcwise connectedness at the point p implies the local connectedness at this point.*

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[1] M. H. A. Newman, *Plane Topology*, p. 75.

* The locally connected metric continua are also called *Peano continua.*

THEOREM 2. *If a connected space is l.a.c., it is also i.a.c.*

Generally speaking, *every region (and in particular every component) of a l.a.c. space is i.a.c.*

Proof. The first part of the theorem is a direct corollary of Theorem 8 of § 46, II, where G_p denotes an i.a.c. neighborhood of p. The second part follows from the first one inasmuch as the components of a l.c. space are regions (§ 49, II, Theorem 4).

THEOREM 3. *In every connected and l.a.c. space the relative distance $\varrho_r(x, y)$ is equal to the greatest lower bound of the diameters of arcs joining x and y. Therefore the condition $\varrho_r(x, y) < \varepsilon$ means that there exists an arc xy with the diameter $< \varepsilon$.*

Proof. This is a corollary of Theorem 2 and of § 49, VII, Theorem 3.

THEOREM 4. *If F is a compact subset of a connected l.a.c. space $\mathscr{X}$, then to each $\varepsilon > 0$ there corresponds an $\eta > 0$ such that for every pair of points x, y in F the condition $|x-y| < \eta$ implies $\varrho_r(x, y) < \varepsilon$, i.e. that x and y can be joined in $\mathscr{X}$ by an arc with diameter $< \varepsilon$.*

Proof. By metrizing the space $\mathscr{X}$ with the distance function $\varrho_r(x, y)$, a continuous transformation of $\mathscr{X}$ is performed (§ 49, VII, Theorem 4), which is therefore uniformly continuous in F.

Inasmuch as F is a non-empty compact set, there exists a continuous f such that (compare § 41, VI, Corollary 2b)

$$f: \mathscr{C} \to F \quad \text{and} \quad f(\mathscr{C}) = F. \tag{1}$$

This statement can be strengthened in the following terms.

THEOREM 5. *If F is a non-empty compact subset of a connected and l.a.c. space $\mathscr{X}$, every continuous mapping f which satisfies conditions* (1) *has an extension $f^*: \mathscr{I} \to \mathscr{X}$ which is a homeomorphism in every interval continguous to $\mathscr{C}$.*

Proof. Let (a_1, b_1), $(a_2, b_2), \ldots$ be the sequence of intervals contiguous to $\mathscr{C}$. Since $\lim\limits_{n=\infty}(b_n - a_n) = 0$, it follows that

$$\lim_{n=\infty} |f(b_n) - f(a_n)| = 0,$$

and according to Theorem 4, there exists in $\mathscr{X}$ a sequence of arcs A_n $(n = 1, 2, \ldots)$ with end-points $f(a_n)$ and $f(b_n)$ such that $\lim\limits_{n=\infty} \delta(A_n) = 0$. Let f_n be a homeomorphic mapping of the interval

$a_n b_n$ onto the arc A_n such that $f_n(a_n) = f(a_n)$ and $f_n(b_n) = f(b_n)$. f^* is defined in the following way

$$f^*(t) = \begin{cases} f(t) & \text{for} \quad t \in \mathscr{C}, \\ f_n(t) & \text{for} \quad a_n \leqslant t \leqslant b_n,\ n = 1, 2, \ldots \end{cases}$$

The following statement is a particular case of Theorem 5.

THEOREM 6. *Every non-empty l.a.c. continuum is a continuous image of an interval.*

Proof. It is sufficient to set $F = \mathscr{X}$.

It will be shown in Section II that Theorem 6 can be generalized replacing the condition of local arcwise connectedness by the condition of local connectedness.

II. Characterization of locally connected continua.

THEOREM 1 (of Mazurkiewicz–Moore–Menger) (1). *Every complete l.c. space is l.a.c.*

Proof. It can be assumed that the space is connected (compare § 49, II, Theorems 3 and 4).

Let $p \neq q$ be two fixed points. Let $\boldsymbol{G}$ be the family of regions of diameter < 1. According to Theorem 9 of § 46, II, this family contains a finite system of regions $R_1, \ldots, R_k$ which is an "irreducible chain" between p and q; this means that

$$p \in R_1, \quad q \in R_k, \quad R_i \cap R_{i+1} \neq 0 \quad (\text{for } i < k), \tag{1}$$

$$R_i \cap R_{i'} = 0 \quad \text{if} \quad |i - i'| > 1. \tag{2}$$

Let $\boldsymbol{G}^*$ be the family of regions S such that $\delta(S) < 1/2$ and $\bar{S} \subset R_1$. Let (compare § 46, II, Theorem 9) $S_1, \ldots, S_{l_1}$ be a chain with links belonging to $\boldsymbol{G}^*$, which is irreducible between p and $R_1 \cap R_2$ (or between p and q if $k = 1$). In a similar way, a chain is constructed which is irreducible between $S_{l_1} \cap R_2$ and $R_2 \cap R_3$, and so on; and finally one — between $S_{j_{k-1}} \cap R_k$ and q.

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(1) S. Mazurkiewicz, *Sur les lignes de Jordan*, Fund. Math. 1 (1920) (new edition 1937, p. 201). See the earlier papers of the same author: C. R. Soc. de Sci. de Varsovie, 6 (1913), p. 305 and 941, vol. 9 (1916), p. 428. R. L. Moore, Trans. Amer. Math. Soc. 17 (1916), p. 135. K. Menger, Mon. Math. Phys. 36 (1929), p. 212. Compare also N. Aronszajn, Fund. Math. 15 (1930), p. 228 and my paper, *ibid.* p. 307.

Thus for $l_{i-1} < j \leqslant l_i$ $(l_0 = 0)$

$$\bar{S}_j \subset R_i \tag{3}$$

and it is easily seen that the chain

$$S_1, \ldots, S_{l_1}, S_{l_1+1}, \ldots, S_{l_2}, \ldots, S_{l_{k-1}}, S_{l_{k-1}+1}, \ldots, S_{l_k} \tag{4}$$

is irreducible between p and q.

Since the chain $R_1, \ldots, R_k$ is irreducible, there exist, for each of its point x, two subscripts a and a' such that

$$a \leqslant a' \leqslant a+1, \quad x \epsilon R_a \cap R_{a'}, \quad x \notin R_i \quad \text{for} \quad a \neq i \neq a'. \tag{5}$$

Similarly, if x belongs to the chain (4), there exist two subscripts β and β' such that $\beta \leqslant \beta' \leqslant \beta+1$, $x \epsilon S_\beta \cap S_{\beta'}$, $x \notin S_j$ for $\beta \neq j \neq \beta'$.

Condition (3) implies the following relation between a and β

$$l_{a-1} < \beta \leqslant \beta' \leqslant l_{a'}. \tag{6}$$

Because, if e.g. $\beta \leqslant l_{a-1}$, it follows by (3) that

$$x \epsilon S_\beta \subset R_1 \cup \ldots \cup R_{a-1},$$

contrary to (5).

Conditions (3) and (6) easily imply that

$$\begin{cases} \bar{S}_1 \cup \ldots \cup \bar{S}_\beta \cup \bar{S}_{\beta'} \subset R_1 \cup \ldots \cup R_a \cup R_{a'}, \\ \bar{S}_\beta \cup \bar{S}_{\beta'} \cup \ldots \cup \bar{S}_{l_k} \subset R_a \cup R_{a'} \cup \ldots \cup R_k. \end{cases} \tag{7}$$

In (generalized) Theorem 3 of § 47, V, let us set

$$A_1 = R_1 \cup \ldots \cup R_{a'}, \quad B_1 = R_a \cup \ldots \cup R_k, \quad C_1 = A_1 \cup B_1,$$
$$A_2 = S_1 \cup \ldots \cup S_{\beta'}, \quad B_2 = S_\beta \cup \ldots \cup S_{l_k}, \quad C_2 = A_2 \cup B_2.$$

It is easily seen that condition (i) of that theorem is satisfied for $n = 1, 2$ and so is condition (iii) for $n = 1$. Moreover,

$$x \epsilon A_1 \cap B_1 = R_a \cup R_{a'} \quad \text{and} \quad x \epsilon A_2 \cap B_2 = S_\beta \cup S_{\beta'},$$

hence

$$\delta(A_1 \cap B_1) \leqslant \delta(R_a) + \delta(R_{a'}) < 2 \quad \text{and} \quad \delta(A_2 \cap B_2) < 1.$$

The procedure, which has produced the chain (4) from the chain $R_1, \ldots, R_k$, can be generalized by induction, so that an infinite sequence of chains is obtained, which are irreducible between p

and q. If C_n is the union of links of the nth one of these chains, then all the hypotheses of the quoted theorem are fulfilled; and in particular

$$\alpha(C_n) < 1/n \quad \text{and} \quad \delta(A_n, B_n) < 2/n.$$

It follows that the set $\bigcap_{n=1}^{\infty} C_n$ is an arc pq.

THEOREM 2 (of Hahn–Mazurkiewicz–Sierpiński (¹)). *If $\mathscr{X}$ is a non-empty continuum, the following conditions are equivalent.*

(i) *$\mathscr{X}$ is a continuous image of an interval,*

(ii) *$\mathscr{X}$ is the union of a finite number of continua of diameter $< \varepsilon$ for every $\varepsilon > 0$,*

(iii) *$\mathscr{X}$ is locally connected.*

Proof. *Condition* (i) *implies* (ii). This is so because, if $f\colon \mathscr{I} \to \mathscr{X}$ is onto and continuous and therefore uniformly continuous, it follows that

$$\mathscr{I} = A_1 \cup \ldots \cup A_n, \quad \mathscr{X} = f(A_1) \cup \ldots \cup f(A_n) \quad \text{and}$$
$$\delta[f(A_i)] < \varepsilon, \quad i = 1, \ldots, n,$$

provided $A_1, \ldots, A_n$ are sufficiently small intervals.

Condition (ii) *implies* (iii). Suppose that $C_1, \ldots, C_n$ are continua such that

$$\mathscr{X} = C_1 \cup \ldots \cup C_n \quad \text{and} \quad \delta(C_i) < \varepsilon,$$

Let $p \in \mathscr{X}$ and let $i_1, \ldots, i_k$ be the system of all subscripts such that

$$p \in C_{i_1}, \ldots, p \in C_{i_k}.$$

Therefore, the set $C_{i_1} \cup \ldots \cup C_{i_k}$ is a connected neighbourhood of p with diameter $< 2\varepsilon$.

(¹) See the quoted papers of S. Mazurkiewicz and of H. Hahn, Jahresb. Deutsch. Math. Ver. 23 (1914), p. 318 and Sgb. Akad. Wiss. Wien 123 (1914), p. 2433, W. Sierpiński, *Sur une condition pour qu'un continu soit une courbe jordanienne*, Fund. Math. 1 (1920) (new edition 1937), p. 44.

A simple proof, due to J. L. Kelley, of the arcwise connectedness of the continuous images of an interval (which follows from Theorem 1 and 2), is reproduced in G. T. Whyburn's *Analytic Topology*, p. 39 (3).

Condition (iii) *implies* (i) (¹). Since the continuum $\mathcal{X}$ is l.c. and therefore l.a.c. (by Theorem 1), it is a continuous image of the interval according to Theorem 6 of Section I.

THEOREM 3. *Every compact l.c. space $\mathcal{X}$ is the union of a finite number of l.c. continua with arbitrarily small diameters.*

Proof. Since every compact l.c. space is the union of a finite number of l.c. continua (§ 49, II, Theorem 7), it is admissible to consider only the case where $\mathcal{X}$ is a l.c. continuum, and hence a continuous image of the interval. Working through the proof of the first part of Theorem 2 ((i) implies (ii)) again, the conclusion is reached that $\mathcal{X}$ is the union of a finite number of continua with diameter $< \varepsilon$ every of which is a continuous image of an interval; and this gives the required result (because (i) implies (iii)).

THEOREM 4. *Let $\mathcal{X}$ be a l.c. continuum and $\varepsilon > 0$ be a given number. There exists an $\eta > 0$ such that every pair of points whose distance is $< \eta$ can be joined by an arc with diameter $< \varepsilon$.*

Proof. This follows from Theorem 4 of Section I because $\mathcal{X}$ can be replaced there by F.

THEOREM 5. *The property of being a l.c. continuum is invariant under continuous mappings.*

Proof. This follows from the equivalence of (i) and (iii).

Theorem 1 implies the following.

THEOREM 6. *Every complete space, which is connected, l.c. and irreducible between two points $a \neq b$, is an arc ab.*

Remark 1. Theorem 6 combined with Theorem 2 of § 49, VI, implies directly Theorem 6 of § 48, VII.

THEOREM 7. *Let $\mathcal{X}$ be a continuum irreducible between two points and let N be the set of points at which $\mathcal{X}$ is not l.c. The space $\mathcal{Y}$ of the decomposition of $\mathcal{X}$ into components of $\overline{N}$ and individual points of $\mathcal{X}-\overline{N}$ is an arc (provided $\overline{N} \neq \mathcal{X}$)* (²).

(¹) This statement contains as a particular case the famous theorem of Peano, according to which the square $\mathcal{I}^2$ is a continuous image of the interval $\mathcal{I}$ (Math. Ann. 36 (1890), p. 157). A more direct proof is given in § 16, II, Corollary 6b.

(²) Theorem of R. L. Moore, *Concerning the prime parts of a continuum*, Math. Zeitschr. 22 (1925), p. 307.

Proof. This is so because $\mathscr{Y}$ is a continuum irreducible between two points (by Theorem 3 of § 48, I) and is l.c. (according to § 49, VI, Theorem 3).

THEOREM 8. *A closed set separates a complete l.c. space between two points if and only if it cuts the space between these points.*

In other words, *if a complete l.c. space is connected between two points, it contains a continuum which joins them.*

Proof. Since the space is connected between a and b, there exists a region R containing these points (namely the component of a and b, compare § 49, II, Theorem 17), and since R is locally connected and topologically complete, it contains a continuum which joins a and b (by Theorem 1).

THEOREM 9. *Let $\mathscr{X}$ be a separable, connected and l.c. space, which is locally compact but not integrally compact. Every point of $\mathscr{X}$ is a vertex of a closed topological ray (that means, of a closed set homeomorphic with the half-line)* (1).

Proof. Let $\mathscr{X}^*$ denote the space $\mathscr{X}$, to which "the point at infinity" ∞ has been added (compare § 41, X, Theorem 5). $\mathscr{X}^*$ is a l.c. continuum (because all points, except the point ∞, are points of local connectedness in $\mathscr{X}^*$, then so is the point ∞ by Theorem 1 of § 49, VI). If A is an arc $p\infty$ (whose existence follows from Theorem 1), $A-(\infty)$ is the required ray.

Let us quote the following theorem (2) without proof.

THEOREM 10. *Let $\mathscr{X}$ be a 1-dimensional locally connected continuum and let $\varepsilon > 0$ be a given number. There exists a continuous mapping f: $\mathscr{X} \to \mathscr{X}$ such that*

(i) *$f(\mathscr{X})$ is homeomorphic to a polygonal line,*

(ii) *$\delta[f^{-1}(y)] < \varepsilon$ for any y.*

Remark 2. Without assuming that the space is metric, one has the following theorems (analogous to Theorem 2).

THEOREM 11. *Let $\mathscr{X}$ be a continuum. $\mathscr{X}$ is locally connected if and only if for each open cover there is a finite refinement composed of continua.*

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(1) See my paper, *Quelques propriétés topologiques de la demi-droite*, Fund. Math. 3 (1922), p. 59.

(2) See S. Mazurkiewicz, Fund. Math. 20 (1933), p. 281.

Proof. 1. The condition is necessary. Let $\{G_t\}$ be an open cover of $\mathscr{X}$. For each $p \in \mathscr{X}$, let $p \in G_{t(p)}$. Since $\mathscr{X}$ is normal (hence regular, see § 41, II, Theorem 3), there is an open H_p such that

$$p \in H_p \quad \text{and} \quad \overline{H}_p \subset G_{t(p)}. \tag{8}$$

Since $\mathscr{X}$ is supposed to be locally connected, therefore (by Theorem 4 of § 49, II) the components of H_p are open. Thus the family of all components of all H_p is an open cover of $\mathscr{X}$. Since $\mathscr{X}$ is compact, this cover contains a finite subcover: $S_1, \ldots, S_n$. Put $C_i = \overline{S}_i$. It follows that

$$\mathscr{X} = C_1 \cup \ldots \cup C_n, \tag{9}$$

and, since S_i is connected, C_i is a continuum. Finally, for each i there is a t such that $C_i \subset G_t$; we put namely $t = t(p)$ where S_i is a component of H_p.

2. The condition is sufficient. Let G be open and $p \in G$. We must define a connected E so that

$$p \in \operatorname{Int}(E) \quad \text{and} \quad E \subset G. \tag{10}$$

Let us consider the cover composed of two members: G and $H = \mathscr{X} - (p)$ (H is open since $\mathscr{X}$ is $\mathscr{T}_2$, hence $\mathscr{T}_1$). By assumption, (9) is true and each C_i is a subcontinuum either of G or of H.

Let $C_{k_1}, \ldots, C_{k_r}$ be those continua which do contain p and $C_{m_1}, \ldots, C_{m_s}$ those which do not. Put

$$E = C_{k_1} \cup \ldots \cup C_{k_r}.$$

Hence $\mathscr{X} - E \subset C_{m_1} \cup \ldots \cup C_{m_s}$ and $\overline{\mathscr{X} - E} \subset C_{m_1} \cup \ldots \cup C_{m_s}$. Thus $p \in \mathscr{X} - \overline{\mathscr{X} - E}$, i.e. $p \in \operatorname{Int}(E)$. The inclusion (10) is also true since $p \in C_{k_i}$; hence $C_{k_i} \not\subset H$ and consequently $C_{k_i} \subset G$ for $i = 1, \ldots, r$.

THEOREM 12. *Let $\mathscr{X}$ and $\mathscr{Y}$ be $\mathscr{T}_2$-spaces. Let $f \colon \mathscr{X} \to \mathscr{Y}$ be continuous and onto. If $\mathscr{X}$ is a locally connected continuum, then so is $\mathscr{Y}$.*

Proof. Let $\{H_t\}$ be an open cover of $\mathscr{Y}$. According to Theorem 11, we must define a finite refinement composed of continua.

Now, since $\{f^{-1}(H_t)\}$ is an open cover of the locally connected continuum $\mathscr{X}$, there are continua $C_1, \ldots, C_n$ such that (9) is true and $C_i \subset f^{-1}(H_i)$. It follows that

$$\mathscr{Y} = f(C_1) \cup \ldots \cup f(C_n) \quad \text{and} \quad f(C_i) \subset ff^{-1}(H_i) \subset H_i.$$

This completes the proof.

Remark 3. Condition (ii) (of Sierpiński), which characterizes locally connected metric continua, follows easily from Theorem 11.

III. Regions and subcontinua of a locally connected continuum $\mathscr{X}$. Let F be a closed set and let G be its complement.

THEOREM 1. *There exists a sequence of closed l.c. sets $F_1, F_2, \dots$ such that*

(i) $F = F_1 \cap F_2 \cap \dots,$

(ii) $F_n \supset F_{n+1},$

(iii) *every component of F_n contains a component of F,*

(iv) *no component of G contains two distinct components of $\mathscr{X}-F_n$.*

(v) *$\mathscr{X}-F_n$ has a finite number of components.*

In particular, *if F is a continuum, then so is F_n; if F does not separate the space, F_n does not separate it either.*

Moreover, *the number of components of F_n does not exceed the number of components of F and the number of components of $\mathscr{X}-F_n$ does not exceed the number of components of G.*

Proof. First we are going to define a sequence of closed l.c. sets $F_1^*, F_2^*, \dots$ which satisfy conditions (i)–(iii).

We proceed by induction. Let $F_1^* = \mathscr{X}$. Assume that F_n^* is closed and l.c. and that $F \subset F_n^*$. We have to define F_{n+1}^*.

According to Theorem 3 of Section II, there exists for every n a system of l.c. continua $K_1^n, \dots, K_{m_n}^n$ such that

$$F_n^* = K_1^n \cup \dots \cup K_{m_n}^n \quad \text{and} \quad \delta(K_i^n) < 1/n.$$

Let F_{n+1}^* be the union of all K_i^n such that $F \cap K_i^n \neq 0$. It follows that

$$F \subset F_{n+1}^* \subset F_n^* \quad \text{and} \quad \operatorname{dist}(F, F_{n+1}^*) < 1/n,$$

which implies conditions (i) and (ii).

Moreover, F_{n+1}^* is l.c. since it is a union of l.c. continua (compare § 49, II, Theorem 1).

Finally, condition (iii) is fulfilled since every component of F_{n+1}^* has some points with F in common, because it is the union of some sets K_i^n which are not disjoint from F.

Thus the sequence $\{F_n^*\}$ is defined; we define now the sequence $\{F_n\}$.

Let $Q_1, Q_2, \ldots$ be the (finite or infinite) sequence of components of G. By Theorem 14 of § 49, II, there exists a double sequence of regions R_k^i such that

$$Q_i = R_1^i \cup R_2^i \cup \ldots, \tag{1}$$

$$\overline{R_k^i} \subset R_{k+1}^i. \tag{2}$$

We rearrange the double sequence $\{R_k^i\}$ into a simple sequence $R_1, R_2, \ldots$ Thus

$$G = R_1 \cup R_2 \cup \ldots, \tag{3}$$

$$\bar{R}_k \subset G. \tag{4}$$

For a fixed n, consider all the regions R_k disjoint from F_n^* and with subscripts $k \leqslant n$. Let S_n be the union of components of $\mathscr{X} - F_n^*$ which contain these regions. Let $F_n = \mathscr{X} - S_n$. It follows that $S_n \subset \mathscr{X} - F_n^*$, which implies $F_n^* \subset \mathscr{X} - S_n = F_n$, and since $F = F_1^* \cap F_2^* \cap \ldots$, we have

$$F \subset F_1 \cap F_2 \cap \ldots.$$

Since (i) is equivalent to identity $G = S_1 \cup S_2 \cup \ldots$, and hence to identity

$$R_1 \cup R_2 \cup \ldots = S_1 \cup S_2 \cup \ldots$$

(according to (3)), our task is to show that, given an integer k, there exists an integer $n \geqslant k$ such that $R_k \cap F_n^* = 0$.

Since $\mathscr{X} - \bar{R}_k$ is a neighbourhood of the set $F = \operatorname{Lim} F_n^*$ (by (4)), it follows for sufficiently large n that $F_n^* \subset \mathscr{X} - \bar{R}_k$, hence $R_k \cap F_n^* = 0$.

Since $F_n^* \supset F_{n+1}^*$, it follows that $S_n \subset S_{n+1}$, which implies (ii).

Since F_n^* is l.c., then so is the set $F_n = \mathscr{X} - S_n$ by Theorem 11 of § 49, II. According to Theorem 12 of § 49, II, every component of F_n contains a component of F_n^*, and therefore contains a component of F (since the set F_n^* satisfies condition (iii)).

Suppose that condition (iv) is not fulfilled; this means that there exist two (distinct) components U and V of S_n which are contained in one component Q_i of G. By the definition of S_n, U and V are two components of $\mathscr{X} - F_n^*$ which contain two distinct terms of the sequence $R_1^i, R_2^i, \ldots$. But this contradicts the inclusion (2).

Finally, condition (v) is fulfilled since the number of components of S_n does not exceed n.

THEOREM 2. *There exists a sequence of closed l.c. sets* $H_1, H_2, \ldots$ *such that*

(i) $G = H_1 \cup H_2 \cup \ldots$,

(ii) $H_n \subset \mathrm{Int}(H_{n+1})$,

(iii) *no component of* G *contains two distinct components of* H_n,

(iv) *every component of* $\mathcal{X} - H_n$ *contains a component of* F, *and therefore*

(v) $\mathcal{X} - H_n$ *has a finite number of components.*

In particular, *if* G *is a region,* H_n *is a continuum; if* F *is a continuum,* $\mathcal{X} - H_n$ *is a region* (1).

Moreover, *the number of components of* H_n *does not exceed the number of components of* G *and the number of components of* $\mathcal{X} - H_n$ *does not exceed the number of components of* F.

Proof. Let $\{Q_i\}$, $\{R_k^i\}$ and $\{R_k\}$ be the sequences considered in the preceding proof. Let

$$U_k = R_1 \cup \ldots \cup R_k, \quad \text{where} \quad k = 1, 2, \ldots. \tag{5}$$

By Theorem 1 there exists a closed l.c. set A_k such that

$$\overline{U}_k \subset A_k \subset G, \quad \text{where} \quad k = 1, 2, \ldots, \tag{6}$$

and every component of A_k contains a component of U_k.

Let H_k denote the union of the set A_k and of all components of $\mathcal{X} - A_k$ contained in G.

By Theorem 11 of § 49, II, the set H_k is closed and l.c.

Condition (i) is a straightforward consequence of (3), (5) and (6). Moreover, since

$$G = U_1 \cup U_2 \cup \ldots, \quad U_k \subset U_{k+1} \quad \text{and} \quad U_k \subset H_k, \tag{7}$$

condition (i) is satisfied (replacing the sequence $\{H_k\}$ by any subsequence $\{H_{m_k}\}$ where $m_1 < m_2 < \ldots$).

In order to satisfy condition (ii), the sequence $\{H_k\}$ will be replaced by the subsequence $\{H_{m_k}\}$ defined by induction in the following way.

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(1) For a particular case and some related statements, see W. M. Kincaid, *On non-cut sets of locally connected continua*, Bull. Amer. Math. Soc. 49 (1943), p. 399.

1) $m_1 = 1$, 2) m_{k+1} is the least subscript r such that $H_{m_k} \subset U_r$ (the existence of such a subscript r follows from identity (7) and from the fact that H_{m_k} is a compact subset of G).

According to (2) no component Q_i of G contains two distinct components of U_k, and therefore of A_k, and hence of H_k (since every component of H_k contains a component of A_k according to § 49, II, Theorem 12).

Since every component C of $\mathcal{X}-H_k$ is a component of $\mathcal{X}-A_k$ not contained in G, it follows that $C-G \neq 0$, i.e. $C \cap F \neq 0$. Thus there exists a component K of F such that $C \cap K \neq 0$. This inequality combined with the inclusion $K \subset \mathcal{X}-H_k$ (which follows from the inclusions $H_k \subset G$ and $K \subset F$) implies that $K \subset C$, since C is a component of the set $\mathcal{X}-H_k$ and K is a subcontinuum.

LEMMA 3. *Let R be a region and $\mathbf{S}$ a system of $n+1$ disjoint continua lying in R. Let $C_0 \in \mathbf{S}$. If the elements of $\mathbf{S}$ are properly numbered, n regions $R_1, \ldots, R_n$ can be found such that for $k = 1, \ldots$ $\ldots, n$ the following holds*

$$\bar{R}_k \subset R_{k-1}-C_k, \qquad C_0 \cup C_{k+1} \cup C_{k+2} \cup \ldots \cup C_n \subset R_k \qquad (\textit{where } R_0 = R). \tag{8}$$

According to Theorem 6 of § 46, III, there exists a continuum $C_1 \in \mathbf{S}-(C_0)$ such that all the elements of $\mathbf{S}-(C_1)$ are situated in a single component Q of $R-C_1$. Therefore, they are contained in a region R_1 such that $\bar{R}_1 \subset Q$ (compare § 49, II, Theorem 15).

We show in a similar way that there exist an element C_2 in $\mathbf{S}-(C_0, C_1)$ and a region R_2 such that all elements of $\mathbf{S}-(C_1, C_2)$ are contained in R_2 and that $\bar{R}_2 \subset R_1-C_2$.

Proceeding this way step by step, the required numbering of the elements of $\mathbf{S}$ is found.

THEOREM 4. *Let F be a closed set consisting of an infinite sequence of components, each of which except one, say C_0, is open in F.*

If these components are properly ordered into an infinite sequence $C_0, C_1, C_2, \ldots$, a sequence of open sets $G_1, G_2, \ldots$ can be found such that G_n consists of $n+1$ components

$$G_n = R_{n,0} \cup \ldots \cup R_{n,n} \tag{9}$$

and that

$$F = \bigcap_{n=1}^{\infty} G_n, \tag{10}$$

$$\bar{G}_{n+1} \subset G_n, \tag{11}$$

$$C_j \subset R_{n,j} \quad \textit{for} \quad 1 \leqslant j \leqslant n, \tag{12}$$

$$C_0 \cup C_{n+1} \cup C_{n+2} \cup \ldots \subset R_{n,0}. \tag{13}$$

Proof. Observe first that there exists a sequence of continua $K_1, K_2, \ldots$ such that

$$C_0 = \bigcap_{n=1}^{\infty} K_n, \tag{14}$$

$$K_{n+1} \subset K_n \tag{15}$$

and that for every n all components of F, except a finite number, are contained in K_n.

Let $F_1, F_2, \ldots$ be a sequence of closed l.c. sets, which satisfy conditions (i) and (ii) of Theorem 1. Let K_n be the component of F_n containing C_0. The intersection $K_1 \cap K_2 \cap \ldots$ coincides with C_0, being a subcontinuum of F. Moreover, since the number of components of F_n is finite (according to Theorem 7 of § 49, II), K_n contains all components of F except a finite number.

Let $\boldsymbol{S}_n$ be a system, whose elements are the continuum K_n and all the components of F disjoint from K_n. From conditions (14) and (15) it easily follows that all components of F distinct from C_0 are the elements of the union $\boldsymbol{S}_1 \cup \boldsymbol{S}_2 \cup \ldots$.

Let l_n+1 be the number of elements in $\boldsymbol{S}_n$. It is admissible to assume that $0 = l_0 < l_1 < l_2 < \ldots$. By Lemma 3 the elements of $\boldsymbol{S}_n - \boldsymbol{S}_{n-1}$ distinct from K_n can be labelled with subscripts $k = l_{n-1}+1, \ldots, l_n$ and a system of $l_n - l_{n-1}$ regions R_k can be defined in such a way that

$$\bar{R}_k \subset R_{k-1} - C_k, \; R_0 = \mathscr{X}, \tag{16}$$

$$K_n \cup C_{k+1} \cup C_{k+2} \cup \ldots \cup C_{l_n} \subset R_k. \tag{17}$$

So, for instance,

$$C_1 \cap \bar{R}_1 = 0, \quad K_1 \cup C_2 \cup C_3 \cup \ldots \cup C_{l_1} \subset R_1, \quad K_1 \subset R_{l_1},$$

$$\bar{R}_{l_1} \subset R_{l_1 - 1} - C_{l_1}.$$

It is also admissible to assume that the region R_{l_n} is contained in a ball Z_n with center K_n and radius $1/n$; this means that

$$\varrho(x, K_n) < 1/n \quad \text{for} \quad x \in R_{l_n}. \tag{18}$$

Because R_{l_n} can be replaced, if necessary, by the component of the set $Z_n \cap R_{l_n}$ which contains K_n.

Now we are going to define the regions $R_{n,0}, \ldots, R_{n,n}$, where $n = 0, 1, \ldots$.

Let n be fixed. Assume $R_{n,0} = R_n$. Supposing that $C_j \subset R_{n-1,j}$ where $1 \leqslant j \leqslant n-1$, let $R_{n,j}$ be a region such that

$$C_j \subset R_{n,j}, \tag{19}$$

$$\bar{R}_{n,j} \subset R_{n-1,j}, \tag{20}$$

$$\varrho(x, C_j) < 1/n \quad \text{for} \quad x \in R_{n,j}. \tag{21}$$

Moreover, let (compare (16))

$$C_n \subset R_{n,n}, \tag{22}$$

$$\bar{R}_{n,n} \subset R_{n-1} - \bar{R}_n. \tag{23}$$

Define G_n by identity (9). Then the regions $R_{n,0}, \ldots, R_{n,n}$ are its components; in other words, they are disjoint. Because, for $0 < i < j \leqslant n$, we have (compare (20), (22) and (16))

$$R_{n,i} \subset R_{i,i} \subset R_{i-1} - R_i \subset R_{i-1} - R_{j-1}, \quad \text{thus} \quad R_{n,i} \cap R_{n,j} = 0,$$

and on the other hand

$$R_{n,0} \cap R_{n,j} \subset R_n - R_j = 0.$$

Condition (10) follows from the identities

$$C_0 = \bigcap_{n=1}^{\infty} R_{n,0} = \bigcap_{n=1}^{\infty} R_n \quad \text{and} \quad C_j = \bigcap_{n=j}^{\infty} R_{n,j} \quad (j > 0),$$

which are consequences of (14), (18) and (16), and of (21).

Next, inclusion (11) follows from the formulas (compare (20) and (8))

$$\bar{R}_{n,j} \subset R_{n-1,j} \ (1 \leqslant j \leqslant n-1), \quad \bar{R}_{n,0} = \bar{R}_n \subset R_{n-1,0}, \quad \bar{R}_{n,n} \subset R_{n-1,0}.$$

Finally, inclusions (12) and (13) follow from (19), (22) and (8).

THEOREM 5. *If $R_1, \ldots R_n$ are regions such that $\mathscr{X} = R_1 \cup \ldots \cup R_n$, there exists a system of l.c. continua $C_1, \ldots, C_n$ such that*

$$\mathscr{X} = C_1 \cup \ldots \cup C_n \quad \textit{and} \quad C_i \subset R_i. \tag{24}$$

Proof. According to Corollary of § 14, III, there exists a system of closed sets $F_1, \ldots, F_n$ such that

$$\mathscr{X} = F_1 \cup \ldots \cup F_n \quad \text{and} \quad F_i \subset R_i.$$

By Theorem 15 of § 49, II, the last inclusion implies the existence of a continuum C_i such that $F_i \subset C_i \subset R_i$. By Theorem 1 it can be assumed that C_i is a l.c. continuum, and (24) follows.

THEOREM 6. *If a l.c. continuum $\mathscr{X}$ is not unicoherent, it is the union of two l.c. continua whose intersection is not connected.*

Proof. By hypothesis there exists two continua K and L such that $\mathscr{X} = K \cup L$ and that $K \cap L$ is not connected. According to Theorem 1, let $K_1, K_2, \ldots$ and $L_1, L_2, \ldots$ be two sequences of l.c. continua such that

$$K = \bigcap_{n=1}^{\infty} K_n, \quad L = \bigcap_{n=1}^{\infty} L_n, \quad K_{n+1} \subset K_n \quad \text{and} \quad L_{n+1} \subset L_n.$$

It follows that

$$\mathscr{X} = K_n \cup L_n \quad \text{and} \quad K \cap L = \bigcap_{n=1}^{\infty} (K_n \cap L_n).$$

For sufficiently large n the set $K_n \cap L_n$ is not connected, because otherwise $K \cap L$ would be connected (according to Theorem 5 of § 47, II).

THEOREM 7. *If R is a subregion of $\mathscr{X}$, then*

(i) *the set of points of* $\mathrm{Fr}(R)$ *accessible from R is dense in* $\mathrm{Fr}(R)$,

(ii) *if $p \in \mathrm{Fr}(R)$ and $R \cup p$ is l.c., then p is accessible from R,*

(iii) *every point p accessible from R is accessible by a l.c. continuum, and therefore by an arc; i.e. there exists an arc L such that $p \in L \subset R \cup p$.*

Proof. Let $p \in \mathrm{Fr}(R)$ and let ε be a positive number. Since $\mathscr{X}$ is l.c., there exists an arc qp such that $q \in R$ and $\delta(qp) < \varepsilon$. Therefore, if r is the first point of $\mathrm{Fr}(R)$ on qp, r is accessible and $|p - r| < \varepsilon$; statement (i) follows.

(ii) follows from Theorem II, 1 because the set $R \cup p$ is a G_δ, and hence is topologically complete (compare § 33, VI).

Finally, let C be a continuum such that

$$p \in C \subset R \cup p \quad \text{and} \quad C \cap R \neq 0.$$

Let

$$A_n = C \cap \underset{x}{E}[1/n \leqslant |x-p| \leqslant 1/(n-1)], \tag{25}$$

thus

$$C - p = \bigcup_{n=1}^{\infty} A_n. \tag{26}$$

Therefore $\underset{n=\infty}{\operatorname{Lim}} A_n = p$. By Theorem 1 there exists a closed l.c. set F_n such that

$$A_n \subset F_n \subset R, \tag{27}$$

$$\underset{n=\infty}{\operatorname{Lim}} F_n = p, \tag{28}$$

and that all components of F_n (whose number is finite, compare § 49, II, Theorem 7) have points in common with A_n. By (26)–(28), the set $C^* = p \cup F_1 \cup F_2 \cup \ldots$ is a continuum. By (27) no point belongs to infinitely many F_n and hence the continuum C^* is l.c. at every point of the union $F_1 \cup F_2 \cup \ldots$, and therefore at every of its points (since p cannot be the single point at which C^* is not l.c., compare § 49, VI, Theorem 1).

Let us quote the following statement without proof.

(iv) *If* $\dim \operatorname{Fr}(R) = 0$ *and* $p \in \operatorname{Fr}(R)$, *then* $R \cup p$ *is l.c.* ([1]).

THEOREM 8. *Let G be an open subset of a l.c. continuum and let $R_1, R_2, \ldots$ be the sequence of its components. If* $\lim\limits_{n=\infty} \delta(R_n) = 0$, *then*

$$d_1(G) = \max \delta(R_n);$$

in other words (compare § 45, IV), *there exists a finite system of open sets $H_1, \ldots, H_m$ such that*

$$G = H_1 \cup \ldots \cup H_m, \; H_i \cap H_j = 0 \textit{ for } i \neq j,$$
$$\textit{and } \delta(H_i) \leqslant \max \delta(R_n).$$

([1]) G. T. Whyburn, *A general notion of accessibility*, Fund. Math. 14 (1929), p. 315.

Proof. Put $\mu = \max \delta(R_n)$. Let k be a subscript such that $\delta(R_n) < \mu/3$ for $n > k$. Since the set G is totally bounded, let $A_1, \ldots, A_r$ be a system of sets such that

$$G = A_1 \cup \ldots \cup A_r \quad \text{and} \quad \delta(A_i) < \mu/3 \quad \text{for} \quad i = 1, \ldots, r.$$

We put $m = k+r$, $H_i = R_i$ for $i \leqslant k$ and H_{k+j} = the union of sets R_n such that $n > k$ and that

$$R_n \cap A_j \neq 0 = R_n \cap A_s \quad \text{for} \quad s < j \quad (1 \leqslant j \leqslant r).$$

IV. Continua hereditarily locally connected (h.l.c.) (1). This is the name used for every continuum which is l.c. as well as each of its subcontinua.

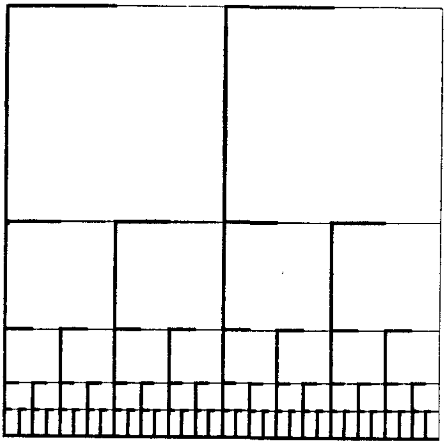

Fig. 7

The arc is an example of a set of that kind and so is the continuum of § 49, VI, Remark. However, the square $\mathscr{I}^2$ is l.c., but not h.l.c. And so is the following continuum, which is the union of two h.l.c. continua.

Let C be the continuum consisting of the segment $(0 \leqslant x \leqslant 1, y = 0)$, of the vertical segments $(x = m/2^{n+1}, 0 \leqslant y \leqslant 1/2^n)$ with $0 \leqslant m \leqslant 2^{n+1}$ and of the level segments $(0 \leqslant x \leqslant 1,\ y = 1/2^n)$, where $n = 0, 1, \ldots$ The continuum C is the union of two h.l.c.

(1) Compare G. T. Whyburn, *Concerning hereditarily locally connected continua*, Amer. Journ. Math. 53 (1931), pp. 374–384.

continua which are symmetric with respect to the line $x = 1/2$ and one of which is marked in the figure with heavy lines.

Theorem 2 of § 49, VI implies the following (compare also § 51, IV, Theorems 3 and 2).

THEOREM 1. *Every continuum which contains no nowhere dense subcontinua (containing more than one point) is h.l.c.*

THEOREM 2 [1]. *A continuum is h.l.c. if and only if it contains no convergence continua (containing more than one point).*

Proof. The condition is sufficient by Theorem 1 of § 49, VI; we are going to prove that it is necessary, i.e. that if $K_1, K_2, \ldots$ are continua such that

$$\operatorname*{Lim}_{n=\infty} K_n = K, \quad K_i \cap K_j = 0 \quad \text{for} \quad i \neq j,$$
$$K \cap K_n = 0, \; p \in K \neq p,$$

there exists a continuum which is not l.c.

Thus, we may assume that the space is l.c. So there exists a continuum Q such that $p \in \operatorname{Int}(Q)$ and $K - Q \neq 0$. Since $p \in \operatorname*{Lim}_{n=\infty} K_n$, there exists n_0 such that $Q \cap K_n \neq 0$ for $n \geqslant n_0$. The continuum

$$C = K \cup Q \cup K_{n_0} \cup K_{n_0+1} \cup \ldots$$

is not l.c. at any point $q \in K - Q$. Because otherwise there would exist a continuum $L \subset C - Q$ containing q in its interior relative to C. Therefore

$$L \cap (K_{n_0} \cup K_{n_0+1} \cup \ldots) \neq 0.$$

But then

$$L = (L \cap K) \cup (L \cap K_{n_0}) \cup (L \cap K_{n_0+1}) \cup \ldots$$

would be a decomposition of the continuum L into a sequence of closed pairwise disjoint sets, at least two of which are non-empty, which contradicts Sierpiński's theorem (§ 47, III, Theorem 6).

THEOREM 3. *If $R_1, R_2, \ldots$ is a sequence of disjoint regions in a h.l.c. continuum, then* $\lim_{n=\infty} \delta(R_n) = 0$.

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[1] See C. Zarankiewicz, Fund. Math. 9 (1927), p. 134 and P. Urysohn, Verh. Akad. Amsterdam 13 (1927), p. 49.

Proof. Otherwise there would exist an $\varepsilon > 0$ and a convergent sequence of continuua $K_{i_1}, K_{i_2}, \ldots$ such that $K_{i_n} \subset R_{i_n}$ and $\delta(K_{i_n}) > \varepsilon$. But then the limit $K = \operatorname*{Lim}_{n=\infty} K_{i_n}$ would be a convergence continuum which contains more than one point since $\delta(K) \geqslant \varepsilon$.

Remark. However, there can exist a sequence of disjoint continua $C_1, C_2, \ldots$ such that $\lim_{n=\infty} \delta(C_n) \neq 0$. The following continuum provides an example (¹).

Let $p_1, p_2, \ldots$ be the sequence of prime numbers starting with 3. Let C be the interval 01 of the X axis in the space XYZ. Let C_n be the arc consisting of half-circles joining successively the points

$$1/p_n, 2/p_n, \ldots, (p_n-1)/p_n$$

and lying in the plane $z = y/n$.

The continuum $C \cup C_1 \cup C_2 \cup \ldots$ is h.l.c., though $\lim_{n=\infty} \delta(C_n) = 1$.

THEOREM 4. *Let G be an open subset of a h.l.c. continuum and $R_1, R_2, \ldots$ be the sequence of components of G. If $\delta(R_n) < \varepsilon$ for every n, then $d_1(G) < \varepsilon$, which means that G admits a decomposition into a finite number of open disjoint sets, of diameter $< \varepsilon$.*

Proof. Theorem 4 follows from Theorem 3 and Theorem 8 of Section III.

Theorems 3 and 4 imply easily the following

THEOREM 5. *Every h.l.c. continuum has the following property.*

(*) *If $G_1, G_2, \ldots$ is a sequence of disjoint open sets, then* $\lim d_1(G_n) = 0$, *which means that to every $\varepsilon > 0$ there corresponds a subscript n_ε such that, for $n > n_\varepsilon$, G_n admits a decomposition into a finite number of disjoint open sets with diameters $< \varepsilon$.*

This can be stated more generally in the following way.

THEOREM 5a. *If A is a subset of a h.l.c. continuum, then A, considered as a space, has property* (*).

This is a corollary of the following statement.

(¹) Compare P. Urysohn, *loc. cit.* p. 46 and G. T. Whyburn, Math. Ann. 102 (1929), p. 333.

It should be noted that the singularity in question does not appear in the plane (compare § 59, II, Theorem 13).

THEOREM 6. *Property* $(*)$ *is hereditary; it means that, if* A *is a subset of a space which has property* $(*)$, *then this property subsists if the sets considered as "open" are the sets open relative to* A.

Proof. If the sets $G_1, G_2, \ldots$ are disjoint and open in A, there exists a sequence of disjoint sets, open in the space, $H_1, H_2, \ldots$ such that $G_n = A \cap H_n$ (compare § 21, XI, Theorem 2). By hypothesis there exists for $n > n_\varepsilon$ a decomposition $H_n = H_n^1 \cup \ldots \cup H_n^{m_n}$ into disjoint open sets with diameters $< \varepsilon$. The identity

$$G_n = (A \cap H_n^1) \cup \ldots \cup (A \cap H_n^{m_n})$$

provides a decomposition of G_n into disjoint sets, open in A and with diameters $< \varepsilon$.

LEMMA 7. *In a separable space with property* $(*)$, *let be given a closed set* A *and a family of closed-open sets* $\{G_t\}$ *such that* $A \subset \bigcup_t G_t$. *There exists an infinite sequence of subscripts* $t_1, t_2, \ldots$ *and an infinite sequence of closed-open sets* $H_1, H_2, \ldots$ *such that*

$$H_n \subset G_{t_n}, \tag{1}$$

$$A \subset \bigcup_n H_n = \overline{\bigcup_n H_n}. \tag{2}$$

Proof. By the Lindelöf theorem (§ 5, XI) there is *a sequence* $t_1, t_2, \ldots$ *such that*

$$A \subset \bigcup_n G_{t_n}.$$

Put

$$F_1 = G_{t_1} \quad \text{and} \quad F_n = G_{t_n} - (G_{t_1} \cup \ldots \cup G_{t_{n-1}}) \quad \text{for} \quad n > 1. \tag{3}$$

Thus the sets $F_1, F_2, \ldots$ are disjoint and closed-open, and it follows that

$$A \subset \bigcup_n F_n. \tag{4}$$

Since $\lim_{n=\infty} d_1(F_n) = 0$, then

$$F_n = F_{n1} \cup \ldots \cup F_{nl_n} \quad \text{where} \quad \delta(F_{nj_n}) < \varepsilon_n \text{ and } \lim_{n=\infty} \varepsilon_n = 0, \tag{5}$$

and the sets $F_{n1}, \ldots, F_{nl_n}$ are closed-open and disjoint. Let H_n be the union of those sets which have common points with A. Conditions (3) and (5) imply inclusion (1). The first part of (2) follows from (3)–(5). In order to prove the second one, let

$$x = \lim_{n=\infty} x_n \quad \text{where} \quad x_n \in H_{m_n} \quad \text{and} \quad m_1 < m_2 < \ldots.$$

Therefore $x_n \epsilon F_{m_n j_{m_n}}$ where $A \cap F_{m_n j_{m_n}} \neq 0$; let $y_n \epsilon A \cap F_{m_n j_{m_n}}$. So $|x_n - y_n| < \delta(F_{m_n j_{m_n}}) < \varepsilon_{m_n}$, and hence $\lim\limits_{n=\infty} y_n = x$. Thus $x \epsilon \bar{A} = A$, and hence $x \epsilon \bigcup\limits_n H_n$, by inclusion (2).

THEOREM 8. *If a separable space has property* (∗), *then the connectedness of the space between two closed sets A and B implies its connectedness between a pair of points $a \epsilon A$ and $b \epsilon B$.*

In other words, *property* (∗) *implies property* (M) (*considered in* § 47, II, *Theorem* 1).

Proof. First we are going to show that the space, which is not connected between B and any point a of A, is not connected between A and B. By hypothesis, a closed-open set $G(a)$ corresponds to every point $a \epsilon A$ so that

$$a \epsilon G(a) \quad \text{and} \quad B \cap G(a) = 0.$$

It follows from the lemma that there exists a closed-open set H such that

$$A \subset H \subset \bigcup_{a \epsilon A} G(a), \quad \text{which yields} \quad B \cap H = 0.$$

Therefore, the space is not connected between A and B.

Thus it has been proved that the connectedness between two closed sets A and B implies the connectedness between B and a point a of A. Applying this implication to the case where $A = (a)$ it follows finally that the space is connected between a and some point b of B.

THEOREM 9. *Every subset E of a h.l.c. continuum* (*every separable space with property* (∗)) *satisfies the following conditions.*

(i) *if E is connected between two sets A and B closed in E, there exists a component C of E such that $A \cap C \neq 0 \neq B \cap C$;*

(ii) *the quasi-components of E are connected and therefore they coincide with the components of E;*

(iii) *if $\dim_p E > 0$, p belongs to a connected set* (*which does not reduce to p*) *contained in E;*

(iv) *for every $\varepsilon > 0$ there exists only a finite number of components of E with diameters $> \varepsilon$;*

(v) (1) *if E is connected, it is locally connected.*

Proof. The statements (i), (ii) and (iii) hold by Theorem 8 or by Theorems 3, 2 and 9 of § 47, II, respectively.

In order to establish (iv), let us consider, according to Theorem 3 of § 46, V, a continuous mapping $f\colon E \to \mathscr{C}$ such that the sets $f^{-1}(y)$ coincide with the quasi-components of E (y varies over $f(E)$). Would there exist infinitely many components, and hence, according to (ii), infinitely many quasi-components of E with diameters $> \varepsilon$, then there would exist an infinite set $A \subset f(E)$ such that $\delta[f^{-1}(y)] > \varepsilon$ for $y \in A$. Let $H_1, H_2, \dots$ be an infinite sequence of disjoint sets, open in $\mathscr{C}$ and such that $A \cap H_n \neq 0$. Therefore no set $G_n = f^{-1}(H_n)$ can be decomposed into disjoint open sets with diameters $< \varepsilon$ (because G_n contains a connected set with diameter $> \varepsilon$). But this contradicts property (*) since the sets $G_1, G_2, \dots$ are disjoint and open.

We proceed to prove (v). Assume E to be the space. Suppose that E is not l.c. at p. Then there exists a closed neighbourhood F of p such that p is not an interior point of its component in F. Thus there exists a sequence of points $p_1, p_2, \dots$ which converge to p and belong to different components $Q_1, Q_2, \dots$ of F. Since $p \in \mathrm{Int}(F)$, then according to Theorem 6 and (iv) there exists n such that $Q_n \subset \mathrm{Int}(F)$, i.e. that $Q_n \cap \mathrm{Fr}(F) = 0$. By (ii), Q_n is the quasi-component of p_n in F. Therefore, F is not connected between p_n and any point of $\mathrm{Fr}(F)$ and hence between p_n and $\mathrm{Fr}(F)$ (by Theorem 8). In other words, F contains a closed set H such that $p_n \in H$, $H \cap \mathrm{Fr}(F) = 0$ and that H is open in F and hence in $\mathrm{Int}(F)$ (since $H \subset \mathrm{Int}(F)$). But then H is closed and open, and the space is not connected.

THEOREM 10. *If every connected subset of a continuum $\mathscr{X}$ is a semi-continuum, then every subset E connected between two points a and b contains an arc ab.*

Proof. According to Theorem 7 of § 49, VI, $\mathscr{X}$ is h.l.c. Since E is connected between two points, it contains a connected set (by Theorem 9, (i)), and therefore a semi-continuum S (by the hypothesis) containing a and b. Thus S contains an arc ab, because every semi-continuum in $\mathscr{X}$ is arcwise connected.

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(1) R. L. Wilder, Proc. Nat. Acad. Sc. 15 (1929), p. 616.

§ 51. Theory of curves. The order of a space at a point

I. Definitions and examples[1]. If $\mathfrak{n}$ is a cardinal number $\leqslant \mathfrak{c}$ or the ordinal number ω, the space $\mathscr{X}$ is said to be of *order* $\leqslant \mathfrak{n}$ *at the point* p,

$$\mathrm{ord}_p \mathscr{X} \leqslant \mathfrak{n},$$

if for every $\varepsilon > 0$ there exists an open set G such that

$$p \in G, \quad \delta(G) < \varepsilon \quad \text{and} \quad \overline{\overline{\mathrm{Fr}(G)}} \leqslant \mathfrak{n}\ ({}^2). \tag{1}$$

Define

$$\mathscr{X}^{[\mathfrak{n}]} = \underset{p}{E}\,(\mathrm{ord}_p \mathscr{X} \leqslant \mathfrak{n}).$$

The identity $\mathrm{ord}_p \mathscr{X} = \mathfrak{n}$ means that $\mathrm{ord}_p \mathscr{X} \leqslant \mathfrak{n}$ and that condition $\mathrm{ord}_p \mathscr{X} \leqslant \mathfrak{m}$ does not hold for any $\mathfrak{m} < \mathfrak{n}$.

Condition $\mathrm{ord}\, \mathscr{X} \leqslant \mathfrak{n}$ means that $\mathrm{ord}_p \mathscr{X} \leqslant \mathfrak{n}$ for each point p.

$\mathscr{X}$ is said to be of *order* $\leqslant \mathfrak{n}$ between two sets A and B,

$$\mathrm{ord}_{A,B} \mathscr{X} \leqslant \mathfrak{n},$$

if there exists an open set G such that

$$A \subset G, \quad \bar{G} \cap B = 0 \quad \text{and} \quad \overline{\overline{\mathrm{Fr}(G)}} \leqslant \mathfrak{n}; \tag{2}$$

in other words, if there exists a closed set F of power $\leqslant \mathfrak{n}$ which separates $\mathscr{X}$ between A and B.

(No meaning can be attributed to the symbol $\mathrm{ord}_{A,B} \mathscr{X}$ if the sets A and B are not separated; then it holds actually $\mathrm{ord}_{A,B} \mathscr{X} > \mathfrak{n}$ for any $\mathfrak{n}$ if one agrees that $>$ means the negation of $\leqslant$.)

The points of order $\leqslant \aleph_0$ are said to be *rational*; the points of order $\leqslant \omega$ are said to be *regular* (these are the points for which the set $\mathrm{Fr}(G)$ is finite). The points of order 1 are called *end points*. It is obvious that the points p of order 0 are identical with those at which $\dim_p \mathscr{X} = 0$.

---

(1) Compare K. Menger, *Kurventheorie*, Teubner, Berlin–Leipzig 1932, where many references will be found. The fundamental definitions and theorems of the theory of curves are contained in the papers of K. Menger, Monatsh. Math.-Phys. 36 (1929), Math. Ann. 95 (1926), *Grundzüge einer Theorie der Kurven*, and of P. Urysohn, *Sur la ramification des lignes cantoriennes*, C. R. Paris 175 (1922), p. 483 and *Mémoire sur la multiplicités cantoriennes* II, Verh. Akad. Amsterdam 13 (1927).

(2) $\overline{\overline{X}}$ denotes the power of the set X. The inequality $\overline{\overline{X}} \leqslant \omega$ means that the set X is finite.

A space which consists exclusively of regular (respectively rational) points, i.e. such that

$$\mathscr{X} = \mathscr{X}^{[\omega]} \quad (\text{respectively } \mathscr{X} = \mathscr{X}^{[\aleph_0]}),$$

is said to be *regular in the sense of the theory of order* (respectively *rational*).

Every 1-dimensional continuum is said to be a *curve*. Thus a rational continuum is a curve.

EXAMPLES. 1. The points 0 and 1 are the end points of the interval $\mathscr{I}$. Clearly

$$\mathscr{S} = \mathscr{S}^{[2]}, \qquad \mathscr{S}^{[1]} = 0,$$

where $\mathscr{S}$ is the circle $x^2+y^2 = 1$.

2. Let A_n be the segment defined in polar coordinates by the conditions

$$a = \pi/n, \qquad 0 \leqslant \varrho \leqslant 1/n$$

and let $\mathscr{X} = A_1 \cup A_2 \cup \ldots$ and $p = (0, 0)$; hence

$$\operatorname{ord}_p \mathscr{X} = \omega.$$

3. In Example (iv) of § 49, I, for every point p of the segment 01 on the y axis we have

$$\operatorname{ord}_p \mathscr{X} = \aleph_0.$$

4. If $\mathscr{X}$ is the continuum of § 46, II, Remark, we have

$$\operatorname{ord}_p \mathscr{X} = \mathfrak{c},$$

for any point p.

The same holds if $\mathscr{X}$ is an *indecomposable continuum*.

5. *The universal Sierpiński curve is a plane nowhere dense and locally connected continuum, and consists exclusively of points of order* $\mathfrak{c}$ (1).

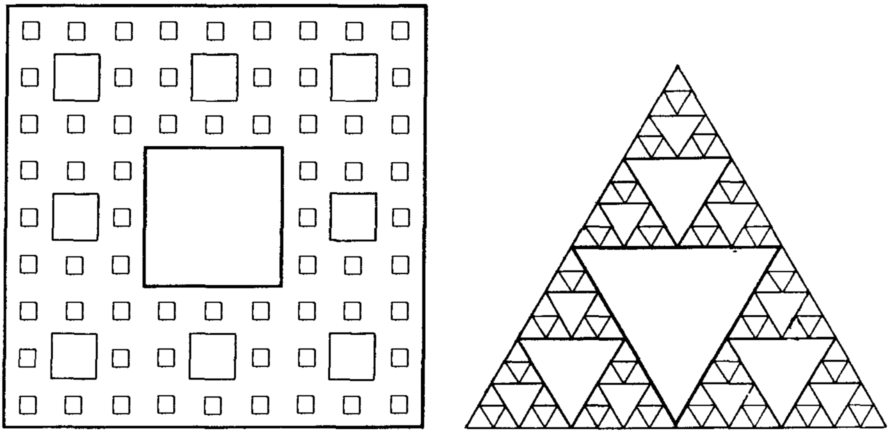
This curve is defined as follows. Let us partition the square $\mathscr{I}^2$ into nine congruent squares and delete the interior of the central square. In a similar way let us partition each of the remaining 8 squares and take out of them the central squares. Let us continue

(1) See C. R. Paris 162 (1916), p. 629.

in this manner step by step. The points which have not been deleted constitute the required curve.

6. The *triangular Sierpiński curve* is defined in the following way([1]).

Let T be an equilateral triangle. Let us partition it into 4 congruent triangles. Let T_0, T_1, T_2 be those which have a vertex with T in common. In a similar way, let us partition each of the



Figs. 8–9

triangles T_0, T_1, T_2 into 4 congruent triangles and let $T_{00}, T_{01}, T_{02}, T_{10}, \ldots, T_{22}$ be those which have a vertex in common with T_0 or with T_1 or with T_2. Let us continue in this manner step by step. Define

$$B_{a_1 a_2 \ldots a_k} = \mathrm{Fr}(T_{a_1 a_2 \ldots a_k}) \quad \text{and} \quad \mathscr{X} = \overline{\bigcup B_{a_1 a_2 \ldots a_k}},$$

where the subscripts a_i take the values 0, 1 and 2, and $k = 0, 1, \ldots$.

The continuum $\mathscr{X}$ has 3 points of order 2 (the vertices of the triangle T), countably many points of order 4 (the vertices of other triangles), and all the other points are of order 3.

In order to construct a continuum which consists exclusively of points of order 3 and 4, it is sufficient to add to $\mathscr{X}$ a topologically equivalent continuum which has in common with $\mathscr{X}$ only the vertices of T.

([1]) Prace Mat.-Fiz. 27 (1915) and C. R. Paris 160 (1915), p. 302.

7. *There exist connected and totally imperfect regular spaces*(1).

Proof. Let us decompose the plane into two totally imperfect and disjoint sets A and B. Let $\mathcal{X}$ denote the continuum of Example 6 and let S be the set of vertices of the triangles $T_{a_1\ldots a_k}$; it can be proved that the set $(A \cap \mathcal{X}) \cup S$ has the required properties.

8. *The union of n arcs $pa_1, \ldots, pa_n$ which are pairwise disjoint, except for p, is of order n at the point p.*

Conversely, the following remarkable theorem holds(2).

Let $\mathcal{X}$ be a locally connected continuum. If $\mathrm{ord}_p \mathcal{X} \geqslant n$, *there exist n arcs $pa_1, \ldots, pa_n$, which are pairwise disjoint except for p.*

II. General properties.

THEOREM 1. *The point p belongs to $(E \cup p)^{[\mathfrak{n}]}$ if and only if for every $\varepsilon > 0$ there exists an open set G such that*

$$p \in G, \quad \delta(G) < \varepsilon \quad \textit{and} \quad \overline{\overline{E \cap \mathrm{Fr}(G)}} \leqslant \mathfrak{n}. \tag{1}$$

A necessary and sufficient condition in order that $\mathrm{ord}_{A,B} E \leqslant \mathfrak{n}$, *where $A, B \subset E$, is the existence of an open set G such that*

$$A \subset G, \quad \bar{G} \cap B = 0 \quad \textit{and} \quad \overline{\overline{E \cap \mathrm{Fr}(G)}} \leqslant \mathfrak{n}. \tag{2}$$

Proof. If $p \in (E \cup p)^{[\mathfrak{n}]}$, there exists a set H open in $E \cup p$ and such that

$$p \in H, \quad \delta(H) < \varepsilon \quad \text{and} \quad \overline{\overline{(E \cup p) \cap \bar{H} - H}} \leqslant \mathfrak{n}. \tag{3}$$

Since the sets H and $E - \bar{H}$ are separated, there exists (according to § 14, V, Theorem 1) an open set G satisfying conditions

$$H \subset G, \quad \bar{G} \cap E - \bar{H} = 0 \quad \text{and} \quad \delta(G) < \varepsilon.$$

Therefore

$$E \cap \bar{G} - G \subset E \cap \bar{H} - H, \text{ hence } \overline{\overline{E \cap \mathrm{Fr}(G)}} \leqslant \overline{\overline{E \cap \bar{H} - H}} \leqslant \mathfrak{n}.$$

(1) See the paper of B. Knaster and myself, *A connected and connected im kleinen point set which contains no perfect subset*, Bull. Amer. Math. Soc. 33 (1927), p. 106.

(2) Theorem of K. Menger ("n-Beinsatz"), Fund. Math. 10 (1927), p. 98. For the proof see K. Menger, *Kurventheorie*, Chapter VI, 1, and G. T. Whyburn, *On n-arc connectedness*, Trans. Amer. Math. Soc. 63 (1948), p. 452. The proof for $n = 2$ will be given in § 52, II, Theorem 15.

Conversely, if the open set G satisfies conditions (1), the set $H = E \cap G \cup p$ is open in $E \cup p$ and satisfies conditions (3), because

$$\begin{aligned}(E \cup p) \cap \bar{H} - H &= (E \cup p) \cap (\overline{E \cap G} \cup p) - (E \cap G \cup p) \\ &= (E \cap \overline{E \cap G} \cup p) - (E \cap G \cup p) \\ &= E \cap \overline{E \cap G} - (E \cap G) - p \subset E \cap \bar{G} - G.\end{aligned}$$

Therefore $p \in (E \cup p)^{[\mathfrak{n}]}$.

Consider the second part of the theorem..

Let $\operatorname{ord}_{A,B} E \leqslant \mathfrak{n}$ and let H be a set open in E such that

$$A \subset H, \quad \bar{H} \cap B = 0 \quad \text{and} \quad \overline{\overline{E \cap \bar{H} - H}} \leqslant \mathfrak{n}. \tag{4}$$

If G is an open set such that

$$H \subset G \quad \text{and} \quad \bar{G} \cap E - \bar{H} = 0,$$

conditions (2) are satisfied.

Conversely, if conditions (2) are satisfied and if $H = E \cap G$, then so are conditions (4), hence $\operatorname{ord}_{A,B} E \leqslant \mathfrak{n}$.

THEOREM 2. *$\mathscr{X}^{[\mathfrak{n}]}$ and, in general, the set*

$$\mathop{E}_{p}[p \in (E \cup p)^{[\mathfrak{n}]}]$$

is a G_δ.

Proof. This set is equal to $G_1 \cap G_2 \cap \ldots$ where G_k consists of the points p for which there exists an open set G satisfying conditions (1) with $\varepsilon = 1/k$.

THEOREM 3. *$E \cap \mathscr{X}^{[\mathfrak{n}]} \subset E^{[\mathfrak{n}]}$, i.e., if $p \in E$, then*

$$\operatorname{ord}_p E \leqslant \operatorname{ord}_p \mathscr{X}.$$

THEOREM 4. *If G is open, then $G \cap \mathscr{X}^{[\mathfrak{n}]} = G \cap G^{[\mathfrak{n}]}$, i.e.*

$$\operatorname{ord}_p G = \operatorname{ord}_p \mathscr{X} \quad \textit{for} \quad p \in G.$$

The proof of Theorems 3 and 4 is straightforward.

THEOREM 5. *If $\mathscr{X} = \mathscr{X}^{[\mathfrak{n}]}$, there exists a base consisting of open sets $R_1, R_2, \ldots$ such that $\overline{\overline{\operatorname{Fr}(R_i)}} \leqslant \mathfrak{n}$.*

Proof. Let $G_k(p)$ be an open set satisfying conditions I (1) for $\varepsilon = 1/k$. By the Lindelöf theorem (§ 5, XI), there exists for every k

a sequence $G_k(p_1), G_k(p_2), \ldots$ such that $\mathscr{X} = G_k(p_1) \cup G_k(p_2) \cup \ldots$. If the double sequence $\{G_k(p_l)\}$, where $k = 1, 2, \ldots$ and $l = 1, 2, \ldots$, is ordered in a simple one, the required base $R_1, R_2, \ldots$ is obtained.

THEOREM 6. *In order that* $\mathrm{ord}_p \mathscr{X} \leqslant \mathfrak{n}$ *it is necessary and sufficient that every closed set* F *such that* $p \in \mathscr{X} - F$ *satisfies the condition* $\mathrm{ord}_{p,F} \mathscr{X} \leqslant \mathfrak{n}$.

Proof. Since this condition is obviously necessary, it remains to show that it is sufficient.

Let $\varepsilon > 0$. Define $F = \underset{x}{E}[|x-p| \geqslant \varepsilon]$. By hypothesis, there exists an open set G such that $p \in G$, $\overline{\overline{\mathrm{Fr}(G)}} \leqslant \mathfrak{n}$ and $\bar{G} \cap F = 0$, which implies $\delta(G) \leqslant \varepsilon$, and hence $\mathrm{ord}_p \mathscr{X} \leqslant \mathfrak{n}$.

III. Order $\aleph_0$ and $\mathfrak{c}$.

THEOREM 1. *Let* $\mathfrak{n} = \aleph_0$ *or* $\mathfrak{n} = \mathfrak{c}$. *If* $\mathscr{X}$ *is a compact space of order* $\mathfrak{n}$ *between two closed sets* A *and* B, *then* $\mathscr{X}$ *is of order* $\mathfrak{n}$ *between two points* $a \in A$ *and* $b \in B$.

In general, *if a subset* E *of a compact space is of order* $\mathfrak{n}$ *between* A *and* B (*where* $A \cup B \subset E$), *there exists a pair of points* a *and* b *such that*

$$a \in \bar{A}, \quad b \in \bar{B} \quad \text{and} \quad \mathrm{ord}_{a,b}(E \cup a \cup b) \geqslant \mathfrak{n}. \tag{1}$$

Proof. This theorem will be proved similarly as Theorem 1 of § 47, II[1].

Let $\boldsymbol{G}$ be the family of all open sets G such that $\overline{\overline{E \cap \mathrm{Fr}(G)}} < \mathfrak{n}$. Suppose that for all points $a \in \bar{A}$ and $b \in \bar{B}$ condition (1) is false, i.e. (compare Theorem 1 of Section II) that there is an open set G such that

$$a \in G, \quad b \notin \bar{G} \quad \text{and} \quad \overline{\overline{(E \cup a \cup b) \cap \mathrm{Fr}(G)}} < \mathfrak{n}, \quad \text{thus} \quad G \in \boldsymbol{G}.$$

Therefore, according to the Lemma of § 37, II, there exists a set

$$H = (G_1^1 \cap \ldots \cap G_{l_1}^1) \cup \ldots \cup (G_1^k \cap \ldots \cap G_{l_k}^k) \quad \text{where} \quad G_j^i \in \boldsymbol{G}$$

---

[1] In what concerns the analogies between the theory of dimension and that of order, it should be mentioned that many theorems of both theories can be derived from the theory of dimensionalizing families (compare § 27, VII).

such that $\bar{A} \subset H$ and $\bar{B} \cap \bar{H} = 0$. Since (compare § 6, II, (8))

$$\mathrm{Fr}(H) \subset \bigcup_{i,j} \mathrm{Fr}(G_j^i), \qquad \text{which yields} \qquad \overline{\overline{E \cap \mathrm{Fr}(H)}} < \mathfrak{n},$$

it follows that $\mathrm{ord}_{A,B} E < \mathfrak{n}$.

THEOREM 2. *Let* $\mathfrak{n} = \aleph_0$ *or* $\mathfrak{n} = \mathfrak{c}$. *If* $\mathcal{X}$ *is compact and* $\mathrm{ord}_p \mathcal{X} = \mathfrak{n}$, *there exists a point* q *such that* $\mathrm{ord}_{p,q} \mathcal{X} = \mathfrak{n}$.

In general, *if* E *is a subset of a compact space* $\mathcal{X}$, *then for every point* p *of order* $\mathfrak{n}$ *in* E *there is a point* q *such that* $\mathrm{ord}_{p,q}(E \cup q) = \mathfrak{n}$.

Proof. On one hand, the inequality $\mathrm{ord}_{p,q}(E \cup q) \leqslant \mathrm{ord}_p E$ holds for every $q \neq p$, and on the other hand, if in Theorem 1 the set B is replaced by a closed set E such that $p \in E - B$ and that $\mathrm{ord}_p E \leqslant \mathrm{ord}_{p,B} E$ (compare II, Theorem 6), it follows that there exists a point $q \in \bar{B}$ such that

$$\mathrm{ord}_{p,B} E \leqslant \mathrm{ord}_{p,q}(E \cup q), \qquad \text{which yields} \qquad \mathrm{ord}_{p,q}(E \cup q) = \mathfrak{n}.$$

THEOREM 3. *If* $\mathcal{X}$ *is compact and* $\mathrm{ord}_{p,x} \mathcal{X} \leqslant \aleph_0$ *for each* $x \neq p$, *then* $\mathrm{ord}_p \mathcal{X} \leqslant \aleph_0$.

Proof. If $\mathrm{ord}_p \mathcal{X} > \aleph_0$, then according to Theorem 6 of Section II, there exists a closed set F such that $p \in \mathcal{X} - F$ and $\mathrm{ord}_{p,F} \mathcal{X} = \mathfrak{c}$. But then there exists (by Theorem 1) a point $x \in F$ such that $\mathrm{ord}_{p,x} \mathcal{X} = \mathfrak{c}$.

THEOREM 4. *Let* $\mathfrak{n} = \aleph_0$ *or* $\mathfrak{n} = \mathfrak{c}$. *If* $\mathcal{X}$ *is a compact space, the set*

$$P = p \cup \underset{x}{E}(\mathrm{ord}_{p,x} \mathcal{X} \geqslant \mathfrak{n}) \text{ [1]}$$

is a continuum.

Proof. P is closed, because $\mathcal{X} - P$ is the union of open sets A such that $p \in \mathcal{X} - \bar{A}$ and $\overline{\overline{\mathrm{Fr}(A)}} < \mathfrak{n}$. In order to prove that P is connected, let G be an open set such that $p \in G$ and $P \cap \mathrm{Fr}(G) = 0$. We have to show that $P \subset G$.

By Theorem 1 the identity $P \cap \mathrm{Fr}(G) = 0$ implies that $\mathrm{ord}_{p,\mathrm{Fr}(G)} \mathcal{X} < \mathfrak{n}$. Hence there exists an open set H such that

$$p \in H, \qquad \bar{H} \cap \mathrm{Fr}(G) = 0 \qquad \text{and} \qquad \overline{\overline{\mathrm{Fr}(H)}} < \mathfrak{n}.$$

---

[1] Clearly, *the quasi-component of* p (§ 46, V) *coincides with* $\underset{x}{E}(\mathrm{ord}_{p,x} \mathcal{X} \geqslant 1)$.

It follows that

$$p \in H \cap G \quad \text{and} \quad \mathrm{Fr}(H \cap G) \subset \mathrm{Fr}(H),$$

because

$$\mathrm{Fr}(H \cap G) = \overline{H \cap G} - (H \cap G) \subset (\bar{H} \cap \bar{G} - H) \cup (\bar{H} \cap \bar{G} - G) \subset \bar{H} - H$$

(since $\bar{H} \cap \bar{G} - G = 0$). Therefore $\overline{\overline{\mathrm{Fr}(H \cap G)}} < \mathfrak{n}$. Since $p \in H \cap G$, it follows (by definition of P) that

$$P \subset \overline{H \cap G} \quad \text{and hence} \quad P \subset \bar{H} \cap \bar{G} = \bar{H} \cap G \subset G.$$

The following theorem is an easy consequence of Theorems 2 and 4.

THEOREM 5. *In every compact space the set of irregular points (i.e. the set $\mathcal{X} - \mathcal{X}^{[\omega]}$) as well as the set of irrational points (i.e. the set $\mathcal{X} - \mathcal{X}^{[\aleph_0]}$) are the unions of continua (which contain more than one point)*(1).

THEOREM 6. *If $\mathcal{X}$ is a compact space, the set of points of order $\mathfrak{c}$*(2) *is irregular at each of its points, i.e.*

$$(\mathcal{X} - \mathcal{X}^{[\aleph_0]})^{[\omega]} = 0 \,(^3). \tag{2}$$

Proof. Suppose that $p \in (\mathcal{X} - \mathcal{X}^{[\aleph_0]})^{[\omega]}$.

Let $\varepsilon > 0$. There exists an open set G such that

$$p \in G, \quad \delta(G) < \varepsilon \quad \text{and} \quad \overline{\overline{\mathrm{Fr}(G) - \mathcal{X}^{[\aleph_0]}}} < \infty. \tag{3}$$

On the other hand, there exists a family $\boldsymbol{G}$ of open sets such that for each $x \in \mathrm{Fr}(G) \cap \mathcal{X}^{[\aleph_0]}$ and each positive integer k there is a set $H \in \boldsymbol{G}$ satisfying conditions

$$x \in H, \quad \delta(H) < \varepsilon/k \quad \text{and} \quad \overline{\overline{\mathrm{Fr}(H)}} \leqslant \aleph_0. \tag{4}$$

Since the set $\mathrm{Fr}(G) - \mathcal{X}^{[\aleph_0]}$ is finite (by (3)), the set $\mathrm{Fr}(G) \cap \mathcal{X}^{[\aleph_0]}$ is an F_σ, and Corollary 5 of § 41, II can be applied. It follows that

(1) See K. Menger, Math. Ann. 95 (1926), p. 287, p. Urysohn, Verh. Akad. Amsterdam 13 (1927), p. 19, and W. Hurewicz, Math. Ann. 96 (1927), p. 759.

(2) This set is called *ordinal kernel* in analogy with dimensional kernel (considered in § 27, V).

(3) See K. Menger, *loc. cit.* p. 289 and P. Urysohn, *loc. cit.* p. 21. The proof of Theorem 6 given here is quite analogous to that of the corresponding theorem on the dimensional kernel (§ 45, V).

there exists a sequence of sets $H_1, H_2, \ldots$ belonging to $\boldsymbol{G}$ and satisfying conditions

$$\mathrm{Fr}(G) \cap \mathscr{X}^{[\aleph_0]} \subset \bigcup_m H_m, \tag{5}$$

$$\overline{\bigcup_m H_m} \subset \bigcup_m \bar{H}_m \cup \mathrm{Fr}(G). \tag{6}$$

Let

$$Q = G \cup \bigcup_m H_m. \tag{7}$$

It follows by (3) and (4) that

$$p \in Q \quad \text{and} \quad \delta(Q) \leqslant 3\varepsilon. \tag{8}$$

Moreover, by (7) and (6) we have

$$\begin{aligned}\mathrm{Fr}(Q) = \bar{Q}-Q &= (\bar{G}-Q) \cup \Big(\overline{\bigcup_m H_m}-Q\Big)\\ &\subset \Big(\bar{G}-G-\bigcup_m H_m\Big) \cup \Big\{\Big[\bigcup_m \bar{H}_m \cup \mathrm{Fr}(G)\Big]-G-\bigcup_m H_m\Big\}\\ &\subset \Big[\mathrm{Fr}(G)-\bigcup_m H_m\Big] \cup \bigcup_m \mathrm{Fr}(H_m)\end{aligned}$$

since

$$\bigcup_m \bar{H}_m - \bigcup_m H_m \subset \bigcup_m (\bar{H}_m - H_m) = \bigcup_m \mathrm{Fr}(H_m).$$

It follows by (4) that

$$\overline{\overline{\mathrm{Fr}(Q)}} \leqslant \aleph_0, \tag{9}$$

because, according to (5) and (3),

$$\overline{\overline{\mathrm{Fr}(G)-\bigcup_m H_m}} \leqslant \overline{\overline{\mathrm{Fr}(G)-\mathscr{X}^{[\aleph_0]}}} < \infty.$$

(8) and (9) imply that $p \in \mathscr{X}^{[\aleph_0]}$ contradicting the hypothesis.

Remark. The term *irregular* cannot be replaced by *irrational*. Actually there exists a compact space $\mathscr{X}$ whose set of points of order $\mathfrak{c}$ is rational (i.e. it is at no point of order $\mathfrak{c}$) and non-empty.

Such is the following example(1).

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(1) See E. Otto, *Über Punkte der Ordnung* $\mathfrak{c}$, Monatsh. Math.-Phys. 40 (1933), p. 88. Compare also my paper *Une application des images de fonctions à la construction de certains ensembles singuliers*, Mathematica 6 (1932), p. 123.

The first example with this property has been found by S. Mazurkiewicz. See *Sur les points d'ordre* $\mathfrak{c}$ *dans les continus*, Fund. Math. 15 (1930), p. 222. For a partial solution, see the paper under the same title of Mazurkiewicz and myself, Fund. Math. 11 (1928).

For $x \in \mathscr{C}$, let

$$x = 2/3^{n_1} + 2/3^{n_2} + \ldots, \quad \text{where} \quad n_1 < n_2 < \ldots < n_k < \ldots,$$

and

$$f(x) = (-1)^{n_1}/(3/2) + (-1)^{n_2}/(3/2)^2 + \ldots + (-1)^{n_k}/(3/2)^k + \ldots,$$
$$f(0) = 0.$$

The space $\mathscr{X}$ consists of segments with end-points (x, i_x) and (x, s_x), where i_x and s_x denote limit inferior and limit superior of the function f at x and where $x \in \mathscr{C}$.

IV. Regular spaces, rational spaces.

THEOREM 1. *Every connected regular space is locally connected.*

More precisely, *if $\mathscr{X}$ is connected and* $\mathrm{ord}_p \mathscr{X} \leqslant \omega$, *then $\mathscr{X}$ is locally connected at p.*

Proof. Let $\varepsilon > 0$ and let G be an open set such that

$$p \in G, \quad \delta(G) < \varepsilon \quad \text{and} \quad \mathrm{Fr}(G) = (q_1, \ldots, q_n). \tag{1}$$

The set $\mathscr{X} - \mathrm{Fr}(G)$ is the union of two separated sets G and $\mathscr{X} - \overline{G}$ and the set $\mathrm{Fr}(G)$ is the union of n connected sets (by (1)). By Theorem 7 of § 46, II, it follows that the set $\overline{G} = G \cup \mathrm{Fr}(G)$ is the union of (at most) n connected separated sets

$$\overline{G} = S_1 \cup \ldots \cup S_k, \quad p \in S_1 \quad \text{and} \quad k \leqslant n. \tag{2}$$

Since the set S_1 is separated from the sets $S_2, S_3, \ldots, S_k$, so it is separated from their union $S_2 \cup \ldots \cup S_k$, and hence S_1 is a neighbourhood of p relative to $\overline{G}$, and therefore relative to $\mathscr{X}$ (since $p \in G$).

Finally, $\delta(S_1) < \varepsilon$, according to (1) and (2). Therefore $\mathscr{X}$ is locally connected at the point p.

THEOREM 2. *Every regular continuum is hereditarily locally connected.*

More precisely, *every connected subset of a regular space is locally connected.*

Proof. Because every subset of a regular space is regular.

Remarks. (i) The converse is not true. *There exist hereditarily locally connected spaces which are not regular.*

Such is the (plane) continuum[1] consisting of the segment $0 \leqslant x \leqslant 1, y = 0$, of the half-circles

$$\left(x - \frac{2k-1}{2^n}\right)^2 + y^2 = \frac{1}{4^n}, \quad y \geqslant 0,$$

where $n = 1, 2, \ldots$ and $k = 1, 2, 3, \ldots, 2^{n-1}$, and of the half-circles

$$\left(x - \frac{2k-1}{2 \cdot 3^n}\right)^2 + y^2 = \frac{1}{4 \cdot 9^n}, \quad y \leqslant 0,$$

where $n = 0, 1, \ldots$ and $k = 1, 2, \ldots, 3^n$.

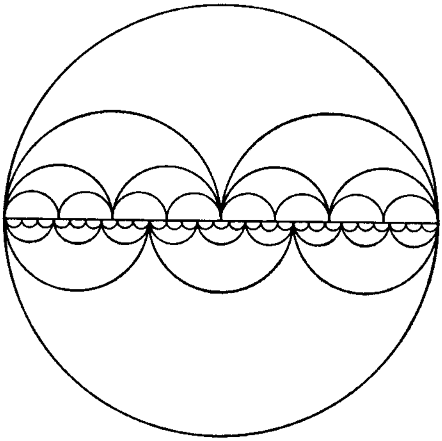

Fig. 10

(ii) However, the following theorem holds: *every hereditarily locally connected space is rational*[2].

Finally, *there exists a rational continuum which is not locally connected*. See Example I, (iii) of § 49.

THEOREM 3. *Every continuum which contains no nowhere dense subcontinua* (*except single points*) *is regular.*

Proof. Let $\mathscr{X}$ be an irregular continuum. Then according to Theorem 5 of Section III, there exists a continuum

$$K \subset \mathscr{X} - \mathscr{X}^{[\omega]} \tag{3}$$

(1) This example is due to B. Knaster; see K. Menger, *Kurventheorie*, p. 258. See also H. M. Gehman, Ann. of Math. 27 (1926), p. 43.

(2) For a proof see K. Menger, *Kurventheorie*, p. 251, or G. T. Whyburn, *Analytic Topology*, p. 94.

containing more than one point. Suppose that $\mathscr{X}$ does not contain nowhere dense continua (containing more than one point). Then $\mathscr{X}$ is hereditarily locally connected (by § 50, IV, Theorem 1). Therefore K is locally connected, and (compare § 50, II, Theorem 1) contains an arc A. This arc is not nowhere dense in $\mathscr{X}$, so let p be an interior point of A. Clearly

$$\operatorname{ord}_p \mathscr{X} = \operatorname{ord}_p A \leqslant 2,$$

contrary to (3).

THEOREM 4. *A space is rational if and only if it is the union of two sets, one of which is at most 0-dimensional and the other is countable* (1).

Proof. Let $\mathscr{X}$ be rational, and (following Theorem 5 of Section II) let $R_1, R_2, \ldots$ be a base of $\mathscr{X}$ such that $\overline{\overline{\operatorname{Fr}(R_i)}} \leqslant \aleph_0$. Assume

$$D = \bigcup_m \operatorname{Fr}(R_m). \tag{4}$$

It follows that

$$\overline{\overline{D}} \leqslant \aleph_0 \quad \text{and} \quad \dim(\mathscr{X} - D) \leqslant 0, \tag{5}$$

since the sets $R_m - D$ are closed-open in $\mathscr{X} - D$ and form a base of $\mathscr{X} - D$.

Conversely, let D be a set satisfying conditions (5). Let $p \in \mathscr{X}$. It follows that

$$\dim_p(\mathscr{X} - D \cup p) = 0, \tag{6}$$

by the inequality (5) and by § 26, III, Corollary 2.

Let $\varepsilon > 0$. According to (6) and § 25, II, Theorem 2_0, there exists an open set G satisfying the following conditions

$$p \in G,\ \ \delta(G) < \varepsilon \text{ and } \operatorname{Fr}(G) \subset D, \text{ thus } \overline{\overline{\operatorname{Fr}(G)}} \leqslant \aleph_0 \tag{7}$$

by condition (5).

The conditions (7) imply that $\operatorname{ord}_p \mathscr{X} \leqslant \aleph_0$.

THEOREM 5. *Every family of disjoint connected subsets (containing more than one point) of a rational space is countable.*

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(1) This theorem is a particular case of Theorem 1 of § 27, VII.

Proof. Suppose that this family is uncountable. Let D be a countable set. Then there exists in this family a connected C disjoint from D (and even uncountably many such sets); i.e. $C \subset \mathcal{X}-D$. Since the set C is connected, $\mathcal{X}-D$ cannot be 0-dimensional, which contradicts Theorem 4.

THEOREM 6. *The union of an infinite series of closed rational sets is a rational set*(1).

Proof. Let $\mathcal{X}$ be the considered union and $A_1, A_2, \dots$ its terms. Set $A_1^* = A_1$ and $A_n^* = A_n-(A_1 \cup \dots \cup A_{n-1})$; it follows by Theorem 4 that

$$A_n^* = B_n \cup D_n,$$

where

$$\dim B_n \leqslant 0, \qquad \overline{\overline{D}}_n \leqslant \aleph_0 \qquad \text{and} \qquad B_n \cap D_n = 0.$$

Therefore

$$\mathcal{X} = \bigcup_n B_n \cup \bigcup_n D_n, \qquad \dim\big(\bigcup_n B_n\big) \leqslant 0 \qquad \text{and} \qquad \overline{\overline{\bigcup_n D_n}} \leqslant \aleph_0, \tag{8}$$

since B_n is an F_σ in $\bigcup_n B_n$ and the union of an infinite series of 0-dimensional F_δ-sets is 0-dimensional (compare § 26, III, Corollary 1).

Conditions (8) and Theorem 4 imply that the space $\mathcal{X}$ is rational.

Remark. The example considered in Remark (i) shows that *the union of two regular continua can be an irregular continuum.* However, the following theorem holds.

THEOREM 7. *Let $\mathcal{X}$ be a compact space. If $\mathcal{X} = A \cup B$, where A and B are closed and regular, then the set $C = \mathcal{X}-\mathcal{X}^{[\omega]}$ is a boundary set both in A and in B.*

Proof. Since $A-B \subset \mathcal{X}^{[\omega]}$ and $B-A \subset \mathcal{X}^{[\omega]}$, so $C \subset A \cap B$.

We must show that

$$C \subset \overline{A-C}. \tag{9}$$

Now

$$\mathcal{X}-\overline{A-C} = [A-\overline{A-C}] \cup [B-\overline{A-C}] \subset A \cap C \cup B \subset B \tag{10}$$

because $C \subset A \cap B$.

(1) Compare § 27, VII, 1.

The set $\mathscr{X}-\overline{A-C}$ is regular as a subset of a regular set B. As an open set, it is a contained in $\mathscr{X}^{[\omega]}$ (compare Theorem 4 of Section II); this yields the inclusion (9).

THEOREM 8. *Suppose that the hypotheses of Theorem 7 are fulfilled. If, moreover, one of the following conditions is satisfied*

(i) $\dim A \cap B = 0$,

(ii) *B does not contain any continuum (with more than one point) which is nowhere dense in B,*

then the space $\mathscr{X}$ is regular.

This is a consequence of Theorems 7 and 5 of Section III.

THEOREM 9. *Let $\mathfrak{m} = \omega$ or $\mathfrak{m} = \aleph_0$. If $\mathscr{X}$ is compact, each of the following conditions is necessary and sufficient in order that $\mathscr{X} = \mathscr{X}^{[\mathfrak{m}]}$, i.e. in order that $\mathscr{X}$ be regular or rational (according to whether $\mathfrak{m}$ is ω or $\aleph_0$).*

(i) $\mathrm{ord}_{x,y}\mathscr{X} \leqslant \mathfrak{m}$ *for any points* $x \neq y$,

(ii) $\mathrm{ord}_{A,B}\mathscr{X} \leqslant \mathfrak{m}$ *for any disjoint closed sets A and B.*

Proof. The conditions are necessary. Because condition (i) is obviously satisfied if $\mathscr{X}$ is regular (rational). Moreover, (i) implies (ii) according to Theorem 1 of Section III.

The conditions are sufficient according to Theorem 6 of Section II.

THEOREM 10 (of decomposition). *Let be given an open cover of a compact regular (or rational) space, $\mathscr{X} = G_1 \cup \ldots \cup G_k$. Then there exists a system of closed sets $F_1, \ldots, F_k$ satisfying conditions*

$$\mathscr{X} = F_1 \cup \ldots \cup F_k, \quad F_i \subset G_i \quad \textit{and} \quad \overline{\overline{F_i \cap F_j}} \leqslant \mathfrak{m} \quad \textit{for} \quad i \neq j, \tag{11}$$

where $\mathfrak{m} = \omega$ (or $\mathfrak{m} = \aleph_0$ respectively).

Proof. Proceed by induction. For $k = 2$ the theorem follows from Theorem 9, (ii). Because, setting

$$\mathscr{X} = G_1 \cup G_2, \quad A = \mathscr{X}-G_1 \quad \text{and} \quad B = \mathscr{X}-G_2,$$

we have $A \cap B = 0$, which yields $\mathrm{ord}_{A,B}\mathscr{X} \leqslant \mathfrak{m}$. Consequently, there exists an open set G satisfying conditions

$$A \subset G, \quad \bar{G} \cap B = 0 \quad \text{and} \quad \overline{\overline{\mathrm{Fr}(G)}} \leqslant \mathfrak{m}.$$

Therefore, the sets $F_1 = \mathscr{X} - G$ and $F_2 = \bar{G}$ satisfy conditions (11).

Now assume that the theorem holds for $k-1$. As we have just proved there exist two closed sets H and F_k such that

$$\mathscr{X} = H \cup F_k, \quad H \subset G_1 \cup \ldots \cup G_{k-1}, \quad F_k \subset G_k \quad \text{and} \quad \overline{\overline{H \cap F_k}} \leqslant \mathfrak{m}. \tag{12}$$

Since the set H is compact and regular (respectively rational), the identity $H = (H \cap G_1) \cup \ldots \cup (H \cap G_{k-1})$ implies by hypothesis the existence of a system of closed sets $F_1, \ldots, F_{k-1}$ satisfying conditions

$$H = F_1 \cup \ldots \cup F_{k-1}, \quad F_i \subset H \cap G_i \quad \text{and} \quad \overline{\overline{F_i \cap F_j}} \leqslant \mathfrak{m} \quad \text{for} \quad i < j < k. \tag{13}$$

Conditions (11) follow easily from (12) and (13).

Theorem 10 implies the following

THEOREM 11. *If $\mathscr{X}$ is a compact regular (or rational) space, then for every $\varepsilon > 0$ there exists a finite system of closed sets $F_1, \ldots, F_k$ satisfying the conditions*

$$\mathscr{X} = F_1 \cup \ldots \cup F_k, \quad \delta(F_i) < \varepsilon, \quad \overline{\overline{F_i \cap F_j}} \leqslant \mathfrak{m}, \quad F_h \cap F_i \cap F_j = 0 \tag{14}$$

for every system of distinct subscripts ($\mathfrak{m}$ *denotes* ω *or* $\aleph_0$).

Proof. Since $\dim X \leqslant 1$, there exists (compare § 45, IV, Theorem 1) a system of open sets $G_1, \ldots, G_k$ satisfying conditions

$$\mathscr{X} = G_1 \cup \ldots \cup G_k, \quad \delta(G_i) < \varepsilon, \quad G_h \cap G_i \cap G_j = 0.$$

Since $F_1, \ldots, F_k$ is a system of closed sets satisfying conditions (11), then conditions (14) are fulfilled.

Remarks. 1. There exists a regular continuum which is not the union of continua with diameters $< \varepsilon$ *such that every pair has* (*at most*) *one point in common*(¹).

2. *If $\mathscr{X}$ is a regular continuum, the sets $F_1, \ldots, F_k$ of Theorem* 11 *can be assumed to be regular continua.*

(¹) J. H. Roberts, *On a problem of Menger concerning regular curves*, Fund. Math. 14 (1929), p. 327.

Proof. Since the union $F_1 \cup \ldots \cup F_k$ is connected and every intersection $F_i \cap F_j$, where $i \neq j$, has a finite number of components, one easily sees that each set F_i has a finite number of components. Let $C_i^1, \ldots, C_i^{m_i}$ be the components of F_i. Replacing the identity

$$\mathscr{X} = \bigcup_{i=1}^{k} F_i \quad \text{by} \quad \mathscr{X} = \bigcup_{i=1}^{k} \bigcup_{r=1}^{m_i} C_i^r$$

we obtain the required decomposition.

THEOREM 12 (1). *Let $\mathscr{X}$ be a regular (or rational) continuum. There exists a continuous function $f\colon \mathscr{X} \to \mathscr{I}$ such that the set of points y for which $\overline{\overline{f^{-1}(y)}} \leqslant \mathfrak{m}$ (where $\mathfrak{m} = \omega$ or $\aleph_0$) is dense in $\mathscr{I}$.*

Proof. Consider the infinite sequence of closed sets $F_1, F_2, \ldots$ defined inductively in the following way.

F_1 is a closed set such that

$$\mathrm{Int}(F_1) \neq 0, \quad F_1 \neq \mathscr{X} \quad \text{and} \quad \overline{\overline{\mathrm{Fr}(F_1)}} \leqslant \mathfrak{m}. \tag{15}$$

Suppose that the system of sets $F_1, F_2, \ldots, F_{3^k}$ (where $k \geqslant 0$) is strictly monotone and satisfies conditions (15) (where the subscript 1 is replaced by $i \leqslant 3^k$); there exists a system of sets $F_{3^k+1}, \ldots \ldots, F_{3^{k+1}}$ satisfying conditions (15) and such that the system $F_1, \ldots, F_{3^{k+1}}$ is strictly monotone.

In order to show this, observe that, if the three closed sets $A \subset F \subset B$ constitute a strictly monotone system, then there exist (by Theorem 9, (ii)) two sets $^*F$ and F^* such that $A \subset {}^*F \subset F \subset F^* \subset B$ and that these five sets constitute a strictly monotone system. Moreover, the sets $^*F$ and F^* satisfy conditions (15) if F does it. Finally, it can be assumed that condition $\varrho(x, \mathscr{X} - F) > 1/k$ implies that $x \in {}^*F$.

Let $\boldsymbol{F}$ be the family of sets F_n, $n = 1, 2, \ldots$. The order type of the family $\boldsymbol{F}$ (ordered by inclusion) is dense, because the conditions $F_m \subset \mathrm{Int}(F_n)$ and $0 \neq F_m \neq \mathscr{X}$ imply that $F_m \neq F_n$.

(1) Theorem of G. T. Whyburn, *Characterizations of certain curves by continuous functions defined upon them*, Amer. Journ. Math. 55 (1933), p. 131. The case where the set $f^{-1}(y)$ is finite for every y has been studied by E. Čech, Fund. Math. 18 (1932), p. 86, S. Mazurkiewicz, *ibid.* p. 89, B. Aitchison, C. R. Soc. Sc. de Varsovie 27 (1934), p. 3, O. G. Harrold, Jr., *Continua of finite sections*, Duke Math. Journ. 8 (1941), p. 682.

By Theorem 1 of § 24, VII, the elements of F can be represented in the form A_r, where the subscript r runs over the (binary) rational numbers between 0 and 1. By the definition of ${}^*F$ and F^* it follows that

$$A_r = \bigcap_{s>r} A_s \quad \text{and} \quad \mathrm{Int}(A_r) = \bigcup_{q<r} \mathrm{Int}(A_q).$$

It follows by § 24, IX, (14), that if $f\colon \mathcal{X} \to \mathcal{I}$ is the mapping considered in § 24, IX, Theorem 3, then

$$\mathrm{Fr}(A_r) \subset f^{-1}(r) \subset \bigcap_{s>r} A_s - \bigcup_{q<r} \mathrm{Int}(A_q) = A_r - \mathrm{Int}(A_r) = \mathrm{Fr}(A_r),$$

hence $f^{-1}(r) = \mathrm{Fr}(A_r)$ for any r.

THEOREM 13 (1). *If $\mathcal{X}$ is compact and rational, there exists an ordinal number $a < \Omega$ such that for every point $p \in \mathcal{X}$ and every $\varepsilon > 0$ there exists an open set G which satisfies conditions*

$$p \in G, \quad \delta(G) < \varepsilon \quad \textit{and} \quad [\mathrm{Fr}(G)]^{(a)} = 0,$$

where $\mathcal{X}^{(a)}$ is the derived set of $\mathcal{X}$ of order a (compare § 24, IV).

Proof. According to Theorem 5 of Section II, let $R_1, R_2, \ldots$ be the base consisting of open sets R_i such that $\overline{\overline{\mathrm{Fr}(R_i)}} \leqslant \aleph_0$. So there exists (compare § 24, IV) an ordinal number $a_i < \Omega$ such that $[\mathrm{Fr}(R_i)]^{(a_i)} = 0$, and it is sufficient to choose $a \geqslant a_i$ for $i = 1, 2, \ldots$.

Remark. Thus *the compact rational spaces can be set into $\aleph_1$ classes* according to the (least possible)(2) number a which just has been defined. It can be proved(3) that none of these classes is empty (in particular, the class 0 is the class of 0-dimensional spaces and the class 1 is the class of regular spaces). It follows that *among the rational spaces no one has the highest topological rank.*

Let us add here three theorems without proof.

THEOREM 15 (of compactification(4)). *Every regular space is topologically contained in a compact regular space.*

(1) See H. Reschovsky, *Über rationale Kurven,* Fund. Math. 15 (1930), p. 19.

(2) Called "Geschlecht" by K. Menger; see Fund. Math. 10 (1927), p. 111. For a more detailed study of this notion, see H. Reschovsky *ibid.* and K. Menger, *Kurventheorie,* Chapter IX.

(3) K. Menger, *ibid.* p. 294.

(4) See G. Nöbeling, *Über regular-eindimensionale Räume,* Math. Ann. 104 (1931), p. 81, and my paper *Quelques théorèmes sur le plongement topologique des espaces,* Fund. Math. 30 (1937), p. 8.

THEOREM 16 (1). *The set of homeomorphisms* $f\colon \mathscr{X} \to \mathscr{I}^{\aleph_0}$ *such that*

$$\operatorname{ord}_{p,q}\mathscr{X} = \operatorname{ord}_{f(p),f(q)}\overline{f(\mathscr{X})}, \quad \textit{where} \quad \operatorname{ord}_{p,q}\mathscr{X} < \infty,$$

is residual in the space $(\mathscr{I}^{\aleph_0})^{\mathscr{X}}$.

THEOREM 17. *Every regular set (lying in an arbitrary space) is contained in a regular* G_δ*-set.*

Theorem 17 is an easy consequence of Theorem 15 combined with the theorem of Lavrentiev (§ 35, II, Corollary).

Remarks. 1. Theorem 15 does not hold for *rational* spaces. Consider the (plane) set consisting of segments joining the point $(\frac{1}{2}, \frac{1}{2})$ with all rational points of the interval $0 \leqslant x \leqslant 1$. It is not possible to compactify this set unless $(\frac{1}{2}, \frac{1}{2})$ becomes the point of order of continuum.

2. Theorem 15 does not hold for regular spaces of a fixed order n. Actually there exists a space of order 4 (i.e. of order $\leqslant 4$ at each point) which is not topologically contained in any compact space of order 4 (2).

3. Among the regular spaces no one has the highest topological rank (3).

V. Points of finite order. Characterization of arcs and simple closed curves.

THEOREM 1. *If* $\operatorname{ord}_p\mathscr{X} \leqslant 1$, *then* p *is not a separation point of* $\mathscr{X}$. *Therefore, if* $\mathscr{X}$ *is connected, then so is* $\mathscr{X}-p$.

Proof. Suppose that $\mathscr{X}-p$ is not connected between two closed sets A and B between which $\mathscr{X}$ is connected (compare § 46, VII). So there exist two open sets A^* and B^* such that

$$\mathscr{X}-p = A^* \cup B^*, \quad A^* \cap B^* = 0,$$

$$A \subset A^* \quad \text{and} \quad B \subset B^*. \tag{1}$$

Since $\operatorname{ord}_p\mathscr{X} \leqslant 1$, there exists an open set G such that

$$p \in G, \quad \bar{G} \cap (A \cup B) = 0 \quad \text{and} \quad \overline{\overline{\operatorname{Fr}(G)}} \leqslant 1. \tag{2}$$

(1) See my paper *Sur la compactification des espaces à connexité n-dimensionelle*, Fund. Math. 30 (1937), p. 246.

(2) See G. Beer, Monatsh. Math.-Phys. 38.

(3) See G. Nöbeling, *loc. cit.* p. 82.

This last inequality implies that

$$\text{either} \quad \mathrm{Fr}(G) \cap A^* = 0 \quad \text{or} \quad \mathrm{Fr}(G) \cap B^* = 0.$$

Assume that $\mathrm{Fr}(G) \cap B^* = 0$. It follows that

$$B^* - \bar{G} = B^* - [G \cup \mathrm{Fr}(G)] = B^* - G,$$

which shows that the set $B^* - G$ is open.

On the other hand, according to (1) and (2),

$$\mathscr{X} = (A^* \cup G) \cup (B^* - G), \quad (A^* \cup G) \cap (B^* - G) = 0,$$
$$A \subset A^* \cup G, \quad B \subset B^* - G.$$

Hence $\mathscr{X}$ is not connected between A and B contrary to our assumption.

THEOREM 2. $\dim \mathscr{X}^{[1]} \leqslant 0$.

Proof. Let S be the set of separation points of $\mathscr{X}$; then $\mathscr{X}^{[1]} \cap S = 0$ by Theorem 1. So it is sufficient to prove that, for every $p \in \mathscr{X}^{[1]}$ and every $\varepsilon > 0$, there exists an open set G such that

$$p \in G, \quad \delta(G) < \varepsilon \tag{3}$$

and

$$\mathrm{Fr}(G) \subset S. \tag{3'}$$

If $\mathrm{ord}_p \mathscr{X} = 0$, then $\dim_p \mathscr{X} = 0$. Consequently, there exists an open set G satisfying conditions (3) and such that $\mathrm{Fr}(G) = 0$, which implies inclusion (3').

If $\mathrm{ord}_p \mathscr{X} = 1$, it follows that $\dim_p \mathscr{X} \neq 0$. Therefore the space $\mathscr{X}$ is connected (compare § 46, IV, Theorem 2) between the point p and a closed set F such that $p \in \mathscr{X} - F$. On the other hand, the identity $\mathrm{ord}_p \mathscr{X} = 1$ implies that there exist a point q and an open set G satisfying conditions (3) and such that $\bar{G} \cap F = 0$ and $\mathrm{Fr}(G) = q$. Thus the point q separates the space between p and F, which proves that $q \in S$, and condition (3') is fulfilled.

THEOREM 3. *If $\mathscr{X}$ is irreducible between a and b, then $\mathscr{X}^{[1]} \subset (a, b)$.*

In other words, *the identity* $\mathrm{ord}_p \mathscr{X} = 1$ *implies that either* $p = a$ *or* $p = b$.

Proof. Suppose that $\mathrm{ord}_p \mathscr{X} = 1$ and $a \neq p \neq b$. Let G be an open set and q a point such that

$$p \in G, \tag{4}$$

$$a, b \in \mathscr{X} - \bar{G}, \tag{4'}$$

$$\mathrm{Fr}(G) = q. \tag{4''}$$

Since the sets G and $\mathscr{X}-\bar{G}$ are separated, the identity

$$\mathscr{X}-q = \mathscr{X}-(\bar{G}-G) = G \cup (\mathscr{X}-\bar{G})$$

implies that the set $q \cup (\mathscr{X}-\bar{G}) = \mathscr{X}-G$ is connected (compare § 46, II, Theorem 4). Thus $\mathscr{X}-G$ is a closed connected set containing the points a and b (by (4')). Since $\mathscr{X}$ is irreducible between these points, it follows that $\mathscr{X}-G = \mathscr{X}$, contrary to (4).

THEOREM 4. *If $\mathscr{X}$ is a continuum, then $\mathscr{X}-\mathscr{X}^{[1]}$ is a semi-continuum.*

Proof. Let $a, b \in \mathscr{X}-\mathscr{X}^{[1]}$ and let C be an irreducible continuum between a and b (compare § 48, I, Theorem 1). According to Theorem 3 of Section II and Theorem 3, it follows that

$$C \cap \mathscr{X}^{[1]} \subset C^{[1]} \subset (a, b), \quad \text{thus} \quad C \cap \mathscr{X}^{[1]} = 0$$
$$\text{and hence} \quad C \subset \mathscr{X}-\mathscr{X}^{[1]}.$$

THEOREM 5. *Every connected space $\mathscr{X}$ containing two points a and b such that*

$$\mathrm{ord}_a \mathscr{X} = 1 = \mathrm{ord}_b \mathscr{X}, \quad a \neq b, \tag{5}$$

$$\mathrm{ord}_x \mathscr{X} = 2 \quad \textit{for} \quad a \neq x \neq b, \tag{6}$$

is an arc ab(1).

Proof. Let $F = S(a, b) \cup a \cup b$, where $S(a, b)$ is the set of points which separate $\mathscr{X}$ between a and b. Since $\mathscr{X}$ is locally connected (according to Theorem 1 of Section IV), it remains to show that $\mathscr{X} = F$ (compare § 49, IV, Theorem 4).

Suppose that $\mathscr{X}-F \neq 0$. Let R be a component of $\mathscr{X}-F$. Therefore $R \neq \mathscr{X}$ and there exists a point $p \in \mathrm{Fr}(R)$, because $\mathscr{X}$ is connected. Since the space is locally connected, it follows that $\mathrm{Fr}(R) \subset \mathrm{Fr}(F)$ (according to Theorem 3 of § 49, III) and since the set F is closed (according to Theorem 3 of § 49, IV), then $\mathrm{Fr}(F) \subset F$. Therefore $p \in F$, i.e.

$$p \in [S(a, b) \cup a \cup b]. \tag{7}$$

On the other hand, $a \neq p \neq b$. Suppose that $p = a$. So there exists by (5) a point c and an open set G sufficiently small in order that

$$a = p \in G, \quad b \in \mathscr{X}-\bar{G}, \quad R-G \neq 0 \quad \text{and} \quad \mathrm{Fr}(G) = c.$$

(1) Compare F. Frankl, *Über die zusammenhängenden Mengen von höchstens zweiter Ordnung*, Fund. Math. 11 (1928), p. 96.

It follows that $c \in S(a, b)$, which yields $c \in F$ and hence $c \in \mathscr{X} - R$, i.e. $R \cap \mathrm{Fr}(G) = 0$.

But the condition $p \in G \cap \mathrm{Fr}(R)$ implies that $R \cap G \neq 0$ and since $R - G \neq 0$, it follows that $R \cap \mathrm{Fr}(G) \neq 0$ because of the connectedness of R (compare § 46, I, Theorem 1). Thus

$$a \neq p \neq b. \tag{8}$$

By (6), $\mathrm{ord}_p \mathscr{X} = 2$. Therefore there exist an open set G and two points q and r such that

$$p \in G, \quad a, b \in \mathscr{X} - \bar{G}, \quad R - G \neq 0 \quad \text{and} \quad \mathrm{Fr}(G) = (q, r). \tag{9}$$

It follows, as before, that $R \cap \mathrm{Fr}(G) \neq 0$ and hence

$$\text{either} \quad q \in R \quad \text{or} \quad r \in R. \tag{10}$$

Moreover, since $p \in S(a, b)$ (by (7) and (8)), there exist two open sets A and B such that

$$a \in A, \quad b \in B, \quad \mathscr{X} - p = A \cup B \quad \text{and} \quad A \cap B = 0. \tag{11}$$

Since the set $A \cup p$ is connected (compare § 46, II, Theorem 4), the conditions $p \in G \cap (A \cup p)$ and $a \in (A \cup p) - G$ (compare (9) and (11)) imply that $(A \cup p) \cap \mathrm{Fr}(G) \neq 0$ (compare § 46, I, Theorem 1), which yields $A \cap \mathrm{Fr}(G) \neq 0$ (compare (9)). Similarly $B \cap \mathrm{Fr}(G) \neq 0$. So let $q \in A$, $r \in B$ and, according to (10),

$$r \in R, \quad \text{hence} \quad r \in \mathscr{X} - F. \tag{12}$$

Let $H = A \cup G$. According to (11) and (9),

$$\bar{A} = A \cup p \quad \text{and} \quad \bar{G} = G \cup q \cup r.$$

Therefore

$$\mathrm{Fr}(H) = (A \cup p \cup G \cup q \cup r) - (A \cup G) = r,$$
$$a \in H \quad \text{and} \quad b \in \mathscr{X} - \bar{H}.$$

And hence $r \in S(a, b) \subset F$ contradicting (12).

THEOREM 6. *Every continuum consisting exclusively of points of order 2 is a simple closed curve.*

Proof. According to Theorem 2 of § 47, V, it is sufficient to show that the considered continuum $\mathscr{X}$ is separated by every pair of its points.

Suppose that $a \neq b$ and that the set $\mathscr{X}-(a,b)$ is connected. Since $\mathscr{X}$ is locally connected (by Theorem 1 of Section IV), let ab be an arc. It follows that $\mathscr{X} \neq ab$ (because otherwise $\operatorname{ord}_a \mathscr{X} = 1$). Let

$$c \in [ab-(a,b)] \quad \text{and} \quad d \in [\mathscr{X}-ab].$$

Since the set $\mathscr{X}-(a,b)$ is open and connected, there exists (according to Theorem 1 of § 50, II) an arc $cd \subset \mathscr{X}-(a,b)$. Let e be the last point of the arc cd belonging to the arc ab. Thus

$$e \neq d, \quad a \neq e \neq b \quad \text{and} \quad ed \cap ab = (e).$$

Clearly e is a point of order 3 of the triode $ab \cup ed$, and hence

$$\operatorname{ord}_e \mathscr{X} \geqslant 3.$$

Let $\mathscr{X}^{[n,k]}$ be the union of open sets G such that

$$\delta(G) < 1/k \quad \text{and} \quad \overline{\overline{\operatorname{Fr}(G)}} \leqslant n. \tag{13}$$

Lemma 7. *Let A and B be two closed sets such that $\mathscr{X} = A \cup B$ and let p be an isolated point of $A \cap B$ such that $\operatorname{ord}_p \mathscr{X} \leqslant 2n-1$ $(n \geqslant 1)$. Then for every $\varepsilon > 0$ and for every integer $k > 0$ there exists an open set P such that*

$$p \in P, \quad \overline{\overline{\operatorname{Fr}(P)}} < \infty, \quad \delta(P) < \varepsilon, \tag{14}$$

and that

$$\textit{either} \quad A \cap \operatorname{Fr}(P) \subset \mathscr{X}^{[n,k]} \quad \textit{or} \quad B \cap \operatorname{Fr}(P) \subset \mathscr{X}^{[n,k]}. \tag{15}$$

Proof. By hypothesis there exists an open set H such that

$$\bar{H} \cap A \cap B = p \in H, \quad \delta(H) < 1/k \quad \text{and} \quad \overline{\overline{\operatorname{Fr}(H)}} \leqslant 2n-1. \tag{16}$$

Since

$$\operatorname{Fr}(H) = [\operatorname{Fr}(H) \cap A] \cup [\operatorname{Fr}(H) \cap B] \quad \text{and}$$
$$\operatorname{Fr}(H) \cap A \cap B = \operatorname{Fr}(H) \cap p = 0,$$

it follows that

$$\overline{\overline{\operatorname{Fr}(H)}} = \overline{\overline{\operatorname{Fr}(H) \cap A}} + \overline{\overline{\operatorname{Fr}(H) \cap B}},$$

hence by (16)

$$\text{either} \quad \overline{\overline{\operatorname{Fr}(H) \cap A}} < n \quad \text{or} \quad \overline{\overline{\operatorname{Fr}(H) \cap B}} < n.$$

Assume that
$$\overline{\overline{\mathrm{Fr}(H) \cap A}} < n. \tag{17}$$
Since
$$\begin{aligned}\mathrm{Fr}(H-B) &= \overline{H-B}-(H-B)\\ &= \overline{H-B}-H \cup [\overline{H-B} \cap B]\\ &\subset (\bar{H}-H) \cap \overline{\mathscr{X}-B} \cup [\bar{H} \cap \overline{\mathscr{X}-B} \cap B]\\ &\subset \big(\mathrm{Fr}(H) \cap A\big) \cup (\bar{H} \cap A \cap B),\end{aligned}$$
it follows by (16) and (17) that $\overline{\overline{\mathrm{Fr}(H-B)}} \leqslant n$, which implies by (13) that
$$H-B \subset \mathscr{X}^{[n,k]}. \tag{18}$$

Let P be an open set satisfying conditions (14) and the inclusion $\bar{P} \subset H$. Hence
$$(\bar{P}-P) \cap A \cap B = 0, \quad \text{i.e.} \quad \bar{P}-P \subset (\mathscr{X}-A) \cup (\mathscr{X}-B).$$
Consequently
$$A \cap \bar{P}-P \subset H-B,$$
because $\bar{P} \subset H$. Condition (15) follows by virtue of (18).

THEOREM 8 (of W. L. Ayres) (1). $\dim(\mathscr{X}^{[2n-1]}-\mathscr{X}^{[n]}) \leqslant 0$ (*for* $n \geqslant 1$).

Proof. Clearly
$$\mathscr{X}^{[n]} = \bigcap_{k=1}^{\infty} \mathscr{X}^{[n,k]}, \quad \text{hence} \quad \mathscr{X}^{[2n-1]}-\mathscr{X}^{[n]} = \bigcup_{k=1}^{\infty} (\mathscr{X}^{[2n-1]}-\mathscr{X}^{[n,k]}).$$
Since the set $\mathscr{X}^{[n,k]}$ is open, it is sufficient (by Theorem 1 of § 21, III) to show that
$$\dim(\mathscr{X}^{[2n-1]}-\mathscr{X}^{[n,k]}) \leqslant 0$$
for any k; in other words, that for every point $q \in \mathscr{X}^{[2n-1]}$ and for every $\varepsilon > 0$ there exists an open set G satisfying conditions
$$q \in G, \quad \delta(G) < \varepsilon \quad \text{and} \quad Q(G) = 0, \tag{19}$$
where for any X
$$Q(X) = \mathscr{X}^{[2n-1]} \cap \mathrm{Fr}(X)-\mathscr{X}^{[n,k]}.$$

(1) Trans. Amer. Math. Soc. 33 (1931), p. 252.

Since $q \in \mathscr{X}^{[2n-1]}$, there exists an open set H such that

$$q \in H, \quad \delta(H) < \varepsilon \quad \text{and} \quad \overline{\overline{\mathrm{Fr}(H)}} < \infty. \tag{20}$$

Put

$$m = \overline{\overline{Q(H)}}. \tag{21}$$

We are going to define the set G by induction.

If $m = 0$, assume $G = H$. So conditions (19) are fulfilled.

Let $m > 0$. Assume that for every open set H_1 such that $\overline{\overline{Q(H_1)}} \leqslant m-1$ and which satisfies conditions (20) (replacing H by H_1), there exists an open set G which satisfies conditions (19). Thus, it remains to establish the existence of a set H_1 of that kind.

Let $p \in Q(H)$. In Theorem 7 let us set $A = \overline{H}$ and $B = \mathscr{X} - H$. By (20) there exists an open set P such that

$$p \in P, \quad q \in \mathscr{X} - \overline{P}, \quad \delta(P) < \varepsilon - \delta(H), \quad \overline{\overline{\mathrm{Fr}(P)}} < \infty, \tag{22}$$

and that

$$\text{either} \quad \overline{H} \cap \mathrm{Fr}(P) \subset \mathscr{X}^{[n,k]} \tag{23}$$

$$\text{or} \quad \mathrm{Fr}(P) - H \subset \mathscr{X}^{[n,k]}. \tag{24}$$

Put

$$\text{either} \quad H_1 = H - \overline{P} \tag{25}$$

$$\text{or} \quad H_1 = H \cup P, \tag{26}$$

according to whether (23) or (24) holds.

In both cases it follows by (20) and (22) that

$$q \in H_1, \quad \delta(H_1) < \varepsilon \quad \text{and} \quad p \in \mathscr{X} - \mathrm{Fr}(H_1), \tag{27}$$

because, since P is open, it follows that (cf. § 5, III and § 6, II, (8))

$$P \cap \mathrm{Fr}(H - \overline{P}) \subset P \cap \overline{H - \overline{P}} \subset \overline{P \cap H - \overline{P}} = 0 \quad \text{and}$$

$$P \cap \mathrm{Fr}(H \cup P) = 0.$$

In both cases it also follows that

$$\mathrm{Fr}(H_1) \subset \mathrm{Fr}(H) \cup \mathscr{X}^{[n,k]}. \tag{28}$$

Because, on one hand, condition (25) implies that (cf. § 6, II (9) and (5))

$$\mathrm{Fr}(H_1) \subset \mathrm{Fr}(H) \cup \big(\bar{H} \cap \mathrm{Fr}(\bar{P})\big) \subset \mathrm{Fr}(H) \cup \big(\bar{H} \cap \mathrm{Fr}(P)\big),$$

and on the other hand, by (26),

$$\mathrm{Fr}(H_1) = \overline{H \cup P} - H - P \subset \mathrm{Fr}(H) \cup [\mathrm{Fr}(P) - H].$$

The inclusion (28) implies that

$$\mathrm{Fr}(H_1) - \mathscr{X}^{[n,k]} \subset \mathrm{Fr}(H) - \mathscr{X}^{[n,k]}, \quad \text{hence} \quad Q(H_1) \subset Q(H),$$

and thus $\overline{\overline{Q(H_1)}} \leqslant m$, by (21).

Therefore $\overline{\overline{Q(H_1)}} \leqslant m-1$, because $p \in [Q(H) - Q(H_1)]$ according to (27). Finally, by (27), H_1 satisfies conditions (20) (in which H has to be replaced by H_1) because

$$\mathrm{Fr}(H_1) \subset \mathrm{Fr}(H) \cup \mathrm{Fr}(P) \quad \text{and} \quad \overline{\overline{\mathrm{Fr}(H) \cup \mathrm{Fr}(P)}} < \infty$$

according to (20) and (22).

Theorem 8 implies the next two theorems.

THEOREM 8'. $\dim(\mathscr{X}^{[n]} - \mathscr{X}^{[n-1]}) \leqslant 0$ *for every* $n \neq 2$.

Proof. For $n = 1$ this is a consequence of Theorem 2.

If $n \geqslant 3$, i.e. if $n \leqslant 2n-3$, it follows that $\mathscr{X}^{[n]} \subset \mathscr{X}^{[2(n-1)-1]}$ which implies the required conclusion by Theorem 8.

THEOREM 8''. *If all points of $\mathscr{X}$ have the same finite order $n > 0$, then $n = 2$.*

Hence, if $\mathscr{X}$ is a continuum, then it is a simple closed curve.

Proof. By hypothesis, $\mathscr{X} = \mathscr{X}^{[n]} - \mathscr{X}^{[n-1]}$. Therefore condition $n \neq 2$ implies by Theorem 8' that $\dim \mathscr{X} \leqslant 0$; but then $n = 0$.

The second part of the theorem follows by Theorem 6.

THEOREM 9. *If $\mathscr{X}$ is a continuum, all its separation points except $\aleph_0$ are of order* 2.(¹)

Proof. By Theorem 1 of § 46, VII, there exists a sequence of points $a_1, a_2, \ldots$ such that every separation point is a separator between a pair (a_i, a_j) properly chosen. Therefore it is sufficient

(¹) See G. T. Whyburn, Trans. Amer. Math. Soc. 30 (1928), p. 606. For an extension of Theorem 6 on the points of *local separation*, see G. T. Whyburn, Monatsh. Math.-Phys. 36 (1929), p. 309.

to show that given a pair of points (a, b), the identity $\operatorname{ord}_p \mathscr{X} = 2$ holds for each $p \in S(a, b)$ perhaps with countably many exceptions.

Imagine the points $p \in S(a, b)$ being labeled with subscripts according to Theorem 2 of § 46, VIII. According to Theorem 3 of § 46, VIII, to every subscript y, with exception of a countable set, there correspond two sequences of subscripts $\{z_n\}$ and $\{u_n\}$ such that

$$p_y = \bigcap_n A(p_{z_n}) \cap \overline{\mathscr{X} - A(p_{u_n})}.$$

Since the space $\mathscr{X}$ is compact, if follows that (compare § 42, V, Theorem 1)

$$\lim_{n=\infty} \delta\{A(p_{z_n}) \cap \overline{\mathscr{X} - A(p_{u_n})}\} = 0.$$

Since the set $A(p_{z_n}) \cap \overline{\mathscr{X} - A(p_{u_n})}$ is a neighbourhood of the point p_y, we have $\operatorname{ord}_{p_y} \mathscr{X} = 2$ by the following formula

$$\begin{aligned} &\operatorname{Fr}\{A(p_{z_n}) \cap \overline{\mathscr{X} - A(p_{u_n})}\} \\ &\quad = A(p_{z_n}) \cap \overline{\mathscr{X} - A(p_{u_n})} \cap \{\overline{\mathscr{X} - A(p_{z_n})} \cup \overline{\mathscr{X} - \overline{\mathscr{X} - A(p_{u_n})}}\} \\ &\quad \subset \operatorname{Fr}[A(p_{z_n})] \cup \operatorname{Fr}[A(p_{u_n})] \subset (p_{z_n}) \cup (p_{u_n}). \end{aligned}$$

THEOREM 10. *Let $\mathscr{X}$ be a locally connected continuum and p a point which is not a separation point. If $\operatorname{ord}_p \mathscr{X} \geqslant 2$, there exists a point $q \neq p$ such that $\operatorname{ord}_{p,q} \mathscr{X} \geqslant 2$.*

Proof. The condition $\operatorname{ord}_p \mathscr{X} \geqslant 2$ implies (compare Theorem 6 of Section II) the existence of a closed set F such that

$$F \subset \mathscr{X} - p \quad \text{and} \quad \operatorname{ord}_{p,F} \mathscr{X} \geqslant 2. \tag{29}$$

Since the set $\mathscr{X} - p$ is connected, there exists, according to § 49, II, Theorem 15, a continuum C such that

$$F \subset C \subset \mathscr{X} - p. \tag{30}$$

Let pq be an arc such that $pq \cap C = q$.

Suppose that $\operatorname{ord}_{p,q} \mathscr{X} = 1$. Hence there exists a point r which separates $\mathscr{X}$ between p and q, and thus $r \in (pq - p - q)$. Since C is a continuum, the conditions $r \in \mathscr{X} - C$ and $q \in C$ imply that r is a separation point between p and C.

Therefore, $\operatorname{ord}_{p,C} \mathscr{X} = 1$, and according to (30), $\operatorname{ord}_{p,F} \mathscr{X} = 1$, contradicting (29).

VI. Dendrites.

DEFINITION. A locally connected continuum which contains no simple closed curve is called a *dendrite.*

EXAMPLES. (i) Every arc is a dendrite. The union of three arcs ab, ac, ad which have pairwise only the point a in common, is a dendrite (called a *triode*).

(ii) Example 2 of Section I is a dendrite containing a point of order ω.

It is easy to condense this singularity.

(iii) The continuum of § 49, VI is a dendrite containing a nowhere dense subcontinuum, namely the segment 01 of the x axis.

(iv) The union of two dendrites does not need be a dendrite. Moreover, as can be seen in the figure of § 50, IV, the union of two dendrites can contain a continuum which is not locally connected. Compare also Remark 1 of Section VII.

(v) In the plane there exists a dendrite which topologically contains every other dendrite([1]).

If $\mathcal{X}$ is a dendrite, the following theorems hold.

THEOREM 1. *$\mathcal{X}$ is unicoherent.*

More precisely, *if K and L are two subcontinua of $\mathcal{X}$, the set $K \cap L$ is a continuum.*

Proof. Since $\mathcal{X}$ is a locally connected continuum, every subcontinuum of $\mathcal{X}$ is the intersection of an infinite decreasing sequence of locally connected continua (compare § 50, III, Theorem 1). Since the intersection of every decreasing sequence of continua is a continuum (compare § 47, II, Theorem 5), the proof is reduced to the case where K and L are locally connected continua.

Suppose that $K \cap L$ is not connected:

$$K \cap L = P \cup Q, \quad P \cap Q = 0, \quad 0 \neq P = \overline{P}, \quad 0 \neq Q = \overline{Q}.$$

Hence there exist (compare § 50, II, Theorem 1) an arc A and two points p and q such that

$$A \subset K, \quad A \cap P = (p) \quad \text{and} \quad A \cap Q = (q).$$

Let B be an arc $pq \subset L$. It follows that

$$A \cap B \subset A \cap K \cap L = A \cap (P \cup Q) = (p) \cup (q).$$

([1]) For a proof see e.g. K. Menger, *Kurventheorie,* Chapter X, 6. Compare T. Ważewski, Ann. Soc. Polon. Math. 2 (1924), p. 57.

Therefore $A \cup B$ is a simple closed curve.

Remark. Conversely, every 1-dimensional, locally connected and unicoherent continuum is a dendrite, see § 57, III, Theorem 8.

COROLLARY 2. *Every pair of points $a \neq b$ in a dendrite can be joined by one and only one arc with end-points a and b.*

THEOREM 3. *If the subcontinua K and L of $\mathscr{X}$ are disjoint, then $\operatorname{ord}_{K,L}\mathscr{X} = 1$, i.e. there exists a point p which separates K and L. In particular,*

$$\operatorname{ord}_{x,y}\mathscr{X} = 1 \quad \textit{for any } x \neq y. \tag{1}$$

Moreover, if K is a continuum and L is the union of n continua and $K \cap L = 0$, then $\operatorname{ord}_{K,L}\mathscr{X} \leqslant n$.

Proof. First let $n = 1$. Let A be an arc qr such that $A \cap K = q$ and $A \cap L = r$. Every point $p \in A - q - r$ separates K and L; would K and L be contained in the same component of $\mathscr{X} - p$, there would exist (compare § 50, II, Theorem 1) an arc $B \subset \mathscr{X} - p$ with end-points q and r; and hence $A \neq B$ contrary to Theorem 2.

Now, let $L = L_1 \cup \ldots \cup L_n$ be the union of n $(n \geqslant 1)$ continua. As just has been proved, there exist a point p_i and an open set G_i (where $i = 1, \ldots, n$) such that

$$K \subset G_i, \qquad \operatorname{Fr}(G_i) = p_i \qquad \text{and} \qquad L_i \subset \mathscr{X} - \bar{G}_i.$$

Put $G = G_1 \cap \ldots \cap G_n$. It follows that

$$K \subset G, \qquad \operatorname{Fr}(G) \subset \operatorname{Fr}(G_1) \cup \ldots \cup \operatorname{Fr}(G_n) = (p_1, \ldots, p_n)$$

and

$$L \subset (\mathscr{X} - \bar{G}_1) \cup \ldots \cup (\mathscr{X} - \bar{G}_n) = \mathscr{X} - (\bar{G}_1 \cap \ldots \cap \bar{G}_n) \subset \mathscr{X} - \bar{G}.$$

The set $\operatorname{Fr}(G)$ separates therefore $\mathscr{X}$ between K and L and consists of n points at most.

According to Theorem 9 of Section IV, condition (1) implies the following

THEOREM 4. *$\mathscr{X}$ is a regular continuum.*

Hence *every subcontinuum of a dendrite is a dendrite.*

Remark. The same implication shows that condition (1) gives a *characterization* of dendrites in the class of continua. Because, on one hand, every regular continuum is locally connected (by Theorem 1 of Section IV) and on the other hand, no space containing a simple closed curve satisfies condition (1).

THEOREM 5 (of decomposition). *For each* $\varepsilon > 0$, *there exists a finite system of dendrites* $F_1, \ldots, F_k$ *which satisfy conditions*

$$\mathscr{X} = F_1 \cup \ldots \cup F_k, \quad \delta(F_i) < \varepsilon, \quad \overline{\overline{F_i \cap F_j}} \leqslant 1 \quad \textit{and} \quad F_h \cap F_i \cap F_j = 0$$

for every system of distinct subscripts.

Proof. Since $\mathscr{X}$ is a regular continuum (by Theorem 4), consider the decomposition (14) of Section IV, Theorem 1, Remark 2. F_i is a dendrite (according to Theorem 4) as a subcontinuum of $\mathscr{X}$. Since the intersection $F_i \cap F_j$ is finite and connected (by Theorem 1), it contains one point at most.

THEOREM 6. *If the number of components of* $\mathscr{X}-p$ *is finite, it is equal to* $\mathrm{ord}_p \mathscr{X}$.

Proof. Let n be this number. Clearly, $\mathrm{ord}_p \mathscr{X} \geqslant n$. Hence it is sufficient to show that $\mathrm{ord}_p \mathscr{X} \leqslant n$; in other words (compare Theorem 6 of Section II), that for every closed set F such that $p \in \mathscr{X}-F$, there exists a set consisting of n points which separates $\mathscr{X}$ between p and F.

Let $\mathscr{X}-p = R_1 \cup \ldots \cup R_n$ be the union of n regions. The set $F \cap R_i$ is closed, because $\overline{F \cap R_i} \subset F \cap (R_i \cup p) = F \cap R_i$, hence there exists a continuum K_i such that $F \cap R_i \subset K_i$ (compare § 49, II, Theorem 15). So there exists by Theorem 3 a set consisting of n points (at most) which separates $\mathscr{X}$ between $K_1 \cup \ldots \cup K_n$ and p, hence between F and p.

THEOREM 7. $\overline{\overline{\mathscr{X}-\mathscr{X}^{[2]}}} \leqslant \aleph_0$.

In other words, *in a dendrite the set of "branching" points is countable.*

Proof. This follows from Theorem 6 combined with Theorem 1 of § 46, IX.

THEOREM 8. $\mathscr{X} = \overline{\mathscr{X}^{[2]} - \mathscr{X}^{[1]}}$ (*provided that* $\mathscr{X}$ *does not consist of a single point*) (1).

Proof. Let G be an open (non-empty) set. Let A be an arc $ab \subset G$. It is clear that

$$A \cap \mathscr{X}^{[1]} \subset (a, b), \quad \text{therefore} \quad \overline{\overline{A \cap \mathscr{X}^{[1]}}} \leqslant 2,$$

(1) See K. Menger, Math. Ann. 96 (1927), p. 576.

and by Theorem 7

$$\overline{\overline{A - \mathscr{X}^{[2]}}} \leqslant \aleph_0 .$$

Therefore

$$\overline{\overline{A \cap \mathscr{X}^{[2]} - \mathscr{X}^{[1]}}} = \mathfrak{c}, \quad \text{and hence} \quad G \cap \mathscr{X}^{[2]} - \mathscr{X}^{[1]} \neq 0 .$$

THEOREM 9. *If $\mathscr{X}$ is a dendrite, then every continuous mapping $f\colon \mathscr{X} \to \mathscr{X}$ has a fixed point*(1).

VII. Local dendrites.

DEFINITION. A continuum is said to be a *local dendrite* if every of its points has a neighbourhood which is a dendrite.

The following is an immediate consequence of Theorem 4 of Section VI.

THEOREM 1. *Every local dendrite is a regular continuum.*

THEOREM 2. *If $\mathscr{X}$ is a local dendrite, there exists a number $\varepsilon > 0$ such that every subcontinuum C of diameter $< \varepsilon$ is a dendrite.*

Proof. By the compactness of $\mathscr{X}$, there exists a finite system of dendrites $D_1, \ldots, D_n$ such that

$$\mathscr{X} = \operatorname{Int}(D_1) \cup \ldots \cup \operatorname{Int}(D_n) .$$

Thus there exists (compare § 41, VI, Corollary 4d) a number $\varepsilon > 0$ such that every set of diameter $< \varepsilon$ is contained in one of the sets $\operatorname{Int}(D_i)$, therefore in a dendrite D_i. Let C be a continuum of diameter $< \varepsilon$. Since every subcontinuum of a dendrite is a dendrite (by Theorem 4 of Section VI), then C is a dendrite.

LEMMA 3. *Let $\mathscr{X}$ be a locally connected continuum and let ε be the greatest lower bound of the diameters of simple closed curves contained in $\mathscr{X}$. If $\varepsilon > 0$, $\mathscr{X}$ is a local dendrite.*

Proof. If $p \in \mathscr{X}$ and C is a locally connected continuum which is a neighbourhood of p such that $\delta(C) < \varepsilon$ (compare § 50, II, Theorem 3), then C does not contain any simple closed curve, and hence is a dendrite.

THEOREM 4. *Each of the following conditions is necessary and sufficient in order that $\mathscr{X}$ be a local dendrite.*

(1) Compare § 53, III, Theorems 8 and 16. The particular case of Theorem 9, where f is a homeomorphism, has been established by W. Scherrer, Math. Zeitschr. 24 (1925), p. 125. See also W. L. Ayres, *Some generalizations of the Scherrer fixed-point theorem*, Fund. Math. 16 (1930), p. 332.

(i) $\mathscr{X}$ *is a locally connected continuum which contains at most a finite number of simple closed curves.*

(ii) $\mathscr{X}$ *is a continuum and is a finite union of dendrites:*

$$\mathscr{X} = D_1 \cup \ldots \cup D_k \quad \textit{where} \quad \overline{\overline{D_i \cap D_j}} < \infty \quad \textit{for} \quad i \neq j. \tag{1}$$

Proof. 1. Condition (i) implies that $\mathscr{X}$ is a local dendrite. This is a straightforward corollary of Theorem 3.

2. If $\mathscr{X}$ is a local dendrite, (ii) is true. Because $\mathscr{X}$ is a regular continuum (by Theorem 1), and therefore there exist (according to IV, 11, Remark 2) continua $D_1, \ldots, D_k$ which satisfy condition (1) and are of diameters $< \varepsilon$, where ε is a positive number given in advance. Assuming that ε is the number considered in Theorem 2, it follows that the continua $D_1, \ldots, D_k$ are dendrites.

3. Condition (ii) implies (i). To show this, set

$$F = \bigcup_{i \neq j} D_i \cap D_j \tag{2}$$

and let $\boldsymbol{A}$ be the family of all arcs A which satisfy the following two conditions

(I) A is contained in one of the dendrites D_i,

(II) the end-points of A belong to F.

Since the set F is finite, then so is the family $\boldsymbol{A}$ (compare Theorem 2 of Section VI).

Let C be a simple closed curve contained in $\mathscr{X}$. It follows that $C \cap F \neq 0$ since C is not contained in any D_i. If R is a component of $C - F$, then $\bar{R} \in \boldsymbol{A}$. It follows that C is the union of some arcs belonging to the family $\boldsymbol{A}$. Since this family is finite, then so is the family of simple closed curves contained in $\mathscr{X}$.

THEOREM 5. *Every locally connected continuum, which contains infinitely many simple closed curves, contains some with arbitrary small diameters* (1).

Proof. It is clear that $\mathscr{X}$ is a local dendrite if the greatest lower bound of simple closed curves contained in $\mathscr{X}$ is positive. But then $\mathscr{X}$ contains only a finite number of simple closed curves (compare (i)).

THEOREM 6. *Theorems* 7 *and* 8 *of Section* VI *remain valid if* $\mathscr{X}$ *is supposed to be a local dendrite.*

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(1) Theorem of C. Zarankiewicz, Fund. Math. 9 (1927), p. 146.

Proof. Apply condition (ii) of Theorem 4. Let F be defined by the identity (2); it follows that $\mathrm{ord}_p(D_i - F) = \mathrm{ord}_p \mathscr{X}$ for any p (compare Theorem 4 of Section II). Hence

$$D_i^{[2]} - F \subset \mathscr{X}^{[2]},$$

thus

$$\mathscr{X} - \mathscr{X}^{[2]} = \bigcup_{i=1}^{k} (D_i - \mathscr{X}^{[2]}) \subset \bigcup_{i=1}^{k} (D_i - D_i^{[2]}) \cup F,$$

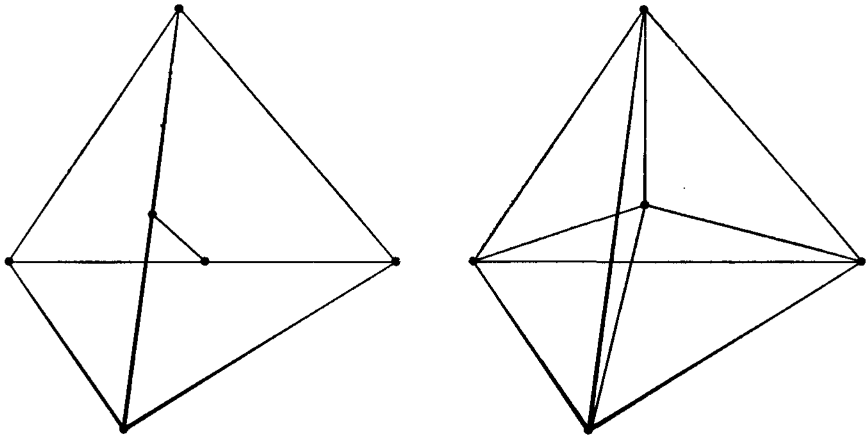

Fig. 11

and therefore, according to Theorem 7 of Section VI,

$$\overline{\overline{\mathscr{X} - \mathscr{X}^{[2]}}} \leqslant \sum_{i=1}^{k} \overline{\overline{(D_i - D_i^{[2]})}} + \overline{\overline{F}} \leqslant \aleph_0.$$

Similarly (compare Theorem 4 of Section II)

$$\mathscr{X}^{[2]} - \mathscr{X}^{[1]} - F = \bigcup_{i=1}^{k} D_i \cap (\mathscr{X}^{[2]} - \mathscr{X}^{[1]}) - F = \bigcup_{i=1}^{k} (D_i^{[2]} - D_i^{[1]}) - F,$$

and since F is finite, it follows according to Theorem 8 of Section VI that

$$\overline{\mathscr{X}^{[2]} - \mathscr{X}^{[1]}} = \bigcup_{i=1}^{k} \overline{D_i^{[2]} - D_i^{[1]}} = \bigcup_{i=1}^{k} D_i = \mathscr{X}.$$

THEOREM 7. *Every local dendrite, skew in the topological sense (that means, which is not topologically contained in the plane), contains topologically one of the two following polygonal curves* (see Fig. 11):

(i) *the union of edges of a tetrahedron and of a segment joining two disjoint edges,*

(ii) *the union of edges of a tetrahedron and of four segments joining its centre with its vertices*(1).

Theorem 7 combined with Theorem 10 of § 50, II implies the following

THEOREM 7'. *Let $\mathcal{X}$ be a 1-dimensional locally connected continuum. If there exists no continuous mapping of $\mathcal{X}$ into the plane, with point inverses of diameters $< \varepsilon$, where ε is a positive number properly chosen, the continuum $\mathcal{X}$ contains topologically one of the two above defined polygonal curves*(2).

THEOREM 8.(3) *A continuum $\mathcal{X}$ is a local dendrite if and only if it can be deformed into a topological polygonal curve by a small continuous displacement; i.e. if for $\varepsilon > 0$ there are a continuous $f\colon \mathcal{X}\times\mathcal{I} \to \mathcal{X}$ and a polygonal curve such that*

$$f(x, 0) = x, \quad |f(x, t)-x| < \varepsilon \quad \text{and} \quad f(\mathcal{X}, 1) \underset{\text{top}}{=} L.$$

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(1) For a proof see my paper, *Sur le problème des courbes gauches en Topologie*, Fund. Math. 15 (1930), p. 271.

If the local dendrite is a polygonal line (a graph), the theorem can be proved in a simpler way. See G. A. Dirac and S. Schuster, *A theorem of Kuratowski*, Indag. Math. 16 (1954), pp. 343–346 and *Corrigendum, ibid.* 23 (1961), p. 360; C. Berge, *Théorie des graphes et ses applications*, Paris (Dunod), 1958, pp. 210–212; R. Halin, *Bemerkungen über ebene Graphen*, Math. Ann. 153 (1964), pp. 38–46; R. G. Busacker and T. L. Saaty, *Finite graphs and networks*, New York (McGraw-Hill) 1965, pp. 70–73; F. Harary and W. T. Tutte, *A dual form of Kuratowski's theorem*, Bull. Amer. Math. Soc. 71 (1965), p. 168.

See also H. Whitney, *Planar graphs*, Fund. Math. 21 (1933), pp. 73–84; S. Mac Lane, *A combinatorial condition for planar graphs*, Fund. Math. 28 (1937), pp. 22–32, and *A structural characterization of planar combinatorial graphs*, Duke Math. Journ. 3 (1937), pp. 460–472; S. Lefschetz, *Planar graphs and related topics*, Proc. Nat. Acad. Sc. 54 (1965), p. 1763; D. W. Hall, *A note on primitive skew-curves*, Bull. Amer. Math. Soc. 49 (1943), pp. 935; R. H. Bing, *Skew sets*, Amer. Journ. Math. 69 (1947), pp. 493–496; K. Wagner, *Über eine Erweiterung eines Satzes von Kuratowski*, Deutsche Math. 2 (1937), pp. 280–285; O. Ore, *Theory of graphs*, Coll. Publ. 1962; F. Harary, *Graph theory and theoretical physics*, Acad. Press 1967.

For other references, see the very extensive bibliography on the Theory of graphs by A. A. Zykov in the Proceedings of the Symposium on that subject held in Smolenice 1963, pp. 171–234.

(2) Theorem of Mazurkiewicz, *Über nichtplättbare Kurven*, Fund. Math. 20 (1933), p. 281.

(3) Theorem of P. Alexandrov, *Über endlich-hoch zusammenhängende stetige Kurven*, Fund. Math. 13 (1929), p. 34.

THEOREM 9. *Every local dendrite is the union of two dendrites*(1).

Remark. 1. The union of two dendrites (even of two arcs) does not need be a local dendrite. Moreover, there exists (in the plane) for each n a regular continuum of order 3 which is the union of n, but of no smaller number of dendrites(2).

Remark 2. In Theorem 7, the assumption of the space being a local dendrite can be replaced by the assumption of being a *locally connected continuum containing no cut point*(3).

In the same paper, S. Claytor considers the general problem of characterizing arbitrary locally connected continua which are topologically contained in the plane.

Remark 3. Every two-dimensional polyhedral skew surface, except the sphere $\mathscr{S}_2$, contains each of the elementary skew graphs, (i) and (ii)(4).

§ 52. Cyclic elements of a locally connected metric continuum(5)

I. Completely arcwise connected sets(6). Let us call so any subset L of a locally connected continuum $\mathscr{X}$ which satisfies the

(1) Theorem of N. E. Steenrod, *Characterization of certain finite curve-sums*, Amer. Journ. Math. 56 (1934), p. 565.

(2) See K. Borsuk, *Sur la décomposition des courbes régulières en dendrites*, Fund. Math. 22 (1934), p. 287.

(3) This statement was proved by S. Claytor. See *Topological immersion of Peanian continua in a spherical surface*, Ann. of Math. 35 (1934), pp. 809–835. For a simple proof, see E. E. Moise, *Remarks on the Claytor imbedding theorem*, Duke Math. Journ. 19 (1952), pp. 199–202.

(4) See my paper cited above, p. 282.

(5) Compare the paper of G. T. Whyburn and myself, *Sur les éléments cycliques et leurs applications*, Fund. Math. 16 (1930), pp. 305–331. For some generalizations of the concept of the cyclic element and for decompositions into finer elements, see G. T. Whyburn, Amer. Journ. Math. 56 (1934), p. 133, D. W. Hall, *On a decomposition of true cyclic elements*, Trans. Amer. Math. Soc. 47 (1940), p. 305, T. Radó and P. Reichelderfer, *Cyclic transitivity*, Duke Math. Journ. 6 (1940), p. 474 and Fund. Math. 34 (1947), p. 14, J. W. T. Youngs, *K-cyclic elements*, Amer. Journ. Math. 62 (1940), p. 449, Albert and J. W. T. Youngs, Trans. Amer. Math. Soc. 51 (1942), p. 637, A. D. Wallace, *Extension sets*, *ibid.* 59 (1946), p. 1. See also B. L. Mc Allister, *Cyclic elements in topology, History*, American Monthly 73 (1966), pp. 337–350, where many references are given.

(6) Compare the concept of an *arc-curve* of W. L. Ayres, Trans. Amer. Math. Soc. 30 (1928), p. 567. This concept is auxiliary for the cyclic element theory which will be studied in Section II.

following condition

$$\textit{if } x, y \in L, \textit{ every arc } xy \textit{ is contained in } L. \tag{1}$$

This definition clearly implies the following two statements.

THEOREM 1. *The intersection of an arbitrary family of completely arcwise connected sets is completely arcwise connected.*

THEOREM 2. *Every completely arcwise connected set is locally and integrally arcwise connected.*

THEOREM 3. *If the union and the intersection of two sets M and N closed in $M \cup N$ are completely arcwise connected, then so are the sets M and N.*

Proof. Let ab be an arc with the end-points $a, b \in M$. We have to show that $ab \subset M$.

By hypothesis

$$ab \subset M \cup N, \quad \text{i.e.} \quad ab = (ab \cap M) \cup (ab \cap N). \tag{2}$$

Let $c \in ab \cap N$. Our task is to prove that $c \in M$.

The sets $ac \cap M$ and $ac \cap N$ are closed in $M \cup N$ and therefore in ac; thus the identity (compare (2)) $ac = (ac \cap M) \cup (ac \cap N)$ implies $ac \cap M \cap N \neq 0$. Let $p \in ac \cap M \cap N$ and $q \in cb \cap M \cap N$. By hypothesis the arc pcq is contained in $M \cap N$. Therefore $c \in M$.

Let L be a *closed, completely arcwise connected set.* The following statements hold.

THEOREM 4. *The boundary of every component R of $\mathcal{X} - L$ consists of a single point (which belongs to L).*

Proof. Otherwise the boundary $\mathrm{Fr}(R)$ would contain two distinct points p and q, which are accessible from R (compare § 50, III, Theorem 6). So there would exist an arc $pq \subset R \cup p \cup q$, hence $pq - L \neq 0$, which contradicts the hypothesis.

THEOREM 5. *Every closed set F which separates L between A and B (where $A \cup B \subset L$) separates the whole space $\mathcal{X}$ between these sets.*

Proof. If F does not separate the space between A and B, there exists an arc ab such that $a \in A$, $b \in B$ and $ab \cap F = 0$. Since $ab \subset L$, the last identity shows that F does not separate the set L between A and B.

THEOREM 5'. *If $A \cup B \subset L \cap Z$ and if Z is connected between A and B, then so is the set $L \cap Z$.*

Proof. Suppose that $L \cap Z$ is not connected between A and B; this means that $L-Z$ separates L between A and B. Hence the set $L-Z$ contains (compare § 46, VII, Theorem 3) a closed set F which separates L between A and B. By Theorem 5, the set F separates the whole space $\mathscr{X}$ between these sets, and hence $\mathscr{X}-F$ is not connected between A and B. The same holds for Z because $Z \subset \mathscr{X}-F$.

Theorem 5 implies immediately the following

THEOREM 6. *If $A \cup B \subset L$, then* $\text{ord}_{A,B} L = \text{ord}_{A,B} \mathscr{X}$.

THEOREM 7. *If the sequence of components $R_1, R_2, \ldots$ of $\mathscr{X}-L$ is infinite, then* $\lim\limits_{n=\infty} \delta(R_n) = 0$.

Proof. Otherwise, a convergent subsequence can be chosen, $\{R_{n_k}\}$, whose limit contains two points $a \neq b$ and, assuming that $Z = \bigcup\limits_k R_{n_k} \cup a \cup b$, we would have $a, b \in L \cap Z$. Thus Z would be connected between a and b (compare § 46, IV, Theorem 9), while $L \cap Z$ is not (since it consists of two points, a and b).

THEOREM 8. *If A and B are two closed, completely arcwise connected sets such that $A \cap B \neq 0$, the set $A \cup B$ is completely arcwise connected.*

Moreover, *the set $A \cap B$ separates the space between $A-B$ and $B-A$.*

Proof. Let ab be an arc whose end-points belong to $A \cup B$. We must show that $ab \subset A \cup B$.

Suppose that this is not true. Let $a_1 b_1$ be an interval of the arc ab, which is contiguous to the set $ab \cap (A \cup B)$. Since both sets A and B are completely arcwise connected, no one of them can contain both points a_1 and b_1. Hence one can assume that

$$a_1 \in A-B \tag{3}$$

and

$$b_1 \in B-A. \tag{4}$$

Consequently

$$a_1 b_1 \cap A = a_1 \tag{5}$$

and

$$a_1 b_1 \cap B = b_1. \tag{6}$$

Let $c \in A \cap B$. Since $b_1, c \in B$, every arc b_1c satisfies condition

$$b_1c \subset B, \quad \text{which yields} \quad a_1b_1 \cap b_1c = b_1 \tag{7}$$

according to (6). Hence $a_1b_1 \cup b_1c$ is an arc whose end-points belong to A. But then $a_1b_1 \cup b_1c \subset A$, and $b_1 \in A$ contrary to (4).

The second part of the theorem is an immediate consequence of Theorem 5, since the set $A \cap B$ is a separator of $A \cup B$ between $A-B$ and $B-A$.

THEOREM 9. *If the space $\mathscr{X}$ is unicoherent, then so is the set L.*

Proof. Let A and B be two continua such that $L = A \cup B$. We have to prove that their intersection $A \cap B$ is a continuum.

Let M be the union of all components R of $\mathscr{X}-L$ such that $\mathrm{Fr}(R) \subset A$. Let S be the union of all other components of $\mathscr{X}-L$. Since (according to Theorem 4) the set $\mathrm{Fr}(R)$ consists of a single point belonging to L, the boundary of components which constitute S is contained in B.

So it follows that

$$\mathscr{X} = (A \cup M) \cup (B \cup S) \tag{8}$$

and

$$(A \cup M) \cap (B \cup S) = A \cap B. \tag{9}$$

Since the sets $A \cup M$ and $B \cup S$ are continua (compare § 49, III, Theorem 1) and the space $\mathscr{X}$ is unicoherent, $A \cap B$ is a continuum.

THEOREM 10. *Let $K, K_1, K_2, \ldots$ be continua such that*

$$\operatorname*{Lim}_{n=\infty} K_n = K \subset L \tag{10}$$

and

$$L \cap K_n \neq 0. \tag{11}$$

Then

$$\operatorname*{Lim}_{n=\infty} L \cap K_n = K. \tag{12}$$

Proof. According to (10) and § 29, IV, 2,

$$\operatorname*{Ls}_{n=\infty} (L \cap K_n) \subset \operatorname*{Ls}_{n=\infty} K_n = K.$$

So we have to show that

$$K \subset \operatorname*{Li}_{n=\infty}(L \cap K_n). \tag{13}$$

Let $p \in K$. We have to define a sequence $p_1, p_2, \ldots$ such that

$$p_n \in L \cap K_n \tag{14}$$

and

$$\lim_{n=\infty} p_n = p. \tag{15}$$

According to (10), $p \in \operatorname*{Li}_{n=\infty} K_n$. Thus there exists a sequence $q_1, q_2, \ldots$ such that

$$q_n \in K_n \tag{16}$$

and

$$\lim_{n=\infty} q_n = p. \tag{17}$$

We define p_n in the following way. If $q_n \in L$, we set $p_n = q_n$; if $q_n \in \mathscr{X} - L$, then according to Theorem 4 we set

$$p_n = \operatorname{Fr}(R_n), \quad \text{where} \quad q_n \in R_n \tag{18}$$

and R_n is a component of $\mathscr{X} - L$.

We claim that condition (14) is satisfied. Since this is an immediate consequence of (16) in case where $q_n \in L$, assume that $q_n \in \mathscr{X} - L$, therefore that $q_n \in K_n - L$ (by (16)). It follows that

$$K_n \cap R_n \neq 0 \neq K_n - R_n \tag{19}$$

since $0 \neq L \cap K_n \subset K_n - R_n$ according to (11).

Condition (19) implies (compare § 46, I, Theorem 1) that

$$K_n \cap \operatorname{Fr}(R_n) \neq 0, \quad \text{therefore} \quad p_n \in K_n$$

(according to (18)) and since $\operatorname{Fr}(R_n) \subset L$ (compare § 49, III, Theorem 3) condition (14) follows.

Suppose that condition (15) is not satisfied. So there exists a sequence $m_1 < m_2 < \ldots$ such that no subsequence of $\{p_{m_n}\}$ converges to p.

Thus it is legitimate to assume by virtue of (17) that for each n $p_{m_n} \neq q_{m_n}$, and therefore by (18) that

$$p_{m_n} = \operatorname{Fr}(R_{m_n}). \tag{20}$$

There are two cases to be considered.

(i) There exists a sequence $i_1 < i_2 < \ldots$ such that the components $R_{m_{i_1}}, R_{m_{i_2}}, \ldots$ are pairwise distinct. Therefore, according to Theorem 7 and (20), it follows that

$$\lim_{n=\infty} \delta(R_{m_{i_n}}) = 0, \quad \text{thus} \quad \lim_{n=\infty} |p_{m_{i_n}} - q_{m_{i_n}}| = 0,$$
$$\text{and hence} \quad \lim_{n=\infty} p_{m_{i_n}} = p$$

by (17).

(ii) There exist a subscript r and a sequence of subscripts $i_1 < i_2 < \ldots$ such that $R_{m_{i_1}} = R_{m_{i_2}} = \ldots = R_r$. Put

$$a = \operatorname{Fr}(R_r). \tag{21}$$

It follows by (20) that

$$p_{m_{i_1}} = p_{m_{i_2}} = \ldots = a, \quad \text{which implies} \quad \lim_{n=\infty} p_{m_{i_n}} = a. \tag{22}$$

On the other hand, $q_{m_{i_n}} \in R_r$ by (18), and it follows that $p \in \bar{R}_r$ according to (17). Since (compare (10)) $p \in L$, so $p \in \operatorname{Fr}(R_r)$, which means that $p = a = \lim_{n=\infty} p_{m_{i_n}}$ by (21) and (22).

Thus in both cases it follows that the sequence $\{p_{m_n}\}$ contains a subsequence convergent to p, contradicting the definition of the sequence $\{m_n\}$.

II. Cyclic elements.

Definition. We call *cyclic element of* $\mathscr{X}$ (1):

(i) every point which separates $\mathscr{X}$,

(ii) every set

$$E_p = \underset{x}{E}(\operatorname{ord}_{p,x}\mathscr{X} \geqslant 2) \tag{1}$$

(i.e. the set of points x such that no point cuts $\mathscr{X}$ between p and x), provided that p does not separate $\mathscr{X}$.

Example. Let the space $\mathscr{X}$ consist of two (externally) tangent circles and a radius. Each of the circles is a cyclic element as well as their common point and each point of the radius.

(1) The concept of the cyclic element is due to G. T. Whyburn; see *Cyclicly connected continuous curves*, Proc. Mat. Acad. Sc. 13 (1927), p. 31. Compare also R. L. Moore, Monatsh. Math.-Phys. 36 (1929), p. 81.

The following statements are easy to establish.

THEOREM 1. *If p does not separate the space, then $p \in E_p$.*

Therefore *the space is the union of its cyclic elements.*

THEOREM 2. *If* $\mathrm{ord}_p \mathscr{X} = 1$, *then* $E_p = p$.

However, Theorem 10 of § 51, V, implies the following

THEOREM 3. *If the point p does not separate the space and if* $\mathrm{ord}_p \mathscr{X} \geqslant 2$, *then* $E_p - p \neq 0$.

THEOREM 4. *If $E_a \neq E_b$, the set $E_a \cap E_b$ is empty or consists of a single point which separates the space.*

Proof. Let $c \in E_a - E_b$. Then there exists a point q which separates the space between c and b. Let $p \in E_a \cap E_b$. We are going to show that $p = q$.

By hypothesis

$$\mathrm{ord}_{a,p} \mathscr{X} \geqslant 2, \quad \mathrm{ord}_{p,b} \mathscr{X} \geqslant 2 \quad \text{and} \quad \mathrm{ord}_{c,a} \mathscr{X} \geqslant 2. \tag{2}$$

Moreover, $q \neq a$, since a is not a separating point.

Suppose that $p \neq q$. Therefore (according to (2)) the following points can be joined by arcs avoiding q: a with p, p with b and c with a, and hence, c with b. But then q does not separate c from b, contrary to the definition of q.

THEOREM 5. *If b does not separate the space and $b \in E_a$, then $E_b = E_a$.*

Because $b \in E_a \cap E_b$ by Theorem 1, and $E_a \cap E_b \neq b$ by Theorem 4.

THEOREM 6. *Each of the following conditions is necessary and sufficient in order that a set E (containing more than one point) be a cyclic element.*

(i) *There exist two points $a \neq b$ such that*

$$\mathrm{ord}_{a,b} \mathscr{X} \geqslant 2 \tag{3}$$

and E is the intersection of all closed sets which are completely arcwise connected and contain a and b.

(ii) *E is connected and saturated with respect to the property of containing no point which separates it.*

Proof. 1. Let E_a be a cyclic element $\neq a$. Let $b \in E_a - a$. Let L be the intersection of all closed sets which are completely arcwise connected and contain a and b. We are going to show that

$$E_a = L. \tag{4}$$

Let $x \in \mathscr{X} - L$. Let R be the component of $\mathscr{X} - L$ which contains x and let $y = \mathrm{Fr}(R)$ according to Theorems 1 of Section I and 4. Since y separates the space, whereas a does not, we have $y \neq a$, and since $a \in L$ and $x \in R$, the point y separates the space between a and x. Therefore $x \in \mathscr{X} - E_a$.

Thus $E_a \subset L$.

Conversely, let $x \in \mathscr{X} - E_a$. Consequently, there exists a point y which separates the space between a and x.

Therefore

$$\mathscr{X} = M \cup N, \quad \overline{M} = M, \quad \overline{N} = N, \quad a \in M, \quad x \in N$$
$$\text{and} \quad M \cap N = y. \tag{5}$$

Since $b \in E_a - a$, the identity (3) follows, and we infer that $b \in M$ according to (5). Let us apply Theorem 3 of Section I; then the conditions (5) imply that M is completely arcwise connected, and it follows from the definition of L that $L \subset M$, which yields $x \in \mathscr{X} - L$ because $x \in \mathscr{X} - M$ according to the last two conditions (5).

Thus it follows that $L \subset E_a$, which proves identity (4).

2. We claim that condition (i) implies (ii). If E satisfies condition (i) and if $x \in E$, the set $E - x$ is connected. Suppose that

$$E = M \cup N, \quad \overline{M} = M, \quad \overline{N} = N \quad \text{and} \quad M \cap N = x. \tag{6}$$

We have to show that either $M = E$ or $N = E$.

Since one of the inequalities $x \neq a$ or $x \neq b$ must be true, assume that $x \neq a$. Moreover, it is legitimate to assume that $a \in M$. By (3) the point x does not separate the space between a and b; consequently neither does the set E separate between these points (compare I, Theorems 1, 4 and 5). Therefore, according to (6) and Theorem 3 of Section I, the set M is closed, completely arcwise connected and contains a and b. Thus $E \subset M$ and $M = E$ by virtue of (6).

Now let C be a connected set such that $E \subset C \neq E$. We have to show that C contains a point which separates it.

Let $p \in C - E$ and let R be the component of $\mathscr{X} - E$ which contains p. According to Theorem 4 of Section I, assume that $q = \mathrm{Fr}(R)$. It is clear that C is separated by the point q.

3. We claim that condition (ii) implies that E is a cyclic element. By hypothesis no point separates the set E. Therefore, the set

of points of E which separate the space is countable (compare § 46, IX, Theorem 3). Let p be the point of E which does not separate the space. We shall prove that $E = E_p$.

By hypothesis, $\operatorname{ord}_{p,x} E \geqslant 2$ for every $x \epsilon E$, therefore $\operatorname{ord}_{p,x} \mathscr{X} \geqslant 2$, and hence $x \epsilon E_p$ according to (1). Thus $E \subset E_p$.

It follows that $E = E_p$, because (as has been proved) E_p satisfies condition (ii), which implies that E_p is connected and does not contain separating points; on the other hand, E is saturated with respect to this property.

Remark. Theorem 4 can be restated as follows.

If A and B are two cyclic elements and p is a point such that $A \cap B = p$, the point p separates the space between $A - B$ and $B - A$.

This is an immediate consequence of Theorem 8 of Section I because the sets A and B are completely arcwise connected (by Theorem 6, (i) and Theorem 1 of Section I).

THEOREM 7. *Every cyclic element is a locally connected continuum.*

Proof. This is an immediate consequence of condition (i) of Theorem 6, of Theorem 1 of Section I and of Theorem 2.

THEOREM 8. *If E is a cyclic element, the set A of its points which belong to other cyclic elements is countable.*

Proof. According to condition (ii) of Theorem 6, no point is a separating point of E. Consequently (compare § 46, IX, Theorem 3), the set B of the separating points of the space which belong to E is countable. Finally $A \subset B$ by Theorem 4.

THEOREM 9. *If $E^1, E^2, \ldots$ is a sequence of distinct cyclic elements, then* $\lim_{n=\infty} \delta(E^n) = 0$.

Therefore, *the family of all cyclic elements which do not reduce to single points is countable.*

Proof. Suppose that this is not true. We can therefore assume that there exist two sequences of points $\{a_n\}$ and $\{b_n\}$ such that

$$a_n, b_n \epsilon E^n, \quad \lim_{n=\infty} a_n = a, \quad \lim_{n=\infty} b_n = b \quad \text{and} \quad a \neq b.$$

This implies that, for sufficiently large values of n, there exist two disjoint arcs $a_n a_{n+1}$ and $b_n b_{n+1}$ (compare § 50, II, Theorem 4).

But then the intersection of the connected set

$$Z = E^{n+1} \cup a_n a_{n+1} \cup b_n b_{n+1}$$

and of E^n is not connected, because

$$Z \cap E^n = (E^n \cap a_n a_{n+1}) \cup \big((E^n \cap b_n b_{n+1}) \cup (E^n \cap E^{n+1})\big)$$

and the set $E^n \cap E^{n+1}$ is empty or reduces to a single point (by Theorem 4). This conclusion contradicts Theorem 5′ of Section I since E^n is completely arcwise connected (by Theorems 6 (i) and 1 of Section I).

THEOREM 10. *Every connected set C which is not separated by any point is contained in a cyclic element.*

More precisely, *if a and b are two (different) points of C, the intersection E of all closed, completely arcwise connected sets which contain these points is the required cyclic element.*

Proof. The set E is a cyclic element by Theorem 6 (i), because $\mathrm{ord}_{a,b}\mathscr{X} \geqslant 2$. On the other hand, if $p \in C - E$ and if R denotes the component of $\mathscr{X} - E$ which contains p, then the point $\mathrm{Fr}(R)$ (compare Theorem 4 of Section I) separates C between a and p or between b and p (according to whether $a \neq \mathrm{Fr}(R)$ or $b \neq \mathrm{Fr}(R)$).

THEOREM 11. *In order that a continuum L (containing more than one point) be completely arcwise connected, it is necessary and sufficient that L be the union of some cyclic elements.*

Proof. 1. The condition is necessary. Suppose that the point $p \in L$ belongs to no cyclic element contained in L. Therefore the point p itself is not a cyclic element. This implies that p does not separate the space, and consequently (compare Theorem 4) E_p is the only cyclic element which contains p.

By hypothesis $E_p - L \neq 0$. Let R be a component of $E_p - L$. Since the set $L \cap E_p$ is completely arcwise connected with respect to the set E_p (considered as the space), the boundary of R relative to E_p consists (by Theorem 4 of Section I) of a single point, say q. Thus the point q separates E_p between $E_p - L$ and $L \cap E_p - q$, which implies, according to Theorem 6 (ii), that $L \cap E_p - q = 0$, i.e. $L \cap E_p = q$; since $p \in L \cap E_p$, it follows that $q = p$ and $L \cap E_p = p$. Therefore (compare I, Theorem 8), p separates the space between $E_p - L$ and $L - E_p$, and hence $L - E_p = 0$, i.e. $L - p = 0$, thus $L = p$ contrary to the hypothesis.

2. The condition is sufficient. Assume that the continuum L is a union of cyclic elements, that $a, b \in L$ and that $ab - L \neq 0$. Therefore the arc ab contains a subarc cd such that

$$L \cap cd = (c, d) \quad \text{and} \quad L - cd \neq 0. \tag{7}$$

It is clear that $\operatorname{ord}_{c,d} \mathcal{X} \geqslant 2$. Thus there exists a cyclic element E such that $c, d \in E$, and hence $E - L \neq 0$ by (7). This inequality leads to a contradiction, because by Theorem 8 it implies that the set $E \cap L$ is countable and by Theorems 6, (i) and 5′ of Section I it implies that $E \cap L$ is a continuum (containing at least two points c and d).

Remark. It can be generally proved that *connected* sets which are unions of cyclic elements coincide with completely arcwise connected sets[1].

THEOREM 12. *Every convergence continuum (containing more than one point) of $\mathcal{X}$ is a convergence continuum of a cyclic element which contains it.*

Proof. Let $K, K_1, K_2, \ldots$ be a sequence of continua and let a, b be a pair of (distinct) points such that

$$\operatorname*{Lim}_{n=\infty} K_n = K, \tag{8}$$

$$a, b \in K, \tag{9}$$

$$K_i \cap K_j = 0 \quad \text{for} \quad i \neq j, \tag{10}$$

$$K \cap K_n = 0 \quad \text{for} \quad n = 1, 2, \ldots. \tag{11}$$

Thus the set $Z = a \cup b \cup K_1 \cup K_2 \cup \ldots$ is connected between a and b (according to § 46, IV, Theorem 9). It follows that

$$\operatorname{ord}_{a,b} \mathcal{X} \geqslant 2. \tag{12}$$

For suppose that p separates $\mathcal{X}$ between a and b; then one of the sets K_n, say K_1, contains p. But then, according to (10), p does not belong to the set $(a \cup b \cup K_2 \cup K_3 \cup \ldots)$, which is also connected between a and b.

Thus formula (12) is established. According to Theorem 6 (i), let E be the cyclic element which contains a and b.

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[1] Compare G. T. Whyburn, *Analytic Topology*, p. 80.

Since the choice of points a and b is arbitrary, it follows

$$K \subset E. \tag{13}$$

We are going to prove that K is a convergence continuum of E; namely that $E \cap K_n$ is a continuum and that

$$\operatorname*{Lim}_{n=\infty}(E \cap K_n) = K, \tag{14}$$

$$K \cap (E \cap K_n) = 0. \tag{15}$$

Now $E \cap K_n$ is a continuum by Theorem 5′ of Section I, and

$$E \cap K_n \neq 0 \quad \text{for sufficiently large } n. \tag{16}$$

For suppose that $m_1 < m_2 < \ldots$ and $E \cap K_{m_i} = 0$; then the set $V = a \cup b \cup K_{m_1} \cup K_{m_2} \cup \ldots$ is connected between a and b (according to § 46, IV, Theorem 9), whereas the set $E \cap V = (a, b)$ is not, contrary to Theorem 5′ of Section I.

Thus condition (16) is established. Condition (14) follows from Theorem 10 of Section I (setting $L = E$). Finally, condition (15) is an immediate consequence of (11).

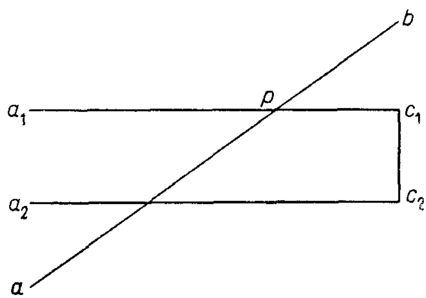

Fig. 13

LEMMA 13. *Let a_1c_1, a_2c_2, ab and c_1c_2 be arcs such that* (see Fig. 13)

$$a_1c_1 \cap a_2c_2 = 0 = ab \cap c_1c_2, \tag{17}$$

$$a_1c_1 \cap c_1c_2 = c_1, \qquad a_2c_2 \cap c_1c_2 = c_2. \tag{18}$$

Then the union $a_1c_1 \cup a_2c_2 \cup ab$ contains two arcs M and N, which join the set (a_1, a_2, a) to b and the set (a_1, a_2) to (c_1, c_2) respectively in such a manner that

$$M \cap (N \cup c_1c_2) = 0. \tag{19}$$

Proof. There are two cases to be considered.

1. $ab \cap (a_1c_1 \cup a_2c_2) = 0$. In this case we set

$$M = ab \quad \text{and} \quad N = a_1c_1 \quad (\text{or } a_2c_2).$$

2. $ab \cap (a_1c_1 \cup a_2c_2) \neq 0$. Let p be the first point of the (oriented) arc ba which belongs to the union $a_1c_1 \cup a_2c_2$. By the symmetry it can be assumed that $p \epsilon a_1c_1$. It follows that

$$a_1p \cap pb = p \neq c_1 \quad \text{and} \quad pb \cap a_2c_2 = 0,$$

and it is sufficient to set

$$M = a_1p \cup pb \quad \text{and} \quad N = a_2c_2.$$

LEMMA 14. *In a locally and integrally arcwise connected space $\mathscr{X}$, let be given two disjoint closed sets A and B such that no point x separates the sets $A-x$ and $B-x$. Then there exist two disjoint arcs which join A to B.*

Proof(1). Let E be the set of points x such that there exist two disjoint arcs connecting A with x and A with B respectively. We have to show that $E \cap B \neq 0$. We are going to prove that $E = \mathscr{X}$.

Since the space is connected, this reduces to show that

$$E \neq 0, \tag{20}$$

$$E \text{ is open}, \tag{21}$$

$$E \text{ is closed}. \tag{22}$$

Let ab be an arc irreducible between A and B. Let $a_1 \epsilon (A-a)$. Since the space is locally arcwise connected, there exists a neighbourhood R of a_1, which is disjoint from ab and is arcwise connected. Therefore $R \subset E$, which implies condition (20).

Let $p \epsilon E$. Then there exist two disjoint arcs M and N which connect A with p and with B respectively. If R is an arcwise connected neighbourhood of p disjoint from N, it is easily seen that $R \subset E$. And hence the set E is open.

In order to prove that $E = \bar{E}$, let $q \epsilon \bar{E}$. It must be shown that $q \epsilon E$. By hypothesis, q does not separate the space between $A-q$ and $B-q$. Thus the set $\mathscr{X}-q$ contains an arc ab irreducible between

(1) Compare G. T. Whyburn, *On the cyclic connectivity theorem*, Bull. Amer. Math. Soc. 37 (1931), p. 429.

A and B. Let R be an arcwise connected neighbourhood of q disjoint from ab. Consequently there exist a point $r \in E \cap R$, and therefore an arc $qr \subset R$, so that

$$qr \cap ab = 0; \tag{23}$$

there are also two arcs P and Q, which join A with r and with B respectively, so that

$$P \cap Q = 0. \tag{24}$$

We may assume that $Q \cap qr \neq 0$, because otherwise an arc S joining A to q could be selected in $P \cup qr$, and we would have $S \cap Q = 0$ and hence $q \in E$.

Now let a_1c_1 and a_2c_2 be two arcs such that

$$a_1c_1 \subset Q, \quad a_1 \in A, \tag{25}$$

$$a_1c_1 \cap qr = c_1, \quad a_2 \in A, \tag{26}$$

$$a_2c_2 \subset P, \tag{27}$$

$$a_2c_2 \cap qr = c_2. \tag{28}$$

The hypotheses of the lemma are fulfilled, namely (17) follows from (25), (27), (24) and (23), and (18) from (26) and (28).

Thus the set $a_1c_1 \cup a_2c_2 \cup ab$ contains two arcs M and N joining A to B and to (c_1, c_2) respectively in such a manner that condition (19) is satisfied, and therefore (compare (26), (28) and (23))

$$M \cap (N \cup c_1c_2 \cup qc_1) = 0. \tag{29}$$

In the set $N \cup c_1c_2 \cup qc_1$ and arc joining A to q can be selected, and condition (29) implies that $q \in E$.

THEOREM 15. *In a locally connected continuum $\mathscr{X}$ every point p of order $\geqslant 2$ is contained in an arc apb (where $a \neq p \neq b$)*(1).

Proof. First we will prove this theorem under the supplementary hypothesis that no point separates $\mathscr{X}$, i.e. that $\mathscr{X}$ is its own cyclic element.

(1) Compare W. L. Ayres, Amer. Journ. Math. 51 (1929), p. 577. For a more direct proof, see the paper of the same author, *A new proof of the cyclic connectivity theorem*, Bull. Amer. Math. Soc. 48 (1948), p. 627.

It is legitimate to assume that p does not separate any locally connected continuum C. Because every arc, which joins in C two points a and b belonging to distinct components of $C-p$, would contain p (which is their common boundary).

Under these hypotheses we are going to define by induction two sequences of points $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$, as well as a sequence of locally connected continua $E_0, E_1, \ldots$ which contain no separating points and which satisfy the following conditions

$$a_n \neq b_n, \tag{30}$$

$$a_n, b_n \in E_n - p, \tag{31}$$

$$p \in E_n \subset E_{n-1}, \tag{32}$$

$$\delta(E_n) < 1/n. \tag{33}$$

Let $E_0 = \mathscr{X}$ and let a_0 and b_0 be two points such that

$$a_0 \neq b_0 \quad \text{and} \quad a_0 \neq p \neq b_0.$$

Assuming that a_{n-1}, b_{n-1} and E_{n-1} satisfy conditions

$$a_{n-1} \neq b_{n-1}, \qquad a_{n-1}, b_{n-1} \in E_{n-1} - p, \tag{34}$$

$$p \in E_{n-1}, \tag{35}$$

we define C_n to be a locally connected continuum such that

$$C_n \subset E_{n-1}, \tag{36}$$

$$a_{n-1}, b_{n-1} \in E_{n-1} - C_n, \tag{37}$$

$$\delta(C_n) < 1/n, \tag{38}$$

and that p is a point of C_n interior relative E_{n-1}, and therefore (compare § 51, II, Theorem 4)

$$\operatorname{ord}_p C_n = \operatorname{ord}_p E_{n-1} \geqslant 2 \tag{39}$$

since E_{n-1} contains no separating points.

According to (39), Theorem 6 (i) and Theorem 10 of § 51, V, the point p belongs to a cyclic element of C_n, say E_n, such that $E_n \neq p$. By (37) it follows that $a_{n-1}, b_{n-1} \in E_{n-1} - E_n$. Since the set E_{n-1} does not contain separating points, therefore by Theorem 14 there exist two arcs $a_{n-1}a_n$ and $b_{n-1}b_n$ such that

$$a_{n-1}a_n \cup b_{n-1}b_n \subset E_{n-1}, \tag{40}$$

$$a_{n-1}a_n \cap b_{n-1}b_n = 0, \quad a_{n-1}a_n \cap E_n = a_n \quad \text{and} \quad b_{n-1}b_n \cap E_n = b_n. \tag{41}$$

It easily follows from (41) and (38) that the conditions (30)-(33) are satisfied ($a_n \neq p$, because otherwise p would separate the continuum $E_n \cup a_{n-1}a_n$).

Let $A = a_0a_1 \cup a_1a_2 \cup \ldots$ and $B = b_0b_1 \cup b_1b_2 \cup \ldots$. We will show that the union $A \cup p \cup B$ is an arc a_0pb_0.

$A \cup p$ (as well as $B \cup p$) is an arc, because $p \in \mathscr{X} - A$ by (32) and (41), and $a_{m-1}a_m \subset (E_{m-1} - E_m) \cup a_m$ by (40) and (41), which implies (by (32))

$$(a_0a_1 \cup \ldots \cup a_{n-1}a_n) \cap a_na_{n+1} \subset (E_0 - E_n \cup a_n) \cap (E_n - E_{n+1} \cup a_{n+1}) = a_n,$$

and according to (32), (33) and (40) it follows that

$$\operatorname*{Lim}_{n=\infty} a_{n-1}a_n = p.$$

Finally, the arcs $A \cup p$ and $B \cup p$ have only the point p in common since $A \cap B = 0$ by (41).

Thus the theorem has been established for the case where $\mathscr{X}$ is a cyclic element.

The general case can be reduced to the former one. As before it can be assumed that p is not a separating point. But then $\operatorname{ord}_p \mathscr{X} \geqslant 2$ implies by Theorems 6 (i) and 10 of § 51, V that p belongs to a cyclic element which contains more than one point.

THEOREM 16. *Every pair of points $p \neq q$ of a locally connected continuum $\mathscr{X}$ which contains no separating points (therefore every pair of points of a cyclic element) is contained in a simple closed curve.*

Proof. Observe first that p belongs to a simple closed curve. By Theorem 15 there exists an arc apb; on the other hand, since the set $\mathscr{X} - p$ is connected by hypothesis, it contains an arc cd

irreducible between ap and pb. The required simple closed curve containing p is obtained by adding to cd the arcs cp and pd (contained in apb).

Now let P and Q be two simple closed curves containing p and q respectively. There are three cases to be considered.

(α) $P \cap Q = 0$. In this case, there exist by Theorem 14, two disjoint arcs A and B irreducible between P and Q. It is easily seen that the union $P \cup Q \cup A \cup B$ contains a simple closed curve which joins p to q.

(β) The intersection $P \cap Q$ consists of a single point r. Let R be an arc contained in $\mathscr{X} - r$ and irreducible between P and Q. Hence the union $P \cup Q \cup R$ contains a simple closed curve joining p to q.

(γ) Finally, if the intersection $P \cap Q$ contains two points (at least), the points p and q belong to a simple closed curve contained in $P \cup Q$.

THEOREM 17. *In order that a set containing more than one point should be a cyclic element, it is necessary and sufficient that it be saturated with respect to the existence, for every pair of its points, of a simple closed curve which contains them.*

Proof. The condition is necessary. If E is a cyclic element, then by Theorem 16 every pair of its points belongs to a simple closed curve contained in E. According to Theorem 6 (ii), the set E is saturated with respect to this property, because every set which has this property contains no separating point.

The condition is sufficient. Every set E which satisfies it contains no separating point. Therefore it is contained (compare Theorem 10) in a cyclic element E^*. Since every pair of points of E^* belongs (as just has been proved) to a simple closed curve contained in E^*, and E is saturated with respect to this property, it follows that $E^* \subset E$, and hence $E = E^*$.

III. Extensible properties(1). The usefulness of the decomposition of a locally connected continuum into cyclic elements stems in general from the dendrite-like structure of the continuum relative

(1) See the quoted paper of G. T. Whyburn and myself, Fund. Math. 15 (1930), p. 322. It contains many examples of extensible properties.

to its cyclic elements(¹). There are numerous properties of the whole space which depend only on analogous properties of its cyclic elements. The properties called *extensible* are of that kind. We shall call so every property (of a set) which is property of the whole space if it is a property of each cyclic element.

We shall say that a property is *reducible* if it is a property of every cyclic element, provided it is a property of the whole space.

THEOREM 1. *Let* $\boldsymbol{F}$ *be a family of closed subsets of a locally connected continuum* $\mathscr{X}$ *such that*

$$(x)\in\boldsymbol{F} \quad \textit{for each } x\in\mathscr{X}. \tag{1}$$

The following property **P** *of* E *is extensible:*

For every pair of (distinct) points $x, y\in E$ *there exists a set* $F\in\boldsymbol{F}$ *which separates them.*

Proof. Assume that every cyclic element has the property **P**. Let a and b be two distinct points. We have to show that there exists a set $F\in\boldsymbol{F}$ which separates them.

If a and b belong to a cyclic element E, there exists by hypothesis a set $F\in\boldsymbol{F}$ which separates E between a and b. And this set separates the whole space between these points (by Theorem 6 (i) of Section II and Theorems 1 and 5 of Section I).

On the other hand, if there is no cyclic element which contains a and b, it follows by Theorem 6 (i) of Section II, that $\mathrm{ord}_{a,b}\mathscr{X}\leqslant 1$, and consequently there exists a point x which separates a and b. We set $F=(x)$ in conformity with (1).

If for $\boldsymbol{F}$ are substituted the families

1. of finite sets,

2. of closed countable sets,

3. of closed sets of dimension $\leqslant n$ ($n=0,1,2,\ldots$),

then Theorem 1, combined with Theorem 9 of § 51, IV and with Corollary 1b of § 27, II, implies the next theorem.

(¹) Compare also A. D. Wallace, *The acyclic elements of a Peano space*, Bull. Amer. Math. Soc. 47 (1941); the author calls so the components of the set of the separating points and of the end-points; the theory of acyclic elements is dual with respect to the theory of cyclic elements.

THEOREM 2. *The following properties of a set E are extensible*

(i) *of being regular,*

(ii) *of being rational,*

(iii) $\dim E \leqslant n$ *(for $n = 1, 2, \ldots$).*

THEOREM 3. *The hereditary local connectedness is an extensible property.*

Proof. Suppose that the locally connected continuum $\mathscr{X}$ is not hereditarily locally connected. Therefore it contains (by § 50, IV, Theorem 2) a convergence continuum K (containing more than one point). According to Theorem 12 of Section II, K is a continuum of convergence of a cyclic element E. It follows (by § 50, IV, Theorem 2) that E is not hereditarily locally connected.

THEOREM 4. *The arcwise connectedness of the connected subsets of E is an extensible property.*

Proof. Let C be a connected set, and let a and b be two (distinct) points of C. Let A be an arc ab and let S be the set of all points which separate $\mathscr{X}$ between a and b. Put $F = S \cup a \cup b$. Clearly, $F \subset C \cap A$.

Since $F = \overline{F}$ (compare § 49, IV, Theorem 3), the set $A - F$ is the union of a sequence of (open) contiguous intervals

$$a_1 b_1 - a_1 - b_1, \ a_2 b_2 - a_2 - b_2, \ \ldots .$$

It follows that

$$\operatorname{ord}_{a_i, b_i} \mathscr{X} \geqslant 2. \tag{2}$$

For otherwise there would exist a point p which separates the space between a_i and b_i, and therefore, between a and b, and consequently the condition $p \in F \cap (a_i b_i - a_i - b_i)$ would hold, which is impossible.

Let E^i be a cyclic element which contains a_i and b_i. Since the set $C \cap E^i$ is connected (by Theorem 5′ of Section I), and therefore arcwise connected (by hypothesis), there is an arc $(a_i b_i)^* \subset C \cap E^i$. Put

$$B = F \cup \bigcup_i (a_i b_i)^*. \tag{3}$$

Since $B \subset C$, it only remains to prove that B is an arc.

Consider the mapping $f\colon A \to B$ such that

(i) $f(x) = x$ for $x \in F$,

(ii) $f | a_i b_i$ is a homeomorphism of $a_i b_i$ onto $(a_i b_i)^*$.

Our aim is to show that f is one-to-one and continuous.

To this end, we are going to prove that

$$A \cap E^i = a_i b_i. \tag{4}$$

The condition $a_i, b_i \in E^i$ implies by Theorem 5′ of Section I that

$$a_i b_i \subset E^i. \tag{5}$$

On the other hand, we will show that

$$(aa_i - a_i) \cap E^i = 0 = (bb_i - b_i) \cap E^i. \tag{6}$$

a_i separates $\mathscr{X}$ between $aa_i - a_i$ and b_i (since $a_i \in F$). Therefore the condition $x \in (aa_i - a_i)$ implies that $\operatorname{ord}_{x,b_i} \mathscr{X} = 1$, and hence $x \in \mathscr{X} - E^i$ by Theorem 6 (ii) of Section II.

Thus the first identity of (6) is established, and the second follows by the symmetry.

Conditions (5) and (6) imply directly (4), and therefore $E^i \neq E^j$ for $i \neq j$.

According to Theorem 6 of Section II (Remark), the intersection $E^i \cap E^j$ is either empty or else reduces to a single point p which separates the space between every pair $x \in E^i - p$ and $y \in E^j - p$. By virtue of (5), it follows that p separates the space between a and b, and hence $p \in F$.

Thus

$$E^i \cap E^j \subset F \quad \text{and} \quad F \cap E^i \subset (a_i, b_i), \tag{7}$$

where the last inclusion is a consequence of (4).

Inclusions (7) imply that the function f is one-to-one, because

$$f(A) = f(F) \cup \bigcup_i f(a_i b_i - a_i - b_i),$$

$$\begin{aligned} f(F) \cap f(a_i b_i - a_i - b_i) &= F \cap (a_i b_i)^* - a_i - b_i \\ &\subset F \cap E^i - (a_i, b_i) = 0, \end{aligned}$$

$$\begin{aligned} &f(a_i b_i - a_i - b_i) \cap f(a_j b_j - a_j - b_j) \\ &\qquad = [(a_i b_i)^* - a_i - b_i] \cap [(a_j b_j)^* - a_j - b_j] \\ &\qquad \subset E^i \cap E^j \cap F - (a_i, b_i, a_j, b_j) = 0. \end{aligned}$$

Finally, the function f is continuous. This is obvious if the family of arcs $a_i b_i$ is finite. If it is infinite, it follows by Theorem 9 of Section II that

$$\lim_{i=\infty} \delta(E^i) = 0, \quad \text{therefore} \quad \lim_{i=\infty} \delta(a_i b_i)^* = 0,$$

which implies the continuity of the function f.

Remark. Applying Theorem 10 of § 50, IV, or slightly changing the proof of Theorem 4, it can be shown that the following properties are extensible:

(i) *every connected subset of E is a semi-continuum,*

(ii) *every subset of E, connected between two points, is arcwise connected between these points.*

THEOREM 5. *The unicoherence is an extensible and reducible property.*

Proof. Assume that the locally connected continuum $\mathscr{X}$ is not unicoherent. Let (according to § 50, III, Theorem 6) K and L be two locally connected continua, and A and B two closed sets satisfying conditions

$$\mathscr{X} = K \cup L, \quad K \cap L = A \cup B, \quad A \cap B = 0, \quad A \neq 0 \neq B. \tag{8}$$

Let ab be an arc such that $ab \subset K$, $ab \cap A = a$ and $ab \cap B = b$. It follows that

$$\operatorname{ord}_{a,b} \mathscr{X} \geqslant 2. \tag{9}$$

Because, if p would separate $\mathscr{X}$ between a and b, then

$$p \in (ab \cap L),$$

therefore

$$p \in (ab \cap K \cap L) = (ab \cap A) \cup (ab \cap B) = (a, b),$$

i.e. either $p = a$ or $p = b$.

Let E be a cyclic element containing a and b. Therefore by (8)

$$E = (E \cap K) \cup (E \cap L),$$

$$(E \cap K) \cap (E \cap L) = (E \cap A) \cup (A \cap B),$$

$$(E \cap A) \cap (E \cap B) = 0, \quad E \cap A \neq 0 \neq E \cap B.$$

Since the sets $E \cap K$ and $E \cap L$ are continua (by Theorem 5′ of Section I) the above conditions show that the cyclic element E is not unicoherent.

Thus, the unicoherence is an extensible property. It is also reducible by Theorem 9 of Section I.

Remarks. A continuum $\mathscr{X}$ is said to be *at most n-coherent* if, whenever $\mathscr{X} = K \cup L$ where K and L are continua, the intersection $K \cap L$ contains at most n components[1]. Let $r(\mathscr{X}) = n-1$ if $\mathscr{X}$ is n-coherent. Then[2]

$$r(\mathscr{X}) = \sum_i r(E^i),$$

where $E^1, E^2, \ldots$ is the sequence of cyclic elements which do not reduce to individual points.

By means of the following formula it is possible to calculate the order of the space at the point p (supposing that this order is finite)[3]

$$\operatorname{ord}_p \mathscr{X} = m(p) - n(p) + \sum_{i=1}^{n(p)} \operatorname{ord}_p E^i,$$

where $m(p)$ is the number of the components of the set $\mathscr{X} - p$ and where $E^1, E^2, \ldots, E^{n(p)}$ is the system of cyclic elements which contain p but do not reduce to p.

IV. θ-curves. A *θ-curve* is a continuum which is the union of three arcs having the same end-points and having pairwise no other point in common (such is the letter θ).

The class of locally connected continua containing no θ-curve is an important generalization of the dendrites (thus, for instance, every continuum of this kind is homeomorphic with the boundary of a region situated in the plane[4]).

THEOREM 1. *Every locally connected continuum $\mathscr{X}$ containing no θ-curve and containing no separating point is a simple closed curve* (*unless it consists of a single point*).

Proof. Since $\mathscr{X}$ is not a dendrite, let C be a simple closed curve contained in $\mathscr{X}$. Suppose that $\mathscr{X} \neq C$. Let R be a component of

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(1) S. Eilenberg, *Sur les espaces multicohérents* I, Fund. Math. 27 (1936), p. 153.

(2) See e.g. G. T. Whyburn, *Analytic Topology*, p. 85.

(3) See the quoted paper of G. T. Whyburn and myself, Fund. Math. 15 (1930), p. 326.

(4) See R. L. Ayres, Fund. Math. 14 (1929), p. 92.

$\mathscr{X}-C$. By hypothesis, $\mathrm{Fr}(R)$ contains at least two points, and consequently there exist (compare § 50, III, Theorem 7) two (distinct) points $p, q \in \mathrm{Fr}(R)$ accessible from R, and therefore there is an arc $pq \subset R \cup p \cup q$. Since $\mathrm{Fr}(R) \subset C$ (compare § 49, III, Theorem 3), it follows that $p, q \in C$. Thus $C \cup pq$ is a θ-curve.

THEOREM 2. *Each of the following conditions is necessary and sufficient in order that a locally connected continuum $\mathscr{X}$ contain no θ-curve.*

(i) *Every cyclic element (which contains more than one point) is a simple closed curve.*

(ii) $\mathrm{ord}_{p,q}\mathscr{X} \leqslant 2$ *for every pair* p, q.

Proof. 1. Condition (i) is necessary by Theorem 1.

2. Condition (i) implies (ii). If the points p and q do not belong to a cyclic element, then $\mathrm{ord}_{p,q}\mathscr{X} = 1$ (by Theorem 6 (i) of Section II). If E is a cyclic element and hence — a simple closed curve — which contains p and q, it follows that $\mathrm{ord}_{p,q}E = 2$, which yields $\mathrm{ord}_{p,q}\mathscr{X} = 2$ by Theorem 6 of Section I.

3. Condition (ii) is sufficient. If p and q are the branching points of a θ-curve, then $\mathrm{ord}_{p,q}\theta = 3$.

Hence, if $\theta \subset \mathscr{X}$, then $\mathrm{ord}_{p,q}\mathscr{X} \geqslant 3$.

THEOREM 3. *Every locally connected continuum $\mathscr{X}$ containing no θ-curve is regular and every connected subset of $\mathscr{X}$ is arcwise connected.*

Proof. This is a consequence of Theorem 2 (i) and Theorems 2 (i) and 4 of Section III.

THEOREM 4. *If K is a not locally connected subcontinuum of a locally connected continuum $\mathscr{X}$, there exists a curve $\theta = (ab)_0 \cup (ab)_1 \cup (ab)_2$ such that for $i = 0, 1, 2$, the arc $(ab)_i$ can be joined with K by a continuum Q_i disjoint from $(ab)_{i+1} \cup (ab)_{i+2}$ (the subscripts are reduced* mod 3)(¹).

Proof. According to Theorem 1 of § 49, VI, the set K contains a convergence continuum (which does not reduce to a point). Thus there exist in K a sequence of disjoint continua $K_1, K_2, \ldots$ and two convergent sequences of points such that

$$\lim_{n=\infty} u_n \neq \lim_{n=\infty} v_n \quad \text{and} \quad u_n, v_n \in K_n.$$

(¹) See my paper in Fund. Math. 15 (1930), p. 180. This theorem will be applied in § 59, II, Theorem 4.

Since the space is locally arcwise connected, it may be assumed that A and B are two locally connected continua such that

$$u_0, u_1, u_2 \in A, \quad v_0, v_1, v_2 \in B \quad \text{and} \quad A \cap B = 0. \tag{1}$$

By Theorem 3 of § 50, II, the space $\mathscr{X}$ is the union of a finite family $\boldsymbol{F}$ of locally connected continua which are so small that, if K_i^*, A^*, B^* denote the union of members of $\boldsymbol{F}$ which intersect K_i, A, B respectively, we have

$$A^* \cap B^* = 0 = K_0^* \cap K_1^* = K_1^* \cap K_2^* = K_2^* \cap K_0^*. \tag{2}$$

Let $p_i q_i$ be an arc such that

$$p_i q_i \cap A = p_i, \quad p_i q_i \cap B = q_i \quad \text{and} \quad p_i q_i \subset K_i^* \quad (i = 0, 1, 2). \tag{3}$$

Let $p_0 p_1$ and $q_0 q_1$ be two arcs contained in A and B respectively. The curve

$$C = p_0 p_1 \cup p_1 q_1 \cup q_1 q_0 \cup q_0 p_0$$

is clearly a simple closed curve. Let us join the arcs $p_0 p_1$ and $q_0 q_1$ in the continuum $A \cup K_2^* \cup B$ by an arc, say $(ab)_2$, which has only the end-points in common with $p_0 p_1$ and $q_0 q_1$ respectively. The curve $C \cup (ab)_2$ is a θ-curve.

For, it follows by (2) and (3) that

$$p_0 q_0 \cap (ab)_2 \subset (p_0, q_0) \quad \text{and} \quad p_1 q_1 \cap (ab)_2 \subset (p_1, q_1).$$

Set $C = (ab)_0 \cup (ab)_1$. Hence

$$(ab)_i \subset A \cup K_i^* \cup B, \quad a \in A, \quad b \in B \quad (i = 0, 1, 2). \tag{4}$$

The conditions $A^* \cap (ab)_i \neq 0 \neq B^* \cap (ab)_i$ and $A^* \cap B^* = 0$ imply that $(ab)_i \not\subset A^* \cup B^*$. Thus, according to (4), there exists a point $c_i \in K_i^* - (A^* \cup B^*)$. According to the definition of K_i^*, the point c_i belongs to a continuum $Q_i \in \boldsymbol{F}$ such that

$$Q_i \cap K_i \neq 0 \quad \text{and} \quad Q_i \subset K_i^*. \tag{5}$$

Since $c_i \in Q_i \not\subset (A^* \cup B^*)$, it follows by the definition of A^* and of B^* that

$$Q_i \cap (A \cup B) = 0. \tag{6}$$

On the other hand, inclusion (5) combined with identities (2) yields

$$Q_i \cap K^*_{i+1} = 0 = Q_i \cap K^*_{i+2}. \tag{7}$$

Conditions (4), (6) and (7) imply that

$$Q_i \cap (ab)_{i+1} = 0 = Q_i \cap (ab)_{i+2}.$$

By (5), Q_i is the required continuum.

ABSOLUTE RETRACTS SPACES CONNECTED IN DIMENSION n CONTRACTIBLE SPACES

§ 53. Extending of continuous functions. Retraction

I. Relations τ and τ_v. Let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces and $\Phi \subset \mathcal{Y}^{\mathcal{X}}$ and $A \subset \mathcal{X}$; denote by $\Phi | A$ the set of all restricted functions $f|A$, where $f \in \Phi$. Thus, in particular, $\mathcal{Y}^{\mathcal{X}} | A$ is the set of functions belonging to $\mathcal{Y}^A$ which admit a continuous extension to $\mathcal{X}$.

By definition the symbol $\mathcal{X} \tau \mathcal{Y}$ shall mean that

$$\mathcal{Y}^F = \mathcal{Y}^{\mathcal{X}} | F \quad \text{for any } F = \overline{F} \subset \mathcal{X},$$

and therefore, that each (continuous) function $f: F \to \mathcal{Y}$ admits a (continuous) extension $f^*: \mathcal{X} \to \mathcal{Y}$ (1).

Assume

$$\mathcal{Y}^{\mathcal{X}} |_v A = \bigcup_E (\mathcal{Y}^E | A),$$

where the variable E ranges over the neighbourhoods of A.

Thus the condition $f \in (\mathcal{Y}^{\mathcal{X}} |_v A)$ means that f admits an extension f^* to a neighbourhood E of A; i.e.

$$f \subset f^* \in \mathcal{Y}^E, \quad \text{where} \quad A \cap \overline{\mathcal{X} - E} = 0.$$

We shall use the symbol $\mathcal{X} \tau_v \mathcal{Y}$ to indicate that $\mathcal{Y}^F = (\mathcal{Y}^{\mathcal{X}} |_v F)$ for any $F = \overline{F} \subset \mathcal{X}$; in other words, it will mean that every function belonging to $\mathcal{Y}^F$ admits an extension to a neighbourhood of F.

Examples and remarks. (i) By the theorem of Tietze (§ 14, IV) the interval $\mathcal{I}$ has the following property:

$\mathcal{X} \tau \mathcal{I}$ for any metric space $\mathcal{X}$.

The spaces $\mathcal{Y}$ which have the said property of interval are called *absolute retracts* (or AR) (see Section III).

(ii) *If $\mathcal{Y}$ is a polyhedron, then $\mathcal{X} \tau_v \mathcal{Y}$ for any $\mathcal{X}$* (see Section III).

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(1) Compare A. D. Wallace, Bull. Amer. Math. Soc. 51 (1945), p. 679.

The spaces $\mathscr{Y}$ which have this property are called *absolute neighbourhood retracts* (or ANR).

(iii) For the n-dimensional ball $\mathscr{Q}_n$ and its surface $\mathscr{S}_{n-1}$, the relation $\mathscr{Q}_n \tau \mathscr{S}_{n-1}$ does not hold (§ 28, III, Corollary 1a).

(iv) The condition $\mathscr{I} \tau \mathscr{X}$ implies that $\mathscr{X}$ is arcwise connected (compare also Section IV). The condition $\mathscr{I}^n \tau_v \mathscr{X}$ characterizes the spaces which are said to be *connected in dimensions* $< n$ (they will be studied in Section IV).

(v) The condition $\mathscr{X} \tau \mathscr{S}_n$ is equivalent to $\dim \mathscr{X} \leqslant n$ (see Section VI).

(vi) The condition $\dim \mathscr{X} \leqslant 0$ is equivalent to the hypothesis that $\mathscr{X} \tau \mathscr{Y}$ for any $\mathscr{Y} \neq 0$.

This condition implies the considered hypothesis according to IV (i). The inverse implication follows from the following statement.

(vii) *If* $\dim \mathscr{X} > 0$ *and* $\mathscr{X} \tau \mathscr{Y}$, *then* $\mathscr{Y}$ *is connected.*

Proof. Suppose that

$$A_0 \cup A_1 \subset \mathscr{X}, \quad A_0 \cap A_1 = 0, \quad A_j = \overline{A}_j \quad (j = 0, 1),$$
$$\mathscr{Y} = B_0 \cup B_1, \quad B_0 \cap B_1 = 0, \quad b_j \in B_j = \overline{B}_j.$$

If $\mathscr{X} \tau \mathscr{Y}$, then the mapping $f: A_0 \cup A_1 \to \mathscr{Y}$, defined by the identities $f(A_j) = b_j$, has an extension $g: \mathscr{X} \to \mathscr{Y}$. Let $F_j = g^{-1}(B_j)$; it follows

$$\mathscr{X} = F_0 \cup F_1, \quad F_0 \cap F_1 = 0 \quad \text{and} \quad A_j \subset F_j = \overline{F}_j.$$

Thus $\dim \mathscr{X} = 0$.

(viii) *The relation* τ *is not transitive.*

Proof. $\mathscr{I}^2 \tau \mathscr{I} \tau \mathscr{S}$ (by (i) and (v)), whereas $\mathscr{I}^2 \tau \mathscr{S}$ does not hold (compare (iii)).

(ix) *The relation* τ *is not reflexive.*

According to (vii) no non-connected $\mathscr{X}$ of positive dimension satisfies the relation $\mathscr{X} \tau \mathscr{X}$.

II. Operations. By definition the function g is an extension of the function f if

$$\underset{xy}{E}[y = f(x)] \subset \underset{xy}{E}[y = g(x)].$$

In this case we write concisely

$$f \subset g.$$

If $f\colon A \to \mathscr{Y}$ and $g\colon B \to \mathscr{Y}$, let

$$f+g = \mathop{E}_{xy}[y = f(x)] \cup \mathop{E}_{xy}[y = g(x)]\ (^1).$$

Let us recall that (compare § 13, V, Theorem 3), if the sets A and B are closed and the continuous functions f and g coincide on $A \cap B$, i.e. $f(x) = g(x)$ for $x \in A \cap B$, then $(f+g)\colon A \cup B \to \mathscr{Y}$ is a continuous function.

THEOREM 1. *Let $F = \overline{F} \subset \mathscr{X}$. The condition $\mathscr{X}\tau_v\mathscr{Y}$ implies $F\tau_v\mathscr{Y}$, and the condition $\mathscr{X}\tau\mathscr{Y}$ implies $F\tau\mathscr{Y}$.*

The proof is immediate.

THEOREM 2. *Let $\mathscr{Y} = A_0 \cup A_1, A_0 = \overline{A}_0$ and $A_1 = \overline{A}_1$. If $\mathscr{X}\tau_v A_0$, $\mathscr{X}\tau_v A_1$ and $\mathscr{X}\tau_v A_0 \cap A_1$, then $\mathscr{X}\tau_v(A_0 \cup A_1)$. If $\mathscr{X}\tau A_0$, $\mathscr{X}\tau A_1$ and $\mathscr{X}\tau A_0 \cap A_1$, then $\mathscr{X}\tau(A_0 \cup A_1)$.*

We shall derive Theorem 2 from the following lemma that will be used also later on.

LEMMA 2'. *Let (for $j = 0, 1$)*

$$\mathscr{Y} = A_0 \cup A_1, \quad A_j = \overline{A}_j, \tag{1}$$

$$\mathscr{X} = B_0 \cup B_1, \quad B_j = \overline{B}_j, \tag{2}$$

$$F = \overline{F} \subset \mathscr{X}, \quad f\colon F \to \mathscr{Y} \textit{ continuous}, \tag{3}$$

$$F \cap B_j = f^{-1}(A_j), \tag{4}$$

$$B_j \tau_v A_j, \tag{5}$$

$$(f|F \cap B_0 \cap B_1) \in [(A_0 \cap A_1)^{B_0 \cap B_1}|_v F \cap B_0 \cap B_1]. \tag{6}$$

Under these hypotheses

$$f \in \mathscr{Y}^{\mathscr{X}}|_v F. \tag{7}$$

Moreover, *if the subscript v is deleted in conditions* (5) *and* (6), *then it can be deleted in* (7).

Proof. It follows by (6) that

$$(f|F \cap B_0 \cap B_1) \subset g \in (A_0 \cap A_1)^H, \tag{8}$$

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(1) In order to avoid confusion we shall write $f \mathring{+} g$ instead of $f+g$, when addition is defined in the space $\mathscr{Y}$.

where H is a closed neighbourhood of $F \cap B_0 \cap B_1$ relative to $B_0 \cap B_1$; therefore

$$F \cap \overline{B_0 \cap B_1 - H} = 0. \tag{9}$$

Put $f_j = f|F \cap B_j$. It follows according to (8) that

$$f_j(x) = f(x) = g(x) \quad \text{for} \quad x \in F \cap B_0 \cap B_1. \tag{10}$$

Since $F \cap B_j \cap H \subset F \cap B_0 \cap B_1$, it follows by (10) that

$$(f_j + g) \in A_j^{F \cap B_j \cup H}. \tag{11}$$

The set $F \cap B_j \cup H$ is a closed subset of B_j, and hence by (5)

$$f_j + g \subset g_j \in A_j^{V_j}, \tag{12}$$

where V_j is a closed neighbourhood of $F \cap B_j \cup H$ relative to B_j; therefore

$$F \cap \overline{B_j - V_j} = 0. \tag{13}$$

Define $E = (V_0 \cup V_1) - V_0 \cap V_1 \cup H$ and $h_j = g_j | E \cap V_j$.

Since $E \cap V_0 \cap V_1 = H$, it follows by (8), (11) and (12) that

$$h_j(x) = g_j(x) = g(x) \quad \text{for} \quad x \in E \cap V_0 \cap V_1,$$

and since $(E \cap V_0) \cup (E \cap V_1) = E$, then $(h_0 + h_1): E \to \mathscr{Y}$ is a continuous function.

It remains to prove that E is a neighbourhood of F, i.e. that

$$F \cap \overline{\mathscr{X} - E} = 0, \tag{14}$$

and that

$$f \subset h_0 + h_1. \tag{15}$$

Now, identity (14) is derived as follows

$$\mathscr{X} - E \subset [\mathscr{X} - (V_0 \cup V_1)] \cup (V_0 \cap V_1 - H),$$

$$F \cap \overline{\mathscr{X} - (V_0 \cup V_1)} = F \cap \overline{B_0 - (V_0 \cup V_1)} \cup F \cap \overline{B_1 - (V_0 \cup V_1)}$$

$$\subset (F \cap \overline{B_0 - V_0}) \cup (F \cap \overline{B_1 - V_1}) = 0 \qquad \text{(compare (13))},$$

$$F \cap \overline{V_0 \cap V_1 - H} \subset F \cap \overline{B_0 \cap B_1 - H} = 0 \qquad \text{(compare (9))}.$$

Finally, since $F \cap B_j \subset E \cap V_j$, it follows by (12) that $f_j \subset h_j$, which implies inclusion (15).

In order to prove the second part of Lemma 2′, one has only to set $H = B_0 \cap B_1$ and $V_j = B_j$ in the foregoing argument. Then it follows that $E = \mathscr{X}$.

Proof of Theorem 2. Assuming that conditions (3) are satisfied, let

$$C_j = f^{-1}(A_j) \quad \text{and} \quad B_j = \underset{x}{E}[\varrho(x, C_j) \leqslant \varrho(x, C_{1-j})].$$

Conditions (2) and (4) follow immediately. The relation $\mathscr{X}\tau_v A_j$ implies (5) by Theorem 1. Finally, the relation $\mathscr{X}\tau_v A_0 \cap A_1$ yields (6). Thus Lemma 2′ implies (7), and the relation $\mathscr{X}\tau_v\mathscr{Y}$ follows.

If the subscript v is deleted in the hypotheses, it may be deleted in the conclusion; the second part of Theorem 2 follows.

THEOREM 3. *Let* $\mathscr{Y} = A_0 \cup A_1$, $A_0 = \bar{A}_0$ *and* $A_1 = \bar{A}_1$.

If $\mathscr{X}\tau_v(A_0 \cup A_1)$ *and* $\mathscr{X}\tau_v(A_0 \cap A_1)$, *then* $\mathscr{X}\tau_v A_j$ *for* $j = 0, 1$.

If $\mathscr{X}\tau(A_0 \cup A_1)$ *and* $\mathscr{X}\tau(A_0 \cap A_1)$, *then* $\mathscr{X}\tau A_j$ *for* $j = 0, 1$.

Proof. Let $F = \bar{F} \subset \mathscr{X}$ and let $f: F \to A_0$ be continuous. We must define a neighbourhood V of F such that $f \in A_0^V | F$. Now, $\mathscr{X}\tau_v(A_0 \cup A_1)$ implies the existence of an extension $h: E \to (A_0 \cup A_1)$ of f, where E is a closed neighbourhood of F. Let $E_j = h^{-1}(A_j)$. It follows that

$$E = E_0 \cup E_1, \quad F \subset E_0 \quad \text{and} \quad h(E_0 \cap E_1) \subset A_0 \cap A_1. \tag{16}$$

The last inclusion implies, by virtue of the relation $\mathscr{X}\tau_v(A_0 \cup A_1)$, the existence of an extension $g: Q \to (A_0 \cap A_1)$ of the mapping $h | E_0 \cap A_1$, where Q is a closed neighbourhood of $E_0 \cap E_1$ in E_1, i.e.

$$\overline{E_1 - Q} \cap E_0 = 0. \tag{17}$$

Let

$$V = E_0 \cup Q \quad \text{and} \quad f^* = (h | E_0) + g. \tag{18}$$

It follows that $f \subset f^* \in A_0^V$, because (compare (16) and (18)) $F \subset V$ and $E_0 \cap Q \subset E_0 \cap E_1$.

Finally, V is a neighbourhood of F. Actually, according to (16) and (17), it follows that

$$\overline{(E - V)} \cap F = \overline{E_1 - (E_0 \cup Q)} \cap F \subset \overline{E_1 - Q} \cap F \cap E_0 = 0,$$

which proves that V is a neighbourhood of F in E and therefore in $\mathscr{X}$ (because E is a neighbourhood of F in $\mathscr{X}$).

In order to prove the second part of Theorem 3, one has to set $E = \mathcal{X}$ and $Q = E_1$, which implies $V = \mathcal{X}$ by (16) and (18).

THEOREM 4. *Let $\mathcal{Y}_1, \mathcal{Y}_2, \ldots$ be a (finite or infinite) sequence of spaces; then $\mathcal{X}\tau(\mathcal{Y}_1\times\mathcal{Y}_2\times\ldots)$ if and only if $\mathcal{X}\tau\mathcal{Y}_k$ for every k. If the sequence is finite, then τ may be replaced by τ_v.*

Proof. Let $\mathcal{Z} = \mathcal{Y}_1\times\mathcal{Y}_2\times\ldots$. Assume that $\mathcal{X}\tau\mathcal{Z}$. We have to show that $\mathcal{X}\tau\mathcal{Y}_1$. Let p_k be a fixed point of $\mathcal{Y}_k$ for $k = 2, 3, \ldots$. Let $F = \overline{F} \subset \mathcal{X}$ and f: $F \to \mathcal{Y}_1$ continuous. Denote by g the function which assigns to $x \in F$ the point $[f(x), p_2, p_3, \ldots]$ of $\mathcal{Z}$. By hypothesis, there exists an extension h: $\mathcal{X} \to \mathcal{Z}$ of g. Set

$$h(x) = [h^1(x), h^2(x), \ldots];$$

it follows that $f \subset h^1 \in \mathcal{Y}_1^{\mathcal{X}}$. Therefore $\mathcal{X}\tau\mathcal{Y}_1$.

Conversely, assume that $\mathcal{X}\tau\mathcal{Y}_k$ for $k = 1, 2, \ldots$. Let $F = \overline{F} \subset \mathcal{X}$ and f: $F \to \mathcal{Z}$ be continuous. Let $f = [f^1, f^2, \ldots]$. For every k there exists by hypothesis a function g^k such that

$$f^k \subset g^k \in \mathcal{Y}_k^{\mathcal{X}}, \qquad \text{which implies} \qquad f \subset g = [g^1, g^2, \ldots] \in \mathcal{Z}^{\mathcal{X}}.$$

Therefore, $\mathcal{X}\tau\mathcal{Z}$.

In order to pass to the relation τ_v, it is necessary only to introduce the following changes in the foregoing argument,

(i) instead of h: $\mathcal{X} \to \mathcal{Z}$ set h: $E \to \mathcal{Z}$, where E is a neighbourhood of F,

(ii) instead of g^k: $\mathcal{X} \to \mathcal{Y}_k$ set g^k: $E_k \to \mathcal{Y}_k$, where E^k is a neighbourhood of F; then the set $E = E_1 \cap \ldots \cap E_n$ is a neighbourhood of F, and it follows that $f \subset g = [g^1, \ldots, g^n] \in \mathcal{Z}^E$.

THEOREM 5. *If $\mathcal{T}$ is compact, the condition $\mathcal{T}\times\mathcal{X}\tau\mathcal{Y}$ implies $\mathcal{X}\tau\mathcal{Y}^{\mathcal{T}}$, and the condition $\mathcal{T}\times\mathcal{X}\tau_v\mathcal{Y}$ implies $\mathcal{X}\tau_v\mathcal{Y}^{\mathcal{T}}$.*

Proof. Let $F = \overline{F} \subset \mathcal{X}$ and let f: $F \to \mathcal{Y}^{\mathcal{T}}$ be continuous. Thus f assigns a continuous function f_x: $\mathcal{T} \to \mathcal{Y}$ to every $x \in F$. Let

$$f_x(t) = g(t, x).$$

Hence g: $\mathcal{T}\times F \to \mathcal{Y}$ is continuous according to Theorem 2 of § 44, IV. Let $\mathcal{T}\times\mathcal{X}\tau\mathcal{Y}$. Then there exists an extension g^*: $\mathcal{T}\times\mathcal{X} \to \mathcal{Y}$ of g. Therefore the function f^*, defined by the identity

$$f_x^*(t) = g^*(t, x),$$

is continuous and $f \subset f^*$: $\mathcal{X} \to \mathcal{Y}^{\mathcal{T}}$.

If $\mathscr{T}\times\mathscr{X}\tau_v\mathscr{Y}$, there exist (compare § 41, IV, Theorem 1) a neighbourhood E of F and an extension g^*: $\mathscr{T}\times E\to\mathscr{Y}$ of g (g is defined as above). Then it follows that $f\subset f^*\epsilon(\mathscr{Y}^{\mathscr{X}})^E$.

Remark. The topology of $\mathscr{Y}^{\mathscr{X}}$ is, as usually, its compact-open topology. It is to be noted that the first part of Theorem 5 is true also for the topology of continuous convergence of $\mathscr{Y}^{\mathscr{X}}$ (compare § 20, VI, Definition 2) without the assumption of compactness of $\mathscr{T}$ (1).

Let us recall (compare § 13, V) that a subset A of $\mathscr{X}$ is said to be a *retract of* $\mathscr{X}$ if there exists a continuous mapping, called a *retraction*, f: $\mathscr{X}\to A$ such that $f(x)=x$ for $x\epsilon A$.

We shall say that A is a *neighbourhood retract of* $\mathscr{X}$ if there exists a retraction of a neighbourhood E of A into A.

THEOREM 6. *If* $\mathscr{X}\tau_v\mathscr{Y}$ *and if* $\mathscr{Z}$ *is a neighbourhood retract of* $\mathscr{Y}$, *then* $\mathscr{X}\tau_v\mathscr{Z}$.

If $\mathscr{X}\tau\mathscr{Y}$ *and if* $\mathscr{Z}$ *is a retract of* $\mathscr{Y}$, *then* $\mathscr{X}\tau\mathscr{Z}$.

Proof. Let $F=\overline{F}\subset\mathscr{X}$ and f: $F\to\mathscr{Z}$ a continuous function. By hypothesis

$$f\subset g\epsilon\mathscr{Y}^E \quad\text{and}\quad r\colon G\to\mathscr{Z},$$

where E is a neighbourhood of F, G a neighbourhood of $\mathscr{Z}$ and r a retraction of G onto $\mathscr{Z}$.

Thus the set $H=E\cap g^{-1}(G)$ is a neighbourhood of F, and it follows that $f\subset rg\epsilon\mathscr{Z}^H$. Therefore $\mathscr{X}\tau_v\mathscr{Z}$.

In order to establish the second part of the theorem, it should be assumed in the foregoing argument that $E=\mathscr{X}$ and $G=\mathscr{Y}$. Then it follows that $H=\mathscr{X}$, which implies $\mathscr{X}\tau\mathscr{Z}$.

THEOREM 7. $\mathscr{Y}$ *is a retract of* $\mathscr{Y}^{\mathscr{T}}$ (*if* $\mathscr{T}$ *is compact metric and* $\neq 0$). *Consequently* (compare Theorem 6), *the condition* $\mathscr{X}\tau_v\mathscr{Y}^{\mathscr{T}}$ *implies* $\mathscr{X}\tau_v\mathscr{Y}$, *and* $\mathscr{X}\tau\mathscr{Y}^{\mathscr{T}}$ *implies* $\mathscr{X}\tau\mathscr{Y}$.

Proof. Let $t_0\epsilon\mathscr{T}$. Let us identify the space $\mathscr{Y}$ with the family of constant functions belonging to $\mathscr{Y}^{\mathscr{T}}$ and let us assign to every function $f\epsilon\mathscr{Y}^{\mathscr{T}}$ the constant function $f(t_0)$. This correspondence is a retraction of $\mathscr{Y}^{\mathscr{T}}$ onto $\mathscr{Y}$. It is continuous, because

$$\lim_{n=\infty} f_n=f \quad\text{implies}\quad \lim_{n=\infty} f_n(t_0)=f(t_0),$$

and it is the identity mapping on $\mathscr{Y}$.

(1) See also Sze-Tsen Hu, *Theory of retracts*, Detroit 1965, p. 187.

III. Absolute retracts.

DEFINITION. The metric space $\mathscr{Y}$ is said to be an *absolute retract* (AR), respectively an *absolute neighbourhood retract* (ANR), if

$$\mathscr{X} \tau \mathscr{Y} \quad \text{respectively} \quad \mathscr{X} \tau_v \mathscr{Y}$$

for any (metric separable) space $\mathscr{X}$ (¹).

EXAMPLES AND REMARKS. (i) According to the Tietze theorem, *the euclidean space* $\mathscr{E}^n$ *and the cube* $\mathscr{I}^n$ *are* AR's.

Moreover, *every convex subset of the space* $\mathscr{E}^{\aleph_0}$ *is an* AR (see § 28, IX, Theorem 1, Remark).

(ii) Clearly

$$0^{\mathscr{X}} = 0 \quad \text{for} \quad \mathscr{X} \neq 0 \quad \text{and} \quad 0^0 = (0),$$

and hence the empty set is not an AR. However it is an ANR.

(iii) Every AR is *arcwise connected* and every ANR is *locally arcwise connected.*

(iv) Every polyhedron is an ANR (²). *Every dendrite is an* AR (see Theorem 16, and for a stronger statement, see § 54, VII, Theorem 7).

This follows easily (by the finite induction) from Theorem 1 below.

(v) *The Betti groups of* (*compact*) ANR's *have finite number of generators* (³).

(vi) There exist (in the space $\mathscr{E}^3$) AR's which are not *finite unions* of (smaller) AR's (⁴).

Moreover, there exists a 2-dimensional AR, no proper closed and 2-dimensional subset of which is an AR (⁵).

(¹) This concept is due to K. Borsuk. See *Sur les rétractes,* Fund. Math. 17 (1931), p. 152. The terminology is connected with the theorem of Section IV. For a more detailed exposition, see the above-mentioned book of S. Hu, and K. Borsuk, *Theory of retracts,* Monografie Matematyczne 44 (Warszawa 1967), and a number of papers of this author in Fund. Math. 37, 46 and 51, and in Bull. Acad. Polon. Sci. 9 (1961). Compare also R. Molski, Fund. Math. 57 (1965), pp. 121–145.

(²) K. Borsuk, Fund. Math. 19 (1932), p. 227.

(³) Compare K. Borsuk, Fund. Math. 21 (1933), p. 97, and S. Lefschetz, Ann. of Math. 35 (1934), p. 129.

(⁴) K. Borsuk and S. Mazurkiewicz, *Sur les rétractes absolus indécomposables,* C. R. Paris 199 (1934), p. 110.

(⁵) K. Borsuk, Acta Szeged, 1950.

(vii) If $\mathcal{Y}$ is a finite dimensional space, the condition $\mathcal{Y}\times\mathcal{I}\tau\mathcal{Y}$ implies that $\mathcal{Y}$ is an AR, and the condition $\mathcal{Y}\times\mathcal{I}\tau_v\mathcal{Y}$ implies that $\mathcal{Y}$ is an ANR (compare § 54, VII, Theorem 6).

Theorems 2, 3 and 4 of Section II imply directly the following ones (1).

THEOREM 1. *Let A_0 and A_1 be two closed sets; if A_0, A_1 and $A_0\cap A_1$ are* AR's *(respectively* ANR's*), then so is $A_0\cup A_1$; if $A_0\cup A_1$ and $A_0\cap A_1$ are* AR's *(respectively* ANR's*), then so are the sets A_0 and A_1.*

Remark. However, the hypothesis that A_0, A_1 and $A_0\cup A_1$ are AR's does not imply that $A_0\cap A_1$ is an AR (even for polyhedra) (2).

THEOREM 2. *The cartesian product $\mathcal{Y}_1\times\mathcal{Y}_2\times\ldots$ (respectively the finite product $\mathcal{Y}_1\times\ldots\times\mathcal{Y}_n$) is an* AR *(respectively an* ANR*) if and only if so is every $\mathcal{Y}_k$.*

Remark. The second part of Theorem 2 concerning ANR's does not admit any extension to *infinite* sequences. For, let all $\mathcal{Y}_k$ be identical and consist of two elements; then the space $\mathcal{Y}_1\times\mathcal{Y}_2\times\ldots$ is homeomorphic to the Cantor set $\mathcal{C}$, and hence is not an ANR whereas each $\mathcal{Y}_k$ is such.

THEOREM 3. *Let $\mathcal{T}$ be compact metric. Then the space $\mathcal{Y}^{\mathcal{T}}$ is an* AR, *respectively an* ANR *if and only if so is $\mathcal{Y}$.*

Consequently, *$\mathcal{I}^{\mathcal{T}}$ and $\mathcal{E}^{\mathcal{T}}$ are* AR's *and, if $\mathcal{Y}$ is a closed polyhedron, $\mathcal{Y}^{\mathcal{T}}$ is an* ANR.

Proof. If $\mathcal{Y}^{\mathcal{T}}$ is an AR (respectively and ANR), then so is $\mathcal{Y}$ by Theorem 7 of Section II.

Conversely, if $\mathcal{Y}$ is an AR, then $\mathcal{T}\times\mathcal{X}\tau\mathcal{Y}$ for every $\mathcal{X}$, and hence $\mathcal{X}\tau\mathcal{Y}^{\mathcal{T}}$ by Theorem 5 of Section II. But this means that $\mathcal{Y}^{\mathcal{T}}$ is an AR.

Similarly, if $\mathcal{Y}$ is an ANR, then $\mathcal{Y}^{\mathcal{T}}$ is an ANR.

Let us quote the following statement without proof.

THEOREM 4. *If $\mathcal{X}$ is a locally connected continuum, the space $2^{\mathcal{X}}$ is an* AR (3).

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(1) Compare N. Aronszajn and K. Borsuk, *Sur la somme et le produit combinatoire des rétractes absolus*, Fund. Math. 18 (1932), p. 193.

(2) See E. G. Begle, *Intersections of contractible polyhedra*, Bull. Amer. Math. Soc. 49 (1943), p. 386.

(3) For a proof, see M. Wojdysławski, *Rétractes absolus et hyperespaces des continus*, Fund. Math. 32 (1939), p. 184.

THEOREM 5. *In order that a (metric separable) space $\mathcal{Y}$ be an absolute retract (respectively an absolute neighbourhood retract), it is necessary and sufficient that for every space $\mathcal{Z}$, which contains $\mathcal{Y}$ as a closed subset, $\mathcal{Y}$ be a retract of $\mathcal{Z}$ (respectively a neighbourhood retract of $\mathcal{Z}$).*

Proof. 1. If $\mathcal{Y}$ is an AR, then $\mathcal{Z}\tau\mathcal{Y}$. Therefore, if $f\colon \mathcal{Y}\to\mathcal{Y}$ is the identity mapping, it follows that $f\in\mathcal{Y}^{\mathcal{Z}}|\mathcal{Y}$, which shows that $\mathcal{Y}$ is a retract of $\mathcal{Z}$. If $\mathcal{Y}$ is an ANR, then $f\in\mathcal{Y}^{E}|\mathcal{Y}$, where E is a neighbourhood of $\mathcal{Y}$ in $\mathcal{Z}$.

2. In order to prove that the condition is sufficient, assume that $F=\overline{F}\subset\mathcal{X}$ and that $f\colon F\to\mathcal{Y}$ is a continuous function. By Corollary 2a of § 28, IX, $\mathcal{Y}$ may be considered as a closed subset of a space $\mathcal{Z}$ such that $f\in\mathcal{Z}^{\mathcal{X}}|F$. Hence there exists a continuous function $g\colon \mathcal{X}\to\mathcal{Z}$ such that $g(x)=f(x)$ for $x\in F$. Assuming that r is a retraction of $\mathcal{Z}$ onto $\mathcal{Y}$ (respectively of an open set E onto $\mathcal{Y}$), it follows that $f\subset rg$, which yields $\mathcal{X}\tau\mathcal{Y}$ (respectively $\mathcal{X}\tau_v\mathcal{Y}$).

THEOREM 6. *A retract of an* AR *is an* AR. *A neighbourhood retract of an* ANR *is an* ANR.

Proof. Let $\mathcal{Y}$ be an ANR, G an open subset of $\mathcal{Y}$, $r\colon G\to A$ a retraction, $F=\overline{F}\subset\mathcal{X}$ and $f\colon F\to A$ a continuous function. Since $\mathcal{Y}$ is an ANR, it follows that $f\subset f^*\in\mathcal{Y}^E$, where E is a neighbourhood of F. Let $H=f^{*-1}(G)$. It follows that $F\subset H$; thus the set H is a neighbourhood of F and $f\subset rf^*\in A^H$. Therefore A is an ANR.

If $\mathcal{Y}$ is an AR, one proceeds in a similar way setting $G=\mathcal{Y}$ and $E=\mathcal{X}$. Then, it follows that $H=\mathcal{X}$ and $f\subset rf^*\in A^{\mathcal{X}}$. Therefore A is an AR.

THEOREM 7. *The compact* AR*'s coincide (topologically) with the retracts of the Hilbert cube $\mathcal{I}^{\aleph_0}$. The compact* ANR*'s coincide with the retracts of open subsets of $\mathcal{I}^{\aleph_0}$* (1).

Proof. On the one hand, since $\mathcal{I}^{\aleph_0}$ is an AR, every retract of $\mathcal{I}^{\aleph_0}$ is an AR and every retract of an open subset of $\mathcal{I}^{\aleph_0}$ is an ANR (by Theorem 6).

On the other hand, if a closed subset $\mathcal{Y}$ of $\mathcal{I}^{\aleph_0}$ is an AR, it is a retract of $\mathcal{I}^{\aleph_0}$ (by Theorem 5); if it is an ANR, it is a retract of an open subset of $\mathcal{I}^{\aleph_0}$.

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(1) K. Borsuk, *loc. cit.* Fund. Math. 19 (1932), p. 223.

THEOREM 8. *Let $\mathscr{Y}$ be a compact space. If $\mathscr{I}^{\aleph_0}\tau\mathscr{Y}$, then $\mathscr{Y}$ is an* AR. *If $\mathscr{I}^{\aleph_0}\tau_v\mathscr{Y}$, then $\mathscr{Y}$ is an* ANR.

Proof. Consider $\mathscr{Y}$ as a subset of $\mathscr{I}^{\aleph_0}$, and let $f\colon \mathscr{Y}\to\mathscr{Y}$ be the identity mapping. If $\mathscr{I}^{\aleph_0}\tau\mathscr{Y}$, then $f\subset r\in\mathscr{Y}^{\mathscr{I}^{\aleph_0}}$, i.e. $\mathscr{Y}$ is a retract of $\mathscr{I}^{\aleph_0}$, and hence is an AR by Theorem 7.

In a similar way, if $\mathscr{I}^{\aleph_0}\tau_v\mathscr{Y}$, then $\mathscr{Y}$ is a retract of an open subset of $\mathscr{I}^{\aleph_0}$, and hence an ANR.

LEMMA 9. *Let $F=\overline{F}\subset\mathscr{X}$. Assume*

$$F^0 = F\times\mathscr{I}\cup\mathscr{X}\times 0. \tag{0}$$

Then

$$\mathscr{Y}^{\mathscr{X}\times\mathscr{I}}|_v F^0 = \mathscr{Y}^{\mathscr{X}\times\mathscr{I}}|F^0;$$

in other words, every continuous function $h\colon F^0\to\mathscr{Y}$, which admits an extension to a neighbourhood E of F^0, also admits an extension to the whole space $\mathscr{X}\times\mathscr{I}$.

Proof. Since $\mathscr{I}$ is compact, there exists an open set G such that $F\subset G$ and $G\times\mathscr{I}\subset E$.

Let(1) $\varphi\colon\mathscr{X}\to\mathscr{I}$ be a continuous function such that

$$\varphi(x)=\begin{cases}1 & \text{if} \quad x\in F,\\ 0 & \text{if} \quad x\in\mathscr{X}-G,\end{cases}$$

for instance,

$$\varphi(x)=\frac{\varrho(x,\mathscr{X}-G)}{\varrho(x,\mathscr{X}-G)+\varrho(x,F)}.$$

According to the hypothesis, let h_0 be an extension of h to $G\times\mathscr{I}\cup\mathscr{X}\times 0$.

Put

$$h^*(x,t)=h_0\big(x,t\cdot\varphi(x)\big).$$

It follows that

$h^*\colon\mathscr{X}\times\mathscr{I}\to\mathscr{Y}$ is continuous and $h^*(x,t)=h(x,t)$ for $(x,t)\in F^0$.

THEOREM 10. *Let $\mathscr{Y}$ be an* ANR. *If the space $\mathscr{Y}^{\mathscr{Y}}$ is arcwise connected, then $\mathscr{Y}$ is an* AR(2).

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(1) Compare the argument of C. H. Dowker, Amer. Journ. Math. 69 (1947), p. 232.

(2) For compact $\mathscr{Y}$, see K. Borsuk, Fund. Math. 19 (1932), p. 229.

Proof. Let $\mathscr{X}$ be a space containing $\mathscr{Y}$ as a closed subset. We have to show (compare Theorem 5) that $\mathscr{Y}$ is a retract of $\mathscr{X}$.

Let $f_i: \mathscr{Y} \to \mathscr{Y}$ $(i = 0, 1)$ be functions such that $f_0(y) = c$ and $f_1(y) = y$ for $y \in \mathscr{Y}$. By hypothesis, there exists an arc joining f_0 to f_1 in the space $\mathscr{Y}^{\mathscr{Y}}$. So let $g: \mathscr{I} \to \mathscr{Y}^{\mathscr{Y}}$ be a continuous function such that $g_0 = f_0$ and $g_1 = f_1$. Assume

$$h(x, t) = g_t(x) \quad \text{for } x \in \mathscr{Y} \quad \text{and} \quad h(x, 0) = c \quad \text{for } x \in \mathscr{X}.$$

Put $\mathscr{Y}^0 = (\mathscr{Y} \times \mathscr{I}) \cup (\mathscr{X} \times 0)$.

It follows that $h: \mathscr{Y}^0 \to \mathscr{Y}$ is continuous according to § 44, IV, Theorem 3. Since $\mathscr{Y}$ is an ANR, the preceding lemma implies that $h \subset h^* \in \mathscr{Y}^{\mathscr{X} \times \mathscr{I}}$.

Let $r(x) = h^*(x, 1)$. Then $f_1 \subset r \in \mathscr{Y}^{\mathscr{X}}$, and hence $r: \mathscr{X} \to \mathscr{Y}$ is a retraction.

THEOREM 11. *Every compact* AR *has the fixed point property*(¹).

Proof. Theorem 11 follows from Theorem 7 and from Theorems 12 and 14 below.

THEOREM 12. *If $\mathscr{Y}$ is a retract of a space $\mathscr{X}$ which has the fixed point property, then so does $\mathscr{Y}$.*

Proof. Let $f: \mathscr{X} \to \mathscr{Y}$ be a retraction. Let $g: \mathscr{Y} \to \mathscr{Y}$ be a continuous function. Since $gf: \mathscr{X} \to \mathscr{Y}$, therefore $gf: \mathscr{X} \to \mathscr{X}$ and there is by hypothesis a point x_0 such that $gf(x_0) = x_0$. Since $gf(x_0) \in \mathscr{Y}$, it follows that $x_0 \in \mathscr{Y}$, which implies $f(x_0) = x_0$, and hence $(gx_0) = x_0$.

LEMMA 13. *Let $\mathscr{X}$ be a compact space. If to every $\varepsilon > 0$ there corresponds a continuous function $f: \mathscr{X} \to \mathscr{X}$ such that*

$$|f(x) - x| < \varepsilon \tag{1}$$

and if the set $f(\mathscr{X})$ has the fixed point property, then so does $\mathscr{X}$.

Proof. Suppose conversely that the continuous function $g: \mathscr{X} \to \mathscr{X}$ has no fixed point. Since $\mathscr{X}$ is compact, there exists an $\varepsilon > 0$ such that

$$|g(x) - x| > \varepsilon \tag{2}$$

(¹) i.e., for every continuous function $f: \mathscr{X} \to \mathscr{X}$ there exists a point x such that $f(x) = x$. Compare *ibid.* p. 230.

for any x. Let f be a function satisfying the hypothesis, and define $h(x) = fg(x)$ and $F = f(\mathscr{X})$. Since $h\colon \mathscr{X} \to F$ is continuous, then so is $(h|F)\colon F \to F$, and there is a point $x_0 \in F$ such that $h(x_0) = x_0$, i.e. $fg(x_0) = x_0$. According to (1), $|fg(x_0) - g(x_0)| < \varepsilon$, and hence $|x_0 - g(x_0)| < \varepsilon$, contrary to (2).

THEOREM 14. *The space* $\mathscr{I}^{\aleph_0}$ *has the fixed point property*(1).

Proof. Write the points $z \in \mathscr{I}^{\aleph_0}$ in the form $z = [z^1, z^2, \ldots]$ where $0 \leqslant z^n \leqslant 1$, and define

$$f_n(z) = [z^1, \ldots, z^n, 0, 0, \ldots].$$

It follows that $|f_n(z) - z| < 1/2^n$, and the set $f_n(\mathscr{I}^{\aleph_0})$, being a homeomorphic image of the closure of an n-dimensional simplex, has the fixed point property (by § 28, III, Theorem 1).

By Lemma 13 the space $\mathscr{I}^{\aleph_0}$ has also the fixed point property.

THEOREM 15. *Every cyclic element (and, more generally, every closed subset completely arcwise connected) of an* AR *is an* AR.

Furthermore, if $\mathscr{X}$ *is a locally connected continuum, every cyclic element (and, more generally, every closed set completely arcwise connected) is its retract.*

Proof. Let F be a closed, completely arcwise connected set. Let $R_1, R_2, \ldots$ be the sequence of components of $\mathscr{X} - F$. By Theorem 4 of § 52, I, the set $\mathrm{Fr}(R_n)$ consists of a single point p_n. Define $f(x) = p_n$ for $x \in R_n$ and $f(x) = x$ for $x \in F$. Since $\lim\limits_{n=\infty} \delta(R_n) = 0$ (compare § 52, I, Theorem 7), the function f is continuous. Therefore F is a retract of $\mathscr{X}$.

On the other hand, the following theorem can be proved(2).

THEOREM 16. *If every cyclic element of a locally connected continuum is an* AR, *the whole continuum is an* AR.

In particular, *every dendrite is an* AR.

Let us mention the following theorems without proof.

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(1) Compare J. Schauder, *Der Fixpunktsatz in Funktionalräumen*, Studia Math. 2 (1930), p. 173. Compare also G. D. Birkhoff and O. D. Kellog, *Invariant point in function space*, Trans. Amer. Math. Soc. 23 (1922), p. 96.

(2) See K. Borsuk, *Einige Sätze über stetige Streckenbilder*, Fund. Math. 18 (1932), p. 211.

THEOREM 17 (of imbedding). *If $\mathscr{Y}$ is a compact space ($\subset \mathscr{I}^{\aleph_0}$), there exists an infinite polyhedron P such that $\mathscr{Y} \cup P$ is an absolute retract* (1).

This theorem is a generalization of Theorem 1 of § 28, IX.

THEOREM 18 (of imbedding). *For each n there is an $n+1$-dimensional* AR *which contains topologically every metric separable n-dimensional space* (2).

THEOREM 19. *Let $\mathscr{X}$ be a compact space, F a closed set and $f\colon \mathscr{X} \to \mathscr{Y}$ a continuous function such that $(f|\mathscr{X}-F)\colon \mathscr{X}-F \to \mathscr{Y}-f(F)$ is a homeomorphic mapping. If $\mathscr{X}$, F and $f(F)$ are* ANR's, *then so is $\mathscr{Y}$* (3).

Remarks on the local characterization of ANR*'s.*

A space is said to be an ANR *at the point p* if there exists a neighbourhood of p which is an ANR.

THEOREM 20 (of Hanner). *A space is an* ANR *if and only if it is locally an* ANR *at each of its points* (4).

The argument is based on the following theorems.

(i) *Every open subset of an* ANR *is an* ANR.

(ii) *A union of open* ANR*'s is an* ANR.

IV. Connectedness in dimension n. The case where $\mathscr{I}^n \tau \mathscr{Y}$. The space $\mathscr{Y}$ is said to be *integrally connected in dimension n* if

$$\mathscr{Y}^{\mathscr{S}_n} = \mathscr{Y}^{\mathscr{Q}_{n+1}} | \mathscr{S}_n,$$

which means that every continuous function $f\colon \mathscr{S}_n \to \mathscr{Y}$ has a continuous extension $f^*\colon \mathscr{Q}_{n+1} \to \mathscr{Y}$ (5).

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(1) For a proof, see K. Borsuk, *Sur le plongement des espaces dans les rétractes absolus*, Fund. Math. 27 (1936), p. 240.

(2) For the proof, see H. G. Bothe, *Eine Einbettung m-dimensionaler Mengen in einen $(m+1)$-dimensionalen absoluten Retrakt*, Fund. Math. 52 (1963), pp. 209–224.

(3) See J. H. C. Whitehead, *Note on a theorem of Borsuk*, Bull. Amer. Math. Soc. 54 (1948), p. 1125, and (in case of finite dimension) K. Borsuk, Fund. Math. 24 (1935), p. 250.

(4) For the proof, see O. Hanner, *Some theorems on absolute neighborhood retracts*, Arkiv f. Mat. 1 (1951), pp. 389–408.

For the case of a compact space, see T. Yajima, *On a local property of absolute neighborhood retracts*, Osaka Math. Journ. 2 (1950), pp. 59–62.

(5) Recall that

$$\mathscr{Q}_{n+1} = \underset{p}{E}[(|p| \leqslant 1)(p \in \mathscr{E}^{n+1})] \quad \text{and} \quad \mathscr{S}_n = \underset{p}{E}[(|p| = 1)(p \in \mathscr{E}^{n+1})].$$

The space $\mathscr{Y}$ is said to be *locally connected in dimension n at the point p* if to every $\varepsilon > 0$ corresponds an $\eta > 0$ such that each continuous function $f\colon \mathscr{S}_n \to \mathscr{Y}$, for which $\delta[p \cup f(\mathscr{S}_n)] < \eta$, has a continuous extension $f^*\colon \mathscr{Q}_{n+1} \to \mathscr{Y}$ such that $f^*(0) = p$ and $\delta[p \cup f^*(\mathscr{Q}_{n+1})] < \varepsilon$(1).

$\mathscr{Y}$ is said to be *locally connected in dimension n* if it has this property at every point.

More concisely, l.c. n space (respectively i.c. n space) will mean space locally (respectively integrally) connected in dimensions $< n$. We shall use also the denomination of *LC^n space* for $\mathscr{Y}$ such that for every $\varepsilon > 0$ there is $\eta > 0$ such that every continuous $f\colon \mathscr{S}_m \to \mathscr{Y}$, $m = -1, 0, \ldots, n$, for which $\delta f(\mathscr{S}_m) < \eta$, admits a continuous extension $f^*\colon \mathscr{Q}_{m+1} \to \mathscr{Y}$ such that $\delta f^*(\mathscr{Q}_{m+1}) \leqslant \varepsilon$.

Clearly, if $\mathscr{Y}$ is compact, then it is LC^n if and only if it is l.c. $n+1$.

In order to establish some conditions which characterize the l.c. n spaces, we shall consider first an auxiliary concept.

Let $f \in \mathscr{Y}^{\mathscr{Q}_{n+1}} | \mathscr{S}_n$. Define

$$\chi(f) = \inf \delta[f^*(\mathscr{Q}_{n+1})],$$

where the variable f^* ranges over $\mathscr{Y}^{\mathscr{Q}_{n+1}}$ satisfying the condition $f \subset f^*$.

Obviously $\chi(f) = 0$, if the function f is constant.

The following lemma is easily proved for a l.a.c. space $\mathscr{Y}$.

LEMMA. *In order that $\mathscr{Y}$ be locally connected in dimension n at the point p, it is necessary and sufficient that χ be defined in a neighbourhood (relative to $\mathscr{Y}^{\mathscr{S}_n}$) of the constant function f with the value p, and that f be its point of continuity.*

In other words, *that the condition* $\lim_{j=\infty} \delta[p \cup f_j(\mathscr{S}_n)] = 0$ (*which means that the sequence $f_1, f_2, \ldots$ converges uniformly to the constant function f*), *implies* $\lim_{j=\infty} \chi(f_j) = 0$.

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(1) This concept is due to J. W. Alexander and S. Lefschetz. See S. Lefschetz, Ann. of Math. 35 (1934), p. 119. See also my paper, *Sur les espaces localement connexes et péaniens en dimension n*, Fund. Math. 24 (1935), p. 269.

THEOREM 1. *The following conditions are equivalent (for $\mathscr{Y} \neq 0$).*

$$\mathscr{Y} \text{ is l.c. } n, \tag{1}$$

$$\mathscr{Y} \text{ is a neighbourhood retract of every space } \mathscr{Z} \text{ such that } \mathscr{Y} \text{ is closed in } \mathscr{Z} \text{ and } \dim(\mathscr{Z}-\mathscr{Y}) \leqslant n, \tag{2}$$

$$\text{if } F = \overline{F} \subset \mathscr{X} \text{ and } \dim(\mathscr{X}-F) \leqslant n, \text{ then } \mathscr{Y}^F = \mathscr{Y}^{\mathscr{X}}|_v F, \tag{3}$$

$$\text{if } \dim\mathscr{X} \leqslant n, \text{ then } \mathscr{X}\tau_v\mathscr{Y}, \tag{4}$$

$$\mathscr{I}^n\tau_v\mathscr{Y}, \tag{5}$$

$$\text{if } T_k = \bigcup_{j=1}^{\infty} \mathop{E}_{p}[(|p| = 1/j)(p \in \mathscr{Q}_{k+1})] \cup \text{the point } 0, \text{ we have } \mathscr{Y}^{T_k} = \mathscr{Y}^{\mathscr{Q}_{k+1}}|_v T_k \text{ for } k < n. \tag{6}$$

THEOREM 1$'$. *The condition*

$$\mathscr{Y} \text{ is l.c. } n \text{ and i.c. } n \tag{1$'$}$$

is equivalent to each of the conditions (denote them (2$'$) through (6$'$)) *which are obtained respectively from* (2) *through* (6) *replacing words "neighbourhood retract" by "retract" and deleting the subscript v.*

Proof of Theorem 1. The proof of Theorem 1$'$ can easily be derived from the following (replacing E and G by $\mathscr{X}$ and $\mathscr{V}$ by $\mathscr{Z}$).

(1) $\to$ (2). If $\overline{\mathscr{Y}} = \mathscr{Y}$ and $\dim(\mathscr{Z}-\mathscr{Y}) \leqslant n$, then by Corollary 2a of § 28, IX, there exist a space $\mathscr{X}$ containing $\mathscr{Y}$, such that $\mathscr{Y}$ is closed in $\mathscr{X}$ and $\mathscr{X}-\mathscr{Y}$ is an infinite polyhedron of dimension $\leqslant n$, and a continuous function g: $\mathscr{Z} \to \mathscr{X}$ such that $g(y) = y$ for $y \in \mathscr{Y}$. Thus it is sufficient to show that $\mathscr{Y}$ is a retract of one of its neighbourhoods E in $\mathscr{X}$ (because $g^{-1}(E)$ is a neighbourhood of $\mathscr{Y}$ in $\mathscr{Z}$). The last statement will be proved by induction; we are going to show

(i) that it holds when $\dim(\mathscr{X}-\mathscr{Y}) = 0$,

(ii) that if it holds for $\dim(\mathscr{X}-\mathscr{Y}) \leqslant k$, then it holds also for $\dim(\mathscr{X}-\mathscr{Y}) = k+1$.

Now, in the case where $\mathscr{X}-\mathscr{Y}$ is a 0-dimensional (infinite) polyhedron, it is a discrete set consisting of a (finite or infinite) sequence of points $p_1, p_2, \ldots$. Denote by $f_0(p_i)$ a point of the set $\mathscr{Y}$ (which is non-void by hypothesis) such that

$$|f_0(p_i)-p_i| < 2\varrho(p_i, \mathscr{Y});$$

Thus a retraction of $\mathscr{X}$ onto $\mathscr{Y}$ is defined.

Now consider (ii). Let the infinite polyhedron $\mathscr{X}-\mathscr{Y}$ (of dimension $k+1$) be presented as the union of simplexes of an infinite complex; denote by R the union of all at most k-dimensional simplexes and by $D_1, D_2, \ldots$ denote the $(k+1)$-dimensional simplexes. Thus the simplexes D_i are disjoint and their boundaries $\bar{D}_i - D_i$ are contained in R. Furthermore we may assume that $\lim_{i=\infty} \delta(D_i) = 0$ (1).

Since $\dim R \leqslant k$, there exist by hypothesis a neighbourhood G of $\mathscr{Y}$ (in $\mathscr{X}$) and a continuous function $f\colon (R \cap G \cup \mathscr{Y}) \to \mathscr{Y}$ which is the identity mapping on $\mathscr{Y}$. Define $f_i = f|\bar{D}_i - D_i$ for $\bar{D}_i \subset G$. Let E be the union of $R \cap G \cup \mathscr{Y}$ and of all simplexes D_i (contained in G) such that the function f_i has an extension $f_i^*\colon D_i \to \mathscr{Y}$. Choose f_i^* so that

$$\delta[f_i^*(\bar{D}_i)] \leqslant 2\chi(f_i). \tag{7}$$

The simplexes D_i are mutually disjoint and disjoint from $\mathscr{Y}$, so let $f^* = f + f_1^* + f_2^* + \ldots$. We have to show that E is a neighbourhood of $\mathscr{Y}$ and that the function f^* is continuous.

Let $p \in \mathscr{Y}$ and $p = \lim_{j=\infty} p_j$, $p_j \in \mathscr{X} - (\mathscr{Y} \cup R)$. Since $\bar{D}_i \cap \mathscr{Y} = 0$, assume, setting $p_j \in D_{i_j}$, that all the subscripts i_j are distinct. Consequently

$$\lim_{j=\infty} \delta(D_{i_j}) = 0 \tag{8}$$

and

$$\lim_{j=\infty} \delta[p \cup D_{i_j}] = 0. \tag{9}$$

Since $p \in G$, it follows for sufficiently large j that

$$\bar{D}_{i_j} \subset G, \quad \text{and hence} \quad \bar{D}_{i_j} - D_{i_j} \subset R \cap G,$$

so that the function f is defined on the set $\bar{D}_{i_j} - D_{i_j}$, homeomorphic to $\mathscr{S}_k$. Since the function f is continuous at the point p, condition (9) implies that

$$\lim_{j=\infty} \delta[f(p) \cup f(\bar{D}_{i_j} - D_{i_j})] = 0. \tag{10}$$

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(1) Compare Alexandrov–Hopf, *Topologie* I, p. 144, "Hilfssatz".

Since the space $\mathscr{Y}$ is locally connected in dimension k at the point p, condition (10), which is obviously equivalent to

$$\lim_{j=\infty}\delta[p\cup f_{i_j}(\overline{D}_{i_j}-D_{i_j})]=0, \tag{11}$$

implies that $\lim\limits_{j=\infty}\chi(f_{i_j})=0$ (compare the lemma).

Therefore, starting with a sufficiently large j, each function f_{i_j} has an extension to $\overline{D}_{i_j}$. This shows that $D_{i_j}\subset E$, thus $p_{i_j}\in E$. Therefore p is an interior point of E.

Moreover, condition (7) implies that

$$\lim_{j=\infty}\delta[f^*_{i_j}(\overline{D}_{i_j})]=0.$$

Taking into account (11) and the condition

$$0\neq f_{i_j}(\overline{D}_{i_j}-D_{i_j})\subset f^*_{i_j}(\overline{D}_{i_j}),$$

we get

$$\lim_{j=\infty}\delta[p\cup f^*_{i_j}(\overline{D}_{i_j})]=0,\quad\text{which implies}\quad \lim_{j=\infty}f^*(p_{i_j})=p.$$

And hence the function f^* is continuous.

(2) $\to$ (3). If $F=\overline{F}\subset\mathscr{X}$, $\dim(\mathscr{X}-F)\leqslant n$ and $f\colon F\to\mathscr{Y}$ is continuous, there exist (by the cited Corollary 2a of § 28, IX) a space $\mathscr{Z}$ containing $\mathscr{Y}$, such that $\dim(\mathscr{Z}-\mathscr{Y})\leqslant n$ and that $\mathscr{Y}$ is closed in $\mathscr{Z}$, and an extension $f_0\colon\mathscr{X}\to\mathscr{Z}$ of f. Assume that condition (2) holds; then there exists a neighbourhood V of $\mathscr{Y}$ (in $\mathscr{Z}$) and a retraction $g\colon V\to\mathscr{Y}$. The composite function $f^*=gf_0$ is the required function, for, it is an extension of f defined on the set $E=f_0^{-1}(V)$, which is a neighborhood of F (in $\mathscr{X}$).

The implications (3) $\to$ (4) $\to$ (6) are obvious.

(6) $\to$ (1). According to the Lemma, we have to show that under the hypothesis (6) the following holds. If $k<n$, $p\in\mathscr{Y}$ and $f_1,f_2,\ldots$ is a sequence of functions belonging to $\mathscr{Y}^{\mathscr{S}_k}$ and converging to p, then $\lim\limits_{j=\infty}\chi(f_j)=0$.

Define $f(x)=f_j(jx)$ for $|x|=1/j$ and $f(0)=p$.

Obviously $f\in\mathscr{Y}^{T_k}$. Therefore, it follows by (6) that $f\subset f^*\in\mathscr{Y}^{E_k}$, where E_k is a neighbourhood of T_k. Since $0\in T_k$, there is a subscript j_0 such that $x\in E_k$ for $|x|\leqslant 1/j_0$. Let

$$f^*_j(x)=f^*\left(\frac{x}{j}\right)\quad\text{for}\quad x\in\mathscr{Q}_{k+1}\quad\text{and}\quad j\geqslant j_0.$$

It follows for $x \in \mathscr{S}_k$ that

$$f_j(x) = f\left(\frac{x}{j}\right) = f^*\left(\frac{x}{j}\right) = f_j^*(x), \quad \text{which implies} \quad f_j \subset f_j^* \in \mathscr{Y}^{\mathscr{Q}_{k+1}},$$

and since $f^*(0) = f(0) = p$, we have

$$\lim_{x=0} f^*(x) = p, \text{ thus } \lim_{j=\infty} \delta[f_j^*(\mathscr{Q}_{k+1})] = 0, \text{ hence } \lim_{j=\infty} \chi(f_j) = 0.$$

COROLLARIES 2 AND 2′. *Each of the following conditions is equivalent to condition* (1) (*for* $\mathscr{Y} \neq 0$).

$$\textit{If } \mathscr{T} \textit{ is compact and } \dim \mathscr{T} = k < n, \textit{ then } \mathscr{I}^{n-k} \tau_v \mathscr{Y}^{\mathscr{T}}. \tag{12}$$

$$\textit{If } k < n, \textit{ then } \mathscr{I} \tau_v \mathscr{Y}^{\mathscr{S}_k}. \tag{13}$$

Condition (1′) *is equivalent to conditions* (12′) *and* (13′) *which are obtained from* (12) *and* (13) *deleting the subscript* v; *in that case, the hypothesis of compactness of the space* $\mathscr{T}$ *may be omitted.*

Proof. Condition (4) implies (12), because if $\dim \mathscr{T} = k < n$, we obtain (compare § 27, VIII)

$$\dim(\mathscr{T} \times \mathscr{I}^{n-k}) \leqslant n, \text{ thus } (\mathscr{T} \times \mathscr{I}^{n-k}) \tau_v \mathscr{Y}, \text{ therefore } \mathscr{I}^{n-k} \tau_v \mathscr{Y}^{\mathscr{T}}$$

by Theorem 5 of Section II.

Since (12) implies (13), it remains to prove that (13) implies (1).

Let $k < n, p \in \mathscr{Y}$ and $\varepsilon > 0$. Since the space $\mathscr{Y}^{\mathscr{S}_k}$ is locally connected in dimension 0 at the point p (by the hypothesis), there exists an $\eta > 0$ such that to every continuous function $h: \mathscr{S}_k \to \mathscr{Y}$, for which $\delta[p \cup h(\mathscr{S}_k)] < \eta$, there corresponds a continuous function $f: \mathscr{I} \to \mathscr{Y}^{\mathscr{S}_k}$ such that $f_0 = p, f_1 = h$ and

$$|f_x - p| < \varepsilon/2, \quad \text{so that} \quad \delta[p \cup f_x(\mathscr{S}_k)] < \varepsilon \quad \text{for each } x \in \mathscr{I}.$$

Let $h^*(t \cdot x) = f_x(t)$ for $t \in \mathscr{S}_k$. It follows that

$$h^* \in \mathscr{Y}^{\mathscr{Q}_{k+1}}, \quad h^*(t) = f_1(t) = h(t), \quad h^*(0) = f_0(t) = p,$$

$$|h^*(t \cdot x) - h^*(0)| = |f_x(t) - p| < \varepsilon/2, \quad \text{thus} \quad \delta[p \cup h^*(\mathscr{Q}_{k+1})] < \varepsilon.$$

Finally, in order to establish the implication (13′) → (1′), the conditions $\varepsilon > 0$ and $\eta > 0$ are deleted in the preceding argument. So, to every continuous function $h: \mathscr{S}_k \to \mathscr{Y}$, is assigned a continuous function $h^*: \mathscr{Q}_{k+1} \to \mathscr{Y}$ such that $h \subset h^*$ (where p is an arbitrary point).

THEOREM 3. *Let $n > 0$. If each subcontinuum of $\mathscr{Y}$ is integrally connected in dimension n, then $\mathscr{Y}$ is locally and integrally connected in dimension n.*

Proof. Let $p \epsilon G \subset \mathscr{Y}$ where G is open, and let $f: \mathscr{S}_n \to G$ be continuous. Since $n > 0$, the sphere $\mathscr{S}_n$ is a continuum and so is $f(\mathscr{S}_n)$. By assumption

$$f \subset f^*: \mathscr{Q}_{n+1} \to f(\mathscr{S}_n) \quad \text{where } f^* \text{ is continuous.}$$

Since $f(\mathscr{S}_n) \subset G$, f^* is a continuous extension of f into G. It follows that $\mathscr{Y}$ is locally connected in dimension n at p.

In order to show that $\mathscr{Y}$ is integrally connected in dimension n, it suffices to replace in the above argument G by $\mathscr{Y}$.

EXAMPLES AND REMARKS.

(i) *If* $\mathscr{Y} \neq 0$, then obviously $\mathscr{I}^0 \tau \mathscr{Y}$ ($\mathscr{I}^0$ reduces to a single point and $\mathscr{S}_{-1} = 0$). Therefore (compare (2′)), *if $\mathscr{Y}$ is a closed subset of a space $\mathscr{X}$ such that* $\dim(\mathscr{X} - \mathscr{Y}) = 0$, *then $\mathscr{Y}$ is a retract of $\mathscr{X}$.*

If $\dim \mathscr{X} = 0$, *then $\mathscr{X} \tau \mathscr{Y}$ for each* $\mathscr{Y} \neq 0$ (compare (4′)).

(ii) The condition $\mathscr{I} \tau_v \mathscr{Y}$ means that $\mathscr{Y}$ is locally arcwise connected, and $\mathscr{I} \tau \mathscr{Y}$ means that, moreover, $\mathscr{Y}$ is integrally arcwise connected.

Therefore, *the local connectedness in dimension* 0 *is equivalent to the local arcwise connectedness.*

(iii) The following space is l.c. n for every n, whereas it is not an ANR.

$\mathscr{Y}$ is the union of an infinite sequence of spheres $\mathscr{S}_0^*, \mathscr{S}_1^*, \ldots$ situated in such a manner that $\mathscr{S}_n^* \cap \mathscr{S}_{n+1}^*$ reduces to a single point, that $\mathscr{S}_n^* \cap \mathscr{S}_{n+i}^* = 0$ for $i > 1$, that $\mathscr{S}_n^* \underset{\text{top}}{=} \mathscr{S}_n$ and that the sequence converges to the point $p \epsilon \mathscr{Y} - (\mathscr{S}_0^* \cup \mathscr{S}_1^* \cup \ldots)$.

(iv) The set $\mathscr{Y}$ of the preceding example may be considered as situated in the Hilbert space in such a manner that the abscissae of its points vanish. Let q be the point $(1, 0, 0, \ldots)$. Let us join q with every point of $\mathscr{Y}$ by a straight line segment. The set which results is l.c. n and i.c. n for every n, but is not an AR.

(v) Let $\dim \mathscr{Y} = n$. If $\mathscr{Y}$ is l.c. $n+1$, then $\mathscr{Y}$ is an ANR. If, moreover, $\mathscr{Y}$ is i.c. $n+1$, then $\mathscr{Y}$ is an AR (compare § 54, VII, Theorem 6).

(vi) A compact l.c. n $(n > 0)$ space is i.c. n if and only if its Betti groups of dimensions $< n$ and its fundamental group are trivial(1).

(vii) If $\mathscr{Y}$ is a compact space ($\subset \mathscr{I}^{\aleph_0}$), then for every n there exists an infinite polyhedron P_n of dimension $\leqslant n$ such that $\mathscr{Y} \cup P_n$ is l.c. n and i.c. n(2).

(viii) A space locally connected in dimension n may fail to be locally connected in dimensions $< n$. Thus, e.g., the set consisting of the points $1, 1/2, 1/3, \ldots, 0$ is locally connected in dimension n for all $n > 0$, but is not locally connected in dimension 0 at the point 0.

(ix) The hypothesis that the space $\mathscr{Y}$ is integrally connected in dimension 1 is equivalent to the assumption that the fundamental group of $\mathscr{Y}$ is trivial.

(x) In the study of local connectedness in dimension n we confine our attention to metric separable spaces. However, it is possible to extend many theorems to arbitrary metric spaces (and even to normal spaces)(3).

V. Operations.

THEOREM 1. *The union of two closed l.c. n sets whose intersection is l.c. $n-1$, is l.c. n.*

The theorem remains true if the term "l.c." is replaced by "l.c. and i.c.".

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(1) See W. Hurewicz, Proc. Akad. Amsterdam 38 (1935), p. 522, and 39 (1936), p. 117.

(2) See K. Borsuk, Fund. Math. 27 (1936), p. 242.

(3) See J. Dugundji, *Absolute neighborhood retracts and local connectedness in arbitrary metric spaces*, Compos. Math. 13 (1958), pp. 229–246. Compare also O. Hanner, *Retraction and extension of mappings of metric and non-metric spaces*, Arkiv for Mat. 2 (1952), pp. 315–360; M. Katětov, *On the dimension of non-separable spaces* (in Russian), Czechoslovak Math. Journ. 6 (1956), pp. 485–516; Y. Kodama, *Note on an absolute neighborhood extensor for metric spaces*, Journ. Math. Soc. Japan 8 (1956), pp. 206–215, and *On* LC^n *metric spaces*, Proc. Japan Akad. 33 (1957), pp. 79–83; E. Michael, *Some extension theorems for continuous functions*, Pacific Journ. Math. 3 (1953), pp. 789–806, and *On a theorem of Kuratowski*, Bull. Amer. Math. Soc. 61 (1955), p. 444; C. H. Dowker, *On a theorem of Hanner*, Ark. for Math. 2 (1952), pp. 307–313.

See also the above-cited monographs *Theory of retracts* of Borsuk and of S. Hu, and *Topology* of J. Dugundji (Chap. VII).

Proof. Let $\mathscr{Y} = A_0 \cup A_1$, $\bar{A}_j = A_j$, $F = \bar{F} \subset \mathscr{I}^n$, $f\colon F \to \mathscr{Y}$ continuous and $C_j = f^{-1}(A_j)$.

Let according to § 27, II, Corollary 1d,

$$\mathscr{I}^n = B_0 \cup B_1, \quad \bar{B}_j = B_j, \quad F \cap B_j = C_j$$

and

$$\dim[(B_0 \cap B_1) - (C_0 \cap C_1)] \leqslant n-1.$$

The last inequality, combined with conditions $\mathscr{I}^{n-1}\tau_v(A_0 \cap A_1)$ and (3) of Section IV, implies condition (6) of Section II. Assuming that $\mathscr{I}^n \tau_v A_0$ and $\mathscr{I}^n \tau_v A_1$, Lemma 2′ of Section II yields immediately the required conclusions (compare IV (5)).

COROLLARY 2. *The sphere $\mathscr{S}_n$ is i.c. n* (but is not i.c. $n+1$, compare I (iii)).

Proof. Let us proceed by induction. The case $n = 0$ is obvious, i.e. $\mathscr{I}^0 \tau \mathscr{S}_0$, because $\mathscr{I}^0$ consists of a single point and $\mathscr{S}_0$ consists of two.

Let $n > 0$. Assume that $\mathscr{I}^{n-1}\tau\mathscr{S}_{n-1}$ and split $\mathscr{S}_n$ into two closed hemi-spheres $\mathscr{S}_n^+$ and $\mathscr{S}_n^-$ so that $\mathscr{S}_n^+ \cap \mathscr{S}_n^- = \mathscr{S}_{n-1}$. Therefore $\mathscr{I}^{n-1}\tau\mathscr{S}_n^+ \cap \mathscr{S}_n^-$; and since $\mathscr{S}_n^+$ and $\mathscr{S}_n^-$ are absolute retracts, then

$$\mathscr{I}^n\tau\mathscr{S}_n^+ \quad \text{and} \quad \mathscr{I}^n\tau\mathscr{S}_n^-, \quad \text{and} \quad \mathscr{I}^n\tau(\mathscr{S}_n^+ \cup \mathscr{S}_n^-) = \mathscr{S}_n$$

by Theorem 1.

Theorems 3 through 6 of Section II imply immediately the following ones.

THEOREM 3. *If the union and the intersection of two closed sets are l.c. n, then so are the sets themselves.*

The theorem remains true if the term "l.c. n" is replaced by "l.c. n and i.c. n".

THEOREM 4. *In order that the cartesian product of a (finite or infinite) sequence of factors be l.c. n and i.c. n, it is necessary and sufficient that the same holds for every factor.*

For the finite sequences the term "i.c. n" may be omitted.

THEOREM 5. *Let* $\dim\mathscr{T} = k$. *If $\mathscr{Y}$ is l.c. $n+k$ and i.c. $n+k$, the space $\mathscr{Y}^{\mathscr{T}}$ is l.c. n and i.c. n. If $\mathscr{T}$ is compact and $\mathscr{Y}$ is l.c. $n+k$, then $\mathscr{Y}^{\mathscr{T}}$ is l.c. n.*

Because the condition $\mathscr{I}^{n+k}\tau\mathscr{Y}$ implies $\mathscr{T} \times \mathscr{I}^n\tau\mathscr{Y}$ (according to IV, Theorem 1′ (4′) and (5′)), and hence $\mathscr{I}^n\tau\mathscr{Y}^{\mathscr{T}}$ by Theorem 5 of Section II.

THEOREM 6. *A neighbourhood retract of an l.c. n space is l.c. n. A retract of an l.c. n and i.c. n space is l.c. n and i.c. n.*

Consequently, the following is true (see Theorem 7 of Section II).

THEOREM 7. *If $\mathscr{Y}^{\mathscr{T}}$ (where $\mathscr{T}$ is compact and non-empty) is l.c. n (respectively l.c. n and i.c. n), then so is $\mathscr{Y}$.*

VI. Characterization of dimension (1).

THEOREM 1. *If $\mathscr{X}$ is metric separable, the conditions* $\dim\mathscr{X} \leqslant n$ *and $\mathscr{X}\tau\mathscr{S}^n$ are equivalent.*

This theorem will be derived from Theorems 2 and 5 below.

THEOREM 2. *The condition* $\dim\mathscr{X} \leqslant n$ *implies that $\mathscr{X}\tau\mathscr{S}_n$.*

Proof. Since $\mathscr{S}_n$ is l.c. n and i.c. n (by Theorem 2 of Section V), it follows by Theorem 1′ (4′) of Section IV that $\mathscr{X}\tau\mathscr{S}_n$, provided $\dim\mathscr{X} \leqslant n$.

THEOREM 3. *The condition $\mathscr{X}\tau\mathscr{S}_n$ implies that $\mathscr{X}\tau\mathscr{S}_l$ for $l > n$.*

Proof. The proof reduces to showing that $\mathscr{X}\tau\mathscr{S}_n$ implies $\mathscr{X}\tau\mathscr{S}_{n+1}$. Let $\mathscr{S}^+_{n+1}$ and $\mathscr{S}^-_{n+1}$ denote as usual two (closed) hemi-spheres of $\mathscr{S}_{n+1}$ such that $\mathscr{S}^+_{n+1} \cap \mathscr{S}^-_{n+1} = \mathscr{S}_n$. It follows that

$$\mathscr{X}\tau\mathscr{S}^+_{n+1}, \quad \mathscr{X}\tau\mathscr{S}^-_{n+1} \quad \text{and} \quad \mathscr{X}\tau\mathscr{S}^+_{n+1} \cap \mathscr{S}^-_{n+1},$$

hence $\mathscr{X}\tau\mathscr{S}_{n+1}$ by Theorem 2 of Section II.

Let S be a non-degenerate m-dimensional simplex $p_0 \dots p_m$. Let A_n be the union of all at most n-dimensional faces of S. Then the following lemma holds.

LEMMA 4. *The condition $\mathscr{X}\tau\mathscr{S}_n$ implies that $\mathscr{X}\tau A_n$, for $n < m$.*

More precisely, *if $\mathscr{X}\tau\mathscr{S}_n$, then for every continuous function $f\colon \mathscr{X} \to \overline{S}$ there exists a continuous function $f^*\colon \mathscr{X} \to A_n$ such that*

$$f(x) \in A_n \quad \textit{implies} \quad f^*(x) = f(x), \tag{1}$$

$$f(x) \in p_{i_0} \dots p_{i_k} \quad \textit{implies} \quad f^*(x) \in \overline{p_{i_0} \dots p_{i_k}}. \tag{2}$$

Proof. Let $d = m - n$ and proceed by induction with respect to d.

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(1) See P. Alexandrov, *Dimensionstheorie*, Math. Ann. 106 (1932), pp. 161–238, § 1. Compare W. Hurewicz, *Über Abbildungen topologischer Räume auf die n-dimensionale Sphäre*, Fund. Math. 24 (1935), p. 144.

For $d = 1$, we have $A_n = A_{m-1} \underset{\text{top}}{=} \mathscr{S}_{m-1}$. Hence, if $f^*\colon \mathscr{X} \to A_n$ is any (continuous) extension of the restriction $f|f^{-1}(A_n)$, conditions (1) and (2) are satisfied.

Assume that the lemma holds for $d-1$. We will prove it for d.

Let $f\colon \mathscr{X} \to \bar{S}$ be a continuous function. By Theorem 3, the condition $\mathscr{X}\tau\mathscr{S}_n$ implies $\mathscr{X}\tau\mathscr{S}_{n+1}$. So, by hypothesis there exists a continuous function $g\colon \mathscr{X} \to A_{n+1}$ such that

$$f(x)\in A_{n+1} \quad \text{implies} \quad g(x) = f(x), \tag{3}$$

$$f(x)\in p_{i_0}\dots p_{i_k} \quad \text{implies} \quad g(x)\in \overline{p_{i_0}\dots p_{i_k}}. \tag{4}$$

According to the definition of A_{n+1}, let $T_1, \dots, T_r$ be the $(n+1)$-dimensional faces of S such that

$$A_{n+1} = \bar{T}_1 \cup \dots \cup \bar{T}_r. \tag{5}$$

Let B_i be the boundary of T_i (i.e. the union of all its faces of dimension $\leqslant n$). Define

$$U_i = g^{-1}(\bar{T}_i), \tag{6}$$

$$C_i = g^{-1}(B_i). \tag{7}$$

It follows that $U_i\tau B_i$, because $B_i \underset{\text{top}}{=} \mathscr{S}_n$ and $\mathscr{X}\tau\mathscr{S}_n$ by hypothesis. Thus $g|C_i$ has a continuous extension $g_i\colon U_i \to B_i$. Let

$$f^* = g_1 + \dots + g_r, \tag{8}$$

i.e. $f^*(x) = g_i(x)$ for $x\in U_i$ where $1 \leqslant i \leqslant r$.

It follows for $i \neq j$ that

$$U_i \cap U_j = g^{-1}(\bar{T}_i \cap \bar{T}_j) = g^{-1}(B_i \cap B_j) = C_i \cap C_j,$$

hence

$$(g_i|U_i \cap U_j) = (g_i|C_i \cap C_j) = (g|C_i \cap C_j)$$
$$= (g_j|C_i \cap C_j) = (g_j|U_i \cap U_j).$$

Thus $f^*\colon \mathscr{X} \to A_n$ is continuous since $B_1 \cup \dots \cup B_r = A_n$.

Let $f(x)\in A_n$. Therefore $f(x) = g(x)$. Assume that $g(x)\in B_i$. Hence $x\in C_i$, and $g(x) = g_i(x) = f^*(x)$. Thus (1) is true.

In order to establish (2), assume that $f(x)\in p_{i_0}\dots p_{i_k}$. Therefore $g(x)\in \overline{p_{i_0}\dots p_{i_k}}$ by (4).

Consequently, it can be assumed that $g(x) \epsilon p_{i_0} \ldots p_{i_j}$ where $j \leqslant k$. If $j \leqslant n$, it follows that $g(x) \epsilon A_n$, therefore $f^*(x) = g(x)$, which implies condition (2).

Now let $j > n$, which means that $j = n+1$ (because $g(x) \epsilon A_{n+1}$). It follows that $p_{i_0} \ldots p_{i_{n+1}} = T_s$ for s conveniently chosen. Therefore $x \epsilon U_s$ by (6) and it follows according to (8) that

$$f^*(x) = g_s(x) \epsilon B_s \subset \overline{T}_s \subset \overline{p_{i_0} \ldots p_{i_k}}.$$

THEOREM 5. *The condition* $\mathscr{X} \tau \mathscr{S}_n$ *implies* $\dim \mathscr{X} \leqslant n$.

Proof. Let $\mathscr{X} \tau \mathscr{S}_n$. Let $G_0, \ldots, G_m$ be a system of open sets such that $\mathscr{X} = G_0 \cup \ldots \cup G_m$ and whose nerve is of dimension $l > n$. According to § 28, VI, Theorem 7 and § 45, VII, Corollary 3, all reduces to defining a continuous function $f\colon \mathscr{X} \to A_{l-1}$ such that

$$f^{-1}(P_i) \subset G_i \quad \text{for} \quad i = 0, 1, \ldots, m, \tag{9}$$

where P_i is the union of the faces of the simplex S which have the vertex p_i.

Consider the transformation $\varkappa$ corresponding to the systems $\{G_0, \ldots, G_m\}$ and $\{p_0, \ldots, p_m\}$,(1)

$$\varkappa(x) = \lambda_0(x) \cdot p_0 + \ldots + \lambda_m(x) \cdot p_m$$

where

$$\lambda_i(x) = \frac{\varrho(x, F_i)}{\varrho(x, F_0) + \ldots + \varrho(x, F_m)} \quad \text{and} \quad F_i = \mathscr{X} - G_i.$$

Since the nerve of the system $\{G_0, \ldots, G_m\}$ is l-dimensional, it follows that $\varkappa\colon \mathscr{X} \to A_l$ is continuous. Since $\mathscr{X} \tau \mathscr{S}_{l-1}$ (by Theorem 3), there exists (by Theorem 4) a continuous function $f\colon \mathscr{X} \to A_{l-1}$ such that

$$\varkappa(x) \epsilon p_{i_0} \ldots p_{i_k} \quad \text{implies} \quad f(x) \epsilon \overline{p_{i_0} \ldots p_{i_k}}. \tag{10}$$

We shall now derive inclusion (9).

Let $x \epsilon f^{-1}(P_i)$. Therefore $f(x) \epsilon P_i$. On the other hand, let

$$\varkappa(x) \epsilon p_{i_0} \ldots p_{i_k}, \quad \text{hence} \quad f(x) \epsilon \overline{p_{i_0} \ldots p_{i_k}}$$

(1) Compare § 28, VI. For covers with arbitrary sets (not necessarily open), see K. Kodaira, *Die Kuratowskische Abbildung und der Hopfsche Erweiterungssatz*, Compos. Math. 7 (1940), pp. 177–184.

according to (10). It follows immediately that i is one of the subscripts $i_0, \ldots, i_k$; thus

$$\varkappa(x) \epsilon P_i, \quad \text{which implies} \quad x \epsilon \varkappa^{-1}(P_i), \quad \text{and hence} \quad x \epsilon G_i,$$

because $\varkappa^{-1}(P_i) \subset G_i$ according to § 28, VI, (10).

COROLLARY 6(1). *In order that* $\dim \mathscr{X} \leqslant n$, *it is necessary and sufficient that to every continuous function* $f\colon \mathscr{X} \to \mathscr{Q}_{n+1}$ *there corresponds a continuous function* $g\colon \mathscr{X} \to (\mathscr{Q}_{n+1} - 0)$ (*where* 0 *stands for the origin*) *such that*

$$f(x) \epsilon \mathscr{S}_n \quad \textit{implies} \quad g(x) = f(x). \tag{11}$$

Proof. Let $f\colon \mathscr{X} \to \mathscr{Q}_{n+1}$ be a continuous function and $F = f^{-1}(\mathscr{S}_n)$. Assuming that $\dim \mathscr{X} \leqslant n$, it follows that $\mathscr{X} \tau \mathscr{S}_n$, therefore $(f|F) \subset g \epsilon \mathscr{S}_n^{\mathscr{X}}$, which yields condition (11).

On the other hand, let $F = \overline{F} \subset \mathscr{X}$ and let $h\colon F \to \mathscr{S}_n$ be a continuous function. Since $\mathscr{Q}_{n+1}$ is an absolute retract, so let $h \subset f \epsilon \mathscr{Q}_{n+1}^{\mathscr{X}}$. Assume that the continuous function $g\colon \mathscr{X} \to (\mathscr{Q}_{n+1} - 0)$ satisfies condition (11) and put

$$f^*(x) = \frac{g(x)}{|g(x)|}.$$

It follows that $h \subset f^* \epsilon \mathscr{S}_n^{\mathscr{X}}$. Therefore $\mathscr{X} \tau \mathscr{S}_n$, and $\dim \mathscr{X} \leqslant n$ by Theorem 5.

Theorem 2 may be restated in the following way.

THEOREM 7 (of duality)(2). *Let* $F = \overline{F} \subset \mathscr{X}$ *and* $\dim(\mathscr{X} - F) = m$. *To every continuous function* $f\colon F \to \mathscr{S}_n$ (*where* $n \leqslant m$) *there correspond a set* Z *such that*

$$Z = \overline{Z} \subset \mathscr{X} - F \quad \textit{and} \quad \dim Z \leqslant m - n - 1,$$

and a continuous extension $f^*\colon (\mathscr{X} - Z) \to \mathscr{S}_n$ *of* f.

Proof. Since the theorem is clearly true for $m = -1$ as well as for $n = -1$, we can assume that it holds for $m - 1$

(1) See P. Alexandrov, *Analyse géométrique de la dimension des ensembles fermés*, C. R. Paris 190 (1930), p. 475.

(2) S. Eilenberg, *Un théorème de dualité*, Fund. Math. 26 (1936), p. 280; compare also W. Hurewicz, *Über Abbildungen von endlichdimensionalen Räumen auf Teilmengen Cartesischer Räume*, Sgb. Preuss. Akad. 34 (1933), p. 765.

and that $n \geqslant 0$. As usual, let $\mathscr{S}_n^+$ and $\mathscr{S}_n^-$ be the (closed) hemispheres into which $\mathscr{S}_{n-1}$ splits the sphere $\mathscr{S}_n$. Let

$f: F \to \mathscr{S}_n$ be a continuous function, $C_0 = f^{-1}(\mathscr{S}_n^+)$, $C_1 = f^{-1}(\mathscr{S}_n^-)$

and according to § 27, II, Corollary 1d,

$$\mathscr{X} = B_0 \cup B_1, \quad \bar{B}_j = B_j, \quad F \cap B_j = C_j,$$
$$\dim[(B_0 \cap B_1) - (C_0 \cap C_1)] \leqslant m-1.$$

Since the theorem holds for $m-1$, then

$$(f|C_0 \cap C_1) \subset g \in \mathscr{S}_{n-1}^{B_0 \cap B_1 - Z} \text{ where } Z = \bar{Z} \subset (B_0 \cap B_1) - (C_0 \cap C_1),$$

$$\dim Z \leqslant \dim[(B_0 \cap B_1) - (C_0 \cap C_1)] - (n-1) - 1,$$

hence $\dim Z \leqslant m-n-1$. It follows that

$$B_0 \cap B_1 \cap F = C_0 \cap C_1 \quad \text{and}$$
$$(B_0 \cap B_1) - (C_0 \cap C_1) = B_0 \cap B_1 - F, \quad \text{thus} \quad Z \subset \mathscr{X} - F.$$

Lemma 2′ of Section II, where one substitutes $\mathscr{S}_n^+$ for A_0, $\mathscr{S}_n^-$ for A, $\mathscr{X} - Z$ for $\mathscr{X}$ and omits the subscript v, yields the existence of a continuous extension f^*: $(\mathscr{X} - Z) \to \mathscr{S}_n$ of f.

Remarks. (i) If the set $\mathscr{X} - F$ is an infinite polyhedron, it can be assumed that Z is a *polyhedron*(1).

(ii) Theorem 7 (completed with Remark (i)) can be generalized replacing $\mathscr{S}_n$ by an arbitrary l.c. m and i.c. n space(2).

VII. The space $LC^n(\mathscr{Y})$. Let us recall (see Section IV) that the space $\mathscr{X}$ is said to be an LC^n if for every $\varepsilon > 0$ there exists an $\eta > 0$ such that every continuous function $f: \mathscr{S}_m \to \mathscr{X}$, where $m = -1, 0, \ldots, n$, for which $\delta f(\mathscr{S}_m) < \eta$, admits a continuous extension f^*: $\mathscr{Q}_{m+1} \to \mathscr{X}$ such that $\delta f^*(\mathscr{Q}_{m+1}) \leqslant \varepsilon$.

Let $a_{\mathscr{X}}(\varepsilon)$ (for a fixed n) be the least upper bound of numbers η ($\leqslant \varepsilon$). Let $LC^n(\mathscr{Y})$ denote the family of all compact subsets of

(1) S. Eilenberg, *ibid.* p. 281.

(2) K. Borsuk, *Un théorème sur les prolongements des transformations*, Fund. Math. 29 (1937), p. 162.

For another generalization (avoiding separability), see T. Akasaki, *The Eilenberg–Borsuk duality theorem for metric spaces*, Duke Math. Journ. 32 (1965), pp. 653–659.

$\mathcal{Y}$ which are LC^n. This family is endowed with a topology by assuming that a sequence of its elements $A_1, A_2, \ldots$ converges to A, in symbols $A_i \xrightarrow[n]{} A$, provided that this sequence converges in the Hausdorff sense, i.e. that

$$\lim_{i=\infty} \operatorname{dist}(A_i, A) = 0,$$

and, moreover, that it is *uniformly* LC^n, i.e. that the number η which corresponds to ε does not depend on the subscript i(1).

THEOREM 1. *If $\mathcal{Y}$ is complete, one can define a distance in the space $LC^n(\mathcal{Y})$ so that it becomes a complete space.*

Namely, *for every pair of elements A and B of $LC^n(\mathcal{Y})$ we put*

$$\sigma(A, B) = \operatorname{dist}(A, B) + \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot \frac{|f_m(A) - f_m(B)|}{1 + |f_m(A) - f_m(B)|}$$

where

$$f_m(X) = 1 : \int_0^{1/m} a_X(t)\, dt.$$

One proves that the distance σ defined that way agrees with the convergence $A_i \xrightarrow[n]{} A$ defined above. The proof(2) leans essentially on the theorem on complete measurability stated in § 33, VI, Remark 2.

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(1) For this concept of convergence, also called *regular convergence in the sense of Curtis*, see M. L. Curtis, *Deformation-free continua*, Ann. of Math. 57 (1953), pp. 231–247. Compare P. A. White, *Regular convergence*, Bull. Amer. Math. Soc. 60 (1954), pp. 431–443. Compare also the concept of *regular convergence of G. T. Whyburn* in his paper *On sequences and limiting sets*, Fund. Math. 25 (1935), pp. 408–426.

(2) See my paper *Sur une méthode de métrisation complète de certains espaces d'ensembles compacts*, Fund. Math. 43 (1956), pp. 114–138. For the homological local connectedness, see E. G. Begle, *Regular convergence*, Duke Journ. 11 (1944), pp. 441–450. For stating the problem and related results, see K. Borsuk, *On some metrization of the hyperspace of compact sets*, Fund. Math. 41 (1955), pp. 168–202.

Concerning complete measurability of the space of locally connected subcontinua of a complete space, see S. Mazurkiewicz, *Sur l'espace des continus péaniens*, Fund. Math. 24 (1935), pp. 118–134, and my paper quoted above.

The next statements concern the space $LC^n(\mathscr{Y})$ for complete $\mathscr{Y}$ (1).

THEOREM 2. *Let* $B_i \underset{n}{\to} A$ *in the space* $LC^n(\mathscr{Y})$. *Let* $\mathscr{X}$ *be a perfect compact space of dimension* $\leqslant n+1$ *and let* $f\colon \mathscr{X} \to A$ *be continuous and onto. Then there exists a sequence of continuous transformations* g_i *uniformly convergent to* f *and such that* $g_i(\mathscr{X}) = B_i$ *for every positive integer* i *greater than a fixed* i_0.

Therefore, *for every* $A \in LC^n(\mathscr{Y})$ *and every* $\varepsilon > 0$ *there exists in the space* $LC^n(\mathscr{Y})$ *a neighbourhood of* A, *whose every element* B *has the form* $B = g(\mathscr{X})$ *where* $|g-f| < \varepsilon$.

Setting $\mathscr{X} = A$ and $f(x) = x$ for all x, we infer

THEOREM 3. *If* $A \in LC^n(\mathscr{Y})$ *is perfect of dimension* $m \leqslant n+1$, *then for every* $\varepsilon > 0$ *there exists a neighbourhood of* A *in* $LC^n(\mathscr{Y})$ *each element* B *of which can be obtained from* A *by a continuous transformation* g *such that* $|g(x)-x| < \varepsilon$.

THEOREM 4. *Under the same hypothesis about* A, *there exists a neighbourhood of* A *in* $LC^n(\mathscr{Y})$ *consisting exclusively of sets of dimension* $\geqslant m$.

Proof. The last theorem follows from the preceding one by virtue of the invariance of the inequality $\dim A \geqslant n$ under transformations with small point inverses. In fact, the dimension condition can be replaced in Theorem 4 by any other property which is also invariant under the said transformations.

§ 54. Homotopy. Contractibility

I. Homotopic functions. Two continuous functions $f_0\colon \mathscr{X} \to \mathscr{Y}$ and $f_1\colon \mathscr{X} \to \mathscr{Y}$ are said to be *homotopic* (with respect to $\mathscr{Y}$), written $f_0 \simeq f_1$, if there exists a continuous function of two variables $h\colon \mathscr{X} \times \mathscr{I} \to \mathscr{Y}$ such that

$$h(x, 0) = f_0(x) \quad \text{and} \quad h(x, 1) = f_1(x). \tag{1}$$

In other words, if the function $g\colon (\mathscr{X} \times 0 \cup \mathscr{X} \times 1) \to \mathscr{Y}$ such that $g|\mathscr{X} \times 0$ is identical with f_0 and $g|\mathscr{X} \times 1$ with f_1, has a continuous extension $h\colon \mathscr{X} \times \mathscr{I} \to \mathscr{Y}$.

(1) See my paper *Quelques propriétés de l'espace des ensembles* LC^n, Bull. Acad. Polon. Sci. 5 (1957), pp. 967–974.

EXAMPLES AND REMARKS[1].

(i) Two homotopic functions f_0 and f_1 are also said to be obtainable one from the other by a (*continuous*) *deformation*. This terminology occurs naturally if the variable $t \in \mathscr{I}$ is considered as the time parameter.

(ii) If $\mathscr{X} \times \mathscr{I} \tau \mathscr{Y}$ (in particular, if $\mathscr{Y}$ is an AR), then $f_0 \simeq f_1$ for every pair f_0, f_1: $\mathscr{X} \to \mathscr{Y}$. Therefore, two continuous functions with real (complex and so forth) values are always homotopic.

(iii) Let $\mathscr{P}$ be the plane $\mathscr{E}_2$ without the point 0. Let f: $\mathscr{X} \to \mathscr{P}$ be a continuous function; then f is homotopic to a constant function if and only if $f(x) = e^{g(x)}$, where g: $\mathscr{X} \to \mathscr{E}^2$ is a continuous function.

The "if" part of the condition is obvious; it is sufficient to set $h(x, t) = e^{(1-t)g(x)}$. The other part will be established in § 56, IX, Theorem 3.

(iv) A locally connected continuum $\mathscr{X}$ is unicoherent if and only if every continuous function f: $\mathscr{X} \to \mathscr{P}$ is homotopic to a constant function (see § 57, III, Theorem 3).

(v) If $\mathscr{X} = \mathscr{S}_n$, the homotopy concept leads in a natural way to the notion of homotopy group in the sense of Hurewicz[2].

THEOREMS 1–3. *The homotopy is a reflexive, symmetric and transitive relation, i.e.* $f \simeq f$, *if* $f \simeq g$, *then* $g \simeq f$, *if* $f \simeq g$ *and* $g \simeq h$, *then* $f \simeq h$.

Proof. In order to establish 3, let u: $\mathscr{X} \times \mathscr{I} \to \mathscr{Y}$ be continuous, $u(x, 0) = f(x)$, $u(x, \frac{1}{2}) = g(x)$ and $u(x, 1) = h(x)$.

THEOREM 4. *If* f: $\mathscr{X} \to \mathscr{Y}$ *and* g_j: $\mathscr{Y} \to \mathscr{Z}$ $(j = 0, 1)$ *are continuous functions and if* $g_0 \simeq g_1$, *then* $g_0 f \simeq g_1 f$.

Proof. Let h: $\mathscr{Y} \times \mathscr{I} \to \mathscr{Z}$ be a continuous function such that $h(y, j) = g_j(y)$. Assume $u(x, t) = h[f(x), t]$. It follows that

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(1) See Sze-Tsen Hu, *Homotopy Theory*, Academic Press 1959, for a detailed study of that theory. For a more algebraic approach, compare P. J. Hilton, *An introduction to homotopy theory*, Cambridge Tracts 1953; G. W. Whitehead, *Homotopy Theory*, Cambridge, Mass., 1966.

(2) See the paper of that author, *Beiträge zur Topologie der Deformationen II, Homotopie und Homologiegruppen*, Proc. Akad. Amsterdam 39 (1935), p. 521.

$u\colon \mathcal{X}\times\mathcal{I}\to\mathcal{Z}$ is continuous and

$$u(x,j) = h[f(x),j] = g_j f(x), \quad \text{thus} \quad g_0 f \simeq g_1 f.$$

THEOREM 4a. *Let $f_0, f_1\colon \mathcal{X}\to\mathcal{Z}$. If $\mathcal{Z}$ is a retract of $\mathcal{Y}$, then*

$$(f_0 \simeq f_1 \operatorname{rel} Y) \equiv (f_0 \simeq f_1 \operatorname{rel} Z).$$

Proof. We have to show the implication from left to right. Let $h\colon \mathcal{X}\times\mathcal{I}\to\mathcal{Y}$ be continuous and $h(x,j) = f_j(x)$. Let $r\colon \mathcal{Y}\to\mathcal{Z}$ be a retraction. Then $g = r\circ h$ is the required homotopy, i.e.

$$g\colon \mathcal{X}\times\mathcal{I}\to\mathcal{Z} \quad \text{and} \quad g(x,j) = r[h(x,j)] = r[f_j(x)] = f_j(x).$$

THEOREM 5. *Let $f_0, f_1\colon \mathcal{X}\to\mathcal{Y}$ be continuous functions and $\{F_t\}$ a family of closed-open sets such that $\mathcal{X} = \bigcup_t F_t$. If $f_0|F_t \simeq f_1|F_t$ for each t, then $f_0\simeq f_1$.*

Proof. Referring to the Lindelöf Theorem (§ 5, XI), let

$$\mathcal{X} = \bigcup_{n=1}^{\infty} F_{t_n} = F_{t_1} \cup (F_{t_2} - F_{t_1}) \cup (F_{t_3} - F_{t_1} - F_{t_2}) \cup \ldots$$

Let $G_{t_1}, G_{t_2}, \ldots$ denote the terms of the preceding union.

By hypothesis, there exists, for $n = 1, 2, \ldots$, a continuous function $h_n\colon G_{t_n}\times\mathcal{I}\to\mathcal{Y}$ such that $h_n(x,j) = f_j(x)$ for $x\in G_{t_n}$ and $j = 0, 1$.

Let h be equal to h_n on $G_{t_n}\times\mathcal{I}$. Since the sets G_{t_n} are open and disjoint, it follows that $h\colon \mathcal{X}\times\mathcal{I}\to\mathcal{Y}$ is a continuous function and that $h(x,j) = f_j(x)$. Hence $f_0\simeq f_1$.

THEOREM 6. *The condition $f_0\simeq f_1$ holds if and only if there exists an arc (or, which is equivalent, a locally connected continuum) joining f_0 to f_1 in the space $\mathcal{Y}^{\mathcal{X}}$.*

Proof. On the one hand, if the continuous function $h\colon \mathcal{X}\times\mathcal{I}\to\mathcal{Y}$ satisfies (1) and if to every $t\in\mathcal{I}$ one assigns the function $g_t\colon \mathcal{X}\to\mathcal{Y}$ defined by the condition

$$g_t(x) = h(x,t), \tag{2}$$

$g\colon \mathcal{I}\to\mathcal{Y}^{\mathcal{X}}$ is continuous (compare § 44, IV, Theorem 1). Since $g(\mathcal{I})$ is a $\mathcal{T}_2$-space (since $\mathcal{Y}^{\mathcal{X}}$ is $\mathcal{T}_2$, compare § 44, I, Remark 1), so $g(\mathcal{I})$ is metrizable (by Theorem 3 of § 41, VI) and hence a locally connected continuum which joins the points $g_0 = f_0$ and $g_1 = f_1$ of $\mathcal{Y}^{\mathcal{X}}$.

On the other hand, if $g\colon \mathcal{I} \to \mathcal{Y}^{\mathcal{X}}$ is continuous and $g_0 = f_0$ and $g_1 = f_1$, the mapping $h\colon \mathcal{X}\times\mathcal{I} \to \mathcal{Y}$ satisfies condition (1) and is continuous by Theorem 3 of § 44, IV.

THEOREM 7. *If the functions $f_j\colon \mathcal{X} \to \mathcal{Y}$ $(j = 0, 1)$ are constant, $f_j(x) \equiv c_j$, the relation $f_0 \simeq f_1$ holds if and only if there exists in $\mathcal{Y}$ an arc joining the points c_0 and c_1.*

Proof. If $h\colon \mathcal{X}\times\mathcal{I} \to \mathcal{Y}$ is a continuous function and $h(x, j) = c_j$, the condition $g(t) = h(x_0, t)$, where x_0 is a fixed point in $\mathcal{X}$, defines a continuous function $g\colon \mathcal{I} \to \mathcal{Y}$ and therefore $g(\mathcal{I})$ is a locally connected continuum. Hence it contains an arc joining $g(0) = c_0$ to $g(1) = c_1$.

Conversely, if the points c_0 and c_1 can be joined by an arc in $\mathcal{Y}$, the functions f_0 and f_1 can be joined by an arc in $\mathcal{Y}^{\mathcal{X}}$, because $\mathcal{Y} \underset{\text{top}}{\subset} \mathcal{Y}^{\mathcal{X}}$.

Let $\mathcal{E}_{-1}^{\aleph_0}$ be the set of sequences $z = [z^1, z^2, \ldots]$ such that $z^1 = -1$ and $z^n \in \mathcal{E}$.

Let $\mathcal{X} \subset \mathcal{E}_{-1}^{\aleph_0}$. Let us denote by $\Lambda(\mathcal{X})$ the union of segments joining the point 0 (the origin of axes) with each point of $\mathcal{X}$[1], i.e. the set of points of the form tx where $x \in \mathcal{X}$ and $0 \leqslant t \leqslant 1$. If $\mathcal{X}$ is empty, we assume that $\Lambda(\mathcal{X})$ is the origin of the axes.

THEOREM 8. *Let $\mathcal{X}$ be a compact space. In order that f be homotopic to a constant c, it is necessary and sufficient that f have a continuous extension $f^*\colon \Lambda(\mathcal{X}) \to \mathcal{Y}$ where $f^*(0) = c$.*

Proof. The function tx is continuous on $\mathcal{X}\times\mathcal{I}$ and, with exception of the points where $t = 0$, it is one-to-one. Consequently, if $f \subset f^* \in \mathcal{Y}^{\Lambda(\mathcal{X})}$ and if $h(x, t) = f^*(tx)$, it follows that $h\colon \mathcal{X}\times\mathcal{I} \to \mathcal{Y}$ is continuous and

$$h(x, 0) = f^*(0) \quad \text{and} \quad h(x, 1) = f^*(x) = f(x),$$

hence $f \simeq f^*(0)$.

Conversely, if $h\colon \mathcal{X}\times\mathcal{I} \to \mathcal{Y}$ is a continuous function, $h(x, 1) = f(x)$ and $h(x, 0) = c$, one has to set $f^*(0) = c$ and $f^*(z) = h(x, t)$ for $z = tx \neq 0$. Then $f \subset f^* \in \mathcal{Y}^{\Lambda(\mathcal{X})}$.

THEOREM 9. *$(x|\mathcal{S}_n)$ non $\simeq 1$ with respect to $\mathcal{S}_n$.*

Proof. This is a consequence of Theorem 8 combined with Corollary 1a of § 28, III.

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(1) Compare S. Lefschetz, *Introduction to Topology*, Princeton 1949, p. 94 (Joins).

Let us add the following obvious formulas

$$\Lambda(A \cup B) = \Lambda(A) \cup \Lambda(B), \tag{3}$$

$$\Lambda(A \cap B) = \Lambda(A) \cap \Lambda(B), \tag{4}$$

$$\dim \Lambda(A) \leqslant \dim A + 1, \tag{5}$$

$$\Lambda(\mathscr{S}_n) \underset{\text{top}}{=} \mathscr{Q}_{n+1}. \tag{6}$$

II. Homotopy with respect to l.c. n spaces. We are going to consider the case where the space $\mathscr{Y}$ is l.c. n. Obviously, all the theorems will hold in particular for absolute neighbourhood retracts; moreover, in the last case the hypotheses concerning the dimension (for instance, $\dim(\mathscr{X}-F) \leqslant n-1$) can be omitted.

THEOREM 1. *Let $\mathscr{Y}$ be an l.c. n space, $F = \overline{F} \subset \mathscr{X}$, $\dim(\mathscr{X}-F) \leqslant n-1$ and $f_0, f_1\colon \mathscr{X} \to \mathscr{Y}$ two continuous functions. If $(f_0|F) \simeq (f_1|F)$, there exists an open set $G \supset F$ such that $(f_0|G) \simeq (f_1|G)$. If, moreover, $\mathscr{Y}$ is i.c. n, then $f_0 \simeq f_1$.*

Proof. Let

$$\mathscr{X}_1 = \mathscr{X} \times \mathscr{I}, \quad F_1 = (F \times \mathscr{I}) \cup (\mathscr{X} \times 0) \cup (\mathscr{X} \times 1)$$

and let $g\colon F_1 \to \mathscr{Y}$ be continuous and such that $g(x, j) = f_j(x)$.

Since $\dim(\mathscr{X}_1 - F_1) \leqslant \dim[(\mathscr{X}-F)\times\mathscr{I}] \leqslant n$ and $\mathscr{Y}$ is l.c. n, there exist, according to § 53, IV, 1 (3), an open set $G_1 \supset F_1$ and an extension $g^*\colon G_1 \to \mathscr{Y}$ of g. Let G be an open set such that $F \subset G$ and $G \times \mathscr{I} \subset G_1$ (compare § 41, IV, Theorem 1). It follows that $(f_0|G) \simeq (f_1|G)$.

If $\mathscr{Y}$ is i.c. n, one has to set $G_1 = \mathscr{X}_1$.

THEOREM 2. *Given: a compact space $\mathscr{X}$ of dimension $\leqslant n-1$, an l.c. n space $\mathscr{Y}$ and two continuous functions $f_0, f_1\colon \mathscr{X} \to \mathscr{Y}$, then the family of closed subsets F of $\mathscr{X}$, such that $f_0|F \simeq f_1|F$, is open in the space $2^{\mathscr{X}}$.*

Because the considered family is the union of families $\boldsymbol{A}_G = \underset{F}{E}(F \subset G)$, where G is an open variable set such that $f_0|G \simeq f_1|G$ (the family $\boldsymbol{A}_G$ is open according to Theorem 1 of § 17, II).

The same hypotheses lead to the following conclusion.

THEOREM 2a. *If $F_1 \supset F_2 \supset \ldots$, $F_n = \overline{F}_n$ and $(f_0|F_n)$ non $\simeq (f_1|F_n)$ for every n, then*

$$(f_0|\bigcap_n F_n) \text{ non } \simeq (f_1|\bigcap_n F_n).$$

THEOREM 3 (1). *Let $\mathscr{Y}$ be an l.c. n space, $F=\overline{F}\subset\mathscr{X}$, $\dim(\mathscr{X}-F)\leqslant n-1$, and let $f_0, f_1\colon F\to\mathscr{Y}$ be two continuous functions such that $f_0\simeq f_1$. If f_0 has an extension f_0^* to $\mathscr{X}$, then f_1 also has an extension f_1^* to $\mathscr{X}$. Moreover, $f_0^*\simeq f_1^*$.*

Proof. Put $F^0=(F\times\mathscr{I})\cup(\mathscr{X}\times 0)$. Since $f_0\simeq f_1$, there exists a continuous function $h\colon F^0\to\mathscr{Y}$ such that

$$h(x,0)=f_0^*(x)\quad\text{for}\quad x\in\mathscr{X},\qquad h(x,1)=f_1(x)\quad\text{for}\quad x\in F.$$

Since $\mathscr{Y}$ is l.c. n, the inequality $\dim(\mathscr{X}\times\mathscr{I}-F^0)\leqslant n$ implies (compare § 53, IV, 1 (3)) that the function h has an extension to a neighbourhood of the set F^0; therefore, by Lemma 9 of § 53, III, it has an extension h^* to the space $\mathscr{X}\times\mathscr{I}$.

It remains only to set $f_1^*(x)=h^*(x,1)$.

THEOREM 4. *Let $\mathscr{X}$ be a compact space, let $\mathscr{Y}$ be an l.c. n space and let $F=\overline{F}\subset\mathscr{X}$. If $\dim\mathscr{X}\leqslant n-1$, the set $\mathscr{Y}^{\mathscr{X}}|F$ (i.e. the set of continuous functions $f\colon F\to\mathscr{Y}$ which are extendable to $\mathscr{X}$) is closed-open in $\mathscr{Y}^F$ (2).*

Proof. Since the space $\mathscr{Y}^F$ is locally arcwise connected according to § 53, IV, (12), every component Φ of $\mathscr{Y}^F$ is arcwise connected (§ 50, I, Theorem 2). Consequently (compare I, Theorem 6), if $f_0, f_1\in\Phi$, then $f_0\simeq f_1$; therefore, if $f_0\in\mathscr{Y}^{\mathscr{X}}|F$, then $\Phi\subset\mathscr{Y}^{\mathscr{X}}|F$ by Theorem 3. Thus $\mathscr{Y}^{\mathscr{X}}|F$ is the union of a family of components of the space $\mathscr{Y}^F$. Since that space is locally connected, every component is closed-open (compare § 49, II, Theorem 4), and so is the set $\mathscr{Y}^{\mathscr{X}}|F$.

Remark. If $\mathscr{Y}$ is a sphere, the following sharper proposition holds.

THEOREM 4a. *Let $\mathscr{X}$ be a compact space and $F=\overline{F}\subset\mathscr{X}$. If $f_0\in\mathscr{S}_m^{\mathscr{X}}|F$ and $|f_1-f_0|<2$, then $f_1\in\mathscr{S}_m^{\mathscr{X}}|F$.*

Proof. By Theorem 3 it is sufficient to show that

$$\text{if } f_0, f_1\colon \mathscr{X}\to\mathscr{S}_m \text{ are continuous and } |f_0-f_1|<2,\quad\text{then } f_0\simeq f_1. \tag{1}$$

(1) K. Borsuk, Ann. Soc. Pol. Math. 16 (1937), p. 218.

(2) K. Borsuk, *Sur les prolongements des transformations continues*, Fund. Math. 28 (1937), p. 106.

Let

$$p(x,t) = f_0(x) + tf_1(x) - tf_0(x) \quad \text{and} \quad h(x,t) = \frac{p(x,t)}{|p(x,t)|}$$

for $0 \leqslant t \leqslant 1$. It follows that $p(x,t) \neq 0$. Because

$$|p(x,t)| \geqslant |f_0(x)| - t|f_1(x) - f_0(x)| > 1 - 2t,$$

and

$$|p(x,t)| \geqslant |f_1(x)| - (1-t)|f_1(x) - f_0(x)| > 1 - 2(1-t).$$

Furthermore, $h(x,0) = f_0(x)$ and $h(x,1) = f_1(x)$.

THEOREM 5. *Let $\mathscr{Y}$ be an l.c. n and i.c. m space (where $m \leqslant n$), and let F, F_0 and F_1 be closed subsets of $\mathscr{X}$ such that*

$$F \subset F_0 \cap F_1, \tag{2}$$

$$\dim(F_j - F) \leqslant n-1, \tag{3}$$

$$\dim(F_0 \cap F_1 - F) \leqslant m-1. \tag{4}$$

Then $(\mathscr{Y}^{F_0}|F) \cap (\mathscr{Y}^{F_1}|F) = (\mathscr{Y}^{F_0 \cup F_1}|F)$, i.e. every continuous function $f\colon F \to \mathscr{Y}$, extendable to F_0 and to F_1, is extendable to $F_0 \cup F_1$.

Proof. Let $f \subset f_j \in \mathscr{Y}^{F_j}$, $j = 0, 1$. Since $\mathscr{Y}$ is l.c. n and i.c. m, the conditions (4) and $f_0|F = f = f_1|F$ imply by Theorem 1 (substituting m for n and $F_0 \cap F_1$ for $\mathscr{X}$) that $(f_0|F_0 \cap F_1) \simeq (f_1|F_0 \cap F_1)$.

We infer by Theorem 3 (substituting F_0 for $\mathscr{X}$, $F_0 \cap F_1$ for F and $f_j|F_0 \cap F_1$ for f_j) that the function $f_1|F_0 \cap F_1$ has a continuous extension $f_2\colon F_0 \to \mathscr{Y}$. Therefore, $f \subset (f_2 + f_1) \in \mathscr{Y}^{F_0 \cup F_1}$.

THEOREM 6. *Let $\mathscr{Y}$ be an l.c. n and i.c. m space (where $m \leqslant n$), $\dim \mathscr{X} \leqslant n-2$, $\mathscr{X} = F_0 \cup F_1$, $\overline{F}_j = F_j$ and $\dim F_0 \cap F_1 \leqslant m-2$.*

If $f_j\colon \mathscr{X} \to \mathscr{Y}$ is a continuous function and $(f_0|F_j) \simeq (f_1|F_j)$ for $j = 0, 1$, then $f_0 \simeq f_1$.

Proof. Let

$$\mathscr{X}^* = \mathscr{X} \times \mathscr{I}, \quad F^* = \big(\mathscr{X} \times (0)\big) \cup \big(\mathscr{X} \times (1)\big), \quad F_j^* = F^* \cup (F_j \times \mathscr{I}).$$

Let $f\colon F^* \to \mathscr{Y}$ be a (continuous) function defined by identities $f(x,j) = f_j(x)$. Since $(f_0|F_j) \simeq (f_1|F_j)$, we have $f \in (\mathscr{Y}^{F_0}|F^*) \cap (\mathscr{Y}^{F_1}|F^*)$. Since the hypotheses of Theorem 5 are fulfilled, it follows that

$$f \in \mathscr{Y}^{\mathscr{X}^*} | F^*, \quad \text{because} \quad \mathscr{X}^* = F_0^* \cup F_1^*, \quad \text{hence} \quad f_0 \simeq f_1.$$

Theorems 3, 5, and 6 imply immediately (compare § 53, III, (iv) and Theorem 3 of Section V) the following three statements.

THEOREM 7. *If* $F = \overline{F} \subset \mathscr{X}$ *and* $\mathscr{Y}$ *is an* ANR, *every continuous function* $f\colon F \to \mathscr{Y}$, *homotopic to a constant, has an extension* $f^*\colon \mathscr{X} \to \mathscr{Y}$.

THEOREM 8. *If* $F = \overline{F}$, $F_j = \overline{F}_j$, $F \subset F_0 \cap F_1$ *and* $\dim(F_0 \cap F_1 - F) \leqslant m-1$, *then*

$$(\mathscr{S}_m^{F_0} | F) \cap (\mathscr{S}_m^{F_1} | F) = (\mathscr{S}_m^{F_0 \cup F_1} | F).$$

THEOREM 9. *If* $F_j = \overline{F}_j$, $\dim F_0 \cap F_1 \leqslant m-2$, $f\colon (F_0 \cup F_1) \to \mathscr{S}_m$ *is a continuous function and if* $f|F_j \simeq 1$ *for* $j = 0, 1$, *then* $f \simeq 1$.

THEOREM 10. *Let* $\mathscr{Y}$ *be a complete l.c.* n *space and* $\mathscr{X}$ *a compact space of dimension* $< n$. *Let* $f, g\colon \mathscr{X} \twoheadrightarrow \mathscr{Y}$ *be continuous functions. Then* $f \simeq g$, *if and only if* f *and* g *belong to the same component of the space* $\mathscr{Y}^{\mathscr{X}}$.

Proof. By Corollary 2 of § 53, IV, the space $\mathscr{Y}^{\mathscr{X}}$ is locally connected. It is complete by Theorem 3 of § 44, V. Hence every component of $\mathscr{Y}^{\mathscr{X}}$ is a connected, locally connected and complete set and therefore is arcwise connected (compare § 50, II, Theorem 1 and I, Theorem 2). Theorem 10 follows by Theorem 6 of Section I.

III. Relation f_0 *irr non* $\simeq f$. This symbol will mean that the space $\mathscr{X}$ is irreducible with respect to the non-homotopy of the functions f_0 and f_1, in other words, that

$$f_0 \, non \simeq f_1, \tag{1}$$

$$F = \overline{F} \neq \mathscr{X} \quad \text{implies} \quad f_0|F \simeq f_1|F. \tag{2}$$

Since $\mathscr{S}_n - p$ is an AR, it follows (compare Section I, Theorem 9 and (ii)) that

$$(x|\mathscr{S}_n) \, irr \, non \simeq 1 \quad \text{with respect to} \quad \mathscr{S}_n. \tag{3}$$

THEOREM 1. *Let* $\mathscr{X}$ *be a compact space,* $\dim \mathscr{X} \leqslant n-1$, *let* $\mathscr{Y}$ *be an l.c.* n *space and* $f_0, f_1\colon \mathscr{X} \to \mathscr{Y}$ *two continuous functions. If* f_0 *non* $\simeq f_1$, $\mathscr{X}$ *contains a closed set* F *such that* $f_0|F$ *irr non* $\simeq f_1|F$.

Proof. This is an immediate consequence of Theorems 2a of Section II and 2 of § 42, IV.

COROLLARY 1a. *If $\mathscr{X}$ is a compact space, $f: \mathscr{X} \to \mathscr{S}_m$ is a continuous function and f non $\simeq 1$, then $\mathscr{X}$ contains a closed set F such that $f|F$ irr non $\simeq 1$.*

THEOREM 2. *Let $\mathscr{Y}$ be a l.c. n and i.c. m space ($m \leqslant n$),* $\dim \mathscr{X} \leqslant n-2$ *and $f_0, f_1: \mathscr{X} \to \mathscr{Y}$ two continuous functions. If f_0 irr non $\simeq f_1$, no closed set of dimension $\leqslant m-2$ separates the space $\mathscr{X}$; in other words, $\mathscr{X}$ is not of the form $\mathscr{X} = F_0 \cup F_1$, where $F_j = \overline{F}_j$ for $j = 0, 1$ and* $\dim F_0 \cap F_1 \leqslant m-2$.

This is an immediate consequence of Theorem 6 of Section II.

COROLLARY 2a. *If $f: \mathscr{X} \to \mathscr{S}_m$ is a continuous function and f irr non $\simeq 1$, then no closed set of dimension $\leqslant m-2$ separates the space $\mathscr{X}$.*

It follows (compare (3)) that $\mathscr{S}_n$ is a Cantor manifold (compare § 46, XI).

THEOREM 3. *If $\mathscr{X}$ is a locally connected continuum, $f: \mathscr{X} \to \mathscr{S}_m$ a continuous function, $m \geqslant 2$* (¹) *and $f|E \simeq 1$ for every cyclic element E of $\mathscr{X}$, then $f \simeq 1$.*

Otherwise, there would exist by Corollary 1a a closed set F such that $f|F$ *irr non* $\simeq 1$. By Corollary 2a the set F is connected and no point separates it. Therefore (compare § 52, II, Theorem 10), F is contained in a single cyclic element E. Since $f|E \simeq 1$, it follows that $f|F \simeq 1$, contrary to the definition of F.

IV. Deformation. Let $\mathscr{X} \subset \mathscr{Y}$. If the continuous function $f: \mathscr{X} \to \mathscr{Y}$ is homotopic to the identity, i.e. if there exists a continuous function $h: \mathscr{X} \times \mathscr{I} \to \mathscr{Y}$ such that

$$h(x, 0) = x \quad \text{and} \quad h(x, 1) = f(x), \tag{1}$$

the set $f(\mathscr{X})$ is said to be obtained from $\mathscr{X}$ by a *deformation* in $\mathscr{Y}$ (namely, h is that deformation).

If the function $f: \mathscr{X} \to \mathscr{X}$ is a retraction homotopic to the identity function, the set $f(\mathscr{X})$ is said to be a *deformation retract of* $\mathscr{X}$ and the function h is said to be a *retracting deformation* (²).

(¹) Theorem 3 holds also for $m = 1$. See § 56, X, 5.

(²) See K. Borsuk, Fund. Math. 21 (1933), p. 91.

THEOREM 1. *Let $\mathscr{X}$ and $\mathscr{X}^*$ be two subsets of $\mathscr{Y}$ and let the latter be obtained by a deformation of the former. Let g_0, g_1: $\mathscr{Y} \to \mathscr{Z}$ be continuous functions. If $g_0|\mathscr{X}^* \simeq g_1|\mathscr{X}^*$, then $g_0|\mathscr{X} \simeq g_1|\mathscr{X}$.*

Proof. Let f: $\mathscr{X} \to \mathscr{X}^*$ and h: $\mathscr{X}\times\mathscr{I} \to \mathscr{Y}$ be two functions satisfying condition (1) and such that $f(\mathscr{X}) = \mathscr{X}^*$. It follows that

$$g_j h\colon\ \mathscr{X}\times\mathscr{I} \to \mathscr{Z}, \quad g_j h(x,0) = g_j(x) \quad \text{and} \quad g_j h(x,1) = g_j f(x),$$

which yields $g_j|\mathscr{X} \simeq g_j f$. Define $g_j^* = g_j|\mathscr{X}^*$. By hypothesis $g_0^* \simeq g_1^*$, therefore $g_0^* f \simeq g_1^* f$ by Theorem 4 of Section I. Since $g_j^* f = g_j f \simeq g_j|\mathscr{X}$, it follows that $g_0|\mathscr{X} \simeq g_1|\mathscr{X}$.

THEOREM 2. *The set $\Lambda(\mathscr{X})$ can be deformed to its vertex (the point 0) in itself.*

Proof. The points z of $\Lambda(\mathscr{X})$ have the form $z = tx$, where $x \in \mathscr{X}$ and $0 \leqslant t \leqslant 1$. Let $h(z,u) = utx$ for $0 \leqslant u \leqslant 1$. It follows that the function h is continuous and that

$$h\colon\ \Lambda(\mathscr{X})\times I \to \Lambda(\mathscr{X}), \quad h(z,0) = 0 \quad \text{and} \quad h(z,1) = z.$$

THEOREM 3. *Every retract R of $\mathscr{X}$, which can be obtained from $\mathscr{X}$ by a deformation, is a deformation retract of $\mathscr{X}$.*

Proof. Let, according to the hypotheses, r: $\mathscr{X} \to R$ be a retraction; thus $r(x) = x$ for $x \in R$; let f: $\mathscr{X} \to R$ and h: $\mathscr{X}\times\mathscr{I} \to \mathscr{X}$ be continuous functions such that

$$h(x,0) = x, \quad h(x,1) = f(x) \quad \text{and} \quad f(\mathscr{X}) = R.$$

Define

$$g(x,t) = \begin{cases} h(x,2t) & \text{for} \quad 0 \leqslant t \leqslant 1/2, \\ r[h(x,2-2t)] & \text{for} \quad 1/2 \leqslant t \leqslant 1. \end{cases}$$

It follows that g: $\mathscr{X}\times\mathscr{I} \to \mathscr{X}$ is continuous since $h(x,1) = r[h(x,1)]$, and that $g(x,0) = x$ and $g(x,1) = r(x)$.

THEOREM 4. *If $\mathscr{X}\times\mathscr{I}\,\tau\,\mathscr{Y}$ (therefore, if $\mathscr{Y}$ is an absolute retract) $\mathscr{X} \subset \mathscr{Y}$ and if f: $\mathscr{X} \to \mathscr{Y}$ is a continuous function, then $\mathscr{X}$ is deformable to $f(\mathscr{X})$.*

Because f is homotopic with the identity function (compare Section I, (ii)).

THEOREM 5. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be continuous. If $\mathscr{Z}$ is a deformation retract of $\mathscr{Y}$, then there is $f^*\colon \mathscr{X} \to \mathscr{Z}$ continuous and such that*

$$f^* \simeq f \quad \textit{and} \quad f^*(x) = f(x) \quad \textit{whenever} \quad f(x) \in \mathscr{Z}.$$

Proof. Let $h\colon \mathscr{Y} \times \mathscr{I} \to \mathscr{Y}$ be continuous, $h(y, 0) = y, h(y, 1) = r(y)$ where $r\colon \mathscr{Y} \to \mathscr{Z}$ is a retraction. Put

$$f^* = rf \quad \text{and} \quad g(x, t) = h(f(x), t).$$

Hence $g(x, 0) = f(x)$ and $g(x, 1) = f^*(x)$. Finally, if $f(x) \in \mathscr{Z}$, then $r[f(x)] = f(x)$, i.e. $f^*(x) = f(x)$.

THEOREM 6(¹). *If $\mathscr{Y}$ is ANR and $\mathscr{Z}$ is a deformation retract of $\mathscr{Y}$, then there is $h\colon \mathscr{Y} \times \mathscr{I} \to \mathscr{Y}$ continuous and such that $h(y, 0) = y$, $h(y, 1) \in \mathscr{Z}$, and $h(y, t) = y$ whenever $y \in \mathscr{Z}$ and $t \in \mathscr{I}$.*

EXAMPLES. (i) If $\mathscr{Y} = \mathscr{E}^n$ or $\mathscr{Y} = \mathscr{I}^n$ $(n \leqslant \aleph_0)$, the deformation of $\mathscr{X}$ onto $f(\mathscr{X})$ in $\mathscr{Y}$ may be defined in the following way

$$h(x, t) = (1-t)\cdot x + t\cdot f(x).$$

(ii) Define $f(x) = x : |x|$. Then the function h is a retracting deformation of the space $(\mathscr{E}^n - 0)$ onto $\mathscr{S}_{n-1}$.

V. Contractibility.

DEFINITION. The space $\mathscr{X}$ is said to be *contractible* with respect to the space $\mathscr{Y}$ if every continuous function $f\colon \mathscr{X} \to \mathscr{Y}$ is homotopic to a constant(²).

THEOREM 1. *If the space $\mathscr{Y}$ is arcwise connected, the contractibility of the space $\mathscr{X}$ with respect to $\mathscr{Y}$ means that every pair of continuous functions $f_0, f_1\colon \mathscr{X} \to \mathscr{Y}$ is homotopic (in $\mathscr{Y}$).*

Because in an arcwise connected space all the constant functions are homotopic (compare Theorem 7 of Section I).

THEOREM 2. *The space $\mathscr{Y}^{\mathscr{X}}$ is arcwise connected if and only if $\mathscr{X}$ is contractible with respect to $\mathscr{Y}$ and $\mathscr{Y}$ is arcwise connected.*

Proof. If $\mathscr{X}$ is contractible with respect to $\mathscr{Y}$, and $\mathscr{Y}$ is arcwise connected, then $f_0 \simeq f_1$ for each pair of continuous functions

(¹) For the proof, see H. Samelson, *Remark on a paper by R. H. Fox*, Ann. of Math. 45 (1944), p. 448.

(²) K. Borsuk, Fund. Math. 24 (1935), p. 250. Compare L. Lusternik and L. Schnirelmann, *Méthodes topologiques dans les problèmes variationnels*, Actual. Scient. 88, p. 25.

f_0, f_1: $\mathcal{X} \to \mathcal{Y}$ (compare Theorem 1). The homotopy $f_0 \simeq f_1$ implies (compare Theorem 6 of Section I) that there exists an arc joining f_0 to f_1 in $\mathcal{Y}^{\mathcal{X}}$. On the other hand, if $\mathcal{Y}^{\mathcal{X}}$ is arcwise connected, then we have $f_0 \simeq f_1$ for every pair of continuous functions f_0, f_1: $\mathcal{X} \to \mathcal{Y}$ (by Theorem 6 of Section I). Therefore $\mathcal{X}$ is contractible with respect to $\mathcal{Y}$ and $\mathcal{Y}$ is arcwise connected by Theorem 7 of Section I.

THEOREM 3. *The contractibility of the space $\mathcal{X}$ with respect to $\mathcal{Y}$ is invariant under retraction of $\mathcal{X}$.*

Proof. Let r be a retraction of $\mathcal{X}$ and f: $r(\mathcal{X}) \to \mathcal{Y}$ be a continuous function. Therefore fr: $\mathcal{X} \to \mathcal{Y}$ and, since $\mathcal{X}$ is contractible with respect to $\mathcal{Y}$, it follows that $fr \simeq$ *constant*. Hence f is homotopic to a constant, because $f(x) = fr(x)$ for $x \in r(\mathcal{X})$.

THEOREM 4. *The non-contractibility of the space $\mathcal{X}$ with respect to $\mathcal{Y}$ is an invariant of deformation of $\mathcal{X}$ onto a subset.*

This follows immediately from Theorem 1 of Section IV.

Theorems 3 and 4 have the following straightforward consequence.

THEOREM 4′. *Let $\mathcal{X}^*$ be a deformation retract of $\mathcal{X}$. The space $\mathcal{X}^*$ is contractible with respect to $\mathcal{Y}$ if and only if so is the space $\mathcal{X}$.*

THEOREM 5. *If the space $\mathcal{Y}$ is an absolute neighbourhood retract, the non-contractibility of the compact space $\mathcal{X}$ with respect to $\mathcal{Y}$ is invariant under transformations with small point-inverses* (1).

Proof. According to Corollary 4c of § 41, VI, we have to show that if $\mathcal{X} = \overline{\mathcal{X}} \subset \mathcal{I}^{\aleph_0}$ and if the continuous function f: $\mathcal{X} \to \mathcal{Y}$ is not homotopic to a constant, there exists an $\alpha > 0$ such that, if g: $\mathcal{X} \to \mathcal{I}^{\aleph_0}$ is any continuous function satisfying the inequality $|g(x) - x| < \alpha$ for each x, the set $g(\mathcal{X})$ is not contractible with respect to $\mathcal{Y}$.

But, since $\mathcal{Y}$ is an ANR, there exists an open neighbourhood G of $\mathcal{X}$ and an extension f^*: $G \to \mathcal{Y}$ of f. Let $\alpha = \varrho(\mathcal{X}, \mathcal{I}^{\aleph_0} - G)$. Therefore the conditions $x \in \mathcal{X}, x' \in \mathcal{I}^{\aleph_0}$ and $|x - x'| < \alpha$ imply that the segment xx' is contained in G. Thus, if g: $\mathcal{X} \to \mathcal{I}^{\aleph_0}$ is a continuous function and $|g(x) - x| < \alpha$, the set $\mathcal{X}^* = g(\mathcal{X})$ can be

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(1) Theorem of K. Borsuk and S. Ulam, *Über gewisse Invarianten der ε-Abbildungen*, Math. Ann. 108 (1933), p. 311. For the proof, see S. Eilenberg, C. R. Paris, 200 (1935), p. 1005. See also P. Alexandrov, *Dimensionstheorie*, Math. Ann. 106 (1932), p. 226.

obtained from $\mathscr{X}$ by the following deformation h in G,

$$h(x, t) = (1-t)\cdot x + t\cdot g(x).$$

It follows that $f^*|\mathscr{X}^*$ is not homotopic to a constant. Because otherwise, by virtue of Theorem 1 of Section IV (replacing $\mathscr{Y}$ by G, $\mathscr{X}$ by $\mathscr{Y}$ and g_0 by f^*), $f^*|\mathscr{X}$ is homotopic to a constant, which means that f is homotopic to a constant, contrary to the hypothesis.

Remark. 1. If $\mathscr{Y} = \mathscr{S}_n$, the following sharper proposition holds(1).

Let $\mathscr{X}$ be a compact space, g: $\mathscr{X} \to \mathscr{S}_n$ a continuous function not homotopic to a constant and η a positive number such that

$$|x-x'| < \eta \quad \textit{implies} \quad |g(x)-g(x')| < \sqrt{2(n+2)/(n+1)}\,(2).$$

If f is a continuous function whose point-inverses are of diameter $< \eta$, the space $f(\mathscr{X})$ is not contractible with respect to $\mathscr{S}_n$.

THEOREM 6. *If $\mathscr{Y}$ is an absolute neighbourhood retract and if A_0 and A_1 are closed sets such that $A_0 \cap A_1$ and $A_0 \cup A_1$ are contractible with respect to $\mathscr{Y}$, then so are the sets A_0 and A_1.*

Proof. Let f_0: $A_0 \to \mathscr{Y}$ be a continuous function. By hypothesis, $f_0|A_0 \cap A_1$ is homotopic to a constant. Then it has (compare Theorem 7 of Section II) an extension f_1: $A_1 \to \mathscr{Y}$. Define $f = f_0 + f_1$. Since $(f_0|A_0 \cap A_1) = (f_1|A_0 \cap A_1)$, f: $(A_0 \cup A_1) \to \mathscr{Y}$ is continuous. By hypothesis, f is homotopic to a constant, and so is f_0.

Theorems 9 of Section II and 2′ of Section III imply Theorems 7 and 8 below.

THEOREM 7. *Let A_0 and A_1 be two closed sets such that* $\dim A_0 \cap A_1 \leqslant m-2$. *If A_0 and A_1 are contractible with respect to $\mathscr{S}_m$ $(m \geqslant 1)$, then so is the set $A_0 \cup A_1$.*

Remark 1. As will be seen in § 58, I, Theorem 5, the condition $\dim(A_0 \cap A_1) \leqslant m-2$ may be replaced in case $m = 1$ by a less restrictive condition of contractibility of $A_0 \cap A_1$ with respect

(1) See my paper in Fund. Math. 20 (1933), p. 208.

(2) This is the length of the edge of a regular $(n+1)$-dimensional simplex inscribed in $\mathscr{S}_n$.

to $\mathcal{S}_{m-1}$ (which is then equivalent to the connectedness of $A_0 \cap A_1$). The case $m = 2$ is different; if $\mathcal{S}_3$ is decomposed into two hemispheres, their intersection $\mathcal{S}_2$ is contractible with respect to $\mathcal{S}_1$, though $\mathcal{S}_3$ is not contractible with respect to $\mathcal{S}_2$. See also Remark 2.

THEOREM 8. *If $\mathcal{X}$ is irreducible relative to the non-contractibility with respect to $\mathcal{S}_m$ (thus, if every proper closed subset of $\mathcal{X}$ is contractible with respect to $\mathcal{S}_m$), then no closed set of dimension $\leqslant m-2$ separates the space $\mathcal{X}$* (¹).

THEOREM 9. *Let $\mathcal{Y}$ be an absolute neighbourhood retract. If each of the compact sets $A_0 \supset A_1 \supset \ldots$ is contractible with respect to $\mathcal{Y}$, then so is their intersection $P = A_0 \cap A_1 \cap \ldots$.*

Proof. Let f: $P \to \mathcal{Y}$ be a continuous function and f^* be its extension to a neighbourhood E of P. Since the sets A_i are compact, so $A_i \subset E$ for sufficiently large values of i. By hypothesis the function $f^* | A_i$ is homotopic to a constant; and hence so is f, because $f = f^* | P \subset f^* | A_i$.

THEOREM 10. *In order that a locally connected continuum be contractible with respect to $\mathcal{S}_m$ $(m \geqslant 2)$* (²), *it is necessary and sufficient that so be all its cyclic elements* (³).

The condition is necessary by Theorems 15 of § 53, III and 3. It is sufficient by Theorem 3 of Section III.

THEOREM 11. *The space $\mathcal{Y}$ is integrally connected in dimension n if and only if $\mathcal{S}_n$ is contractible with respect to $\mathcal{Y}$.*

Because the integral connectedness in dimension n means that

$$\mathcal{Y}^{\mathcal{S}_n} = \mathcal{Y}^{\mathcal{Q}_{n+1}} | \mathcal{S}_n, \quad \text{therefore that} \quad \mathcal{Y}^{\mathcal{S}_n} = \mathcal{Y}^{\Lambda(\mathcal{S}_n)} | \mathcal{S}_n,$$

since $\mathcal{Q}_{n+1} \underset{\text{top}}{=} \Lambda(\mathcal{S}_n)$, and hence that $\mathcal{S}_n$ is contractible with respect to $\mathcal{Y}$ (compare I, Theorem 8).

Remark 2. The problem arises to determine the subscripts m and n such that the sphere $\mathcal{S}_m$ be contractible with respect to the sphere $\mathcal{S}_n$.

Let us mention, for example, that $\mathcal{S}_5$ is contractible with re-

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(¹) See P. Alexandrov, *Dimensionstheorie*, Math. Ann. 106 (1932), p. 161.

(²) Theorem 10 holds also for $m = 1$. See § 56, X, Theorem 5.

(³) Borsuk, Fund. Math. 18 (1932), p. 206.

spect to $\mathscr{S}_2$, whereas $\mathscr{S}_3$ and $\mathscr{S}_4$ are not. $\mathscr{S}_{n+2}$ is contractible with respect to $\mathscr{S}_n$ for $n \geqslant 3$, $\mathscr{S}_{2n-1}$ is not for even n [1].

The contractibility of (arbitrary) spaces with respect to $\mathscr{S}_1$ will be studied in § 58.

VI. Spaces contractible in themselves. We shall call so every space which is contractible with respect to itself.

THEOREM 1. *Every space contractible in itself is arcwise connected* [2].

Proof. Since by the hypothesis the identity mapping is homotopic to a constant, assume that

$$h\colon \mathscr{X}\times\mathscr{I} \to \mathscr{X} \text{ is continuous and } h(x,0)=x, \quad h(x,1)=c. \tag{0}$$

Let $p \in \mathscr{X}$. Define $f(t) = h(p,t)$. It follows that f is continuous and that

$$f\colon \mathscr{I} \to \mathscr{X}, \quad f(0) = p \quad \text{and} \quad f(1) = c.$$

Thus the point c can be joined to every point p of $\mathscr{X}$ by a locally connected continuum, and hence by an arc.

THEOREM 2. *The following five conditions are equivalent.*

$$\mathscr{X} \text{ is contractible in itself,} \tag{1}$$

$$\mathscr{X} \text{ is deformable to one point,} \tag{2}$$

$$\mathscr{X} \text{ is contractible with respect to every space } \mathscr{Y}, \tag{3}$$

$$\text{every space } \mathscr{T} \text{ is contractible with respect to } \mathscr{X}, \tag{4}$$

$$\mathscr{X}^{\mathscr{X}} \text{ is arcwise connected.} \tag{5}$$

Proof. (1) implies (2), because condition (1) means that the identity mapping (of $\mathscr{X}$) is homotopic to a constant.

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(1) The non-contractibility of $\mathscr{S}_3$ with respect to $\mathscr{S}_2$ and of $\mathscr{S}_{2n-1}$ with respect to $\mathscr{S}_n$ (for even n) has been established by H. Hopf. See of that author, *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche*, Math. Ann. 104 (1931), p. 639, and *Über die Abbildungen von Sphären auf Sphären niedriger Dimension*, Fund. Math. 25 (1935), p. 427.

For other results and for more details, see S. Eilenberg, *Lectures in Topology* edited by Wilder and Ayres, Ann Arbor 1941, p. 57, and Ann. of Math. 41 (1940), p. 662, H. Freudenthal, Comp. Math. 5 (1937), p. 299, and Proc. Akad. Amsterdam 42 (1939), p. 139, W. Hurewicz, Proc. Akad. Amsterdam 38 (1935), p. 112, L. Pontrjagin, C. R. Acad. Sc. U.R.S.S. 19 (1938), p. 147 and 361, and Matem. Sbornik 9 (1941), p. 331.

(2) This is a particular case of Theorem 4.

(2) implies (3). Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous function. If the function h satisfies conditions (0), it follows that $f \simeq f(c)$, because $fh\colon \mathcal{X} \times \mathcal{I} \to \mathcal{Y}$ is continuous, $fh(x, 0) = f(x)$ and $fh(x, 1) = f(c)$.

(2) implies (4). Let $f\colon \mathcal{T} \to \mathcal{X}$ be a continuous function. If the function h satisfies conditions (0), define

$$g(t, u) = h[f(t), u], \quad \text{where} \quad u \in \mathcal{I}.$$

It follows that $f \simeq c$. Because

$$g\colon \mathcal{T} \times \mathcal{I} \to \mathcal{X} \text{ is continuous}, \quad g(t, 0) = f(t) \quad \text{and} \quad g(t, 1) = c.$$

The implications $(3) \to (1)$ and $(4) \to (1)$ are obvious. Finally the equivalence $(5) \equiv (1)$ is an immediate consequence of Theorems 2 of Section V and 1.

THEOREM 3. *Every absolute retract and every set $\Lambda(\mathcal{X})$ is con. tractible in itself.*

This follows from (2) by virtue of Theorems 4 and 2 of Section IV-

THEOREM 4. *Every contractible in itself space $\mathcal{X}$ is integrally connected in every dimension $n = 0, 1, 2, \ldots$*

Proof. According to Theorem 2 (4), the sphere $\mathcal{S}_n$ is contractible with respect to $\mathcal{X}$, which implies the required conclusion by Theorem 11 of Section V.

VII. Local contractibility. The space $\mathcal{Y}$ is said to be *contractible in itself at the point y_0* if to every $\varepsilon > 0$ corresponds an $\eta > 0$ such that each set A, for which $\delta(y_0 \cup A) < \eta$, can be deformed to y_0 in the ball Q with the center y_0 and the radius ε, i.e. if the identity mapping of A is homotopic to y_0 with respect to Q (1).

The space $\mathcal{Y}$ is said to be *locally contractible in itself* if it is contractible in itself at each point.

THEOREM 1. *If $\mathcal{Y}$ is contractible in itself at the point y_0, then to every $\varepsilon > 0$ corresponds an $\eta > 0$ such that, for any compact space $\mathcal{X}$, every continuous function $f\colon \mathcal{X} \to \mathcal{Y}$, for which $\delta[y_0 \cup f(\mathcal{X})] < \eta$, has an extension $f^*\colon \Lambda(\mathcal{X}) \to \mathcal{Y}$ such that*

$$f^*(0) = y_0 \quad \textit{and} \quad \delta\{y_0 \cup f^*[\Lambda(\mathcal{X})]\} < \varepsilon. \tag{i}$$

(1) This concept is due to K. Borsuk. See Fund. Math. 19 (1932), p. 236.

Proof. Since the identity mapping of the set $f(\mathscr{X})$ is homotopic to y_0 with respect to the ball Q with center y_0 and radius $\varepsilon/2$, so $f \simeq y_0$ with respect to Q. Conditions (i) follow by Theorem 8 of Section I (for $\mathscr{Y} = Q$).

THEOREM 2. *The contractibility in itself at the point y_0 implies the local connectedness at this point in all dimensions $n = 0, 1, 2, \ldots$*

Proof. It is necessary only to substitute $\mathscr{S}_n$ for $\mathscr{X}$ in Theorem 1 by virtue of the homeomorphism $\Lambda(\mathscr{S}_n) \underset{\text{top}}{=} \mathscr{Q}_{n+1}$.

THEOREM 3. *Every space locally contractible in itself is l.c. n, and every space locally and integrally contractible in itself, is l.c. n and i.c. n, for each $n = 0, 1, 2, \ldots$.*

Proof. This is a consequence of Theorem 2 combined with Theorem 4 of Section VI.

THEOREM 4. *The condition $(\mathscr{Y}\times\mathscr{I})\tau_v\mathscr{Y}$ implies that $\mathscr{Y}$ is locally contractible in itself.*

The condition $(\mathscr{Y}\times\mathscr{I})\tau\mathscr{Y}$ implies that $\mathscr{Y}$ is integrally and locally contractible in itself.

Proof. Let $y_0 \in \mathscr{Y}$ and let $F_1, F_2, \ldots$ be a sequence of closed neighbourhoods of y_0 such that

$$\lim_{n=\infty} \delta(F_n) = 0. \tag{ii}$$

Let $\varepsilon > 0$. We have to prove that there exists, for a fixed and sufficiently large n, a continuous deformation $h\colon F_n \times \mathscr{I} \to \mathscr{Y}$ of F_n to y_0 such that $\delta[h(F_n \times \mathscr{I})] < \varepsilon$. Let

$$F = \big(F_1 \times (0)\big) \cup \left(F_2 \times \left(\frac{1}{2}\right)\right) \cup \ldots \cup \left(F_n \times \left(\frac{n-1}{n}\right)\right) \cup \ldots \cup \big(\mathscr{Y} \times (1)\big).$$

Therefore $F = \overline{F} \subset \mathscr{Y} \times \mathscr{I}$. Define

$$f\left(y, \frac{n-1}{n}\right) = y \text{ for } y \in F_n, \qquad \text{and} \qquad f(y, 1) = y_0 \text{ for } y \in \mathscr{Y}.$$

It follows by (ii) that $f\colon F \to \mathscr{Y}$ is continuous.

Since $(\mathscr{Y}\times\mathscr{I})\tau_v\mathscr{Y}$, there exists an open neighbourhood G of F and an extension $f^*\colon G \to \mathscr{Y}$ of f. Since G is open, there exists (compare § 41, IV, Theorem 1) an open neighbourhood H of y_0

(in $\mathscr{Y}$) and a closed interval $J = \langle t_0, 1\rangle$ such that $H \times J \subset G$. Moreover, we may assume that $\delta[f^*(H\times J)] < \varepsilon$ (because f^* is continuous).

Consider n such that $(n-1)/n > t_0$ and that $F_n \subset H$ (compare (ii)). Setting $J_n = \langle (n-1)/n, 1\rangle$, one gets

$$F_n \times J_n \subset H\times J, \quad \text{which implies} \quad \delta[f^*(F_n\times J_n)] < \varepsilon.$$

Define

$$h(y,t) = f^*\left(y, \frac{t+n-1}{n}\right) \quad \text{where} \quad y \in F_n \text{ and } t \in \mathscr{I}.$$

It follows that

$$h(y,0) = f^*\left(y, \frac{n-1}{n}\right) = f\left(y, \frac{n-1}{n}\right) = y, \quad h(y,1) = f^*(y,1) = y_0,$$

$h\colon F_n \times \mathscr{I} \to \mathscr{Y}$ is continuous and

$$\delta[h(F_n\times\mathscr{I})] = \delta[f^*(F_n\times J_n)] < \varepsilon.$$

Thus the local contractibility of $\mathscr{Y}$ is established.

Finally, the condition $(\mathscr{Y}\times\mathscr{I})\,\tau\,\mathscr{Y}$ implies the existence of a continuous function $f\colon \mathscr{Y}\times\mathscr{I}\to\mathscr{Y}$ such that $f(y,0) = y$ and $f(y,1) = y_0$, which proves the integral contractibility of $\mathscr{Y}$ (compare VI (2)).

It follows that

THEOREM 5. *Every absolute neighbourhood retract is locally contractible in itself and every absolute retract is locally and integrally contractible in itself*(1).

For the finite dimensional spaces the following theorem holds, which is a converse of Theorems 3 and 5.

THEOREM 6. *Let* $\dim\mathscr{Y} = n$. *The following conditions are equivalent*

$$\mathscr{Y} \textit{ is l.c. } n+1, \tag{1}$$

$$(\mathscr{Y}\times\mathscr{I})\,\tau_v\,\mathscr{Y}, \tag{2}$$

$$\mathscr{Y} \textit{ is locally contractible in itself}, \tag{3}$$

$$\mathscr{Y} \textit{ is an absolute neighbourhood retract.} \tag{4}$$

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(1) See K. Borsuk, Fund. Math. 19 (1932), p. 237, and S. Lefschetz, *Topics in Topology*, Princeton 1942, p. 93.

Similarly, *the following conditions are equivalent*

$$\mathscr{Y} \textit{ is l.c. } n+1 \textit{ and i.c. } n+1, \tag{1'}$$

$$(\mathscr{Y}\times\mathscr{I})\,\tau\,\mathscr{Y}, \tag{2'}$$

$$\mathscr{Y} \textit{ is locally and integrally contractible in itself}, \tag{3'}$$

$$\mathscr{Y} \textit{ is an absolute retract.} \tag{4'}$$

Proof. (1) $\to$ (2). Condition (1) combined with the inequality $\dim(\mathscr{Y}\times\mathscr{I}) \leqslant n+1$, implies (2) by Theorem 1 (4) of § 53, IV.

The implication (2) $\to$ (3) follows from Theorem 4.

(3) $\to$ (4). According to Theorem 1 of § 45, VII, assume that

$$\mathscr{Y} \subset \mathscr{S}_{2n+1} \quad \text{and} \quad R = \mathscr{Y} \cup (\mathscr{Q}_{2n+2} - \mathscr{S}_{2n+1}).$$

Since $\mathscr{Y}$ is locally connected in dimensions $< 2n+2$ (compare Theorem 2), $\mathscr{Y}$ is a neighbourhood retract of the space R (compare § 53, IV, Theorem 1 (2)). Since the latter is an absolute retract (compare § 53, III, (i)), then $\mathscr{Y}$ is an ANR (by § 53, III, Theorem 6).

The implication (4) $\to$ (1) is obvious, and so the first part of Theorem 6 is established. The proof of the second one is analogous.

Remark. Theorem 6 does not apply to infinite dimension. There exists indeed a (compact) space, locally and integrally contractible in itself, which is not an absolute neighbourhood retract [1].

COROLLARY 7. *Every connected subset E of a dendrite is an absolute retract.*

Proof. First, E is integrally and locally arcwise connected (compare Theorem 3 of § 52, IV, and Corollary 2 of § 51, VI). Next, E is i.c. n and l.c. n for each n (by Theorem 3 of § 53, IV), since every subcontinuum of a dendrite is a dendrite, hence an AR (by Theorem 16 of § 53, III). Thus, E satisfies condition (1′), hence — condition (4′).

VIII. The components of $\mathscr{Y}^{\mathscr{X}}$ where $\mathscr{Y}$ is ANR. In Sections VIII–X, *$\mathscr{X}$ is assumed to be a metric space and $\mathscr{Y}$ a separable* ANR. C denotes an arbitrary compact subset of $\mathscr{X}$, i.e. $C \in \boldsymbol{C}(X)$.

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[1] See K. Borsuk, *Sur un espace compact localement contractile qui n'est pas un rétracte absolu de voisinage*, Fund. Math. 35 (1948), p. 175.

THEOREM 0. *If $\mathcal{X}$ is compact, the space $\mathcal{Y}^{\mathcal{X}}$ (with the c.o. topology) is* ANR. *Consequently its components are closed-open and arcwise connected subsets of $\mathcal{Y}^{\mathcal{X}}$, and hence every two elements of each component are homotopic.*

This follows immediately from Theorem 3 of § 53, III and Theorem 6 of Section I.

Denote, as in § 44, III, by ϱ_C the *restriction operation*, i.e. $\varrho_C(f) = f|C$ for $C \subset \mathcal{X}$ and $f \in \mathcal{Y}^{\mathcal{X}}$. Thus

$$\varrho_C\colon \mathcal{Y}^{\mathcal{X}} \to \mathcal{Y}^{C}.$$

Furthermore, *the restriction operation is continuous* (by Theorem 1 of § 44, III).

Let $\Gamma \subset \mathcal{Y}^{\mathcal{X}}$. As usually, $\varrho_C(\Gamma)$, written more concisely, $\Gamma|C$, is the image of C under the restriction operation, i.e. it is the set of all restricted mappings $f|C$ where $f \in \Gamma$.

THEOREM 1(¹). *If Γ is the component of f in $\mathcal{Y}^{\mathcal{X}}$, then $\Gamma|C$ is the component of $f|C$ in $\mathcal{Y}^{C}$.*

Proof. Since the restriction operation is continuous and Γ is connected, then so is $\varrho_C(\Gamma) = \Gamma|C$.

Denote by Δ the component of $f|C$ in $\mathcal{Y}^{C}$. Since $(f|C) \in (\Gamma|C)$, we have $(\Gamma|C) \subset \Delta$. We will show that $\Delta = \Gamma|C$. Thus, we have to prove that

$$(g \in \Delta) \Rightarrow \big(g \in (\Gamma|C)\big).$$

Since C is compact, the elements of Δ are mutually homotopic (by Theorem 0). In particular $g \simeq (f|C)$, and by the Theorem of Borsuk (see II, 3), there is $f^* \in \mathcal{Y}^{\mathcal{X}}$ such that $g = (f^*|C)$ and $f^* \simeq f$. It follows (by Theorem 6 of Section I) that there is an arc in $\mathcal{Y}^{\mathcal{X}}$ joining f^* to f. Hence $f^* \in \Gamma$ and since $g = f^*|C$, we have finally $g \in (\Gamma|C)$.

THEOREM 2. *Let Γ be a component of $\mathcal{Y}^{\mathcal{X}}$. Then*

$$(f \in \Gamma) \equiv [(f|C) \in (\Gamma|C) \text{ for each } C \in \boldsymbol{C}(\mathcal{X})].$$

This follows from § 44, III (2), because Γ and $\Gamma|C$ are closed subsets of the spaces $\mathcal{Y}^{\mathcal{X}}$ and $\mathcal{Y}^{C}$ respectively.

(¹) See my paper *Un critère de coupure de l'espace euclidien par un sous-ensemble arbitraire*, Math. Zeitschr. 72 (1959), pp. 88–94.

COROLLARY 2a. *Let Γ_0 and Γ_1 be two components of $\mathcal{Y}^{\mathcal{X}}$. Then*

$$(\Gamma_0 = \Gamma_1) \equiv [(\Gamma_0|C) = (\Gamma_1|C) \text{ for each } C \in \boldsymbol{C}(\mathcal{X})].$$

THEOREM 3. *Let $\Phi \subset \mathcal{Y}^{\mathcal{X}}$ be closed. Then Φ is a component of $\mathcal{Y}^C$ if and only if $\Phi|C$ is a component of $\mathcal{Y}^C$ for each $C \in \boldsymbol{C}(\mathcal{X})$.*

Proof. In view of Theorem 1, it remains to show that the condition is sufficient. So let $f \in \Phi$, let Γ be its component in $\mathcal{Y}^{\mathcal{X}}$ and let $C \subset \mathcal{X}$ be compact. We shall show that $\Phi = \Gamma$. According to Corollary 2a of § 44, III, it is sufficient to show that $\Phi|C = \Gamma|C$.

But $\Phi|C$ and $\Gamma|C$ are components of $\mathcal{Y}^C$ (the first by assumption, and the second by Theorem 1) and both contain $f|C$. Hence they are identical.

THEOREM 4. *The quasi-components of the space $\mathcal{Y}^{\mathcal{X}}$ are connected, and therefore they are identical with its components.*

Proof. Let f_0 and f_1 be two elements of $\mathcal{Y}^{\mathcal{X}}$, let Γ be the component of f_0 and let $f_1 \notin \Gamma$. We have to show that f_1 does not belong to the quasi-component of f_0, i.e. that there is a closed-open $\Phi \subset \mathcal{Y}^{\mathcal{X}}$ such that $f_0 \in \Phi$ and $f_1 \notin \Phi$.

Since $f_1 \notin \Gamma$, there is by Theorem 2 a compact $C \subset \mathcal{X}$ such that $(f_1|C) \notin (\Gamma|C)$. Put

$$\Phi = \varrho_C^{-1}(\Gamma|C), \quad \text{i.e.} \quad (f \in \Phi) \equiv [(f|C) \in (\Gamma|C)].$$

Hence $f_0 \in \Phi$ and $f_1 \notin \Phi$. Furthermore, Φ is closed-open in $\mathcal{Y}^{\mathcal{X}}$, because ϱ_C is continuous and $\Gamma|C$ is (by Theorem 1) a component of the locally connected space $\mathcal{Y}^C$, hence a closed-open subset of that space.

IX. The space $\mathfrak{C}(\mathcal{Y}^{\mathcal{X}})$ of components of $\mathcal{Y}^{\mathcal{X}}$. We denote (compare § 46, Va) by $\mathfrak{C}(\mathcal{Y}^{\mathcal{X}})$ the decomposition space of $\mathcal{Y}^{\mathcal{X}}$ into its quasi-components (hence — by Theorem 4 of Section VIII — into its components). That means that $\mathfrak{Z} \subset \mathfrak{C}(\mathcal{Y}^{\mathcal{X}})$ is open in $\mathfrak{C}(\mathcal{Y}^{\mathcal{X}})$ if and only if $S(\mathfrak{Z})$ (i.e. the union of components belonging to $\mathfrak{Z}$) is open in $\mathcal{Y}^{\mathcal{X}}$.

Denote by $P(f)$ the component of f in $\mathcal{Y}^{\mathcal{X}}$. Thus P is the (natural) *projection* of $\mathcal{Y}^{\mathcal{X}}$ onto $\mathfrak{C}(\mathcal{Y}^{\mathcal{X}})$. By Theorem 1 of § 19, II,

P is bicontinuous,

i.e. $P^{-1}(\mathfrak{Z})$ is open in $\mathcal{Y}^{\mathcal{X}}$ if and only if $\mathfrak{Z}$ is open in $\mathfrak{C}(\mathcal{Y}^{\mathcal{X}})$ (compare § 13, XV).

Similarly, for compact $C \subset \mathscr{X}$, denote by $Q_C(g)$ the component Δ of g in $\mathscr{Y}^C$, and put

$$S_{C_0C_1}(\Delta) = \Delta | C_0 \quad \text{for} \quad C_0 \subset C_1 \quad \text{and} \quad \Delta \in \mathfrak{C}(\mathscr{Y}^{C_1}).$$

Finally, put $R_C(\Gamma) = \Gamma | C$ for $\Gamma \in \mathfrak{C}(\mathscr{Y}^{\mathscr{X}})$. By Theorem 1 of Section I, we have $(\Gamma | C) \in \mathfrak{C}(\mathscr{Y}^C)$. Thus

$$R_C\colon \mathfrak{C}(\mathscr{Y}^{\mathscr{X}}) \to \mathfrak{C}(\mathscr{Y}^C) \quad \text{and} \quad R\colon \mathfrak{C}(\mathscr{Y}^{\mathscr{X}}) \to \boldsymbol{P}\mathfrak{C}(\mathscr{Y}^C)$$

where $R(\Gamma) = \{R_C(\Gamma)\}$.

THEOREM 1. *The following diagram is commutative.*

$$\begin{tikzcd}
\mathscr{Y}^{\mathscr{X}} \arrow[r, "P"] \arrow[d, "\varrho_C"'] & \mathfrak{C}(\mathscr{Y}^{\mathscr{X}}) \arrow[d, "R_C"] \\
\mathscr{Y}^C \arrow[r, "Q_C"'] & \mathfrak{C}(\mathscr{Y}^C)
\end{tikzcd}$$

Consequently R_C is continuous, and so is R.

Proof. Let $f \in \mathscr{Y}^{\mathscr{X}}$ and $f \in \Gamma \in \mathfrak{C}(\mathscr{Y}^{\mathscr{X}})$, i.e. $\Gamma = P(f)$. Then $R_C[P(f)] = P(f) | C = \Gamma | C$ and by Theorem 1 of Section I, $\Gamma | C$ is the component of $f | C$ in $\mathscr{Y}^C$. Thus $\Gamma | C = Q_C(f | C) = Q_C[\varrho_C(f)]$.

This completes the proof of the commutativity of the diagram.

The continuity of R_C follows from Theorem 3 of § 13, XV in virtue of the bicontinuity of P and the continuity of $Q_C \circ \varrho_C$ (by Theorem 1 of § 44, III, ϱ_C is continuous).

THEOREM 2. *If $\mathscr{X}$ is compact, then $\mathfrak{C}(\mathscr{Y}^{\mathscr{X}})$ is discrete.*

Because the elements of $\mathfrak{C}(\mathscr{Y}^{\mathscr{X}})$ are closed-open sets (by Theorem 0 of Section I).

THEOREM 3. *Let $\mathfrak{Z} \subset \mathfrak{C}(\mathscr{Y}^{\mathscr{X}})$. Then*

$$(\Gamma \in \overline{\mathfrak{Z}}) \equiv [R_C(\Gamma) \in R_C(\mathfrak{Z}) \textit{ for each } C], \tag{0}$$

i.e.

$$(\Gamma \in \overline{\mathfrak{Z}}) \equiv [(\Gamma | C) \in (\mathfrak{Z} | C) \textit{ for each } C].$$

Proof. The implication from left to right follows from the continuity of R_C (Theorem 1) and from the fact that $\mathfrak{Z}|C$ is closed in the space $\mathfrak{C}(\mathscr{Y}^C)$ (since the last one is discrete by Theorem 2).

Now assume that the right side of (0) is true. We are going to show that $\Gamma \in \overline{\mathfrak{Z}}$.

Let $f \in \Gamma$. It follows that

$$(f|C) \in (\Gamma|C) \in (\mathfrak{Z}|C), \quad \text{i.e.} \quad (f|C) \in \mathrm{S}(\mathfrak{Z}|C).$$

Clearly $\mathrm{S}(\mathfrak{Z}|C) = \mathrm{S}(\mathfrak{Z})|C$, so that $(f|C) \in [\mathrm{S}(\mathfrak{Z})|C]$ and therefore $f \in \overline{\mathrm{S}(\mathfrak{Z})}$ (by § 44, III (2)); since f is an arbitrary element of Γ, it follows that $\Gamma \subset \overline{\mathrm{S}(\mathfrak{Z})}$. But $\overline{\mathrm{S}(\mathfrak{Z})} \subset \mathrm{S}(\overline{\mathfrak{Z}})$ (because $\overline{\mathrm{S}(\mathfrak{Z})} \subset \overline{\mathrm{S}(\overline{\mathfrak{Z}})} = \mathrm{S}(\overline{\mathfrak{Z}})$ by definition of quotient-topology), and it follows that $\Gamma \subset \mathrm{S}(\overline{\mathfrak{Z}})$, i.e. $\Gamma \in \overline{\mathfrak{Z}}$.

THEOREM 4. *The projection* P: $\mathscr{Y}^{\mathscr{X}} \to \mathfrak{C}(\mathscr{Y}^{\mathscr{X}})$ *is an open mapping.*

Proof. Let $G \subset \mathscr{Y}^{\mathscr{X}}$ be open and let $\Gamma \in P(G)$, i.e. $\Gamma \cap G \neq 0$; so let $f \in \Gamma \cap G$.

Suppose, contrary to our assertion, that $P(G)$ is not open, namely that $\Gamma \in \overline{\mathfrak{Z}}$ where $\mathfrak{Z} = \mathfrak{C}(\mathscr{Y}^{\mathscr{X}}) - P(G)$. It follows by Theorem 3 that $(\Gamma|C) \in (\mathfrak{Z}|C)$ for each compact C. Since $(f|C) \in (\Gamma|C)$, so $(f|C) \in \mathrm{S}(\mathfrak{Z}|C)$. Now $\mathrm{S}(\mathfrak{Z}|C) = \mathrm{S}(\mathfrak{Z})|C$, hence $(f|C) \in [\mathrm{S}(\mathfrak{Z})|C]$, which yields $f \in \overline{\mathrm{S}(\mathfrak{Z})}$. Consequently, $\overline{\mathrm{S}(\mathfrak{Z})} \cap G \neq 0$ and hence $\mathrm{S}(\mathfrak{Z}) \cap G \neq 0$ (since G is open) which means that there is a component of $\mathfrak{C}(\mathscr{Y}^{\mathscr{X}})$ which belongs to $\mathfrak{Z}$ and has point in common with G.

But this contradicts the definition of $\mathfrak{Z}$.

Remark. The decomposition of $\mathscr{Y}^{\mathscr{X}}$ in components is *lower semicontinuous*. Because the property of P of being an open mapping is equivalent to the openness of the set $\underset{\Gamma}{E}(\Gamma \cap G \neq 0)$ (provided $G \subset \mathscr{Y}^{\mathscr{X}}$ is open).

Clearly, $S_{C_0C_1} \circ S_{C_1C_2} = S_{C_0C_2}$ for $C_0 \subset C_1 \subset C_2$. Therefore the spaces $\mathfrak{C}(\mathscr{Y}^C)$ and the mappings $S_{C_0C_1}$ form an *inverse system* (the variables C, C_0, C_1 ranging over the family $\boldsymbol{C}(\mathscr{X})$ which is obviously directed relative the inclusion $C_0 \subset C_1$, compare § 44, III).

THEOREM 5. $\mathfrak{C}(\mathscr{Y}^{\mathscr{X}}) \underset{\text{top}}{\subset} \underset{C, C_0 \subset C_1}{\mathrm{Lim}} \{\mathfrak{C}(\mathscr{Y}^C), S_{C_0C_1}\}$.

Namely, the mapping R: $\mathfrak{C}(\mathscr{Y}^{\mathscr{X}}) \to \underset{C}{\boldsymbol{P}} \mathfrak{C}(\mathscr{Y}^C)$ is the required homeomorphism of the space $\mathfrak{C}(\mathscr{Y}^{\mathscr{X}})$ into the above considered inverse limit.

Proof. Denote, concisely, this limit by $\mathfrak{L}$. We have first to show that

$$\Gamma \in \mathfrak{C}(\mathscr{Y}^{\mathscr{X}}) \Rightarrow R(\Gamma) \in \mathfrak{L}. \tag{1}$$

By definition, $R(\Gamma) = \{R_C(\Gamma)\}$. Thus to show (1), it suffices to prove that $S_{C_0C_1}[R_{C_1}(\Gamma)] = R_{C_0}(\Gamma)$. But this means that $(\Gamma|C_1)|C_0 = \Gamma|C_0$, which is obvious.

It remains to be shown that R is a homeomorphism, i.e. that

$$(\Gamma \in \overline{\mathfrak{Z}}) \equiv [R(\Gamma) \in \overline{R(\mathfrak{Z})}] \quad \text{for each } \mathfrak{Z} \subset \mathfrak{C}(\mathscr{Y}^{\mathscr{X}}). \tag{2}$$

Now, since $R(\Gamma) \in \mathfrak{L}$ (by (1)), we have the equivalence (see § 16, VI, Theorem 3)

$$[R(\Gamma) \in \overline{R(\mathfrak{Z})}] \equiv [R_C(\Gamma) \in \overline{R_C(\mathfrak{Z})} \text{ for each compact } C], \tag{3}$$

which implies (2) by virtue of Theorem 3.

Corollary. *The space $\mathfrak{C}(\mathscr{Y}^{\mathscr{X}})$ is completely regular.*

Because this space is homeomorphic to a subset of the product $\underset{C}{\boldsymbol{P}}\mathfrak{C}(\mathscr{Y}^{C})$ of completely regular (in fact, discrete) spaces.

We are going now to strengthen Theorem 5 for the case of $\mathscr{X}$ separable and locally compact.

First, let us recall (see Theorem 8 of § 41, X) that in the case under consideration there is a sequence $C_1 \subset C_2 \subset \ldots$ of compact subsets of $\mathscr{X}$ such that each compact subset of $\mathscr{X}$ is a subset of some C_n.

Theorem 6. *If $\mathscr{X}$ is separable and locally compact, then*

$$\mathfrak{C}(\mathscr{Y}^{\mathscr{X}}) \underset{\text{top}}{=} \underset{n, n \leqslant k}{\text{Lim}} \{\mathfrak{C}(\mathscr{Y}^{C_n}), S_{C_nC_k}\}.$$

Namely, denote by $R(\Gamma)$ the element $\{R_{C_n}(\Gamma)\}$ of the product $\overset{\infty}{\underset{n=1}{\boldsymbol{P}}} \mathfrak{C}(\mathscr{Y}^{C_n})$; then R is the required homeomorphism.

Proof. As in the proof of Theorem 5, it follows easily that R is a mapping *into* the above inverse limit. In order to show that R is a *homeomorphism*, it is sufficient (according to (3) and (0)) to prove that if

$$(\Gamma|C_n) \in (\mathfrak{Z}|C_n) \quad \text{for each } n = 1, 2, \ldots, \tag{4}$$

then

$$(\Gamma|C) \in (\mathfrak{Z}|C) \quad \text{for each } C \in \boldsymbol{C}(\mathscr{X}). \tag{5}$$

Let $C \in \boldsymbol{C}(\mathscr{X})$ and $C \subset C_n$. By assumption

$$(\Gamma | C_n) \in (\mathfrak{Z} | C_n), \quad \text{hence} \quad [(\Gamma | C_n) | C] \in [(\mathfrak{Z} | C_n) | C].$$

But $(\Gamma | C_n) | C = \Gamma | C$ and $(\mathfrak{Z} | C_n) | C = \mathfrak{Z} | C$.

Thus $(\Gamma | C) \in (\mathfrak{Z} | C)$. This completes the proof of the implication (4) $\Rightarrow$ (5).

Now we are going to prove the essential part of the theorem, namely that R is *onto*. Thus, it is to be shown that, for each sequence $\Delta_n \in \mathfrak{C}(\mathscr{Y}^{C_n})$, $n = 1, 2, \ldots$, such that

$$\Delta_k | C_n = \Delta_n \quad \text{whenever} \quad n < k, \tag{6}$$

there is $\Gamma \in \mathfrak{C}(\mathscr{Y}^{\mathscr{X}})$ such that

$$\Gamma | C_n = \Delta_n \quad \text{for} \quad n = 1, 2, \ldots \tag{7}$$

For this purpose we shall define by induction a sequence $f_1, f_2, \ldots$ such that

$$f_i \in \Delta_i \quad \text{and} \quad f_i | C_{i-1} = f_{i-1}. \tag{8}$$

Let f_1 be an arbitrary element of Δ_1. Suppose that $f_i \in \Delta_i$ for some $i \geqslant 1$; since $\Delta_{i+1} | C_i = \Delta_i$ by (6), there is $f_{i+1} \in \Delta_{i+1}$ such that $f_{i+1} | C_i = f_i$. Therefore condition (8) is satisfied for $i+1$.

The sequence $f_1, f_2, \ldots$ yields a mapping f: $\mathscr{X} \to \mathscr{Y}$ defined as follows:

$$f(x) = f_i(x) \textit{ where } i \textit{ is the least subscript such that } x \in C_i. \tag{9}$$

f is continuous (by Theorem 3 of § 44, III), since it is continuous on every C_n, hence on every compact subset C of $\mathscr{X}$ (for C is contained in some C_i with a proper subscript i).

Let Γ be the component of f in $\mathscr{Y}^{\mathscr{X}}$. Identities (8) and (9) imply that $f | C_i = f_i$. Therefore $f_i \in \Delta_i \cap (\Gamma | C_i)$, which proves (7) since Δ_n and $\Gamma | C_n$ are components of the same space, namely of $\mathfrak{C}(\mathscr{Y}^{C_n})$ (compare Theorem 1 of Section VIII).

COROLLARY 6a. *If $\mathscr{X}$ is separable and locally compact, then the space $\mathfrak{C}(\mathscr{Y}^{\mathscr{X}})$ is homeomorphic to a closed subset of the space of irrational numbers $\mathscr{N}$, hence to a complete separable* 0-*dimensional space.*

Because $\mathfrak{C}(\mathscr{Y}^{C_n})$ is discrete and countable, and hence $\mathop{\boldsymbol{P}}\limits_{n=1}^{\infty} \mathfrak{C}(\mathscr{Y}^{C_n})$ is homeomorphic to $\mathscr{N}$.

CHAPTER EIGHT

GROUPS $\mathscr{G}^{\mathscr{X}}$, $\mathscr{S}^{\mathscr{X}}$ AND $\mathfrak{M}(\mathscr{X})$

§ 55. Groups $\mathscr{G}^{\mathscr{X}}$ and $\mathfrak{B}_0(\mathscr{X})$

I. General properties of commutative groups. First we shall collect some elementary properties of commutative groups for later use.

A set $\mathscr{X}$ of arbitrary elements is said to be a *commutative* (or *abelian*) *group* with respect to the operation $a+b$, which assigns an element of $\mathscr{X}$ to every pair $a, b \in \mathscr{X}$, if the following conditions are satisfied

(i) $(a+b)+c = a+(b+c)$,

(ii) $a+b = b+a$,

(iii) there exists one (and only one) element 0 such that $a+0 = a$(1),

(iv) to each $a \in \mathscr{X}$ there exists one (and only one) element $-a$ such that $a+(-a) = 0$(2).

If m is an integer, the product $m \cdot a$ is defined in the following way

$$0 \cdot a = 0, \quad m \cdot a = (m-1) \cdot a + a$$
$$\text{and} \quad (-m) \cdot a = m \cdot (-a) \quad \text{for} \quad m > 0.$$

An element $a \neq 0$ is said to be *of finite order* (of order m) if there exists an integer $m \neq 0$ such that $m \cdot a = 0$.

A subset of the group $\mathscr{X}$ is said to be a *subgroup* if it contains $a+b$ provided that it contains a and b, and if it contains $-a$ provided that it contains a. Therefore any subgroup contains 0; and in particular it may reduce to that element.

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(1) This element is said to be the *neutral element* of the group $\mathscr{X}$. In Sections I through VIII of § 55 zero will stand exclusively for the neutral element.

(2) The group operation (called *composition*) is denoted here by the sign $+$. It is sometimes more convenient to consider the composition as multiplication; in such a case 0 is replaced by 1 and $-a$ by $1:a$ (and hence $a-b$ by $a:b$).

The term "group" will always mean a *commutative group* here.

DEFINITION. If in a group $\mathcal{X}$ a topology is defined so that the operations $a+b$ and $-a$ are continuous, then $\mathcal{X}$ is called a *topological group*.

II. Homomorphism. Isomorphism. Let $\mathcal{X}$ and $\mathcal{Y}$ be two groups. Assign to each element x of $\mathcal{X}$ an element $h(x)$ of $\mathcal{Y}$. The correspondence h is said to be a *homomorphism* (or an additive operation) if

$$h(a+b) = h(a)+h(b).$$

Then it follows

$$h(0) = 0 \quad \text{and} \quad h(-a) = -h(a),$$

because $h(0) = h(0+0) = h(0)+h(0)$ and $0 = h(0) = h(a-a) = h(a)+h(-a)$.

Furthermore, $h(\mathcal{X})$ *is a subgroup of* $\mathcal{Y}$, and if G is a subgroup of $\mathcal{Y}$, $h^{-1}(G)$ *is a subgroup of* $\mathcal{X}$. The group $h^{-1}(0)$ is said to be the *kernel* of the homomorphism h.

A homomorphism h such that the condition $h(a)+h(b) = h(c)$ implies $a+b = c$, in other words, such that

$$[a+b = c] \equiv [h(a)+h(b) = h(c)],$$

is said to be an *isomorphism* between $\mathcal{X}$ and $h(\mathcal{X})$.

In order that a homomorphism be an isomorphism, it is necessary and sufficient that it be *one-to-one*, i.e. that its kernel shall reduce to the element 0 (therefore, that the condition $h(x) = 0$ implies $x = 0$).

From the group-theoretical point of view two isomorphic groups cannot be distinguished (like from the topological point of view no two homeomorphic spaces can be distinguished). We shall write in such a case

$$\mathcal{X} \underset{\mathrm{gr}}{=} \mathcal{Y}.$$

III. Factor groups. If G is a subgroup of the group $\mathcal{X}$, the relation "$a \sim b \bmod G$" means that $(a-b) \in G$. In particular,

$$(a \sim 0 \bmod G) \equiv (a \in G).$$

It is easily verified that the relation $a \sim b \bmod G$ is *reflexive, symmetric, and transitive.*

Consequently, if two elements a and b are assumed to belong to the same set if and only if $a \sim b \bmod G$, then the group $\mathscr{X}$ is decomposed into mutually disjoint subsets. The family of these sets is said to be the *factor group* $\mathscr{X}/G$ and the composition of elements of the factor group is defined in the following way. The set C is the sum of A and B, $C = A \dotplus B$, if $c \sim a+b \bmod G$ for any $a \epsilon A$, $b \epsilon B$, $c \epsilon C$. By this definition it is easily seen that $\mathscr{X}/G$ is a commutative group. G is its neutral element; we denote it also by the symbol O.

Clearly,

$$\mathscr{X}/\mathscr{X} \underset{\mathrm{gr}}{=} (0), \tag{1}$$

$$\mathscr{X}/(0) \underset{\mathrm{gr}}{=} \mathscr{X}. \tag{2}$$

THEOREM 1. *The function P: $\mathscr{X} \to \mathscr{X}/G$ (called projection) which assigns to every element x of $\mathscr{X}$ the element $P(x)$ of $\mathscr{X}/G$ such that $x \epsilon P(x)$, is a homomorphism from $\mathscr{X}$ onto $\mathscr{X}/G$.*

Besides, it follows that

$$[x \sim x' \bmod G] = [P(x) = P(x')], \tag{3}$$

$$P^{-1}(O) = G, \tag{4}$$

$$[x \sim 0 \bmod G] \equiv [P(x) = O]. \tag{5}$$

THEOREM 2. *If A is a group such that $G \subset A \subset \mathscr{X}$, then $A = P^{-1}(A/G)$, so that $P(A) = A/G$.*

The following theorem is converse to Theorem 1.

THEOREM 3. *If h: $\mathscr{X} \to h(\mathscr{X})$ is a homomorphism, then the groups $h(\mathscr{X})$ and $\mathscr{X}/h^{-1}(0)$ are isomorphic:*

$$h(\mathscr{X}) \underset{\mathrm{gr}}{=} \mathscr{X}/h^{-1}(0), \tag{6}$$

and the factor group $\mathscr{X}/h^{-1}(0)$ coincides with the family of point-inverses of the function h (i.e. with the family of sets $h^{-1}(y)$ where y varies over the group $h(\mathscr{X})$).

Proof. The transformation h^{-1}: $h(\mathscr{X}) \to \mathscr{X}/h^{-1}(0)$ is just the isomorphism yield by relation (6). Actually, it is one-to-one and it is a homomorphism, because the conditions $x_1 \epsilon h^{-1}(y_1)$ and $x_2 \epsilon h^{-1}(y_2)$ imply that $(x_1+x_2) \epsilon h^{-1}(y_1+y_2)$.

THEOREM 4. *If h: $\mathscr{X} \to h(\mathscr{X})$ is a homomorphism and A is a group such that $h^{-1}(0) \subset A \subset \mathscr{X}$, then $A = h^{-1}h(A)$.*

Proof. Let $F(x) = h^{-1}h(x)$ and $G = h^{-1}(0)$. By Theorem 2 $F(A) = A/G$, hence $F(a) \subset A$ for $a \epsilon A$ and therefore

$$h^{-1}h(A) = \bigcup_{a\epsilon A} h^{-1}h(a) = \bigcup_{a\epsilon A} F(a) = A.$$

THEOREM 5. *Let $\mathscr{X}$ and $\mathscr{Y}$ be two groups and A and B two subgroups of $\mathscr{X}$ and $\mathscr{Y}$, respectively. Let $h\colon \mathscr{X} \to \mathscr{Y}$ be a mapping such that*

(i) *$x \sim 0 \bmod A$ implies $h(x) \sim 0 \bmod B$ (i.e. $h(A) \subset B$),*

(ii) *$h(x_1 + x_2) \sim h(x_1) + h(x_2) \bmod B$.*

(iii) *for every $y \epsilon \mathscr{Y}$ there is an $x \epsilon \mathscr{X}$ such that $y \sim h(x) \bmod B$.*

Let us assign to every $X \epsilon \mathscr{X}/A$ an element $e(X) \epsilon X$ as well as the element $H(X)$ of the factor group $\mathscr{Y}/B$ which contains $h[e(X)]$. The mapping $H\colon (\mathscr{X}/A) \to (\mathscr{Y}/B)$ (which is "induced" by h) does not depend on the definition of the operation $e\colon (\mathscr{X}/A) \to \mathscr{X}$ and is an onto homomorphism.

H is an isomorphism if, moreover,

(iv) *$h(x) \sim 0 \bmod B$ implies $x \sim 0 \bmod A$ (i.e. $h^{-1}(B) \subset A$).*

Proof. According to Theorem 1, let

$$y \epsilon F(y) \epsilon \mathscr{Y}/B \quad \text{and} \quad [y \sim y'] \equiv [F(y) = F(y')]\text{(1)}.$$

It follows that $H(X) = F\{h[e(X)]\}$.

In order to prove that H does not depend on e, assume that $e'(X) \epsilon X$. We have to show that

$$F\{h[e(X)]\} = F\{h[e'(X)]\}.$$

But, since $e(X) \sim e'(X)$, it follows that

$$e(X) - e'(X) \sim 0, \quad \text{so that} \quad h[e(X) - e'(X)] \sim 0$$

by (i). Since $h[e(X) - e'(X)] \sim h[e(X)] - h[e'(X)]$ by (ii), we have

$$F\{h[e(X)] - h[e'(X)]\} = F(0) = 0.$$

H is a homomorphism, because

$$e(X_1 \dot{+} X_2) \sim e(X_1) + e(X_2),$$

$$\text{thus} \quad he(X_1 \dot{+} X_2) \sim he(X_1) + he(X_2),$$

(1) We omit here the terms "mod A" and "mod B".

therefore

$$F\{h[e(X_1 \dot{+} X_2)]\} = F\{h[e(X_1)]+h[e(X_2)]\}$$
$$= F\{h[e(X_1)]\} \dot{+} F\{h[e(X_2)]\}.$$

Furthermore, there corresponds to every $Y \in \mathscr{Y}/B$ an $X \in \mathscr{X}/A$ such that $Y = H(X)$. For, let $y \in Y$, therefore $Y = F(y)$. Let (compare (iii)) $y \sim h(x)$, where $x \in X$. Thus $x \sim e(X)$, which yields (compare (i) and (ii)) $h(x) \sim he(X)$, and hence $y \sim he(X)$ and

$$Y = F(y) = F\{h[e(X)]\} = H(X).$$

Finally, condition (iv) implies that H is one-to-one. For let $F\{h[e(X_1)]\} = F\{h[e(X_2)]\}$. Therefore

$$h[e(X_1)] \sim h[e(X_2)], \text{ so that } h[e(X_1)] - h[e(X_2)] \sim 0,$$

thus

$$h[e(X_1) - e(X_2)] \sim 0$$

and by (iv),

$$e(X_1) - e(X_2) \sim 0, \quad \text{i.e.} \quad e(X_1) \sim e(X_2), \quad \text{which implies}$$
$$X_1 = X_2.$$

Theorems 4 and 5 imply (for $\mathscr{Y} = h(\mathscr{X})$ and $B = h(A)$) the following statement.

THEOREM 6. *Under the hypotheses of Theorem 4, one has the isomorphism*

$$\mathscr{X}/A \underset{\text{gr}}{=} h(\mathscr{X})/h(A).$$

THEOREM 7. *If A, G and $\mathscr{X}$ are groups such that $G \subset A \subset \mathscr{X}$, then*

$$\mathscr{X}/A \underset{\text{gr}}{=} [\mathscr{X}/G]/[A/G]. \tag{7}$$

Because, if F is the homomorphism defined in Theorem 1, then it follows by Theorem 2 that

$$F(\mathscr{X}) = \mathscr{X}/G \quad \text{and} \quad F(A) = A/G,$$

which implies (7) by Theorem 6.

IV. Operation $\widehat{A}$. If A is a subset of a group $\mathscr{X}$, let $\widehat{A}$ denote the least subgroup of $\mathscr{X}$ containing A. The group $\widehat{A}$ does always exist, because the intersection of an arbitrary family of groups (of the family of groups containing A, in the considered case) is a group. The following theorems can easily be established.

THEOREMS 1. $A \subset \widehat{A}$. 2. $\mathscr{X} = \widehat{\mathscr{X}}$. 3. $\widehat{\widehat{A}} = \widehat{A}$.

THEOREM 4. *$A \subset B$ implies $\widehat{A} \subset \widehat{B}$,* so that $\widehat{A} \cup \widehat{B} \subset \widehat{A \cup B}$ *and* $\widehat{A \cap B} \subset \widehat{A} \cap \widehat{B}$.

THEOREM 5. $(A = \widehat{A}) \equiv (A$ *is a group*$)$.

THEOREM 6. $[(x) = \widehat{(x)}] \equiv [x = 0]$.

THEOREM 7. *$\widehat{A}$ is the set of all elements of the form*

$$m_1 \cdot a_1 + \ldots + m_n \cdot a_n,$$

where $a_1, \ldots, a_n \in A$ and $m_1, \ldots, m_n$ are integers.

Because, on one hand, the set of these elements is a group and, on the other hand, the group which contains A contains also all these elements.

It follows

THEOREM 8. *If A and B are two subgroups of $\mathscr{X}$, then $\widehat{A \cup B}$ is the set of all the elements x of the form*

$$x = a + b, \quad \text{where} \quad a \in A \quad \text{and} \quad b \in B. \tag{1}$$

Moreover, if $A \cap B = (0)$, i.e. if the groups A and B have only the neutral element in common, then every $x \in \widehat{A \cup B}$ has one and only one representation of the form (1).

In the latter case, $\widehat{A \cup B}$ is said to be the *direct sum* of A and B; then the correspondence, which assigns a (resp. b) to every $x \in \widehat{A \cup B}$ is a homomorphism of $\widehat{A \cup B}$ onto A (respectively onto B).

THEOREM 9. *If h: $\mathscr{X} \to h(\mathscr{X})$ is a homomorphism, then*

$$h(\widehat{A}) = \widehat{h(A)}, \quad \textit{for any} \quad A \subset \mathscr{X}.$$

This is an immediate consequence of Theorem 7, because

$$\begin{aligned} h(m_1 \cdot a_1 + \ldots + m_n \cdot a_n) &= h(m_1 \cdot a_1) + \ldots + h(m_n \cdot a_n) \\ &= m_1 \cdot h(a_1) + \ldots + m_n \cdot h(a_n). \end{aligned}$$

THEOREM 10. *If A and B are two groups, then*

$$\widehat{A \cup B}/A \cap B = \widehat{(A/A \cap B) \cup (B/A \cap B)}.$$

Proof. According to Theorem 1 of Section III, let F: $\widehat{A \cup B} \to \widehat{A \cup B}/A \cap B$ be the natural homomorphism. Therefore (compare Theorem 1 of Section III and Theorem 9)

$$\begin{aligned} \widehat{A \cup B}/A \cap B &= F(\widehat{A \cup B}) = \widehat{F(A \cup B)} \\ &= \widehat{F(A) \cup F(B)} = \widehat{(A/A \cap B) \cup (B/A \cap B)}. \end{aligned}$$

THEOREM 11. *If $\widehat{A \cup B}$ is the direct sum of the groups A and B, then*

$$\widehat{A \cup B}/A \underset{\text{gr}}{=} B. \tag{2}$$

Proof. There exist (compare Theorem 8) g and h such that

$$g\colon \widehat{A \cup B} \to A, \qquad h\colon \widehat{A \cup B} \to B$$

and

$$x = g(x) + h(x) \qquad \text{for all } x \in \widehat{A \cup B}.$$

Since h is a homomorphism and since $h^{-1}(0) = A$, Theorem 3 of Section III implies formula (2) (replacing $\mathscr{X}$ by $\widehat{A \cup B}$).

V. Linear independence, rank, basis. A subset A of the group $\mathscr{X}$ is said to be *linearly independent*, provided that for every finite system $a_1, \ldots, a_n$ of elements of A the relation

$$m_1 \cdot a_1 + \ldots + m_n \cdot a_n = 0,$$

with integer coefficients $m_1, \ldots, m_n$, holds if and only if

$$m_1 = \ldots = m_n = 0.$$

The maximal number of linearly independent elements in $\mathscr{X}$, if it exists, is said to be the *rank* of $\mathscr{X}$; if there is no such number, $\mathscr{X}$ is said to be of *infinite rank*.

So, for example, the group $\mathscr{G}$ of integers has the rank 1, the group $\mathscr{G}^2$ of complex integers has the rank 2, the group $\mathscr{G}^\omega$ of infinite sequences of integers, every of which contains only a finite number of terms distinct from 0, has the infinite rank. And so is the rank of the group $\mathscr{G}^{\aleph_0}$ of all infinite sequences of integers.

If the set A is linearly independent and if $\widehat{A} = \mathscr{X}$, then A is said to be the *basis* (in the group-theoretical sense) of $\mathscr{X}$. Then every $x \in \mathscr{X}$ admits one and only one representation of the form

$$x = m_1 \cdot a_1 + \ldots + m_n \cdot a_n \quad \text{where} \quad a_1, \ldots, a_n \in A.$$

THEOREM 1. *If $\mathscr{X}$ has a basis consisting of n elements, then $\mathscr{X} \underset{\text{gr}}{=} \mathscr{G}^n$. If $\mathscr{X}$ has a countably infinite basis, then $\mathscr{X} \underset{\text{gr}}{=} \mathscr{G}^\omega$.*

Proof. Let $A = a_1, a_2, \ldots$ be a (finite or infinite) basis of $\mathscr{X}$. Define the function $f\colon \mathscr{X} \to \mathscr{G}^n$ (or $\mathscr{G}^\omega$) so that

$$f(x) = [m_1, m_2, \ldots] \quad \text{for} \quad x = m_1 \cdot a_1 + m_2 \cdot a_2 + \ldots .$$

f is the required isomorphism.

THEOREM 2. *If $h\colon \mathscr{X} \to h(\mathscr{X})$ is a homomorphism, then*

$$\operatorname{rank} \mathscr{X} = \operatorname{rank} h^{-1}(0) + \operatorname{rank} h(\mathscr{X}) \text{ (}^1\text{)}.$$

Proof. Let $a_1, \ldots, a_k$ be a system of elements linearly independent in $h^{-1}(0)$, and $h(a_{k+1}), \ldots, h(a_n)$ a system of elements linearly independent in $h(\mathscr{X})$. We shall show that the elements $a_1, \ldots, a_k, a_{k+1}, \ldots, a_n$ are linearly independent. Assume

$$m_1 \cdot a_1 + \ldots + m_k \cdot a_k + \ldots + m_n \cdot a_n = 0. \tag{1}$$

Since $(m_1 \cdot a_1 + \ldots + m_k \cdot a_k) \in h^{-1}(0)$, it follows that

$$h(m_1 \cdot a_1 + \ldots + m_k \cdot a_k) = 0,$$

hence (compare (1))

$$m_{k+1} \cdot h(a_{k+1}) + \ldots + m_n \cdot h(a_n) = 0,$$

which implies that $m_{k+1} = \ldots = m_n = 0$. Therefore (compare (1))

$$m_1 \cdot a_1 + \ldots + m_k \cdot a_k = 0, \quad \text{so that} \quad m_1 = \ldots = m_k = 0.$$

---

(1) Compare Alexandroff–Hopf, *Topologie I*, p. 573, and Alexandrov, *Combinatorial Topology* (Russian), p. 634.

So the linear independence of the system $(a_1, \ldots, a_n)$ is established. It follows that

$$\operatorname{rank}\mathscr{X} \geqslant \operatorname{rank} h^{-1}(0) + \operatorname{rank} h(\mathscr{X}).$$

In order to prove the inverse inequality, it is legitimate to assume that the ranks of the groups $h^{-1}(0)$ and $h(\mathscr{X})$ are finite. Let

$$\operatorname{rank} h^{-1}(0) = k, \tag{2}$$

$$\operatorname{rank} h(\mathscr{X}) = n - k. \tag{3}$$

We have to show that $\operatorname{rank}\mathscr{X} \leqslant n$, which means that, if $p_1, \ldots, p_{n+1}$ is a system of elements in $\mathscr{X}$, there exists a system of integers $r_1, \ldots, r_{n+1}$, not all of them being zeros, such that

$$r_1 \cdot p_1 + \ldots + r_{n+1} \cdot p_{n+1} = 0. \tag{4}$$

First we shall show that every $x \in \mathscr{X}$ has the following form

$$m \cdot x = m_1 \cdot a_1 + \ldots + m_n \cdot a_n \quad \text{where} \quad m \neq 0. \tag{5}$$

Since the elements $h(a_{k+1}), \ldots, h(a_n)$ are linearly independent, condition (3) implies that there exists a system of integers $s, s_{k+1}, s_{k+2}, \ldots, s_n$ such that

$$s \cdot h(x) - s_{k+1} \cdot h(a_{k+1}) - \ldots - s_n \cdot h(a_n) = 0 \quad \text{and} \quad s \neq 0.$$

It follows that $sx - s_{k+1} a_{k+1} - \ldots - s_n a_n$ belongs to $h^{-1}(0)$, therefore that there exists, according to (2), a system $m_0, m_1, \ldots, m_n$ of integers such that

$$m_0(sx - s_{k+1} a_{k+1} - \ldots - s_n a_n) = m_1 a_1 + \ldots + m_k a_k.$$

Moreover, $m_0 \neq 0$, because the elements $a_1, \ldots, a_k$ are linearly independent. Put

$$m = m_0 s, \quad m_{k+1} = m_0 s_{k+1}, \quad \ldots, \quad m_n = m_0 s_n.$$

Then condition (5) is satisfied.

It follows that to every p_j (where $j = 1, \ldots, n+1$) a system of integers $m_j, m_{j_1}, \ldots, m_{j_n}$ corresponds such that

$$m_j p_j = m_{j_1} a_1 + \ldots + m_{j_n} a_n \tag{6}$$

and

$$m_j \neq 0. \tag{7}$$

Let $c_1, \ldots, c_{n+1}$ be a system of integer solutions, not all of them being zeros, of the n homogeneous equations

$$m_{1i}x_1 + \ldots + m_{n+1,i}x_{n+1} = 0, \quad i = 1, \ldots, n,$$

i.e.

$$\sum_{j=1}^{n+1} m_{ji}c_j = 0, \quad \text{where} \quad i = 1, \ldots, n. \tag{8}$$

Let $r_j = c_j m_j$, $j = 1, \ldots, n+1$. It follows (compare (6) and (8)) that

$$\sum_{j=1}^{n+1} r_j p_j = \sum_{j=1}^{n+1} c_j m_j p_j = \sum_{j=1}^{n+1} \sum_{i=1}^{n} c_j m_{ji} a_i = \sum_{i=1}^{n} a_i \sum_{j=1}^{n+1} m_{ji} c_j = 0.$$

Moreover, at least one of the numbers $r_j \neq 0$ by (7).

THEOREM 3. *If G is a subgroup of the group $\mathscr{X}$, then*

$$\operatorname{rank} \mathscr{X} = \operatorname{rank} G + \operatorname{rank}(\mathscr{X}/G).$$

This is a consequence of Theorem 2 combined with formula (4) of Section III.

THEOREM 4. *If $\widehat{A \cup B}$ is the direct sum of the groups A and B, then*

$$\operatorname{rank} \widehat{A \cup B} = \operatorname{rank} A + \operatorname{rank} B.$$

This is a consequence of Theorem 3 and of Theorem 11 of Section IV.

VI. Linear independence $\bmod G$. Let G be a subgroup of $\mathscr{X}$. A subset A of $\mathscr{X}$ is said to be *linearly independent* $\bmod G$ if, for every finite system $a_1, \ldots, a_n$ of elements of A,

$$m_1 a_1 + \ldots + m_n a_n \sim 0 \bmod G \quad \textit{implies} \quad m_1 = \ldots = m_n = 0.$$

If

$$x \sim m_1 a_1 + \ldots + m_n a_n, \quad \text{where} \quad a_1, \ldots, a_n \in A,$$

for every element x of $\mathscr{X}$, then A is said to be the *set of generators* $\bmod G$ of $\mathscr{X}$.

Moreover, if the set A is linearly independent $\bmod G$, it is said to be the *basis* $\bmod G$ of $\mathscr{X}$.

Let P be the function considered in Theorem 1 of Section III. By formula (5) of Section III, A is linearly independent if and only if the family $P(A)$ (i.e. the family of elements $P(a)$ of the factor group $\mathscr{X}/G$, where $a \epsilon A$) is linearly independent. Thus, we have the following equivalences

$$\{A \text{ is a basis mod} G \text{ of } \mathscr{X}\} \equiv \{P(A) \text{ is a basis of } \mathscr{X}/G\},$$

$$\{A \text{ is a set of generators mod} G \text{ of } \mathscr{X}\}$$
$$\equiv \{P(A) \text{ is a set of generators of } \mathscr{X}/G\}.$$

The rank of the group $\mathscr{X}/G$ is equal to the maximal number of elements of $\mathscr{X}$ which are linearly independent mod G.

VII. Cartesian products. If $\mathscr{X}$ and $\mathscr{Y}$ are two groups, their cartesian product $\mathscr{Z} = \mathscr{X}\times\mathscr{Y}$ becomes a group, the composition being defined in the following way

$$(x, y)+(x', y') = (x+x', y+y'). \tag{1}$$

Thus $(0, 0)$ is the neutral element of the group $\mathscr{Z}$, and

$$-(x, y) = (-x, -y), \qquad m\cdot(x, y) = (mx, my). \tag{2}$$

The *projection* h: $\mathscr{Z} \to \mathscr{X}$, defined by the condition $h(x, y) = x$, is an onto homomorphism.

It is clear that

$$\mathscr{X}\times(0) \underset{\mathrm{gr}}{=} \mathscr{X}, \qquad (0)\times\mathscr{Y} \underset{\mathrm{gr}}{=} \mathscr{Y}, \tag{3}$$

$$\mathscr{X}\times\mathscr{Y} \underset{\mathrm{gr}}{=} \overline{\mathscr{X}\times(0) \cup (0)\times\mathscr{Y}}. \tag{4}$$

The group $\mathscr{X}\times\mathscr{Y}$ is the *direct sum* of the groups $\mathscr{X}\times(0)$ and $(0)\times\mathscr{Y}$, and it follows by Theorem 4 of Section V that

$$\operatorname{rank}(\mathscr{X}\times\mathscr{Y}) = \operatorname{rank}\mathscr{X}+\operatorname{rank}\mathscr{Y}. \tag{5}$$

If f: $\mathscr{X} \to f(\mathscr{X})$ and g: $\mathscr{Y} \to g(\mathscr{Y})$ are homomorphisms, the function h: $\mathscr{X}\times\mathscr{Y} \to f(\mathscr{X})\times g(\mathscr{Y})$, defined by the condition $h(x, y) = [f(x), g(y)]$, is an *onto homomorphism*.

If A and B are two subgroups of $\mathscr{X}$ and $\mathscr{Y}$ respectively, then

$$(\mathscr{X}\times\mathscr{Y})/(A\times B) \underset{\mathrm{gr}}{=} (\mathscr{X}/A)\times(\mathscr{Y}/B). \tag{6}$$

Because, if F is the homomorphism which assigns to every $x \in \mathscr{X}$ an element $F(x)$ of $\mathscr{X}/A$ such that $x \in F(x)$, and if G is the homomorphism defined in a similar way so that $y \in G(y) \in \mathscr{Y}/B$, we put $H(x, y) = [F(x), G(y)]$. The function H is an onto homomorphism $H\colon \mathscr{X}\times\mathscr{Y} \to (\mathscr{X}/A)\times(\mathscr{Y}/B)$, and $A\times B$ is its kernel. The isomorphism (6) follows by Theorem 3 of Section III.

In particular (compare (3) and III (1) and (2))

$$(\mathscr{X}\times\mathscr{Y})/\mathscr{X} \underset{\mathrm{gr}}{=} \mathscr{Y} \tag{7}$$

(we identify the group $\mathscr{X}\times(0)$ with $\mathscr{X}$).

The above statements concerning the product of two groups $\mathscr{X}\times\mathscr{Y}$ can easily be extended to *generalized products* $\mathfrak{Z} = \underset{t \in T}{\boldsymbol{P}} \mathscr{X}_t$ where $\mathscr{X}_t$ is an (abelian) group and T is an arbitrary set. Namely, we assume for every pair $\mathfrak{z} = \{\mathfrak{z}^t\}$ and $\mathfrak{y} = \{\mathfrak{y}^t\}$ of elements of $\mathfrak{Z}$ that

$$\mathfrak{z}+\mathfrak{y} = \{\mathfrak{z}^t+\mathfrak{y}^t\} \quad \text{where} \quad t \in T.$$

Thus $\mathfrak{Z}$ becomes an (abelian) group.

Moreover, *if each $\mathscr{X}_t$ is a topological group, then so is $\mathfrak{Z}$.*

Now suppose that T is a directed set, $\{\mathscr{X}_t, f_{t_0t_1}\}$ is an inverse system, $\mathscr{X}_t$ is a group (for each $t \in T$) and $f_{t_0t_1}\colon \mathscr{X}_{t_1} \to \mathscr{X}_{t_0}$ is a homomorphism (for $t_0 \leqslant t_1$). Put

$$\mathscr{X}_\infty = \underset{t, t_0 \leqslant t_1}{\mathrm{Lim}} \{\mathscr{X}_t, f_{t_0t_1}\}.$$

THEOREM 1. *$\mathscr{X}_\infty$ is a subgroup of $\underset{t}{\boldsymbol{P}} \mathscr{X}_t$.*

Moreover, *if each $\mathscr{X}_t$ is a topological group, then so is $\mathscr{X}_\infty$.*

Proof. Let $\mathfrak{w} = \mathfrak{z}+\mathfrak{y}$ where $\mathfrak{z}, \mathfrak{y} \in \mathscr{X}_\infty$. Let $t_0 \leqslant t_1$. We have to show that $f_{t_0t_1}(\mathfrak{w}^{t_1}) = \mathfrak{w}^{t_0}$. But

$$f_{t_0t_1}(\mathfrak{w}^{t_1}) = f_{t_0t_1}(\mathfrak{z}^{t_1}+\mathfrak{y}^{t_1}) = f_{t_0t_1}(\mathfrak{z}^{t_1})+f_{t_0t_1}(\mathfrak{y}^{t_1}) = \mathfrak{z}^{t_0}+\mathfrak{y}^{t_0} = \mathfrak{w}^{t_0}.$$

The second part is an obvious consequence of the property of $\underset{t}{\boldsymbol{P}} \mathscr{X}_t$ of being a topological group.

Let us add the following statement, analogous to Corollary 4a of § 16, VI, and which can be easily proved.

THEOREM 2. *Let, as before, $\{\mathscr{X}_t, f_{t_0t_1}\}$ and $\{\mathscr{Y}_t, g_{t_0t_1}\}$ be two inverse group-systems and suppose that h assigns to each $t \in T$ a map h_t: $\mathscr{X}_t \to \mathscr{Y}_t$ such that commutativity holds in the diagram (for $t_0 \leqslant t_1$)*

$$\begin{tikzcd}
\mathscr{X}_{t_0} \arrow[d, "h_{t_0}"'] & \mathscr{X}_{t_1} \arrow[l, "f_{t_0t_1}"'] \arrow[d, "h_{t_1}"] \\
\mathscr{Y}_{t_0} & \mathscr{Y}_{t_1} \arrow[l, "g_{t_0t_1}"]
\end{tikzcd}$$

Let the mapping h_∞: $\mathscr{X}_\infty \to \mathscr{Y}_\infty$ be such that $h^t(\mathfrak{z}) = h_t(\mathfrak{z}^t)$. If h_t is an isomorphism (for each $t \in T$), then so is h_∞.

VIII. Group $\mathscr{Y}^{\mathscr{X}}$. Let $\mathscr{X}$ and $\mathscr{Y}$ be two metric spaces (or, more generally, two $\mathscr{L}^*$ spaces). Assume that $\mathscr{Y}$ is a *topological abelian group*. The family $\mathscr{Y}^{\mathscr{X}}$ of continuous functions f: $\mathscr{X} \to \mathscr{Y}$ is endowed with a group structure by defining the composition of the elements f_1, f_2 of $\mathscr{Y}^{\mathscr{X}}$ as follows:

$$\{f_3 = f_1 + f_2\} \equiv \{f_3(x) = f_1(x) + f_2(x), \text{ for every } x \in \mathscr{X}\}. \tag{1}$$

Since the group $\mathscr{Y}$ is commutative, then so is $\mathscr{Y}^{\mathscr{X}}$. The neutral element of the group $\mathscr{Y}^{\mathscr{X}}$ is the constant function with the value 0.

The *constant* functions form a subgroup of $\mathscr{Y}^{\mathscr{X}}$ which is isomorphic with $\mathscr{Y}$. It will be often useful to denote this subgroup by the same symbol $\mathscr{Y}$.

The group $\mathscr{Y}^{\mathscr{X}}$, considered as an $\mathscr{L}^*$ space (compare § 20, VI), is a topological group.

In order to show this statement, put

$$h_n = f_n + g_n, \quad \lim_{n=\infty} f_n = f, \quad \lim_{n=\infty} g_n = g \quad \text{and} \quad h = f + g.$$

Let $\lim_{n=\infty} x_n = x$. It follows that

$$\lim_{n=\infty} h_n(x_n) = \lim_{n=\infty} [f_n(x_n) + g_n(x_n)] = \lim_{n=\infty} f_n(x_n) + \lim_{n=\infty} g_n(x_n)$$

$$= f(x) + g(x) = h(x), \quad \text{thus} \quad \lim_{n=\infty} h_n = h.$$

Let A be a fixed subset of $\mathscr{X}$; define

$$\zeta(f) = f|A. \tag{2}$$

Thus the operation ζ assigns to every element $f \in \mathscr{Y}^{\mathscr{X}}$ an element of the group $\mathscr{Y}^A$, namely the restriction $f|A$. Then, if Φ is a subset of $\mathscr{Y}^{\mathscr{X}}$, it follows (compare § 53, I) that

$$\zeta(\Phi) = \Phi|A. \tag{3}$$

THEOREM 1. *The operation ζ is a homomorphism,*

$$\zeta(f_1+f_2) = \zeta(f_1)+\zeta(f_2). \tag{4}$$

Therefore the following is true.

THEOREM 2. *If Φ is a subgroup of $\mathscr{Y}^{\mathscr{X}}$, then $\Phi|A$ is a subgroup of $\mathscr{Y}^A$.*

In particular, *the continuous functions g: $A \to \mathscr{Y}$ which have (continuous) extensions to $\mathscr{X}$ constitute a subgroup of $\mathscr{Y}^A$* (namely the subgroup $\mathscr{Y}^{\mathscr{X}}|A$).

Define

$$\varDelta(A) = \zeta^{-1}(0), \quad \text{i.e.} \quad \varDelta(A) = \mathop{E}_{f}[f \in \mathscr{Y}^{\mathscr{X}}][f(A) = (0)]. \tag{5}$$

The following statement follows from (3) and Theorem 3 of Section III.

THEOREM 3. *If Φ is a subgroup of $\mathscr{Y}^{\mathscr{X}}$, then*

$$\Phi/\varDelta(A) \underset{\text{gr}}{=} \Phi|A.$$

THEOREM 4. *If Φ_1 and Φ_2 are two subgroups of $\mathscr{Y}^{\mathscr{X}}$ such that $\varDelta(A) \subset \Phi_1 \subset \Phi_2$, then*

$$\Phi_2/\Phi_1 \underset{\text{gr}}{=} (\Phi_2|A)/(\Phi_1|A).$$

This is a consequence of Theorem 3 combined with Theorem 7 of Section III (where $\varDelta(A)$ is substituted for A).

THEOREM 5. *Let f: $\mathscr{X} \to Z$ be a continuous mapping of $\mathscr{X}$ onto Z. If Φ is the subgroup of $\mathscr{Y}^{\mathscr{X}}$ consisting of compound functions gf where g: $Z \to \mathscr{Y}$ are continuous functions, then*

$$\mathscr{Y}^Z \underset{\text{gr}}{=} \Phi, \tag{6}$$

$$\mathscr{Y}^Z/\mathscr{Y} \underset{\text{gr}}{=} \Phi/\mathscr{Y}. \tag{7}$$

Proof. Put $h_g(x) = gf(x)$, where $g \in \mathcal{Y}^Z$ and $x \in \mathcal{X}$; then h is an isomorphism between $\mathcal{Y}^Z$ and Φ since

$$h_{g_1+g_2}(x) = g_1 f(x) + g_2 f(x) = h_{g_1}(x) + h_{g_2}(x)$$

and

$$h_g = 0 \quad \text{implies} \quad g = 0,$$

(because, if $h_g = 0$, then $gf(x) = 0$ for any x, and hence $g(z) = 0$ for any $z \in Z$).

Formula (7) follows from formula (6) and from the equivalence $(h_g = \text{constant}) \equiv (g = \text{constant})$ (compare III, 5).

IX. Group $\mathcal{G}^{\mathcal{X}}$. Let $\mathcal{G}$ denote, as usually, the group of integers. Define

$$\mathfrak{B}_0(\mathcal{X}) = \mathcal{G}^{\mathcal{X}}/\mathcal{G} \quad \text{and} \quad b_0(\mathcal{X}) = \operatorname{rank} \mathfrak{B}_0(\mathcal{X}).$$

Thus, the elements of the factor group are obtained "identifying" the elements of $\mathcal{G}^{\mathcal{X}}$ which differ by a constant.

Let us agree to write $b_0(\mathcal{X}) = 0$ if $\mathcal{X}$ is empty.

THEOREM 1. *If $\mathcal{X}$ is connected, then $\mathcal{G}^{\mathcal{X}} = \mathcal{G}$, and hence $b_0(\mathcal{X}) = 0$.*

Because every continuous function $d: \mathcal{X} \to \mathcal{G}$ is constant.

Theorem 3 below is a generalization of Theorem 1.

THEOREM 2. *Let $F_0, \ldots, F_n$ be a system of closed, disjoint and non-empty sets such that $\mathcal{X} = F_0 \cup \ldots \cup F_n$. The system of characteristic functions $d_1, \ldots, d_n$ of the sets $F_1, \ldots, F_n$ is linearly independent* mod $\mathcal{G}$.

Proof. Let $(a_{kl}) = d_k(F_l)$. Therefore $a_{ll} = 1$ and $a_{kl} = 0$ for $k \neq l$. Since the determinant of the system of $n+1$ homogeneous equations

$$m_0 + a_{1l} m_1 + \ldots + a_{nl} m_n = 0 \quad (l = 0, \ldots, n) \tag{i}$$

is equal to 1, it follows that $m_0 = m_1 = \ldots = m_n = 0$.

THEOREM 3. *If the (non-empty) space $\mathcal{X}$ has a finite number, say $n+1$, of components $F_0, \ldots, F_n$, then*

$$\mathfrak{B}_0(\mathcal{X}) = \mathcal{G}^n \quad \textit{and} \quad b_0(\mathcal{X}) = n.$$

Proof. The characteristic functions $d_1, \ldots, d_n$ of the sets $F_1, \ldots$ $\ldots, F_n$ are the generators mod $\mathcal{G}$ of $\mathcal{G}^{\mathcal{X}}$. For, let d be an arbitrary element of $\mathcal{G}^{\mathcal{X}}$ and $(m_l) = d(F_l)$. It follows that

$$d = m_0 + (m_1 - m_0) d_1 + \ldots + (m_n - m_0) d_n.$$

THEOREM 4. *Let $d_1, \ldots, d_n$ be a system of elements of $\mathscr{G}^{\mathscr{X}}$. If*

$$\mathscr{X} = F_1 \cup \ldots \cup F_n$$

and $d_k | F_l$ is constant for every pair k, l, then the functions $d_1, \ldots, d_n$ are linearly dependent mod $\mathscr{G}$.

Proof. Let $(a_{kl}) = d_k(F_l)$. The required integers $m_0, \ldots, m_n$, the last n of which are not all equal to zero, are determined by the system of n equations (i) for $l = 1, \ldots, n$.

THEOREM 5. *If the functions $d_1, \ldots, d_n \in \mathscr{G}^{\mathscr{X}}$ are linearly independent* mod $\mathscr{G}$, *then $\mathscr{X} = F_0 \cup \ldots \cup F_n$ where $F_0, \ldots, F_n$ are closed-open, non-empty sets such that for every pair of distinct subscripts l, r there exists a subscript k such that $d_k(F_l) \cdot d_k(F_r) = 0$.*

Proof. For that purpose the following (set-theoretical) lemma will be established.

Let $\mathfrak{z}_0, \ldots, \mathfrak{z}_m$ be a system of $m+1$ distinct points of the cartesian product $A = A^{(1)} \times \ldots \times A^{(n)}$. Then we have $A = B_0 \cup \cup \ldots \cup B_m$ where

(i) $\mathfrak{z}_i \in B_i$ for $i = 0, \ldots, m$,

(ii) for every pair $l \neq r$ there exists k such that $B_l^{(k)} \cap B_r^{(k)} = 0$, where $X^{(k)}$ denotes the projection of X into the axis $A^{(k)}$.

Proceed by induction. For $m = 0$ assume $B_0 = A$. For $n = 1$ and $m > 0$, put $B_i = \mathfrak{z}_i$ if $i < m$, and $B_m = A - (B_0 \cup \ldots \cup B_{m-1})$.

Assume that the lemma holds for subscripts $j < m$ (and for each n). Let $n > 1$. Since the points $\mathfrak{z}_0, \ldots, \mathfrak{z}_m$ are distinct, we may assume that there exists a subscript $j < m$ such that $\mathfrak{z}_i^{(1)} = \mathfrak{z}_0^{(1)}$ for $i \leqslant j$ and $\mathfrak{z}_i^{(1)} \neq \mathfrak{z}_0^{(1)}$ for $i > j$.

Since $j < m$, we have by hypothesis the formula

$$\mathfrak{z}_0^{(1)} \times A^{(2)} \times \ldots \times A^{(n)} = B_0 \cup \ldots \cup B_j$$

and conditions (i) and (ii) (replacing m by j) are fulfilled. The same hypothesis implies that $A = C_{j+1} \cup \ldots \cup C_m$ where $\mathfrak{z}_i \in C_i$ for $i > j$ and that condition (ii) is fulfilled (where B is replaced by C).

Let $B_i = C_i - \mathfrak{z}_0^{(1)} \times A^{(2)} \times \ldots \times A^{(n)}$ for $i > j$. The sets B_i are the required sets. Because, if $l, r \leqslant j$ or if $l, r > j$, condition (ii) is fulfilled according to the definition of B_i $(i \leqslant j)$ and of C_i $(i > j)$. If $l \leqslant j < r$, then $B_l^{(1)} = \mathfrak{z}_0^{(1)}$ and $B_r^{(1)} = C_r^{(1)} - \mathfrak{z}_0^{(1)}$.

The lemma being established, let $\delta(x)$ be the point of $\mathcal{G}^n$ such that $\delta^{(k)}(x) = d_k(x)$, $k = 1, \ldots, n$ (i.e. the point of $\mathcal{E}^n$ with coordinates $d_1(x), \ldots, d_n(x)$). Consider the decomposition $\mathcal{X} = \bigcup \delta^{-1}(\mathfrak{z})$ into closed-open sets where $\mathfrak{z}$ varies over $\mathcal{G}^n$. Since each of the restricted functions $d_k | \delta^{-1}(\mathfrak{z})$ is constant $(= \mathfrak{z}^{(k)})$, there exist by Theorem 4, $n+1$ distinct elements $\mathfrak{z}_0, \ldots, \mathfrak{z}_n$ in $\mathcal{G}^n$ such that

$$\delta^{-1}(\mathfrak{z}_i) \neq 0, \quad i = 0, \ldots, n.$$

Let us set in the lemma $A^{(1)} = \ldots = A^{(n)} = \mathcal{G}$ and $m = n$, and consider the sets $F_i = \delta^{-1}(B_i)$, $i = 0, \ldots, n$. Since $\mathcal{G}^n = B_0 \cup \cup \ldots \cup B_n$, so $\mathcal{X} = F_0 \cup \ldots \cup F_n$. Since $\mathfrak{z}_i \in B_i$, it follows that $\delta^{-1}(\mathfrak{z}_i) \subset \delta^{-1}(B_i)$, so that $F_i \neq 0$. Finally, by the identity $d_k(F_i) = d_k[\delta^{-1}(B_i)] = B_i^{(k)}$, the condition $B_l^{(k)} \cap B_r^{(k)} = 0$ implies that $d_k(F_l) \cdot d_k(F_r) = 0$.

Using the following theorem, the study of the group $\mathcal{G}^{\mathcal{X}}$ for compact spaces $\mathcal{X}$ can be reduced to the case of $\mathcal{X}$ of dimension 0.

THEOREM 6. *If $\mathcal{X}$ is a compact space and $\boldsymbol{D}$ is the space of the decomposition of $\mathcal{X}$ into components, then* $\mathcal{G}^{\mathcal{X}} \underset{\text{gr}}{=} \mathcal{G}^{\boldsymbol{D}}$.

Proof. Let $f \colon \mathcal{X} \to \boldsymbol{D}$ be the continuous mapping having as point-inverses the components of $\mathcal{X}$ (compare § 47, VI, Theorem 1). According to VIII, 5 (6), we have to show that for every $d \in \mathcal{G}^{\mathcal{X}}$ there is a function $g \in \mathcal{G}^{\boldsymbol{D}}$ such that $d(x) = gf(x)$.

Let $n_1, \ldots, n_k$ be the values of the function d. Define $F_i = d^{-1}(n_i)$, where $i = 1, \ldots, k$. Then the sets F_i are disjoint, closed-open and $\mathcal{X} = F_1 \cup \ldots \cup F_k$.

Therefore, each component C of $\mathcal{X}$ is contained in only one set F_i. Put $g(C) = n_i$. It follows that $gf(x) = d(x)$ for $x \in C$, and since $f(F_i)$ is closed, the identity $g^{-1}(n_i) = f(F_i)$ implies that g is continuous.

X. Addition theorems. Let $\mathcal{X} = A_0 \cup A_1$ and let A_0 and A_1 be two closed or two open sets. Define

$$\Theta_0(A_0, A_1) = \overline{\mathcal{G}^{A_0} | A_0 \cap A_1 \cup \mathcal{G}^{A_1} | A_0 \cap A_1},$$

which means (compare IV, Theorem 8) that $d \in \Theta_0(A_0, A_1)$ if there exist two functions $d_j \in \mathcal{G}^{A_j}$, where $j = 0, 1$, such that $d(x) = d_0(x) - d_1(x)$ for each $x \in A_0 \cap A_1$(1).

(1) Compare the paper of S. Eilenberg and myself, Fund. Math. 32 (1939), p. 193.

Therefore $\Theta_0(A_0, A_1)$ is a subgroup of $\mathscr{G}^{A_0 \cap A_1}$ and $\mathscr{G} \subset \Theta_0(A_0, A_1)$. Define

$$\mathfrak{D}_0(A_0, A_1) = \Theta_0(A_0, A_1)/\mathscr{G}$$

$$\text{and} \quad d_0(A_0, A_1) = \operatorname{rank} \mathfrak{D}_0(A_0, A_1).$$

THEOREM 1. *If A_0 is connected, then $\Theta_0(A_0, A_1) = \mathscr{G}^{A_1} | A_0 \cap A_1$.*

Because $\mathscr{G}^{A_0} | A_0 \cap A_1 = \mathscr{G} | A_0 \cap A_1 = \mathscr{G}$ since A_0 is connected.

COROLLARY 2. *If A_0 and A_1 are connected, then $\Theta_0(A_0, A_1) = \mathscr{G}$, and therefore $\mathfrak{D}_0(A_0, A_1)$ reduces to the neutral element.*

THEOREM 3. *Let $p_j \in A_0 \cap A_1$, where $j = 0, 1$. If the sets A_0 and A_1 are connected between the points p_0 and p_1, whereas $A_0 \cap A_1$ is not, then*

$$\Theta_0(A_0, A_1) \neq \mathscr{G}^{A_0 \cap A_1}.$$

Proof. By hypothesis there exists a closed-open set F in $A_0 \cap A_1$ such that

$$p_0 \in F \quad \text{and} \quad p_1 \in A_0 \cap A_1 - F. \tag{1}$$

Let $d \in \mathscr{G}^{A_0 \cap A_1}$ be defined by the following conditions

$$d(F) = 0 \quad \text{and} \quad d(A_0 \cap A_1 - F) = 1. \tag{2}$$

If $d \in \Theta_0(A_0, A_1)$, then

$$d = d_0 | A_0 \cap A_1 - d_1 | A_0 \cap A_1 \quad \text{and} \quad d_j \in \mathscr{G}^{A_j}.$$

Therefore, in particular,

$$d(p_0) = d_0(p_0) - d_1(p_0) \quad \text{and} \quad d(p_1) = d_0(p_1) - d_1(p_1),$$

and hence (by (1) and (2))

$$d_0(p_0) - d_1(p_0) = 0 \quad \text{and} \quad d_0(p_1) - d_1(p_1) = 1;$$

thus

$$\text{either} \quad d_0(p_0) \neq d_0(p_1) \quad \text{or} \quad d_1(p_0) \neq d_1(p_1),$$

and hence either A_0 or A_1 is not connected between p_0 and p_1.

DEFINITIONS 1. Let $\Pi_0(A_0, A_1)$ be the subgroup of $\mathscr{G}^{A_0 \cup A_1}$ consisting of functions which are constant on A_0 and on A_1. Let

$$\mathfrak{P}_0(A_0, A_1) = \Pi_0(A_0, A_1)/\mathscr{G}$$

$$\text{and} \quad p_0(A_0, A_1) = \operatorname{rank} \mathfrak{P}_0(A_0, A_1).$$

The following statements can easily be shown.

THEOREM 4. *If either* $A_0 \cap A_1 \neq 0$ *or* $A_0 = 0$ *or else* $A_1 = 0$, *then*

$$\Pi_0(A_0, A_1) = \mathscr{G}, \quad \textit{and therefore} \quad p_0(A_0, A_1) = 0.$$

THEOREM 5. *If* $A_0 \cap A_1 = 0$ *and* $A_0 \neq 0 \neq A_1$, *then* $\Pi_0(A_0, A_1) \underset{\mathrm{gr}}{=} \mathscr{G}^2$, *hence* (compare VII (7))

$$\mathfrak{P}_0(A_0, A_1) \underset{\mathrm{gr}}{=} \mathscr{G}, \quad \textit{and therefore} \quad p_0(A_0, A_1) = 1.$$

DEFINITIONS 2. Let $\Lambda_0(A_0, A_1)$ be the subgroup of the cartesian product $\mathscr{G}^{A_0} \times \mathscr{G}^{A_1}$ consisting of the pairs d_0, d_1 such that

$$d_0 | A_0 \cap A_1 - d_1 | A_0 \cap A_1 = \text{constant}.$$

Let

$$\mathfrak{L}_0(A_0, A_1) = \Lambda_0(A_0, A_1)/\mathscr{G}^2$$
$$\text{and} \quad l_0(A_0, A_1) = \operatorname{rank} \mathfrak{L}_0(A_0, A_1).$$

THEOREM 6.

$$\mathscr{G}^{A_0} \times \mathscr{G}^{A_1} / \Lambda_0(A_0, A_1) \underset{\mathrm{gr}}{=} \mathfrak{D}_0(A_0, A_1)$$
$$\underset{\mathrm{gr}}{=} \mathfrak{B}_0(A_0) \times \mathfrak{B}_0(A_1) / \mathfrak{L}_0(A_0, A_1).$$

THEOREM 7.

$$\mathscr{G}^{A_0 \cup A_1} / \Pi_0(A_0, A_1) \underset{\mathrm{gr}}{=} \mathfrak{L}_0(A_0, A_1) \underset{\mathrm{gr}}{=} \mathfrak{B}_0(A_0 \cup A_1) / \mathfrak{P}_0(A_0, A_1).$$

Proof. In order to establish the first isomorphisms of Theorems 6 and 7, let us set respectively

$$h_{d_0, d_1} = (d_0 | A_0 \cap A_1) - (d_1 | A_0 \cap A_1) \quad \text{where} \quad d_j \in \mathscr{G}^{A_j},$$
$$h_d = (d | A_0, d | A_1) \quad \text{where} \quad d \in \mathscr{G}^{A_0 \cup A_1}.$$

In the first case, the first isomorphism of Theorem 6 is deduced by virtue of Theorem 5 of Section III (replacing the group $\mathscr{X}$ by $\mathscr{G}^{A_0} \times \mathscr{G}^{A_1}$, $\mathscr{Y}$ by $\Theta_0(A_0, A_1)$, A by $\Lambda_0(A_0, A_1)$ and B by $\mathscr{G}$). In the second case, $\mathscr{X}$ is replaced by $\mathscr{G}^{A_0 \cup A_1}$, $\mathscr{Y}$ by $\Lambda_0(A_0, A_1)$, A by $\Pi_0(A_0, A_1)$ and B by $\mathscr{G}^2$, and one refers to the fact that, if $(d_0, d_1) \in \Lambda_0(A_0, A_1)$, therefore if $d_0(x) - d_1(x) = c$ for $x \in A_0 \cap A_1$, the function d defined by the conditions $d(x) = d_0(x)$ on A_0 and $d(x) = d_1(x) + c$ on A_1, satisfies the formula $[h_d - (d_0, d_1)] \in \mathscr{G}^2$ (which implies condition (iii) of Theorem 5 of Section III).

The remaining parts of Theorems 6 and 7 follow from Theorem 7 of Section III and from (6) of Section VII.

THEOREM 8.

$$b_0(A_0 \cup A_1)+d_0(A_0, A_1) = b_0(A_0)+b_0(A_1)+p_0(A_0, A_1).$$

Because Theorems 6 and 7 imply, according to Theorem 3 of Section V and formula (5) of Section VII, that

$$b_0(A_0)+b_0(A_1) = d_0(A_0, A_1)+l_0(A_0, A_1),$$
$$b_0(A_0 \cup A_1) = l_0(A_0, A_1)+p_0(A_0, A_1),$$

and Theorem 8 is obtained by adding these two identities (for $l_0(A_0, A_1) \leqslant \infty$).

Theorem 8 combined with Theorems 4 and 5 implies the following ones.

THEOREM 9. *If either* $A_0 \cap A_1 \neq 0$ *or* $A_0 = 0$ *or else* $A_1 = 0$, *then*

$$b_0(A_0 \cup A_1)+d_0(A_0, A_1) = b_0(A_0)+b_0(A_1).$$

THEOREM 10. *If* $A_0 \cap A_1 = 0$ *and* $A_0 \neq 0 \neq A_1$, *then*

$$b_0(A_0 \cup A_1) = b_0(A_0)+b_0(A_1)+1.$$

Because in the latter case, we have

$$d_0(A_0, A_1) = 0 \quad \text{and} \quad p_0(A_0, A_1) = 1.$$

Using these formulas one can calculate the number of components of the union of two closed or two open sets.

XI. Relations to the connectedness between sets(1). Define

$$\Xi(Z, \mathscr{X}) = \bigcap \mathscr{G}^{Z \cup F}|Z,$$

where F ranges over the family of all sets $F = \overline{F} \neq \mathscr{X}$.

In other words, the function $f \in \mathscr{G}^Z$ belongs to $\Xi(Z, \mathscr{X})$ if it has an extension $f_F \in \mathscr{G}^{Z \cup F}$ for every $F = \overline{F} \neq \mathscr{X}$.

THEOREM 1. *If* $\mathscr{X}$ *is irreducibly connected between closed-open, disjoint sets* A *and* B, *and if the function* $f\colon A \cup B \to \mathscr{G}$ *equals* 0 *on* A *and* 1 *on* B, *then* $f \in \Xi(A \cup B, \mathscr{X})$.

(1) Compare my paper in Fund. Math. 31 (1938), p. 231.

Because, if F is closed and differs from $\mathscr{X}$, the set $F \cup A \cup B$ is not connected between A and B. So, there exists an extension $f_F \in \mathscr{G}^{F \cup A \cup B}$ according to § 46, IV, Theorem 5.

THEOREM 2. *If $\mathscr{X}$ is connected between $A = \bar{A}$ and $B = \bar{B}$, and if $f \in \Xi(A \cup B, \mathscr{X})$ and $f(A) \cap f(B) = 0$, the space $\mathscr{X}$ is irreducibly connected between A and B.*

Proof. Assume that $F = \bar{F} \neq \mathscr{X}$ and that $f \subset f^* \in \mathscr{G}^{F \cup A \cup B}$. Since $f^*(A) \cap f^*(B) = f(A) \cap f(B) = 0$, the set $F \cup A \cup B$ is not connected between A and B according to § 46, IV, Theorem 5.

THEOREM 3. *Let $\mathscr{X} = A_0 \cup A_1$ and let A_0 and A_1 be two closed connected sets. Let*

$$\Xi_j = \Xi(A_0 \cap A_1, A_j). \tag{1}$$

In order that there exist a system $F_0, \ldots, F_n$ $(n \geqslant 1)$ of sets such that

(i) $A_0 \cap A_1 = F_0 \cup \ldots \cup F_n$, $F_k = \bar{F}_k$, $F_l \cap F_r = 0$ *for* $l \neq r$,

(ii) A_j $(j = 0, 1)$ *is irreducibly connected between* F_k *and* $H_k = A_0 \cap A_1 - F_k$ *for* $k = 0, \ldots, n$,

it is necessary and sufficient that

$$\operatorname{rank}[(\Xi_0 \cap \Xi_1)/\mathscr{G}] \geqslant n. \tag{2}$$

Proof. First, let $F_0, \ldots, F_n$ be a system of sets satisfying conditions (i) and (ii). According to § 46, IV, Theorem 1a, $F_k \neq 0$. Let d_k be a function belonging to $\mathscr{G}^{A_0 \cap A_1}$ such that $d_k(F_k) = 1$ and $d_k(H_k) = 0$. By Theorem 1, $d_k \in \Xi_0 \cap \Xi_1$, and by Theorem 2 of Section IX, the functions $d_1, \ldots, d_n$ are linearly independent mod $\mathscr{G}$. The inequality (2) follows.

Conversely, assume inequality (2), let $d_1, \ldots, d_n$ be a system of elements of $\Xi_0 \cap \Xi_1$ linearly independent mod $\mathscr{G}$.

By Theorem 5 of Section IX, there exists a system $F_0, \ldots, F_n$ of non-empty sets satisfying condition (i) and such that for every pair of subscripts $l \neq r$ there exist k such that $d_k(F_l) \cap d_k(F_r) = 0$. Since $d_k \in \Xi_j$ and $F_l \cup F_r \subset A_0 \cap A_1$, it follows that

$$(d_k | F_l \cup F_r) \in \Xi(F_l \cup F_r, A_j).$$

By Theorem 2 the (connected) set A_j is irreducibly connected between F_l and F_r for every $r \neq l$ (one has to replace $\mathscr{X}$ by A_j,

A by F_l and B by F_r). Therefore, by Theorem 4 of § 48, VIII, the set A_j is irreducibly connected between the sets F_l and

$$F_1 \cup \ldots \cup F_{l-1} \cup F_{l+1} \cup \ldots \cup F_n = H_l.$$

§ 56*. The groups $\mathscr{S}^{\mathfrak{X}}$ and $\mathscr{P}^{\mathfrak{X}}$

I. General properties. Let $\mathscr{E}$ be the group of real numbers with addition being the group operation. Let $\mathscr{S}$ be the circle $|z| = 1$ of the complex plane, considered as a group of complex numbers with multiplication being the group operation ("the group of rotations"). Assume

$$e(t) = e^{2\pi i t} \quad \text{for} \quad t \in \mathscr{E}. \tag{1}$$

The function e is a continuous onto homomorphism $e\colon \mathscr{E} \to \mathscr{S}$, and the group $\mathscr{G}$ of integers is its kernel,

$$e(t+t') = e(t)+e(t') \quad \text{and} \quad [e(t) = 1] \equiv [t \in \mathscr{G}]. \tag{2}$$

If $\mathscr{X}$ is an arbitrary space and $\varphi \in \mathscr{E}^{\mathfrak{X}}$ (compare § 55, VIII), let e_φ be the element of the group $\mathscr{S}^{\mathfrak{X}}$ defined by the condition

$$e_\varphi(x) = e[\varphi(x)] = e^{2\pi i \varphi(x)}. \tag{3}$$

The operation e is a homomorphism of $\mathscr{E}^{\mathfrak{X}}$ onto a subgroup, denote it by $\Psi(\mathscr{X})$, of the group $\mathscr{S}^{\mathfrak{X}}$. Thus the group $\Psi(\mathscr{X})$ consists of those elements f of $\mathscr{S}^{\mathfrak{X}}$ which have the form $f = e_\varphi$ where $\varphi \in \mathscr{E}^{\mathfrak{X}}$, i.e.

$$f(x) = e^{2\pi i \varphi(x)} \quad \text{for} \quad x \in \mathscr{X}. \tag{4}$$

The kernel of this homomorphism is the group $\mathscr{G}^{\mathfrak{X}}$.

The following statements can be easily established.

THEOREM 1. *If* $f(x) = e_\varphi(x) = e_\psi(x)$ *and* $|\varphi(x) - \psi(x)| < 1$, *then* $\varphi(x) = \psi(x)$.

THEOREM 2. *If* $\mathscr{X}$ *is connected,* $\varphi, \psi \in \mathscr{E}^{\mathfrak{X}}$ *and* $e_\varphi = e_\psi$, *the function* $\varphi - \psi$ *is constant.*

This may be expressed more generally in the following way.

THEOREM 3. *If* $\mathscr{X}$ *is connected between* a *and* b, *and if* $e_\varphi = e_\psi$, *where* $\varphi, \psi \in \mathscr{E}^{\mathfrak{X}}$, *then* $\varphi(a) - \psi(a) = \varphi(b) - \psi(b)$.

* The fundamental theorems of § 56 are contained in the Thesis of S. Eilenberg, *Transformations continues en circonférence et la topologie du plan*, Fund. Math. 26 (1936), pp. 61–112.

Because setting $d = \varphi - \psi$, we infer that d is a continuous function $d\colon \mathcal{X} \to \mathcal{G}$ and the set of x's such that $d(x) = d(a)$ is closed-open and contains a, and hence contains b.

THEOREM 4. *Let $f \in \Psi(\mathcal{X})$ and $x_0 \in \mathcal{X}$. Then the function φ of formula* (4) *can be chosen so as to satisfy the ("initial") condition $\varphi(x_0) = c_0$, where $2\pi i c_0$ is any value of* $\log f(x_0)$.

Consequently (compare Theorem 2), *if $\mathcal{X}$ is connected, the above condition uniquely determines the function φ.*

Proof. It follows by hypothesis that $f(x) = e^{2\pi i \psi(x)}$ where $\psi \in \mathcal{E}^{\mathcal{X}}$. Let $k = \psi(x_0) - c_0$; therefore k is an integer. It is sufficient to set $\varphi(x) = \psi(x) - k$.

Remark. Let $\mathcal{P}$ denote the plane $\mathcal{E}^2$ without the point 0. The operation $e\colon\ \mathcal{E}^2 \to \mathcal{P}$ defined by condition (1) for $t \in \mathcal{E}^2$ is a continuous homomorphism of the (additive) group $\mathcal{E}^2$ onto the (multiplicative) group $\mathcal{P}$. Similarly, the operation e_φ defined by formula (3) is a homomorphism of the group $(\mathcal{E}^2)^{\mathcal{X}}$ onto the subgroup of $\mathcal{P}^{\mathcal{X}}$ which consists of those elements of $\mathcal{P}^{\mathcal{X}}$ that have the form (4). Therefore this subgroup consists of functions whose *logarithm has a unique continuous branch*; the function $2\pi i \varphi(x)$ is just this branch of the logarithm.

The theorems of §§ 56 and 57 are stated for groups $\mathcal{E}$ and $\mathcal{S}$, but it is easily seen that they can also be applied to groups $\mathcal{E}^2$ and $\mathcal{P}$ (1).

II. Group $\Gamma(A)$. Let $A \subset \mathcal{X}$. Following the notations of § 55, III, "$f \sim 1 \bmod \Psi(A)$" will mean that $f \in \Psi(A)$. More concisely the symbol $f \sim 1$ (2), where $f \in \mathcal{S}^A$, will mean that f has the form

$$f = e_\varphi \quad \text{where} \quad \varphi \in \mathcal{E}^A. \tag{1}$$

Define

$$\Gamma(A) = \mathop{E}_{f}(f \in \mathcal{S}^{\mathcal{X}})\,(f|A \sim 1). \tag{2}$$

The following statements can easily be established (compare § 55, VIII, Theorem 2).

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(1) For the generalizations to topological groups, see my paper *Sur les espaces des transformations continues en certains groupes abeliens*, Fund. Math. 31 (1938), p. 231.

(2) As will be seen in Section IX, the relation $f \sim 1$ is equivalent to $f \simeq 1$.

THEOREM 1. $\Gamma(A) = \zeta^{-1}[\Psi(A)]$.

THEOREM 2. $\Gamma(\mathscr{X}) = \Psi(\mathscr{X})$.

THEOREM 3. *$\Gamma(A)$ is a subgroup of $\mathscr{S}^{\mathscr{X}}$.*

THEOREM 4. *$A \subset B$ implies $\Gamma(B) \subset \Gamma(A)$.*

THEOREM 5. *If $A = (p)$, then $\Gamma(A) = \mathscr{S}^{\mathscr{X}}$.*

THEOREM 6. *Let R be the ray with the origin at 0. If $f \in (\mathscr{E}^2 - R)^{\mathscr{X}}$, then $f \sim 1$.*

Clearly $\log z$ can be defined so that it be continuous in the set $\mathscr{E}^2 - R$.

THEOREM 7. *For every continuous function $f: \mathscr{X} \to \mathscr{S}$ ($f: \mathscr{X} \to \mathscr{P}$) there exist two open sets A_0 and A_1 such that $f|A_j \sim 1$ for $j = 0, 1$, and $\mathscr{X} = A_0 \cup A_1$.*

Proof. Let R_0 and R_1 be the half-circles with abscissae positive and negative respectively, and let $A_j = f^{-1}(\mathscr{S} - R_j)$ (or $A_j = f^{-1}(\mathscr{P} - R_j)$).

THEOREM 8. *If $F = \overline{F}$, then $\Psi(F) = \Gamma(\mathscr{X})|F$.*

In other words, *if $f \in \mathscr{S}^F$ and $f \sim 1$, there exists a function f^* such that $f \subset f^* \in \mathscr{S}^{\mathscr{X}}$ and $f^* \sim 1$.*

Proof. By hypothesis (compare (1))

$$f = e_\varphi \quad \text{and} \quad \varphi \in \mathscr{E}^F, \quad \text{hence} \quad \varphi \subset \varphi^* \in \mathscr{E}^{\mathscr{X}} \tag{3}$$

by the theorem of Tietze (§ 14, IV).

It only remains to set $f^* = e_{\varphi^*}$.

THEOREM 9. *$\Gamma(A) = \bigcup \Gamma(G)$, where G varies over the family of open sets containing A.*

In other words, *for every $f \in \Gamma(A)$ there exists an open set G such hat*

$$A \subset G \quad \textit{and} \quad f \in \Gamma(G). \tag{4}$$

First we shall consider the case where A reduces to a single point a. Namely, we shall establish the following statement.

THEOREM 10. *Let $f: \mathscr{X} \to \mathscr{S}$ be a continuous function. For every point $a \in \mathscr{X}$ there exists an open set G_a such that $a \in G_a$ and $f|G_a \sim 1$.*

Proof. Let H be the plane $\mathscr{E}^2$ without a ray, which starts at 0 and does not pass through a. It is sufficient (compare Theorem 6) to set $G = f^{-1}(H)$.

Proof of Theorem 9. Let

$$A \subset \mathcal{X}, \quad f \in \mathcal{S}^{\mathcal{X}}, \quad f|A = e_\varphi \quad \text{and} \quad \varphi \in \mathcal{E}^A. \tag{5}$$

By Theorem 10, for every $a \in A$ there exist an open set G_a and a continuous function ψ_a: $G_a \to \mathcal{E}$ such that

$$a \in G_a, \quad f(x) = e_{\psi_a}(x) \quad \text{for} \quad x \in G_a, \tag{6}$$

and (compare (5) and Theorem 4 of Section I)

$$\psi_a(a) = \varphi(a). \tag{7}$$

Diminishing the set G_a if necessary, we may assume that

$$|\psi_a(x) - \psi_a(a)| < 1/3 \quad \text{if} \quad x \in G_a, \tag{8}$$

$$|\varphi(x) - \varphi(a)| < 1/3 \quad \text{if} \quad x \in A \quad \text{and} \quad |x-a| < 2\delta(G_a). \tag{9}$$

We will show that

the conditions $x \in G_a \cap G_b$, $a \in A$ and $b \in A$ imply $\psi_a(x) = \psi_b(x)$. (10)

By (7) and (8) we have

$$|\psi_a(x) - \varphi(a)| < 1/3 \quad \text{and} \quad |\psi_b(x) - \varphi(b)| < 1/3, \tag{11}$$

and assuming (by the symmetry) that $\delta(G_b) \leqslant \delta(G_a)$, we infer that $|\varphi(b) - \varphi(a)| < 1/3$ (by virtue of (9)). This inequality combined with inequalities (11) implies that $|\psi_a(x) - \psi_b(x)| < 1$, and thus $\psi_a(x) = \psi_b(x)$ by (6) and Theorem 1 of Section I.

Let G be the union of all sets G_a for $a \in A$. In accordance with (10), let ψ be the function identical with $\psi_a(x)$ for $x \in G_a$. It follows that ψ: $G \to \mathcal{E}$ is continuous and that $f = e_\varphi$, according to (6). Therefore $f|G \sim 1$.

III. Group $\mathfrak{B}_1(\mathcal{X})$.

DEFINITION. $\mathfrak{B}_1(\mathcal{X})$ denotes the factor group $\mathcal{S}^{\mathcal{X}}/\Psi(\mathcal{X})$ and $b_1(\mathcal{X})$ denotes its rank(1). Let us agree that $b_1'(0) = 0$.

THEOREM 1. *The group* $\mathfrak{B}(\mathcal{X})$ *contains no element of finite order* (*has no "torsion"*).

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(1) If $\mathcal{X}$ is a polyhedron, the group $\mathfrak{B}_1(\mathcal{X})$ is isomorphic to the first reduced Betti group of $\mathcal{X}$. See N. Bruschlinsky, *Stetige Abbildungen und Bettische Gruppen der Dimensionszahlen* 1 *und* 3, Math. Ann. 109 (1934), p. 525, where the more general case of compact $\mathcal{X}$ is considered. It will be seen in § 60, III, Theorem 5, that if $\mathcal{X}$ is a compact subset of the plane, $b_1(X)+1$ is the number of components of its complement.

In other words, *if* $f^n \sim 1$ *and* $n \neq 0$, *then* $f \sim 1$.

Proof. Let

$$f^n(x) = e^{2\pi i\varphi(x)}, \quad \varphi_k(x) = \frac{1}{n}[\varphi(x)+k] \quad \text{where} \quad 0 \leqslant k \leqslant n-1,$$

and

$$F_k = \underset{x}{E}[f(x) = e^{2\pi i\varphi_k(x)}].$$

We will show that $\mathscr{X} = F_0 \cup \ldots \cup F_{n-1}$.

Let $x \in \mathscr{X}$. We have to define a number k such that $x \in F_k$. Let

$$t \in \mathscr{E} \quad \text{such that} \quad f(x) = e^{2\pi it},$$

$$q \in \mathscr{G} \quad \text{such that} \quad -q \leqslant \frac{1}{n}[nt-\varphi(x)] < -q+1.$$

Since $e^{2\pi i\varphi(x)} = f^n(x) = e^{2\pi int}$, then $[nt-\varphi(x)] \in \mathscr{G}$. Therefore the integer $k = [nt-\varphi(x)+qn]$ satisfies the conditions $0 \leqslant k \leqslant n-1$ and $\varphi_k(x) = t+q$, thus $f(x) = e^{2\pi i\varphi_k(x)}$, and hence $x \in F_k$.

Since the sets $F_0, \ldots, F_{n-1}$ are closed and disjoint, the function ψ, defined by the conditions $\psi(x) = \varphi_k(x)$ for $x \in F_k$, is continuous and $f = e_\psi$.

THEOREM 2. *The conditions* $b_1(\mathscr{X}) = 0, \mathscr{S}^{\mathscr{X}} = \Psi(\mathscr{X})$ *and* $\mathfrak{B}_1(\mathscr{X}) = (0)$ *are equivalent* (*for* $\mathscr{X} \neq 0$).

This is a direct consequence of § 55, III (1).

THEOREM 3. $\mathfrak{B}_1(\mathscr{I}) = (0)$, *thus* $b_1(\mathscr{I}) = 0$.

In other words, *if* $f\colon \mathscr{I} \to \mathscr{S}$ *is a continuous function, then* $f \sim 1$.

Proof. Following Theorem 10 of Section II, let us assign to every point a of $\mathscr{I}$ an open set G_a such that $a \in G_a$ and $f|G_a \sim 1$. The family $\{G_a\}$ may obviously be replaced by a finite family. In other terms, there exists a finite system $a_0 < a_1 < a_2 < \ldots < a_n$, where $a_0 = 0$ and $a_n = 1$, such that for $k = 1, \ldots, n$, $f|a_{k-1}a_k \sim 1$, therefore $f(x) = e^{2\pi i\varphi_k(x)}$ for $a_{k-1} \leqslant x \leqslant a_k$ (where φ_k is continuous). Besides, it can be assumed, by Theorem 4 of Section I, that $\varphi_{k+1}(a_k) = \varphi_k(a_k)$ for $k = 1, 2, \ldots, n-1$.

Therefore the functions $\varphi_1, \ldots, \varphi_n$ define a continuous function $\varphi\colon \mathscr{I} \to \mathscr{E}$ such that $\varphi|a_{k-1}a_k = \varphi_k$. It follows that

$$f(x) = e^{2\pi i\varphi(x)} \quad \text{for} \quad x \in \mathscr{I}, \quad \text{which implies} \quad f \sim 1.$$

THEOREM 4. $\mathfrak{B}_1(\mathcal{S}) \underset{\text{gr}}{=} \mathcal{G}$, *therefore* $b_1(\mathcal{S}) = 1$.

More precisely: *the identity transformation constitutes a basis* $\operatorname{mod} \Psi(\mathcal{S})$.

In other words,

(i) x non ~ 1 *on* $\mathcal{S}$,

(ii) *for every continuous function* $f\colon \mathcal{S} \to \mathcal{S}$ *there exist an integer* n (*namely the increment of its logarithm*) *and a continuous function* $\varphi\colon \mathcal{S} \to \mathcal{E}$ *such that*

$$f(x) = x^n e^{2\pi i \varphi(x)}. \tag{1}$$

Proof. In order to establish (i), let A denote the circle $\mathcal{S}$ without the point $(1, 0)$ and let $0 < a(x) < 2\pi$ be the argument of x. Therefore $a\colon A \to \mathcal{E}$ is a continuous function and $x = e^{ia(x)}$ for $x \in A$; moreover, the function a cannot be extended on the point $(1, 0)$ in a continuous way.

Suppose that (i) is not true, i.e. that $x \sim 1$ on $\mathcal{S}$. Then $x = e^{i\beta(x)}$, where $\beta\colon \mathcal{S} \to \mathcal{E}$ is a continuous function. But then it would follow by Theorem 2 of Section I (since A is connected) that

$$a(x) = \beta(x) + 2k\pi \quad \text{for} \quad x \in A,$$

and the function $\beta(x) + 2k\pi$ would be a continuous extension of $a(x)$ to the whole circle $\mathcal{S}$.

Now consider statement (ii). Since $f(e^{2\pi i t})$ is defined for every $t \in \mathcal{I}$, it follows by Theorem 3 that

$$f(e^{2\pi i t}) = e^{2\pi i \psi(t)}, \quad \text{where} \quad \psi\colon \mathcal{I} \to \mathcal{E} \text{ is continuous.} \tag{2}$$

Hence $\psi(1) - \psi(0)$ is an integer; let

$$n = \psi(1) - \psi(0). \tag{3}$$

Assign to every $x \in \mathcal{S}$ the point

$$y = \psi(t) - nt, \tag{4}$$

provided that $2\pi t$ is the argument of the point x. Although two values of t correspond to the point $(1, 0)$, namely 0 and 1, the value of y is uniquely determined by means of (3) and (4). Setting $y = \varphi(x)$, we easily infer that $\varphi\colon \mathcal{S} \to \mathcal{E}$ is continuous, and assuming that $x = e^{2\pi i t}$, condition (2) implies that

$$f(x) = e^{2\pi i \psi(t)} \quad \text{and} \quad x^n = e^{2n\pi i t},$$

thus

$$f(x) = x^n e^{2\pi i[\psi(t)-nt]} = x^n e^{2\pi i\varphi(x)}.$$

IV. Addition theorems(¹). Let A_0 and A_1 be two closed or two open sets such that $\mathscr{X} = A_0 \cup A_1$. Define

$$\Theta_1(A_0, A_1) = \overline{\mathscr{S}^{A_0}|A_0 \cap A_1 \cup \mathscr{S}^{A_1}|A_0 \cap A_1}. \tag{0}$$

THEOREM 1. *To every continuous function φ: $A_0 \cap A_1 \to \mathscr{E}$ there correspond two continuous functions φ_j: $A_j \to \mathscr{E}$, where $j = 0, 1$, such that $\varphi_1(x) - \varphi_0(x) = \varphi(x)$ for $x \in A_0 \cap A_1$.*

That means

$$\mathscr{E}^{A_0 \cap A_1} = \overline{\mathscr{E}^{A_0}|A_0 \cap A_1 \cup \mathscr{E}^{A_1}|A_0 \cap A_1}.$$

Namely let (see § 14, III) A_0^* and A_1^* be two closed sets such that $\mathscr{X} = A_0^* \cup A_1^*$ and $A_j^* \subset A_j$ (and $A_j^* = A_j$ if $A_0 = \bar{A}_0$ and $A_1 = \bar{A}_1$) and let φ^* be an extension of $\varphi|A_0^* \cap A_1^*$ to $\mathscr{X}$ (compare the Theorem of Tietze, § 14, IV), and finally let

$$\varphi_0(x) = \begin{cases} 0 & \text{if } x \in A_0^*, \\ \varphi^*(x) - \varphi(x) & \text{if } x \in A_0 \cap A_1^*, \end{cases}$$

$$\varphi_1(x) = \begin{cases} \varphi^*(x) & \text{if } x \in A_1^*, \\ \varphi(x) & \text{if } x \in A_1 \cap A_0^*. \end{cases}$$

Observe that $\varphi_0 = 0$ and $\varphi_1 = \varphi^*|A_1$ for closed A_j.

THEOREM 2. *For every function $f \in \Gamma(A_0 \cap A_1)$ there exist two functions $f_j \in \Gamma(A_j)$, where $j = 0, 1$, such that $f_1 : f_0 = f$.*

That means

$$\Gamma(A_0 \cap A_1) = \overline{\Gamma(A_0) \cup \Gamma(A_1)}.$$

Proof. Let $f|A_0 \cap A_1 = e_\varphi$, where φ: $A_0 \cap A_1 \to \mathscr{E}$ is a continuous function. Using notation of Theorem 1, define

$$f_0(x) = \begin{cases} 1 & \text{if } x \in A_0^*, \\ e_{\varphi^*}(x) : f(x) & \text{if } x \in A_1^*, \end{cases} \qquad f_1(x) = \begin{cases} f(x) & \text{if } x \in A_0^*, \\ e_{\varphi^*}(x) & \text{if } x \in A_1^*. \end{cases}$$

(¹) For Sections IV and V, see the paper of S. Eilenberg and myself, *Théorèmes d'addition concernant le groupe des transformations en circonférence*, Fund. Math. 32 (1939), p. 193.

It follows that $f_j(x) = e_{\varphi_j}(x)$ for $x \in A_j$, which implies the required conclusion.

THEOREM 3. *For every pair of continuous functions f_j: $A_j \to \mathscr{S}$, where $j = 0, 1$, bound by the relation $f_0|A_0 \cap A_1 \sim f_1|A_0 \cap A_1$, there exists a continuous function f: $A_0 \cup A_1 \to \mathscr{S}$ such that $f|A_j \sim f_j$ for $j = 0, 1$.*

Proof. By hypothesis (for $x \in A_0 \cap A_1$)

$$f_0(x):f_1(x) = e_\varphi(x), \quad \text{where} \quad \varphi\colon (A_0 \cap A_1) \to \mathscr{E},$$

and by Theorem 1,

$$\varphi(x) = \varphi_1(x) - \varphi_0(x), \quad \text{where} \quad \varphi_j\colon A_j \to \mathscr{E}.$$

Referring to the identity

$$[f_0(x)\cdot e_{\varphi_0}(x)]:[f_1(x)\cdot e_{\varphi_1}(x)] = e_\varphi(x)\cdot e_{\varphi_0}(x):e_{\varphi_1}(x) = 1,$$

which holds in $A_0 \cap A_1$, we define f by setting $f|A_j = f_j\cdot e_{\varphi_j}$ for $j = 0, 1$.

THEOREM 4. $\Psi(A_0 \cap A_1) \subset \Theta_1(A_0, A_1)$. *That means that every continuous function f: $(A_0 \cap A_1) \to \mathscr{S}$ such that $f \sim 1$ has the form $f(x) = f_1(x):f_0(x)$, where f_j: $A_j \to \mathscr{S}$, $j = 0, 1$, are continuous functions.*

Proof. By hypothesis and by Theorem 1, it follows that $f = e_\varphi$, φ: $(A_0 \cap A_1) \to \mathscr{E}$ is continuous, $\varphi(x) = \varphi_1(x) - \varphi_0(x)$ and φ_j: $A_j \to \mathscr{E}$ is continuous.

It is sufficient to set $f_j = e_{\varphi_j}$.

THEOREM 5. *Let φ_j: $A_j \to \mathscr{E}$, $j = 0, 1$, be continuous functions such that $(\varphi_1 - \varphi_0)$: $(A_0 \cap A_1) \to \mathscr{G}$. Then there exists a continuous function f: $(A_0 \cup A_1) \to \mathscr{S}$ such that $f|A_j = e_{\varphi_j}$.*

Moreover, if φ_j': $A_j \to \mathscr{E}$ is continuous, $\varphi_1' - \varphi_0' = \varphi_1 - \varphi_0$, and f': $A_0 \cup A_1 \to \mathscr{S}$ and $f'|A_j = e_{\varphi_j'}$, then $f' \sim f$.

Proof. The condition $(\varphi_1 - \varphi_0)$: $A_0 \cap A_1 \to \mathscr{G}$ implies that $e_{\varphi_1 - \varphi_0} = 1$, therefore f: $(A_0 \cup A_1) \to \mathscr{S}$ is continuous, where $f(x) = e^{2\pi i \varphi_j(x)}$ for $x \in A_j$ and $j = 0, 1$.

In order to establish the relation $f' \sim f$, let $\psi_j = \varphi_j' - \varphi_j$. Since $(\psi_1 - \psi_0)|(A_0 \cap A_1) = 0$, there exists a continuous function ψ: $\mathscr{X} \to \mathscr{E}$ such that $\psi_j = \psi|A_j$. It follows that

$$e_{\varphi_j'}:e_{\varphi_j} = e_{\varphi_j' - \varphi_j} = e_{\psi_j}, \text{ thus } f'\colon f = e_\psi, \text{ so that } f' \sim f.$$

THEOREM 6. *Following Theorem 1, assign to every continuous function* $d\colon (A_0 \cap A_1) \to \mathscr{G}$ *two continuous functions* $\varphi_{d,j}$, $j = 0, 1$, *such that*

$$\varphi_{d,j}\colon A_j \to \mathscr{E} \quad \text{and} \quad \varphi_{d,1}(x) - \varphi_{d,0}(x) = d(x) \text{ for } x \in A_0 \cap A_1. \tag{1}$$

Let h_d *be a continuous function such that (compare Theorem 5)*

$$h_d\colon (A_0 \cup A_1) \to \mathscr{S} \quad \text{and} \quad h_d | A_j = e_{\varphi_{d,j}}, \quad j = 0, 1. \tag{2}$$

Then

(i) $\Theta_0(A_0, A_1) = h^{-1}[\Psi(A_0 \cup A_1)]$

(*i.e.* $h_d \sim 1$ *if and only if*

$$d(x) = d_1(x) - d_0(x) \text{ for } x \in A_0 \cap A_1$$
$$\text{and } d_j\colon A_j \to \mathscr{G} \text{ is continuous}), \tag{3}$$

(ii) $h_{d+d'} \sim h_d \cdot h_{d'}$,

(iii) *for every* $f \in \Gamma(A_0) \cap \Gamma(A_1)$ *there exists* d *such that*

$$f \sim h_d.$$

Proof. Let $h_d \sim 1$. Therefore $h_d = e_\psi$ and $\psi\colon A_0 \cup A_1 \to \mathscr{E}$ is continuous. The functions $d_j = \varphi_{d,j} - \psi$, $j = 0, 1$, satisfy condition (3), because

$$e_{\varphi_{d,j} - \psi} = e_{\varphi_{d,j}} : e_\psi = 1.$$

Conversely, assume that conditions (3) are satisfied and that (according to Theorem 5)

$$h_d'\colon (A_0 \cup A_1) \to \mathscr{S} \quad \text{is continuous and} \quad h_d' | A_j = e_{d_j}.$$

Since $e_{d_j} = 1$ and $h_d \sim h_d'$ (by Theorem 5), it follows $h_d \sim 1$. Thus condition (i) is established.

Let $\varphi'_{(d+d'),j} = \varphi_{d,j} + \varphi_{d',j}$. By (1) it follows

$$\varphi'_{(d+d'),1} - \varphi'_{(d+d'),0} = d + d' = \varphi_{(d+d'),1} - \varphi_{(d+d'),0}.$$

So there exists by Theorem 5 a continuous $h'_{d+d'}\colon A_0 \cup A_1 \to \mathscr{S}$ such that

$$h'_{d+d'} | A_j = e_{\varphi'(d+d'),j}, \tag{4}$$

$$h'_{d+d'} \sim h_{d+d'}. \tag{5}$$

Since

$$e_{\varphi'(d+d'),j} = e_{\varphi d,j}\cdot e_{\varphi d',j} = (h_d|A_j)\cdot(h_{d'}|A_j),$$

formula (4) implies that $h'_{d+d'} = h_d\cdot h_{d'}$, thus (ii) follows by (5).

Let $f\in\Gamma(A_j)$. Assume that $f|A_j = e_{\psi_j}$, $\psi_j\colon A_j\to\mathscr{E}$ is continuous and $d = \psi_1-\psi_0$. It follows that $d\colon(A_0\cap A_1)\to\mathscr{G}$ is a continuous function. Therefore, conditions (1) and (2) imply that $f\sim h_d$ by Theorem 5.

V. Relations between factor groups. Let A_0 and A_1 be two closed or two open sets such that $\mathscr{X} = A_0\cup A_1$ (as in Section IV). Consider the following groups (which correspond to the groups studied in § 55, X).

$$\Pi_1(A_0, A_1) = \mathop{E}_{f}(f\in\mathscr{S}^{A_0\cup A_1})(f|A_j\sim 1, j=0,1)$$

$$= \Gamma(A_0)\cap\Gamma(A_1), \tag{1}$$

$$\Lambda_1(A_0, A_1) =$$

$$\mathop{E}_{f_0f_1}(f_j\in\mathscr{S}^{A_j}, j=0,1)(f_0|A_0\cap A_1\sim f_1|A_0\cap A_1), \tag{2}$$

$$\mathfrak{D}_1(A_0, A_1) = \Theta_1(A_0, A_1)/\Psi(A_0\cap A_1), \tag{3}$$

$$\mathfrak{P}_1(A_0, A_1) = \Pi_1(A_0, A_1)/\Psi(A_0\cup A_1), \tag{4}$$

$$\mathfrak{L}_1(A_0, A_1) = \Lambda_1(A_0, A_1)/\Psi(A_0)\times\Psi(A_1)]. \tag{5}$$

The following three isomorphism theorems hold[1].

THEOREM 1.

$$\mathscr{S}^{A_0}\times\mathscr{S}^{A_1}/\Lambda_1(A_0, A_1) \underset{\mathrm{gr}}{=} \mathfrak{D}_1(A_0, A_1)$$

$$\underset{\mathrm{gr}}{=} \mathfrak{B}_1(A_0)\times\mathfrak{B}_1(A_1)/\mathfrak{L}_1(A_0, A_1).$$

THEOREM 2.

$$\mathscr{S}^{A_0\cup A_1}/\Pi_1(A_0, A_1) \underset{\mathrm{gr}}{=} \mathfrak{L}_1(A_0, A_1) \underset{\mathrm{gr}}{=} \mathfrak{B}_1(A_0\cup A_1)/\mathfrak{P}_1(A_0, A_1).$$

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(1) For Theorems 1 and 2, see the paper of S. Eilenberg and myself, Fund. Math. 32 (1939), p. 197; for Theorem 3, see my paper in Fund. Math. 31 (1938), p. 239. These three theorems correspond to formulas of L. Vietoris, Mon. f. Math. u. Phys. 37 (1930), p. 162. Compare Alexandroff–Hopf, *Topologie I*, Chapter VII, § 2; the reduced groups $N^j(A_0\cdot A_1)$ and $S^j(A_0+A_1)$ ("Nahtzyklen" and "Summenzyklen") are isomorphic to the groups $\mathfrak{B}_j(A_0\cap A_1)/\mathfrak{D}_j(A_0, A_1)$ and $\mathfrak{B}_j(A_0\cup A_1)/\mathfrak{P}_j(A_0, A_1)$, respectively. Compare also E. Čech, Fund. Math. 19 (1932), p. 149.

THEOREM 3.

$$\mathscr{G}^{A_0 \cap A_1}/\Theta_0(A_0, A_1) \underset{\mathrm{gr}}{=} \mathfrak{P}_1(A_0, A_1) \underset{\mathrm{gr}}{=} \mathfrak{B}_0(A_0 \cap A_1)/\mathfrak{D}_0(A_0, A_1).$$

Proof. The proofs of Theorems 1 and 2 are quite similar to those of Theorems 6 and 7 of § 55, X. We set respectively

$$h_{f_0,f_1} = (f_0|A_0 \cap A_1):(f_1|A_0 \cap A_1) \quad \text{where} \quad f_j \in \mathscr{S}^{A_j},$$
$$h_f = (f|A_0, f|A_1) \quad \text{where} \quad f \in \mathscr{S}^{A_0 \cup A_1},$$

and one refers to Theorems 5 and 7 of § 55, III and to (6) in § 55, VII.

Theorem 3 is an immediate consequence of Theorem 6 of Section IV and of Theorem 5 of § 55, III, where $\mathscr{X}$ is replaced by $\mathscr{G}^{A_0 \cap A_1}$, $\mathscr{Y}$ by $\Pi_1(A_0, A_1)$, A by $\Theta_1(A_0, A_1)$ and B by $\Psi(A_0 \cup A_1)$.

Let $d_1(A_0, A_1)$ and $p_1(A_0, A_1)$ be the *ranks* of groups $\mathfrak{D}_1(A_0, A_1)$ and $\mathfrak{P}_1(A_0, A_1)$, respectively. They are related in the following way.

THEOREM 4.

$$p_1(A_0, A_1) \leqslant b_1(A_0 \cup A_1).$$

THEOREM 5.

$$b_1(A_0 \cup A_1) + d_1(A_0, A_1) = b_1(A_0) + b_1(A_1) + p_1(A_0, A_1)\text{(1)}.$$

THEOREM 6.

$$p_1(A_0, A_1) + d_0(A_0, A_1) = b_0(A_0 \cap A_1).$$

Proof. Theorem 4 follows from the inclusion $\mathfrak{P}_1(A_0, A_1) \subset \subset \mathfrak{B}_1(A_0 \cup A_1)$, Theorem 5 follows from Theorems 1 and 2 (compare the implication $6, 7 \Rightarrow 8$ of § 55, X) and Theorem 6 is a consequence of Theorem 3 of § 55, V.

THEOREM 7. *If* $A_0 \cap A_1 \neq 0$, *then*

$$b_0(A_0) + b_0(A_1) + p_1(A_0, A_1) = b_0(A_0 \cup A_1) + b_0(A_0 \cap A_1), \tag{7.1}$$

$$b_1(A_0 \cup A_1) + b_0(A_0) + b_0(A_1) + d_1(A_0, A_1) = b_0(A_0 \cup A_1) + b_1(A_0) + b_1(A_1) + b_0(A_0 \cap A_1). \tag{7.2}$$

(1) Compare a similar formula of W. Mayer concerning Betti numbers, Mon. f. Math. u. Phys. 36 (1929), p. 40.

Proof. Condition (7.1) follows from Theorem 6 and from Theorem 9 of § 55, X; condition (7.2) is derived by combining Theorem 5 with (7.1).

In the case where the considered ranks are finite, we put

$$\operatorname{ind}(A) = b_0(A) - b_1(A)\text{(1)}. \tag{$*$}$$

It follows from (7.2) that

if $A_0 \cap A_1 \neq 0$, *then* $\operatorname{ind}(A_0 \cup A_1) + \operatorname{ind}(A_0 \cap A_1)$
$$= \operatorname{ind}(A_0) + \operatorname{ind}(A_1) - b_1(A_0 \cap A_1) + d_1(A_0, A_1). \tag{7.3}$$

THEOREM 8. *If* $A_0 \cap A_1 = 0$, *then* $\mathfrak{B}_1(A_0 \cup A_1) \underset{\mathrm{gr}}{=} \mathfrak{B}_1(A_0) \times \times \mathfrak{B}_1(A_1)$, *hence* $b_1(A_0 \cup A_1) = b_1(A_0) + b_1(A_1)$.

Proof. Assigning to every $f \in \mathcal{S}^{A_0 \cup A_1}$ a pair $(f|A_0, f|A_1)$, we obtain an isomorphism between $\mathcal{S}^{A_0 \cup A_1}$ and $\mathcal{S}^{A_0} \times \mathcal{S}^{A_1}$. This isomorphism maps $\Psi(A_0 \cup A_1)$ onto $\Psi(A_0) \times \Psi(A_1)$. Therefore (compare § 55, III, Theorem 5 and VII (6))

$$\mathcal{S}^{A_0 \cup A_1}/\Psi(A_0 \cup A_1) \underset{\mathrm{gr}}{=} [\mathcal{S}^{A_0} \times \mathcal{S}^{A_1}]/[\Psi(A_0) \times \Psi(A_1)]$$
$$\underset{\mathrm{gr}}{=} [\mathcal{S}^{A_0}/\Psi(A_0)] \times [\mathcal{S}^{A_1}/\Psi(A_1)].$$

VI. Relations to connectedness. Following the former notation, let A_0 and A_1 be two closed or two open sets such that

$$\mathcal{X} = A_0 \cup A_1.$$

THEOREM 1. *If* A_0 *is connected, then*

$$\mathfrak{P}_1(A_0, A_1) \underset{\mathrm{gr}}{=} \mathcal{G}^{A_0 \cap A_1}/(\mathcal{G}^{A_1} | A_0 \cap A_1). \tag{1}$$

THEOREM 2. *If the two sets* A_0 *and* A_1 *are connected, then*

$$\mathfrak{P}_1(A_0, A_1) \underset{\mathrm{gr}}{=} \mathfrak{B}_0(A_0 \cap A_1),$$
$$\textit{thus } p_1(A_0, A_1) = b_0(A_0 \cap A_1) \leqslant b_1(A_0 \cup A_1). \tag{2}$$

THEOREM 3. *If* $A_0 \cap A_1$ *is connected, then* $\Gamma(A_0) \cap \Gamma(A_1) = \Gamma(A_0 \cup A_1)$, *i.e. conditions* $f|A_j \sim 1$, $j = 0, 1$, *imply* $f \sim 1$.

THEOREM 4. *If the sets* A_0 *and* A_1 *are connected whereas* $A_0 \cap A_1$ *is not, then* $\Gamma(A_0) \cap \Gamma(A_1) \neq \Gamma(A_0 \cup A_1)$.

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(1) For polyhedrons this index is identical with the *Euler-Poincaré characteristic* provided that all Betti numbers, beginning with the second one, vanish.

More generally, *if the set* $A_0 \cap A_1$ *is not connected between the points* p_0 *and* p_1 *whereas* A_0 *and* A_1 *are connected between these points, then* $\Gamma(A_0) \cap \Gamma(A_1) \neq \Gamma(A_0 \cup A_1)$.

Proof. Theorem 1 follows from Theorems 1 of § 55, X and 3 of Section V. Theorem 2 follows from Theorems 2 of § 55, X and 3 and 4 of Section V. In order to prove Theorem 3, observe that $\mathscr{G}^{A_0 \cap A_1} = \mathscr{G} = \Theta_0(A_0, A_1)$ if $A_0 \cap A_1$ is connected; therefore, according to the first isomorphism of Theorem 3 of Section V, $\mathfrak{P}_1(A_0, A_1) = (0)$, and hence $\Gamma(A_0) \cap \Gamma(A_1) = \Gamma(A_0 \cup A_1)$. The same isomorphism implies Theorem 4 by Theorem 3 of § 55, X.

Remarks. 1. The isomorphisms (1) and (2) are determined by the transformation h considered in Theorem 6 of Section IV.

2. The first part of Theorem 3 can be proved more directly in the following manner.

Let $f|A_j = e_{\varphi_j}$, where φ_j: $A_j \to \mathscr{E}$ is continuous. The identity $e_{\varphi_0} = e_{\varphi_1}$ holds on $A_0 \cap A_1$. Thus, by Theorem 2 of Section I, there exists a number $k \in \mathscr{G}$ such that $(\varphi_0 - \varphi_1)|(A_0 \cap A_1) = k$. Since the functions φ_0 and $\varphi_1 + k$ are identical on $A_0 \cap A_1$, there exists a continuous function ψ: $\mathscr{X} \to \mathscr{E}$ such that $\psi|A_0 = \varphi_0$ and $\psi|A_1 = \varphi_1 + k$. Since $k \in \mathscr{G}$, it follows that

$$e_{\varphi_1+k} = e_{\varphi_1} \cdot e_k = e_{\varphi_1}, \quad \text{so that} \quad f = e_\psi, \quad \text{and hence} \quad f \sim 1.$$

3. In the case where $\bar{A}_j = A_j$, Theorem 4 can be proved more directly by setting

$$A_0 \cap A_1 = P_0 \cup P_1, \quad \bar{P}_j = P_j, \quad P_0 \cap P_1 = 0, \quad P_0 \neq 0 \neq P_1,$$

$$\varphi(x) = \frac{\varrho(x, P_0)}{\varrho(x, P_0) + \varrho(x, P_1)} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } x \in A_0, \\ e^{2\pi i \varphi(x)} & \text{if } x \in A_1. \end{cases}$$

It follows that f non ~ 1.

THEOREM 5. *If* C_0, C_1 *and* C_2 *are three connected sets such that* $C_0 \cap C_1 \cap C_2 \neq 0$ *and* $\mathscr{X} = C_0 \cup C_1 \cup C_2$, *then*

$$\Gamma(C_0 \cup C_1) \cap \Gamma(C_1 \cup C_2) \cap \Gamma(C_2 \cup C_0) = \Gamma(C_0 \cup C_1 \cup C_2),$$

i.e. the condition $f|C_k \cup C_{k+1} \sim 1$, *where* $k = 0, 1, 2$ (*the subscripts being reduced* mod 3), *implies that* $f \sim 1$.

Proof. Let $x_0 \in C_0 \cap C_1 \cap C_2$. Assume that $f|C_k \cup C_{k+1} = e_{\varphi_{k-1}}$ where φ_{k-1}: $C_k \cup C_{k+1} \to \mathscr{E}$ is a continuous function. We may assume of course that $\varphi_0(x_0) = \varphi_1(x_0) = \varphi_2(x_0)$. Hence, by Theorem 2 of Section I (replacing $\mathscr{X}$ by C_k), $\varphi_{k-1}(x)-\varphi_{k+1}(x) = 0$ for $x \in C_k$. Thus the functions φ_0, φ_1 and φ_2 are consistent and hence determine a continuous function $\varphi\colon \mathscr{X} \to \mathscr{E}$ such that $\varphi | C_k \cup C_{k+1} = \varphi_{k-1}$. It follows that $f = e_\varphi$, which implies $f \sim 1$.

THEOREM 6. *If C_0, C_1 and C_2 are three connected sets such that $\mathscr{X} = C_0 \cup C_1 \cup C_2$, then*

$$\operatorname{rank}[\Gamma(C_0 \cup C_1) \cap \Gamma(C_1 \cup C_2) \cap \Gamma(C_2 \cup C_0)/\Gamma(C_0 \cup C_1 \cup C_2)] \leqslant 1.$$

In other words, *if $f, g\colon \mathscr{X} \to \mathscr{S}$ are two continuous functions such that*

$$f | C_k \cup C_{k+1} \sim 1 \quad \textit{and} \quad g | C_k \cup C_{k+1} \sim 1 \quad (k = 0, 1, 2),$$

there exist two integers m and n which do not vanish simultaneously and such that $f^m \cdot g^n \sim 1$.

Proof. By hypothesis

$$f | C_k \cup C_{k+1} = e_{\varphi_{k-1}} \quad \text{and} \quad g | C_k \cup C_{k+1} = e_{\psi_{k-1}}. \tag{3}$$

Since the set C_k is connected, there exist two integers a_k and b_k such that

$$\varphi_{k+1}(x)-\varphi_{k-1}(x) = a_k \quad \text{and} \quad \psi_{k+1}(x)-\psi_{k-1}(x) = b_k \text{ for } x \in C_k. \tag{4}$$

Let

$$a = a_0+a_1+a_2 \quad \text{and} \quad b = b_0+b_1+b_2$$

and define the integers m and j so that

$$ma+jb = 0 \quad \text{and} \quad |m|+|j| \neq 0. \tag{5}$$

Let

$$\begin{aligned} \chi_0(x) &= m\varphi_0(x)+j\psi_0(x), && x \in C_1 \cup C_2, \\ \chi_1(x) &= m[\varphi_1(x)+a_2]+j[\psi_1(x)+b_2], && x \in C_2 \cup C_0, \\ \chi_2(x) &= m[\varphi_2(x)+a_2+a_0]+j[\psi_2(x)+b_2+b_0], && x \in C_0 \cup C_1. \end{aligned} \tag{6}$$

It is easy to show by (4) and (5) that

$$\chi_{k-1}(x)-\chi_{k+1}(x) = 0 \quad \text{for} \quad x \in C_k.$$

Consequently, there exists a continuous function $\chi\colon \mathscr{X} \to \mathscr{E}$ such that

$$\chi | C_k \cup C_{k+1} = \chi_{k-1}. \tag{7}$$

Therefore, $f^m \cdot g^j \sim 1$, in fact

$$f^m \cdot g^j = e_\chi.$$

For let $x \in C_k \cup C_{k+1}$. By (6) and (7) it follows that

$$f^m(x) \cdot g^j(x) = e^{2\pi i[m\varphi_{k-1}(x)+j\psi_{k-1}(x)]} = e^{2\pi i\chi_{k-1}(x)} = e^{2\pi i\chi(x)}.$$

Remark. Theorems 5 and 6 can be generalized as follows(1). Let $\mathscr{X} = A_0 \cup \ldots \cup A_{n-1}$. Put $C_k = A_{k+1} \cup \ldots \cup A_{k+n-2}$ (the subscripts being reduced mod n).

If the sets $C_0, \ldots, C_{n-1}$ are connected, the following statements hold.

THEOREM 5′. *Let f: $\mathscr{X} \to \mathscr{S}$ be a continuous function. If $C_0 \cap \cap \ldots \cap C_{n-1} \neq 0$ and if $f \sim 1$ on $A_{k+1} \cup \ldots \cup A_{k+n-1}$ for $k = 0, 1, \ldots, n-1$, then $f \sim 1$ on $\mathscr{X}$.*

THEOREM 6′. *Let f, g: $\mathscr{X} \to \mathscr{S}$ be continuous functions. If $f \sim 1$ and $g \sim 1$ in $A_{k+1} \cup \ldots \cup A_{k+n-1}$, there exist two integers m and j which do not vanish simultaneously and such that $f^m \cdot g^j \sim 1$ on $\mathscr{X}$.*

THEOREM 7. *If C is connected, then*

$$\bigcap_x \Gamma(C \cup x) = \Gamma(\bar{C});$$

i.e. (assuming $\bar{C} = \mathscr{X}$), *if f: $\mathscr{X} \to \mathscr{S}$ is a continuous function, $f|C \sim 1$ and f* non ~ 1, *then there exists a point p such that $(f|C \cup p)$* non ~ 1; more precisely: *if $f|C = e_\varphi$, then the oscillation of the function φ does not vanish at the point p*(2).

Proof. First, there exists a point p at which the oscillation of the function φ does not vanish. Because otherwise there would exist a (continuous) extension ψ: $\mathscr{X} \to \mathscr{E}$ of φ (compare § 35, I, Theorem 1); but then $f = e_\psi$, since conditions $\lim\limits_{n=\infty} x_n = x$ and $x_n \in C$ imply that $\lim\limits_{n=\infty} \varphi(x_n) = \psi(x)$, so that

$$f(x) = \lim_{n=\infty} f(x_n) = \lim_{n=\infty} e_\varphi(x_n) = e_\psi(x), \quad \text{therefore} \quad f \sim 1.$$

On the other hand, assume that $(f|C \cup p) \sim 1$, so that $f|C \cup p = e_\chi$, where χ: $C \cup p \to \mathscr{E}$ is a continuous function. This condition combined with the condition $f|C = e_\varphi$, implies by connectedness

(1) For the proof, see my paper in Fund. Math. 36 (1949), p. 277.

(2) See S. Eilenberg, Fund. Math. 24 (1935), p. 171.

of C that $\chi|C = \varphi+k$, where $k \in \mathscr{G}$. Therefore the function $\chi-k$ is a continuous extension of φ to $C \cup p$ and hence the oscillation of the function φ vanishes at the point p contradicting the hypothesis.

THEOREM 8. *Let $C_0, C_1, \ldots$ be a sequence of connected sets such that*

$$\mathscr{X} = C_0 \cup C_1 \cup \ldots \quad \textit{and} \quad C_n \subset \operatorname{Int}(C_{n+1}).$$

If $f: \mathscr{X} \to \mathscr{S}$ is a continuous function and $f|C_n \sim 1$ for each n, then $f \sim 1$.

Proof. It is legitimate to assume that $C_0 \neq 0$. Let $a \in C_0$. Let $f|C_n = e_{\varphi_n}$, let $\varphi_n: C_n \to \mathscr{E}$ be continuous and $\varphi_n(a) = \varphi_0(a)$.

Since C_n is connected, the condition $\varphi_{n+1}(a) = \varphi_n(a)$ implies that $\varphi_n = \varphi_{n+1}|C_k$. Therefore the functions φ_n are consistent and define a continuous function $\varphi: \mathscr{X} \to \mathscr{E}$ such that $\varphi|C_n = \varphi_n$. It follows that $f = e_\varphi$, so that $f \sim 1$.

VII. Relation *f irr non* ~ 1. A continuous function $f: \mathscr{X} \to \mathscr{S}$ is said to be *irreducibly non* ~ 1, in symbols, *f irr non* ~ 1, if f non ~ 1, whereas $f|F \sim 1$ for every proper closed subset F of $\mathscr{X}$.

Put

$$\Omega(\mathscr{X}) = \bigcap_F \Gamma(F) \quad \text{where} \quad F = \overline{F} \neq \mathscr{X}; \tag{1}$$

thus the conditions *f irr non* ~ 1 and $f \in \Omega(\mathscr{X}) - \Gamma(\mathscr{X})$ are equivalent.

THEOREM 1. *If f irr non* ~ 1, *then $\mathscr{X}$ is a connected discoherent space.*

Proof. If $\mathscr{X}$ is not discoherent, there exists a closed connected set C (empty or not) which separates $\mathscr{X}$ (compare § 46, X, Theorem 1).

$$\mathscr{X} = A_0 \cup A_1, \quad A_0 \cap A_1 = C, \quad A_j = \overline{A}_j \neq \mathscr{X}, \qquad \text{where} \quad j = 0, 1. \tag{2}$$

But then $f|A_j \sim 1$, and hence $f \sim 1$ by Theorem 3 of Section VI.

THEOREM 2. *If f irr non* ~ 1 *and $\mathscr{X}$ is locally connected, then $\mathscr{X}$ is a simple closed curve.*

This is a consequence of Theorem 1 combined with Theorem 6 of § 49, IV.

Theorem 1 implies the following statement by Theorems 2 of § 46, X and 2 of § 48, V.

THEOREM 3. *If f irr non ~ 1 and $\mathscr{X}$ is decomposable, then there are two connected sets A_0 and A_1 such that $\mathscr{X} = A_0 \cup A_1$ and*

$$A_0 = \overline{\mathscr{X} - A_1} \neq \mathscr{X} \quad \textit{and} \quad A_1 = \overline{\mathscr{X} - A_0} \neq \mathscr{X}. \tag{3}$$

In particular (compare § 48, VII, Theorem 7), *if $\mathscr{X}$ is irreducible between two points, the sets A_0 and A_1 are indecomposable.*

THEOREM 4. *If A_0 and A_1 are two closed connected sets such that $\mathscr{X} = A_0 \cup A_1$, then* (compare § 55, XI (1))

$$(\Xi_0 \cap \Xi_1)/G \underset{\mathrm{gr}}{=} \Big[\bigcap_{K_0, K_1} \Gamma(A_0 \cup K_1) \cap \Gamma(A_1 \cup K_0)\Big]/\Gamma(X) \subset \Omega(X)/\Gamma(X), \tag{4}$$

where K_j varies over the family of proper closed subsets of A_j.

Moreover, *if conditions* (3) *are fulfilled the inclusion may be replaced by identity.*

Proof. Let $K = \bar{K} \subset A_1$. Using the notation of Theorem 6 of Section IV, put

$$k_d = (h_d | A_0 \cup K). \tag{5}$$

Since

$$A_0 \cup (A_0 \cap A_1 \cup K) = A_0 \cup K, \; A_0 \cap (A_0 \cap A_1 \cup K) = A_0 \cap A_1,$$

it follows from Theorem 6 (i) of Section IV (replacing $\mathscr{X}$ by $A_0 \cup K$) that

$$\Theta_0(A_0, A_0 \cap A_1 \cup K) = k^{-1}[\Psi(A_0 \cup K)]. \tag{6}$$

But, since A_0 is connected, we deduce from Theorem 1 of § 55, X that

$$\Theta_0(A_0, A_0 \cap A_1 \cup K) = \mathscr{G}^{A_0 \cap A_1 \cup K} | A_0 \cap A_1. \tag{7}$$

On the other hand, by (5) $(k_d \sim 1) \equiv (h_d | A_0 \cup K \sim 1)$, which implies

$$[k_d \in \Psi(A_0 \cup K)] \equiv [h_d \in \Gamma(A_0 \cup K)],$$

therefore

$$k^{-1}[\Psi(A_0 \cup K)] = h^{-1}[\Gamma(A_0 \cup K)]. \tag{8}$$

So, it follows by conditions (6) through (8) that

$$\mathscr{G}^{A_0 \cap A_1 \cup K} | A_0 \cap A_1 = h^{-1}[\Gamma(A_0 \cup K)],$$

Taking the intersection of all these groups for all $K \neq A_1$, we infer that

$$\Xi_1 = \bigcap_K (\mathscr{G}^{A_0 \cap A_1 \cup K} | A_0 \cap A_1) = h^{-1}\big[\bigcap_K \Gamma(A_0 \cup K)\big],$$

therefore

$$\Xi_0 \cap \Xi_1 = h^{-1}\big[\bigcap_{K_0, K_1} \Gamma(A_0 \cup K_1) \cap \Gamma(A_1 \cup K_0)\big],$$

$$\text{where} \quad K_j \subset A_j \neq K_j.$$

The isomorphism (4) follows from Theorem 5 of § 55, III and from the twofold identity (compare IV, 6 (i) and § 55, X, Theorem 2) $\mathscr{G} = \Theta_0(A_0, A_1) = h^{-1}[\Gamma(\mathscr{X})]$.

The proof of inclusion (4) reduces to showing that

$$\bigcap_{K_0, K_1} \Gamma(A_0 \cup K_1) \cap \Gamma(A_1 \cup K_0) \subset \bigcap_F \Gamma(F),$$

$$\text{where} \quad F = \overline{F} \neq \mathscr{X}. \tag{9}$$

But the inequality $F \neq \mathscr{X}$ implies that either $F \cap A_0 \neq A_0$ or $F \cap A_1 \neq A_1$. Assume $K_1 = F \cap A_1 \neq A_1$. Since $F \subset A_0 \cup K_1$, it follows (compare II, Theorem 4) that $\Gamma(A_0 \cup K_1) \subset \Gamma(F)$, which implies inclusion (9).

Now assume condition (3) and let $K_1 \neq A_1$. It follows that $A_0 \cup K_1 \neq \mathscr{X}$, because otherwise $\mathscr{X} - A_0 \subset K_1$, so that $A_1 = \overline{\mathscr{X} - A_0} \subset K_1$ and $K_1 = A_1$. Let $f \in \Omega(\mathscr{X})$. Define $F = A_0 \cup K_1$. Therefore $f \in \Gamma(A_0 \cup K_1)$. It follows that

$$\Omega(\mathscr{X}) \subset \bigcap_{K_0, K_1} \Gamma(A_0 \cup K_1) \cap \Gamma(A_1 \cup K_0),$$

which implies the required identity by (9).

THEOREM 5. *Let $\mathscr{X}$ be a decomposable space. There exists a continuous function f: $\mathscr{X} \to \mathscr{S}$ such that f irr non ~ 1 if and only if there exists a decomposition of $\mathscr{X}$ into two closed connected sets A_0 and A_1, and a decomposition of $A_0 \cap A_1$ into two closed sets F_0 and F_1 between which A_j $(j = 0, 1)$ is irreducibly connected.*

More precisely, *for $n \geqslant 1$ the condition*

$$\operatorname{rank}[\Omega(\mathscr{X})/\Gamma(\mathscr{X})] \geqslant n \tag{10}$$

is equivalent to the existence of a representation of $\mathscr{X} = A_0 \cup A_1$, *where* A_0 *and* A_1 *are closed connected sets* (*which can be chosen so as to satisfy condition* (3)) *and of a decomposition of* $A_0 \cap A_1$ *into* $n+1$ *disjoint closed sets* $F_0, \ldots, F_n$ *such that* A_j *is irreducibly connected between* F_k *and* $H_k = A_0 \cap A_1 - F_k$ *for* $j = 0, 1$ *and* $k = 0, 1, \ldots, n$.

Proof. If f *irr non* ~ 1, there exist by Theorem 3 two connected sets A_0 and A_1 satisfying condition (3), and it follows by Theorem 4 that

$$\Omega(\mathscr{X})/\Gamma(\mathscr{X}) \underset{\mathrm{gr}}{=} (\Xi_0 \cap \Xi_1)/\mathscr{G}. \tag{11}$$

Conditions (10), (11) and condition (2) of § 55, XI imply that there exists a system of sets $F_0, \ldots, F_n$ which satisfies the required conditions.

Conversely, if there exists a system of sets $F_0, \ldots, F_n$ satisfying the said conditions, then condition (2) of § 55, XI combined with formulas (4) implies condition (10).

THEOREM 6. *If* $\mathscr{X}$ *is decomposable and the rank of the group* $\Omega(\mathscr{X})/\Gamma(\mathscr{X})$ *is* $\geqslant 2$, *then* $\mathscr{X}$ *is the union of two closed indecomposable sets.*

Moreover, *if* $\mathscr{X}$ *is compact, then* $\mathscr{X}$ *is a continuum irreducible between two points* (¹).

Proof. Assume that $n = 2$ in Theorem 5, and select a point p_k from F_k $(k = 0, 1, 2)$. Then A_j is irreducible between p_0 and p_1, between p_1 and p_2, and between p_0 and p_2. Therefore A_j is indecomposable.

The second part of Theorem 6 follows directly from Theorem 4 of § 48, IX.

Remark. The theory of irreducible spaces leads in a very natural manner to a *linear* stratification of the (non-"monostratic") irreducible-between-two-points spaces into closed "layers" (§ 48, IV). This linear stratification, in its turn, implies, by Theorem 5, a cyclic stratification of every (non-"monostratic") space which admits a transformation f *irr non* ~ 1 (²). In the case where the space

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(¹) Compare an analogous theorem of P. Alexandrov, Math. Ann. 96 (1926), p. 534.

(²) In the same direction, see my paper *Sur la structure des frontières communes à deux régions*, Fund. Math. 12 (1928), p. 20.

is a continuum, its layers are continua (compare § 48, IV, Theorem 2); in the particular case where the space is a simple closed curve, the layers reduce to single points.

VIII. Compact sets. Let $\mathcal{X}$ be a compact space.

THEOREM 1. *Let f: $\mathcal{X} \to \mathcal{S}$ be a continuous function. The family of closed sets F such that $f|F \sim 1$ is open (in the space $2^{\mathcal{X}}$).*

Proof. By Theorem 9 of Section II,

$$\underset{F}{E}(f|F \sim 1) = \bigcup_G \underset{F}{E}(F \subset G),$$

where G is a variable open set such that $f|G \sim 1$.

Since the family $\underset{F}{E}(F \subset G)$ is open (by § 17, II, Theorem 1), then so is its union.

The next theorem follows directly from Theorem 1.

THEOREM 2. *If the sets $F_1 \supset F_2 \supset \dots$ are closed, then*

$$\Gamma(\bigcap_n F_n) = \bigcup_n \Gamma(F_n),$$

i.e., if $(f|F_n)$ non ~ 1 for $n = 1, 2, \dots$, then $(f|\bigcap_n F_n)$ non ~ 1.

THEOREM 3. *If f non ~ 1, then $\mathcal{X}$ contains a continuum C such that $f|C$ irr non ~ 1.*

Proof. It follows from Theorem 1 and Theorem 2 of § 42, IV, that there exists a closed set C such that $f|C$ *irr non* ~ 1. By Theorem 1 of Section VII the set C is a continuum.

The following statement is a direct consequence of the foregoing one.

THEOREM 4. *If $f|Q \sim 1$ for each component Q of $\mathcal{X}$, then $f \sim 1$.*

Remark. It follows easily from Theorem 3 that if $\mathcal{X}$ is an arc, then $f \sim 1$ for every continuous function f: $\mathcal{X} \to \mathcal{S}$ (compare III, Theorem 3). Because, otherwise $\mathcal{X}$ would contain a discoherent continuum (by Theorem 3 and Theorem 1 of Section VII).

IX. Cartesian products. Relations to homotopy.

LEMMA 1. *Let $\mathcal{X}$ be a connected space, $\mathcal{Y}$ an arbitrary space, f: $\mathcal{X} \times \mathcal{Y} \to \mathcal{S}$ continuous, $f \sim 1$, and finally let φ: $\mathcal{X} \times \mathcal{Y} \to \mathcal{E}$.*

If ψ is continuous with respect to x for every y, and continuous with respect to y for some point a of $\mathscr{X}$, and if $e_\psi = f$, then ψ is continuous (in $\mathscr{X}\times\mathscr{Y}$).

Proof. Let $f = e_\varphi$, where $\varphi\colon \mathscr{X}\times\mathscr{Y}\to\mathscr{E}$ is a continuous function. Since the difference $\psi(x, y)-\varphi(x, y)$ is constant for y fixed (by Theorem 2 of Section I), put $\psi(x, y)-\varphi(x, y) = k(y)$. In particular

$$\psi(a, y)-\varphi(a, y) = k(y),$$

$$\text{thus}\qquad \psi(x, y) = \varphi(x, y)+\psi(a, y)-\varphi(a, y),$$

which implies the required conclusion.

LEMMA 2. *Let $\mathscr{X}$ be a continuum, $\mathscr{Y}$ an arbitrary space, $a\in\mathscr{X}$ and $f\colon \mathscr{X}\times\mathscr{Y}\to\mathscr{S}$ continuous. If $(f|a\times\mathscr{Y})\sim 1$ and if $(f|\mathscr{X}\times\mathscr{Y})\sim 1$ for every $y\in\mathscr{Y}$, then $f\sim 1$.*

Proof. By hypothesis, there exists a continuous function $\chi\colon \mathscr{Y}\to\mathscr{E}$ and, for every y, a continuous function $\psi_y\colon \mathscr{X}\to\mathscr{E}$ such that

$$f(a, y) = e_\chi(y) \quad\text{and}\quad f(x, y) = e_{\psi_y}(x) \quad \text{for any } x \text{ and } y.$$

Define

$$\psi(x, y) = \psi_y(x)-\psi_y(a)+\chi(y). \tag{1}$$

It follows that

$$e_\psi(x, y) = e_{\psi_y}(x) : e_{\psi_y}(a)\cdot e_\chi(y) = f(x, y).$$

We have to show that the function ψ is continuous. This reduces to proving that, if $y_1, y_2, \ldots$ is a sequence converging to y_0 and if $\mathscr{Y}^* = (y_0, y_1, y_2, \ldots)$, then the restricted function $\psi|\mathscr{X}\times\mathscr{Y}^*$ is continuous.

But, since the set $\mathscr{X}\times\mathscr{Y}^*$ is compact and the sets $\mathscr{X}\times y_n$ are its components, it follows by Theorem 4 of Section VIII that $(f|\mathscr{X}\times\mathscr{Y}^*)\sim 1$. Since $f = e_\psi$, the function $\psi|\mathscr{X}\times\mathscr{Y}^*$ is continuous by Theorem 1.

Lemmas 1 and 2 imply the following important statement.

THEOREM 3 (of Eilenberg(1)). *Let $\mathscr{Y}$ be an arbitrary space and*

(1) Fund. Math. 26 (1936), p. 68.

$g\colon \mathcal{Y} \to \mathcal{S}$ be a continuous function. In order that $g \sim 1$, it is necessary and sufficient that $g \simeq 1$, i.e. that g be homotopic to 1.

Consequently, *the conditions $\mathcal{S}^{\mathcal{Y}} = \Psi(\mathcal{Y})$, $b_1(\mathcal{Y}) = 0$ and the contractibility of $\mathcal{Y}$ with respect to $\mathcal{S}$ are equivalent.*

Proof. Let $g \sim 1$, therefore $g = e_\varphi$, where $\varphi\colon \mathcal{Y} \to \mathcal{E}$ is a continuous function. Assume that

$$f(x, y) = e^{2\pi i x\varphi(y)} \quad \text{for} \quad x \in \mathcal{I} \text{ and } y \in \mathcal{Y}.$$

It follows

$$f\colon \mathcal{I} \times \mathcal{Y} \to \mathcal{S}, \quad f(0, y) = 1 \quad \text{and} \quad f(1, y) = g(y). \tag{2}$$

Therefore g is homotopic to 1.

Conversely, assume that $g \simeq 1$; thus conditions (2) are satisfied. Setting $\mathcal{X} = \mathcal{I}$ and $a =$ the point 0, in Lemma 2, we infer (by Theorem 3 of Section III) that $f \sim 1$, which implies $g \sim 1$.

Theorem 3 can be generalized as follows.

THEOREM 4. *The conditions $f \sim g$ and $f \simeq g$ are equivalent.*

Proof. The relation $f \sim g$ is equivalent to $f{:}g \sim 1$, hence to $(f{:}g) \simeq 1$ (by Theorem 3); but this means, that it is equivalent to the existence of a continuous function $h\colon \mathcal{X} \times \mathcal{I} \to \mathcal{S}$ such that $h(x, 0) = f(x){:}g(x)$ and $h(x, 1) = 1$.

Put $u(x, t) = g(x)\cdot h(x, t)$; it follows that $u(x, 0) = f(x)$ and $u(x, 1) = g(x)$.

THEOREM 5. *If $\mathcal{X}$ is compact, the group $\mathfrak{B}_1(\mathcal{X})$ coincides with the family of components of the space $\mathcal{S}^{\mathcal{X}}$. Its neutral element coincides with the component containing constant functions.*

Proof. Since $\mathcal{S}$ is an ANR, the relation $f \simeq g$ (therefore $f \sim g$ where $f, g\colon \mathcal{X} \to \mathcal{S}$ are continuous functions) holds if and only if f and g belong to the same component of $\mathcal{S}^{\mathcal{X}}$ (compare § 54, II, Theorem 10).

THEOREM 6. *Let $\mathcal{X}$ be a continuum, $\mathcal{Y}$ a connected space, $x_0 \in \mathcal{X}$, $y_0 \in \mathcal{Y}$ and $f\colon \mathcal{X} \times \mathcal{Y} \to \mathcal{S}$ a continuous function. If $f|x_0 \times \mathcal{Y} \sim 1$ and $f|\mathcal{X} \times y_0 \sim 1$, then $f \sim 1$.*

Proof. Let $g_y(x) = f(x, y)$. According to Theorem 3 of § 44, IV, the function g, which assigns to every y an element g_y of $\mathcal{S}^{\mathcal{X}}$, is continuous. Therefore $\Phi = g(\mathcal{Y})$ is a connected subset of $\mathcal{S}^{\mathcal{X}}$.

By hypothesis $g_{y_0} \simeq 1$, therefore $g_{y_0} \sim 1$. This shows that Φ is a subset of that component of $\mathcal{S}^{\mathcal{X}}$ which contains the constant functions. By Theorem 5, it follows that $h \sim 1$ for every $h \in \Phi$. Therefore $g_y \sim 1$, which implies $f|\mathscr{X}\times y \sim 1$, for each y. And hence $f \sim 1$ by Lemma 2.

THEOREM 7. *If $\mathscr{X}$ is a continuum and $\mathscr{Y}$ is a connected space, then*

$$\mathfrak{B}_1(\mathscr{X}\times\mathscr{Y}) \underset{\mathrm{gr}}{=} \mathfrak{B}_1(\mathscr{X})\times\mathfrak{B}_1(\mathscr{Y}),$$

$$\textit{hence} \quad b_1(\mathscr{X}\times\mathscr{Y}) = b_1(\mathscr{X})+b_1(\mathscr{Y}). \tag{3}$$

Proof. This isomorphism is defined by assigning to every pair $f\colon \mathscr{X}\to\mathscr{S}$, $g\colon \mathscr{Y}\to\mathscr{S}$ of continuous functions the function $h = f\cdot g$ defined by the identity

$$h(x, y) = f(x)\cdot g(y).$$

In order to show this statement, assume $f^*(x, y) = f(x)$, $g^*(x, y) = g(y)$, $x_0 \in \mathscr{X}$ and $y_0 \in \mathscr{Y}$.

The conditions $f \sim 1$ and $g \sim 1$ imply that $f^* \sim 1$ and $g^* \sim 1$, hence $h \sim 1$.

Conversely, if $h \sim 1$, then $(f^*\cdot g^*)|(\mathscr{X}\times y_0) \sim 1$ and since $g^*|(\mathscr{X}\times \times y_0) \sim 1$, it follows that $f^*|(\mathscr{X}\times y_0) \sim 1$, so that $f \sim 1$. Similarly, $g \sim 1$.

Finally, to every continuous function $l\colon \mathscr{X}\times\mathscr{Y}\to\mathscr{S}$ there corresponds a pair of continuous functions $f\colon \mathscr{X}\to\mathscr{S}$ and $g\colon \mathscr{Y}\to\mathscr{S}$ such that $l \sim f\cdot g$, namely

$$f(x) = l(x, y_0) \quad \text{and} \quad g(y) = l(x_0, y).$$

Because

$$(l{:}fg)|(\mathscr{X}\times y_0) \sim 1 \quad \text{and} \quad (l{:}fg)|(x_0\times\mathscr{Y}) \sim 1,$$

thus $l{:}fg \sim 1$, by Theorem 6.

Finally, condition (3) follows from Theorem 5 of § 55, III and of formula (4) of Section VII.

X. Locally connected sets.

THEOREM 1. *If $f|Q \sim 1$ for every component Q of a locally connected space $\mathscr{X}$, then $f \sim 1$.*

This is a consequence of Theorem 5 of § 54, I, since Q is a closed-open set (compare § 49, II, Theorem 4).

THEOREM 2. *Let $\mathscr{X}$ be a connected and locally connected space, $\mathscr{Y}$ an arbitrary space, $a \in \mathscr{X}$ and $f\colon \mathscr{X}\times\mathscr{Y}\to\mathscr{S}$ a continuous function. If $(f|a\times\mathscr{Y}) \sim 1$ and if $(f|\mathscr{X}\times y) \sim 1$ for every y, then $f \sim 1$.*

Proof. ψ and $\mathscr{Y}^*$ are defined as in the proof of Theorem 2 of Section IX. We have to show that the restricted function $\psi_0 = \psi|\mathscr{X}\times\mathscr{Y}^*$ is continuous. Let C be the set of x's such that ψ_0 is continuous at the point (x, y_0). Since $\mathscr{X}$ is connected, it is sufficient to show that C is closed, open and non-empty.

Let $x_1 \in \mathscr{X}$. By Theorem 10 of Section II, there exist two open sets G in $\mathscr{X}$ and H in $\mathscr{Y}^*$ such that $x_1 \in G$ and $y_0 \in H$ and $f|G\times H \sim 1$ (one has to substitute the point (x_1, y_0) for a in Theorem 10 of Section II). Moreover, since $\mathscr{X}$ is locally connected, we can assume that G is connected.

There are two cases to be considered. If there exists a point $x_0 \in G$ such that, for $x = x_0$, the function ψ_0 is continuous with respect to the variable y, then it is continuous in $G\times H$, by Theorem 1 of Section IX (replacing there $\mathscr{X}$ by G, $\mathscr{Y}$ by H and a by x_0), and it follows that $G \subset C$. In particular, $a \in C$, because for $x = a$ the function ψ_0 is continuous with respect to y (by the identity $\psi(a, y) = \chi(y)$, compare IX (1)).

In the case where G does not contain any point x_0 of this kind, the function ψ_0, considered as a function of the variable y, is discontinuous for any $x \in G$; therefore it is discontinuous at the point y_0 (since y_0 is the only accumulation point of $\mathscr{Y}^*$) and it follows that $x \notin C$, which implies $x_1 \in G \subset \mathscr{X} - C$.

Thus the set C is open, non-empty and closed.

THEOREM 3. *Let* $A \subset B \subset A \cup L(A)$ *and* $g\colon B \to \mathscr{S}$ *a continuous function. If* $g|A \sim 1$, *then* $g \sim 1$.

Proof. Let $g|A = e_\varphi$ where $\varphi\colon A \to \mathscr{E}$ is a continuous function. Let $p \in B - A$. We will show that the oscillation $\omega_\varphi(p)$ vanishes.

Let (compare § 49, II) $p \in G$, where G is an open set in B such that $G \cap A$ is connected and $g(G) \neq \mathscr{S}$. Therefore $g|G \sim 1$ (see Theorem 6 of Section II). Let $g|G = e_\psi$ where $\psi\colon G \to \mathscr{E}$ is continuous. Since the set $G \cap A$ is connected, the identity $e_\varphi(x) = e_\psi(x)$ for $x \in G \cap A$ implies that the difference $\varphi(x) - \psi(x)$ is constant in $G \cap A$ (Theorem 2 of Section I). Since ψ is continuous at the point p, then $\omega_\psi(p) = 0$, which implies $\omega_\varphi(p) = 0$ (because G is a neighbourhood of p in B).

Consequently, there exists (according to § 35, I, Theorem 1) a continuous extension $\varphi^*\colon B \to \mathscr{E}$ of φ. Since $B \subset \bar{A}$ and $g|A = e_\varphi$, it follows that $g = e_{\varphi^*}$.

THEOREM 4. *Let $\mathcal{X}$ be an arcwise connected space and $f\colon \mathcal{X} \to \mathcal{S}$ a continuous function. If $f|C \sim 1$ for every simple closed curve C, then $f \sim 1$.*

Proof. By Theorem 1, it is legitimate to assume that $\mathcal{X}$ is connected. It can also be assumed that there exists a point a such that $f(a) = 1$.

Assign to every point x an arc $A_x = ax$ and a continuous function ψ_x such that

$$\psi_x\colon A_x \to \mathcal{E}, \quad f|A_x = e_{\psi_x} \quad \text{and} \quad \psi_x(a) = 0, \tag{1}$$

according to Theorems 3 of Section III and 4 of Section I.

Define $\varphi(x) = \psi_x(x)$. It follows that $e_\varphi = f$.

We have to show that the function φ is continuous. Let $p \in \mathcal{X}$ and $\varepsilon > 0$. By Theorem 10 of Section II, $p \in G$, where G is an open set such that $f|G = e_\lambda$ and $\lambda\colon G \to \mathcal{E}$ is a continuous function. Moreover, one can suppose (diminishing the set G if necessary) that $\delta[\lambda(G)] < \varepsilon$.

Since the space is locally connected, there exists $\eta > 0$ such that for every point q with $|q-p| < \eta$ there is assigned an arc $Q = pq \subset G$. It follows that

$$f|Q = e_\lambda \quad \text{and} \quad |\lambda(q)-\lambda(p)| < \varepsilon. \tag{2}$$

We must show that $|\varphi(q)-\varphi(p)| < \varepsilon$.

The continuum $K = A_p \cup A_q \cup Q$ is hereditarily locally connected, because it consists of three arcs (compare § 51, IV, Theorems 2 and 8). Therefore

$$f|K \sim 1, \tag{3}$$

for otherwise K would contain by Theorem 3 of Section VIII a subcontinuum, and hence a locally connected subcontinuum C such that $f|C\,irr\,non \sim 1$. By Theorem 2 of Section VII, the set C is a simple closed curve, contradicting the hypothesis.

Following (3) we define

$$f|K = e_\chi, \quad \chi\colon K \to \mathcal{E} \quad \text{and} \quad \chi(a) = 0, \tag{4}$$

where χ is a continuous function.

Conditions (1) and (4) imply that $\psi_p(a) = \psi_q(a) = \chi(a)$, so that by Theorem 2 of Section I

$$\chi(x) = \begin{cases} \lambda(x)+k & \text{for} \quad x \in Q, \\ \psi_p(x) & \text{for} \quad x \in A_p, \\ \psi_q(x) & \text{for} \quad x \in A_q. \end{cases}$$

It follows that

$$\varphi(q)-\varphi(p) = \psi_q(q)-\psi_p(p) = \chi(q)-\chi(p) = \lambda(q)-\lambda(p),$$

hence by (2)

$$|\varphi(q)-\varphi(p)| < \varepsilon.$$

THEOREM 5. *Let $\mathfrak{X}$ be a locally connected continuum and $f\colon \mathfrak{X} \to \mathscr{S}$ a continuous function. If $f|E \sim 1$ for every cyclic element E of $\mathfrak{X}$, then $f \sim 1$.*

Proof. If f *non* ~ 1, then $\mathfrak{X}$ contains a simple closed curve C such that $f|C$ *non* ~ 1. Thus, if E denotes the cyclic element of $\mathfrak{X}$ which contains C (compare § 52, II, Theorem 10), then $f|E$ *non* ~ 1.

THEOREM 6. *Let $\mathfrak{X}$ be a locally connected continuum, $F = \overline{F} \subset \mathfrak{X}$, $f_k\colon \mathfrak{X} \to \mathscr{S}$ a continuous function and $f_k|F \sim 1$ for $k = 1, \ldots, n$. Then there exists a locally connected continuum C such that $F \subset C$ and that $f_k|C \sim 1$ for $k = 1, \ldots, n$*[1].

Proof. Since F is the intersection of a sequence of sets, every of which consists of a finite number of locally connected continua (§ 50, III, Theorem 1 (i) and § 49, II, Theorem 7), the proof can be reduced to the case (by Theorem 9 of Section III) where $F = C_1 \cup \ldots \cup C_m$ and where the sets $C_1, \ldots, C_m$ are disjoint, locally connected continua.

We proceed by induction. Clearly, the theorem holds for $m = 1$. Thus assume that it holds for the number $m-1$. Let L be an arc ab such that $L \cap F$ consists only of two points a and b, which are contained in different sets C_n, say, in C_{m-1} and C_m.

The conditions (compare Theorem 3 of Section III)

$$f_k|C_1 \cup \ldots \cup C_{m-1} \sim 1, \quad f_k|L \sim 1$$
$$\text{and} \quad (C_1 \cup \ldots \cup C_{m-1}) \cap L = a$$

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(1) S. Eilenberg, Fund. Math. 27 (1936), p. 172.

imply by Theorem 3 of Section VI that

$$f_k|C_1 \cup \ldots \cup C_{m-1} \cup L \sim 1,$$

$$\text{thus} \quad f_k|C_1 \cup \ldots \cup C_{m-1} \cup L \cup C_m \sim 1,$$

because $(C_1 \cup \ldots \cup C_{m-1} \cup L) \cap C_m = b$ and $f|C_m \sim 1$.

Since the set $C_{m-1} \cup L \cup C_m$ is a locally connected continuum, the proof has been reduced to the case $m-1$.

THEOREM 7. *Let $\mathscr{X}$ be a locally connected continuum and $f\colon \mathscr{X} \to \mathcal{S}$ a continuous function. There exist two locally connected continua C_0 and C_1 such that $\mathscr{X} = C_0 \cup C_1$ and $f|C_j \sim 1$ for $j = 0, 1$.*

Proof. Let A_0 and A_1 be the halves of the circle $\mathcal{S}$ with the ordinates $\geqslant 0$ and $\leqslant 0$, respectively. Let $F_j = f^{-1}(A_j)$. It follows that $\mathscr{X} = F_0 \cup F_1$ and $f|F_j \sim 1$ by Theorem 6 of Section II, which implies the required conclusion according to Theorem 6 (the case where $n = 1$).

THEOREM 8. *Let $\mathscr{X}$ be a locally arcwise connected space, let $f\colon \mathscr{X} \to \mathcal{S}$ be a continuous function and let $G_1, G_2, \ldots$ be a sequence of open sets such that*

$$\mathscr{X} = G_1 \cup G_2 \cup \ldots \quad \textit{and} \quad G_n \subset G_{n+1}.$$

If $f|G_n \sim 1$ for every number n, then $f \sim 1$.

This is a consequence of Theorem 4.

XI. Mappings.

THEOREM 1. *If A is a retract of $\mathscr{X}$, then $\mathfrak{B}_1(A)$ is a homomorphic image of $\mathfrak{B}_1(\mathscr{X})$, thus $b_1(A) \leqslant b_1(\mathscr{X})$.*

THEOREM 2. *If $\mathscr{X}$ is deformable onto A, then*

$$\mathfrak{B}_1(\mathscr{X}) \underset{\text{gr}}{=} (\mathcal{S}^{\mathscr{X}}|A)/\Psi(A), \quad \textit{thus} \quad b_1(\mathscr{X}) \leqslant b_1(A).$$

THEOREM 3. *If A is a deformation retact of $\mathscr{X}$, then*

$$\mathfrak{B}_1(A) \underset{\text{gr}}{=} \mathfrak{B}_1(X), \quad \textit{thus} \quad b_1(A) = b_1(\mathscr{X}).$$

Proof. The retraction operation $\zeta\colon \mathcal{S}^{\mathscr{X}} \to \mathcal{S}^A$, i.e. $\zeta(f) = f|A$, is a homomorphism (compare § 55, VIII, Theorem 1) of $\mathcal{S}^{\mathscr{X}}$ onto $\mathcal{S}^{\mathscr{X}}|A$, and the condition $f \sim 1$ implies $f|A \sim 1$. Theorem 1 follows, because A is a retract of $\mathscr{X}$ and hence $\mathcal{S}^{\mathscr{X}}|A = \mathcal{S}^A$.

On the other hand, if $\mathscr{X}$ is deformable onto A, the condition $f|A \sim 1$ implies $f \sim 1$ (by § 54, IV, Theorem 1), which proves Theorem 2.

Theorem 3 follows from Theorem 2 by the identity $\mathcal{S}^{\mathcal{X}}|A = \mathcal{S}^A$.

THEOREM 4[1]. *Let $\mathcal{X}$ be a compact space; let $g\colon \mathcal{X} \to g(\mathcal{X})$ and $f\colon g(\mathcal{X}) \to \mathcal{S}$ be two continuous mappings. In the following two cases the condition $fg \sim 1$ implies $f \sim 1$,*

(i) *g is monotone,*

(ii) *g is open.*

Therefore, in both cases the group $\mathfrak{B}_1[g(\mathcal{X})]$ is isomorphic to a subgroup of $\mathfrak{B}_1(\mathcal{X})$ and $b_1[g(\mathcal{X})] \leqslant b_1(\mathcal{X})$.

Proof. Let $fg(x) = e_\psi(x)$ where $\psi\colon \mathcal{X} \to \mathcal{E}$ is a continuous function.

Case (i). Let $y_0 \epsilon g(\mathcal{X})$ and $x_0 \epsilon g^{-1}(y_0)$.

For all $x \epsilon g^{-1}(y_0)$ we have

$$e_\psi(x) = fg(x) = f(y_0) = fg(x_0) = e_\psi(x_0).$$

Therefore the function $\psi(x) - \psi(x_0)$ assumes integer values (on $g^{-1}(y_0)$), thus it is constant, because $g^{-1}(y)$ is connected; this implies

$$\psi(x) - \psi(x_0) = \psi(x_0) - \psi(x_0) = 0.$$

Thus, the condition $\varphi(y) = \psi(x)$ for $x \epsilon g^{-1}(y)$ uniquely determines the function $\varphi\colon g(\mathcal{X}) \to \mathcal{E}$. This function is continuous, because (compare § 20, V, Theorem 8)

$$[t = \varphi(y)] \equiv \bigvee_x [y = g(x)][t = \psi(x)].$$

Moreover, $e_\varphi(y) = e_\psi(x) = fg(x) = f(y)$, so that $f \sim 1$.

Case (ii). If g is open, then $g^{-1}\colon\ g(\mathcal{X}) \to 2^{\mathcal{X}}$ is a continuous function (compare § 43, V, Theorem 1). Therefore the function ψg^{-1} is also continuous, and so is the function $\varphi\colon g(\mathcal{X}) \to \mathcal{E}$ defined by the condition

$$\varphi(y) = \text{greatest lower bound of the set } \psi[g^{-1}(y)]$$

(compare § 42, II, Remark 2); if x is a point of $g^{-1}(y)$ such that $\varphi(y) = \psi(x)$, it follows, as before, that $e_\varphi = f$.

Remark. The part (ii) of Theorem 4 remains true if $\mathcal{S}$ is replaced by $\mathcal{P} = \mathcal{E}^2 - (0)$.

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[1] Compare S. Eilenberg, Fund. Math. 24 (1935), p. 165 and 174.

Since we have $fg(x) = |fg(x)|e_\psi(x)$, where ψ: $\mathscr{X} \to \mathscr{E}$ is continuous, it follows $f(y) = |f(y)|e_\psi(y)$; to show this, one assigns to y an $x \in g^{-1}(y)$ where $\psi[g^{-1}(y)]$ has its lower bound.

§ 57. Spaces contractible with respect to $\mathscr{S}$. Unicoherent spaces

I. Contractibility with respect to $\mathscr{S}$. By definition (§ 54, V) a space $\mathscr{X}$ is contractible with respect to $\mathscr{S}$ (briefly, is c.r. $\mathscr{S}$) if every continuous function f: $\mathscr{X} \to \mathscr{S}$ is homotopic to a constant.

The next statement follows from Theorems 2 of § 54, V, 3 of § 56, IX, and 1 of § 50, II.

THEOREM 1. *The following conditions are equivalent.*

(i) *$\mathscr{X}$ is c.r. $\mathscr{S}$,*

(ii) *the space $\mathscr{S}^{\mathscr{X}}$ is arcwise connected,*

(iii) *$f \sim 1$ for every continuous function f: $\mathscr{X} \to \mathscr{S}$,*

(iv) $b_1(\mathscr{X}) = 0$.

Moreover, *if the space $\mathscr{X}$ is compact, the arcwise connectedness may be replaced by connectedness* (in (ii)).

Theorems 3 of § 54, V, and 4 of § 56, XI, imply the following

THEOREM 2. *If $\mathscr{X}$ is c.r. $\mathscr{S}$, then so is every set which can be obtained from $\mathscr{X}$*

(i) *by a retraction,*

(ii) *by a continuous monotone transformation (if $\mathscr{X}$ is compact),*

(iii) *by an open transformation (if $\mathscr{X}$ is compact).*

Theorems 4 and 5 of § 54, V imply the following

THEOREM 3. *The non-contractibility of a compact $\mathscr{X}$ with respect to $\mathscr{S}$ is invariant under deformations of $\mathscr{X}$, and also under transformations with small point-inverses.*

Theorems of § 56, 2 of Section IX, and 2 of Section X imply the statement below.

THEOREM 4. *If $\mathscr{X}$ and $\mathscr{Y}$ are c.r. $\mathscr{S}$ and if, moreover, $\mathscr{X}$ is a continuum or a connected and locally connected space, then $\mathscr{X} \times \mathscr{Y}$ is c.r. $\mathscr{S}$.*

The next two theorems of addition are implied by Theorems 3 and 8 of § 56, VI.

THEOREM 5. *Let A_0 and A_1 be two closed (or open) sets whose common part, $A_0 \cap A_1$, is connected. If A_0 and A_1 are c.r. $\mathscr{S}$, then so is their union $A_0 \cup A_1$.*

THEOREM 6. *Let $C_0, C_1, \ldots$ be a sequence of connected sets such that*

$$\mathscr{X} = C_0 \cup C_1 \cup \ldots \quad \textit{and} \quad C_n \subset \operatorname{Int}(C_{n+1}).$$

If every set C_n is c.r. $\mathscr{S}$, then so is $\mathscr{X}$.

THEOREM 7. *If each of the compact sets $A_0 \supset A_1 \supset \ldots$ is c.r. $\mathscr{S}$, then so is their intersection $A_0 \cap A_1 \cap \ldots$.*

Proof. This is a consequence of Theorem 9 of § 54, V.

THEOREM 8. *If $\mathscr{X}$ is compact or locally connected and if every component of $\mathscr{X}$ is c.r. $\mathscr{S}$, then so is $\mathscr{X}$.*

Proof. This is a direct corollary of Theorems 4 of Section VIII and 1 of Section X of § 56.

THEOREM 9. *The following spaces are c.r. $\mathscr{S}$:*

(i) *every space contractible in itself, therefore every absolute retract; in particular, $\mathscr{I}^n$ and $\mathscr{E}^n$ for every $n \leqslant \aleph_0$,*

(ii) *the sphere $\mathscr{S}_n$ for $n \neq 1$,*

(iii) *the projective space of dimension $\neq 1$,*

(iv) *every subset A of $\mathscr{I}$.*

Proof. (i) is a direct consequence of Theorem 3 and of Theorem 2 (3) of § 54, VI (compare also § 56, III, Theorem 3).

(ii) follows from Theorem 5 (and from Theorem 2 (ii) as well).

(iii) follows from Theorem 2 (iii), because the projective space is an image of the sphere under an open transformation (see § 43, VI, Theorem 2).

(iv). Let $f\colon A \to \mathscr{S}$ be a continuous function. Since every component C of A is c.r. $\mathscr{S}$ (compare (i)), it follows that $f|C \sim \mathbf{1}$. Therefore C is contained (compare § 56, II, Theorem 9) in an open interval $p_C q_C$ such that $f|F_C \sim 1$ for $F_C = A \cap p_C q_C$. Clearly, it may be assumed that the points p_C and q_C do not belong to A. Thus the set F_C is closed-open in A, and the conditions

$$A = \bigcup_C F_C \quad \text{and} \quad f|F_C \sim \mathbf{1}$$

imply that $f \sim 1$ by Theorem 5 of § 54, I.

Remark. The property of $\mathscr{S}_2$ of being c.r. $\mathscr{S}$ (Theorem 9 (ii)) yields a very simple proof of the Theorem on antipodes (see § 41, VII) for $n = 2$(1).

Proof. We have to show that for every continuous function $g: \mathscr{S}_2 \to \mathscr{E}^2$ there exists a point $x_0 \in \mathscr{S}_2$ such that $g(x_0) = g(-x_0)$.

Suppose not. In other words, we assume that setting

$$f(x) = g(x) - g(-x) \tag{1}$$

the condition $f(x) \neq 0$ holds for each $x \in \mathscr{S}_2$. This means that f is a continuous function $f: \mathscr{S}_2 \to \mathscr{P}$. Since the sphere $\mathscr{S}_2$ is c.r. $\mathscr{S}$, then $f \sim 1$ and so there exists a continuous function $u: \mathscr{S}_2 \to \mathscr{E}^2$ such that

$$f(x) = e^{u(x)} \quad \text{for any} \quad x \in \mathscr{S}_2. \tag{2}$$

Replace x by $-x$; it follows by (2) that

$$f(-x) = e^{u(-x)}$$

and by (1)

$$f(-x) = g(-x) - g(x) = -f(x) = -e^{u(x)} = e^{u(x)+\pi i},$$

which implies that $e^{u(-x)} = e^{u(x)+\pi i}$. Since the sphere $\mathscr{S}_2$ is connected, one infers (compare Theorem 2 of § 56, I) that

$$u(-x) = u(x) + \pi i + 2k\pi i, \tag{3}$$

where k is an integer which does not depend on x.

Replace x by $-x$ in (3); it follows that $u(x) = u(-x) + (2k+1)\pi i$, which combined with (3) yields $4k+2 = 0$ which is a contradiction.

II. Properties of c.r. $\mathscr{S}$ spaces(2). Let $\mathscr{X}$ be a c.r. $\mathscr{S}$ space. Let A_0 and A_1 be two closed or two open sets such that $\mathscr{X} = A_0 \cup A_1$.

THEOREM 1. *$\mathscr{S}^{\mathscr{X}} = \Gamma(A_0) = \Gamma(A_1) = \Gamma(\mathscr{X})$, therefore $\mathfrak{P}_1(A_0, A_1) = (0)$ and $p_1(A_0, A_1) = 0$.*

This is a direct consequence of Theorem 1 (iii) of Section I (see also (1) and (4) of § 56, V).

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(1) A similar proof has been given by H. Steinhaus.

(2) Compare S. Eilenberg, Fund. Math. 26 (1936), I, § 3. Compare also (for locally connected continua) my papers in Fund. Math. 13 (1929), § 1 and in Fund. Math. 8 (1925), § 3.

THEOREM 2. *If the sets A_0 and A_1 are connected, then so is their intersection $A_0 \cap A_1$.*

In other words, (assuming $B_j = \mathscr{X} - A_j$), if B_0 and B_1 are two open or two closed, disjoint sets, none of which separates the (connected) space $\mathscr{X}$, then their union $B_0 \cup B_1$ does not separate $\mathscr{X}$ either.

Consequently, every connected c.r. $\mathscr{S}$ space is unicoherent.

This is a consequence of the inequality $b_0(A_0 \cap A_1) \leqslant b_1(A_0 \cup A_1)$ (compare § 56, VI, Theorem 2) and of Theorem 1 (iv) of Section I.

Theorem 2 admits two generalizations.

THEOREM 3. *Theorem 2 remains true if the connectedness of the sets A_0, A_1 and $A_0 \cap A_1$ is replaced by their connectedness between two points p_0 and p_1 (and also the property of separating the space by the property of separating it between p_0 and p_1).*

Otherwise $\Gamma(A_0) \cap \Gamma(A_1) \neq \Gamma(\mathscr{X})$ (by Theorem 4 of § 56, VI), contradicting Theorem 1.

THEOREM 4. $b_0(A_0) + b_0(A_1) = b_0(A_0 \cup A_1) + b_0(A_0 \cap A_1)$, *if* $A_0 \cap A_1 \neq 0$.

This is a consequence of Theorem 1 and of Theorem (7.1) of § 56, V.

THEOREM 5. *Every closed set F which irreducibly separates $\mathscr{X}$ between two points p_0 and p_1 is connected.*

Proof. Suppose conversely that

$$F = B_0 \cup B_1, \quad B_0 \cap B_1 = 0 \quad \text{and} \quad B_j = \bar{B}_j \neq F.$$

It follows by hypothesis that B_j does not separate the space between p_0 and p_1 (for $j = 0, 1$); then $B_0 \cup B_1$ does not separate it between those points either (by Theorem 3).

THEOREM 6. *If A and $\mathscr{X}$ are connected and C is a component of $\mathscr{X} - A$, then the boundary* $\mathrm{Fr}(C)$ *is connected.*

Proof. $\mathscr{X} - C$ is connected by Theorem 5 of § 46, III. Put $A_0 = \bar{C}$ and $A_1 = \overline{\mathscr{X} - C}$; by Theorem 2 the set $\mathrm{Fr}(C) = \bar{C} \cap \overline{\mathscr{X} - C}$ is connected.

THEOREM 7. *If R is a region in the space $\mathscr{X}$, then no quasi-component of $\mathscr{X} - R$ contains two different quasi-components of* $\mathrm{Fr}(R)$.

If $\mathscr{X}$ is a continuum, then

$$b_0[\mathrm{Fr}(R)] = b_0(\mathscr{X} - R). \tag{1}$$

Proof. Let p_0 and p_1 be two points of $\mathrm{Fr}(R)$ which belong to some quasi-component of $\mathscr{X}-R$. Let $A_0 = \mathscr{X}-R$ and $A_1 = \bar{R}$. Theorem 3 implies that $\mathrm{Fr}(R)$ is connected between p_0 and p_1, and therefore these two points belong to the same quasi-component of $\mathrm{Fr}(R)$.

If $\mathscr{X}$ is a continuum, then $C \cap \mathrm{Fr}(R) \neq 0$ for every component C of $\mathscr{X}-R$ (by Theorem 1 of § 47, III). Therefore C contains one, and, as just has been proved, only one component of $\mathrm{Fr}(R)$ (since the quasi-components of a compact set coincide with its components, compare § 47, II, Theorem 2). This implies identity (1).

III. Local connectedness and unicoherence.

THEOREM 1. *Let $\mathscr{X}$ be a connected, locally connected and c.r. $\mathscr{S}$ space. Every separator E between the points a and b contains a closed connected set which separates these points.*

Proof. The set E contains a closed set F which separates a and b (compare § 14, V, Theorem 1), in turn F contains a closed set C which irreducibly separates $\mathscr{X}$ between a and b (by § 49, V, Theorem 3). Hence C is connected by Theorem 5 of Section II.

THEOREM 2. *Let $\mathscr{X}$ satisfy the same hypotheses as in Theorem* 1. *Let*

$$p_j \in F_j = \bar{F}_j \quad \textit{and} \quad F_0 \cap F_1 = 0.$$

There exists a separator C between p_0 and p_1 which is closed, connected and disjoint from $F_0 \cup F_1$.

Moreover, *if $\mathscr{X}$ is compact, the separator C can be assumed to be a locally connected continuum.*

Proof. This is a direct consequence of Theorem 4 of § 49, V, combined with Theorem 5 of Section II and Theorem 1 of § 50, III.

THEOREM 3. *In order that a locally connected continuum be c.r. $\mathscr{S}$, it is necessary and sufficient that it be unicoherent.*

Proof. The condition is necessary by Theorem 2 of Section II.

On the other hand, assume that a locally connected continuum $\mathscr{X}$ is unicoherent and that $f\colon \mathscr{X} \to \mathscr{S}$ is a continuous function.

By Theorem 7 of § 56, X, there are two continua C_0 and C_1 such that $\mathscr{X} = C_0 \cup C_1$ and $f|C_j \sim 1$ for $j = 0, 1$. By hypothesis $C_0 \cap C_1$ is connected. By Theorem 3 of § 56, VI, it follows that $f \sim 1$.

Remark. Theorem 3 can be generalized by replacing the term *continuum* by *connected space*(1).

THEOREM 4. *Every locally connected continuum $\mathscr{X}$ which is not unicoherent contains a simple closed curve which is a retract of $\mathscr{X}$*(2).

Proof. By hypothesis there exist a continuous function $f\colon \mathscr{X} \to \mathscr{S}$ and (compare § 56, X, Theorem 7) two locally connected continua C_0 and C_1 such that

$$f \, non \sim 1, \quad \mathscr{X} = C_0 \cup C_1 \quad \text{and} \quad f|C_j \sim 1 \quad \text{for} \quad j = 0, 1.$$

According to Theorem 3 of § 56, VI, the set $C_0 \cap C_1$ is not connected. Thus

$$C_0 \cap C_1 = F_0 \cup F_1, \quad F_0 \cap F_1 = 0, \quad \overline{F}_j = F_j \neq 0 \quad \text{for} \quad j = 0, 1. \tag{1}$$

Let A_0 be an arc contained in C_0 and irreducibly connected between F_0 and F_1. Put $A_0 \cap F_j = p_j$. Let A_1 be an arc p_0p_1 contained in C_1. It follows that

$$A_0 \cap A_1 \subset A_0 \cap C_0 \cap C_1 = (A_0 \cap F_0) \cup (A_0 \cap F_1) = (p_0, p_1).$$

Therefore the set $A_0 \cup A_1$ is a simple closed curve.

Let f_0 be a retraction of $F_0 \cup A_0 \cup F_1$ onto A_0 such that

$$f_0(F_0) = p_0 \quad \text{and} \quad f_0(F_1) = p_1. \tag{2}$$

According to the Tietze theorem (§ 14, IV), let

$$f_0 \subset g_0 \in A_0^{C_0}. \tag{3}$$

Let q_0 and q_1 be two distinct points belonging to $A_0 - (p_0, p_1)$ and q_0q_1 and q_1q_0 the two subarcs of $A_0 \cup A_1$, where $q_0q_1 \subset A_0$. Let

$$D_0 = g_0^{-1}(q_0q_1), \tag{4}$$

$$D_1 = g_0^{-1}(p_0q_0 \cup q_1p_1) \cup C_1. \tag{5}$$

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(1) For a proof, see S. Eilenberg, Fund. Math. 26 (1936), p. 70. For a partial generalization, see K. Borsuk, Fund. Math. 17 (1931), p. 190. With regard to the equivalence between the unicoherence and the vanishing of the first Betti number, see K. Borsuk, Fund. Math. 20 (1933), p. 230, and E. Čech, *ibid.*, p. 233.

(2) K. Borsuk, Fund. Math. 17 (1931), p. 184.

It follows that

$$\mathscr{X} = D_0 \cup D_1, \tag{6}$$

$$D_0 \cap D_1 = g_0^{-1}(q_0) \cup g_0^{-1}(q_1), \tag{7}$$

$$g_0^{-1}(q_0) \cap g_0^{-1}(q_1) = 0, \tag{8}$$

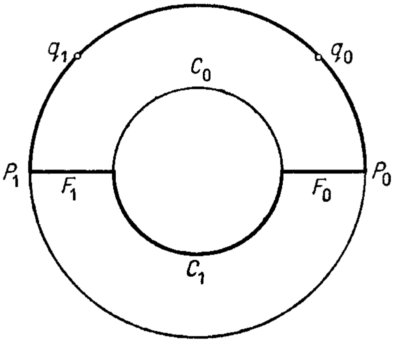

Fig. 14

because (compare (1) through (3)) the formula

$$C_0 \cap C_1 = F_0 \cup F_1 \subset g_0^{-1} g_0(F_0 \cup F_1) = g_0^{-1}(p_0, p_1)$$

implies that

$$g_0^{-1}(q_0 q_1) \cap C_1 \subset g_0^{-1}[q_0 q_1 \cap (p_0, p_1)] = 0.$$

On the other hand

$$q_1 q_0 \cap g_0^{-1}(q_j) = q_j, \tag{9}$$

because g_0 is a retraction and hence (see (1) and (2)) the conditions

$$q_1 q_0 \subset q_1 p_1 \cup C_1 \cup p_0 q_0 \quad \text{and} \quad C_1 \cap g_0^{-1}(q_j) = C_1 \cap C_0 \cap g_0^{-1}(q_j) = 0$$

imply that

$$\begin{aligned} q_1 q_0 \cap g_0^{-1}(q_j) &= (q_1 p_1 \cup p_0 q_0) \cap g_0^{-1}(q_j) \\ &= g_0[(q_1 p_1 \cup p_0 q_0) \cap g_0^{-1}(q_j)] \\ &\subset g_0(q_1 p_1 \cup p_0 q_0) \cap g_0 g_0^{-1}(q_j) \\ &= (q_1 p_1 \cup p_0 q_0) \cap q_j = q_j. \end{aligned}$$

Conditions (8) and (9) imply immediately that there exists a retraction f_1 of $g_0^{-1}(q_0) \cup q_1q_0 \cup g_0^{-1}(q_1)$ onto q_1q_0 such that

$$f_1g_0^{-1}(q_j) = q_j \quad \text{for} \quad j = 0, 1. \tag{10}$$

Let

$$f_1 \subset g_1 \epsilon (q_1q_0)^{D_1}. \tag{11}$$

It follows that

$$g_0(x) = g_1(x) \quad \text{for} \quad x \epsilon g_0^{-1}(q_j), \tag{12}$$

because the condition $x \epsilon g_0^{-1}(q_j)$ implies, on the one hand, that $g_0(x) = q_j$, and on the other hand, that $f_1(x) = q_j$ (by (10)), thus $g_1(x) = q_j$ (compare (11)).

Conditions (6), (7) and (12) imply that the function $g_0|D_0 + g_1$ is a retraction of $\mathcal{X}$ onto $A_0 \cup A_1$.

COROLLARY 5. *If $\mathcal{X}$ is a not unicoherent locally connected continuum, there exists a continuous mapping f: $\mathcal{X} \to \mathcal{X}$ which has no fixed point*(1).

Remark. Theorems 4 and 5 can be generalized by replacing the term *locally connected continuum* by *locally and integrally arcwise connected space*(2).

THEOREM 6. *Let $\mathcal{X}$ be a locally connected, unicoherent continuum which is not separated by any point. If R is a region and* $\dim E = 0$, *the set $R - E$ is connected*(3).

Proof. Suppose that $R - E$ is not connected between p and q. There exists therefore (by Theorem 1) a continuum $C \subset E \cup (\mathcal{X} - R)$ which separates between p and q. Since $\dim E = 0$, it follows that $C \subset \mathcal{X} - R$ (recall that C does not reduce to a single point by hypothesis). But this is not possible, because the set $\mathcal{X} - R$ does not separate the points p and q.

COROLLARY 7. *Under the same hypotheses we have* $\dim \mathcal{X} \geqslant 2$ (*provided $\mathcal{X}$ contains more than one point*).

More precisely, $\mathrm{dc}\, \mathcal{X} \geqslant 2$ (compare § 46, XI).

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(1) See my paper *Sur quelques théorèmes fondamentaux de l'Analysis situs*, Fund. Math. 14 (1929), p. 307.

(2) See K. Borsuk, *loc. cit.*, pp. 184 and 188.

(3) Compare R. L. Wilder, Trans. Amer. Math. Soc. 31 (1929), p. 349. For the case where $\mathcal{X} = \mathcal{S}_2$, compare E. Phragmèn, Acta Math. 7 (1885), p. 43 and L. E. J. Brouwer, Math. Ann. 69 (1910), p. 169.

COROLLARY 8. *Every locally connected unicoherent continuum of dimension* 1 *is a dendrite*(1).

Otherwise, the continuum would contain a cyclic element C (containing more than one point), and since C is unicoherent (by § 52, III, Theorem 5), it would follow that $\dim C \geqslant 2$.

IV. Remarks on extending homeomorphisms in c.r. $\mathcal{S}$ continua. Let us mention without proof the following theorem.

THEOREM 1. *Let $\mathcal{Y}_0$ and $\mathcal{Y}_1$ be two c.r. $\mathcal{S}$ metric continua, and let $\mathcal{X}_0 \subset \mathcal{Y}_0$ and $\mathcal{X}_1 \subset \mathcal{Y}_1$ be two connected sets whose complements $\mathcal{Y}_0 - \mathcal{X}_0$ and $\mathcal{Y}_1 - \mathcal{X}_1$ are hereditarily discontinuous* (*i.e. contain no continua except single points*).

Then, if $\mathcal{X}_0$ and $\mathcal{X}_1$ are homeomorphic, so are $\mathcal{Y}_0$ and $\mathcal{Y}_1$.

Moreover, each homeomorphism h of $\mathcal{X}_0$ onto $\mathcal{X}_1$ can be extended to a homeomorphism $\bar{h}$ of $\mathcal{Y}_0$ onto $\mathcal{Y}_1$(2).

Identifying x with $h(x)$, we can state equivalently the above statement as follows

THEOREM 2. *Let $\mathcal{Y}_0$ and $\mathcal{Y}_1$ be two compactifications of a connected space $\mathcal{X}$ and let $\mathcal{Y}_0 - \mathcal{X}$ and $\mathcal{Y}_1 - \mathcal{X}$ be totally discontinuous.*

If $\mathcal{Y}_0$ and $\mathcal{Y}_1$ are c.r. $\mathcal{S}$ continua, they are homeomorphic.

Moreover, there is a homeomorphism of $\mathcal{Y}_0$ onto $\mathcal{Y}_1$ which is the identity on $\mathcal{X}$(3).

One can easily deduce from Theorem 1 the following invariance Theorem.

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(1) Compare L. Vietoris, Proc. Akad. Amsterdam 1926, p. 446.

(2) For the proof, see E. Sklyarenko, *On extending homeomorphisms*, Dokl. Akad. Nauk SSSR 141 (1961), p. 1045, and the paper of R. Engelking and myself *On extending homeomorphisms in continua contractible relative to the circle*, Rendic. di Mat. 21 (1962), pp. 305–311.

Theorem 1 is a generalization of earlier results by H. Freudenthal, Ann. of Math. 43 (1942), pp. 261–279, and *Über die Enden diskreter Räume und Gruppen*, Comment. Math. Helvet. 17 (1944/45), pp. 1–38, by J. de Groot, *Sätze über topologische Erweiterung von Abbildungen*, Indag. Math. 3 (1944), p. 419 and by R. Duda, *Sur le prolongement des homéomorphies*, Fund. Math. 46 (1959), p. 175, and *Sur les prolongements ponctiformes des homéomorphies*, Indag. Math. 22 (1960), p. 132.

(3) For the case of $\mathcal{X}$ connected and open, see the paper of S. Eilenberg and myself *A remark on duality*, Fund. Math. 50 (1962), pp. 515–517.

THEOREM 3. *Let R be connected open $\subset \mathscr{S}_n$. Then the cardinality of the set of components of $\mathscr{S}_n - R$ is invariant under homeomorphic mappings of R.*

Because there is a continuous mapping h of $\mathscr{S}_n$ onto a c.r. $\mathscr{S}$ continuum $\mathscr{Y}$ such that $h|R$ is a homeomorphism and the point-inverses $h^{-1}(y)$ are either single points of R or components of $\mathscr{S}_n - R$ (compare I, Theorem 2 (ii), and Theorem 4 of § 46, V)(1).

Let us add that the condition of connectedness in Theorem 3 cannot be omitted (compare § 62, X).

§ 58. The group $\mathfrak{M}(\mathscr{X})$

0. Introduction. The family $(0,1)^{\mathscr{X}}$**.** We denote by this symbol the *family of all closed-open subsets* of $\mathscr{X}$. This notation is motivated by the fact that the closed-open subsets of $\mathscr{X}$ can be identified with their characteristic functions (defined on $\mathscr{X}$).

DEFINITIONS. Let $\boldsymbol{C}$ denote an arbitrary cover of $\mathscr{X}$ composed of open, non-empty disjoint sets. Thus

$$\boldsymbol{C} \subset (0,1)^{\mathscr{X}}, \quad \mathrm{S}\boldsymbol{C} = \mathscr{X},$$
$$[(Z_0, Z_1 \in \boldsymbol{C})(Z_0 \neq Z_1) \Rightarrow (Z_0 \cap Z_1 = 0)]. \tag{1}$$

$\mathbf{R}(\mathscr{X})$ denotes the totality of those covers.

The family $\boldsymbol{A}(\boldsymbol{C})$ of all unions of members of $\boldsymbol{C}$ is said to be *generated* by $\boldsymbol{C}$ and $\boldsymbol{C}$ is its *base*. Clearly, they are closed-open.

THEOREM 1. $\mathbf{R}(\mathscr{X})$ *is directed by the refinement relation*: $\boldsymbol{C}_1$ *is a refinement of* $\boldsymbol{C}_0$ (*denoted* $\boldsymbol{C}_0 \leqslant \boldsymbol{C}_1$ *or equivalently* $\boldsymbol{A}(\boldsymbol{C}_0) \subset \boldsymbol{A}(\boldsymbol{C}_1)$).

Namely, if $\boldsymbol{C}_0, \boldsymbol{C}_1 \in \mathbf{R}(\mathscr{X})$ and if $\boldsymbol{C}_2$ denotes the cover composed of (non-empty) intersections $Z_{0,t} \cap Z_{1,s}$ where $\boldsymbol{C}_0 = \{Z_{0,t}\}$ and $\boldsymbol{C}_1 = \{Z_{1,s}\}$, then $\boldsymbol{C}_0 \leqslant \boldsymbol{C}_2$ and $\boldsymbol{C}_1 \leqslant \boldsymbol{C}_2$.

THEOREM 2. $\mathbf{R}(\mathscr{X})$ *has a last member* (*which means that there exists the finest cover of* $\mathscr{X}$) *if and only if the components of* $\mathscr{X}$ *are open. Such is the case of the family* $\mathfrak{C}(\mathscr{X})$ *of all components of* $\mathscr{X}$.

(1) For a direct proof, see M. K. Fort, Jr., *The complements of bounded, open, connected subsets of euclidean space*, Bull. Polish Acad. Sci. 9 (1961), p. 457. Much earlier the case $n = 2$ was considered by L. E. J. Brouwer; compare § 61, V.

Proof. 1. Suppose that $\boldsymbol{C}$ is the last member of $\mathbf{R}(\mathscr{X})$. We have to show that each $Z \in \boldsymbol{C}$ is connected. Now, would Z be not connected, then $Z = P \cup Q$ where P and Q are open, non-empty and disjoint; consequently, $\boldsymbol{C} \leqslant \boldsymbol{C}^* \in \mathbf{R}(\mathscr{X})$ where $\boldsymbol{C}^*$ is derived from $\boldsymbol{C}$ by replacing Z by the pair P, Q.

2. If the components of $\mathscr{X}$ are open, then $\mathfrak{C}(\mathscr{X})$ is the last member of $\mathbf{R}(\mathscr{X})$ since every closed-open set Z is the union of some components of $\mathscr{X}$.

LEMMA. *Let $F \subset \mathscr{I}$ be infinite, non-dense and closed. Then there is a system of closed sets $F_{\alpha_1 \dots \alpha_n}$ defined for systems $\alpha_1 \dots \alpha_n$ (may be, not for all) of numbers* 0 *and* 1, *such that*

$$F = F_0 \cup F_1, \quad F_0 \cap F_1 = 0, \quad F_\alpha \neq 0, \quad \delta(F_\alpha) < 1,$$

and—provided that $F_{\alpha_1 \dots \alpha_n}$ contains more than one point—

$$\left.\begin{array}{c} F_{\alpha_1 \dots \alpha_n} = F_{\alpha_1 \dots \alpha_n 0} \cup F_{\alpha_1 \dots \alpha_n 1}, \quad F_{\alpha_1 \dots \alpha_n 0} \cap F_{\alpha_1 \dots \alpha_n 1} = 0, \\ F_{\alpha_1 \dots \alpha_{n+1}} \neq 0, \quad \delta(F_{\alpha_1 \dots \alpha_{n+1}}) < 1/2^n. \end{array}\right\}$$

One defines the sets $F_{\alpha_1 \dots \alpha_n}$ without difficulty.

Call *sets of order* n the sets $F_{\alpha_1 \dots \alpha_n}$ as well as the one-element sets $F_{\alpha_1 \dots \alpha_m}$ with $m < n$. One sees easily that *every closed-open set in F is, for sufficiently large n, the union of some sets of order n.*

THEOREM 3. *If $\mathscr{X}$ is compact metric, then $\mathbf{R}(\mathscr{X})$ is countable and each $\boldsymbol{C} \in \mathbf{R}(\mathscr{X})$ is finite.*

Moreover, $\mathbf{R}(\mathscr{X})$ *is cofinal with an infinite sequence $\boldsymbol{C}_1 < \boldsymbol{C}_2 < \dots$ (provided $\mathscr{X}$ has an infinite number of components).*

Proof. The first part of the theorem is obvious. The proof of the second part can be reduced to the particular case where $\mathscr{X} \subset \mathscr{I}$ and $\dim \mathscr{X} = 0$, because there is a continuous mapping of $\mathscr{X}$ into the Cantor discontinuum whose point-inverses are the components of $\mathscr{X}$ (see Theorem 3 of § 46, V).

Now, in this particular case, we denote by $\boldsymbol{C}_n$ the family of sets of order n considered in the Lemma.

THEOREM 4. *Let $\mathscr{Y}$ be metric and $\mathscr{X} \subset \mathscr{Y}$. Denote by $\mathscr{X} \circ \boldsymbol{D}$, for $\boldsymbol{D} \in \mathbf{R}(\mathscr{Y})$, the family of all (non-empty) sets $\mathscr{X} \cap V$ where $V \in \boldsymbol{D}$. Then $(\mathscr{X} \circ \boldsymbol{D}) \in \mathbf{R}(\mathscr{X})$.*

Furthermore, *for each $\boldsymbol{C} \in \mathbf{R}(\mathscr{X})$ there is a G open in $\mathscr{Y}$ and a $\boldsymbol{D} \in \mathbf{R}(G)$ such that $\boldsymbol{C} = \mathscr{X} \circ \boldsymbol{D}$.*

Proof. The first part of the Theorem is obvious. In view of proving its second part, put $\boldsymbol{C} = \{Z_t\}$. Since the sets Z_t are disjoint and open in $\mathscr{X}$, there exist sets $\{V_t\}$ disjoint and open in $\mathscr{Y}$ such that $\mathscr{X} \cap V_t = Z_t$ (by Theorem 2 of § 21, XI). Put $G = \bigcup\limits_t V_t$ and $\boldsymbol{D} = \{V_t\}$.

Definition. Let $\mathscr{X} \subset \mathscr{Y}$ and suppose that the components R_t of $\mathscr{Y}$ are open. Then $\boldsymbol{Q}_{\mathscr{Y}}$ denotes the family of all (non-empty) sets $\mathscr{X} \cap R_t$. In other words,

$$\boldsymbol{Q}_{\mathscr{Y}} = \mathscr{X} \circ \mathfrak{C}(Y).$$

Theorem 5. *Let $\mathscr{X} \subset \mathscr{Y}$ and let $\mathscr{Y}$ be locally connected and metric. Then $\mathbf{R}(\mathscr{X})$ is cofinal with the set of covers $\boldsymbol{Q}_G$ where G ranges over the family of open subsets of $\mathscr{Y}$ such that $\mathscr{X} \subset G$.*

Proof. Let $\boldsymbol{C} \epsilon \mathbf{R}(\mathscr{X})$. It follows by Theorem 4 that $\boldsymbol{C} = \mathscr{X} \circ \boldsymbol{D}$ and $\boldsymbol{D} \epsilon \mathbf{R}(G)$ for some G open in $\mathscr{Y}$. Since G is locally connected, it follows by Theorem 2 that

$$\boldsymbol{D} \leqslant \mathfrak{C}(G), \qquad \text{hence} \qquad \mathscr{X} \circ \boldsymbol{D} \leqslant \mathscr{X} \circ \mathfrak{C}(G), \qquad \text{i.e.} \qquad \boldsymbol{C} \leqslant \boldsymbol{Q}_G.$$

Theorem 6. *Let $\mathscr{Y}$ and $\mathscr{Z}$ be two locally connected spaces containing $\mathscr{X}$. Then we have*

$$(\mathscr{Y} \subset \mathscr{Z}) \Rightarrow (\boldsymbol{Q}_{\mathscr{Z}} \leqslant \boldsymbol{Q}_{\mathscr{Y}}). \tag{2}$$

Proof. If $\mathscr{Y} \subset \mathscr{Z}$, then every component R of $\mathscr{Y}$ is contained in a component S of $\mathscr{Z}$ and hence $\mathscr{X} \cap R \subset \mathscr{X} \cap S$. Consequently, $\boldsymbol{Q}_{\mathscr{Z}} \leqslant \boldsymbol{Q}_{\mathscr{Y}}$.

Theorem 7. *Let $\mathscr{Y}$ be compact, metric and locally connected, let $\mathscr{X} \subset \mathscr{Y}$ and let $\mathscr{Y} - \mathscr{X}$ be locally compact; thus*

$$\mathscr{Y} - \mathscr{X} = F_1 \cup F_2 \cup \ldots$$

$$\textit{where} \qquad F_i \textit{ is compact} \subset \mathrm{Int}_{\mathscr{Y}-\mathscr{X}}(F_{i+1}) \tag{3}$$

($\mathrm{Int}_{\mathscr{Y}-\mathscr{X}}$ denotes the interior relative $\mathscr{Y} - \mathscr{X}$).

Put $G_i = \mathscr{Y} - F_i$. Then $\mathbf{R}(\mathscr{X})$ is cofinal with

$$\boldsymbol{Q}_{G_1} \leqslant \boldsymbol{Q}_{G_2} \leqslant \ldots. \tag{4}$$

Proof. Let $\mathscr{X} \subset G \subset \mathscr{Y}$ where G is open (in $\mathscr{Y}$). Hence

$$\mathscr{Y} - G \subset \mathscr{Y} - \mathscr{X} = \bigcup_i \mathrm{Int}_{\mathscr{Y}-\mathscr{X}}(F_{i+1}).$$

Since $\mathscr{Y}-G$ is compact, there is i such that $\mathscr{Y}-G \subset F_i$, i.e. $G_i \subset G$. Thus $\mathscr{X} \subset G_i \subset G$ (since $F_i \subset \mathscr{Y}-\mathscr{X}$) and hence $\boldsymbol{Q}_G \leqslant \boldsymbol{Q}_{G_i}$ (by (2)). Therefore, by Theorem 5, $\mathbf{R}(\mathscr{X})$ is cofinal with $\{\boldsymbol{Q}_{G_i}\}$. Formula (4) is true, because the inclusions $F_i \subset F_{i+1}$ and $G_{i+1} \subset G_i$ are equivalent.

I. $\mathfrak{M}(\mathscr{X})$ as a topological space.

DEFINITIONS. Let $\mathscr{X}$ be a metric space(¹). $\mathfrak{M}(\mathscr{X})$ denotes the family of completely additive functions μ (called *measures*) which assign to each $Z \in (0,1)^{\mathscr{X}}$ an integer number $\mu(Z)$; this means that, whenever the sets $Z_t \in (0,1)^{\mathscr{X}}$ are disjoint and $(\bigcup_t Z_t) \in (0,1)^{\mathscr{X}}$, then

$$\mu(\bigcup_t Z_t) = \sum_t \mu(Z_t). \tag{0}$$

Clearly, in that case *only a finite number of terms Z_t has a non-zero measure.*

If $\boldsymbol{C} \in \mathbf{R}(\mathscr{X})$ denotes (as in Section 0) an arbitrary cover of $\mathscr{X}$ consisting of non-empty disjoint, open sets, let $\mathfrak{F}(\boldsymbol{C})$ be the set of completely additive integer-valued functions defined on the elements of $\boldsymbol{A}(\boldsymbol{C})$.

Denote for $\boldsymbol{C}_0 \leqslant \boldsymbol{C}_1$ (which means, as in Section 0, that $\boldsymbol{C}_1$ is a refinement of $\boldsymbol{C}_0$, or—equivalently—that $\boldsymbol{A}(\boldsymbol{C}_0) \subset \boldsymbol{A}(\boldsymbol{C}_1)$), by $\sigma_{\boldsymbol{C}_0\boldsymbol{C}_1}$ the restriction operation

$$\sigma_{\boldsymbol{C}_0\boldsymbol{C}_1}\colon \mathfrak{F}(\boldsymbol{C}_1) \to \mathfrak{F}(\boldsymbol{C}_0)$$

defined by the condition

$$\sigma_{\boldsymbol{C}_0\boldsymbol{C}_1}(\nu) = \nu \,|\, \boldsymbol{A}(\boldsymbol{C}_0) \quad \text{for} \quad \nu \in \mathfrak{F}(\boldsymbol{C}_1).$$

It follows easily that $\{\mathfrak{F}(\boldsymbol{C}), \sigma_{\boldsymbol{C}_0\boldsymbol{C}_1}\}$, where $\boldsymbol{C} \in \mathbf{R}(\mathscr{X})$ and $\boldsymbol{C}_0 \leqslant \boldsymbol{C}_1$, is an *inverse system.*

Denote finally by $\sigma_{\boldsymbol{C}}$ the restriction operation

$$\sigma_{\boldsymbol{C}}\colon \mathfrak{M}(\mathscr{X}) \to \mathfrak{F}(\boldsymbol{C}), \quad \text{where} \quad \sigma_{\boldsymbol{C}}(\mu) = \mu \,|\, \boldsymbol{A}(\boldsymbol{C}),$$

and put $\sigma = \{\sigma_{\boldsymbol{C}}\}$. Thus

$$\sigma\colon \mathfrak{M}(\mathscr{X}) \to \mathop{\boldsymbol{P}}_{\boldsymbol{C}} \mathfrak{F}(\boldsymbol{C}) \quad \text{where} \quad \boldsymbol{C} \in \mathbf{R}(\mathscr{X}).$$

(¹) Under more general assumptions, see the paper of R. Engelking and myself *Quelques théorèmes de l'Algèbre de Boole et leurs applications topologiques*, Fund. Math. 50 (1962), pp. 519–535.

THEOREM 1. *σ is a one-to-one mapping of $\mathfrak{M}(\mathscr{X})$ onto $\mathfrak{L} = \operatorname*{Lim}_{C, C_0 \leqslant C_1} \{\mathfrak{F}(C), \sigma_{C_0 C_1}\}$.*

Consequently, *$\mathfrak{M}(\mathscr{X})$ becomes a topological space (in fact, a completely regular and 0-dimensional space) if we define the closure $\overline{\mathfrak{Y}}$ of $\mathfrak{Y} \subset \mathfrak{M}(\mathscr{X})$ by the equivalence*

$$(\mu \in \overline{\mathfrak{Y}}) \equiv [(\mu | C) \in (\mathfrak{Y} | C) \text{ for each } C \in \mathbf{R}(\mathscr{X})], \tag{1}$$

which means that for each C there is $\nu \in \mathfrak{Y}$ such that $\nu(Z) = \mu(Z)$ for $Z \in C$.

Furthermore, *σ: $\mathfrak{M}(\mathscr{X}) \to \mathfrak{L}$ is a homeomorphism onto.*

Proof. First, the mapping σ is *into*. This means that $\sigma_{C_0 C_1}(\mu | A(C_1)) = \mu | A(C_0)$, in other terms

$$[\mu | A(C_1)] | A(C_0) = \mu | A(C_0),$$

but this follows at once from the transitivity of the restriction operation.

Next, σ is *one-to-one*. For suppose that $\mu_0(Z) = \mu_1(Z)$ for a given $Z \in (0, 1)^{\mathscr{X}}$, and denote by C the two-elements cover $Z^* = (Z, \mathscr{X} - Z)$. Then

$$\mu_0 | A(C) \neq \mu_1 | A(C), \text{ i.e. } \sigma_C(\mu_0) \neq \sigma_C(\mu_1), \text{ hence } \sigma(\mu_0) \neq \sigma(\mu_1).$$

We are now going to show that σ is *onto*. In other terms: we assume that to each $C \in \mathbf{R}(\mathscr{X})$ there corresponds $\nu_C \in \mathfrak{F}(C)$ such that

$$\nu_{C_0} = \nu_{C_1} | A(C_0) \quad \text{provided} \quad C_0 \leqslant C_1, \tag{2}$$

and we shall define $\mu \in \mathfrak{M}(\mathscr{X})$ so that

$$\mu | A(C) = \nu_C \quad \text{for each } C \in \mathbf{R}(\mathscr{X}). \tag{3}$$

We shall show that such is the measure μ defined by the condition[1]

$$\mu(Z) = \nu_{Z^*}(Z) \quad \text{where} \quad Z^* = (Z, \mathscr{X} - Z). \tag{4}$$

First, it is to be proved that $\mu \in \mathfrak{M}(\mathscr{X})$, i.e. that $\mu(Z) = \sum_t \mu(Z_t)$ provided that $Z = \bigcup_t Z_t$ and that the sets Z_t are disjoint.

---

[1] We denote usually by Z, C and μ variables ranging over $(0, 1)^{\mathscr{X}}$, $\mathbf{R}(\mathscr{X})$ and $\mathfrak{M}(\mathscr{X})$, respectively.

Let $\boldsymbol{C}$ be the cover of $\mathscr{X}$ which consists of all Z_t and of the set $\mathscr{X}-Z$ (which is omitted if $Z = \mathscr{X}$).

It is clear that $\boldsymbol{C} \in \mathbf{R}(\mathscr{X})$, $\boldsymbol{Z}^* \leqslant \boldsymbol{C}$ and $\boldsymbol{Z}_t^* \leqslant \boldsymbol{C}$. By (2) it follows that

$$\nu_{\boldsymbol{Z}^*} = \nu_{\boldsymbol{C}} | \boldsymbol{A}(\boldsymbol{Z}^*) \quad \text{and} \quad \nu_{\boldsymbol{Z}_t^*} = \nu_{\boldsymbol{C}} | \boldsymbol{A}(\boldsymbol{Z}_t^*), \tag{5}$$

hence

$$\nu_{\boldsymbol{Z}^*}(Z) = \nu_{\boldsymbol{C}}(Z) \quad \text{and} \quad \nu_{\boldsymbol{Z}^*}(Z_t) = \nu_{\boldsymbol{C}}(Z_t). \tag{6}$$

Since $\nu_{\boldsymbol{C}} \in \mathfrak{F}(\boldsymbol{C})$, it follows from (4) and (6) that

$$\mu(Z) = \nu_{\boldsymbol{Z}^*}(Z) = \nu_{\boldsymbol{C}}(Z) = \sum_t \nu_{\boldsymbol{C}}(Z_t) = \sum_t \nu_{\boldsymbol{Z}_t^*}(Z_t) = \sum_t \mu(Z_t).$$

It remains to prove (3). Let $Z \in \boldsymbol{A}(\boldsymbol{C})$. It is to be proved that $\mu(Z) = \nu_{\boldsymbol{C}}(Z)$. But, since $\boldsymbol{Z}^* \leqslant \boldsymbol{C}$, this identity follows directly from (4) and (6).

This completes the proof of the first part of the Theorem.

The second part follows from Theorem 3 of § 16, VI, assuming that $\mathfrak{F}(\boldsymbol{C})$ has the discrete topology (which implies that $\underset{\boldsymbol{C}}{\boldsymbol{P}} \mathfrak{F}(\boldsymbol{C})$ is completely regular and $\mathfrak{L}$ as well).

THEOREM 2. *If the components of $\mathscr{X}$ are open (in particular if $\mathscr{X}$ is locally connected), the space $\mathfrak{M}(\mathscr{X})$ is discrete.*

Proof. Let $\boldsymbol{C}$ be the family of components of $\mathscr{X}$. The family $(0, 1)^{\mathscr{X}}$ is generated by $\boldsymbol{C}$, and therefore the function μ is uniquely determined by its values at the elements Z of $\boldsymbol{C}$. Thus the condition $(\mu | \boldsymbol{C}) \in (\mathfrak{Y} | \boldsymbol{C})$ implies that $\mu \in \mathfrak{Y}$, which yields $\mathfrak{Y} = \overline{\mathfrak{Y}}$ according to (1).

THEOREM 3. *The mapping $\sigma_{\boldsymbol{C}}: \mathfrak{M}(\mathscr{X}) \to \mathfrak{F}(\boldsymbol{C})$ is continuous, as well as $\varphi_Z: \mathfrak{M}(\mathscr{X}) \to \mathscr{G}$ for each Z, where $\varphi_Z(\mu) = \mu(Z)$.*

The first part of Theorem 3 follows directly from Theorem 1, and to derive the second we put $\boldsymbol{C} = (Z, \mathscr{X}-Z)$.

THEOREM 4. *If $\mathscr{X}$ is compact, $\mathfrak{M}(\mathscr{X})$ is homeomorphic to a complete separable 0-dimensional space.*

Because $\mathbf{R}(\mathscr{X})$ is countable in this case (compare § 54, IX, 6a).

Remark. Under the assumption of compactness of $\mathscr{X}$, *complete* additivity of functions $\mu \in \mathfrak{M}(\mathscr{X})$ can obviously be replaced by *finite* additivity (because each cover $\boldsymbol{C}$ of $\mathscr{X}$ is finite).

Under the same assumption, one shows easily that

$$(\mu = \lim_{k=\infty} \mu_k) \equiv [(\mu \mid C) = \lim_{k=\infty} (\mu_k \mid C) \text{ for each } C]$$
$$\equiv [\mu(Z) = \lim_{k=\infty} \mu_k(Z) \text{ for each } Z]. \tag{7}$$

Theorems 5 and 6 are direct consequences of Theorems § 1, VII, 5 and 7.

THEOREM 5. *Let $\mathscr{X} \subset \mathscr{Y}$ where $\mathscr{Y}$ is a metric locally connected space. Then for every $\mu \in \mathfrak{M}(\mathscr{X})$ and every $\mathfrak{Y} \subset \mathfrak{M}(\mathscr{X})$ we have*

$$(\mu \in \overline{\mathfrak{Y}}) \equiv [(\mu \mid \mathcal{Q}_G) \in (\mathfrak{Y} \mid \mathcal{Q}_G) \text{ for each open } G \text{ such that } \mathscr{X} \subset G \subset \mathscr{Y}], \tag{8}$$

$$(\mu_0 = \mu_1) \equiv [(\mu_0 \mid \mathcal{Q}_G) = (\mu_1 \mid \mathcal{Q}_G) \text{ for each open } G \text{ such that } \mathscr{X} \subset G \subset \mathscr{Y}]. \tag{9}$$

THEOREM 6. *Under the hypotheses of Theorem 7 of Section 0, we have*

$$(\mu \in \overline{\mathfrak{Y}}) \equiv [(\mu \mid \mathcal{Q}_{G_i}) \in (\mathfrak{Y} \mid \mathcal{Q}_{G_i}) \text{ for } i = 1, 2, \ldots]. \tag{10}$$

Moreover, *$\mathfrak{M}(\mathscr{X})$ is metrizable and*

$$(\mu_0 = \mu_1) \equiv [(\mu_0 \mid \mathcal{Q}_{G_i}) = (\mu_1 \mid \mathcal{Q}_{G_i}) \text{ for } i = 1, 2, \ldots]. \tag{11}$$

II. $\mathfrak{M}(\mathscr{X})$ as a topological group. Put, for μ_0, μ_1 and μ_2 in $\mathfrak{M}(\mathscr{X})$,

$$(\mu_0 = \mu_1 + \mu_2) \equiv [\mu_0(Z) = \mu_1(Z) + \mu_2(Z) \text{ for each } Z \in (0, 1)^{\mathscr{X}}]. \tag{1}$$

Then $\mathfrak{M}(\mathscr{X})$ becomes an *abelian group*.

More generally, $\mathfrak{F}(C)$ becomes an abelian group if we assume that μ_0, μ_1 and μ_2 belong to $\mathfrak{F}(C)$ and we replace $(0, 1)^{\mathscr{X}}$ in (1) by $A(C)$.

Clearly,

$$(\mu_0 = \mu_1 + \mu_2) \equiv \big[\big(\mu_0 \mid A(C)\big) = \big(\mu_1 \mid A(C)\big) + \big(\mu_2 \mid A(C)\big) \text{ for each } C\big]$$

and one shows easily the next statement (compare also Section I).

THEOREM 1. *The following mappings are homomorphisms:*

$$\varphi_Z\colon \mathfrak{M}(\mathscr{X}) \to \mathscr{G}, \qquad \sigma_C\colon \mathfrak{M}(\mathscr{X}) \to \mathfrak{F}(C), \qquad \sigma_{C_0 C_1}\colon \mathfrak{F}(C_1) \to \mathfrak{F}(C_0).$$

The last statement implies (compare § 55, VII, and Theorem 1 of Section I, by which σ is one-to-one) the following

THEOREM 1′.

$$\mathfrak{M}(\mathscr{X}) \underset{\mathrm{gr}}{=} \underset{C_1C_0 \leqslant C_1}{\mathrm{Lim}} \{\mathfrak{F}(C), \sigma_{C_0C_1}\}.$$

Namely, *the mapping* $\sigma = \{\sigma_C\}$, *where* $\sigma_C(\mu) = \mu \mid A(C)$, *is the required isomorphism.*

It follows also that:

THEOREM 2. $\mathfrak{M}(\mathscr{X})$ *is a topological group.*

In other words, the addition defined by formula (1) and subtraction are continuous (with respect to the topology defined by I(1)).

DEFINITION. Let Q be a quasi-component of $\mathscr{X}$ and let

$$a_Q(Z) = \begin{cases} 1 & \text{if} \quad Q \subset Z, \\ 0 & \text{if} \quad Q \cap Z = 0. \end{cases} \tag{2}$$

Then a_Q is called the *characteristic measure of* Q.

LEMMA. $a_Q \in \mathfrak{M}(\mathscr{X})$.

Proof. Every closed-open subset Z of $\mathscr{X}$ such that $Q \cap Z \neq 0$ satisfies the condition $Q \subset Z$; therefore formula (2) determines the function a_Q for every Z. Moreover, the function a_Q is completely additive, because if $Q \subset Z = \bigcup_t Z_t$, where the sets Z_t are disjoint, there is only one Z_{t_0} which contains Q, and all other Z_t are disjoint from Q; hence the identity I(0) is fulfilled.

THEOREM 3. *If the quasi-components* $Q_1, \ldots, Q_m$ *are pairwise distinct, their characteristic measures are linearly independent, i.e.*

$$(k_1 a_{Q_1} + \ldots + k_m a_{Q_m} = 0) \Rightarrow (k_1 = 0, \ldots, k_m = 0). \tag{3}$$

Proof. By hypothesis there exist disjoint closed-open sets $Z_1, \ldots, Z_m$ such that $Q_i \subset Z_i$ for $i = 1, \ldots, m$. Suppose that the right-hand term of implication (3) does not hold, for instance let $k_1 \neq 0$. Since $a_{Q_1}(Z_1) = 1$ and $a_{Q_i}(Z_1) = 0$ for $i > 1$, it follows that

$$k_1 a_{Q_1}(Z_1) + \ldots + k_m a_{Q_m}(Z_1) = k_1.$$

Therefore the left-hand term of (3) also fails.

THEOREM 4. *If the components of $\mathcal{X}$ are open (in particular, if $\mathcal{X}$ is locally connected), their characteristic measures are the generators of the group $\mathfrak{M}(\mathcal{X})$.*

More generally, *if $\boldsymbol{C} = \{Z_t\} \in \mathbf{R}(\mathcal{X})$, where $\mathcal{X}$ is an arbitrary space, and if $Q_t \subset Z_t$ where Q_t is a quasi-component of $\mathcal{X}$, then the functions $\gamma_{Q_t} = (\alpha_{Q_t} | \boldsymbol{A}(\boldsymbol{C}))$ are generators of the group $\mathfrak{F}(\boldsymbol{C})$.*

Proof. Let $\mu \in \mathfrak{F}(\boldsymbol{C})$. Let us set $k_t = \mu(Z_t)$. We have to show that

$$\mu = \sum_t k_t \gamma_{Q_t}, \quad \text{i.e. that} \quad \mu(Z_t) = \sum_s k_s \alpha_{Q_s}(Z_t) \quad \text{for all } t \tag{4}$$

(the sum is finite since there is only a finite number of non-zero k_t's).

It follows according to (2) that $\alpha_{Q_t}(Z_t) = 1$ and $\alpha_{Q_s}(Z_t) = 0$ for $s \neq t$, hence

$$\sum_s k_s \alpha_{Q_s}(Z_t) = k_t \alpha_{Q_t}(Z_t) = k_t = \mu(Z_t).$$

The first part of Theorem 4 follows for $\boldsymbol{C} = \mathcal{X}$, because

$$\boldsymbol{A}(\boldsymbol{C}) = (0,1)^{\mathcal{X}} \quad \text{and} \quad \mathfrak{F}(\boldsymbol{C}) = \mathfrak{M}(\mathcal{X}).$$

Theorems 3 and 4 immediately imply the following one.

THEOREM 5. *If the space $\mathcal{X}$ has a finite number, say m, of components, the group $\mathfrak{M}(\mathcal{X})$ is isomorphic with the group $\mathcal{G}^m$, i.e.*

$$\mathfrak{M}(\mathcal{X}) \underset{\mathrm{gr}}{=} \mathcal{G}^m. \tag{5}$$

If the components of $\mathcal{X}$ are open and their family is countably infinite, then

$$\mathfrak{M}(\mathcal{X}) \underset{\mathrm{gr}}{=} \mathcal{G}^\omega \tag{5'}$$

(the group of infinite sequences of integers each of which contains only a finite number of non-zero terms).

Now we are going to deduce the following important theorem from Theorem 4.

THEOREM 6. *Let $\{Q_s\}$ be a family of quasi-components of an (arbitrary) space $\mathcal{X}$ such that*

$$Q \cap \overline{\bigcup_s Q_s} \neq 0 \tag{6}$$

for all quasi-components Q of $\mathcal{X}$.

Let $\mathfrak{A}$ be the subgroup of $\mathfrak{M}(\mathcal{X})$ generated by the measures a_{Q_s} (i.e. $\mathfrak{A}$ is the set of their linear combinations), then

$$\overline{\mathfrak{A}} = \mathfrak{M}(\mathcal{X}). \tag{7}$$

Proof. Let $C = \{Z_t\} \in \mathbf{R}(\mathcal{X})$. Let Q_t be a quasi-component of $\mathcal{X}$ and $Q_t \subset Z_t$. Then $Z_t \cap \bigcup_s Q_s \neq 0$ according to (6). So, there exists (since the set Z_t is open) a number $s(t)$ such that $Z_t \cap Q_{s(t)} \neq 0$, therefore such that $Q_{s(t)} \subset Z_t$. It follows by Theorem 4 that the functions $a_{Q_{s(t)}} | \boldsymbol{A}(\boldsymbol{C})$ are generators of the group $\mathfrak{F}(\boldsymbol{C})$, and hence $\mathfrak{F}(\boldsymbol{C}) \subset (\mathfrak{A} \mid \boldsymbol{A}(\boldsymbol{C}))$.

Therefore $(\mu \mid \boldsymbol{C}) \in (\mathfrak{A} \mid \boldsymbol{C})$ for all $\mu \in \mathfrak{M}(\mathcal{X})$, and it follows that $\mu \in \overline{\mathfrak{A}}$ by I(1).

COROLLARY 7. *If the space $\mathcal{X}$ is separable, then so is the space $\mathfrak{M}(\mathcal{X})$.*

Because under that hypothesis, the subscript s in Theorem 6 can be supposed to be a positive integer; therefore the set $\mathfrak{A}$ is countable.

THEOREM 8. *If $\mathcal{X}$ is a compact space with infinitely many components, then*

$$\mathfrak{M}(\mathcal{X}) \underset{\text{gr.t.}}{=} \mathcal{G}^{\aleph_0}, \tag{8}$$

i.e. there exists an isomorphism $\varkappa$ of the group $\mathfrak{M}(\mathcal{X})$ onto the (additive) group $\mathcal{G}^{\aleph_0}$, which is also a homeomorphism.

Proof. In proving that $\mathfrak{M}(\mathcal{X})$ is isomorphic to $\mathcal{G}^{\aleph_0}$ we shall use the lemma of Section 0(1). With no loss of generality, we may assume that $\mathcal{X} (= \mathfrak{F}) \subset \mathcal{I}$ is infinite non-dense and closed (compare Section 0). Reconsider the sets $F_{\alpha_1 \ldots \alpha_n}$ of the Lemma and range all the sets $F_{\alpha_1 \ldots \alpha_n 0}$ (where $n \geqslant 0$) into an infinite sequence $A_1, A_2, \ldots$ Let $A_0 = \mathcal{X}$.

For $\mu \in \mathfrak{M}(\mathcal{X})$ put

$$\varkappa(\mu) = \{\mu(A_0), \mu(A_1), \ldots\}, \qquad \text{thus} \qquad \varkappa\colon \mathfrak{M}(\mathcal{X}) \to \mathcal{G}^{\aleph_0}. \tag{9}$$

We are going to show that $\varkappa$ is the required isomorphism.

First, $\varkappa$ is a homomorphism, because

$$\begin{aligned} \varkappa(\mu_1 + \mu_2) &= \{\mu_1(A_0) + \mu_2(A_0), \mu_1(A_1) + \mu_2(A_2), \ldots\} \\ &= \varkappa(\mu_1) + \varkappa(\mu_2). \end{aligned}$$

---

(1) This proof is due to A. Mostowski.

Obviously, the condition $\varkappa(\mu) = \{0, 0, \ldots\}$ implies $\mu(A_m) = 0$ for $m = 1, 2, \ldots$. It will follow that $\mu(Z) = 0$ for every closed-open subset Z of $\mathscr{X}$.

Clearly, it is sufficient to show that $\mu(F_{\alpha_1 \ldots \alpha_n}) = 0$. We shall proceed by induction. Since $\mathscr{X} = A_0$, so $\mu(\mathscr{X}) = 0$, and since $\mu(F_0) = 0$, we have $\mu(F_1) = 0$ as well. Next, assuming that $\mu(F_{\alpha_1 \ldots \alpha_n}) = 0$ and that $F_{\alpha_1 \ldots \alpha_n}$ contains more than one point, we have $\mu(F_{\alpha_1 \ldots \alpha_n 0}) + \mu(F_{\alpha_1 \ldots \alpha_n 1}) = 0$, and since $F_{\alpha_1 \ldots \alpha_n 0}$ belongs to the sequence $\{A_m\}$, it follows that

$$\mu(F_{\alpha_1 \ldots \alpha_n 0}) = 0 = \mu(F_{\alpha_1 \ldots \alpha_n 1}).$$

It remains to prove that for every sequence of integers $k_0, k_1, \ldots$ there exists $\mu \in \mathfrak{M}(\mathscr{X})$ such that

$$\mu(A_m) = k_m \quad \text{for} \quad m = 0, 1, \ldots. \tag{9'}$$

Since the function μ is defined for $Z = A_m$ (according to (9′)), we can define $\mu(F_{\alpha_1 \ldots \alpha_n 1})$ by induction (with respect to n) as follows: $\mu(F_1) = \mu(A_0) - \mu(F_0)$ and

$$\mu(F_{\alpha_1 \ldots \alpha_n 1}) = \mu(F_{\alpha_1 \ldots \alpha_n}) - \mu(F_{\alpha_1 \ldots \alpha_n 0}). \tag{9''}$$

Finally, if Z is the union of some sets of order n (least possible n), $Z = P_1 \cup \ldots \cup P_r$, we put

$$\mu(Z) = \mu(P_1) + \ldots + \mu(P_r), \qquad \text{let} \qquad \mu(0) = -\mu(A_0).$$

It follows from (9″) that, if $P_1 = F_{\alpha_1 \ldots \alpha_n 0} \cup F_{\alpha_1 \ldots \alpha_n 1}$, then

$$\mu(F_{\alpha_1 \ldots \alpha_n 0}) + \mu(F_{\alpha_1 \ldots \alpha_n 1}) + \mu(P_2) + \ldots + \mu(P_r) = \mu(F).$$

Consequently, if Z is represented as the union of sets of order $n+1$ (or more generally: of order $p > n$), $Z = R_1 \cup \ldots \cup R_s$, then

$$\mu(R_1) + \ldots + \mu(R_s) = \mu(F).$$

It follows easily that the function μ is additive, i.e. $\mu \in \mathfrak{M}(\mathscr{X})$.

Thus, it has been proved that $\varkappa$ is an (algebraic) *isomorphism* of $\mathfrak{M}(\mathscr{X})$ onto $\mathscr{G}^{\aleph_0}$.

We are going now to prove that $\varkappa$ is a *homeomorphism*, i.e. that

$$(\mu = \lim_{k=\infty} \mu_k) \equiv [\varkappa(\mu) = \lim_{k=\infty} \varkappa(\mu_k)],$$

or (compare (9)) that

$$(\mu = \lim_{k=\infty} \mu_k) \equiv [\mu(A_m) = \lim_{k=\infty} \mu_k(A_m) \text{ for } m = 0, 1, \ldots]. \tag{10}$$

The implication from left to right is a consequence of the continuity of φ_Z (where $\varphi_Z(\mu) = \mu(Z)$) for Z fixed (compare Theorem 3 of Section I).

On the other hand, assume that the right term of equivalence (10) is true. Thus all is reduced to showing (compare I (7)) that, for every $Z \in (0, 1)^{\mathscr{X}}$,

$$\mu(Z) = \lim_{k=\infty} \mu_k(Z). \tag{11}$$

Moreover, since μ is additive, we may assume that the sets Z are of the form $Z = F_{a_1 \ldots a_m}$.

Since identity (11) is satisfied for $m = 0$, i.e. for $Z = \mathscr{X} = A_0$, we may proceed by induction assuming that it is satisfied for some $m \geqslant 0$.

It follows that $F_{a_1 \ldots a_m 1} = F_{a_1 \ldots a_m} - F_{a_1 \ldots a_m 0}$, and hence

$$\begin{cases} \mu(F_{a_1 \ldots a_m 1}) = \mu(F_{a_1 \ldots a_m}) - \mu(F_{a_1 \ldots a_m 0}), \\ \mu_k(F_{a_1 \ldots a_m 1}) = \mu_k(F_{a_1 \ldots a_m}) - \mu_k(F_{a_1 \ldots a_m 0}). \end{cases}$$

In the limit we get identity (11) for $Z = F_{a_1 \ldots a_m 1}$, because $F_{a_1 \ldots a_m 0}$ is a term of the sequence $A_0, A_1, \ldots$ (and therefore satisfies identity (11) by hypothesis).

III. Normed measures.

DEFINITION. A measure $\mu \in \mathfrak{M}(\mathscr{X})$ is said to be *normed* if $\mu(\mathscr{X}) = 0$. $\mathfrak{N}(\mathscr{X})$ denotes the *set of normed measures*.

Clearly, $\mathfrak{N}(\mathscr{X})$ can be identified with the factor group $\mathfrak{M}(\mathscr{X})/\mathscr{G}$ (where $\mathscr{G}$ stands for the group of constant measures).(1)

One easily sees that *$\mathfrak{N}(\mathscr{X})$ is a subgroup of $\mathfrak{M}(\mathscr{X})$* and a *closed subset of $\mathfrak{M}(\mathscr{X})$* (because the mapping $\varphi_{\mathscr{X}}$: $\mathfrak{M}(\mathscr{X}) \to \mathscr{G}$ is continuous, compare Theorem 3 of Section I).

THEOREM 1. *Let ∞ be a fixed point of $\mathscr{X}$*(2). *Put for $\mu \in \mathfrak{M}(\mathscr{X})$:*

$$\mu^*(Z) = \begin{cases} \mu(Z) & \text{if} \quad \infty \in \mathscr{X} - Z, \\ \mu(Z) - \mu(\mathscr{X}) & \text{if} \quad \infty \in Z. \end{cases} \tag{1}$$

Then μ^*: $\mathfrak{M}(\mathscr{X}) \to \mathfrak{N}(\mathscr{X})$ is a homomorphism and a continuous retraction.

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(1) As shown by W. Nikolajshvili, the group $\mathfrak{N}(\mathscr{X})$ is isomorphic to the Alexandrov–Čech homology group $H_0(\mathscr{X})$. See *On a duality theorem of Kuratowski*, Bull. Acad. Sc. Georgian SSR 35 (1964), p. 514.

(2) This notation is motivated by the applications considered in § 60.

Proof. Clearly $\mu^*(\mathcal{X}) = 0$. Next, assume that $Z = \bigcup_t Z_t$ where the sets Z_t are disjoint. We have to show that $\mu^*(Z) = \sum_t \mu^*(Z_t)$. This is obvious if $\infty \in \mathcal{X} - Z$. So suppose that $\infty \in Z_{t_0}$. Hence

$$\mu^*(Z) - \mu^*(Z_{t_0}) = \mu(Z) - \mu(Z_{t_0}) = \mu\Big(\sum_{t \neq t_0} Z_t\Big) = \sum_{t \neq t_0} \mu(Z_t) = \sum_{t \neq t_0} \mu^*(Z_t),$$

which proves the required identity.

Thus, it is established that $\mu^* \in \mathfrak{N}(\mathcal{X})$. The obvious implication

$$[\mu_1(Z) + \mu_2(Z) = \mu_3(Z)] \Rightarrow [\mu_1^*(Z) + \mu_2^*(Z) = \mu_3^*(Z)]$$

shows that μ^* is a homomorphism.

Finally, we shall prove that μ^* is continuous. Let $\mu \in \overline{\mathfrak{Z}}$, where $\mathfrak{Z} \subset \mathfrak{M}(\mathcal{X})$; we have to prove that $\mu^* \in \overline{\mathfrak{Z}^*}$, i.e. that $\mu^* | \boldsymbol{C} \in \mathfrak{Z}^* | \boldsymbol{C}$. By hypothesis, $\mu | \boldsymbol{C} \in \mathfrak{Z} | \boldsymbol{C}$; i.e. there exists $\mu_{\boldsymbol{C}} \in \mathfrak{Z}$ such that

$$\mu_{\boldsymbol{C}} | \boldsymbol{C} = \mu | \boldsymbol{C}, \quad \text{hence} \quad \mu_{\boldsymbol{C}}^* | \boldsymbol{C} = \mu^* | \boldsymbol{C}, \quad \text{thus} \quad \mu^* | \boldsymbol{C} \in \mathfrak{Z}^* | \boldsymbol{C}$$

(because $\mu_{\boldsymbol{C}}^* \in \mathfrak{Z}^*$).

Write concisely

$$\beta_Q = \alpha_Q^*. \tag{2}$$

It can easily be seen that

$$\textit{if} \quad \infty \in Q, \quad \textit{then} \quad \beta_Q = 0, \tag{3}$$

and if $\infty \in \mathcal{X} - Q$, then

$$\beta_Q(Z) = \\ = \begin{cases} 1 & \text{if } Q \subset Z \text{ and } \infty \in \mathcal{X} - Z, \\ 0 & \text{if } Q \subset Z \text{ and } \infty \in Z \text{ or } Q \cap Z = 0 \text{ and } \infty \in \mathcal{X} - Z, \\ -1 & \text{if } Q \cap Z = 0 \text{ and } \infty \in Z. \end{cases} \tag{4}$$

Theorems 3 through 8 of Section II lead to the following

THEOREM 2. *If the quasi-components $Q_0, \ldots, Q_m$ of $\mathcal{X}$ are distinct and $\infty \in Q_0$, the measures $\beta_{Q_1}, \ldots, \beta_{Q_m}$ are linearly independent.*

The argument is similar to that of Theorem 3 of Section II.

THEOREM 3. *Let $\{Q_t\}$ be the family of components of $\mathcal{X}$; let $\infty \in Q_0$. If these components are open, the measures β_{Q_t} with $t \neq 0$ are the generators of the group $\mathfrak{N}(\mathcal{X})$.*

Proof. Since μ^* is a homomorphism of $\mathfrak{M}(\mathscr{X})$ onto $\mathfrak{N}(\mathscr{X})$, Theorem 3 follows from Theorem 4 of Section II.

THEOREM 4. *If the space $\mathscr{X}$ has $m+1$ components, then*

$$\mathfrak{N}(\mathscr{X}) \underset{\text{gr}}{=} \mathscr{G}^m. \tag{5}$$

If $\mathscr{X}$ is compact with infinitely many components, then $\mathfrak{N}(\mathscr{X}) = \mathscr{G}^{\aleph_0}$.

THEOREM 5. *Under the hypotheses of Theorem 6 of Section II the subgroup $\mathfrak{B}$ of $\mathfrak{N}(\mathscr{X})$ generated by the measures β_{Q_s} is dense in $\mathfrak{N}(\mathscr{X})$.*

Because the operation μ^* is continuous.

THEOREM 6. *If the space $\mathscr{X}$ is separable, then so is $\mathfrak{N}(\mathscr{X})$.*

THEOREM 7. *Theorem 8 of Section II remains true if $\mathfrak{M}(\mathscr{X})$ is replaced by $\mathfrak{N}(\mathscr{X})$.*

Proof. One has only to delete the term $\mu(A_0)$ in formula (9) of Section II.

THEOREM 8. *If $\mathscr{X}$ is a compact space consisting of an infinite sequence of components $Q_0, Q_1, \ldots$ all of which, except Q_0, are open, every measure $\mu \in \mathfrak{N}(\mathscr{X})$ can be expanded in the infinite series*

$$\mu = \sum_{i=1}^{\infty} k_i \beta_i, \quad \textit{where} \quad k_i = \mu(Q_i) \textit{ and } \beta_i = \beta_{Q_i}. \tag{6}$$

Proof. Let $\mu \in \mathfrak{N}(\mathscr{X})$ and $\mu_j = k_1\beta_1 + \ldots + k_j\beta_j$. We have to show that $\mu = \lim_{j=\infty} \mu_j$, and hence that (compare I (7)), if Z is a given closed-open set, then

$$\mu(Z) = \lim_{j=\infty} \mu_j(Z) = \sum_{i=1}^{\infty} k_i \beta_i(Z). \tag{7}$$

It is easy to see that either Z has the form

$$Z = Q_{i_1} + \ldots + Q_{i_s} \quad (\text{where } 0 < i_1 < \ldots < i_s)$$

or $\mathscr{X} - Z$ has this form (according to whether $Q_0 \cap Z = 0$ or $Q_0 \subset Z$). In the first case

$$\mu(Z) = \mu(Q_{i_1}) + \ldots + \mu(Q_{i_s}) = k_{i_1}\beta_{i_1}(Z) + \ldots + k_{i_s}\beta_{i_s}(Z),$$

which proves (7), because for the subscripts i' distinct from $i_1, \ldots, i_s$ we have $\beta_{i'}(Z) = 0$; in the second case

$$\mu(\mathscr{X}-Z) = \sum_{i=1}^{\infty} k_i \beta_i(\mathscr{X}-Z),$$

thus

$$\mu(Z) = -\mu(\mathscr{X}-Z) = \sum_{i=1}^{\infty} k_i \beta_i(Z),$$

because $\beta_i(\mathscr{X}-Z)+\beta_i(Z) = \beta_i(\mathscr{X}) = 0$.

IV. Extension of measures. In this section we assume that $\mathscr{Y}$ *is a metric locally connected space containing* $\mathscr{X}$.

DEFINITION. Let $\mu^{\mathscr{Y}}$ denote the *extension* of the measure $\mu \in \mathfrak{M}(\mathscr{X})$ defined in the following way

$$\mu^{\mathscr{Y}}(H) = \mu(\mathscr{X} \cap H) \quad \textit{for each } H \in (0,1)^{\mathscr{Y}}. \tag{1}$$

THEOREM 1. $\mu^{\mathscr{Y}} \in \mathfrak{M}(\mathscr{Y})$.

Proof. Let $H = \bigcup_t H_t$ where H_t are disjoint. It follows that

$$\mu^{\mathscr{Y}}\big(\bigcup_t H_t\big) = \mu\big(\mathscr{X} \cap \bigcup_t H_t\big) = \mu\big(\bigcup_t \mathscr{X} \cap H_t\big) = \sum_t \mu(\mathscr{X} \cap H_t)$$
$$= \sum_t \mu^{\mathscr{Y}}(H_t).$$

Recall that $\boldsymbol{Q}_{\mathscr{Y}}$ denotes by definition (see § 1, VII) the family of sets $\mathscr{X} \cap Q$ where Q is a component of $\mathscr{Y}$. Since every $H \in (0,1)^{\mathscr{Y}}$ is generated by components of $\mathscr{Y}$, it follows that

$$\bigwedge_{Z \in \boldsymbol{Q}_{\mathscr{Y}}} [\mu_0(Z) = \mu_1(Z)] \equiv \bigwedge_H [\mu_0(\mathscr{X} \cap H) = \mu_1(\mathscr{X} \cap H)]$$
$$\equiv \bigwedge_H [\mu_0^{\mathscr{Y}}(H) = \mu_1^{\mathscr{Y}}(H)]$$

according to (1); therefore

$$[(\mu_0 | \boldsymbol{Q}_{\mathscr{Y}}) = (\mu_1 | \boldsymbol{Q}_{\mathscr{Y}})] \equiv (\mu_0^{\mathscr{Y}} = \mu_1^{\mathscr{Y}}). \tag{2}$$

THEOREM 2. *Under the same hypotheses,*

$$(\mu_0 = \mu_1) \equiv [(\mu_0^G = \mu_1^G) \textit{ for each open } G \textit{ such that } \mathscr{X} \subset G \subset \mathscr{Y}], \tag{3}$$

$$(\mu \in \overline{\overline{\mathfrak{Y}}}) \equiv \bigwedge_G (\mu^G \in \mathfrak{Y}^G). \tag{4}$$

Proof. Equivalence (3) follows directly from (2) and I (9), and (4) is obtained by applying (2) and I (8),

$$(\mu \in \overline{\mathfrak{Y}}) \equiv \bigwedge_G [(\mu \mid \boldsymbol{Q}_G) \in (\mathfrak{Y} \mid \boldsymbol{Q}_G)] \equiv \bigwedge_G \bigvee_{\mu_1} (\mu_1 \in \mathfrak{Y})[(\mu_1 \mid \boldsymbol{Q}_G) = (\mu \mid \boldsymbol{Q}_G)]$$
$$\equiv \bigwedge_G \bigvee_{\mu_1} (\mu_1 \in \mathfrak{Y})(\mu_1^G = \mu^G) \equiv \bigwedge_G (\mu^G \in \mathfrak{Y}^G).$$

THEOREM 3. *The extension $\mu^{\mathscr{Y}}$ is a continuous homomorphism of $\mathfrak{M}(\mathscr{X})$ into $\mathfrak{M}(\mathscr{Y})$.*

Proof. The continuity of $\mu^{\mathscr{Y}}$ is an immediate consequence of (4); and $\mu^{\mathscr{Y}}$ is a homomorphism because

$$(\mu_0 + \mu_1)^{\mathscr{Y}}(H) = (\mu_0 + \mu_1)(\mathscr{X} \cap H) = \mu_0(\mathscr{X} \cap H) + \mu_1(\mathscr{X} \cap H)$$
$$= \mu_0^{\mathscr{Y}}(H) + \mu_1^{\mathscr{Y}}(H).$$

THEOREM 4. *The preceding transformation is also a continuous homomorphism of $\mathfrak{N}(\mathscr{X})$ into $\mathfrak{N}(\mathscr{Y})$.*

Because $\mu^{\mathscr{Y}}(\mathscr{X}) = \mu(\mathscr{X} \cap \mathscr{Y}) = \mu(\mathscr{X})$.

THEOREM 5. *If G is an open set in $\mathscr{Y}$ such that $\mathscr{X} \subset G \subset \mathscr{Y}$, then for every $\mu \in \mathfrak{M}(\mathscr{X})$ we have*

$$(\mu^G)^{\mathscr{Y}} = \mu^{\mathscr{Y}}. \tag{5}$$

Proof. Let $H \in (0,1)^{\mathscr{Y}}$. By Theorem 1, $\mu^G \in \mathfrak{M}(G)$, and it follows by (1) that

$$(\mu^G)^{\mathscr{Y}}(H) = \mu^G(G \cap H) = \mu(\mathscr{X} \cap G \cap H) = \mu(\mathscr{X} \cap H) = \mu^{\mathscr{Y}}(H).$$

THEOREM 6. *If to every open set G in $\mathscr{Y}$ and containing $\mathscr{X}$ corresponds a measure $\nu_G \in \mathfrak{M}(G)$ such that*

$$\nu_{G_0}^{G_1} = \nu_{G_1} \quad \textit{provided that} \quad G_0 \subset G_1, \tag{6}$$

there exists a measure $\mu \in \mathfrak{M}(\mathscr{X})$ such that

$$\mu^G = \nu_G \quad \textit{for each } G. \tag{7}$$

Proof. Let Z be a fixed set closed-open in $\mathscr{X}$. Let H and H^* be two disjoint sets open in $\mathscr{Y}$ and such that $Z = \mathscr{X} \cap H$ and $\mathscr{X} - Z = \mathscr{X} \cap H^*$. Let us set $G = H \cup H^*$. Therefore $H \in (0,1)^G$. Assume by definition that

$$\mu(Z) = \nu_G(H). \tag{8}$$

First, we are going to show that this definition is correct (i.e. it does not depend on the choice of H and H^* for a fixed Z) and next that μ defined that way satisfies the conclusion of Theorem 6, i.e., that $\mu \in \mathfrak{M}(\mathscr{X})$ and that condition (7) is fulfilled.

For this purpose assume that G_1 and G_2 are two sets containing $\mathscr{X}$ and that

$$H_1 \in (0,1)^{G_1}, \qquad H_2 \in (0,1)^{G_2} \qquad \text{and} \qquad \mathscr{X} \cap H_1 = \mathscr{X} \cap H_2. \tag{9}$$

We have to show that

$$\nu_{G_1}(H_1) = \nu_{G_2}(H_2). \tag{10}$$

Let G_0 be the union of components of $G_1 \cap G_2$ which intersect $\mathscr{X}$. By (6) it follows

$$\nu_{G_0}^{G_1} = \nu_{G_1} \qquad \text{and} \qquad \nu_{G_0}^{G_2} = \nu_{G_2},$$

which implies (according to (1) where μ is replaced by ν_{G_0} and $\mathscr{Y}$ by G_1 and G_2 respectively)

$$\nu_{G_1}(H_1) = \nu_{G_0}^{G_1}(H_1) = \nu_{G_0}(G_0 \cap H_1),$$
$$\nu_{G_2}(H_2) = \nu_{G_0}^{G_2}(H_2) = \nu_{G_0}(G_0 \cap H_2).$$

Therefore, identity (10) will follow from the identity $G_0 \cap H_1 = G_0 \cap H_2$. Now suppose that $G_0 \cap H_1 - H_2 \neq 0$; then this set contains (at least) one component of G_0 (as a closed-open subset of G_0); since this component intersects $\mathscr{X}$, $\mathscr{X} \cap G_0 \cap H_1 - H_2 \neq 0$, contradicting (9).

Now we are going to show that μ is completely additive. Let us set $Z_0 = \mathscr{X} - Z$ and $Z = \bigcup_t Z_t$ where Z_t are disjoint and open in $\mathscr{X}$ ($t \neq 0$). So, there exists (compare Theorem 2 of Volume I, § 21, XI) a family $\{H_t\}$ (where t takes also the value 0) of disjoint sets open in $\mathscr{Y}$ and such that $\mathscr{X} \cap H_t = Z_t$.

Let $G = \bigcup_t H_t$. Then $\mathscr{X} \subset G$ and $H_t \in (0,1)^G$. By (8) it follows that

$$\mu(Z) = \nu_G\Big(\bigcup_{t \neq 0} H_t\Big) = \sum_{t \neq 0} \nu_G(H_t) = \sum_{t \neq 0} \mu(Z_t).$$

Finally, identity (7) is true, i.e.

$$\mu^G(H) = \nu_G(H) \qquad \text{for each } H \in (0,1)^G.$$

Because, let $Z = \mathscr{X} \cap H$; it follows by (1) and (8) that

$$\mu^G(H) = \mu(\mathscr{X} \cap H) = \mu(Z) = \nu_G(H).$$

COROLLARY 7. *If* $\nu_G \in \mathfrak{N}(G)$ *under the preceding hypotheses, then the function* μ *defined by condition* (8) *belongs to* $\mathfrak{N}(\mathscr{X})$.

Proof. By (8) it follows that $\mu(\mathscr{X}) = \nu_G(G) = 0$.

THEOREM 8. *Let* $\mathrm{ext}_{G_0G_1}$ *be the extension operation defined for every pair* $G_0 \supset G_1$ *in the following way*

$$\mathrm{ext}_{G_0G_1}(\nu) = \nu^{G_0} \quad \textit{for each} \quad \nu \in \mathfrak{M}(G_1). \tag{11}$$

Since the family of sets G open in $\mathscr{Y}$ and such that $G \supset \mathscr{X}$ is directed by the relation $G_0 \supset G_1$, the set $\mathfrak{M}(G)$ and the relations $\mathrm{ext}_{G_0G_1}$ form an inverse system.

Note that

$$\mathrm{ext}_{G_0G_1}[\mathrm{ext}_{G_1G_2}(\nu)] = \mathrm{ext}_{G_0G_1}(\nu^{G_1}) = (\nu^{G_1})^{G_0} = \nu^{G_0} = \mathrm{ext}_{G_0G_2}(\nu)$$

according to Theorem 5 and (11).

Therefore Theorem 6 implies the following.

THEOREM 9. $\mathfrak{M}(\mathscr{X}) \underset{\text{gr.t.}}{=} \underset{G, G_0 \supset G_1}{\mathrm{Lim}} \{\mathfrak{M}(G), \mathrm{ext}_{G_0G_1}\}$ *where G is open and* $\mathscr{X} \subset G \subset \mathscr{Y}$.

The required isomorphism, which is a homeomorphism at the same time, is obtained by assigning to every $\mu \in \mathfrak{M}(\mathscr{X})$ the element

$$\{\mu^G\} \in \underset{G}{\boldsymbol{P}} \mathfrak{M}(G).$$

THEOREM 10. *Theorem 9 remains true if $\mathfrak{M}$ is replaced by $\mathfrak{N}$.*

This is a consequence of Corollary 7.

SOME THEOREMS ON THE DISCONNECTION OF THE SPHERE $\mathscr{S}_n$

§ 59. Qualitative problems

I. Polygonal arcs in $\mathscr{E}^n$. Every arc consisting of a finite number of straight line segments is said to be a *polygonal arc.*

THEOREM 1. *Every pair of points in a region $R \subset \mathscr{E}^n$ can be joined in R by a polygonal arc.*

Proof. Let p be a fixed point of R. Our problem is to show that the set P of points which can be joined with p by a polygonal arc contained in R is closed-open in R (and hence coincides with R). But this is an easy corollary of the fact, that if x and y belong to an (n-dimensional) ball contained in R, then one of them belongs to P, provided that the other does.

LEMMA 2. *Let c be a fixed number such that $0 < c < 1$. Let*

$$R_1 \text{ be the rectangle} \quad c-1 \leqslant x \leqslant 1, \quad |y| \leqslant 1-c,$$

$$R_c \text{ the rectangle} \quad c(c-1) \leqslant x \leqslant c, \quad |y| \leqslant c(1-c),$$

L the segment $0 \leqslant x \leqslant c$ of the axis X, and let M and N be the segments joining the point $(0, 0)$ with the boundary F_c of R_c and the boundary F_1 of R_1 respectively and making with the axis X a given angle θ or an angle $\theta+\pi$, where $0 < \theta < \pi$. Then there exists a homeomorphism h such that

$$h(R_1) = R_1, \quad h(L) = M \quad \textit{and} \quad h(x, y) = (x, y) \textit{ for } (x, y) \in F_1 \cup N.$$

Proof. Let φ be an increasing continuous function such that

$$\varphi(0) = \theta, \quad \varphi(\theta+\pi) = \theta+\pi, \quad \varphi(2\pi) = \theta+2\pi. \tag{1}$$

Consider the function g defined in the following way:

$$g(a, u) = \begin{cases} \varphi(a) & \text{for} \quad 0 \leqslant u \leqslant c, \\ \dfrac{1}{1-c}[a(u-c)+\varphi(a)(1-u)] & \text{for} \quad c \leqslant u \leqslant 1. \end{cases} \tag{2}$$

It is easy to show that

(i) $g(2\pi, u) - g(0, u) = 2\pi$,

(ii) $g(\alpha, u)$ is an increasing function of α for u fixed,

(iii) $g(\alpha, 0) = \varphi(\alpha)$, $g(\alpha, 1) = \alpha$,

(iv) $g(\theta+\pi, u) = \theta+\pi$.

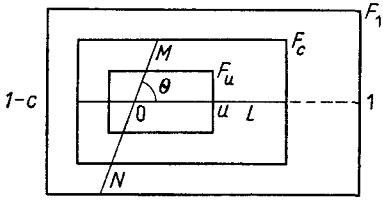

Fig. 15

Let R_u denote in general the rectangle

$$u(c-1) \leqslant x \leqslant u, \qquad |y| \leqslant u(1-c)$$

and let F_u be its boundary. It follows that

$$R_1 = \bigcup_{0 \leqslant u \leqslant 1} F_u, \quad \text{where} \quad F_u \cap F_{u'} = 0 \quad \text{for} \quad u \neq u'. \tag{3}$$

The function h is defined in the following way. Let (x, y) be a point of R_1 different from the origin; let α be its argument $(= \arctan y/x)$ and let $(x, y) \in F_u$. Let $h(x, y)$ be the point of F_u with the argument $g(\alpha, u)$. Besides, assume that $h(0, 0) = (0, 0)$.

It follows from (ii) and (i) that h is a homeomorphism of F_u onto F_u, and hence of R_1 onto R_1 (by (3)).

Next, $h(L) = M$. Because $\alpha = 0$ and $u \leqslant c$ for $(x, y) \in L$. Therefore, according to (2), $g(0, u) = \varphi(0) = \theta$ by (1).

Finally, if $(x, y) \in F_1$, then $u = 1$, therefore $g(\alpha, 1) = \alpha$ by (iii), which implies $h(x, y) = (x, y)$.

The same identity holds if $(x, y) \in N$, because in that case $\alpha = \theta + \pi$, thus $g(\alpha, u) = \alpha$ by (iv).

Generalized Lemma 3. *In $\mathscr{E}^n$, let P be an n-dimensional rectangle*

$$c-1 \leqslant x_1 \leqslant 1, \quad |x_2| \leqslant 1-c, \quad \ldots, \quad |x_n| \leqslant 1-c.$$

Let L, M and N have the same meaning as in Lemma 2 *(replacing there x by x_1 and y by x_2). Then there exists a homeomorphism f such that*

$$f(P) = P, \quad f(L) = M \quad \textit{and} \quad f(x) = x \quad \textit{for} \quad x \in \operatorname{Fr}(P) \cup N$$

(x stands for the point $x_1, \ldots, x_n$).

Proof. Let v_x be the greatest among the $n-1$ numbers

$$u, |x_3|:(1-c), \ldots, |x_n|:(1-c),$$

where u is defined by the condition $(x_1, x_2) \in F_u$.

The point $(y_1, \ldots, y_n) = f(x_1, \ldots, x_n)$ is defined in the following way. (y_1, y_2) is the point of F_u whose argument is $g(\alpha, v_x)$ where α is the argument of (x_1, x_2); besides $y_3 = x_3, \ldots, y_n = x_n$; finally $f(0) = 0$.

Given a system of $n-2$ numbers $x_3^0, \ldots, x_n^0$, denote by $P_{x_3^0 \ldots x_n^0}$ the cross section of P consisting of the points $x_1, \ldots, x_n$ such that $x_3 = x_3^0, \ldots, x_n = x_n^0$.

The set of points of $P_{x_3^0 \ldots x_n^0}$ such that $(x_1, x_2) \in F_u$ (where u is given in advance) will be denoted by $P_{x_3^0 \ldots x_n^0 u}$.

It is easy to see that v_x is constant when x ranges over the set $P_{x_3^0 \ldots x_n^0 u}$. Consequently (compare (i) and (ii)), f is a homeomorphism of this set onto itself. On the other hand,

$$P = \bigcup P_{x_3 \ldots x_n u},$$

where the union is taken for all the systems $x_3 \ldots x_n u$ such that

$$|x_3| \leqslant 1-c, \quad \ldots, \quad |x_n| \leqslant 1-c, \qquad 0 \leqslant u \leqslant 1.$$

Since the terms of that union are disjoint, it follows that f is a homeomorphism such that $f(P) = P$.

The identity $v_x = u$ holds in the plane X_1X_2, so that the function f is identical with h. Therefore $f(L) = M$ and $f(x) = x$ for $x \in N$.

Finally, let $x \in \mathrm{Fr}(P)$. We have to show that $f(x) = x$. Now, the boundary of P consists of the points x such that $(x_1, x_2) \in F_1$, and of the points x for which either $|x_3| = 1-c$, or $\ldots$, or else $|x_n| = 1-c$. If $(x_1, x_2) \in F_1$, then $u = 1$, which implies $v_x = 1$, and hence $f(x) = x$ by (iii). The same conclusion is reached if $|x_j| = 1-c$ for a subscript $j \geqslant 3$.

THEOREM 4[1]. *In $\mathscr{E}^n$ all the polygonal arcs are topologically equivalent; this means that for every pair of polygonal arcs there exists a homeomorphism of $\mathscr{E}^n$ onto $\mathscr{E}^n$ which maps one of these arcs onto the other.*

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[1] For a generalization of this theorem to arbitrary arcs (for $n = 2$) see § 60, V, Theorem 2.

Proof. Proceed by induction with respect to the number m of sides of the considered polygonal arc. Since the theorem is obvious for $m = 1$ (straight line segment), assume that it is true for $m-1$ $(m \geqslant 2)$. Let $A = a_0 a_1 \ldots a_{m-1} a_m$ be a polygonal arc. It is clearly admissible to assume that the point a_{m-1} coincides with the origin of axes, that the side $a_{m-1}a_m$ lies on the axis X_1, that $a_{m-2}a_{m-1}$ lies in the plane X_1X_2 and that $\theta = \sphericalangle\, a_{m-2}a_{m-1}a_m < \pi$. Let $c = |a_m - a_{m-1}|$. Let $\varepsilon > 0$ be a number such that the (n-dimensional) rectangle P, defined by the conditions

$$-\varepsilon \leqslant x \leqslant c+\varepsilon, \quad |x_2| \leqslant \varepsilon, \quad \ldots, \quad |x_n| \leqslant \varepsilon,$$

is disjoint from the arc $a_0 a_1 \ldots a_{m-2}$.

Let us assume that $c+\varepsilon = 1$; so, we may set in Lemma 3, $L = a_{m-1}a_m$ and $N = P \cap a_{m-2}a_{m-1}$, and we denote by M the straight line prolongation of the segment $a_{m-2}a_{m-1}$ from a_{m-1} to F_c. Let f be the homeomorphism considered in Lemma 3, and denote by h the mapping of $\mathscr{E}^n$ onto $\mathscr{E}^n$ defined in the following way

$$h(x) = f(x) \quad \text{for} \quad x \in P, \qquad h(x) = x \quad \text{for} \quad x \in \mathscr{E}^n - P.$$

Since $f(x) = x$ on the boundary of P, h is continuous and, since $f(P) = P$, h is a homeomorphism of $\mathscr{E}^n$ onto $\mathscr{E}^n$. Moreover, h maps the arc A onto the polygonal arc $B = a_0 a_1 \ldots a_{m-1} \cup M$ with $m-1$ sides, because the condition $f(x) = x$ for $x \in N$ implies that $h(x) = x$ for $x \in a_0 \ldots a_{m-1}$, and since $f(a_{m-1}a_m) = M$, it follows that $h(A) = B$.

Since the theorem holds for B by assumption, it holds also for A.

THEOREM 5. *If A is a polygonal arc, then*

$$A = \bigcap_{k=1}^{\infty} F_k, \quad \textit{where} \quad F_{k+1} \subset \operatorname{Int}(F_k) \quad \textit{and}$$

$$F_k \underset{\text{top}}{=} \mathop{E}_{x}(x_1^2 + \ldots + x_n^2 \leqslant 1).$$

Proof. By Theorem 4 the proof is reduced to the case where A is a straight line segment. But in that case the proof is obvious; actually, every segment is an intersection of an infinite decreasing sequence of ellipsoids.

THEOREM 6. *If A is a polygonal arc, then $\mathscr{E}^n - A \underset{\text{top}}{=} \mathscr{E}^n - (0)$. Moreover, it may be assumed that the considered homeomorphism is the identity mapping outside of a neighbourhood of A.*

Proof. Assume that A is a straight line segment and that the sets F_k of Theorem 5 are ellipsoids. Let K_m be the ball $x_1^2 + \ldots + x_n^2 \leqslant 1/m$. It is easy to define the mapping of $\mathscr{E}^n - F_1$ onto $\mathscr{E}^n - K_1$ and, in general, of $F_m - F_{m+1}$ onto $K_m - K_{m+1}$, and the homeomorphism of $\mathscr{E}^n - A$ onto $\mathscr{E}^n - (0)$.

THEOREM 7 (linear accessibility). *Let A be a polygonal arc, R a region in $\mathscr{E}^n$, $p \in A \subset R$ and $q \in R - A$. There exists a polygonal arc B such that $p, q \in B \subset R$ and $A \cap B = (p)$.*

Proof. Enclose the point p in a sufficiently small ball; it is easy to see that there exists in that ball a straight line segment $pq' \subset R$ having with A only the point p in common. By Theorem 6 the set $\mathscr{E}^n - A$ is a region and so is the set $R - A$ (because $R - A = R \cap (\mathscr{E}^n - A)$ and $R \cup (\mathscr{E}^n - A) = \mathscr{E}^n$, compare § 57, II, Theorem 2). Therefore there exists a polygonal arc $q'q \subset R - A$ (by Theorem 1). We denote by B an arc pq contained in $pq' \cup q'q$.

Theorems 5 through 7, applied to the space $\mathscr{S}_n$, lead to the following statement.

THEOREM 8. *Let $a_0, \ldots, a_m$ be a system of points in a region $R \subset \mathscr{S}_n$. Then there exist an arc A and a region R^* such that*

(i) $a_0, \ldots, a_m \in A \subset R^*$, $\overline{R^*} \subset R$,

(ii) $\mathscr{S}_n - A \underset{\text{top}}{=} \mathscr{E}^n$,

(iii) $\overline{R^*} \underset{\text{top}}{=} \underset{x}{E}\,(x_1^2 + \ldots + x_n^2 \leqslant 1)$, $R^* \underset{\text{top}}{=} \mathscr{E}^n$ *and* $\operatorname{Fr}(R^*) \underset{\text{top}}{=} \mathscr{S}_{n-1}$.

Proof. Let $a_0, \ldots, a_m$ be a system of points in a region $R \subset \mathscr{E}^n$; referring to Theorems 1 and 7, it is easy to prove (by induction) that there exists in R a polygonal arc containing these points. Considering $\mathscr{S}_n$ as the space $\mathscr{E}^n$ augmented by the point at infinity, we infer that there exist an arc A such that $a_0, \ldots, a_m \in A \subset R$ and a homeomorphism h of $\mathscr{E}^n - A$ onto $\mathscr{E}^n - (0)$, which leaves invariant the point at infinity according to Theorem 6; therefore h determines a homeomorphism of $\mathscr{S}_n - A$ onto $\mathscr{S}_n - (0)$; since the latter set is homeomorphic with $\mathscr{E}^n$, condition (ii) is established.

The rest of Theorem 8 follows from Theorem 5.

II. Cuts of $\mathscr{S}_n$(1). According to Theorem 2′ of § 54, III, the dimension of connectedness of $\mathscr{S}_n$ and of $\mathscr{E}^n$ is n. In other words, no closed subset of dimension $\leqslant n-2$ separates these spaces. Now we will show that no set of dimension $\leqslant n-2$ (closed or not) cuts them. More precisely, the following is true.

THEOREM 1 (of Mazurkiewicz) (2). *Let R be a region in $\mathscr{E}^n$ (or in $\mathscr{S}_n$). If* $\dim A \leqslant n-2$, *then $R-A$ is a semi-continuum.*

LEMMA 2. *Let $S = p_0 \dots p_n$ be an n-dimensional simplex. If $E \subset \bar{S} - \overline{p_1 \dots p_n} - p_0$ and* $\dim E \leqslant n-2$, *there exists a subcontinuum of $\bar{S}-E$ joining p_0 to $p_1 \dots p_n$.*

Proof. Let

$$Q_i = p_0 \dots p_{i-1} p_{i+1} \dots p_n \quad \text{and} \quad G_i = \bar{S} - \bar{Q}_i - \bar{Q}_0.$$

Since $G_1 \cup \dots \cup G_n = \bar{S} - p_0 - \bar{Q}_0 \supset E$, the inequality $\dim E \leqslant n-2$ implies by Theorem 4 of § 27, III, that there exists a system of open sets $H_1, \dots, H_n$ such that

$$E \subset H_1 \cup \dots \cup H_n, \quad H_1 \cap \dots \cap H_n = 0,$$
$$H_i \subset G_i, \quad \text{thus} \quad H_i \cap \bar{Q}_i = 0.$$

It follows by Theorem 8 of § 28, II, that the set $H_1 \cup \dots \cup H_n$ does not separate $\bar{S}$ between p_0 and Q_0. Therefore there exists a continuum joining p_0 to Q_0 in $\bar{S} - (H_1 \cup \dots \cup H_n) \subset \bar{S} - E$.

Thus the lemma is established. Let $\mathscr{Q}_n$ be the ball $\underset{x}{E}[|x| \leqslant 1]$ in $\mathscr{E}^n$ and p_N and p_S its poles. We shall show that no subset A of $\mathscr{Q}_n - p_N - p_S$ of dimension $\leqslant n-2$ cuts $\mathscr{Q}_n$ between p_N and p_S.

Let $f\colon \bar{S} \to \mathscr{Q}_n$ be continuous and such that

$$f(\bar{S}) = \mathscr{Q}_n, \quad f(p_0) = p_N, \quad f^{-1}(p_S) = \bar{Q}_0$$

and that $f | \bar{S} - p_0 - \bar{Q}_0$ is a homeomorphism; then the set $E = f^{-1}(A)$ satisfies to the conditions of Lemma 2. Let C be a continuum such that

$$p_0 \in C \subset \bar{S} - E \quad \text{and} \quad C \cap Q_0 \neq 0.$$

Then $f(C)$ is a continuum joining p_N to p_S in $\mathscr{Q}_n - A$.

(1) For Theorems 3 through 11, compare K. Borsuk, *Über Schnitte der n-dimensionalen Euklidischen Räume*, Math. Ann. 106 (1932), p. 239. Compare also the paper by the same author in Monatsh. Math.-Phys. 38 (1931), p. 381, and P. Alexandroff, *Dimensionstheorie*, § 5, Math. Ann. 106 (1932), p. 218.

(2) Fund. Math. 13 (1929), p. 211.

It follows that no set of dimension $\leqslant n-2$ cuts $\mathscr{E}^n$ between two points a and b since it does not cut the ball with center $\frac{1}{2}(a+b)$ and diameter $|a-b|$ between a and b.

Finally, let R be a region in $\mathscr{E}^n, a, b \in R$ and $\dim A \leqslant n-2$. By Theorem 8 of Section I, there exists a region $R^* \subset R$ homeomorphic with $\mathscr{E}^n$ and containing a and b. As just has been proved there exists a continuum joining a to b in R^*-A, and hence in $R-A$.

Remark. K. Sitnikov has shown for $n=3$ that *there exists a set A of dimension $n-1$ such that $R-A$ is a semi-continuum whatever the region R may be*(1).

The considered set is a G_δ. Let us mention that no F_σ-set has this property.

There is no example of that kind in the plane either.

THEOREM 3. *Let $F = \overline{F} \subset \mathscr{S}_n \neq F$ and $f\colon F \to \mathscr{S}_{n-1}$ continuous. The following conditions are equivalent*

(i) *$f \simeq 1$ (i.e. f is homotopic to unity),*

(ii) *f admits an extension to $\mathscr{S}_n$,*

(iii) *f admits an extension to $\mathscr{S}_n$ minus one point.*

Proof. The implication (i) $\Rightarrow$ (ii) directly follows from Theorem 7 of § 54, II, and the implication (ii) $\Rightarrow$ (iii) is obvious. It remains to prove that (iii) $\Rightarrow$ (i). Let $p \in \mathscr{S}_n - F$ and let $g\colon \mathscr{S}_n - p \to \mathscr{S}_{n-1}$ be a continuous extension of f. Since the set $\mathscr{S}_n - p$ is contractible in itself (being homeomorphic with $\mathscr{E}^n$), it follows that $g \simeq 1$ (compare § 54, VI, 2 (3)), which implies $f \simeq 1$.

THEOREM 4. *Let $p \neq q, F = \overline{F} \subset \mathscr{S}_n - p - q$ and $f\colon \mathscr{S}_n - p - q \to \mathscr{S}_{n-1}$ continuous. If F is not a cut between p and q, then $f|F \simeq 1$.*

Proof. Let L be a polygonal arc such that $p, q \in L \subset \mathscr{S}_n - F$. Since $\mathscr{S}_n - L$ is homeomorphic to $\mathscr{E}^n$ (compare I, 8 (ii)), it is contractible in itself. Therefore $(f|\mathscr{S}_n - L) \simeq 1$, and hence $f|F \simeq 1$.

THEOREM 5. *Let $p \neq q$, $f\colon \mathscr{S}_n - p - q \to \mathscr{S}_{n-1}$ continuous, and f non $\simeq 1$. If X separates p from q, then $f|X$ non $\simeq 1$.*

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(1) *An example of a 2-dimensional set in the 3-dimensional Euclidean space which does not cut any region* (Russian), Dokl. Acad. Nauk SSSR 94 (1954), p. 1007. See also of the same author, *Combinatorial topology of non-closed sets* II (Russian), Mat. Sb. 37 (1955), pp. 385–434.

Proof. Since X separates p and q, it contains a closed F which separates these points (§ 14, V, Theorem 1). Therefore

$$\mathscr{S}_n = A \cup B, \quad p \in A = \bar{A}, \quad q \in B = \bar{B} \quad \text{and} \quad A \cap B = F.$$

With no loss of generality, it can be assumed that p is the north pole of $\mathscr{S}_n$, q is the south pole and A is contained in the north closed hemisphere. Suppose that

$$f|X \simeq 1, \quad \text{hence} \quad f|F \simeq 1, \quad \text{and therefore} \quad (f|F) \subset g \in \mathscr{S}_{n-1}^{\mathscr{S}_n},$$

by Theorem 3. Put

$$f^*(x) = \begin{cases} g(x) & \text{for} \quad x \in A, \\ f(x) & \text{for} \quad x \in B - q. \end{cases}$$

It follows that $f^*\colon \mathscr{S}_n - q \to \mathscr{S}_{n-1}$ is a continuous function and $(f|\mathscr{S}_{n-1}) \subset f^*$ since $\mathscr{S}_{n-1} \subset B$. By Theorem 3 we have $f|\mathscr{S}_{n-1} \simeq 1$. Since the set $\mathscr{S}_n - p - q$ is deformable onto $\mathscr{S}_{n-1}$ in itself (compare § 54, IV, Example 2), it follows that $f \simeq 1$ (by Theorem 1 of § 54, IV).

COROLLARY 6. *Let p_{N} and p_{S} be the north and south pole of $\mathscr{S}_n$ and $r\colon \mathscr{S}_n - p_{\mathrm{N}} - q_{\mathrm{S}} \to \mathscr{S}_{n-1}$ be the projection onto the equator $\mathscr{S}_{n-1}$ effected along the meridians. A closed*[1] *subset F of $\mathscr{S}_n - p_{\mathrm{N}} - p_{\mathrm{S}}$ is a cut between the poles if and only if $r|F$ non $\simeq 1$.*

Proof. Since $r(x) = x$ for $x \in \mathscr{S}_{n-1}$, we have (compare § 54, I, Theorem 9) $r|\mathscr{S}_{n-1}$ *non* $\simeq 1$ and hence r *non* $\simeq 1$.

LEMMA 7. *Every continuous function $f\colon \mathscr{S}_{n-1} \to \mathscr{Y}$ admits a (continous) extension $f^*\colon \mathscr{Q}_n - (0) \to \mathscr{Y}$.*

Proof. One has only to set $f^*(x) = f(x/|x|)$ for $0 \neq x \in \mathscr{Q}_n$.

LEMMA 8. *Let R be a region in $\mathscr{S}_n$, $p, q \in R$ and $f\colon \bar{R} - p \to \mathscr{Y}$ continuous. There exists a continuous function $g\colon \bar{R} - q \to \mathscr{Y}$ such that $f|\mathrm{Fr}(R) \subset g$.*

Proof. By Theorem 8 of Section I, there exists a region R^* such that

$$p, q \in R^*, \quad \overline{R^*} \subset R, \quad \overline{R^*} \underset{\text{top}}{=} \mathscr{Q}_n \quad \text{and} \quad \mathrm{Fr}(R^*) \underset{\text{top}}{=} \mathscr{S}_{n-1}.$$

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[1] The hypothesis that F is closed may be omitted for $n = 2$, see § 60, II Theorem 1.

According to Theorem 7, let $h\colon \overline{R^*}-q \to \mathscr{Y}$ be a continuous extension of $f|\mathrm{Fr}(R^*)$. It remains to set

$$g(x) = \begin{cases} h(x) & \text{for} \quad x \in \overline{R^*}-q, \\ f(x) & \text{for} \quad x \in \overline{R}-R^*. \end{cases}$$

THEOREM 9. *Let $F = \overline{F} \subset \mathscr{S}_n$ and $f\colon F \to \mathscr{S}_{n-1}$ continuous. There exist a finite set $A \subset \mathscr{S}_n - F$ and a (continuous) extension $f^*\colon \mathscr{S}_n - A \to \mathscr{S}_{n-1}$ of f*(1).

More precisely, *there exists a finite system (empty or not) of (different) components $R_0, \ldots, R_k$ of $\mathscr{S}_n - F$ such that*

(i) $f|\mathrm{Fr}(R_j)$ *non* $\simeq 1$ *for* $j \leqslant k$,

(ii) $k \neq 0$,

(iii) *if E denotes a system of points $p_0, \ldots, p_k$, where $p_j \in R_j$, there exists a continuous extension $f^*\colon \mathscr{S}_n - E \to \mathscr{S}_{n-1}$ of f.*

Proof. Since $\mathscr{S}_{n-1}$ is an absolute neighbourhood retract, f has an extension to a neighbourhood of F, and so to a polyhedron containing F. Thus, there exists a (closed non-singular) complex $T_1, \ldots, T_r$ such that $\mathscr{S}_n = \overline{T}_1 \cup \ldots \cup \overline{T}_m \cup \ldots \cup \overline{T}_r$ and $\dim T_j = n$, and a continuous extension $f_1\colon P \to \mathscr{S}_{n-1}$ of f, where $P = \overline{T}_1 \cup \ldots \cup \overline{T}_m$. Let $Q = \mathrm{Fr}(T_{m+1}) \cup \ldots \cup \mathrm{Fr}(T_r)$. Therefore $\dim Q \leqslant n-1$. Since the sphere $\mathscr{S}_{n-1}$ is locally and integrally connected in dimensions $< n-1$ (compare § 53, V, Theorem 2), f_1 admits a continuous extension $f_2\colon P \cup Q \to \mathscr{S}_{n-1}$ (by § 53, IV, Theorem 1′). Let a_j be the center of T_j. Define $A = (a_{m+1}, \ldots, a_r)$. By Theorem 8 the function $f_2|\mathrm{Fr}(T_j)$ admits (for every $j > m$) an extension $f_{2,j}\colon \overline{T}_j - a_j \to \mathscr{S}_{n-1}$. The function f_3 identical with f_2 on $P \cup Q$ and with $f_{2,j}$ on $\overline{T}_j - a_j$, where $j = m+1, \ldots, r$, is a continuous extension of f and $f_3\colon \mathscr{S}_n - A \to \mathscr{S}_{n-1}$.

Thus, the first part of Theorem 9 is established.

Let $B = (b_0, \ldots, b_k)$ be a subset of $\mathscr{S}_n - F$ such that the function f has a continuous extension $g\colon \mathscr{S}_n - B \to \mathscr{S}_{n-1}$ and that B is irreducible with respect to this property, which means that f admits no extension to $\mathscr{S}_n - B \cup b_j$, for any $j \leqslant k$.

Let R_j be the component of $\mathscr{S}_n - F$ which contains b_j. We will show that $R_i \neq R_j$ for $i \neq j$.

(1) See e.g. S. Eilenberg, Fund. Math. 26 (1936), p. 280. Cf. p. 357, Theorem 7.

Suppose on the contrary that $b_0, b_1 \in R_0$. Since the difference $R_0-(b_2, \ldots, b_k)$ is a region (compare Theorem 1), let R^* be (according to Theorem 8) a region such that

$$b_0, b_1 \in R^*, \quad \overline{R^*} \subset R_0-(b_2, \ldots, b_k), \quad \overline{R^*} \underset{\text{top}}{=} \mathcal{Q}_n \quad \text{and}$$
$$\operatorname{Fr}(R^*) \underset{\text{top}}{=} \mathcal{S}_{n-1}.$$

It follows by Theorem 7 that

$$[g \mid \operatorname{Fr}(R^*)] \subset g_1 \in \mathcal{S}_{n-1}^{\overline{R^*}-b_1}.$$

But then the function g^* identical with g on $\mathcal{S}_n-R^*-B$ and with g_1 on $\overline{R^*}-b_1$ is an extension of f to $\mathcal{S}_n-(b_1, \ldots, b_k)$, contradicting the definition of B.

Would we suppose contrary to (i) that $f \mid \operatorname{Fr}(R_j) \simeq 1$, there would exist by Theorem 3 an extension $h\colon \mathcal{S}_n \to \mathcal{S}_{n-1}$ of $f \mid \operatorname{Fr}(R_j)$. But then the function g^* identical with g on $\mathcal{S}_n-R_j-B$ and with h on $\bar{R}_j$ would be an extension of f to $\mathcal{S}_n-B \cup b_j$, which is impossible.

Would we suppose that $k=0$, it would follow $B=b_0$, and hence $g\colon \mathcal{S}_n-b_0 \to \mathcal{S}_{n-1}$ would be continuous, but then f would admit an extension to the whole space $\mathcal{S}_n$ (by Theorem 3).

In order to establish (iii), assume, according to Theorem 8, that

$$g \mid \operatorname{Fr}(R_j) \text{ has an extension } f_j\colon \bar{R}_j-p_j \to \mathcal{S}_{n-1}.$$

It remains to denote by f^* the function identical with g on $\mathcal{S}_n-(R_0 \cup \ldots \cup R_k)$ and with f_j on $\bar{R}_j-p_j$ for $j=0, \ldots, k$.

THEOREM 10 (of Borsuk[1]). *Let $F=\bar{F} \subset \mathcal{S}_n \neq F$. The following conditions are equivalent.*

(i) *$\mathcal{S}_n-F$ is connected,*

(ii) *$\mathcal{S}_{n-1}^F$ is connected,*

(iii) *F is contractible with respect to $\mathcal{S}_{n-1}$,*

(iv) *$\mathcal{S}_{n-1}^F=\mathcal{S}_{n-1}^{\mathcal{S}_n} \mid F$.*

Proof. The equivalence (ii) $\equiv$ (iii) follows from Theorem 2 of § 54, V. The equivalence (iii) $\equiv$ (iv) follows from Theorem 3. The

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(1) See p. 466, footnote 1.

implication (iii) $\Rightarrow$ (i) follows from Theorem 6. It remains to show that (i) $\Rightarrow$ (iv). Thus assume that F is not a cut and that $f\colon F \to \mathscr{S}_{n-1}$ is a continuous function. Since the set $\mathscr{S}_n - F$ is a region, then f admits an extension $f^*\colon \mathscr{S}_n \to \mathscr{S}_{n-1}$ (by Theorem 9, (ii)).

THEOREM 11. *If none of the closed sets F_0 and F_1 cuts $\mathscr{S}_n$ between the points p and q and if* $\dim(F_0 \cap F_1) \leqslant n-3$, *their union $F_0 \cup F_1$ does it neither*(¹).

Proof. Assume that $p = p_N$ and $q = p_S$; we infer by Theorem 6 that $r|F_0 \simeq 1$ and $r|F_1 \simeq 1$, and it follows by Theorem 9 of § 54, II, that $r|F_0 \cup F_1 \simeq 1$. This implies the required conclusion by Theorem 6.

III. Irreducible cuts.

THEOREM 1. *Let $F = \overline{F} \subset \mathscr{S}_n - p - q$, $f\colon \mathscr{S}_n - p - q \to \mathscr{S}_{n-1}$ continuous and f non $\simeq 1$. The set F is an irreducible cut of $\mathscr{S}_n$ between p and q if and only if $f|F$ irr non $\simeq 1$.*

In other words, *the condition: F is a cut between p and q whereas no subset $A = \overline{A} \subset F \neq A$ is one, is equivalent to the condition: $f|F$ non $\simeq 1$ whereas $f|A \simeq 1$ for every $A = \overline{A} \subset F \neq A$.*

This is a direct consequence of Theorems of Section II, 4 and 5.

Remark. If $p = p_N$ and $q = p_S$, the function f may be replaced by the function r of Corollary 6 of Section II.

THEOREM 2(²). *Let $F = \overline{F} \neq \mathscr{S}_n$. The following conditions are equivalent.*

(i) *F is an irreducible cut between two points,*

(ii) *F is the common boundary of two components of $\mathscr{S}_n - F$,*

(iii) *there exists a continuous function $f\colon F \to \mathscr{S}_{n-1}$ such that f irr non $\simeq 1$.*

Proof. The equivalence (i) $\equiv$ (ii) follows from Theorem 1 of § 49, V and the implication (i) $\Rightarrow$ (iii) follows from Theorem 1. It remains to show that (iii) $\Rightarrow$ (ii). So, let $f\colon F \to \mathscr{S}_{n-1}$ be a con-

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(¹) For $n = 2$, the hypothesis of the set $F_0 \cap F_1$ being empty may be replaced by the hypothesis of its connectedness. See § 59, I, Theorem 6. For $n = 3$, Theorem 11 follows from the unicoherence of $\mathscr{S}_3$, see § 57, II, Theorem 3.

(²) S. Eilenberg, Fund. Math. 26 (1936), p. 104.

tinuous function such that f *irr non* $\simeq 1$. Since f *non* $\simeq 1$, there exist two components $R_0 \neq R_1$ of $\mathcal{S}_n - F$ such that $f|\mathrm{Fr}(R_j)$ *non* $\simeq 1$ for $j = 0, 1$ (compare II, 9, (ii) and II, 3, (ii)). It follows that $F = \mathrm{Fr}(R_j)$, because otherwise we would have $f|\mathrm{Fr}(R_j) \simeq 1$ by virtue of the condition f *irr non* $\simeq 1$.

THEOREM 3. *If F is a closed cut irreducible between p and q, no closed subset of dimension $\leqslant n-3$ separates F*(¹).

This is a direct consequence of Theorem 11 of Section II.

THEOREM 4. *Let C be a continuum in $\mathcal{S}_n$, R a component of $\mathcal{S}_n - C$ and $F = \overline{F} \subset \mathrm{Fr}(R)$. If $\dim F \leqslant n-3$ and if $C-F$ is connected, then so is $\mathrm{Fr}(R)-F$.*

Proof. Suppose that $\mathrm{Fr}(R)-F$ is not connected, so that

$$\mathrm{Fr}(R)-F = M \cup N, \quad (\overline{M} \cap N) \cup (\overline{N} \cap M) = 0, \quad M \neq 0 \neq N.$$

Let $K = \mathcal{S}_n - R$. It follows that

$$K = \mathrm{Fr}(R) \cup (\mathcal{S}_n - \overline{R}) \quad \text{and} \quad \mathcal{S}_n - \overline{R} = B_1 \cup B_2 \cup \ldots$$

is an (infinite, finite or empty) series of components of $\mathcal{S}_n - \overline{R}$. By Theorem 2 of § 49, V, the set $\mathrm{Fr}(B_i)$ is an irreducible cut of $\mathcal{S}_n$. The formula

$$\mathrm{Fr}(B_i) \subset \mathrm{Fr}(\overline{R}) \subset \mathrm{Fr}(R) = M \cup F \cup N,$$

implies that either $M \cap \mathrm{Fr}(B_i) = 0$ or $N \cap \mathrm{Fr}(B_i) = 0$, because the set F does not separate $\mathrm{Fr}(B_i)$ (by Theorem 3) since $\dim F \leqslant n-3$.

Denote by M^* the union of the set $M \cup F$ and of all B_i such that $N \cap \mathrm{Fr}(B_i) = 0$, and by N^* the union of the set $N \cup F$ and all other sets B_i. It follows that (compare § 49, III, Theorem 3)

$$K = M^* \cup N^*, \quad \overline{M^*} = M^*, \quad \overline{N^*} = N^* \quad \text{and} \quad M^* \cap N^* = F.$$

(¹) Compare P. Alexandroff, *Dimensionstheorie*, Math. Ann. 106 (1932), p. 227, and my paper in Ann. Soc. Polon. Math. 16 (1937), p. 220. Theorem 3 is due to P. Urysohn (compare his paper in Fund. Math. 7 (1925), p. 96 and 8 (1926), p. 312).

Since

$$M \cup N \subset \mathrm{Fr}(R) \subset C \subset \mathcal{S}_n - R = K,$$

it follows that

$$C = (C \cap M^*) \cup (C \cap N^*), \quad F = (C \cap M^*) \cap (C \cap N^*)$$
$$\text{and} \quad C \cap M^* \neq C \neq C \cap N^*,$$

which shows that $C - F$ is not connected.

IV. Invariants. The concepts of the closed cut of $\mathcal{S}_n$ and of the closed irreducible cut *are topological invariants*, because they are characterized by intrinsic properties (see Theorem 10 of Section II and Theorem 2, (iii) of Section III). It follows in particular that *a subset of $\mathcal{S}_n$ homeomorphic with $\mathcal{S}_{n-1}$ is an irreducible cut of $\mathcal{S}_n$.*

More precisely, the following theorem holds.

THEOREM 1. *The concept of a closed cut is invariant under transformations with small point-inverses.*

This is a consequence of Theorem 5 of § 54, V combined with Theorem 10, (iii) of Section II.

THEOREM 2. *Let $F = \overline{F} \subset \mathcal{S}_n - p - q$. If F is a cut between p and q, every set obtained from F by a deformation in $\mathcal{S}_n - p - q$, is a cut between p and q.*

This is a consequence of Theorem 1 of § 54, IV combined with Theorem 6 of Section II.

Since the concept of closed irreducible cut is an intrinsic invariant, the following is a consequence of Theorem 2.

THEOREM 3. *No closed irreducible cut admits a deformation onto a proper subset.*

Theorem 2 implies also the following statement.

THEOREM 4. *A closed cut between p and q cannot be deformed onto a single point without passing in the course of this deformation either through p or through q.*

Here is an other form of Theorem 4(1):

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(1) K. Borsuk, Monatsh. Math.-Phys. 38 (1931), p. 383. In a similar direction, see S. Gołąb, *Un théorème de balayage*, Fund. Math. 12 (1928), p. 4, and F. Leja, Fund. Math. 10 (1927), p. 421.

THEOREM 4a. *If F is a closed subset of $\mathscr{E}^n$, every bounded component of $\mathscr{E}^n - F$ is swept out in the course of every deformation of F onto a single point.*

Proof. It may be assumed in Theorem 4 that p is an arbitrary point of a bounded component and that q is the point at infinity.

THEOREM 5. *Every set $F = \bar{F} \subset \mathscr{S}_n - p - q$, which is not a cut between p and q, can be reduced to a point by a deformation effected in $\mathscr{S}_n - p - q$.*

Proof. Let A be a polygonal arc $pq \subset \mathscr{S}_n - F$. Then the set $\mathscr{S}_n - A$ is contractible in itself since it is a homeomorphic image of $\mathscr{E}^n$ (compare Theorem 8 of Section I).

THEOREM 6. *The concept of a separator is an intrinsic invariant; in other words, if A and B are two homeomorphic subsets of $\mathscr{S}_n$ and $\mathscr{S}_n - A$ is connected, then so is $\mathscr{S}_n - B$.*

Proof. If $\mathscr{S}_n - B$ is not connected, there exists by Theorem 7 of § 49, IV, a closed set $F \subset B$ such that the conditions $F \subset H = \bar{H} \subset B$ imply that H is a cut of $\mathscr{S}_n$. Since the concept of a closed cut is an intrinsic invariant, it follows by the same Theorem that A is a separator of $\mathscr{S}_n$.

Remark. The concept of a (non-closed) cut of $\mathscr{S}_2$ is not an intrinsic invariant.

This is seen on the following example.

$$A = \mathop{E}_{xy}\left\{\left(y = \sin\frac{1}{x}\right)(0 < |x| < 1)\right\} \cup (0, 1) \cup (0, -1),$$

$$B = \mathop{E}_{xy}\left\{\left(y = \sin\frac{1}{x}\right)(0 < x < 1)\right\} \cup$$
$$\cup \mathop{E}_{xy}\left\{\left(y = x + \sin\frac{1}{x}\right)(0 < x < 1)\right\} \cup (0, 1) \cup (0, -1).$$

LEMMA 7. *If G is a bounded open subset of $\mathscr{E}^n$, there is no retraction of $\bar{G}$ onto* $\mathrm{Fr}(G)$(1).

(1) K. Borsuk, Fund. Math. 17 (1931), p. 161.

Proof. It is admissible to assume that $G \subset \mathcal{Q}_n$ and that the point 0 belongs to G. Let $r\colon \bar{G} \to \mathrm{Fr}(G)$ be a retraction. Setting

$$f(x) = \begin{cases} r(x):|r(x)| & \text{for} \quad x \in \bar{G}, \\ x:|x| & \text{for} \quad x \in \mathcal{Q}_n - G, \end{cases}$$

we define a retraction of $\mathcal{Q}_n$ onto $\mathcal{S}_{n-1}$, contradicting Theorem 2 of § 28, III.

THEOREM 8. *If A is a retract of a compact subset F of $\mathcal{E}^n$, the number of components of $\mathcal{E}^n - A$ does not exceed the number of components of $\mathcal{E}^n - F$.*

If A is a deformation retract of F, these numbers are equal.

Proof. Let $r\colon F \to A$ be a retraction. Let R be a component of $\mathcal{E}^n - A$. Then $\mathrm{Fr}(R) \subset A$. It follows that $R - F \neq 0$, because, otherwise, $r|\bar{R}$ would be a retraction of $\bar{R}$ onto $A \cap \bar{R} = \mathrm{Fr}(R)$, contradicting Theorem 7.

Thus, a point $p_R \in R - F$ can be assigned to R. Let Q_R be the component of $\mathcal{E}^n - F$ containing p_R. Since $\mathcal{E}^n - F \subset \mathcal{E}^n - A$, it follows that $Q_R \subset R$, which proves the first part of Theorem 8.

The second one follows from the first by Theorem 2.

THEOREM 9. *If A is a compact absolute neighbourhood retract lying in $\mathcal{E}^n$, the number of regions in $\mathcal{E}^n - A$ is finite.*

If A is an absolute retract, this number is 1, i.e. A does not cut the space.

Proof. Since the set A is a retract of a bounded open subset of $\mathcal{E}^n$, A is a retract of a polyhedron and so of a compact set which cuts the space into a finite number of regions.

It may be assumed that this polyhedron is an n-dimensional cube if A is an absolute retract.

THEOREM 10. *For the subsets of $\mathcal{S}_n$, the concept of an interior point is an intrinsic invariant, and so is the concept of an open set and of a boundary set*(1).

(1) Theorem established by Schönflies for $n = 2$; compare Jahresber. D. Math. Ver. 1908. For the general case, compare H. Lebesgue, *Sur les correspondances entre les points de deux espaces*, Fund. Math. 2 (1921), p. 270 and *Sur le théorème de Schönflies*, *ibid.* 6 (1924), p. 96, E. Sperner, Abh. Math. Seminar Hamburg 6 (1928), p. 265, and S. Eilenberg, Fund. Math. 26 (1936), p. 94.

Proof. Let $A \subset \mathscr{S}_n$, G open, $p \in G \subset A$ and h a homeomorphism defined on A. We have to show that $h(p)$ is an interior point of $h(A)$.

Let F be a closed set and P and R two (non-empty) regions such that

$$F \subset G-p, \quad \mathscr{S}_n - F = P \cup R, \quad p \in P \subset G \quad \text{and} \quad P \cap R = 0 \tag{1}$$

(F may be defined for instance as the intersection of $\mathscr{S}_n$ with a sufficiently small sphere with the center p).

Since F is a cut of $\mathscr{S}_n$ (between P and R), then so is $h(F)$ (compare Theorem 6); therefore, there exist two open sets M and N such that

$$\mathscr{S}_n - h(F) = M \cup N, \quad M \cap N = 0, \quad M \neq 0 \neq N.$$

Since the set $h(P)$ is connected, then either $h(P) \subset M$ or $h(P) \subset N$. Assume that

$$h(P) \subset M, \quad \text{hence} \quad N \cap h(P) = 0. \tag{2}$$

It follows that

$$\mathscr{S}_n - h(F \cup P) = \mathscr{S}_n - [h(F) \cup h(P)] = [M - h(P)] \cup [N - h(P)]$$
$$= [M - h(P)] \cup N.$$

The set $F \cup P$ is not a cut (because the region R is its complement), and $h(F \cup P)$ is it neither. In other words, the set $[M - h(P)] \cup N$ is connected. Since the sets $M - h(P)$ and N are separated (because so are M and N), one of them is empty. Since $N \neq 0$, it follows that $M - h(P) = 0$, which implies $M \subset h(P)$ and hence $M = h(P)$ by (2). Thus the set $h(P)$ is open.

Since $h(p) \in h(P) \subset h(G) \subset h(A)$, $h(p)$ is an interior point of $h(A)$.

Remark. *For locally connected continua $\mathscr{X}$ the invariance of the concept of a closed cut implies the invariance of the concept of an interior point.*

Proof. Since $\mathscr{X}$ contains a point which does not cut it (compare § 47, IV, Theorem 5), the invariance of the cut implies that no point of $\mathscr{X}$ is a cut. So there exists (compare § 50, III, Theorem 1) a closed set H such that $p \in \operatorname{Int}(H) \subset G$ and that the set $R = \mathscr{X} - H$ is a region. Let P denote the component of the point p in $\operatorname{Int}(H)$

and let $F = H - P$; then conditions (1) are satisfied and the preceding proof remains valid (replacing $\mathcal{S}_n$ by $\mathcal{X}$).

Let us add without proof the following statement[1].

THEOREM 11. *Every countable set dense in $\mathcal{E}^n$ is topologically equivalent to the set of rational points of $\mathcal{E}^n$.*

It follows that

THEOREM 12. *In $\mathcal{E}^n$ every boundary set A has dimension $< n$.*

Proof. The argument reduces to the case where A consists of points every of which has at least one irrational coordinate. Denoting by A_k the set of points of $\mathcal{E}^n$ which have k rational coordinates and $n-k$ irrational ones, we infer that

$$A = A_0 \cup \ldots \cup A_{n-1}, \quad \text{thus} \quad \dim A_j = 0\,[2],$$

therefore $\dim A \leqslant n-1$ by Theorem 3 of § 27, I.

V. Remarks connected with the Borsuk-Ulam Theorem. By this theorem (quoted in § 41, VIII), known also as the theorem on antipodes, *for every continuous function $f\colon \mathcal{S}_n \to \mathcal{E}^n$ there exists a point $p \in \mathcal{S}_n$ such that $f(p) = f(-p)$*[3].

Let us quote (without proof) some theorems concerning similar problems.

1. For every continuous function $f\colon \mathcal{S}_n \to \mathcal{E}$ there exists a system of n orthogonal points $p_1, \ldots, p_n$ such that $f(p_1) = f(-p_1) = \ldots = f(p_n) = f(-p_n)$[4].

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(1) See, for instance, K. Menger, *Dimensionstheorie*, p. 263, or Hurewicz–Wallman, *Dimension Theory*, p. 44.

(2) *Ibid.* p. 147, respectively p. 29.

(3) For a simple proof of that theorem in the case $n = 2$, see § 57, I. In that case the theorem on antipodes can be interpreted in the following way: at every moment there exist on the earth surface two antipodal points at which the temperature is the same and the pressure is the same.

(4) See C. T. Yang, *On theorems of Borsuk–Ulam, Kakutani–Yamade–Yujobo and Dyson*, I, Ann. of Math. 60 (1954), pp. 262–282.

Using homological methods Yang has generalized his theorem and the theorem on antipodes. Compare also G. R. Livesay, *On a theorem of F. J. Dyson*, Ann. of Math. 59 (1954), pp. 227–229; D. G. Bourgin, *Modern algebraic topology*, New York–London 1963, p. 338.

In the case of $n = 2$ this theorem was proved earlier by F. J. Dyson(¹).

2. For every continuous function $f\colon \mathscr{S}_n \to \mathscr{E}$ there exists a system of $n+1$ orthogonal points $p_0, \ldots, p_n$ such that $f(p_0) = \ldots = f(p_n)$(²).

3. If $\mathscr{S}_n$ is covered with $n+1$ closed sets, at least one of these sets contains an antipodal pair of points(³).

The aim of other generalizations of the above-mentioned theorems is:

(i) to replace the sphere $\mathscr{S}_n$ by a more general space (an antipodal transformation being understood as a continuous involution)(⁴);

(¹) *Continuous functions defined on spheres*, Ann. of Math. 54 (1951), pp. 534–536. A simple proof of Dyson's theorem has been given by K. Zarankiewicz, *Un théorème sur l'uniformisation des fonctions continues et son application à la démonstration du théorème de F. J. Dyson sur les transformations de la surface sphérique*, Bull. Acad. Pol. Sci. 2 (1954), pp. 117–120.

(²) H. Yamabe and Z. Yujobo, *On the continuous functions defined on a sphere*, Osaka Math. Journ. 2 (1950), pp. 19–22. In the case $n = 2$ this theorem has been proved by S. Kakutani, *A proof that there exists a circumscribing cube around any bounded closed convex set in R^3*, Ann. of Math. 43 (1942), pp. 739–741. Kakutani's theorem has served as the starting point for many investigations (see the formerly quoted papers). The object of a generalization of Kakutani's theorem is to extend it to arbitrary triangles (without the hypothesis of orthogonality of vertices; compare a problem of H. Steinhaus which is mentioned in Colloq. Math. 1 (1947), p. 30); see E. E. Floyd, *Real valued mappings of spheres*, Proc. Amer. Math. Soc. 6 (1955), pp. 957–959. Compare also A. de Mira Fernandes, *Funzioni continue sopra una superficie sferica*, Portug. Math. 5 (1946), pp. 132–134, and D. G. Bourgin, *Some mappings theorems*, Rendic. di Matem. 15 (1956), pp. 177–189.

(³) K. Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. 20 (1933), p. 177.

(⁴) In what concerns the generalization of the theorem on the antipodes, see (in addition to the quoted paper of C. T. Yang) J. W. Jaworowski, *On antipodal sets on the sphere and on continuous involutions*, Fund. Math. 43 (1956), pp. 241–254; M. A. Krasnosielskii, Dokl. Akad. Nauk SSSR 101 (1955), p. 401. For the case where $\mathscr{S}_n$ is replaced by an n-dimensional closed manifold, see H. Hopf, *Eine Verallgemeinerung bekannter Abbildungs- und Überdeckungssätze*, Portug. Math. 4 (1943–45), pp. 129–139. Compare also D. G. Bourgin, *On some separation and mapping theorems*, Comment. Math. Helvet. 29 (1955), pp. 199–214; Ph. Bacon, Can. J. M. 18, p. 492.

(ii) to replace the (one-valued) transformation f by a multi-valued one (1);

(iii) to apply these theorems (e.g. the theorem on antipodes) to topological linear spaces (of infinite dimension) (2);

(iv) to extend the theorems on the space $(\mathscr{E}^n)^{\mathscr{S}_n}$ to the spaces $(\mathscr{E}^k)^{\mathscr{S}_n}$ with $k \leqslant n$ (problem of B. Knaster) (3).

§ 60. Quantitative problems. Cohomotopic multiplication Duality theorems

I. Introduction. Let us set $\mathscr{P}_n = \mathscr{E}^n - (0)$, $X \subset \mathscr{E}^n$ and $n \geqslant 2$. The cohomotopic multiplication, introduced by Borsuk (for X compact), is an operation on the elements of $\mathfrak{C}(\mathscr{P}_n^X)$ (the set of components of $\mathscr{P}_n^X$) such that $\mathfrak{C}(\mathscr{P}_n^X)$ becomes a topological group.

In the particular case where $n = 2$, the multiplication of elements of $\mathfrak{C}(\mathscr{P}_2^X)$ is induced by the multiplication of functions $f_1, f_2 \in \mathscr{P}_2^X$ (compare § 62, I) and that one in turn — by the multiplication of complex numbers:

$$(f_0 = f_1 \cdot f_2) \equiv [f_0(x) = f_1(x) \cdot f_2(x) \textit{ for any } x \in X], \tag{1}$$

$$(\Gamma_0 = \Gamma_1 \cdot \Gamma_2) \equiv [\textit{there exist } f_1 \in \Gamma_1, f_2 \in \Gamma_2 \textit{ such that } (f_1 \cdot f_2) \in \Gamma_0], \tag{2}$$

similarly, the division is induced by the division of non-zero complex numbers.

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(1) Compare J. W. Jaworowski, *Theorem on antipodes for multi-valued mappings and a fixed point theorem*, Bull. Acad. Pol. Sci. 4 (1956), pp. 187–192; A. Granas, *Theorem on antipodes and theorem on fixed points for a certain class of multi-valued mappings in Banach spaces*, *ibid.* 7 (1959), pp. 271–275; A. Granas and J. W. Jaworowski, *Some theorems on multi-valued mappings of subsets of the Euclidean space*, *ibid.*, pp. 277–283.

(2) Compare M. A. Krasnosielskii, Dokl. Akad. Nauk SSSR 73 (1950), p. 13; M. Altman, *An extension to locally convex spaces of Borsuk's theorem on antipodes*, Bull. Acad. Pol. Sci. 6 (1958), pp. 271–275; A. Granas *loc. cit.* and *Extension homotopy theorem in Banach spaces and some of its applications to the theory of nonlinear equations*, *ibid.* 7 (1959), pp. 387–394.

(3) Colloq. Math. 1 (1947), Problem 4, p. 30. For partial solutions, see G. R. Livesay, *On maps of the three-sphere into the plane*, Michigan Math. Journ. 4 (1957), pp. 157–159, and D. G. Bourgin, *Deformation and mapping theorems*, Fund. Math. 46 (1959), pp. 258–303.

If $n > 2$, this procedure cannot be applied since there is no multiplication of points in $\mathscr{E}^n$. However, formula (1) can be applied in the particular case where for every $x \in X$ either $f_1(x) = 1$ or $f_2(x) = 1$ (the case where the functions f_1 and f_2 are said to be "multipliable") and where we assume that

$$1 = (1, 0, 0, \ldots, 0) \quad \text{and} \quad p \cdot 1 = p = 1 \cdot p \quad \text{for} \quad p \in \mathscr{E}^n;$$

the division $1 : f(x)$ is derived from the following rule

$$\text{if} \quad p = (x_1, x_2, \ldots, x_n), \quad \text{then}$$
$$1 : p = (x_1, x_2, \ldots, x_{n-1}, -x_n) : |p|^2$$

(clearly this division depends on n).

In the sequel we are going to show how the multiplication of the elements of $\mathfrak{C}(\mathscr{P}_n^X)$ can be derived from the multiplication of multipliable functions (1).

Remark. If X is compact, the components of $\mathscr{P}_n^X$ are arcwise connected (Theorem 3 of § 53, III), in other words, they coincide with the homotopy classes. In that case the group $\mathfrak{C}(\mathscr{S}_n^X)$ is called the *n-th cohomotopy group of X* (denoted by $\pi^n(X)$) (2).

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(1) In this exposition we follow the general idea of K. Borsuk and extend it to sets $X \subset \mathscr{E}^n$ not necessarily compact.

See of that author *Sur les groupes des classes des transformations continues*, C. R. Paris 202 (1936), p. 1400, and *Set-theoretical approach to the disconnection theory of the Euclidean spaces*, Fund. Math. 37 (1950), pp. 217–241.

The ideas of Borsuk have been developed by E. Spanier, *Borsuk's cohomotopy groups*, Ann. of Math. 50 (1949), pp. 203–245, K. Morita, *Cohomotopy groups for fully normal spaces*, Sci. Reports Tokyo Bunrika Daigaku S. A. 4 (1953), pp. 251–261, A. Granas, *On the theory of the cohomotopy groups of Borsuk* (Russian), Fund. Math. 44 (1957), pp. 159–164.

With regard to some generalizations, see K. Gęba, *Sur les groupes de cohomotopie dans les espaces de Banach*, C. R. Paris 254 (1962), p. 3293, and J. W. Jaworowski, *Some remarks on Borsuk generalized cohomotopy groups*, Fund. Math. 50 (1961–62), pp. 257–264, and *Generalized cohomotopy groups as limit groups*, *ibid.*, pp. 393–402.

For historical reasons, let us mention H. Freudenthal, *Die Hopfsche Gruppe*, Compos. Math. 2 (1935), pp. 134–162.

(2) For the relations of the group $\pi^n(X)$ to the nth cohomology group, see E. Spanier, *loc. cit.*, and F. P. Peterson, *Some results on cohomotopy groups* and *Generalized cohomotopy groups*, Amer. Journ. Math. 78 (1956), pp. 243–280.

Furthermore, if X is compact, $\mathfrak{E}(\mathscr{P}_2^X)$ coincides with the quotient group of $\mathscr{P}_2^X$ by the group of functions homotopic to the unity (compare § 62, I). See also Remark 2 of Section VII.

II. Formulation of the problem. Let Ω be a set of arbitrary elements and $f\cdot g$ (or briefly fg) an operation defined on some pairs f, g of elements of Ω. In all cases where this operation can be performed (i.e. where the elements in question are "multipliable") the following usually required conditions are fulfilled(¹):

(α) $fg \in \Omega$,

(β) $fg = gf$,

(γ) $f(gh) = (fg)h$,

(δ) there exists an element 1 such that $f\cdot 1 = f$.

Next assume that $f \approx g$ is an equivalence relation defined for every pair of elements of Ω, that $1:f$ is a mapping of Ω onto Ω, and that the following conditions are fulfilled:

(i) *to every pair f, g there corresponds a multipliable pair f', g' such that $f' \approx f$ and $g' \approx g$; and generally, to every triple f, g, h there corresponds a triple f', g', h' for which there exist all the products indicated by condition* (γ) *and that $f' \approx f$, $g' \approx g$, $h' \approx h$;*

(ii) *if $f_0 \approx g_0$ and $f_1 \approx g_1$, then $f_0 f_1 \approx g_0 g_1$, provided that these products exist;*

(iii) *for every f there exists a pair g, h such that*

$$g \approx f, \quad h \approx 1:f, \quad gh \approx 1;$$

(iv) *$f \approx g$ implies that $(1:f) \approx (1:g)$.*

Under these conditions, *the multiplication fg induces a multiplication of equivalence classes so that the set of these classes, denoted by $\mathfrak{E}(\Omega)$, is an abelian group.*

More precisely, the multiplication of equivalence classes is defined in the following way.

$$(\Gamma_0 = \Gamma_1\cdot\Gamma_2) \equiv [\textit{there exist } f_1 \in \Gamma_1, f_2 \in \Gamma_2 \textit{ such that } (f_1\cdot f_2) \in \Gamma_0]. \quad (1)$$

(¹) Compare the concept of "groupoid" by H. Brandt, *Über eine Verallgemeinerung des Gruppenbegriffes*, Math. Ann. 96 (1927), pp. 360–366.

Compare also the concept of the multiplicative system by S. Eilenberg and N. Steenrod, *op. cit.*, p. 108.

The unity of this group is the class which contains the unity of Ω. The element $1:\Gamma$ is the class which contains $1:f$ where $f\epsilon\Gamma$.

We omit the easy proof of this theorem.

Notations. For $f\epsilon\ \Omega$ denote by $\hat{f}$ the equivalence class containing f, i.e.

$$f\epsilon\hat{f}\epsilon\mathfrak{C}(\Omega).$$

We assume the following notation

$$f_0 \approx f_1\cdot f_2 \quad \text{if} \quad \hat{f}_0=\hat{f}_1\cdot\hat{f}_2, \tag{2}$$

$$f_1\cdot f_2 \approx f_3\cdot f_4 \quad \text{if} \quad \hat{f}_1\cdot\hat{f}_2=\hat{f}_3\cdot\hat{f}_4 \tag{3}$$

(and similarly for an arbitrary number of factors), and also in the case where the products are not defined for the pairs $(f_1\cdot f_2)$, (f_3, f_4), and so on.

It is easy to show that

1) to every pair $f, g\epsilon\ \Omega$ there corresponds an element $h\epsilon\ \Omega$ such that $h\approx f\cdot g$,

2) the relations (β) and (γ) hold for all elements of Ω when the symbol "$=$" is replaced by "$\approx$",

3) statement (ii) holds without respect to the existence of involved products; in other words, the equivalences can be multiplied (and also divided),

4) $f\cdot(1:f)\approx 1$.

Assuming that $\Omega=\mathscr{P}_n^X$, we can easily show that conditions (α) through (δ) are fulfilled, the multiplication of f and g being understood in the sense of the multiplication of multipliable functions (defined in Section I). On the other hand, let us agree that "$f\approx g$" means that the functions f and g belong to the same component of $\mathscr{P}_n^X$ (i.e. to the same element of $\mathfrak{C}(\mathscr{P}_n^X)$). Our problem is to show that if the multiplication $f\cdot g$ and the relation $f\approx g$ are understood that way, conditions (i) through (iv) are satisfied for each $X\subset\mathscr{E}^n$ ($n\geqslant 2$).

In particular, conditions (i) and (ii) can be formulated in the following way (the space $\mathscr{S}_n^X$ with $X\subset\mathscr{E}^{n+1}$ being substituted for $\mathscr{P}_n^X$ which does not mean any essential difference, compare V, Theorem 6). Let us agree that

$$\mathscr{S}_n \vee \mathscr{S}_n = [\mathscr{S}_n \times (1)] \cup [(1) \times \mathscr{S}_n], \tag{4}$$

$$\mathscr{S}_n \vee \mathscr{S}_n \vee \mathscr{S}_n = [\mathscr{S}_n \times \mathscr{S}_n \times (1)] \cup [\mathscr{S}_n \times (1) \times \mathscr{S}_n] \cup [(1) \times \mathscr{S}_n \times \mathscr{S}_n], \tag{5}$$

$$\mathscr{U}_n = [\mathscr{S}_n \times (1) \times (1)] \cup [(1) \times \mathscr{S}_n \times (1)] \cup [(1) \times (1) \times \mathscr{S}_n]. \tag{6}$$

Then the existence of the product $f \cdot g$ with $f, g \in \mathscr{S}_n^X$ means that the "complex" function $\varphi = (f, g)$ is an element of $(\mathscr{S}_n \vee \mathscr{S}_n)^X$. Similarly, the hypothesis that the functions f, g and h are multipliable is equivalent to the condition $(f, g, h) \in \mathscr{U}_n^X$.

Thus conditions (i) and (ii) are equivalent (in the case considered) to the following ones

(i′) *to every* $\varphi \in (\mathscr{S}_n \times \mathscr{S}_n \times \mathscr{S}_n)^X$ *there corresponds a* $\varphi' \in \mathscr{U}_n^X$ *such that* $\varphi' \approx \varphi$,

(ii′) *if* $\varphi' = (f', g') \in (\mathscr{S}_n \vee \mathscr{S}_n)^X$, $\varphi'' = (f'', g'') \in (\mathscr{S}_n \vee \mathscr{S}_n)^X$ *and* $\varphi' \approx \varphi''$, *then* $f'g' \approx f''g''$.

III. Auxiliary homotopy properties.

THEOREM 1. *Let* $\mathscr{X}$ *and* $\mathscr{Y}$ *be metric separable and let* $\mathscr{Y}$ *be locally euclidean at some point; in other words,* $\mathscr{Y}$ *contains an open set* G *such that* $\bar{G}$ *is homeomorphic to* $\mathscr{I}^r$. *Let* $\dim \mathscr{X} \leqslant r-1$ *and let* $f\colon \mathscr{X} \to \mathscr{Y}$ *be a continuous function. Then there exists a continuous function* $f^*\colon \mathscr{X} \to (\mathscr{Y} - G)$ *such that*

$$f^*(x) = f(x) \quad \text{if} \quad f(x) \in \mathscr{Y} - G; \quad \text{hence} \quad f^* \simeq f.$$

Proof[1]. Let $\mathscr{X}^* = f^{-1}(\bar{G})$ and $F = \mathscr{X}^* - f^{-1}(G)$. Since $\dim(\mathscr{X}^* - F) \leqslant r-1$, there exists (compare § 28, IX, Theorem 1) a continuous extension $g\colon \mathscr{X}^* \to \bar{G}$ of $f|F$ such that $\dim g(\mathscr{X}^* - F) \leqslant r-1$. So, there exists a point $p \in G - g(\mathscr{X}^* - F)$. Let $r(y)$ denote the projection of the point $y \in G$ into $\bar{G} - G$ performed from the point p ($\bar{G}$ is identified with $\mathscr{I}^r$). We set $f^*(x) = rg(x)$ for $x \in \mathscr{X}^*$ and $f^*(x) = f(x)$ for $x \in \mathscr{X} - f^{-1}(G)$.

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[1] Compare also W. Hurewicz and H. Wallman, *Dimension theory*, Chapter VI, § 1.

THEOREM 2. *Under the same hypotheses about $\mathcal{X}$, $\mathcal{Y}$ and f, let $P = (p_1, \ldots, p_m)$ be a finite set of points at which $\mathcal{Y}$ is locally euclidean (in dimension r). Then there exists a continuous $f^*\colon \mathcal{X} \to (\mathcal{Y} - P)$ such that $f^* \simeq f$.*

Proof. This statement follows immediately from Theorem 1 for $m = 1$. Assuming that it holds for $m-1$, we derive it from Theorem 1 replacing $\mathcal{Y}$ by $\mathcal{Y} - (p_1, \ldots, p_{m-1})$.

IV. Auxiliary properties of the sphere. Using the notation (4) through (6) of Section II one obtains the following theorems (1).

THEOREM 1. *The set $\mathcal{S}_n \vee \mathcal{S}_n$ is a deformation retract of the set $\mathcal{S}_n \times \mathcal{S}_n - (p)$ for every point $p \in (\mathcal{S}_n \times \mathcal{S}_n) - (\mathcal{S}_n \vee \mathcal{S}_n)$.*

THEOREM 2. *The set $\mathcal{S}_n \vee \mathcal{S}_n \vee \mathcal{S}_n$ is a deformation retract of the set $\mathcal{S}_n \times \mathcal{S}_n \times \mathcal{S}_n - (p)$ for every point $p \in (\mathcal{S}_n \times \mathcal{S}_n \times \mathcal{S}_n) - (\mathcal{S}_n \vee \vee \mathcal{S}_n \vee \mathcal{S}_n)$.*

THEOREM 3 (2). *If F_1, F_2, F_3 are the three terms of the right-hand side of formula (5) of Section II, the set $\mathcal{U}_n$ is a deformation retract of the set $\mathcal{S}_n \vee \mathcal{S}_n \vee \mathcal{S}_n - (p_1, p_2, p_3)$ with $p_i \in F_i - \mathcal{U}_n$.*

Proof. Referring to Theorem 1, it is easy to show that $F_i \cap \cap (F_j \cup F_k)$ is a deformation retract of $F_i - p_i$ (where $i \neq j \neq k \neq i$). Since $F_i - p_i$ is an absolute neighbourhood retract, let (compare Theorem 6 of § 54, IV)

$$h_i\colon (F_i - p_i) \times \mathcal{I} \to (F_i - p_i)$$

be a continuous function such that

$$h_i(x, 0) = x, \qquad h_i(x, 1) \in F_i \cap (F_j \cup F_k) \quad \text{for every } x,$$
$$h_i(x, t) = x \quad \text{for} \quad x \in F_i \cap (F_j \cup F_k) \text{ and } t \in \mathcal{I}.$$

Set $h(x, t) = h_i(x, t)$ for $x \in F_i - p_i$; the required conclusion follows, since

$$\mathcal{U}_n = \bigcup F_i \cap (F_j \cup F_k) \quad \text{where} \quad i \neq j \neq k \neq i.$$

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(1) For the proof of Theorems 1 and 2 see E. Spanier, *loc. cit.* p. 209. Compare K. Borsuk, *Sur l'addition homologique des types de transformations continues en surfaces sphériques*, Ann. of Math. 38 (1937), p. 733.

(2) Compare K. Borsuk, *On the concept of dependence for continuous mappings*, Fund. Math. 43 (1956), p. 95.

V. The group $\mathfrak{C}(\mathcal{S}_n^X)$ for $\dim X \leqslant 2n-2$. In Theorem 1 (and in 3 and 4 as well) it will be possible to replace this inequality by the less restrictive condition

$$\dim X \leqslant 2n-1. \tag{1}$$

The notations of Section IV will be used. According to the theorem of Section II, it remains to show that conditions (i′), (ii′), (iii) and (iv) of Section II are fulfilled provided that $f \approx g$ means that f and g belong to the same element of $\mathfrak{C}(\mathcal{S}_n^X)$.

THEOREM 1. *Let* $\varphi = (f, g, h) \in (\mathcal{S}_n \times \mathcal{S}_n \times \mathcal{S}_n)^X$. *There exists a function* $\varphi' \in \mathcal{U}_n^X$ *such that* $\varphi' \simeq \varphi$.

Proof. Let $p \in (\mathcal{S}_n \times \mathcal{S}_n \times \mathcal{S}_n) - (\mathcal{S}_n \vee \mathcal{S}_n \vee \mathcal{S}_n)$. By Theorem 1 of Section III (where $r = 3n$), there exists a continuous function $\psi\colon X \to [\mathcal{S}_n \times \mathcal{S}_n \times \mathcal{S}_n - (p)]$ such that $\psi \simeq \varphi$. Since $\mathcal{S}_n \vee \mathcal{S}_n \vee \mathcal{S}_n$ is a deformation retract of $\mathcal{S}_n \times \mathcal{S}_n \times \mathcal{S}_n - (p)$ (by Theorem 2 of Section IV), there exists by Theorem 5 of § 54, IV, a continuous function $\psi^*\colon X \to \mathcal{S}_n \vee \mathcal{S}_n \vee \mathcal{S}_n$ such that $\psi^* \simeq \psi$; thus $\psi^* \simeq \varphi$.

Let $p_i \in F_i - \mathcal{U}_n$, $i = 1, 2, 3$. By Theorem 2 of Section III (where $r = 2n$) there exists a continuous function $\psi'\colon X \to [\mathcal{S}_n \vee \vee \mathcal{S}_n \vee \mathcal{S}_n - (p_1, p_2, p_3)]$ such that $\psi' \simeq \psi^*$. Finally, by Theorem 3 of Section IV it follows from Theorem 5 of § 54, IV, that there exists a continuous function $\varphi'\colon X \to \mathcal{U}_n$ such that $\varphi' \simeq \psi'$.

THEOREM 2. *Let* $\varphi = (f_0, f_1) \in (\mathcal{S}_n \vee \mathcal{S}_n)^X$, $\psi = (g_0, g_1) \in (\mathcal{S}_n \vee \mathcal{S}_n)^X$ *and* $\varphi \approx \psi$ *(relative to* $\mathcal{S}_n \times \mathcal{S}_n$*). Then* $f_0 f_1 \simeq g_0 g_1$ *(relative to* $\mathcal{S}_n$*).*

Proof. First assume that $\varphi \simeq \psi$. We will show that $f_0 f_1 \simeq g_0 g_1$. By hypothesis there exists a continuous function $h\colon X \times \mathcal{I} \to \mathcal{S}_n \times \mathcal{S}_n$ such that $h(x, 0) = \varphi(x)$ and $h(x, 1) = \psi(x)$. Let $p \in (\mathcal{S}_n \times \mathcal{S}_n) - (\mathcal{S}_n \vee \mathcal{S}_n)$. Since $\dim(X \times \mathcal{I}) \leqslant 2n-1$, there exists by Theorem 1 of Section III, a continuous $h^*\colon X \times \mathcal{I} \to [\mathcal{S}_n \times \times \mathcal{S}_n - (p)]$ with $h^*(x, t) = h(x, t)$ provided that $h(x, t) \in (\mathcal{S}_n \vee \mathcal{S}_n)$. It follows that $h^*(x, 0) = h(x, 0)$ and $h^*(x, 1) = h(x, 1)$. So, the functions φ and ψ are homotopic relative to the set $\mathcal{S}_n \times \mathcal{S}_n - (p)$, and therefore relative (compare § 54, I, Theorem 4a) to the set $\mathcal{S}_n \vee \mathcal{S}_n$ which is its retract (by Theorem 1 of Section IV).

Consequently, there exists a continuous function $\alpha\colon X \times \mathcal{I} \to \mathcal{S}_n \vee \mathcal{S}_n$ such that

$$\alpha(x, 0) = \varphi(x) = [f_0(x), f_1(x)],$$
$$\alpha(x, 1) = \psi(x) = [g_0(x), g_1(x)].$$

Let α^0 and α^1 be the coordinates of α, i.e.,

$$\alpha(x, t) = [\alpha^0(x, t), \alpha^1(x, t)].$$

Since the functions α^0 and α^1 are multipliable, we can set

$$\beta(x, t) = \alpha^0(x, t) \cdot \alpha^1(x, t).$$

Since the multiplication of points $p \cdot q$ is a continuous function on $\mathscr{S}_n \vee \mathscr{S}_n$, it follows that the function β: $X \times \mathscr{I} \to \mathscr{S}_n$ is continuous. Moreover,

$$\beta(x, 0) = f_0(x) \cdot f_1(x) \quad \text{and} \quad \beta(x, 1) = g_0(x) \cdot g_1(x),$$

which means that $f_0 \cdot f_1 \simeq g_0 \cdot g_1$.

Consider the general case where $\varphi \approx \psi$. If F is an arbitrary compact subset of X, it follows that $\varphi|F \approx \psi|F$, so that $\varphi|F \simeq \psi|F$, and therefore $(f_0 \cdot f_1)|F \simeq (g_0 \cdot g_1)|F$ as we have just shown. It follows that $f_0 \cdot f_1 \approx g_0 \cdot g_1$ by Corollary 2a of § 54, VIII.

THEOREM 3. *To every function $f \in \mathscr{S}_n^X$ there corresponds a pair of functions $(g, h) \in (\mathscr{S}_n \vee \mathscr{S}_n)^X$ such that*

$$g \simeq f, \quad h \simeq 1{:}f \quad \textit{and} \quad g \cdot h \simeq 1.$$

Proof. Let $\mathscr{S}_n^+$ and $\mathscr{S}_n^-$ denote as usual the two (closed) hemispheres of $\mathscr{S}_n$ and define

$$k(x) = \begin{cases} f(x) & \text{if} \quad f(x) \in \mathscr{S}_n^+, \\ 1{:}f(x) & \text{if} \quad f(x) \in \mathscr{S}_n^-. \end{cases}$$

Thus, k: $X \to \mathscr{S}_n$ is a continuous function since $p = 1{:}p$ for $p \in \mathscr{S}_n^+ \cap \mathscr{S}_n^- = \mathscr{S}_{n-1}$.

More precisely, k: $X \to \mathscr{S}_n^+$ is a continuous function since $p \in \mathscr{S}_n^-$ implies $(1{:}p) \in \mathscr{S}_n^+$. And since the set $\mathscr{S}_n^+$ is an absolute retract, it follows that $k \simeq 1$.

Let φ: $\mathscr{S}_n \to \mathscr{S}_n$ be such that $\varphi(\mathscr{S}_n^-) = 1$ and $\varphi(y) \simeq y$. Setting

$$g = \varphi(f) \quad \text{and} \quad h = \varphi(1{:}f)$$

we have

$$g \simeq f \quad \text{and} \quad h \simeq 1{:}f.$$

Next, (g, h): $X \to \mathscr{S}_n \vee \mathscr{S}_n$ is continuous. Because

$$[f(x) \in \mathscr{S}_n^+] \Rightarrow [(1{:}f(x)) \in \mathscr{S}_n^-] \Rightarrow [\varphi(1{:}f(x)) = 1] \Rightarrow [h(x) = 1],$$

$$[f(x) \in \mathscr{S}_n^-] \Rightarrow [\varphi f(x) = 1] \Rightarrow [g(x) = 1].$$

It remains to show that $gh \simeq 1$, and hence that $gh \simeq k$.

In view of $\varphi(y) \simeq y$, let $\alpha\colon \mathscr{S}_n \times \mathscr{I} \to \mathscr{S}_n$ be a continuous function such that

$$\alpha(y, 0) = \varphi(y), \qquad \alpha(y, 1) = y,$$

and let

$$u(x, t) = \begin{cases} \alpha[f(x), t] & \text{if} \quad f(x) \in \mathscr{S}_n^+, \\ \alpha\big[\big(1{:}f(x)\big), t\big] & \text{if} \quad f(x) \in \mathscr{S}_n^-. \end{cases}$$

Clearly $u\colon X \times \mathscr{I} \to \mathscr{S}_n$ is continuous. Moreover,

(i) if $f(x) \in \mathscr{S}_n^+$, then $u(x, 0) = \alpha[f(x), 0] = \varphi f(x) = g(x) = g(x) \cdot h(x)$ since $h(x) = 1$; furthermore $u(x, 1) = \alpha[f(x), 1] = f(x) = k(x)$;

(ii) if $f(x) \in \mathscr{S}_n^-$, then $u(x, 0) = \alpha\big[\big(1{:}f(x)\big), 0\big] = \varphi[1{:}f(x)] = h(x) = g(x) \cdot h(x)$ since $g(x) = 1$; moreover, $u(x, 1) = \alpha\big[\big(1{:}f(x)\big), 1\big] = 1{:}f(x) = k(x)$.

Therefore, in both cases $u(x, 0) = g(x) \cdot h(x)$ and $u(x, 1) = k(x)$, which shows that $gh \simeq k$.

THEOREM 4. *$(f \approx g)$ implies $(1{:}f) \approx (1{:}g)$.*

Proof. First consider the case where $f \simeq g$. Then there exists a continuous function $h\colon X \times \mathscr{I} \to \mathscr{S}_n$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. Set $k(x, t) = 1{:}h(x, t)$; it follows that k is a continuous function and $k\colon X \times \mathscr{I} \to \mathscr{S}_n$, $k(x, 0) = 1{:}f(x)$ and $k(x, 1) = 1{:}g(x)$. Thus $(1{:}f) \simeq (1{:}g)$.

In the general case where $f \approx g$, let F be a compact subset of X. It follows that $(f|F) \simeq (g|F)$, which shows, as we have just proved, that $(1{:}f)|F \simeq (1{:}g)|F$. Since F is an arbitrary compact subset of X, it follows by Corollary 2a of § 54, VIII, that $(1{:}f) \approx (1{:}g)$.

THEOREM 5. *If $\dim X \leqslant 2n-2$, then $\mathfrak{C}(\mathscr{S}_n^X)$ is an abelian group with respect to the cohomotopic multiplication of its elements defined by condition* II (1).

THEOREM 6. *In the preceding theorem, the space $\mathscr{S}_n$ can be replaced by $\mathscr{P}_{n+1}$.*

Proof. It is to be shown that this substitution can be done in Theorems 1 through 4.

1. To begin with, notice that for every continuous function $f\colon X \to \mathscr{P}_{n+1}$ we have (compare § 54, IV, Examples)

$$f \simeq \vec{f} \quad \text{where} \quad \vec{f}(x) = f(x) : |f(x)|. \tag{2}$$

Now, assume that $\varphi \in (\mathscr{P}^3_{n+1})^X$; it follows that $\varphi \simeq \vec{\varphi} \in (\mathscr{S}^3_n)^X$ and $\vec{\varphi} \simeq \varphi' \in \mathscr{U}^X_n$.

2. Let us notice first that, if $F \subset X$ and f_0 and f_1 are multipliable, then

$$(f_2 = f_0 \cdot f_1) \Rightarrow (f_2 | F) = (f_0 | F) \cdot (f_1 | F). \tag{3}$$

Let $\varphi = (f_0, f_1) \in (\mathscr{P}_{n+1} \vee \mathscr{P}_{n+1})^X$, $\psi = (g_0, g_1) \in (\mathscr{P}_{n+1} \vee \mathscr{P}_{n+1})^X$ and $\varphi \simeq \psi$; it follows that $\vec{\varphi} \simeq \vec{\psi}$. Therefore, by Theorem 2,

$$\vec{f_0} \cdot \vec{f_1} \simeq \vec{g_0} \cdot \vec{g_1}, \quad \text{i.e.} \quad \frac{f_0(x)}{|f_0(x)|} \cdot \frac{f_1(x)}{|f_1(x)|} \simeq \frac{g_0(x)}{|g_0(x)|} \cdot \frac{g_1(x)}{|g_1(x)|},$$

so that $f_0 \cdot f_1 \simeq g_0 \cdot g_1$.

Now consider the general case where we assume that $\varphi \approx \psi$ instead of assuming that $\varphi \simeq \psi$. Since for every compact subset F of X we have $\varphi | F \simeq \psi | F$, we infer that

$$(f_0 | F) \cdot (f_1 | F) \simeq (g_0 | F) \cdot (g_1 | F)$$

and $(f_0 \cdot f_1) | F \simeq (g_0 \cdot g_1) | F$ by (3). Hence $f_0 \cdot f_1 \approx g_0 \cdot g_1$ by Corollary 2a of § 54, VIII.

3 and 4. The proofs follow easily by (2).

VI. The group $\mathfrak{C}(\mathscr{P}^X_n)$ for $X \subset \mathscr{E}^n$ and $n \geqslant 2$. We shall prove in this section that $\mathfrak{C}(\mathscr{P}^X_n)$ is a *topological group*, the multiplication of its elements (called *cohomotopic multiplication*) being defined by formula (1) of Section II.

First, we shall show the following statement.

THEOREM 1. *$\mathfrak{C}(\mathscr{P}^X_n)$ is an abelian group.*

Proof. For $n \geqslant 4$ and an arbitrary subset X of $\mathscr{E}^n$, as well as for $n = 3$ and $\dim X \leqslant 2$, Theorem 1 follows from Theorem 6 of Section V (which requires that $\dim X \leqslant 2n-4$).

The proof reduces to show Theorem 2 of Section V for $n = 2$, X being supposed a 3-dimensional subset of $\mathscr{E}^3$.

First, consider the case where X is an *elementary continuum*, i.e., X has the following form

$$X = \mathscr{S}_3 - (R_0 \cup \ldots \cup R_m), \quad \bar{R}_i \cap \bar{R}_j = 0 \quad \text{for} \quad i \neq j, \tag{0}$$

where $R_0, \ldots, R_m$ are (3-dimensional) open balls.

Let C be the union of the surfaces $\mathrm{Fr}(R_1), \ldots, \mathrm{Fr}(R_m)$ and of pairwise disjoint arcs which join each of them with the following one. Let

$$\varphi = (f_0, f_1) \in (\mathcal{S}_2 \vee \mathcal{S}_2)^X, \quad \psi = (g_0, g_1) \in (\mathcal{S}_2 \vee \mathcal{S}_2)^X \quad \text{and} \quad \varphi \simeq \psi. \tag{1}$$

We have to show that

$$f_0 \cdot f_1 \simeq g_0 \cdot g_1. \tag{2}$$

Since $\dim C = 2$, homotopy (2) holds on C by Theorem 5 of Section V. Therefore it holds also on the whole continuum X because (see Theorem 1 of § 54, IV) C is its deformation retract[1].

Now consider the case where X is an arbitrary *compact* subset of $\mathcal{E}^3$. Assume that relations (1) hold. In order to establish (2) it is sufficient (as we have just proved) to show that there exist an elementary continuum $X^* \supset X$ and functions φ^* and ψ^* such that

$$\varphi \subset \varphi^* \in (\mathcal{S}_2 \vee \mathcal{S}_2)^{X^*}, \quad \psi \subset \psi^* \in (\mathcal{S}_2 \vee \mathcal{S}_2)^{X^*} \quad \text{and} \quad \varphi^* \simeq \psi^* \quad (\text{relative } \mathcal{S}_2 \times \mathcal{S}_2). \tag{3}$$

By Theorem 9 of § 59, II, each of the functions f_0 and f_1 has a continuous extension to the sphere $\mathcal{S}_3$ minus a finite number of points.

Therefore, there exists an elementary continuum $X^* \supset X$ such that

$$\varphi \subset \varphi' \in (\mathcal{S}_2 \times \mathcal{S}_2)^{X^*}. \tag{4}$$

The same holds for ψ. Moreover, since $\psi \simeq \varphi$ (by (1)), the function ψ can be extended without affecting this homotopy (see Theorem 3, of § 54, II), i.e. in the following way

$$\psi \subset \psi' \in (\mathcal{S}_2 \times \mathcal{S}_2)^{X^*} \quad \text{and} \quad \psi' \simeq \varphi'. \tag{5}$$

Let $p \in (\mathcal{S}_2 \times \mathcal{S}_2) - (\mathcal{S}_2 \vee \mathcal{S}_2)$. Since $\dim X^* = 3$ and $\dim(\mathcal{S}_2 \times \mathcal{S}_2) = 4$ and since $\varphi(X) \subset \mathcal{S}_2 \vee \mathcal{S}_2$, there exists by Theorem 1 of Section III (where $\mathcal{Y}$ is replaced by $\mathcal{S}_2 \times \mathcal{S}_2$, $\mathcal{X}$ by X^*, f by φ' and where $r = 4$) a continuous function $\varphi^*\colon X^* \to [\mathcal{S}_2 \times \mathcal{S}_2 - (p)]$

[1] For a simple proof of this statement see K. Borsuk, Fund. Math. 37, *op. cit.*, p. 229.

such that $\varphi \subset \varphi^* \simeq \varphi'$. Moreover, since $\mathscr{S}_2 \vee \mathscr{S}_2$ is a deformation retract of $\mathscr{S}_2 \times \mathscr{S}_2 - (p)$ (by Theorem 1 of Section IV), it can be assumed (compare § 54, IV, Theorem 5) that $\varphi^*(X^*) \subset \mathscr{S}_2 \vee \mathscr{S}_2$.

Therefore

$$\varphi \subset \varphi^* \in (\mathscr{S}_2 \vee \mathscr{S}_2)^{X^*}, \quad \varphi^* \simeq \varphi', \tag{6}$$

and similarly

$$\psi \subset \psi^* \in (\mathscr{S}_2 \vee \mathscr{S}_2)^{X^*}, \quad \psi^* \simeq \psi'. \tag{7}$$

Conditions (5) through (7) immediately yield (3).

Thus, our theorem is proved for compact sets X.

Finally, assume that X is an *arbitrary* subset of $\mathscr{E}^3$, that the first two conditions (1) are fulfilled, and that $\varphi \approx \psi$. Then $\varphi|F \simeq \psi|F$ for every compact subset F of X. As was just proved, we have

$$(f_0|F)\cdot(f_1|F) \simeq (g_0|F)\cdot(g_1|F),$$

thus

$$(f_0\cdot f_1)|F \simeq (g_0\cdot g_1)|F,$$

by formula (3) of Section V. And hence $f_0\cdot f_1 \approx g_0\cdot g_1$ by Corollary 2a of § 54, VIII.

THEOREM 2. *Let F be a compact subset of X. Put $R_F(\Gamma) = \Gamma|F$. Then the mapping R_F: $\mathfrak{C}(\mathscr{P}_n^{\mathscr{X}}) \to \mathfrak{C}(\mathscr{P}_n^F)$ is a homomorphism.*

In other words,

$$(\Gamma_1\cdot\Gamma_2)|F = (\Gamma_1|F)\cdot(\Gamma_2|F) \quad \textit{and} \quad (\Gamma_1:\Gamma_2)|F = (\Gamma_1|F):(\Gamma_2|F), \tag{8}$$

i.e. (compare the notations of II (2) and (3))

$$(f_1\cdot f_2)|F \approx (f_1|F)\cdot(f_2|F) \quad \textit{and} \quad (f_1:f_2)|F \approx (f_1|F):(f_2|F). \tag{9}$$

Proof. Let $g \in (\Gamma_1\cdot\Gamma_2)|F$. Hence there is $f \in (\Gamma_1\cdot\Gamma_2)$ such that $g = f|F$. Therefore $f \approx f_0 = f_1\cdot f_2$ where $f_1 \in \Gamma_1$ and $f_2 \in \Gamma_2$. It follows that (compare V (3))

$$f|F \simeq f_0|F = (f_1|F)\cdot(f_2|F), \quad \text{thus} \quad f|F \in (\Gamma_1|F)\cdot(\Gamma_2|F).$$

So, the inclusion of the left-hand side of (8) in the right one has been established.

Conversely, let $g \in (\Gamma_1|F) \cdot (\Gamma_2|F)$. Let f_1 and f_2 be two multipliable functions which belong to Γ_1 and to Γ_2 respectively. It follows that

$$g \approx (f_1|F)\cdot(f_2|F) = (f_1 \cdot f_2)|F, \quad \text{so that} \quad g \in (\Gamma_1 \cdot \Gamma_2|F).$$

This completes the proof of Theorem 2.

Now, let us recall the notation $S_{F_0F_1}$ used in § 54, IX (for F_0 and F_1 compact):

$$S_{F_0F_1}(\Delta) = \Delta|F_0 \quad \text{for} \quad F_0 \subset F_1 \quad \text{and} \quad \Delta \in \mathfrak{C}(\mathscr{P}_n^{F_1}).$$

By Theorem 2 the mapping $S_{F_0F_1}$: $\mathfrak{C}(\mathscr{P}_n^{F_1}) \to \mathfrak{C}(\mathscr{P}_n^{F_0})$ is a *homomorphism*. Moreover, by Theorem 1 of § 57, IX, this mapping is *continuous*.

This yields, by Theorem 5 of § 57, IX, the following theorem.

THEOREM 3. $\mathfrak{C}(\mathscr{P}_n^X) \underset{\text{gr.t.}}{\subset} \operatorname*{Lim}_{F,F_0\subset F_1} \{\mathfrak{C}(\mathscr{P}_n^F), S_{F_0F_1}\}$, *where F runs over all compact subsets of F and where $\underset{\text{gr.t.}}{\subset}$ denotes an isomorphic and homeomorphic imbedding.*

Namely the mapping $R = \{R_F\}$ is the required iso-homeomorphism; it assigns to each $\Gamma \in \mathfrak{C}(\mathscr{P}_n^X)$ the element $\{\Gamma|F\}$ of the inverse limit under consideration.

THEOREM 4. *$\mathfrak{C}(\mathscr{P}_n^X)$ is a topological group.*

This follows from Theorem 1 of § 55, VII (because the above inverse limit is a subgroup of the product $\underset{F}{P}\mathfrak{C}(\mathscr{P}_n^F)$ of topological (discrete) groups).

THEOREM 5. *Let X be locally compact, and hence*

$$X = F_1 \cup F_2 \cup \ldots \quad \textit{where } F_i \textit{ is compact and } F_i \subset \operatorname{Int}_X(F_{i+1}).$$

Then

$$\mathfrak{C}(\mathscr{P}_n^X) \underset{\text{gr.t.}}{=} \operatorname*{Lim}_{i,i\leqslant k} \{\mathfrak{C}(\mathscr{P}_n^{F_i}, S_{F_iF_k}\}.$$

Namely the mapping R is onto.

This follows from Theorem 6 of § 54, IX.

VII. The group $\mathfrak{C}(\mathscr{P}_n^X)$ where X is a compact subset of $\mathscr{E}^n$. The relation $\approx$ will have the same meaning as in Section II.

THEOREM 1. *For each continuous function f: $\mathscr{S}_{n-1} \to \mathscr{P}_n$, there is an integer k such that $f(x) \approx x^k$.*

Besides, *if* $x^m \approx 1$, *then* $m = 0$ [1].

It follows immediately

THEOREM 2. *If X is the surface of an n-dimensional ball with the center p, then $f(x) \approx (x-p)^k$ for every continuous function $f\colon X \to \mathcal{P}_n$.*

Moreover, $(x-p)^m \approx 1$ *only if* $m = 0$.

THEOREM 3. *Let X be an arbitrary compact subset of $\mathcal{E}^n$ and let $R_0, R_1, \ldots$ be the sequence (finite or infinite) of components of $\mathcal{S}_n - X$. Let $p_i \in R_i$ for $i = 1, 2, \ldots$ and let $\infty \in R_0$.*

To every function $f \in \mathcal{P}_n^X$ there corresponds a finite system of integers $k_1, \ldots, k_m$ such that ($\hat{f}$ denotes the component of $\mathcal{P}_n^X$ which contains f)

$$\hat{f} = \widehat{(x-p_1)}^{k_1} \cdot \ldots \cdot \widehat{(x-p_m)}^{k_m}. \tag{1}$$

In other terms [2],

$$f(x) \approx (x-p_1)^{k_1} \cdot \ldots \cdot (x-p_m)^{k_m}. \tag{2}$$

Moreover, *if* $\mathcal{S}_n - X = R_0$, *then* $f \approx 1$ (and the group $\mathfrak{C}(\mathcal{P}_n^X)$ reduces to the neutral element).

Proof. Since the function f can be extended to $\mathcal{S}_n$ minus a finite number of points p_i (Theorem 9 of § 59, II), and hence to an elementary continuum, the argument reduces (in view of VI (9)) to the case where X is an elementary continuum, i.e. where X has form (0) of Section VI (replacing 3 by n).

Let us set $X = C_m$ $(m \geqslant 0)$ and proceed by induction with respect to m.

1) If $m = 0$, then $\mathcal{S}_n - X = R_0$ and hence the complement of X is connected. Then $f \simeq 1$ by Theorem 10 of § 59, II.

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[1] For the proof of the first part, see K. Borsuk, *ibid.*, p. 231. For the second one, which essentially leans on the Brouwer theorem (on the degree of a mapping), see e.g. Eilenberg–Steenrod, *op. cit.*, p. 304, or P. J. Hilton, *An introduction to homotopy theory*, Cambridge Tracts 43, 1953, p. 29, Theorems 2 through 6. All known proofs of that part of the Brouwer theorem use algebraic topology.

[2] Compare my paper *Sur l'extension de la notion de fonction rationnelle à l'espace euclidien n-dimensionnel*, Bull. Acad. Pol. Sci. 6 (1958), pp. 281–287. Compare A. Granas, *On local disconnection of Euclidean spaces*, Fund. Math. 41 (1954), p. 46.

2) Let $m > 0$ and assume that the theorem is true for $m-1$. Let

$$A_m = \bar{R}_m - R_m \quad \text{and} \quad C_{m-1} = \mathscr{S}_n - (R_0 \cup \ldots \cup R_{m-1}).$$

By Theorem 2, there exists an integer k_m such that

$$f(x) \approx (x - p_m)^{k_m} \quad \text{on } A_m. \tag{3}$$

Define $\Gamma = \hat{f} \cdot \widehat{(x - p_m)}^{-k_m}$. Let $g \in \Gamma$. In other words,

$$g(x) \approx f(x) \cdot (x - p_m)^{-k_m} \quad \text{on } C_m. \tag{4}$$

It follows by (3) that $g \approx 1$ on A_m, and therefore $g \simeq 1$ on A_m. Consequently, there exists by Theorem 9 of § 59, II, a continuous function $g^*\colon C_m \cup R_m \to \mathscr{P}_n$ such that $g \subset g^*$. Since $C_m \cup R_m = C_{m-1}$, it follows that $g^*\colon C_{m-1} \to \mathscr{P}_n$. Since our theorem is true for $m-1$ (by hypothesis), we have

$$g^*(x) \approx (x - p_1)^{k_1} \cdot \ldots \cdot (x - p_{m-1})^{k_{m-1}} \quad \text{on } C_{m-1}. \tag{5}$$

Since $g^*(x) = g(x)$ for $x \in C_m$, it follows from (5) and from (9) of Section VI that

$$g(x) \approx (x - p_1)^{k_1} \cdot \ldots \cdot (x - p_{m-1})^{k_{m-1}} \quad \text{on } C_m. \tag{6}$$

Thus, (2) is obtained by dividing (4) by (6).

THEOREM 4. *Let F be a closed subset of X which separates every pair of points of $\mathscr{S}_n - X$ which are separated by X. Let $f\colon X \to \mathscr{P}_n$ be a continuous function. If $f \simeq 1$ on F, then $f \simeq 1$ on X.*

In particular, *if $f \simeq 1$ on* $\operatorname{Fr}(X)$, *then $f \simeq 1$ on X.*

Proof. Let R be an arbitrary component of $\mathscr{S}_n - X$. Since $\mathscr{S}_n - X \subset \mathscr{S}_n - F$, there exists a component T of $\mathscr{S}_n - F$ such that $R \subset T$. It follows that

$$R = T - X. \tag{7}$$

Because, if we would suppose that $p \in T - X - R$ and $q \in R$, then the points p and q would be separated by X but not by F, contrary to our hypothesis.

Since $f|F \simeq 1$ by hypothesis, there exists (according to Theorem 7 of § 54, II) a function g such that

$$(f|F) \subset g \in \mathscr{P}_n^{\mathscr{S}_n}. \tag{8}$$

Define

$$h(x) = \begin{cases} g(x) & \text{for} \quad x \in \mathcal{S}_n - T, \\ f(x) & \text{for} \quad x \in X \cap \overline{T}. \end{cases} \tag{9}$$

These conditions are consistent, because

$$(\mathcal{S}_n - T) \cap (X \cap \overline{T}) \subset \overline{T} - T = \mathrm{Fr}(T) \subset F.$$

Since, on the other hand (compare (7))

$$(\mathcal{S}_n - T) \cup (X \cap \overline{T}) = (\mathcal{S}_n - T) \cup X = \mathcal{S}_n - R,$$

it follows that $h: \mathcal{S}_n - R \to \mathcal{P}_n$ is continuous and, since the complement of $\mathcal{S}_n - R$ is connected (it coincides with R), then $h \simeq 1$ by Theorem 10 of § 59, II.

It follows that

$$f \simeq 1 \quad \text{on } \mathrm{Fr}(R), \tag{10}$$

because the inclusions $\mathrm{Fr}(R) \subset X$ and $\mathrm{Fr}(R) \subset \overline{R} \subset \overline{T}$ imply by (9) that $f(x) = h(x)$ for $x \in \mathrm{Fr}(R)$.

Since the homotopy (10) holds for every component R of $\mathcal{S}_n - X$, it follows by Theorem 9 of § 59, II, that $f \simeq 1$ on X.

THEOREM 5. *Under the hypotheses of Theorem 3 (with $\mathcal{S}_n - X \neq R_0$), the elements $\widehat{x - p_1}, \widehat{x - p_2}, \ldots$ of the group $\mathfrak{E}(\mathcal{P}_n^X)$ are linearly independent.*

In other words, *the following implication holds*

$$[(x-p_1)^{k_1} \cdot \ldots \cdot (x-p_m)^{k_m} \approx 1 \ \ on \ \ X] \Rightarrow [k_1 = 0, \ldots, k_m = 0]. \tag{11}$$

Proof. First, consider the case where X is an elementary continuum (compare VI (0)). Since $A_m = \overline{R}_m - R_m \subset X$, it follows by hypothesis that

$$(x-p_1)^{k_1} \cdot \ldots \cdot (x-p_m)^{k_m} \approx 1 \quad \text{on } A_m. \tag{12}$$

On the other hand, since the points $p_1, p_2, \ldots, p_{m-1}$ belong to the (connected) set $\mathcal{S}_n - \overline{R}_m$, then (compare Theorem 10 of § 59, II)

$$x - p_1 \simeq 1, \ldots, x - p_{m-1} \simeq 1 \quad \text{on } \overline{R}_m, \text{ hence on } A_m. \tag{13}$$

One infers by (12) and (13) that $(x-p_m)^{k_m} \approx 1$ on A_m, hence $k_m = 0$ by Theorem 2 (since A_m is the surface of the ball R_m).

Thus, $k_i = 0$ for $i = 1, \ldots, m$.

This completes the proof in the case where X is an elementary continuum. In the general case, enclose every point p_i, where $i = 1, 2, \ldots, m$ (for a given m), in the ball $Q_i \subset R_i$ and consider the elementary continuum $X^* = \mathscr{S}_n - (Q_1 \cup \ldots \cup Q_m)$. It follows that $X \subset X^*$ and that X separates every pair of points of $\mathscr{S}_n - X^*$ which is separated by X^*. If the left-hand term of implication (11) holds on X, so does it on X^* (by Theorem 4 where one has to substitute X for F and X^* for X). It follows, as we have just proved, that the right-hand term of (11) holds too.

Theorems 3 and 5 imply the following

THEOREM 6. *The group* $\mathfrak{C}(\mathscr{P}_n^X)$ *is generated by the finite or infinite sequence* $\widehat{x-p_1}, \widehat{x-p_2}, \ldots$ (*unless it reduces to the neutral element; case where* $\mathscr{S}_n - X = R_0$).

In other words, *in formula* (2) *the exponents* $k_i \neq 0$ *are uniquely determined. Moreover, they do not depend on the choice of the points* p_i *in* R_i.

Proof. The last part of theorem follows from the fact that $x - p_i \simeq x - q_i$ on X if p_i and q_i belong to R_i (and hence to an arc in R_i).

It follows immediately from Theorem 6 that

$$\mathfrak{C}(\mathscr{P}_n^X) \underset{\text{gr}}{=} \mathscr{G}^m \quad \textit{respectively} \quad \mathfrak{C}(\mathscr{P}_n^X) \underset{\text{gr}}{=} \mathscr{G}^\omega \tag{14}$$

according to whether the number of components is finite, say $m+1$, *or infinite.*

The invariance theorem follows (compare for $n = 2$, Theorem 4 of § 62, IV).

THEOREM 7. *The number of* (*non-empty*) *separated parts into which* $\mathscr{S}_n$ *is decomposed by an arbitrary set* $A \subset \mathscr{E}^n$ *is an intrinsic invariant*(1).

THEOREM 8. *If* X *is an arbitrary subset* (*compact or not*) *of* $\mathscr{E}^n$, *the group* $\mathfrak{C}(\mathscr{P}_n^X)$ *contains a countable* (*finite or infinite*) *set* $\mathfrak{G}$ *such*

(1) For a proof of this theorem (in the case where A is compact) which makes no use of the mentioned Brouwer theorem nor of the homology theory, see K. Borsuk, Fund. Math. 37, *op. cit.*, for the case of an arbitrary set A (which is derived of it), see my *Remarque, ibid.*, pp. 251–252. See also S. Eilenberg, *An invariance theorem for subsets of* S^n, Bull. Amer. Math. Soc. 47 (1941), pp. 73–75.

Theorem 7 (for A compact) is a particular case of the duality theorem of J. W. Alexander.

that the group $\widehat{\mathfrak{G}}$ *generated by* $\mathfrak{G}$ *is dense in* $\mathfrak{C}(\mathcal{P}_n^X)$. *Thus the space* $\mathfrak{C}(\mathcal{P}_n^X)$ *is separable.*

More precisely, *let* $Q_0, Q_1, \ldots$ *be a sequence (finite or infinite) of (distinct) quasi-components of the set* $Y = \mathcal{S}_n - X$ *satisfying condition* (6) *of* § 58, II (*where* $\mathcal{X}$ *is to be replaced by* $\mathcal{S}_n - X$), *let* $p_i \in Q_i$ *for* $i > 0$ *and let* $\infty \in Q_0$. *Then*

$$\mathfrak{G} = (\widehat{x-p_1}, \widehat{x-p_2}, \ldots)$$

is the required set.

Proof. By Theorem 3 of § 54, IX, it is sufficient to show that $\widehat{\mathfrak{G}|F} = \mathfrak{C}(\mathcal{P}_n^X)$ for every compact subset F of X. But this is an easy consequence of Theorem 6 since the components $\widehat{(x-p_i)|F}$, $i = 1, 2, \ldots,$ of $\mathcal{P}_n^F$ constitute the set $\mathfrak{G}|F$.

Remarks. 1. The following statement (which is an extension of Borsuk Theorem (§ 59, II, 10) to non-compact sets) is a particular case of Theorem 6.

The set $\mathcal{S}_n - X$ *is connected if and only if* $\mathcal{P}_n^X$ *is connected* (1).

2. As we have seen (Theorem 3 of § 53, III), if $\mathcal{X}$ is compact and $\mathcal{Y}$ is an absolute neighbourhood retract, the components of the space $\mathcal{Y}^{\mathcal{X}}$ are arcwise connected. It should be remarked that the hypothesis of compactness cannot be omitted even if $\mathcal{Y} = \mathcal{P}_2$.

Moreover, *if* X *is a subset of the plane such that* $\mathcal{S}_n - X$ *is connected without being a semi-continuum, the space* $\mathcal{P}_2^{\mathcal{X}}$ *is connected without being arcwise connected.*

This follows easily from the theorem formulated in the preceding Remark combined with the theorem of S. Eilenberg, Theorem 1 of § 62, II.

3. By Theorem 4 of § 56, X (combined with Theorem IX, 3 and Theorem 6 of § 54, I), if X is an open subset of $\mathcal{E}^2$, the components of $\mathcal{P}_2^X$ are arcwise connected. But this is not the case for $\mathcal{E}^3$. In other words, *there exists an open subset* X *of* $\mathcal{E}^3$ *such that* $\mathcal{P}_3^X$ *contains two functions which are not homotopic, although they are homotopic on every compact subset of* X (2).

(1) For a more direct proof, see my paper *Un critère de coupure de l'espace euclidien par un sous-ensemble arbitraire*, Math. Zeitschr. 72 (1959), p. 88.

(2) See N. E. Steenrod, *Regular cycles of compact metric spaces*, Ann. of Math. 41 (1940), pp. 833–851.

It follows easily that *the relation of homotopy is not closed.*

4. Let us call *rational* each component of the form

$$\Gamma_1^{k_1}\cdot\ldots\cdot\Gamma_m^{k_m} \quad \text{where} \quad \Gamma_i = \widehat{(x-p_i)\,|\,X}.$$

By Theorem 8, the family of rational components is dense in the space $\mathfrak{C}(\mathscr{P}_n^X)$.

In the particular case where X is an *open* subset of $\mathscr{E}^n$ (or, more generally, a locally compact subset), the space $\mathfrak{C}(\mathscr{P}_n^X)$ is metrizable (compare Corollary 6a of § 54, IX) and hence *every component of $\mathscr{P}_n^X$ is a limit of rational components.*

This is a remarkable extension of Runge's theorem to the space $\mathscr{E}^n$ (for the case $n = 2$, see a more precise statement, Theorem 2 of § 62, VII).

5. The invariance theorem (Theorem 7) can be extended, in the case where A is compact, to mappings which are *more general than homeomorphisms* (1).

Namely, let A and B be compact subsets of $\mathscr{E}^n$ and let $h\colon A \to B$ be a continuous onto transformation such that (2)

(i) $h[\operatorname{Fr}(A)] = \operatorname{Fr}(B)$,

(ii) $h[\operatorname{Int}(A)] = \operatorname{Int}(B)$,

(iii) the restriction $h|\operatorname{Fr}(A)$ is one-to-one.

Moreover, assume that the set $\mathscr{S}_n - \operatorname{Fr}(A)$ has a finite number of components.

Then the sets $\mathscr{S}_n - B$ and $\mathscr{S}_n - A$ have the same number of components (3).

6. In the same direction one has the following theorem (4).

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(1) See my paper, *Sur quelques invariants topologiques dans l'espace euclidien*, Journ. de Math. 36 (1957), pp. 191–200.

(2) Compare these conditions with the conditions which define the "vanishing" transformations (of open sets) in the sense of K. A. Sitnikov. See the paper of that author, *On the continuous transformations of open sets of the Euclidean space* (Russian), Mat. Sbornik 31 (1952), pp. 439–458.

(3) If A and B are topological polyhedra, this invariance extends to all Betti numbers of dimension $\leqslant n-1$ (with rational coefficients).

For some extensions to algebraic topology, see K. Borsuk and A. Kosiński, *On connections between the homology properties of a set and of its frontier*, Bull. Acad. Polon. Sci. 4 (1956), pp. 331–333, and A. Kosiński, *On mappings which satisfy certain conditions on the boundary*, *ibid.*, pp. 335–340.

(4) See my paper quoted above, p. 196.

Let A and B be two compact subsets of $\mathscr{E}^n$, let F be a closed subset of A and let $h\colon A \to B$ be a continuous onto transformation which maps F onto $\mathrm{Fr}(B)$ in a one-to-one manner.

Under these conditions, *the number of components of* $\mathscr{S}_n - B$ *is less or equal to the number of components of* $\mathscr{S}_n - A$ (¹).

7. Let us add that conditions (i) through (iii) are equivalent to condition (iii) combined with the following one

(iv) $h[\mathrm{Fr}(A)] \cap h[\mathrm{Int}(A)] = 0$ (²).

VIII. Duality theorems for compact $X \subset \mathscr{E}^n$ $(n \geqslant 2)$. Let $Y = \mathscr{S}_n - X$. We assign to every component Γ of $\mathscr{P}_n^X$ a measure $\mu \in \mathfrak{N}(Y)$ which we call the *multiplicity* of Γ (see § 58, III).

Let $f\colon X_n \to \mathscr{P}_n$ be a continuous function. Referring to Theorem 3 of Section VII and representing f in form (2) of Section VII, we assume by definition that

$$\mu_{R_i} f = k_i \quad \text{for} \quad i > 0, \quad \mu_{R_0} f = -(k_1 + k_2 + \ldots), \tag{1}$$

and we call $\mu_{R_i} f$ *the multiplicity of* R_i *relative* f.

Since the multiplication is obviously invariant with respect to the relation $f_0 \approx f_1$, it is legitimate to write $\mu_{R_j} \Gamma = k_j$ $(j \geqslant 0)$ for $f \in \Gamma$.

Since every closed-open subset H of Y can be (uniquely) represented in the form $H = R_{j_1} \cup R_{j_2} \cup \ldots$ where $j_i \geqslant 0$, we extend the concept of multiplicity to all $H \in (0,1)^Y$ assuming that

$$\mu_H \Gamma = \mu_{R_{j_1}} \Gamma + \mu_{R_{j_2}} \Gamma + \ldots . \tag{2}$$

The multiplicity of Γ, *written* $\mu\Gamma$, *will mean the function which assigns to every* $H \in (0,1)^Y$ *an integer* $\mu_H \Gamma$ *defined by condition* (2).

It is easy to prove

THEOREM 1. $\mu_H \Gamma$ *is the algebraic number of zeros and poles of the right side of formula* (2) *of Section* VII *which belong to* H, *where* f *is an arbitrary element of* Γ.

(¹) An analogous remark as in the second footnote to Remark 5 applies here.

(²) For the proof, see J. H. Michael, *Continuous mappings of subsets of the Euclidean n-sphere*, Bull. Acad. Polon. Sci. 5 (1957), pp. 133–137.

For additional conditions under which (i)-(iii) imply that h is a homeomorphism, see G. T. Whyburn, *Topological Analysis*, Princeton 1958, p. 98; L. F. Mc Auley, *Conditions under which light open mappings are homeomorphisms*, Duke Journ. 33 (1966), pp. 445–452; G.H. Meisters and C. Olech, *ibid.* 30 (1963), pp. 63–80. For higher dimensions, see J. Cronin and L. F. Mc Auley, Proc. Nat. Acad. Sc. 56 (1966), p. 405.

THEOREM 2. *If X is a compact subset of $\mathscr{E}^n$, then*

$$\mathfrak{C}(\mathscr{P}_n^X) \underset{\mathrm{gr}}{=} \mathfrak{N}(Y). \tag{3}$$

More precisely, *an isomorphism of the group $\mathfrak{C}(\mathscr{P}_n^X)$ onto $\mathfrak{N}(Y)$ is defined by assigning to every component Γ of $\mathscr{P}_n^X$ its multiplicity $\mu\Gamma$.*

Proof. It is easy to show that

(i) $(\mu\Gamma) \in \mathfrak{N}(Y)$,

(ii) μ is a *homomorphism*, i.e.

$$\mu(\Gamma_0 \cdot \Gamma_1) = \mu(\Gamma_0) + \mu(\Gamma_1),$$

(iii) μ is *one-to-one*, i.e.

$$(\Gamma_0 = \Gamma_1) \equiv (\mu\Gamma_0 = \mu\Gamma_1),$$

in other words, *every Γ is determined by its multiplicity.*

It remains to be proved that for a given measure $\nu \in \mathfrak{N}(Y)$ there exists $\Gamma \in \mathfrak{C}(\mathscr{P}_n^X)$ such that $\mu\Gamma = \nu$. Now, it is sufficient for this purpose to set

$$\Gamma = (\widehat{x - p_1})^{k_1} \cdot (\widehat{x - p_2})^{k_2} \cdot \ldots,$$

where $k_i = \nu(R_i)$ and where $\widehat{x - p_i}$ denotes the component of $\mathscr{P}_n^X$ which contains the function $x - p_i$.

THEOREM 3. *If F is a closed subset of a compact set $X \subset \mathscr{E}^n$, then the following identity holds for every $\Gamma \in \mathfrak{C}(\mathscr{P}_n^X)$,*

$$(\mu\Gamma)^G = \mu(\Gamma | F), \quad \textit{where} \quad G = \mathscr{S}_n - F. \tag{4}$$

Proof. We have for $H \in (0, 1)^G$ (compare § 58, IV (1))

$$(\mu\Gamma)^G(H) = (\mu\Gamma)(H \cap Y) = \mu_{H \cap Y} = \mu_H(\Gamma | F), \tag{5}$$

since (compare Theorem 1) the algebraic number of zeros and poles of every function $f \in \mathscr{P}_n^X$ which belongs to H and to $H \cap Y$ is the same (because none of them belongs to X).

It follows immediately

THEOREM 4. *Under the same hypotheses the following diagram is commutative*

$$\begin{array}{ccc}
\mathfrak{C}(\mathscr{P}_n^F) & \xleftarrow{\Gamma|F} & \mathfrak{C}(\mathscr{P}_n^X) \\
{\scriptstyle \mu\Delta}\downarrow & & \downarrow{\scriptstyle \mu\Gamma} \\
\mathfrak{N}(G) & \xleftarrow[\mu^G]{} & \mathfrak{N}(Y)
\end{array}$$

IX. Duality theorems for arbitrary $X \subset \mathscr{E}^n$**.** Let $Y = \mathscr{S}_n - X$.

THEOREM 1. *If* $X \subset \mathscr{E}^n$, *then*

$$\operatorname*{Lim}_{F, F_0 \subset F_1} \{\mathfrak{C}(\mathscr{P}_n^F), S_{F_0F_1}\} \underset{\text{gr.t.}}{=} \operatorname*{Lim}_{G, G_0 \supset G_1} \{\mathfrak{N}(G), \mathrm{ext}_{G_0G_1}\}, \tag{1}$$

where as usual, F is a compact subset of X, $G = \mathscr{S}_n - F$ and the operations $S_{F_0F_1}$ and $\mathrm{ext}_{G_0G_1}$ are defined as in Section VI and in § 58, IV (11).

Namely, the required isomorphism is defined by setting

$$\nu_G = \mu \Delta_F \tag{2}$$

for every family $\{\Delta_F\}$ belonging to the left-hand side of (1), i.e. such that

$$\Delta_F \in \mathfrak{C}(\mathscr{P}_n^F) \quad \text{and} \quad \Delta_{F_1} | F_0 = \Delta_{F_0} \quad \text{if} \quad F_0 \subset F_1. \tag{3}$$

Proof. We have only to apply Theorems 4 of Section VIII and 2 of § 55, VII, referring to the following statements:

(i) $S_{F_0F_1}$ is a homomorphism of the group $\mathfrak{C}(\mathscr{P}_n^{F_1})$ into $\mathfrak{C}(\mathscr{P}_n^{F_0})$ (Theorem 2 of Section VI),

(ii) $\mathrm{ext}_{G_0G_1}$ is a homomorphism of the group $\mathfrak{N}(G_1)$ into $\mathfrak{N}(G_0)$ (§ 58, IV, Theorem 4),

(iii) for every F the operation μ is an isomorphism of the group $\mathfrak{C}(\mathscr{P}_n^F)$ onto $\mathfrak{N}(G)$ (Theorem 2 of Section VIII),

(iv) the spaces $\mathfrak{N}(\mathscr{P}_n^X)$ and $\mathfrak{N}(G)$ are discrete.

THEOREM 2. *If* $X \subset \mathscr{E}^n$, *then*

$$\operatorname*{Lim}_{F, F_0 \subset F_1} \{\mathfrak{C}(\mathscr{P}_n^F), S_{F_0F_1}\} \underset{\text{gr.t.}}{=} \mathfrak{N}(Y). \tag{4}$$

More precisely, *if* $\{\Delta_F\}$ *belongs to the left-hand side of* (4), *the required isomorphism* μ *is determined by setting for every* $Z \in (0,1)^Y$

$$\mu(Z) = \mu_H \Delta_F, \tag{5}$$

where

$$Z = H \cap Y, \qquad H \in (0,1)^G \quad \textit{and} \quad G = \mathscr{S}_n - F. \tag{6}$$

Proof. The isomorphism between the right sides of (1) and of (4) (compare § 55, VIII, Theorem 1 and § 58, IV, Theorem 6 (8)) is defined by the formula

$$\mu(Z) = \nu_G(H). \tag{7}$$

Conditions (7) and (2) yield immediately (5).

The following definition allows us to extend the concept of multiplicity, defined in Section VIII for compact sets X, to arbitrary subsets X of $\mathscr{E}_n$.

DEFINITION. Let $\Gamma \in \mathfrak{C}(\mathscr{P}_n^X)$. By the *multiplicity* of Γ we understand the function $\mu\Gamma$ defined by the condition

$$\mu_Z \Gamma = \mu_H(\Gamma | F), \quad \text{where} \quad Z \in (0,1)^Y, \tag{8}$$

and where H and F satisfy conditions (6).

$\mu_Z \Gamma$ is said to be *the multiplicity of Z with respect to Γ.*

As was proved (compare § 58, IV (8)), $\mu_Z \Gamma$ does not depend on the choice of H and F provided they satisfy conditions (6).

THEOREM 3. *If $X \subset \mathscr{E}^n$, then*

$$\mathfrak{C}(\mathscr{P}_n^X) \underset{\text{gr.t.}}{\subset} \mathfrak{N}(Y). \tag{9}$$

More precisely, *a topological isomorphism of the group $\mathfrak{C}(\mathscr{P}_n^X)$ onto a subgroup of $\mathfrak{N}(Y)$ is defined by assigning to every component Γ of $\mathscr{P}_n^X$ its multiplicity $\mu\Gamma$.*

Proof. By Theorem 3 of Section VI, an isomorphism is defined of the left-hand side of (9) into the left-hand side of (4) by assigning to Γ the element $\{\Gamma|F\}$ of $\underset{F}{\boldsymbol{P}}\mathfrak{C}(\mathscr{P}_n^F)$ (setting $\Delta_F = \Gamma|F$). Next, applying Theorem 2 and conditions (5) and (8), it is easily seen that μ is the required isomorphism.

THEOREM 4. *Let $p \in Y$, let $\hat{p}$ be the quasi-component of Y which contains p and $\widehat{x-p}$ that one of $\mathscr{P}_n^X$ which contains the function $x-p$. Then*

$$\mu(\widehat{x-p}) = \beta_{\hat{p}} \tag{10}$$

(for the notation, see § 58, III (2); for $p = \infty$, we agree that $x-p \equiv 1$).

Proof. Let $Z \in (0,1)^Y$ and let us assume that conditions (6) are fulfilled. Then it follows by (8) that

$$\mu_Z(\widehat{x-p}) = \mu_H[(\widehat{x-p})|F]. \tag{11}$$

Let $p \in Z$ (and therefore $\hat{p} \subset Z$) and $\infty \in Y - Z$; it follows that $p \in H$ and $\infty \notin H$. So, the algebraic number of zeros and poles of the function $x-p$ which belong to H is 1, i.e.

$$\mu_H[\widehat{(x-p)|F}] = 1, \quad \text{thus} \quad \mu_Z(\widehat{x-p}) = 1.$$

This number is 0 if $p, \infty \in Z$.

The required conclusion follows by (4) of § 58, III.

Theorem 4 allows to complete Theorem 3 in the following way:

$$\overline{\mu\mathfrak{C}(\mathscr{P}_n^X)} = \mathfrak{N}(Y). \tag{12}$$

Proof. Let $\mathfrak{G}$ be the family of all components of the form $\widehat{x-p}$, where $p \in Y$, and let $\mathfrak{Y}$ be the set of measures β_Q where Q runs over all the quasi-components of Y. By Theorem 4, $\mu(\mathfrak{G}) = \mathfrak{Y}$. Let $\widehat{\mathfrak{G}}$ and $\widehat{\mathfrak{Y}}$ be the groups generated by $\mathfrak{G}$ and $\mathfrak{Y}$ respectively; it follows that $\mu(\widehat{\mathfrak{G}}) = \widehat{\mathfrak{Y}}$ (since μ is a homomorphism). By Theorem 5 of § 58, III, $\overline{\widehat{\mathfrak{Y}}} = \mathfrak{N}(Y)$, which implies (12).

Theorem 5. *If the points $p_0 = \infty, p_1, \ldots, p_m$ belong to different quasi-components of Y, the elements $\widehat{x-p_1}, \ldots, \widehat{x-p_m}$ of $\mathfrak{C}(\mathscr{P}_n^X)$ are linearly independent.*

Because the measures $\beta_{\hat{p}_1}, \ldots, \beta_{\hat{p}_m}$ are linearly independent (by Theorem 2 of § 58, III; compare Theorems 5 and 6 of Section VII).

Theorem 6. *If Y consists of $m+1$ components, then*

$$\mathfrak{C}(\mathscr{P}_n^X) \underset{\text{gr}}{=} \mathscr{G}^m.$$

If $Q_0, \ldots, Q_m$ are these components and if $p_i \in Q_i$, where $p_0 = \infty$, then $\widehat{x-p_1}, \ldots, \widehat{x-p_m}$ are the generators of the group $\mathfrak{C}(\mathscr{P}_n^X)$.

This is a consequence of Theorem 4 of § 58, III.

X. Duality theorems for locally compact $X \subset \mathscr{E}^n$. As usually, let us represent X in the form

$$X = F_1 \cup F_2 \cup \ldots \quad \text{with} \quad F_i \text{ being compact} \subset \operatorname{Int}_X(F_{i+1}),$$

where Int_X denotes the interior with respect to X.

Then formula (1) of Section IX and Theorem 9 of § 58, IV, can be replaced by the following ones.

$$\operatorname*{Lim}_{i,i<k}\{\mathfrak{C}(\mathscr{P}_n^{F_i}),\, S_{F_iF_k}\} \underset{\text{gr.t.}}{=} \operatorname*{Lim}_{i,i>k}\{\mathfrak{N}(G_i),\, \mathrm{ext}_{G_iG_k}\}, \tag{1}$$

$$\operatorname*{Lim}_{i,i>k}\{\mathfrak{N}(G_i),\, \mathrm{ext}_{G_iG_k}\} \underset{\text{gr.t.}}{=} \mathfrak{N}(Y), \tag{2}$$

where $G_i = \mathscr{S}_n - F_i$.

These formulas combined with Theorem 5 of Section VI and with Theorem 3 of Section II imply the following statement.

THEOREM 1. *If X is a locally compact subset of $\mathscr{E}^n$, then*

$$\mathfrak{C}(\mathscr{P}_n^X) \underset{\text{gr.t.}}{=} \mathfrak{N}(Y);\ ^{(1)} \tag{3}$$

the multiplicity μ is the required isomorphism.

THEOREM 2. *If X is open and Y has infinitely many components, then*

$$\mathfrak{C}(\mathscr{P}_n^X) \underset{\text{gr.t.}}{=} \mathscr{G}^{\aleph_0}. \tag{4}$$

Proof. This is a direct consequence of Theorem 1 combined with Theorem 7 of § 58, III.

The following theorem corresponds to the Weierstrass theorem on the decomposition of a function into prime factors (compare also § 62, VIII (iii) for $n = 2$).

THEOREM 3. *If X is open and Y consists of an infinite sequence of components $Q_0, Q_1, \ldots,$ each of which except Q_0 (which contains the point ∞) is open in Y, then every $\Gamma \in \mathfrak{C}(\mathscr{P}_n^X)$ is an infinite product*

$$\Gamma = \prod_{i=1}^{\infty} (\widehat{x - p_i})^{k_i} \quad \textit{where} \quad k_i = \mu_{Q_i}\Gamma \ \textit{and}\ \ p_i \in Q_i. \tag{5}$$

Proof. Since $(\mu\Gamma) \in \mathfrak{N}(Y)$, it follows by Theorem 8 of § 58, III, that

$$\mu\Gamma = \sum_{i=1}^{\infty} k_i \beta_i, \quad \text{where} \quad k_i = \mu_{Q_i}\Gamma \text{ and } \beta_i = \beta_{Q_i}; \tag{6}$$

---

(1) For a proof based on the Alexander duality theorem and Hopf classification theorem, see the reference to W. Nikolajshvili in § 58, III.

in other words,

$$\mu\Gamma = \lim_{j=\infty} \sigma_j \quad \text{where} \quad \sigma_j = k_1\beta_{Q_1} + \ldots + k_j\beta_{Q_j}. \tag{7}$$

By Theorem 4 of Section IX, $\beta_{Q_i} = \mu(\widehat{x-p_i})$; and since μ is a homomorphism, then

$$\sigma_j = \mu[(\widehat{x-p_1})^{k_1} \cdot \ldots \cdot (\widehat{x-p_j})^{k_j}]. \tag{8}$$

Since μ is also a homeomorphism (by Theorem 1), it follows from (7) and (8) that

$$\Gamma = \operatorname*{Lim}_{j=\infty} (\widehat{x-p_1})^{k_1} \cdot \ldots \cdot (\widehat{x-p_j})^{k_j},$$

which implies identity (7).

TOPOLOGY OF THE PLANE

§ 61. Qualitative problems

I. Janiszewski spaces. $\mathscr{X}$ is said to be a *Janiszewski space* if $\mathscr{X}$ is a locally connected continuum having the following property.

(J) *If C_0 and C_1 are two continua whose intersection $C_0 \cap C_1$ is not connected, the union $C_0 \cup C_1$ is a cut of the space.*

THEOREM 1. *For a locally connected continuum $\mathscr{X}$ the property (J) is equivalent to the following one.*

(J_0) *If R is a region, $\mathscr{X}-R$ is contractible with respect to $\mathscr{S}$.*

Proof. $(J) \Rightarrow (J_0)$. Let R be a region. By Theorem 8 of § 57, I it is sufficient to show that every component C of $\mathscr{X}-R$ is contractible with respect to $\mathscr{S}$.

By Theorem 5 of § 46, III, the set $\mathscr{X}-C$ is a region. Following Theorem 1 of § 50, III, let

$$C = \bigcap_{n=1}^{\infty} (\mathscr{X}-R_n), \qquad R_1 \subset R_2 \subset \ldots,$$

where R_n is a region and $\mathscr{X}-R_n$ is a locally connected continuum.

According to (J) the continuum $\mathscr{X}-R_n$ is unicoherent, and therefore contractible with respect to $\mathscr{S}$ (by § 57, III, Theorem 3).

It follows by Theorem 7 of § 57, I, that C is contractible with respect to $\mathscr{S}$.

$(J_0) \Rightarrow (J)$. If C_0 and C_1 are two continua such that $\mathscr{X}-(C_0 \cup C_1)$ is a region, the set $C_0 \cup C_1$ is contractible with respect to $\mathscr{S}$ according to (J_0), and hence it is unicoherent (by § 57, II, Theorem 2). But then $C_0 \cap C_1$ is connected.

Setting $R = 0$ in (J_0) yields the following statement

THEOREM 1'. *Every Janiszewski space is contractible with respect to $\mathscr{S}$, and hence is unicoherent* (compare § 57, II, Theorem 2).

According to Theorem 10 ((i) and (iii)) of § 59, II, the following holds.

THEOREM 2. *The sphere $\mathcal{S}_2$ is a Janiszewski space*(1).

Assume for the present that *$\mathcal{X}$ is a Janiszewski space*. The following theorems hold (3 through 9).

THEOREM 3. *If C is a continuum, $\mathcal{X}-C$ is contractible with respect to $\mathcal{S}$.*

Proof. Let $R_1, R_2, \ldots$ be the sequence of components of $\mathcal{X}-C$ and let $f\colon \mathcal{X}-C \to \mathcal{S}$ be a continuous function. By Theorem 5 of § 46, III, the set $\mathcal{X}-R_m$ is connected. Therefore it follows by Theorem 2 of § 50, III, that

$$R_m = C_{m,1} \cup C_{m,2} \cup \ldots \quad \text{and} \quad C_{m,n} \subset \operatorname{Int}(C_{m,n+1}),$$

where $C_{m,n}$ is a continuum and $\mathcal{X}-C_{m,n}$ a region. Since $C_{m,n}$ is contractible with respect to $\mathcal{S}$ in virtue of (J_0), then so is R_m by Theorem 6 of § 57, I, and hence so is the set $\mathcal{X}-C$, by Theorem 8 of § 57, I.

Condition (J_0) and Theorem 3 combined with Theorem 2 of § 57, II imply the following statements.

THEOREM 4. *If A is a continuum and $\mathcal{X}-A$ is a region, then A and $\mathcal{X}-A$ are unicoherent.*

THEOREM 5. *Let A_0 and A_1 be two closed or two open sets. If these sets are connected, whereas their intersection $A_0 \cap A_1$ is not, then their union $A_0 \cup A_1$ is a cut of the space $\mathcal{X}$.*

The same statements combined with Theorem 3 of § 57, II imply the following

THEOREM 6. *Let A_0 and A_1 be two closed or two open sets. If $A_0 \cup A_1$ is not a cut and if the sets A_0 and A_1 are connected between p_0 and p_1, then so is the set $A_0 \cap A_1$.*

COROLLARY 6′. *Let A_0 and A_1 be two closed or two open sets. Let C_0 and C_1 be components of A_0 and A_1 respectively. If the intersection $C_0 \cap C_1$ is not connected, the union $A_0 \cup A_1$ is a cut.*

Proof. Let p_0, p_1 be a pair of points of the set $C_0 \cap C_1$. The sets A_0 and A_1 are therefore connected between these points. Assuming that $A_0 \cup A_1$ is not a cut, $A_0 \cap A_1$ is connected between p_0 and p_1, so that (compare § 47, II, Theorem 3 and § 49, II, Theo-

(1) Property (J) of $\mathcal{S}_2$ has been established by Z. Janiszewski. This is the *second theorem of Janiszewski*. See of that author *Sur les coupures du plan*, Prace Mat.-Fiz. 26 (1913), p. 55. Compare also L. E. J. Brouwer, Proc. Akad. Amsterdam 1911.

rem 17) there exists a component Q of $A_0 \cap A_1$ which contains p_0 and p_1. It follows that $Q \subset A_j$, thus $Q \subset C_j$, and $Q \subset C_0 \cap C_1$, which shows that every pair of points of $C_0 \cap C_1$ can be joined in $C_0 \cap C_1$ by a connected set. Hence $C_0 \cap C_1$ is connected.

COROLLARY 6″. *If G is open, C is a component of $\mathscr{X}-G$ and K is a continuum such that $K \cap C$ is not connected, then $G-K$ is not connected either.*

Proof. In Corollary 6′ put $A_0 = \mathscr{X}-G$, $A_1 = K = C_1$ and $C_0 = C$. Since the set $C_0 \cap C_1 = K \cap C$ is not connected, it follows that the set $\mathscr{X}-(A_0 \cup A_1) = G-K$ is not connected either.

Setting $B_j = \mathscr{X}-A_j$ in Theorem 6 yields the following statement.

THEOREM 7. *Let B_0 and B_1 be two closed or two open sets. If none of these sets is a cut between p_0 and p_1 and if $B_0 \cap B_1$ is connected, then $B_0 \cup B_1$ is not a cut between p_0 and p_1 either*(¹).

Theorem 7 implies the following

THEOREM 8. *Every closed set F, which irreducibly separates $\mathscr{X}$ between two points p_0 and p_1, is discoherent.*

If, moreover, F is locally connected (and does not reduce to a point), it is a simple closed curve (compare § 49, IV, Theorem 6).

Replacing Theorem 3 of § 59, III by Theorem 8 in the proof of Theorem 4 of § 59, III, we deduce the following

THEOREM 8′. *Let C be a continuum, R a component of $\mathscr{X}-C$ and $F = \overline{F} \subset \mathrm{Fr}(R)$. If the sets F and $C-F$ are connected, then so is the set $\mathrm{Fr}(R)-F$.*

THEOREM 9. *The concept of a Janiszewski space is invariant under monotone continuous transformations.*

In other words, *the space of every semi-continuous decomposition of $\mathscr{X}$ into continua is a Janiszewski space.*

Proof. Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a monotone continuous mapping of $\mathscr{X}$ onto $\mathscr{Y}$. Let R be a region in the space $\mathscr{Y}$. By Theorem 4 of § 47, VI, $f^{-1}(R)$ is a region in $\mathscr{X}$. Therefore, the set $\mathscr{X}-f^{-1}(R) = f^{-1}(\mathscr{Y}-R)$ is contractible with respect to $\mathscr{S}$ (by virtue of the property (J_0)) and so is the set $ff^{-1}(\mathscr{Y}-R) = \mathscr{Y}-R$ by Theorem 2, (ii) of § 57, I.

(¹) In case B_0 and B_1 are closed and $X = \mathscr{S}_2$, this is the *first theorem of Janiszewski*; compare op. cit.

Both theorems of Janiszewski are *equivalent* (if $\mathscr{X}$ is a locally connected continuum). Compare my paper in Fund. Math. 13 (1929), p. 311.

THEOREM 10. *The concept of a Janiszewski space is reducible and extensible*(1).

Proof. First assume that $\mathscr{X}$ is a Janiszewski space. Let C be a cyclic element, R a region relative to C, and K and L two continua such that $C-R = K \cup L$. We have to show that $K \cap L$ is connected.

Denote by R_1 and K_1 the union of all components S of $\mathscr{X}-C$ such that $\mathrm{Fr}(S) \subset R$ or $\mathrm{Fr}(S) \subset K$, respectively. Let L_1 be the union of the components S which are contained neither in R_1 nor in K_1; then $\mathrm{Fr}(S) \subset L$ (by § 52, I, Theorem 4). It follows that

$$R \cup R_1 = \mathscr{X}-(K \cup K_1 \cup L \cup L_1).$$

By the definition of K_1 and Theorem 1 of § 49, III, it follows that $\mathrm{Fr}(K_1) \subset K$. Therefore the set $K \cup K_1$ is closed, and so is $L \cup L_1$. Thus the set $R \cup R_1$ is open.

Since the sets $R \cup R_1$, $K \cup K_1$ and $L \cup L_1$ are connected (compare § 46, II, Theorem 2), condition (J) implies that the intersection

$$(K \cup K_1) \cap (L \cup L_1) = K \cap L$$

is connected.

Now assume that $\mathscr{X}$ is a locally connected continuum which does not satisfy condition (J). We have to define a cyclic element C which does not satisfy that condition either.

So let R be a region, K and L two continua and M and N two closed sets such that

$$\mathscr{X}-R = K \cup L, \tag{1}$$

$$K \cap L = M \cup N, \tag{2}$$

$$M \cap N = 0, \tag{3}$$

$$M \neq 0 \neq N. \tag{4}$$

Since C is a cyclic element, it follows from Theorem 5′ of § 52, I that $C \cap R$ is a region relative to C and that $C \cap K$ and $C \cap L$ are continua. Moreover, it follows by (1) through (3) that

$$C-(C \cap R) = (C \cap K) \cup (C \cap L), \qquad (C \cap K) \cap (C \cap L)$$
$$= (C \cap M) \cup (C \cap N) \quad \text{and} \quad (C \cap M) \cap (C \cap N) = 0.$$

(1) Compare my paper *Quelques applications d'éléments cycliques de M. Whyburn*, Fund. Math. 14 (1929), p. 139.

Therefore, in order to establish that the Janiszewski property is extensible, it is sufficient to show that there exists a cyclic element C such that

$$C \cap M \neq 0 \neq C \cap N. \tag{5}$$

Denote by A and B the union of all cyclic elements C such that $C \cap M \neq 0$ or $C \cap N \neq 0$ respectively.

If C_1 and C_2 are two different cyclic elements such that $C_1 \cap C_2 \neq 0$, the intersection $C_1 \cap C_2$ consists of a single point p which cuts $\mathscr{X}$ between $C_1 - C_2$ and $C_2 - C_1$ (compare § 52, II, Theorem 4). Consequently, if we suppose that

$$C_1 \cap M \neq 0 \neq C_2 \cap N,$$

it follows from (2) that $C_1 \cap K \neq 0 \neq C_2 \cap K$, whence $p \in K$, and similarly $p \in L$. Thus it follows by (2) that either $p \in M$ or $p \in N$, hence

$$\text{either} \quad C_2 \cap M \neq 0 \quad \text{or} \quad C_1 \cap N \neq 0.$$

Thus, one of the cyclic elements C_1 and C_2 satisfies condition (5). Therefore, it only remains to show that

$$A \cap B \neq 0. \tag{6}$$

Denote by V and W the union of all cyclic elements C such that $C \cap K \neq 0$ or $C \cap L \neq 0$ respectively. By Theorem 9 of § 52, II, the sets V and W are continua, and therefore they are completely arcwise connected according to Theorem 11 of § 52, II; hence $V \cap W$ is a continuum (compare § 52, I, Theorem 1).

Repeating the foregoing argument, we infer that $V \cap W$ consists of all cyclic elements C such that

$$C \cap K \neq 0 \neq C \cap L. \tag{7}$$

Now, since $K \cup L$ is a continuum (by (2) and (4)), so is $C \cap (K \cup L)$ (according to § 52, I, Theorem 5′). Then it follows from (7) and (2) that $C \cap (M \cup N) \neq 0$. In other words, $V \cap W = A \cup B$. Since the sets A and B are non-empty (by (4)) and closed according to Theorem 9 of § 52, II and since $V \cap W$ is a continuum, formula (6) follows.

COROLLARY 11. *Every dendrite is a Janiszewski space.*

This corollary follows also from Theorem 1 of § 51, VI.

II. Locally connected subcontinua of $\mathcal{S}_2$. The space $\mathcal{X}$ considered in this Section is *a Janiszewski space containing no separating points* (and containing more than one point). By virtue of Theorem 2 of Section I all theorems of this Section are applicable to $\mathcal{S}_2$ (1).

DEFINITION. Every region whose boundary is a simple closed curve is called a *disk*.

THEOREM 1 (of Jordan)(2). *Every simple closed curve cuts the space $\mathcal{X}$ into two regions and is their common boundary.*

In other words, *the complement of every simple closed curve consists of two disjoint disks.*

We shall prove first the following statement.

THEOREM 1'. *No arc cuts the space $\mathcal{X}$.*

Proof. Suppose conversely that an arc L cuts $\mathcal{X}$ between p_0 and p_1. By Theorems 1' of Section I and 5 of § 57, II, L contains a continuum, and therefore an arc L^*, which irreducibly separates the space between p_0 and p_1. But this is inconsistent with Theorem 8 of Section I.

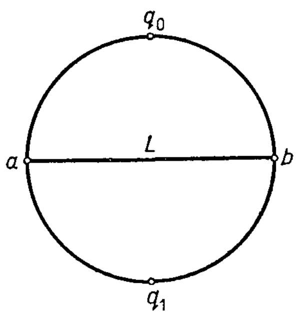

Fig. 16

Thus Theorem 1' is established. Now assume that C is a simple closed curve.

By condition (J) the curve C is a cut of the space.

C is the common boundary of components of $\mathcal{X}-C$, because C is an irreducible cut of $\mathcal{X}$ by Theorem 1' (compare § 49, V, Theorem 1).

It remains to prove that $\mathcal{X}-C$ contains at most two components(3).

Suppose conversely that there are three of them, R_0, R_1 and R_2. Let L be an arc which joins two points a and b of C and which is accessible from R_2 (compare § 50, III, Theorem 7), $L \subset R_2 \cup a \cup b$. Let aq_jb $(j = 0, 1)$ be two arcs determined in C by a and b,

$$aq_0b \cup aq_1b = C \quad \text{and} \quad aq_0b \cap aq_1b = (a, b).$$

(1) It will be seen in Section III that $\mathcal{X} \underset{\text{top}}{=} \mathcal{S}_2$.

(2) C. Jordan, *Cours d'Analyse*, Paris 1893, p. 92. See L. E. J. Brouwer, *Beweis des Jordanschen Kurvensatzes*, Math. Ann. 69 (1910), p. 169.

(3) In fact, this is a direct consequence of Theorem 6 of § 62, III for $\mathcal{X} = \mathcal{S}_2$.

Let $p_j \in R_j$. Define $A_j = aq_jb \cup L$. It follows that

$$A_0 \cap A_1 = L. \tag{1}$$

Since $C = \mathrm{Fr}(R_j)$, the set $R_0 \cup q_j \cup R_1$ is connected; therefore A_{1-j} (which is disjoint from it) does not cut between p_0 and p_1. According to Theorem 7 of Section I and to (1) the union $A_0 \cup A_1$ does not cut there either. But this implies a contradiction, since $C \subset A_0 \cup A_1$ and C cuts between p_0 and p_1 by the definition of these points.

THEOREM 2 (about the θ-curve). *If C is a θ-curve consisting of three arcs L_0, L_1, L_2 having, pairwise, only their end-points in common, then*

$$\mathscr{X} - C = D_0 \cup D_1 \cup D_2, \tag{2}$$

$$\mathrm{Fr}(D_j) = L_j \cup L_{j+1}, \tag{3}$$

where the disks D_0, D_1 and D_2 are the components of $\mathscr{X} - C$ (the subscripts being reduced mod 3).

Proof. By the Jordan theorem the simple closed curve $L_j \cup L_{j+1}$ cuts the space $\mathscr{X}$ into two disks one of which, say Q_j, contains the arc L_{j+2} (without its end-points) and the other, say D_j, is disjoint from L_{j+2}. Moreover,

$$\mathrm{Fr}(D_j) = L_j \cup L_{j+1} \quad \text{and} \quad \mathscr{X} - \bar{D}_j = Q_j.$$

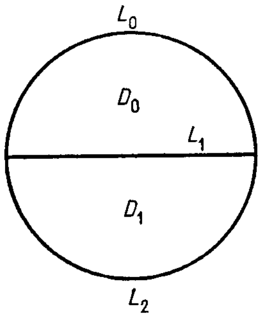

Fig. 17

D_j is a component of $\mathscr{X} - C$, because on the one hand $D_j \subset \mathscr{X} - C$, and on the other the inclusion $\mathrm{Fr}(D_j) \subset C$ implies that $\bar{D}_j - C = D_j - C$, so that D_j is a closed-open set in $\mathscr{X} - C$.

It remains to prove that $\mathscr{X} - (C \cup D_0 \cup D_1) = D_2$.

But the identities

$$\bar{D}_0 \cap \bar{D}_1 = (L_0 \cup L_1) \cap (L_1 \cup L_2) = L_1 \quad \text{and} \quad \mathscr{X} - \bar{D}_j = Q_j$$

imply by Theorem 7 of Section I that the set

$$\mathscr{X} - (\bar{D}_0 \cup \bar{D}_1) = \mathscr{X} - (C \cup D_0 \cup D_1)$$

is connected. And this set coincides with D_2 as a connected subset of $\mathscr{X} - C$ containing the component D_2 of $\mathscr{X} - C$.

COROLLARY 3. *Let $L_0, L_1, \ldots$ be a converging infinite sequence of arcs having only their end-points a and b in common; then there are at most two terms L_n such that*

$$L_n \cap \operatorname*{Lim}_{m=\infty} L_m \neq (a, b). \tag{4}$$

Proof. Suppose conversely that there exist three subscripts n, say 0, 1 and 2, which satisfy condition (4). Consider the curve $\theta = L_0 \cup L_1 \cup L_2$. For every $n > 2$, the set $L_n - a - b$ is contained by Theorem 2 in one of the three regions D_0, D_1 or D_2. Therefore it is admissible to assume that there exists an infinity of subscripts k such that $L_k - a - b \subset D_0$. Consequently,

$$\operatorname*{Lim}_{m=\infty} L_m \subset \bar{D}_0 = D_0 \cup L_0 \cup L_1, \quad \text{thus} \quad (L_2 - a - b) \cap \operatorname*{Lim}_{m=\infty} L_m = 0,$$

contrary to the hypothesis.

THEOREM 4. *If C is a locally connected continuum, every component R of $\mathscr{X} - C$ has the following properties:*

(i) $\mathrm{Fr}(R)$ *is a regular continuum containing no θ-curve*(1),

(ii) *if C contains no cut point, then R is a disk and therefore* $\mathrm{Fr}(R)$ *is a simple closed curve,*

(iii) $\bar{R}$ *is a locally connected continuum.*

Proof. Since $\mathscr{X}$ is contractible with respect to $\mathscr{S}$ (compare I, Theorem 1′), the set $K = \mathrm{Fr}(R)$ is a continuum (§ 57, II, Theorem 6). Would K be not locally connected, then there would exist a curve $\theta = L_0 \cup L_1 \cup L_2$ in C and three continua Q_0, Q_1 and Q_2 such that

$$Q_j \cap L_j \neq 0 \neq Q_j \cap K, \tag{5}$$

$$Q_j \cap (L_{j+1} \cup L_{j+2}) = 0, \tag{6}$$

where the subscripts are reduced mod 3 (compare § 52, IV, Theorem 4).

The same conclusion is obtained, if we suppose that K contains a curve θ; then we take for Q_j a point of $L_j - (L_{j+1} \cup L_{j+2})$.

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(1) See M. Torhorst, Math. Zeitschr. 9 (1921), p. 44. Compare also B. v. Kerékjártó, Abh. Math. Seminar Hamburg 4 (1925), p. 167 and G. T. Whyburn, Fund. Math. 12 (1928), p. 267.

Let D_0, D_1 and D_2 be the disks which are components of $\mathscr{X}-\theta$ (compare Theorem 2). Since $R \cap \theta = 0$, then R is contained in one of these disks, say $R \subset D_0$. Therefore

$$K \subset \overline{D}_0, \quad \text{which implies} \quad Q_2 \cap \overline{D}_0 \neq 0$$

by (5). On the other hand,

$$Q_2 \cap (L_2 - L_0 - L_1) \neq 0, \quad \text{and hence} \quad Q_2 - \overline{D}_0 \neq 0.$$

Thus $Q_2 \cap \mathrm{Fr}(\overline{D}_0) \neq 0$, so that (compare (3)) $Q_2 \cap (L_0 \cup L_1) \neq 0$ contradicting (6).

Thus, K is a regular continuum by Theorem 3 of § 52, IV, being a locally connected continuum containing no θ-curve.

If C contains no cut point, the set $\mathrm{Fr}(R)$ does not contain one either (compare I, Theorem 8′). Hence $\mathrm{Fr}(R)$ is a simple closed curve by Theorem 1 of § 52, IV, being a locally connected continuum which contains no cut point and no θ-curve.

Finally, since $\mathrm{Fr}(R)$ is locally connected, then so is $\bar{R}$ by Theorem 4 of § 49, III.

Remark. The boundary of a locally connected continuum may fail to be a locally connected continuum(¹).

THEOREM 5. *Every locally connected continuum C which separates $\mathscr{X}$ between two continua A and B contains a simple closed curve which separates $\mathscr{X}$ between these continua*(²).

Proof. Let Q be the component of $\mathscr{X}-C$ which contains A, and R the component of $\mathscr{X}-\bar{Q}$ which contains B; then $\mathrm{Fr}(R)$ is an irreducible separator between A and B (compare § 49, V, Theorem 2). Since $\bar{Q}$ is locally connected (by Theorem 4 (iii)), then so is $\mathrm{Fr}(R)$ (by Theorem 4 (i)). And hence $\mathrm{Fr}(R)$ is a simple closed curve being a locally connected irreducible separator (compare Theorem 8 of Section I).

Theorem 5 combined with Theorem 2 of § 57, III implies the following

THEOREM 5′. *Every pair of disjoint continua can be separated by a simple closed curve.*

(¹) A. Rosenthal, Math. Zeitschr. 10 (1921), p. 102.

(²) Compare R. L. Wilder, Fund. Math. 7 (1925), p. 354.

THEOREM 6. *Every continuum C which is not a cut of $\mathscr{X}$ is the intersection of a sequence of disks,*

$$C = D_1 \cap D_2 \cap \ldots \quad \textit{where} \quad \overline{D}_n \subset D_{n-1}. \tag{7}$$

Proof. We have to show that, if G is an open set such that $C \subset G$, there exists a disk D which satisfies the conditions

$$C \subset D \subset \overline{D} \subset G. \tag{8}$$

By Theorem 2 of § 50, III (with $\mathscr{X}-C$ replacing G), there exists a continuum H such that

$$\mathscr{X}-G \subset H \subset \mathscr{X}-C. \tag{9}$$

By Theorem 5′ we may find a disk D such that

$$C \subset D \subset \overline{D} \subset \mathscr{X}-H. \tag{10}$$

Formula (8) follows from (10) and (9).

COROLLARY 7. *The space $\mathscr{X}$ has a base consisting of disks.*

Proof. By Theorem 6 to every point p and to every number $\varepsilon > 0$ there corresponds a disk D such that $p \in D$ and $\delta(D) < \varepsilon$. The corollary follows from the compactness of $\mathscr{X}$.

COROLLARY 8. *If R is a region which does not cut the space, there exists a sequence of disks $\{D_n^*\}$ such that*

$$R = D_1^* \cup D_2^* \cup \ldots, \quad \textit{where} \quad \overline{D_{n-1}^*} \subset D_n^*. \tag{11}$$

Consequently, *if $F = \overline{F} \subset R$, there exists a disk D such that*

$$F \subset D \subset \overline{D} \subset R. \tag{12}$$

Proof. It is sufficient to set $R = \mathscr{X}-C$ and $D_n^* = \mathscr{X}-\overline{D}_n$ in Theorem 6.

THEOREM 9. *Every pair of closed disjoint sets A and B which do not cut the space can be separated by a simple closed curve.*

Proof. By Theorem 1 of § 50, III, there exists a finite system of disjoint continua $C_1, \ldots, C_n$ such that $A \subset C_1 \cup \ldots \cup C_n \subset \mathscr{X}-B$ and such that the set $C_1 \cup \ldots \cup C_n$ is not a cut of the space. By Theorem 5 of § 46, III, none of the continua $C_1, \ldots, C_n$ is a cut.

Therefore, the theorem reduces to the case where A is the union of a finite system of continua $A = C_1 \cup \ldots \cup C_n$, none of which is a cut.

Proceed by induction. Since the theorem is true for $n = 1$ (by Theorem 6), assume that it is true for $n-1$.

Since the set $C_n \cup B$ is not a cut (compare § 57, II, Theorem 2), there exists by hypothesis a simple closed curve K which separates the space between the sets $C_1 \cup \ldots \cup C_{n-1}$ and $C_n \cup B$. Let D be a disk such that

$$C_1 \cup \ldots \cup C_{n-1} \subset D \quad \text{and} \quad \mathrm{Fr}(D) = K.$$

Let L be an arc disjoint from B and irreducible between $\overline{D}$ and C_n. It is easy to infer from Janiszewski's theorem (I, 7) that the continuum $E = \overline{D} \cup L \cup C_n$ does not cut the space (compare Theorems 1 and 1'). Consequently there exists by Theorem 6 a simple closed curve which separates the space between E and B, and hence between A and B.

Theorem 9 admits the following generalization.

THEOREM 9'. *Let $F_0, \ldots, F_n$ be a system of disjoint closed sets none of which cuts the space. Then there exists a system of disks $D_0, \ldots, D_n$ such that*

$$F_j \subset D_j \quad \textit{and} \quad \overline{D}_i \cap \overline{D}_j = 0 \quad \textit{for} \quad i \neq j.$$

Proof. Since the union of a finite system of disjoint closed sets, none of which cuts the space, is not a cut (compare § 58, II, Theorem 2), the disks $D_0, \ldots, D_n$ can be defined successively so as to satisfy the following conditions

$$\begin{cases} F_0 \subset D_0 \quad \text{and} \quad \overline{D}_0 \cap (F_1 \cup \ldots \cup F_n) = 0, \\ F_1 \subset D_1 \quad \text{and} \quad \overline{D}_1 \cap (\overline{D}_0 \cup F_2 \cup \ldots \cup F_n) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_n \subset D_n \quad \text{and} \quad \overline{D}_n \cap (\overline{D}_0 \cup \ldots \cup \overline{D}_{n-1}) = 0. \end{cases}$$

THEOREM 10 (of Schönflies)(1). *If the set C is closed and locally connected and if the sequence $R_1, R_2, \ldots$ of components of $\mathscr{X} - C$ is infinite, then*

$$\lim_{n=\infty} \delta(R_n) = 0. \tag{13}$$

(1) A. Schönflies, *Bericht über die Entwickelung der Mengenlehre* II, Leipzig 1908, p. 199.

Proof. Suppose conversely that $\varepsilon > 0$, $i_1 < i_2 < \ldots$ and that $\delta(R_{i_n}) > \varepsilon$ for $n = 1, 2, \ldots$. It can be assumed (compare § 29, VIII) that the sequence $\{R_{i_n}\}$ is convergent. Let $r \in \operatorname{Lim}_{n=\infty} R_{i_n}$. Therefore $r \in C$. Referring to Theorem 7, let U and V be two disks such that

$$r \in V, \qquad \overline{V} \subset U \qquad \text{and} \qquad \delta(U) < \varepsilon.$$

Let $K = \operatorname{Fr}(U)$ and $L = \operatorname{Fr}(V)$. It follows that, for sufficiently large values of n,

$$V \cap R_{i_n} \neq 0, \qquad \text{which implies} \qquad K \cap R_{i_n} \neq 0 \neq L \cap R_{i_n}$$

since $R_{i_n} - U \neq 0$.

So, there exists an arc $A_{i_n} \subset R_{i_n}$ which is irreducible between K and L. This means that only the end-points of A_{i_n}, say k_{i_n}, l_{i_n}, belong to K and L respectively.

We can assume that the sequence $\{A_{i_n}\}$ is convergent,

$$Z = \operatorname{Lim}_{n=\infty} A_{i_n}, \qquad \text{hence} \qquad Z \subset C \qquad \text{and} \qquad Z \cap K \neq 0 \neq Z \cap L.$$

Since Z is a continuum, there exists a point $p \in Z - (K \cup L)$. We are going to show that C is not locally connected at p. It is sufficient to prove that every region G such that

$$p \in G \qquad \text{and} \qquad G \cap (K \cup L) = 0 \tag{14}$$

contains a point of C which cannot be joined with p by a subcontinuum of $C - (K \cup L)$.

The set G has points in common with infinitely many terms of the sequence $\{A_{i_n}\}$ since G is a neighbourhood of p. For the sake of simplicity we may assume that

$$G \cap A_j \neq 0 \qquad \text{for} \qquad j = 0, 1, 2. \tag{15}$$

Consider the curve θ consisting of arcs A_0, A_1, A_2 and of arcs $k_0k_1k_2$ and $l_0l_1l_2$ contained in K and L respectively.

Let D_0, D_1 and D_2 be the disks, which are components of $\mathscr{X} - \theta$; then (compare Theorem 2)

$$\operatorname{Fr}(D_j) = A_j \cup A_{j+1} \cup k_j k_{j+1} \cup l_j l_{j+1}.$$

One of these disks, say D_2, contains p (since $p \in C - K - L$). It follows from (15) that the region G contains an arc M which is irreducible between the sets A_1 and $A_0 \cup A_2$, so that the set $M \cap (A_0 \cup A_1 \cup A_2)$ consists of the end-points of M, one of which is $M \cap A_1$ and the other is $M \cap (A_0 \cup A_2)$. Since the set $N = M - (A_0 \cup A_1 \cup A_2)$ is connected and disjoint from $K \cup L$

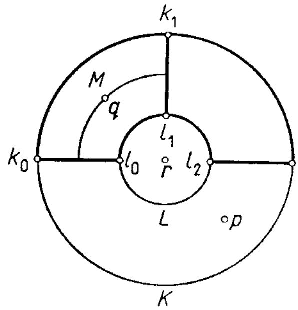

Fig. 18

(by (14)), N is contained in one of the disks D_0, D_1 or D_2; and since $M \cap A_1 \neq 0$, then either $N \subset D_0$ or $N \subset D_1$. Assume that

$$N \subset D_0. \tag{16}$$

Therefore $M \subset \overline{D}_0$. Since

$$0 \neq M \cap A_1 \subset M \cap R_1 \quad \text{and} \quad 0 \neq M \cap (A_0 \cup A_2) \subset M \cap (R_0 \cup R_2),$$

it follows that $M \cap C \neq 0$. Let $q \in M \cap C$. Thus, $q \in G \cap N$ and by (16)

$$q \in G \cap D_0. \tag{17}$$

It remains to be shown that every continuum H such that $p, q \in H \subset C$ intersects the set $K \cup L$.

It follows from (17) that

$$0 \neq H \cap \operatorname{Fr}(D_0) = H \cap (A_0 \cup A_1 \cup k_0k_1 \cup l_0l_1) \subset C \cap (A_0 \cup A_1) \cup \big(H \cap (K \cup L)\big),$$

which implies the required conclusion, since $C \cap (A_0 \cup A_1) = 0$.

THEOREM 11. *If C is a closed locally connected set and R is a component of $\mathscr{X} - C$, every point $p \in \operatorname{Fr}(R)$ is accessible from R.*

More precisely, $R \cup p$ *is locally connected*(1).

Proof. In order to establish the second part of Theorem 11 it is sufficient to show that, if D is a disk containing p (compare Theorem 7), there exists a connected set T open in $R \cup p$ and such that $p \in T \subset D$. But, since the sets $\mathscr{X}-D$ and $\mathscr{X}-R$ are closed and locally connected (§ 49, II, Theorem 1), then so is their union $\mathscr{X}-(D \cap R)$. Thus, by Theorem 10, the sequence $R_1, R_2, \ldots$ of components of $D \cap R$ satisfies condition (13) (unless it is finite).

We may suppose that the sequence $\{R_n\}$ does not reduce to a single term (because in that case one has only to set $T = D \cap R \cup p$). Since R is connected and R_n is open in R, it follows that the boundary of R_n relative to R (let us denote it by $\mathrm{Fr}_R(R_n)$) is not empty, so that $R \cap \bar{R}_n - R_n \neq 0$. By Theorem 3 of § 49, III,

$$\mathrm{Fr}_R(R_n) \subset \mathrm{Fr}_R(R \cap D) = R \cap \overline{R \cap D} - R \cap D \subset \mathscr{X}-D,$$
$$\text{thus} \quad \bar{R}_n - D \neq 0.$$

It follows that $\delta(R_n) \geqslant \varrho(p, \mathscr{X}-D)$ if $p \in \bar{R}_n$. Therefore, the number of regions R_n such that $p \in \bar{R}_n$ is finite in consequence of identity (13). Let $R_1, \ldots, R_k$ be these regions. Then the set $T = R_1 \cup \ldots \cup R_k \cup p$ is connected.

It remains to be proved that T is open in $R \cup p$, i.e. that $p \notin \mathrm{Ls}\, R_n$. But, if $p = \lim\limits_{i=\infty} p_i$ where $p_i \in R_{n_i}$, it would follow for sufficiently large values of i that

$$\delta(R_{n_i}) \geqslant \varrho(p_i, \mathscr{X}-D) > \tfrac{1}{2}\varrho(p, \mathscr{X}-D),$$

contradicting (13).

Thus the local connectedness of $R \cup p$ has been established, and p can be joined to every point of R by an arc in $R \cup p$. Because $R \cup p$ is connected, locally connected and topologically complete (compare § 50, II, Theorem 1).

THEOREM 12 (the converse of the Jordan theorem)(2). *Let C be a continuum and R_0 and R_1 two components of $\mathscr{X}-C$ such that every*

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(1) This theorem is due to Schönflies. Compare G. T. Whyburn, Bull. Amer. Math. Soc. 34 (1928), p. 504 and *Analytic Topology*, Chapter VI, § 4.

(2) Compare A. Schönflies, *op. cit.* p. 180. See also F. Riesz, Math. Ann. 59 (1914), p. 409.

point of C is accessible from R_0 and from R_1. Then C is a simple closed curve.

Proof. By Theorem 2 of § 47, V, it is sufficient to show that every pair of points $a, b \in C$ separates C. Now, since a and b are accessible from R_j ($j = 0, 1$), there exists an arc $(ab)_j$ such that setting $A_j = (ab)_j - a - b$, we have

$$A_j \subset R_j. \tag{18}$$

Let D_0 and D_1 be the disks into which the simple closed curve $A_0 \cup a \cup b \cup A_1$ cuts the space. It remains to show that $D_0 \cap C \neq 0 \neq D_1 \cap C$.

But, since the set $S_j = D_j \cup A_0 \cup A_1$ is connected, the inequalities $S_j \cap R_0 \neq 0 \neq S_j \cap R_1$ (which follow from (18)) imply that

$$S_j \cap \mathrm{Fr}(R_0) \neq 0, \quad \text{so that} \quad S_j \cap C \neq 0,$$
$$\text{therefore} \quad D_j \cap C \neq 0,$$

because $(A_0 \cup A_1) \cap C = 0$ by (18).

THEOREM 13 (of Gehman)(1). *If C is a hereditarily locally connected subcontinuum of $\mathscr{X}$, and if $K_0, K_1, \ldots$ is an infinite sequence of subcontinua of C which are pairwise disjoint, then*

$$\lim_{n=\infty} \delta(K_n) = 0.$$

Proof. Suppose conversely that there exists in C an infinite sequence $\{K_n\}$ of subcontinua such that

$$\delta(K_n) > \eta \quad \text{where} \quad \eta > 0, \tag{19}$$

$$K_n \cap K_m = 0 \quad \text{for} \quad n \neq m. \tag{20}$$

In view of (19), let $p_n, q_n \in K$ and $|p_n - q_n| \geqslant \eta$. Since K_n is locally connected, so let A_n be an arc $p_n q_n \subset K_n$. It is admissible to assume that the sequences $\{p_n\}$ and $\{q_n\}$ are convergent,

$$\lim_{n=\infty} p_n = p \quad \text{and} \quad \lim_{n=\infty} q_n = q.$$

Since the points p and q are distinct, let P and Q be two disjoint continua which do not cut $\mathscr{X}$ and such that $p \in \mathrm{Int}(P)$ and $q \in \mathrm{Int}(Q)$ (compare Theorem 7).

(1) Annals of Math. 27 (1926), p. 39.

It follows that $P \cap A_n \neq 0 \neq Q \cap A_n$ for sufficiently large values of n. It is admissible to assume that, in fact, this is so for each value of n. Take out of A_n an arc B_n only the end-points of which belong to P and to Q respectively. It can be also assumed that the sequence $\{B_n\}$ is convergent.

The decomposition of the space $\mathscr{X}$ in single points of $\mathscr{X}-P-Q$ and the sets P and Q is upper semi-continuous. By Theorem 9 of Section I, there exists a continuous mapping f of $\mathscr{X}$ such that

(i) $f(\mathscr{X})$ is a Janiszewski space without cut points,

(ii) $f(P)$ and $f(Q)$ reduce to two distinct points a and b,

(iii) f is a homeomorphism of $\mathscr{X}-P-Q$ onto $f(\mathscr{X})-a-b$.

Set $L_n = f(B_n)$. It follows that $L_0, L_1, \ldots$ is a convergent sequence of arcs which have pairwise only their end-points (a and b) in common (compare (20)). According to Theorem 3 we have $L_n \cap \operatorname*{Lim}_{m=\infty} L_m = (a, b)$ for each n with the exception of a pair of subscripts (say 0 and 1). In other words

$$(L_n - a - b) \cap \operatorname*{Lim}_{m=\infty}(L_m - a - b) = 0 \quad \text{for} \quad n > 1.$$

Denoting the arc B_n without end-points by B_n^* we have

$$B_n^* = f^{-1}(L_n - a - b), \quad \text{therefore}$$

$$B_n^* \cap \operatorname*{Lim}_{m=\infty} B_m^* = 0 \quad \text{for} \quad n > 1.$$

But then $\operatorname*{Lim}_{m=\infty} B_m^*$ is a convergence continuum containing more than one point and consequently C is not hereditarily locally connected (compare § 50, IV, Theorem 2).

Remarks. The theorem does not hold in $\mathscr{E}^3$. See § 50, IV, 3 Remark.

Theorem 13 implies immediately the following

THEOREM 13′. *If a hereditarily locally connected subcontinuum of $\mathscr{X}$ is decomposed into disjoint continua, the decomposition is semi-continuous.*

THEOREM 14[1]. *If A and B are two separated sets, there exists an open set G such that*

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[1] See my paper *Sur la séparation d'ensembles situés sur le plan,* Fund. Math. 12 (1928), p. 217.

$$A \subset G, \tag{21}$$

$$\bar{G} \cap B = 0, \tag{22}$$

$$\mathrm{Fr}(G) - (\bar{A} \cap \bar{B}) \subset [\mathrm{Fr}(G)]^{[\omega]}, \tag{23}$$

i.e. the set $\mathrm{Fr}(G)$ *is regular at every point of* $\mathrm{Fr}(G) - (\bar{A} \cap \bar{B})$.

Proof. Define

$$A_n = \mathop{E}_{x}(x \in \bar{A})\left[\frac{1}{n+1} \leqslant \varrho(x, \bar{B}) \leqslant \frac{1}{n}\right].$$

It follows that

$$\bar{A} - \bar{B} = A_0 \cup A_1 \cup \ldots. \tag{24}$$

Since the set A_n is compact, there exists a finite system of disks $D_1^n, \ldots, D_{k_n}^n$ such that

$$A_n \cap D_i^n \neq 0, \tag{25}$$

$$\overline{D_i^n} \cap \bar{B} = 0, \tag{26}$$

$$\delta(D_i^n) < 1/n \tag{27}$$

and

$$A_n \subset D_1^n \cup \ldots \cup D_{k_n}^n. \tag{28}$$

Let

$$G = \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{k_n} D_i^n. \tag{29}$$

Since A and B are separated, it follows that $A \subset \bar{A} - \bar{B}$, which implies the inclusion (21) in view of conditions (24), (28) and (29).

Before we go over to (22), observe that every point p of $\bar{G} - \bar{A}$ belongs to some $\overline{D_{i_0}^{n_0}}$ and that in the neighbourhood of p there is only a finite number of disks D_i^n.

Suppose that

$$p = \lim_{m=\infty} p_m, \quad \text{where} \quad p_m \in D_{i_m}^{n_m} \quad \text{and} \quad n_m < n_{m+1}.$$

It follows by (25), (27) and (24) that

$$p \in \overline{A_1 \cup A_2 \cup \ldots} \subset \bar{A},$$

contradicting the hypothesis.

It follows immediately that there exists a pair of subscripts (n_0, i_0) such that $p \in \overline{D_{i_0}^{n_0}}$.

This being established, it follows, on the one hand, by (26) that

$$\bar{G} \cap \bar{B} - \bar{A} = 0, \tag{30}$$

which yields identity (22), since $B \cap \bar{A} = 0$.

On the other hand, since every point p of the set $\mathrm{Fr}(G) - (\bar{A} \cap \bar{B}) = \mathrm{Fr}(G) - \bar{A}$ is contained in a neighbourhood relative to $\mathrm{Fr}(G)$ which consists of a finite number of simple closed curves, inclusion (23) follows (because every finite union of arcs is a regular set by Theorem 8 of § 51, IV).

COROLLARY 15. *If A and B are two separated sets, there exists for every pair $a \epsilon A$, $b \epsilon B$ a closed C such that $C \cap (A \cup B) = 0$, which is irreducibly separating $\mathscr{X}$ between a and b, and which is locally an arc at every point of $C - \bar{A} \cap \bar{B}$.*

Moreover, *if* $\dim \bar{A} \cap \bar{B} = 0$, *then C is a simple closed curve*(1).

Proof. Since $\mathrm{Fr}(G)$ is a cut between a and b, it contains an irreducible cut C between these points (compare § 49, V, Theorem 3). By Theorem 8 of Section I, the set C is discoherent and therefore, according to Theorem 5 of § 49, VI, it is locally an arc at every point of $C - (\bar{A} \cap \bar{B})$.

Finally, since the set of points at which C is not locally connected is either empty or of a positive dimension (compare § 49, VI, Theorem 1), the second part of the Corollary follows from the discoherence of C by Theorem 8 of Section I.

THEOREM 16. *If A and B are two continua such that*

(i) *$A - B$ is connected,*

(ii) $\dim A \cap B = 0$,

(iii) *A is not a cut between any pair of points of $B - A$,*

then there exists a simple closed curve which is a cut between $A - B$ and $B - A$(2).

THEOREM 17. *If R is a region and A is an arc pq such that $A - R = (p, q)$, then the set $R - A$ is connected if and only if the end-points of A belong to different components of $\mathscr{X} - R$ (and hence of* $\mathrm{Fr}(R)$)(3).

(1) Compare R. L. Moore, *Concerning the separation of point sets by curves*, Proc. Nat. Ac. Sc. 11 (1925), p. 469, and R. G. Lubben, Trans. Amer. Math. Soc. 30 (1928), p. 668.

(2) For the proof see my paper, already cited, in Fund. Math. 12, p. 232. Compare also R. L. Moore, *op. cit.* p. 470.

(3) Compare F. Hausdorff, *Grundzüge der Mengenlehre*, p. 350.

Proof. If the points p and q belong to the same component C of $\mathscr{X}-R$, the set $R-A$ is not connected (by Theorem 6″ of Section I).

Now, assume that the points p and q belong to different components of $\mathscr{X}-R$. Thus, the set $\mathscr{X}-R$ is not connected between these points, so that there exist two closed sets P and Q such that

$$\mathscr{X}-R = P\cup Q, \quad P\cap Q = 0, \quad p\in P, \quad q\in Q. \tag{31}$$

Let u and v be two arbitrary points of the set $R-A$. We have to prove that the set $\mathscr{X}-R\cup A$ does not separate them.

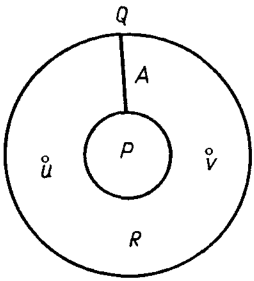

Fig. 19

It follows by (31) that

$$\mathscr{X}-R\cup A = (P\cup A)\cup(Q\cup A)$$
$$\text{and} \quad (P\cup A)\cap(Q\cup A) = A. \tag{32}$$

Applying Theorem 1′, we deduce from Theorem 7 of Section I that neither $P\cup A$ nor $Q\cup A$ separates the points u and v. By (32) and Theorem 7 of Section I, it follows that the set $\mathscr{X}-R\cup A$ does not separate them either.

III. Elementary sets. As before, let $\mathscr{X}$ be a Janiszewski space containing no cut points.

Let $D_0, \ldots, D_n$ be a system of disks such that

$$\overline{D}_i\cap\overline{D}_j = 0 \quad \text{for} \quad i\neq j. \tag{1}$$

The sets

$$\mathscr{X}-(D_0\cup\ldots\cup D_n) \quad \text{and} \quad \mathscr{X}-(\overline{D}_0\cup\ldots\cup\overline{D}_n)$$

are said to be an *elementary continuum* and an *elementary region* respectively. The space $\mathscr{X}$ and the set 0 are considered both as elementary continua and as elementary regions.

We shall establish a number of properties of elementary sets which will be used subsequently (Section V and § 62, XII).

The finite union of disjoint elementary continua is said to be *an elementary closed set*. The finite union of elementary regions whose closures are disjoint is called an *elementary open set*.

It is easy to establish the following five statements.

THEOREM 1. *The interior of an elementary continuum is an elementary region. The interior of an elementary closed set is an elementary open set.*

THEOREM 2. *The closure of an elementary region is an elementary continuum. The closure of an elementary open set is an elementary closed set.*

THEOREM 3. *Every elementary closed (open) set is a closed (open) domain.*

THEOREM 4. *If $A_1, \ldots, A_m$ are the components of an elementary set A, then*

$$\mathrm{Fr}(A) = \mathrm{Fr}(A_1) \cup \ldots \cup \mathrm{Fr}(A_m) \quad \textit{and} \quad \mathrm{Int}(A) = \mathrm{Int}(A_1) \cup \ldots \cup \mathrm{Int}(A_m).$$

THEOREM 5. *If R is an elementary region and D is a disk such that $\mathrm{Fr}(D) \subset R$, the set $R \cap D$ is an elementary region.*

Similarly, *if C is an elementary continuum and D is a disk such that $\mathrm{Fr}(D) \subset \mathrm{Int}(C)$, the set $C \cap \bar{D}$ is an elementary continuum.*

THEOREM 6. *If R is an elementary region and C is an elementary continuum such that $C \subset R$, the set $R - C$ is elementary.*

Similarly, *if C is an elementary continuum and R is an elementary region such that $\bar{R} \subset \mathrm{Int}(C)$, then $C - R$ is an elementary set.*

Proof. Set $X' = \mathscr{X} - X$, and let

$$C = (D_0 \cup \ldots \cup D_n)' \quad \text{and} \quad K_j = \mathrm{Fr}(D_j).$$

It follows that $R - C = (R \cap D_0) \cup \ldots \cup (R \cap D_n)$. Therefore the inclusion $K_j \subset R$ implies by Theorem 5 that the set $R \cap D_j$ is an elementary region.

Since $\overline{R \cap D_i} \cap \overline{R \cap D_j} = 0$ according to (1), it follows that $R - C$ is an elementary set.

The proof of the second part of Theorem 6 is similar.

THEOREM 7. *The complement of an elementary closed (open) set is an elementary open (closed) set.*

Proof. Let $F = C_0 \cup \ldots \cup C_n$ and $C_i \cap C_j = 0$, where the sets $C_0, \ldots, C_n$ are elementary continua.

Proceed by induction. Since the theorem is obvious for $n = 0$, assume that it holds for $n-1$. It follows that

$$F' = G - C_n, \quad \text{where} \quad G = C_0' \cap \ldots \cap C_{n-1}'.$$

Since the open set G is elementary (by hypothesis), it follows that $G = R_0 \cup \ldots \cup R_m$ and $\bar{R}_i \cap \bar{R}_j = 0$, where the sets $R_0, \ldots, R_m$ are elementary regions.

Now $C_n \cap (C_0 \cup \ldots \cup C_{n-1}) = 0$, so that $C_n \subset G$. Therefore, the continuum C_n is contained in one of the components of G, say $C_n \subset R_0$. Consequently,

$$F' = (R_0 - C_n) \cup R_1 \cup \ldots \cup R_m. \tag{2}$$

Since $R_0 - C_n$ is an elementary region (by Theorem 6), F' is an elementary open set.

The argument is analogous in the case of an open set.

THEOREM 8. *If $G_0, \ldots, G_n$ are elementary open sets such that $G_i \cup G_j = \mathscr{X}$ for $i \neq j$, the intersection $G_0 \cap \ldots \cap G_n$ is an elementary set. If $F_0, \ldots, F_n$ are elementary closed sets such that* $\mathrm{Int}(F_i) \cup \mathrm{Int}(F_j) = \mathscr{X}$ *for $i \neq j$, the intersection $F_0 \cap \ldots \cap F_n$ is an elementary set.* Moreover,

$$\left.\begin{aligned} b_0(G_0 \cap \ldots \cap G_n) &= b_0(G_0) + \ldots + b_0(G_n), \\ b_0(F_0 \cap \ldots \cap F_n) &= b_0(F_0) + \ldots + b_0(F_n). \end{aligned}\right\} \tag{3}$$

Proof. $G_i' \cap G_j' = 0$ implies that $G_0' \cup \ldots \cup G_n'$ is an elementary set, as the union of disjoint and closed elementary sets (compare Theorem 7).

Similarly, the condition $\overline{F_i'} \cap \overline{F_j'} = 0$ implies that the open set $F_0' \cup \ldots \cup F_n'$ is elementary.

Formulas (3) follow immediately from Theorem 4 of § 57, II (generalized to the case of n terms).

THEOREM 9. *If E is an elementary set and K is a component of $\mathscr{X} - E$, the set $E \cup K$ is elementary.*

Proof. Let $K_0, \ldots, K_n$ be the components of E',

$$E' = K_0 \cup \ldots \cup K_n, \quad \bar{K}_i \cap \bar{K}_j = 0.$$

It follows that $(E \cup K_0)' = K_1 \cup \ldots \cup K_n$.

Since the components of an elementary set are elementary sets, it follows from Theorem 7 that $E \cup K_0$ is elementary.

THEOREM 10. *If F is a closed subset of an open set G, there exists an elementary closed set A such that*

$$F \subset \operatorname{Int}(A) \subset A \subset G, \tag{4}$$

$$b_0(A) \leqslant b_0(F), \tag{5}$$

$$b_0(A') \leqslant b_0(G'). \tag{6}$$

Proof. Let $R_0, \ldots, R_k$ be the components of G which contain points of F. Define $F_i = F \cap R_i$. It follows that

$$F = F_0 \cup \ldots \cup F_k \quad \text{and} \quad 0 \neq F_i = \overline{F}_i, \tag{7}$$

because $\overline{F \cap R_i} - (F \cap R_i) \subset F \cap \overline{R}_i - R_i \subsetneq F - G = 0$.

Moreover,

$$k \leqslant b_0(F). \tag{8}$$

Let $R_{i1}, \ldots, R_{im_i}$ (where $0 \leqslant i \leqslant k$) be the components of F_i' which contain points of R_i'. Define $H_{ij} = R_{ij} - R_i$. It follows as before that

$$R_i' = H_{i0} \cup \ldots \cup H_{im_i} \quad \text{and} \quad \overline{H}_{ij} = H_{ij}. \tag{9}$$

Since H_{ij} is closed-open in R_i', it is the union of a family of components of R_i'. Since R_i is a region, it follows that H_{ij} is not a cut, because otherwise there would exist a component of H_{ij} (therefore a component of R_i') which would be a cut (by § 57, III, Theorem 1), which is not consistent with Theorem 5 of § 46, III.

Therefore there exists by Theorem 9 of Section II a disk D_{ij} such that

$$H_{ij} \subset D_{ij} \subset \overline{D}_{ij} \subset R_{ij}. \tag{10}$$

Conditions (9) and (10) imply that

$$R_i' \subset D_{i0} \cup \ldots \cup D_{im_i} \subset \overline{D}_{i0} \cup \ldots \cup \overline{D}_{im_i} \subset F_i' \quad \text{and} \quad \overline{D}_{ij} \cap \overline{D}_{ij'} = 0. \tag{10'}$$

Moreover,

$$m_i \leqslant b_0(R_i'). \tag{11}$$

By (10′) the elementary continuum $C_i = (D_{i0} \cup \ldots \cup D_{im_i})'$ satisfies the condition

$$F_i \subset \operatorname{Int}(C_i) \subset C_i \subset R_i, \quad \text{where} \quad i = 0, \ldots, k, \tag{12}$$

and consequently (compare (7)), the elementary closed set

$$A = C_0 \cup \ldots \cup C_k \tag{13}$$

satisfies conditions (4).

Formula (5) follows from (8) and (13). Finally,

$$\begin{aligned} b_0(A') &= b_0(C_0') + \ldots + b_0(C_k') = m_0 + \ldots + m_k \\ &\leqslant b_0(R_0') + \ldots + b_0(R_k') = b_0(R_0' \cap \ldots \cap R_k') \leqslant b_0(G'), \end{aligned}$$

by (13), (11), (3) and Theorem 4 of § 57, II (compare also § 49, II, Theorem 4).

Theorem 10 immediately implies (compare § 50, III, Theorem 2) the following

THEOREM 11. *For every closed set F there exists a sequence of elementary closed sets $F_1, F_2, \ldots$ satisfying the following conditions*

$$F = F_1 \cap F_2 \cap \ldots, \tag{14}$$

$$F_n \subset \operatorname{Int}(F_{n-1}), \tag{15}$$

$$b_0(F_n) \leqslant b_0(F), \tag{16}$$

$$b_0(\mathscr{X} - F_n) \leqslant b_0(\mathscr{X} - F). \tag{17}$$

In particular, *if F is a continuum, then so is F_n; if F does not separate the space, then F_n does not separate it either.*

IV. Topological characterization of $\mathscr{S}_2$(¹)**. Consequences.** Let $\mathscr{X}$ be a Janiszewski space without cut points. We will establish

(¹) Concerning this question compare R. L. Moore, *On the foundations of plane analysis situs,* Trans. Amer. Math. Soc. 17 (1916), p. 131, J. Gawehn, Math. Ann. 98 (1927). See also H. Whitney, *A characterization of the closed 2-cell,* Trans. Amer. Math. Soc. 35 (1933), p. 26, and E. R. v. Kampen, *On some characterizations of 2-manifolds,* Duke Math. Journ. 1 (1935), p. 74, containing many references.

the homeomorphism

$$\mathscr{X} \underset{\text{top}}{=} \mathscr{S}_2 \text{ (}^1\text{)}. \tag{1}$$

To this end we will introduce an auxiliary concept and we shall establish a number of its properties.

DEFINITION. The union $L_0 \cup \ldots \cup L_n$ of $n+1$ arcs $(n \geqslant 1)$ is said to be a *network* if

(i) $L_0 \cup L_1$ is a simple closed curve,

(ii) $L_k \cap (L_0 \cup \ldots \cup L_{k-1})$ consists of the end-points of L_k.

The network $L_0 \cup \ldots \cup L_n$ is said to be an *extension* of the network $L_0 \cup \ldots \cup L_m$ if $m < n$.

The following statement is easily established by finite induction.

THEOREM 1. *The set* $\mathscr{X} - (L_0 \cup \ldots \cup L_n)$ *consists of* $n+1$ *disjoint disks (which are its components).*

THEOREM 2. *Let* D *be a disk,* A *and* B *two subcontinua of* $\bar{D}$ *and* L *an arc such that the set* $L \cap \bar{D}$ *separates* $\bar{D}$ *between* A *and* B. *Then there exists a component* M *of* $D \cap L$ *which separates* D *between* $A \cap D$ *and* $B \cap D$.

Proof. It is legitimate to assume that $A \cap D \neq 0 \neq B \cap D$. Let $a \in A \cap D$ and $b \in B \cap D$. Then the set $L \cap D$ separates D between a and b. Since $\mathscr{X} - D$ is a continuum (by Theorem of Jordan), D is contractible with respect to $\mathscr{S}$ (by Theorem 3 of Section I); thus $L \cap D$ contains (compare § 58, III, Theorem 1) a component M which separates D between a and b. Obviously $\mathrm{Fr}(D) \cup M$ is a θ curve. Let U and V denote the components of $\mathscr{X} - \theta$ which contain a and b respectively; it follows that

$$A \subset \bar{U} \quad \text{and} \quad B \subset \bar{V}, \quad \text{thus}$$

$$A \cap D \subset U \quad \text{and} \quad B \cap D \subset V.$$

THEOREM 3. *Let* R_0 *be a network and* A *and* B *two disjoint continua which do not separate the space. Then there exists a network* R_1 *which is an extension of the network* R_0 *and which separates* $\mathscr{X}$

(1) See my paper, *Une caractérisation topologique de la surface de la sphère*, Fund. Math. 13 (1929), p. 307. Compare also *Un système d'axiomes pour la Topologie de la surface de la sphère*, Atti del Congr. Int. dei Matemat. Bologna 1928, VI, p. 239.

between $A-R_1$ *and* $B-R_1$ *(so that there is no component of* $\mathscr{X}-R_1$ *which contains points both of* A *and of* B*).*

Proof. Let D be a component of $\mathscr{X}-R_0$ such that $D \cap A \neq 0 \neq D \cap B$. Let us assume first that

$$A \cap \mathrm{Fr}(D) \neq 0 \neq B \cap \mathrm{Fr}(D). \tag{2}$$

Let C be a simple closed curve which separates A from B (see Theorem 5′ of Section II). Then the set $C \cap \bar{D}$ separates $\bar{D}$ between $\bar{D} \cap A$ and $\bar{D} \cap B$, and $C-\bar{D} \neq 0$ by (2). Therefore there exists a subarc L of C which contains $C \cap \bar{D}$ and whose end-points belong to $\mathrm{Fr}(D)$. Let (compare § 50, III, Theorem 1) $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$ be two systems of continua such that

$$\bar{D} \cap A \subset F \subset \bar{D}, \quad \bar{D} \cap B \subset H \subset \bar{D},$$
$$F \cap H = 0, \quad F \cap L = 0 = H \cap L,$$

where $F = A_1 \cup \ldots \cup A_m$ and $H = B_1 \cup \ldots \cup B_n$.

By Theorem 2, for every pair i, j (where $i \leqslant m$ and $j \leqslant n$) there exists a component M_{ij} of $D \cap L$ which separates D between $D \cap A_i$ and $D \cap B_j$. Put

$$R(D) = \mathrm{Fr}(D) \cup \bigcup_{ij} M_{ij}.$$

Then no component of $\bar{D}-R(D)$ contains points both of A and of B.

If formula (2) does not hold, say $A \cap \mathrm{Fr}(D) = 0$, then $A \subset D$ and there exists (see Theorem 6 of Section II) a simple closed curve C which separates A and B and which is contained in D. In that case $R(D)$ is defined as a network which is obtained joining the curves $\mathrm{Fr}(D)$ and C by two arcs which are irreducible between these curves.

Finally, put $R_1 = \bigcup R(D)$, where the summation runs over all the components D of $\mathscr{X}-R_0$.

THEOREM 4. *For every* $\varepsilon > 0$ *there exists a network* R *such that the components of* $\mathscr{X}-R$ *have diameters* $\leqslant \varepsilon$.

Moreover, *the network* R *can be chosen to be an extension of a network* R_0 *given in advance.*

Proof. In accordance with Corollary 7 of Section II, let $K_1, \ldots, K_n$ be continua which do not cut the space and such that

$$\mathscr{X} = K_1 \cup \ldots \cup K_n \quad \text{and} \quad \delta(K_i) < 1/2. \tag{3}$$

Let

$$(j_1, m_1), (j_2, m_2), \ldots, (j_r, m_r) \tag{4}$$

be the system of all pairs of subscripts such that $K_{j_i} \cap K_{m_i} = 0$.

Let R_0 be a network (a simple closed curve, for instance). Applying successively Theorem 3 to the pairs

$$(K_{j_1}, K_{m_1}), \ldots, (K_{j_r}, K_{m_r}),$$

a network R is obtained such that no component D of $\mathscr{X} - R$ satisfies for any $i \leqslant r$ the condition

$$D \cap K_{j_i} \neq 0 \neq D \cap K_{m_i}. \tag{5}$$

We claim that $\delta(D) \leqslant \varepsilon$. Suppose conversely that $a, b \in D$ and $|a \dot{-} b| > \varepsilon$. Let $a \in K_\alpha$ and $b \in K_\beta$. Then $K_\alpha \cap K_\beta = 0$ by (3) and hence the pair α, β belongs to the system (4), say $\alpha = j_i$ and $\beta = m_i$. Hence condition (5) is satisfied. But this is impossible, as just has been proved.

For the sake of brevity, a homeomorphism $h\colon R \to R^*$ of the network R onto the network R^* will be called *regular* if the components of $\mathscr{X} - R$ and of $\mathscr{X} - R^*$ can be ordered into two systems

$$D_1, \ldots, D_n \quad \text{and} \quad D_1^*, \ldots, D_n^*$$

in such a manner that

$$h[\mathrm{Fr}(D_i)] = \mathrm{Fr}(D_i^*) \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{6}$$

THEOREM 5. *Let R_0 and R_1 be two networks, the latter of which is an extension of the former. Every regular homeomorphism h of R_0 admits an extension to a regular homeomorphism h_1 of R_1.*

Proof. Clearly, it is sufficient to establish the theorem in the case where $R_1 = R_0 \cup L$, where L is an arc ab such that $L \cap R_0 = (a, b)$.

Assume for the sake of simplicity of notation that $L \subset D_n \cup (a, b)$. In view of (6), $h(a), h(b) \in \mathrm{Fr}(D_n^*)$. Let (compare Theorem 11 of Section II) L^* be an arc $h(a)h(b)$ such that $L^* \subset D_n^* \cup h(a) \cup h(b)$.

The homeomorphism h can be immediately extended to $R_0 \cup L$ in such a manner that L is mapped onto L^*. Let h_1 denote the homeomorphism extended in that way.

It remains only to set $E_i = D_i$ and $E_i^* = D_i^*$ for $i < n$ and to denote by E_n and E_{n+1} (or by E_n^* and E_{n+1}^* respectively) the two components of $D_n - L$ (respectively of $D_n^* - L^*$) labeled in such a manner as to satisfy the condition (compare II, Theorem 2)

$$h_1[\mathrm{Fr}(E_j)] = \mathrm{Fr}(E_j^*) \quad \text{for } j = n \text{ and for } j = n+1.$$

Remark. In the case considered, where $R_1 = R_0 \cup L$, it is clear that

$$h_1(R_1 \cap \bar{D}_i) \subset \overline{D_i^*} \quad \text{for} \quad i = 1, \ldots, n. \tag{7}$$

It now follows by induction that (7) holds in the general case where $R_1 = R_0 \cup L_2 \cup L_3 \cup \ldots \cup L_m$, where the arcs $L_2, \ldots, L_m$ satisfy condition (ii) (replacing $L_0 \cup L_1$ by R_0).

Fundamental theorem, 6. *If $\mathcal{X}$ and $\mathcal{X}^*$ are two Janiszewski spaces which contain no separation points and which do not consist of single points, then $\mathcal{X} \underset{\text{top}}{=} \mathcal{X}^*$.*

In particular (compare Theorem 2 of Section I), *$\mathcal{X}$ is homeomorphic to the sphere $\mathcal{S}_2$.*

Proof. Assume that $\delta(\mathcal{X}) < 1$ and $\delta(\mathcal{X}^*) < 1$. Let R_0 and R_0^* be two simple closed curves lying in $\mathcal{X}$ and $\mathcal{X}^*$ respectively (compare Theorem 7 of Section II). Let h_0: $R_0 \to R_0^*$ be a homeomorphism of R_0 onto R_0^*. Denote by h_0^* the homeomorphism inverse to h_0.

In view of Theorem 4, let R_1^* be an extension of the network R_0^* such that the components of $\mathcal{X}^* - R_1^*$ have diameters $< 1/2$. In accordance with Theorem 5, let h_1^* be a regular homeomorphism such that $h_0^* \subset h_1^*$. Put $R_1 = h_1^*(R_1^*)$ and denote by h_1 the homeomorphism inverse to h_1^*.

Similarly, let R_2 be an extension of the network R_1 such that the components of $\mathcal{X} - R_2$ have diameters $< 1/3$, and let h_2 be a regular homeomorphism such that $h_1 \subset h_2$. Define $R_2^* = h_2(R_2)$ and denote by h_2^* the homeomorphism inverse to h_2.

Proceeding this way step by step, we define:

(i) two sequences of networks

$$R_0 \subset R_1 \subset \ldots \subset \mathcal{X}, \quad R_0^* \subset R_1^* \subset \ldots \subset \mathcal{X}^*,$$

such that the components of $\mathcal{X} - R_n$ and of $\mathcal{X} - R_n^*$ have diameters $< 1/n$;

(ii) two sequences of regular homeomorphisms

$$h_0 \subset h_1 \subset \ldots, \quad h_0^* \subset h_1^* \subset \ldots,$$

such that h_n^* is the inverse to h_n and that $h_n(R_n) = R_n^*$.

Define

$$R = \bigcup_{n=0}^{\infty} R_n, \quad R^* = \bigcup_{n=0}^{\infty} R_n^*, \quad h = \sum_{n=0}^{\infty} h_n, \quad h^* = \sum_{n=0}^{\infty} h_n^*.$$

It follows that

$$h(R) = R^*, \quad h^*(R^*) = R,$$

and the mappings h and h^*, inverse to each other, are one-to-one.

We will show that they are continuous.

Let $x_0 \in R$. Let $\varepsilon > 0$ and $n > 1/\varepsilon$. Then $\delta(D^*) < \varepsilon$ for every component D^* of $\mathscr{X}^* - R_n^*$.

Let $D_1, \ldots, D_k$ be the system of all components D_i of $\mathscr{X} - R_n$ such that $x_0 \in \bar{D}_i$. Therefore the set $E = \bar{D}_1 \cup \ldots \cup \bar{D}_k$ is a neighbourhood of x_0.

It follows by (7) that

$$h(R \cap \bar{D}_i) \subset \overline{D_i^*},$$

therefore $h(x_0) \subset \overline{D_1^*} \cap \ldots \cap \overline{D_k^*}$ and $h(R \cap E) \subset \overline{D_1^*} \cup \ldots \cup \overline{D_k^*}$, so that $\delta[h(R \cap E)] < 2\varepsilon$. And hence the function h is continuous at the point x_0.

Similarly, the function h^* is continuous on R^*.

Now we are going to prove that the oscillation of the function h vanishes at the points of $\mathscr{X} - R$ (and that the oscillation of the function h^* vanishes at the points of $\mathscr{X}^* - R^*$).

Let $x_0 \in \mathscr{X} - R$. The subscript n being defined as before, let D be the component of $\mathscr{X} - R_n$ which contains x_0. It follows that

$$h(R \cap \bar{D}) \subset \overline{D^*}, \quad \text{hence} \quad \delta[h(R \cap D)] < \varepsilon,$$

and this proves that the oscillation of the function h vanishes at the point x_0 (compare § 21, III).

Therefore, there exists a continuous extension g of h to the whole space $\mathscr{X}$ (compare § 35, I, Theorem 1). It follows similarly that $h^* \subset g^*$, where $g^*\colon \mathscr{X}^* \to \mathscr{X}$ is a continuous function.

It remains to prove that the function g^* is inverse to g, i.e. that $g^*g(x) = x$.

Let $x = \lim_{n=\infty} x_n$ where $x_n \in R$. Therefore

$$g(x) = \lim_{n=\infty} g(x_n) = \lim_{n=\infty} h(x_n),$$

so that

$$g^*g(x) = \lim_{n=\infty} g^*h(x_n) = \lim_{n=\infty} h^*h(x_n) = \lim_{n=\infty} x_n = x.$$

Remarks. (i) If R_0 is an arbitrary network and $R_0^* = h_0(R_0)$, where h_0 is a *regular* homeomorphism, it can be proved as before that the homeomorphism $g\colon \mathscr{X} \to \mathscr{X}^*$ onto $\mathscr{X}^*$ is an extension of the homeomorphism h_0.

(ii) $\mathscr{S}_2$ can be characterized as follows:

$\mathscr{X}$ is a locally connected continuum containing at least one simple closed curve and every simple closed curve is an irreducible separator of $\mathscr{X}$(1).

COROLLARY 7(2). *A locally connected continuum is a Janiszewski space if and only if each of its cyclic elements, which does not reduce to a point, is homeomorphic to $\mathscr{S}_2$.*

Proof. If $\mathscr{X}$ is a Janiszewski space, then so is every one of its cyclic elements E (by Theorem 10 of Section I). Since E has no separating points, then $E \underset{\text{top}}{=} \mathscr{S}_2$ by Theorem 6 (provided that E does not reduce to a point).

Thus the considered condition is necessary. It is also sufficient, because the Janiszewski property is extensible (compare Theorem 10 of Section I) and $\mathscr{S}_2$ possesses it (compare Theorem 2 of Section I).

THEOREM 8 (of R. L. Moore)(3). *If f is a continuous mapping of $\mathscr{S}_2$ such that, for each y, $f^{-1}(y)$ is a continuum which does not cut the space, then $f(\mathscr{S}_2) \underset{\text{top}}{=} \mathscr{S}_2$.*

In other words, *the space of a semi-continuous decomposition of $\mathscr{S}_2$ into continua, which do not cut $\mathscr{S}_2$, is homeomorphic to $\mathscr{S}_2$.*

Proof. According to Theorem 9 of Section I, $f(\mathscr{S}_2)$ is a Janiszewski space and it contains no cut points (by hypothesis). Therefore, this space is homeomorphic to $\mathscr{S}_2$ by Theorem 6.

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(1) See L. Zippin, *On continuous curves and the Jordan curve theorem,* Amer. Journ. of Math. 52 (1930), p. 331.

(2) Compare L. Zippin, Trans. Amer. Math. Soc. 31 (1929), p. 744.

(3) *Concerning upper semi-continuous collections of continua,* Trans. Amer. Math. Soc. 27 (1925), p. 416. See also the paper by the same author in Monatsh. Math.-Phys. 36 (1929), p. 80, where a more general problem is considered (without the hypothesis that $f^{-1}(y)$ does not cut the space).

COROLLARY 9. *Every region* $R \subset \mathcal{S}_2$ *is homeomorphic to the sphere* $\mathcal{S}_2$ *without a closed set of dimension* 0, *namely without a set homeomorphic to the space of the decomposition of* $\mathcal{S}_2 - R$ *into components* (compare § 47, VI, Theorem 1).

In particular, *if the region* R $(\neq \mathcal{S}_2)$ *does not cut* $\mathcal{S}_2$, *then* $R \underset{\text{top}}{=} \mathcal{E}^2$.

Proof. On the one hand, the decomposition of $\mathcal{S}_2$ into components of $\mathcal{S}_2 - R$ and individual points of R is semi-continuous, and on the other hand, the components of $\mathcal{S}_2 - R$ do not cut $\mathcal{S}_2$ (by Theorem 5 of § 46, III).

THEOREM 10. *If* D *is a disk, then* $\overline{D} \underset{\text{top}}{=} \mathcal{J}^2$.

Proof. This is a direct consequence of Remark (i).

THEOREM 11. *If* $C \neq \mathcal{S}_2$ *is a locally connected continuum which does not cut* $\mathcal{S}_2$, *then* C *is an* AR. *If, moreover,* C *contains no cut points, then* $C \underset{\text{top}}{=} \mathcal{J}^2$ *(provided* C *contains more than one point).*

Proof. If C contains no cut points, then $R = \mathcal{S}_2 - C$ is a disk by II, 4 (ii), and so is $D = \mathcal{S}_2 - \overline{R}$. Hence $C = \mathcal{S}_2 - R = \overline{D} \underset{\text{top}}{=} \mathcal{J}^2$. To show the first part of the theorem, it is sufficient (by Theorem 16, p. 344), to prove that each cyclic element E of C is an AR. But this follows (as was just shown) from the fact that E does not cut $\mathcal{S}_2$ (by Theorems 15, p. 344 and 8, p. 475).

Remarks. Let U be the universal continuum of Sierpiński (see § 51, I, Example 5). Let $D_0, D_1, \ldots$ be the sequence of components of $\mathcal{S}_2 - U$. The decomposition of $\mathcal{S}_2$ in continua $\overline{D}_0, \overline{D}_1, \ldots$ and in the individual points of the set $V = U - (\overline{D}_0 \cup \overline{D}_1 \cup \ldots)$, is semi-continuous. Its space is homeomorphic to $\mathcal{S}_2$ by Theorem 8. The set V is therefore homeomorphic to the plane from which a countable dense set has been deleted.

Thus (see § 59, Theorem 11) the following statement holds.

THEOREM 12. *Every boundary subset of the plane is topologically contained in the continuum* U.

THEOREM 13 [1]. *If* C *is a one-dimensional continuum, the set of continuous functions* $f: C \to \mathcal{E}^2$ *such that* $f(C) \underset{\text{top}}{=} U$ *is residual in the space* $(\mathcal{E}^2)^C$.

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[1] For the proof, see S. Mazurkiewicz, Fund. Math. 31 (1938), p. 247. Compare also the paper by the same author, Fund. Math. 25 (1935), p. 253, and V. Jarník, Monatsh. Math.-Phys. 41 (1934), p. 408.

V. Extensions of homeomorphisms. Topological equivalence.

THEOREM 1. *If A ($\subset \mathcal{S}_2$) is an arc, or a simple closed curve, or a θ-curve, every homeomorphism $h: A \to \mathcal{S}_2$ can be extended to $\mathcal{S}_2$.*

In other words, *there exists a homeomorphism h^* such that*

$$h \subset h^* \quad \textit{and} \quad h^*(\mathcal{S}_2) = \mathcal{S}_2.$$

Proof. If A is a simple closed curve or a θ-curve, the theorem follows from Remark (i) to Theorem 6 of Section IV since in this case each homeomorphism of A is regular.

Let A be an arc ab and $h(A) = a_1 b_1$. Since the points a and b are accessible from $\mathcal{S}_2 - A$ (compare Theorem 11 of Section II), the arc A can be completed to a simple closed curve C. Similarly, $a_1 b_1$ is contained in a simple closed curve C_1. Obviously there exist a homeomorphism h_1 such that

$$h \subset h_1 \quad \text{and} \quad h_1(C) = C_1$$

and, as we have just stated, a homeomorphism h^* such that

$$h^*(\mathcal{S}_2) = \mathcal{S}_2 \quad \text{and} \quad h_1 \subset h^*, \quad \text{whence} \quad h \subset h^*.$$

COROLLARY 2. *All arcs lying in $\mathcal{S}_2$ are topologically equivalent. So are all simple closed curves and all θ-curves contained in $\mathcal{S}_2$.*

Remarks. (i) In $\mathcal{E}^3$ the corollary does not hold. Actually *there exists an arc which is not topologically equivalent to $\mathcal{I}$*(¹).

(ii) Let Φ denote the space of all homeomorphisms $f: \mathcal{S}_1 \to \mathcal{S}_3$. Then the set of all f such that $f(\mathcal{S}_1)$ is topologically equivalent to $\mathcal{S}_1$ is of the first category in Φ(²).

(iii) *There exists in $\mathcal{S}_3$ a surface homeomorphic with $\mathcal{S}_2$ which is not topologically equivalent with $\mathcal{S}_2$*(³).

(¹) L. Antoine, *Sur l'homéomorphie de figures et de leurs voisinages*, Journ. de Math. 1921, p. 221. See also the paper by that author in Fund. Math. 5 (1924), p. 265. Compare E. Artin and R. H. Fox, Ann. of Math. 49 (1948), p. 979; W. A. Blankinssip, *Generalization of a construction of Antoine*, Ann. of Math. 53 (1951), pp. 276–297; R. Y. T. Wong, *A wild Cantor set in the Hilbert cube*, Pacific Journ. Math. 24 (1968), p. 189, and V. Klee, Proc. Amer. Math. Soc. 7 (1956), p. 673.

(²) See J. Milnor, *Most knots are wild*, Fund. Math. 54 (1964), p. 335.

(³) Compare J. W. Alexander, Proc. Nat. Acad. Sc. 10 (1924) (three reports).

THEOREM 3. *Let* $D_0, \ldots, D_n$ *and* $D_0^*, \ldots, D_n^*$ *be two systems of disks such that*

$$\bar{D}_i \cap \bar{D}_j = 0 = \overline{D_i^*} \cap \overline{D_j^*} \quad \textit{for} \quad i \neq j.$$

Let $C_i = \mathrm{Fr}(D_i)$ *and* $C_i^* = \mathrm{Fr}(D_i^*)$. *Then every homeomorphism* $h\colon C_0 \to C_0^*$ *admits a homeomorphic extension* h^* *such that*

$$h^*(\mathcal{S}_2) = \mathcal{S}_2, \tag{1}$$

$$h^*(D_i) = D_i^* \quad \textit{for} \quad i = 0, \ldots, n. \tag{2}$$

Proof. Proceed by induction. Let $n = 1$. Let $A = a_0a_1$ and $B = b_0b_1$ be two disjoint arcs which are irreducible between $\bar{D}_0$ and $\bar{D}_1$, and similarly, let $A^* = a_0^*a_1^*$ and $B^* = b_0^*b_1^*$ be two disjoint arcs irreducible between $\overline{D_0^*}$ and $\overline{D_1^*}$, where $a_0^* = h(a_0)$ and $b_0^* = h(b_0)$ (compare Theorem 11 of Section II).

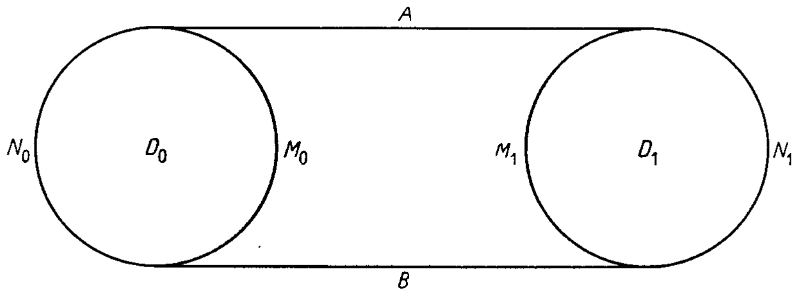

Fig. 20

The set $\mathcal{S}_2 - (C_0 \cup C_1 \cup A \cup B)$ consists of 4 disks D_0, D_1, D_2 and D_3. In order to see this, one has only to apply the Jordan theorem to the space which is obtained from $\mathcal{S}_2$ by considering $\bar{D}_0$ and $\bar{D}_1$ as single points.

Let M_j and N_j be the arcs a_jb_j of C_j labeled in such a way that

$$\mathrm{Fr}(D_2) = M_0 \cup A \cup M_1 \cup B,$$

$$\text{so that} \quad \mathrm{Fr}(D_3) = N_0 \cup A \cup N_1 \cup B. \tag{3}$$

Similarly, let

$$\mathcal{S}_2 - (C_0^* \cup C_1^* \cup A^* \cup B^*) = D_0^* \cup D_1^* \cup D_2^* \cup D_3^*,$$

$$M_0^* = h(M_0), \quad N_0^* = h(N_0), \tag{4}$$

where the sets M_1^* (and N_1^*) and D_2^* (and D_3^*) are labeled so as to satisfy the conditions

$$\mathrm{Fr}(D_2^*) = M_0^* \cup A^* \cup M_1^* \cup B^*,$$
$$\mathrm{Fr}(D_3^*) = N_0^* \cup A^* \cup N_1^* \cup B^*. \qquad (5)$$

Clearly, h admits an extension to a homeomorphism between $C_0 \cup C_1 \cup A \cup B$ and $C_0^* \cup C_1^* \cup A^* \cup B^*$ and hence to a homeomorphism h^* such that $h^*(D_j) = D_j^*$ for $j = 0, 1, 2, 3$ (by virtue of conditions (3) through (5)).

The case $n = 1$ is thus established; so let us assume that the theorem holds for $n-1$. We will show that it holds for n.

Let a and b be two points of C_0, and L_1 and L_2 two subarcs of C_0 with end-points a and b. One easily sees that there exists an arc ab such that $\bar{D}_0 \cap ab = (a, b)$ and that denoting by R_i the component of $\mathscr{S}_2 - C_0 - ab$ with the boundary $L_i \cup ab$ (where $i = 1, 2$), we have

$$\bar{D}_n \subset R_1 \quad \text{and} \quad \bar{D}_1 \cup \ldots \cup \bar{D}_{n-1} \subset R_2.$$

(In order to show this, one has to consider $\bar{D}_1, \ldots, \bar{D}_n$ as single points and apply the Moore Theorem 8 of Section IV.)

Similarly, let $h(a)h(b)$ be an arc such that

$$\overline{D_0^*} \cap h(a)h(b) = [h(a), h(b)]$$

and that, denoting by R_i^* the component of $\mathscr{S}_2 - C_0^* - h(a)h(b)$ with the boundary $h(L_i) \cup h(a)h(b)$, we have

$$\overline{D_n^*} \subset R_1^* \quad \text{and} \quad \overline{D_1^*} \cup \ldots \cup \overline{D_{n-1}^*} \subset R_2^*.$$

Let h_0 be an extension of the homeomorphism h on the set $\bar{D}_0 \cup ab$ such that

$$h_0(D_0) = D_0^* \quad \text{and} \quad h_0(ab) = h(a)h(b).$$

Applying Theorem 3 to the cases of two disks, $\mathscr{S}_2 - \bar{R}$ and D_n, and of n disks, $\mathscr{S}_2 - \bar{R}_2, D_1, D_2, \ldots, D_{n-1}$, we can first extend the homeomorphism $h_0|L_1 \cup ab$ to $\bar{R}_1$ and then the homeomorphism $h_0|L_2 \cup ab$ to $\bar{R}_2$ so as to satisfy conditions (2) first for $i = n$ and then for $i = 1, 2, \ldots, n-1$.

THEOREM 4. *Every homeomorphism between two closed 0-dimensional sets can be extended to a homeomorphism of the whole sphere $\mathscr{S}_2$.*

In particular, *on the sphere $\mathscr{S}_2$ every perfect 0-dimensional set is topologically equivalent with the Cantor discontinuum $\mathscr{C}$.*

Proof. Let

$$F = \overline{F}, \quad \dim F = 0 \quad \text{and} \quad F^* = h(F),$$

where h is a homeomorphism.

We are going to define two sequences of closed sets $\{F_n\}$ and $\{F_n^*\}$, and a sequence of homeomorphisms $h_n\colon \mathscr{S}_2 \to \mathscr{S}_2$ which satisfy the following conditions

$$F \subset F_n \subset \operatorname{Int}(F_{n-1}), \quad F^* \subset F_n^* \subset \operatorname{Int}(F_{n-1}^*), \tag{i}$$

$$F_n = \overline{D}_{n1} \cup \ldots \cup \overline{D}_{nk_n}, \quad \overline{D}_{ni} \cap \overline{D}_{nj} = 0, \quad F \cap D_{ni} \neq 0, \tag{ii}$$

$$F_n^* = \overline{D_{n1}^*} \cup \ldots \cup \overline{D_{nk_n}^*}, \quad \overline{D_{ni}^*} \cap \overline{D_{nj}^*} = 0, \quad F^* \cap D_{ni}^* \neq 0, \tag{iii}$$

$$h_n(D_{ni}) = D_{ni}^*, \tag{iv}$$

$$h(F \cap D_{ni}) = F^* \cap D_{ni}^*, \tag{v}$$

$$h_{n-1}|\overline{\mathscr{S}_2 - F_{n-1}} \subset h_n, \tag{vi}$$

$$\delta(D_{ni}) < 1/n \quad \text{for even values of } n, \tag{vii}$$

$$\delta(D_{ni}^*) < 1/n \quad \text{for odd values of } n, \tag{viii}$$

where D_{ni} and D_{ni}^* are disks.

Proceed by induction. Let D_0 and D_0^* be two disks such that $F \subset D_0$ and $F^* \subset D_0^*$. Define $F_0 = \overline{D}_0$ and $F_0^* = \overline{D_0^*}$. Let $h_0\colon \mathscr{S}_2 \to \mathscr{S}_2$ be a homeomorphism such that $h_0(D_0) = D_0^*$.

Let $n > 0$ be even. Assume that conditions (i) through (vii) are satisfied for $n-1$.

Since F is not a cut (by § 57, III, Theorem 6), let F_n be a set satisfying conditions (i), (ii) and (vii) in accordance with Theorem 11 of Section III.

Define the disks $D_{n1}^*, \ldots, D_{nk_n}^*$ successively in the following way. It is admissible to assume by virtue of (i) through (iii) that

$$\overline{D}_{n1} \cup \ldots \cup \overline{D}_{ns_n} \subset D_{n-1,1} \quad \text{and} \quad \overline{D}_{nm} \not\subset D_{n-1,1} \quad \text{for} \quad s_n < m \leqslant k_n.$$

Hence $F \cap D_{n-1,1} = F \cap D_{n1} \cup \ldots \cup F \cap D_{ns_n}$ and it follows by (v) (for $n-1$) that

$$h(F \cap D_{n1}) \cup \ldots \cup h(F \cap D_{ns_n}) = h(F \cap D_{n-1,1}) = F^* \cap D_{n-1,1}^*.$$

Therefore, by Theorem 9′ of Section II, there exists a system of disks $D^*_{n1}, \ldots, D^*_{ns_n}$ such that

$$h(F \cap D_{ni}) \subset D^*_{ni}, \quad \overline{D^*_{ni}} \subset D^*_{n-1,1} \quad \text{and}$$
$$\overline{D^*_{ni}} \cap \overline{D^*_{nj}} = 0 \quad \text{for} \quad i \neq j.$$

Define the disks D^*_{nm} in a similar way for $m > s_n$ and define the sets F^*_n by the first identity (iii). Then conditions (i), (iii) and (v) are fulfilled.

Finally, the homeomorphism h_n is defined in accordance with Theorem 3 so as to satisfy conditions (iv) and (vi).

Thus conditions (i) through (vii) are satisfied.

For odd values of n, one proceeds in a similar way, replacing F and F_n by F^* and F^*_n, and D^*_{ni} and F^*_{n-1} by D_{ni} and F_{n-1}. A homeomorphism h_n^{-1} is obtained, which is inverse to h_n.

Define
$$h^*(x) = \lim_{n=\infty} h_n(x).$$

Since the convergence is uniform, the function h^* is continuous and $h^*(\mathscr{S}_2) = \mathscr{S}_2$.

It follows that $h \subset h^*$, since by (iv), (v) and (viii) we have

$$|h(x) - h_n(x)| \leqslant \delta(D^*_{ni}) < 1/(n-1) \quad \text{for} \quad x \in F \cap D_{ni}.$$

Since $(h^* | \mathscr{S}_2 - F)\colon \mathscr{S}_2 - F \to \mathscr{S}_2 - F^*$ is a homeomorphism onto (by (vi)), it follows that h^* is a homeomorphism of the whole space $\mathscr{S}_2$.

Remark. In $\mathscr{E}^3$ Theorem 4 does not hold(1).

THEOREM 5 (of Denjoy–Riesz). *Every closed 0-dimensional set F is contained in an arc*(2).

Proof. Let A be a subset of the interval $\mathscr{I}$ and $h\colon A \to F$ a homeomorphism onto (comp. § 26, IV, Theorem 2); it is sufficient to extend this homeomorphism to $\mathscr{I}$ in order to obtain an arc containing F.

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(1) See L. Antoine, op. cit. Compare the paper by the same author *Sur les voisinages de deux figures homéomorphes*, Fund. Math. 5 (1924), p. 265. See also R. H. Bing, *Tame Cantor Sets in $\mathscr{E}^3$*, Pacific Jour. of Math. 11 (1961), pp. 435–446, A. Kirkor, *Wild 0-dimensional sets and the fundamental group*, Fund. Math. 45 (1958), pp. 228–236, D. R. Mc Millan, Jr., *Taming Cantor sets in $\mathscr{E}^n$*, Bull. Amer. Math. Soc. 70 (1964), pp. 706–708.

(2) See F. Riesz, C. R. Paris 141 (1906), p. 650 and A. Denjoy, *ibid.* 151 (1910), p. 138. For a generalization, see J. R. Kline and R. L. Moore, Ann. of Math. 20 (1918), p. 218.

Remark. This theorem remains true if $\mathscr{S}_2$ is replaced by a locally connected continuum which contains no point of local separation (that is, no point separates any region)(1).

Theorem 5 can be sharpened as follows.

THEOREM 6. *If A and B are two compact 0-dimensional subsets of $\mathscr{E}^2$, there exists an arc L such that*

$$L \cap B \subset A \subset L. \tag{6}$$

Moreover, *if a_0 and a_1 are any two points of A given in advance, it can be assumed that they are the end-points of L.*

Proof. Since $A \cup B \underset{\text{top}}{\subset} \mathscr{C}$ and since $\mathscr{C}$ is homogeneous, there exists by Theorem 4 a homeomorphism h: $\mathscr{E}^2 \to \mathscr{E}^2$ such that

$$h(A \cup B) \subset \mathscr{C}, \quad h(\mathscr{E}^2) = \mathscr{E}^2, \quad h(a_0) = 0 \quad \textit{and} \quad h(a_1) = 1.\ )$$

Let

$$K = \underset{xy}{E}\{y = \varrho[x, h(A)]\} \quad \text{where} \quad x \in \mathscr{I}, \text{ and let } L = h^{-1}(K).$$

Since K is an arc (compare § 21, IV, (5) and § 15, V, Theorem 1 with end-points 0 and 1, L is an arc $a_0 a_1$. Moreover,

$$K \cap \mathscr{I} = h(A), \quad \text{so that} \quad L \cap h^{-1}(\mathscr{I}) = A,$$

which implies condition (6), because $B \subset h^{-1}(\mathscr{I})$.

THEOREM 7. *If R_0 and R_1 are two homeomorphic regions such that* $\dim(\mathscr{S}_2 - R_0) = 0 = \dim(\mathscr{S}_2 - R_1)$, *every homeomorphism h of R_0 onto R_1 can be extended to a homeomorphism of $\mathscr{S}_2$ onto $\mathscr{S}_2$*(2).

Proof. Let $p \in \mathscr{S}_2 - R_0$. We will show that the oscillation $\omega_h(p)$ vanishes.

Suppose that $\omega_h(p) > 0$.

Then there exists a sequence of points $a_n \in R_0$ such that

$$\lim_{n=\infty} a_n = p, \tag{7}$$

$$|h(a_n) - h(a_{n+1})| > \eta \qquad \text{where} \qquad \eta > 0. \tag{8}$$

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(1) Theorem of G. T. Whyburn, Fund. Math. 18 (1932), p. 57.

(2) Theorem 7 is a particular case of Theorem 1 of § 57, IV (which was stated without proof).

Let A be the set consisting of the point p and the points a_n, where $n = 1, 2, \ldots$. Since $\bar{A} = A$ and $\dim A = 0$, there exists by Theorem 6 (where $B = \mathscr{S}_2 - R_0$) an arc $L = a_1 p$ such that $A \subset L$ and $L - R_0 = p$.

Define

$$C_n = h(a_n a_{n+1}) \quad \text{where} \quad a_n a_{n+1} \subset L. \tag{9}$$

Let $\{C_{k_n}\}$ be a convergent subsequence of the sequence $\{C_n\}$; let

$$C = \operatorname*{Lim}_{n=\infty} C_{k_n}.$$

It follows from (8) that $\delta(C) \geqslant \eta$. Thus the continuum C contains more than one point. Therefore $C \cap R_1 \neq 0$ since $\dim(\mathscr{S}_2 - R_1) = 0$. But this is impossible, because it follows from (9) that $p = \operatorname*{Lim}_{n=\infty}(a_n a_{n+1})$, so that $R_0 \cap \operatorname*{Lim}_{n=\infty}(a_{k_n} a_{k_n+1}) = 0$, and hence

$$R_1 \cap \operatorname*{Lim}_{n=\infty} C_{k_n} = 0.$$

Thus it has been established that $\omega_h(p) = 0$ for every $p \in \mathscr{S}_2 - R_0$, and similarly, $\omega_{h^{-1}}(q) = 0$ holds for every $q \in \mathscr{S}_2 - R_1$.

Therefore

$$h \subset f, \quad \text{where} \quad f\colon \mathscr{S}_2 \to \mathscr{S}_2 \text{ is continuous and } f(\bar{R}_0) = \mathscr{S}_2,$$

$$h^{-1} \subset g, \quad \text{where} \quad g\colon \mathscr{S}_2 \to \mathscr{S}_2 \text{ is continuous and } g(\bar{R}_1) = \mathscr{S}_2.$$

It now easily follows that $g = f^{-1}$ (compare the final part of the proof of Theorem 6 of Section IV).

Theorem 9 of Section IV can be reformulated more precisely in the following way.

THEOREM 8 (1). *Let R_j be a region and H_j the space of the decomposition of $\mathscr{S}_2 - R_j$ into components* ($j = 0, 1$). *Then the following equivalence holds*

$$(R_0 \underset{\text{top}}{=} R_1) \equiv (H_0 \underset{\text{top}}{=} H_1). \tag{10}$$

Proof. Consider H_j as a subset of $\mathscr{S}_2$; then by Theorem 9 of Section IV

$$R_j \underset{\text{top}}{=} \mathscr{S}_2 - H_j. \tag{11}$$

Since $\dim H_j = 0$, it follows from Theorem 7 that

$$(\mathscr{S}_2 - H_0 \underset{\text{top}}{=} \mathscr{S}_2 - H_1) \Rightarrow (H_0 \underset{\text{top}}{=} H_1)$$

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(1) See B. v. Kerékjártó, *Topologie I*, p. 123.

and from Theorem 4 that

$$(H_0 \underset{\text{top}}{=} H_1) \Rightarrow (\mathcal{S}_2 - H_0 \underset{\text{top}}{=} \mathcal{S}_2 - H_1).$$

These implications combined with (11) give (10).

THEOREM 9 (of invariance). *If R is a region in $\mathcal{S}_2$, the cardinality of the family of components of $\mathcal{S}_2 - R$ is an intrinsic invariant of R.*

Proof. This is a direct consequence of Theorem 8.(1)

The similar question concerning open subsets of $\mathcal{S}_2$ will be considered in § 62, X, Remark 1.

§ 62. Quantitative problems. The group $\mathcal{P}^A$

I. General properties and notation(2). According to the final Remark of § 56, I, all the theorems of § 56 and § 57 remain true if the circle $\mathcal{S}$ is replaced by the set $\mathcal{P}$ (the plane $\mathcal{E}^2$ without the point 0) and the set $\mathcal{E}$ by $\mathcal{E}^2$. We agree henceforth that $f \sim 1$, where $f\colon \mathcal{X} \to \mathcal{P}$ is a continuous function, means that there exists a continuous function $u\colon \mathcal{X} \to \mathcal{E}^2$ such that

$$f(x) = e^{u(x)} \quad \text{for} \quad x \in \mathcal{X}.$$

$\Psi(\mathcal{X})$ will stand for the set of functions $f \in \mathcal{P}^{\mathcal{X}}$ such that $f \sim 1$. $\mathfrak{B}_1(\mathcal{X})$ will denote the factor group $\mathcal{P}^{\mathcal{X}}/\Psi(\mathcal{X})$ and $b_1(\mathcal{X})$ — its rank.

Let us mention some statements which can easily be proved and which we shall use frequently.

THEOREM 1. *Let $f\colon \mathcal{X} \to \mathcal{P}$ be a continuous function. Define $\vec{f}$ by the following condition*

$$\vec{f}(x) = \frac{f(x)}{|f(x)|}.$$

The necessary and sufficient condition in order that $f(x) = e^{u(x)}$ where $u\colon \mathcal{X} \to \mathcal{E}^2$ is a continuous function, is that $\vec{f}(x) = e^{i\varphi(x)}$ where $\varphi\colon \mathcal{X} \to \mathcal{E}$ is a continuous function.

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(1) Compare the footnote to Theorem 7.

(2) A large number of theorems of § 62 can be deduced from the corresponding theorems of § 60 (substituting $n = 2$). However, it seemed to the author desirable to give here direct and more elementary proofs of the theorems under consideration without using the cohomotopic multiplication (which can be avoided for $n = 2$ since $\mathcal{P}_2$ is an abelian group and consequently every pair of functions with values in $\mathcal{P}_2$ is multipliable).

Consequently, *the following conditions*: $f \sim 1$, f *is homotopic to* 1 *with respect to* $\mathcal{P}$, $\vec{f} \sim 1$, $\vec{f}$ *is homotopic to* 1 *with respect to* $\mathcal{S}$, *are equivalent.*

THEOREM 2. x *non* ~ 1 *on* $\mathcal{P}$.

Compare § 56, III, 4 (i). More generally

THEOREM 3. $x-p$ *non* ~ 1 *on* $\mathcal{E}^2-p$ *and* $\dfrac{x-p}{x-q}$ *non* ~ 1 *on* $\mathcal{S}_2-p-q$.

Proof. The homographic function $f(x) = \dfrac{x-p}{x-q}$ maps $\mathcal{S}_2-p-q$ onto $\mathcal{P}$. Supposing that $f(x) = e^{u(x)}$ for $x \in \mathcal{S}_2-p-q$, we would have $y = e^{uf^{-1}(y)}$ for $y \in \mathcal{P}$, contradicting Theorem 2.

Remark. If $q = \infty$, we agree to write $x-q \equiv 1$.

THEOREM 4. *If* C *is the circle* $|x-p| = r$ *where* $p \neq \infty$, *then for every continuous function* $f\colon C \to \mathcal{P}$ *there exists one and only one* n *such that* $f(x) \sim (x-p)^n$.

In other words, *the function* $x-p$ *is the basis of the group* $\mathcal{P}^C$ mod $\Psi(C)$.

Proof. This is an easy consequence of Theorem 4 of § 56, III.

THEOREM 5. $|q-p| > r$ *implies that* $(x-q) \sim 1$ *on* C.

Because the ray R issued out from the point 0 and parallel to the vector $q-p$ does not contain any value of the function $x-q$ for $x \in C$.

THEOREM 6. *If* Q *is the closed disk* $|x-p| \leqslant r$, *every continuous function* $f\colon Q \to \mathcal{P}$ *is homotopic to unity, i.e.* $f \sim 1$.

This is a direct consequence of Theorem 9, (i) of § 58, I.

Remark. Theorems 4 and 6 yield a very simple proof of the Fundamental Theorem of Algebra (1).

Suppose, namely, that the polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

does not vanish for any x. Let $g(x) = a_{n-1}x^{n-1} + \ldots + a_0$. Let r be a real number such that

$$|x^n| > |g(x)| \quad \text{for} \quad |x| \geqslant r. \tag{1}$$

(1) Compare also Alexandroff–Hopf, *Topologie I*, p. 469.

Let Q be the closed disk $|x| \leqslant r$ and C its boundary. By Theorem 6, $f \sim 1$ on Q and therefore on C.

We will show that $f(x) \sim x^n$ on C, which will imply a contradiction, because x^n *non* ~ 1 on C by Theorem 4.

Let

$$h(x, t) = x^n + t \cdot g(x) \quad \text{where} \quad 0 \leqslant t \leqslant 1;$$

if follows that f is homotopic to x^n on C. Because

$$h(x, 0) = x^n, \quad h(x, 1) = f(x) \quad \text{and} \quad h(x, t) \neq 0 \quad \text{for} \quad |x| = r,$$

since, supposing that $h(x, t) = 0$, we would have

$$x^n = -t \cdot g(x), \quad \text{which implies } |x^n| = t \cdot g(x), \quad \text{thus} \quad |x^n| \leqslant |g(x)|,$$

contradicting (1).

II. Cuts of $\mathscr{S}_2$.

THEOREM 1 (of Eilenberg)(1). *The set $A \subset \mathscr{S}_2 - p - q$ does not cut $\mathscr{S}_2$ between p and q if and only if*

$$\frac{x-p}{x-q} \sim 1 \quad on\ A. \tag{1}$$

Proof. Define

$$f(x) = \frac{x-p}{x-q}. \tag{2}$$

First, assume that C is a continuum such that $p, q \in C \subset \mathscr{S}_2 - A$. It follows by Theorem 3 of § 61, I that $f | \mathscr{S}_2 - C \sim 1$, which implies relation (1), because $A \subset \mathscr{S}_2 - C$.

Conversely, assume that condition (1) holds. By Theorem 9 of § 56, II, there exists an open set G such that

$$A \subset G \subset \mathscr{S}_2 - p - q \quad \text{and} \quad f | G \sim 1. \tag{3}$$

If we suppose that A is a cut between p and q, then so is G. Therefore G contains a closed separator F between p and q. But then $f | F$ *non* ~ 1 (by Theorem 3 of Section I and § 59, II, Theorem 5), contradicting (3).

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(1) *Op. cit.* Fund. Math. 26 (1936), p. 75.

THEOREM 2. *No set contractible with respect to $\mathscr{S}$ is a cut of $\mathscr{S}_2$. In particular, no set homeomorphic to a subset of an interval is a cut of $\mathscr{S}_2$* (compare § 58, I, 9, (iv)).

THEOREM 3. *If C is a connected set and D is a cut between p and q such that $C \subset D \subset \bar{C}-p-q$, there exists a point $a \epsilon D$ such that $C \cup a$ is a cut between p and q.*

Proof. If f is defined by formula (2), the conditions $f|D$ *non* ~ 1 and $f|C \sim 1$ imply by Theorem 7 of § 56, VI (setting $\mathscr{X} = D$) the existence of a point a such that $(f|C \cup a)$ *non* ~ 1.

THEOREMS 4 and 5. *If in the statements of Theorems* 2 *and* 5 *of* § 59, IV, *$\mathscr{S}_n$ is replaced by $\mathscr{S}_2$, the hypothesis that F is closed may be omitted.*

Proof. In order to establish Theorem 4, we have only to replace Corollary 6 of § 59, II by Theorem 1 in the proof of Theorem 2 of 59, IV.

In order to prove Theorem 5, assume that $p = 0$ and $q = \infty$ (to simplify the notation). Then $x|F \sim 1$ by Theorem 1, and this means that the set F can be deformed to a point in $\mathscr{P}$ (compare § 54, IV).

THEOREM 6 (generalized theorem of Eilenberg). *Let $p_0, \ldots, p_n$ be $n+1$ distinct points and let $A \subset \mathscr{S}_2-(p_0, \ldots, p_n)$. In order that A be a cut between every pair p_i, p_j (where $i \neq j$), it is necessary and sufficient that the homographic functions*

$$f_1(x) = \frac{x-p_1}{x-p_0}, \quad \ldots, \quad f_n(x) = \frac{x-p_n}{x-p_0} \tag{4}$$

be linearly independent on $A \bmod \Psi(A)$; this means that the conditions

$$(x-p_0)^{k_0} \cdot \ldots \cdot (x-p_n)^{k_n} \sim 1 \quad \text{on } A \tag{5}$$

and

$$k_0 + \ldots + k_n = 0 \tag{6}$$

imply that $k_0 = 0, \ldots, k_n = 0$.

Proof. The condition is *necessary*. In order to show that, proceed by induction. The statement is true for $n = 1$, because f_1 *non* ~ 1 by Theorem 1. Assume that it is true for $n-1$ and each A.

Let $k_0, \ldots, k_n$ be a system of integers satisfying conditions (5) and (6).

Following Theorem 9 of § 56, II, let G be an open set such that

$$A \subset G \subset \mathscr{S}_2 - (p_0, \ldots, p_n) \quad \text{and}$$
$$(x-p_0)^{k_0} \cdot \ldots \cdot (x-p_n)^{k_n} \sim 1 \quad \text{on } G. \tag{7}$$

Denote by C_j the component of $\mathscr{S}_2 - G$ which contains p_j. Let L be an irreducible arc between C_0 and $C_1 \cup \ldots \cup C_n$, thus an arc whose one end-point, say a, belongs to C_0, and the other, say b, to one of the sets $C_1, \ldots, C_n$, to C_n for instance, and $L \cap C_j = 0$ for $0 \neq j \neq n$. It is easy to see that the sets $C_0 \cup L \cup C_n, C_1, C_2, \ldots, C_{n-1}$ are components of the set $\mathscr{S}_2 - H$ where $H = G - L$. Therefore, H is a cut between every pair p_i, p_j where $i < j < n$, while it is not a cut between p_0 and p_n. It follows from Theorem 1 that $f_n \sim 1$ on H, so that $f_n^{k_n} \sim 1$, which implies by (7) that

$$(x-p_0)^{k_0+k_n} \cdot (x-p_1)^{k_1} \cdot \ldots \cdot (x-p_{n-1})^{k_{n-1}} \sim 1 \quad \text{on } H,$$

and this implies by hypothesis that $k_0 + k_n = 0, k_1 = 0, \ldots, k_{n-1} = 0$. These identities, combined with (5), give $f_n^{k_n} \sim 1$ on A, and thus $k_n = 0$, because A is a cut between p_0 and p_n.

The condition is *sufficient*. Assume that A is not a cut between p_0 and p_1. By Theorem 1, $f_1 \sim 1$ on A. Set $k_0 = -1, k_1 = 1, k_2 = 0, \ldots, k_n = 0$; then conditions (5) and (6) are fulfilled.

THEOREM 7. *If* $A = \mathscr{S}_2 - (p_0, \ldots, p_n)$ *where* $p_i \neq p_j$ *for* $i \neq j$, *the homographic functions* (4) *form a basis for the group* $\mathscr{P}^A \bmod \Psi(A)$.

In other words (referring to Theorem 6), *every continuous function* $f\colon A \to \mathscr{P}$ *has the following form*

$$f(x) = e^{u(x)}(x-p_0)^{k_0} \cdot \ldots \cdot (x-p_n)^{k_n} \tag{8}$$

where $u\colon A \to \mathscr{E}^2$ *is continuous and* $k_0 + \ldots + k_n = 0$.

Proof. Proceed by induction.

(i) Let $n = 0$. Since $\mathscr{S}_2 - p_0 \underset{\text{top}}{=} \mathscr{E}^2$, we have $f \sim 1$ (compare § 57, I, 9, (i)). Therefore $k_0 = 0$.

(ii) Let $n > 0$ and assume that the theorem holds for $n-1$. We can also assume that $p_n \neq \infty$. Let $K \subset \mathscr{S}_2 - (p_0, \ldots, p_{n-1})$ be a closed disk with center p_n. Let $C = \mathrm{Fr}(K)$. By Theorems 4 and 5 of Section I, it follows that on C

$$f(x) \sim (x-p_n)^{k_n} \quad \text{and} \quad x - p_0 \sim 1;$$

therefore, setting

$$h(x) = f(x)(x-p_0)^{k_n}(x-p_n)^{-k_n}, \tag{9}$$

we have $h|C \sim 1$.

According to Theorem 8 of § 56, II, let

$$h|C \subset f^*, \quad \text{where} \quad f^*\colon K \to \mathscr{P} \text{ is continuous and } f^* \sim 1. \tag{10}$$

Define

$$g(x) = \begin{cases} f^*(x) & \text{if} \quad x \in K, \\ h(x) & \text{if} \quad x \in A - \mathrm{Int}(K). \end{cases} \tag{11}$$

By (10) and (11), $g\colon A \cup p_n \to \mathscr{P}$ is continuous. Thus, it follows by hypothesis that

$$g(x) \sim (x-p_0)^{l_0} \cdot \ldots \cdot (x-p_{n-1})^{l_{n-1}} \quad \text{on } A \cup p_n$$
$$\text{and} \quad l_0 + \ldots + l_{n-1} = 0. \tag{12}$$

On the other hand, we have

$$g \sim h \quad \text{on } A - \mathrm{Int}(K) \text{ and on } K - p_n, \tag{13}$$

because $g = h$ on $A - \mathrm{Int}(K)$ by (11), therefore $g = h$ on C, which implies $g \sim h$ on $K - p_n$ since $K - p$ is deformable onto C (compare § 54, IV, Theorem 1).

Since the sets $A - \mathrm{Int}(K)$ and $K - p_n$ are closed in their union A, and their intersection C is connected, it follows by (13) that $g \sim h$ on A (compare § 56, VI, Theorem 3).

It follows by (9) and (12) that

$$f(x) \sim (x-p_0)^{l_0-k_n}(x-p_1)^{l_1} \cdot \ldots \cdot (x-p_{n-1})^{l_{n-1}} \cdot (x-p_n)^{k_n} \text{ on } A.$$

Set $k_0 = l_0 - k_n$, $k_1 = l_1$, ..., $k_{n-1} = l_{n-1}$; then formulas (8) follow.

THEOREM 8. *Let $g\colon \mathscr{X} \to \mathscr{S}_2$ be a continuous function. If the set $g(\mathscr{X})$ does not cut $\mathscr{S}_2$ between p and q, then*

$$\frac{g(x)-p}{g(x)-q} \sim 1 \quad \textit{on } \mathscr{X}. \tag{14}$$

Proof. It follows by Theorem 1 that

$$\frac{y-p}{y-q} = e^{v(y)} \quad \text{where} \quad v\colon g(\mathscr{X}) \to \mathscr{E}^2 \text{ is continuous.}$$

Hence

$$\frac{g(x)-p}{g(x)-q} = e^{vg(x)} \quad \text{for} \quad x \in \mathscr{X}.$$

THEOREM 9. *If the function* $g\colon \mathscr{X} \to \mathscr{S}_2$ *is a homeomorphism satisfying condition* (14), *the set* $g(\mathscr{X})$ *does not cut* $\mathscr{S}_2$ *between* p *and* q.

Proof. Let $h\colon g(\mathscr{X}) \to \mathscr{X}$ be the inverse function to g. Assuming condition (14), we have

$$\frac{g(x)-p}{g(x)-q} = e^{u(x)}, \quad \text{which implies} \quad \frac{y-p}{y-q} = e^{uh(y)} \quad \text{for} \quad y \in g(\mathscr{X}).$$

By Theorem 1, $g(\mathscr{X})$ is not a cut between p and q.

III. Groups $\mathscr{P}^F$ and $\mathfrak{B}_1(F)$ for $F = \overline{F} \subset \mathscr{S}_2$.

THEOREM 1. *Every continuous function* $f\colon F \to \mathscr{P}$ *is homotopic to a rational function whose zeros and poles belong to* $\mathscr{S}_2 - F$.

Proof. By Theorem 9 of § 59, II, there exist a finite set $p_0, \ldots, p_n$ contained in $\mathscr{S}_2 - F$ and a continuous function f^* such that

$$f \subset f^*\colon A \to \mathscr{P} \quad \text{where} \quad A = \mathscr{S}_2 - (p_0, \ldots, p_n).$$

By Theorem 7 of Section II, $f^*(x) \sim (x-p_0)^{k_0} \cdot \ldots \cdot (x-p_n)^{k_n}$, so that

$$f(x) \sim (x-p_0)^{k_0} \cdot \ldots \cdot (x-p_n)^{k_n} \quad \text{on } F \quad \text{and} \quad k_0 + \ldots + k_n = 0. \tag{1}$$

Remark. Let $R_0, R_1, \ldots$ be a (finite or infinite) sequence of components of $\mathscr{S}_2 - F$. According to Theorem 9 of § 58, II, we can assume that

$$p_j \in R_j \quad \text{where} \quad j = 0, 1, \ldots, \tag{2}$$

$$k_j = 0 \quad \text{if} \quad f|\mathrm{Fr}(R_j) \sim 1. \tag{3}$$

In view of (2), Theorem 1 can be completed in the following way.

THEOREM 2. *If r is an arbitrary rational function homotopic to f on F, the exponent k_j is the algebraic number of zeros and poles of r which belong to R_j, i.e. the number of zeros and poles, each counted with its multiplicity* (the multiplicity of a pole is negative by definition).

Therefore, *the exponents* $k_0, k_1, \ldots$ *are uniquely determined* (*they do not depend on the choice of the points* $p_j \in R_j$).

Proof. Let $r(x) = c(x-q_0)^{l_0} \cdot \ldots \cdot (x-q_m)^{l_m}$, where $q_0 = \infty$ if F is bounded. Therefore

$$f(x) \sim (x-q_0)^{l_0} \cdot \ldots \cdot (x-q_m)^{l_m} \quad \text{and} \quad l_0 + \ldots + l_m = 0. \tag{3'}$$

Define $Q_j = R_j \cap (q_0, \ldots, q_m)$. For a fixed j, let $Q_j = (q_{t_1}, \ldots, q_{t_v})$. By Theorem 1 of Section II, it follows that on F

$$\frac{x-q_{t_1}}{x-p_j} \sim 1, \quad \ldots, \quad \frac{x-q_{t_v}}{x-p_j} \sim 1,$$

hence

$$(x-q_{t_1})^{l_{t_1}} \cdot \ldots \cdot (x-q_{t_v})^{l_{t_v}} \cdot (x-p_j)^{-k_j'} \sim 1, \tag{4}$$

where $k_j' = l_{t_1} + \ldots + l_{t_v}$ (and $k_j' = 0$ if $Q_j = 0$).

Thus k_j' is the algebraic number of zeros and poles of the function r which belong to R_j.

It follows from (3′) and (4) that

$$f(x) \sim (x-p_0)^{k_0'} \cdot (x-p_1)^{k_1'} \cdot \ldots$$
$$\text{and} \quad k_0' + k_1' + \ldots = l_0 + \ldots + l_m = 0. \tag{5}$$

Set $k_j = 0$ for $j > n$; then (1) and (5) yield

$$1 \sim (x-p_0)^{k_0'-k_0} \cdot (x-p_1)^{k_1'-k_1} \cdot \ldots \text{ on } F,$$
$$\text{and} \quad (k_0'-k_0) + (k_1'-k_1) + \ldots = 0,$$

hence $k_j' = k_j$ for $j = 0, 1, \ldots$ by (2) and Theorem 6 of Section II.

Corollary 2′. *In order that* $k_j = 0$, *it is necessary and sufficient that the function* f *admits a continuous extension* $f^*\colon F \cup R_j \to \mathscr{P}$.

Proof. By (1) it follows for $x \in F$ that

$$f(x) = e^{u(x)} \cdot (x-p_0)^{k_0} \cdot \ldots \cdot (x-p_n)^{k_n} \tag{6}$$

where $u\colon \mathscr{S}_2 \to \mathscr{E}^2$ is continuous.

Assuming that $k_j = 0$ and setting

$$f^*(x) = e^{u(x)} \cdot (x-p_0)^{k_0} \ldots (x-p_{j-1})^{k_{j-1}} \cdot$$
$$\cdot (x-p_{j+1})^{k_{j+1}} \cdot \ldots \cdot (x-p_n)^{k_n} \tag{7}$$

for $x \in F \cup R_j$, we infer that there is a continuous f^* such that

$$f \subset f^*\colon\ F \cup R_j \to \mathscr{P}. \tag{8}$$

On the other hand, assuming conditions (8), we infer that identity (7) holds in $F \cup R_j$. Thus it is legitimate to delete the asterisk in (7) and hence $k_j = 0$.

THEOREM 3. *If* (2) *is fulfilled, the homographic functions*

$$\frac{x-p_1}{x-p_0},\ \frac{x-p_2}{x-p_0},\ \ldots, \quad \textit{where} \quad x \in F, \tag{9}$$

constitute a basis of $\mathscr{P}^F \bmod \Psi(F)$.

Proof. By Theorem 1 it is sufficient to show that the functions (9) are linearly independent on $F \bmod \Psi(F)$; in other words, that the conditions

$$(x-p_0)^{m_0}\cdot(x-p_1)^{m_1}\cdot\ldots \sim 1 \quad \text{on } F \quad \text{and} \quad m_0+m_1+\ldots = 0$$

imply $m_0 = 0, m_1 = 0, \ldots$.

But this is a direct consequence of Theorem 6 of Section II.

The next theorem about the *characterization of the group* $\mathfrak{B}_1(F)$ follows immediately.

THEOREM 4.

$$\mathfrak{B}_1(F) \underset{\mathrm{gr}}{=} \mathscr{G}^n,$$

where $n = \omega$ *if the number* $b_0(\mathscr{S}_2 - F)$ *of components of* $\mathscr{S}_2 - F$ *is infinite, and* $n = b_0(\mathscr{S}_2 - F)$ *if that number is finite.*

More precisely, *the required isomorphism is defined by assigning to every* $f \in \mathscr{P}^F$ *the point* $(k_1, k_2, \ldots) \in \mathscr{G}^n$, *determined by conditions* (1) *and* (2).

Theorem 4 implies directly the following.

THEOREM 5 (of duality)(1). $b_1(F) = b_0(\mathscr{S}_2 - F)$.

THEOREM 6 (of invariance). *The number of (non-empty) separated parts into which a set* $A \subset \mathscr{S}_2$ *decomposes* $\mathscr{S}_2$ *is an intrinsic invariant.*

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(1) Compare J. W. Alexander theorem of duality in Algebraic Topology, *A proof and extension of the Jordan-Brouwer separation theorem*, Trans. Amer. Math. Soc. 23 (1922), p. 343. Compare also N. Bruschlinsky, Math. Ann. 109 (1934), p. 525, and H. Freudenthal, Comp. Math. 2 (1935), p. 134.

Proof. For $A = \bar{A}$ this theorem follows from the preceding one. The general case is obtained from this particular case combined with Theorem 7 of § 49, IV (compare the proof of Theorem 6 of § 59, IV).

IV. Addition theorems.

THEOREM 1. *If none of the sets A_0 and A_1, which are both closed (or both open) in $A_0 \cup A_1$, cuts $\mathscr{S}_2$ between no pair of points $p_0, \ldots, p_n$ $(n > 0)$, whereas $A_0 \cup A_1$ cuts $\mathscr{S}_2$ between every pair of these points, the set $A_0 \cap A_1$ contains $n+1$ components at least* (¹).

Proof. By Theorem 1 of Section II

$$\frac{x-p_k}{x-p_0} \sim 1 \quad \text{on } A_0 \text{ and on } A_1 \text{ for } k = 1, 2, \ldots, n,$$

and by Theorem 6 of Section II, the functions

$$\frac{x-p_1}{x-p_0}, \ \ldots, \ \frac{x-p_n}{x-p_0}$$

are linearly independent on $A_0 \cup A_1 \bmod \Psi(A_0 \cup A_1)$.

Therefore, the rank $p_1(A_0, A_1)$ of the group $\mathfrak{P}_1(A_0, A_1)$ (see § 56, V (4) substituting $\mathscr{X} = A_0 \cup A_1$) is $\geqslant n$. Since, on the other hand, $p_1(A_0, A_1) \leqslant b_0(A_0 \cap A_1)$ (by § 56, V, Theorem 6), it follows that $b_0(A_0 \cap A_1) \geqslant n$.

LEMMA 2. *If A_0 and A_1 are two sets such that*

$$A_0 \cup A_1 \neq \mathscr{S}_2 \quad \textit{and} \quad A_0 \cap A_1 = \overline{A_0 \cap A_1}, \tag{0}$$

then for every continuous function $f\colon A_0 \cap A_1 \to \mathscr{P}$ there exist two continuous functions $f_0\colon A_0 \to \mathscr{P}$ and $f_1\colon A_1 \to \mathscr{P}$ such that

$$f(x) = f_0(x) \cdot f_1(x) \quad \textit{for} \quad x \in A_0 \cap A_1. \tag{1}$$

This means that (compare § 56, IV (0))

$$\Theta_1(A_0, A_1) = \mathscr{P}^{A_0 \cap A_1}, \quad \textit{hence} \quad d_1(A_0, A_1) = b_1(A_0 \cap A_1), \tag{2}$$

under the hypothesis (0).

(¹) Theorem of S. Straszewicz (for A_0 and A_1 closed). See *Über die Zerschneidung der Ebene durch abgeschlossene Mengen,* Fund. Math. 7 (1925), p. 168.

Proof. Let $p_0, p_1, \ldots$ be a sequence containing precisely one point of each component of $\mathscr{S}_2-(A_0 \cap A_1)$; assume, moreover, that $p_0 \in \mathscr{S}_2-(A_0 \cup A_1)$. Setting $F = A_0 \cap A_1$ in Theorem 1 of Section III, we have

$$f(x) = e^{u(x)}\left(\frac{x-p_1}{x-p_0}\right)^{k_0}\cdot\ldots\cdot\left(\frac{x-p_n}{x-p_0}\right)^{k_n} \quad \text{for} \quad x \in A_0 \cap A_1,$$

where $u\colon \mathscr{S}_2 \to \mathscr{E}^2$ is continuous.

Since $\mathscr{S}_2-(A_0 \cap A_1) = (\mathscr{S}_2-A_0) \cup (\mathscr{S}_2-A_1)$, we can assume that

$$(p_1, \ldots, p_m) \subset \mathscr{S}_2-A_0 \quad \text{and} \quad (p_{m+1}, \ldots, p_n) \subset \mathscr{S}_2-A_1.$$

Set

$$f_0(x) = e^{u(x)}\left(\frac{x-p_1}{x-p_0}\right)^{k_0}\cdot\ldots\cdot\left(\frac{x-p_m}{x-p_0}\right)^{k_m}$$

and

$$f_1(x) = \left(\frac{x-p_{m+1}}{x-p_0}\right)^{k_{m+1}}\cdot\ldots\cdot\left(\frac{x-p_n}{x-p_0}\right)^{k_n}.$$

Then condition (1) is fulfilled.

THEOREM 3. *If A_0 and A_1 are two closed (or two open) sets in $A_0 \cup A_1$ such that*

$$\overline{A_0 \cap A_1} = A_0 \cap A_1 \neq 0 \neq \mathscr{S}_2-(A_0 \cup A_1),$$

then, setting $\operatorname{ind}(A) = b_0(A)-b_1(A)$, *we have*

$$\operatorname{ind}(A_0 \cup A_1)+\operatorname{ind}(A_0 \cap A_1) = \operatorname{ind}(A_0)+\operatorname{ind}(A_1). \tag{3}$$

Proof. Formula (3) follows from (2) and from (7.3) of § 56, V.

THEOREM 4 (of Straszewicz)(1). *Let* $A_j = \overline{A}_j$, $B_j = \mathscr{S}_2-A_j$ $(j = 0, 1)$, *and*

$$A_0 \cap A_1 \neq 0 \neq B_0 \cap B_1. \tag{4}$$

Then

$$\begin{aligned} b_0(A_0 \cup A_1)+b_0(A_0 \cap A_1)-b_0(A_0)-b_0(A_1)& \\ = b_0(B_0 \cup B_1)+b_0(B_0 \cap B_1)-b_0(B_0)-b_0(B_1)&. \end{aligned}$$

Proof. This is a consequence of Theorem 3 and of Theorem 5 of Section III.

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(1) *Op. cit.* 184.

Remark. If one of the inequalities (4) does not hold, then, by Theorem 4 of § 57, II and of Theorem 10 of § 55, X, the following implications hold

(i) if $A_0 \cap A_1 = 0$, then $b_0(B_0 \cap B_1) = b_0(B_0) + b_0(B_1)$(¹),

(ii) if $A_0 \cup A_1 = \mathscr{S}_2$ and $B_j \neq 0$, then $b_0(B_0 \cup B_1) = b_0(B_0) + b_0(B_1) + 1$.

THEOREM 5. *If A_0 and A_1 are continua, then*

$$b_0(A_0 \cap A_1) \leqslant b_0[\mathscr{S}_2 - (A_0 \cup A_1)]. \tag{5}$$

If, moreover, A_0 and A_1 do not cut $\mathscr{S}_2$, then

$$b_0(A_0 \cap A_1) = b_0[\mathscr{S}_2 - (A_0 \cup A_1)]. \tag{6}$$

Proof. Inequality (5) easily follows from Theorem 4 (by virtue of the unicoherence of $\mathscr{S}_2$ and of Theorem 9 of § 55, X). The second part can be derived from (5) by Theorem 1 (which implies the converse inequality).

THEOREM 6. *If the intersection $A_0 \cap A_1$ of two continua A_0 and A_1 is not connected, there exists a pair of points $(p, q) \subset \mathscr{S}_2 - (A_0 \cup A_1)$ such that $A_0 \cup A_1$ cuts $\mathscr{S}_2$ between them, but A_0 does not*(²).

Proof. By Theorem 4 of § 56, VI, there exists a continuous function $f: A_0 \cup A_1 \to \mathscr{P}$ such that

$$f \text{ non} \sim 1, \tag{7}$$

$$f|A_0 \sim 1. \tag{8}$$

Referring to Theorem 1 of Section III, let

$$f(x) \sim (x-p_0)^{k_0} \cdot \ldots \cdot (x-p_n)^{k_n} \quad \text{on } A_0 \cup A_1, \tag{9}$$

$$k_0 + \ldots + k_n = 0, \tag{10}$$

$$|k_0| + \ldots + |k_n| \neq 0, \tag{11}$$

where the points $p_0, \ldots, p_n$ belong to different components of $\mathscr{S}_2 - (A_0 \cup A_1)$.

(¹) In that direction see also A. H. Stone, Trans. Amer. Math. Soc. 65 (1949), p. 427, and for further addition theorems, R. H. Bing, *Generalizations of two theorems of Janiszewski*, Bull. Amer. Math. Soc. 51 (1945), pp. 954–960, and vol. 52 (1946), pp. 478–480.

(²) Theorem of S. Straszewicz and myself, *Généralisation d'un théorème de Janiszewski*, Fund. Math. 12 (1928), p. 154.

Conditions (8) and (9) imply that

$$(x-p_0)^{k_0}\cdot\ldots\cdot(x-p_n)^{k_n} \sim 1 \quad \text{on } A_0.$$

Hence by (10) and (11) the functions

$$\frac{x-p_1}{x-p_0}, \quad \ldots, \quad \frac{x-p_n}{x-p_0}$$

are not linearly independent $\bmod \Psi(A_0)$. Therefore, there exists by Theorem 6 of Section II a pair p_i, p_j, where $0 \leqslant i < j \leqslant n$, between which A_0 does not cut $\mathscr{S}_2$.

Remark. The example of two circles having two points in common shows that it is not possible to conclude in Theorem 6 that there exists a pair of points between which $A_0 \cup A_1$ is a cut whereas neither A_0 nor A_1 is one.

The following two theorems generalize the *theorem about three continua* (1).

THEOREM 7. *Let C_0, C_1 and C_2 be three connected sets and p_0, p_1 two points of the set $\mathscr{S}_2-(C_0 \cup C_1 \cup C_2)$.*

If none of the sets $C_k \cup C_{k+1}$ cuts $\mathscr{S}_2$ between the points p_0 and p_1, and if $C_0 \cap C_1 \cap C_2 \neq 0$, then the set $C_0 \cup C_1 \cup C_2$ does not cut $\mathscr{S}_2$ between these points either (here $k = 0, 1, 2$ and the subscripts are reduced mod 3).

Proof. By hypothesis (compare Theorem 1 of Section II)

$$\frac{x-p_1}{x-p_0} \sim 1 \quad \text{on } C_k \cup C_{k+1} \quad \text{for} \quad k = 0, 1, 2.$$

Therefore, by Theorem 5 of § 56, VI,

$$\frac{x-p_1}{x-p_0} \sim 1 \quad \text{on } C_0 \cup C_1 \cup C_2,$$

which implies the required conclusion (by Theorem 1 of Section II).

THEOREM 8. *Let C_0, C_1 and C_2 be three connected sets and p_0, p_1, p_2 three points of the set $\mathscr{S}_2-(C_0 \cup C_1 \cup C_2)$. If none of the sets $C_k \cup C_{k+1}$ cuts $\mathscr{S}_2$ between any of the pairs (p_0, p_1), (p_1, p_2) and (p_2, p_0), then*

(1) Compare my paper *Théorème sur trois continus*, Monatsh. Math.-Phys. 36 (1929), p. 77, E. Čech, Public. Univ. Masaryk 19 (1931), p. 20 and S. Eilenberg, *op. cit.* p. 78.

$C_0 \cup C_1 \cup C_2$ does not cut $\mathcal{S}_2$ between one of these pairs at least (k is as in Theorem 7).

Proof. Let $\mathcal{X} = C_0 \cup C_1 \cup C_2$. If the points p_0, p_1 and p_2 belong to three different components of $\mathcal{S}_2 - \mathcal{X}$, the functions

$$f_1(x) = \frac{x-p_1}{x-p_0} \quad \text{and} \quad f_2(x) = \frac{x-p_2}{x-p_0}$$

are linearly independent on $\mathcal{X} \bmod \Psi(\mathcal{X})$ by Theorem 6 of Section II.

It follows by Theorem 6 of § 56, VI, that one of the six functions $f_j | C_k \cup C_{k+1}$, where $j = 1, 2$ and $k = 0, 1, 2$, is *non* ~ 1. For instance, let

$$\frac{x-p_1}{x-p_0} \ non \sim 1 \quad \text{on } C_0 \cup C_1.$$

It follows by Theorem 1 of Section II that $C_0 \cup C_1$ cuts $\mathcal{S}_2$ between p_0 and p_1.

Remark. Theorems 5′ and 6′ of § 56, VI imply the following generalization of Theorems 7 and 8(1) (the subscripts should be reduced mod n).

THEOREM 7′. *Let $A_0, \ldots, A_{n-1}$ be n ($n \geqslant 3$) arbitrary sets and let p_0, p_1 be two points of the set $\mathcal{S}_2 - (A_0 \cup \ldots \cup A_{n-1})$. If none of the sets $A_{k+1} \cup \ldots \cup A_{k+n-1}$ cuts $\mathcal{S}_2$ between p_0 and p_1, if all the sets $C_k = A_{k+1} \cup \ldots \cup A_{k+n-2}$ are connected and if $C_0 \cap \ldots \cap C_{n-1} \neq 0$, then the set $A_0 \cup \ldots \cup A_{n-1}$ does not cut $\mathcal{S}_2$ between p_0 and p_1.*

THEOREM 8′. *Let $(p_0, p_1, p_2) \subset \mathcal{S}_2 - (A_0 \cup \ldots \cup A_{n-1})$. If none of the sets $A_{k+1} \cup \ldots \cup A_{k+n-1}$ cuts $\mathcal{S}_2$ between any pair (p_0, p_1), (p_1, p_2) and (p_2, p_0), and if all the sets $C_k = A_{k+1} \cup \ldots \cup A_{k+n-2}$ are connected, then the set $A_0 \cup \ldots \cup A_{n-1}$ does not cut $\mathcal{S}_2$ between one of these three pairs at least.*

THEOREM 9. *Let C be a locally connected continuum and F a closed set disjoint from C. Then there exist two locally connected continua C_0 and C_1 such that $C = C_0 \cup C_1$ and that none of them is a cut of $\mathcal{S}_2$ between any pair of points of F.*

Proof. Since F is compact, there exists a finite system $R_0, \ldots, R_n$ of components of $\mathcal{S}_2 - C$ such that $F \subset R_0 \cup \ldots \cup R_n$. If $n = 0$,

(1) For the proof, see my paper in Fund. Math. 36 (1949), p. 277.

we can set $C_0 = C = C_1$. So assume $n \geqslant 1$(1). Let $p_j \in R_j$ for $j = 0, \ldots$ $\ldots, n$. Let L be an arc joining the points $p_0, \ldots, p_n$. By Theorem 9 (iv) of § 58, I, the set $L \cap C$ is contractible with respect to $\mathscr{S}$. Therefore

$$\frac{x-p_j}{x-p_0} \sim 1 \quad \text{on } L \cap C \quad \text{for} \quad j = 1, \ldots, n.$$

By Theorem 9 of § 56, II, there is a closed neighbourhood F_0 of $L \cap C$ in C such that

$$\frac{x-p_j}{x-p_0} \sim 1 \quad \text{on } F_0 \quad \text{for} \quad j = 1, \ldots, n. \tag{12}$$

Define $F_1 = \overline{C - F_0}$. Therefore $F_1 \cap L = 0$, and consequently F_1 is not a cut between p_0 and p_j. It follows (by II, Theorem 1) that

$$\frac{x-p_j}{x-p_0} \sim 1 \quad \text{on } F_1 \quad \text{for} \quad j = 1, \ldots, n. \tag{13}$$

We derive from (12) and (13) by means of Theorem 6 of § 56, X (setting $\mathscr{X} = C$) that there exist two locally connected continua C_0 and C_1 such that for $m = 0, 1$,

$$F_m \subset C_m \subset C \quad \text{and} \quad \frac{x-p_j}{x-p_0} \sim 1 \quad \text{on } C_m \quad \text{for} \quad j = 1, \ldots, n.$$

Thus C_m is not a cut between any pair p_0, p_j, so that there exists a component Q_m of $\mathscr{S}_2 - C_m$ such that $(p_0, \ldots, p_n) \subset Q_m$, hence $R_0 \cup \ldots \cup R_n \subset Q_m$, therefore $F \subset Q_m$.

Besides, since $C = F_0 \cup F_1$, it follows that $C = C_0 \cup C_1$.

Corollary 10(2). *Every locally connected continuum C, which cuts $\mathscr{S}_2$ into a finite number of regions, is the union of two locally connected continua which do not cut $\mathscr{S}_2$.*

Proof. Let E be a finite set containing one point of every component of $\mathscr{S}_2 - C$.

(1) For the particular case where $n = 1$, see my paper in Fund. Math. 8 (1926), p. 137.

(2) Theorem of Borsuk, Fund. Math. 24 (1935), p. 135.

By Theorem 9 there exist two locally connected continua C_0 and C_1 and two components R_0 of $\mathcal{S}_2-C_0$ and R_1 of $\mathcal{S}_2-C_1$ such that

$$C = C_0 \cup C_1, \tag{14}$$

$$E \subset R_m, \quad \text{hence} \quad \mathcal{S}_2 - C \subset R_m \quad \text{for} \quad m = 0, 1. \tag{15}$$

Therefore, the set $C_m^* = \mathcal{S}_2 - R_m$ is a locally connected continuum which does not cut $\mathcal{S}_2$ (compare § 49, II, Theorem 1). Furthermore it follows by (15) that

$$\mathcal{S}_2 - C \subset R_m \subset \mathcal{S}_2 - C_m, \quad \text{so that} \quad C_m \subset \mathcal{S}_2 - R_m = C_m^* \subset C,$$

and hence $C = C_0^* \cup C_1^*$ by (14).

V. Irreducible cuts. Theorem 1 of Section II implies the following.

THEOREM 1. *A set $A \subset \mathcal{S}_2 - p - q$ irreducibly cuts $\mathcal{S}_2$ between p and q if and only if*

$$\frac{x-p}{x-q} \; irr\,non \sim 1 \quad on\ A.$$

This can be generalized as follows(1).

THEOREM 2. *Let $A \subset \mathcal{S}_2 - (p_0, \ldots, p_n)$. The set A irreducibly cuts $\mathcal{S}_2$ between every pair p_i, p_j (where $i \neq j$) if and only if the homographic functions*

$$\frac{x-p_1}{x-p_0}, \ldots, \frac{x-p_n}{x-p_0}$$

are linearly independent on $A \operatorname{mod} \Psi(A)$ and if

$$\frac{x-p_k}{x-p_0} \; irr\,non \sim 1 \quad on\ A \quad for \quad k = 1, \ldots, n.$$

Proof. This is a direct consequence of Theorems 1 and 6 of Section II.

THEOREM 3. *Let $F = \overline{F} \subset \mathcal{S}_2$. Let $R_0, \ldots, R_m$ (m finite or infinite) be the components of $\mathcal{S}_2 - F$ such that $\operatorname{Fr}(R_j) = F$. Then (compare § 56, VIII (1)) we have*

$$\operatorname{rank}[\Omega(F)/\Psi(F)] = m. \tag{1}$$

(1) Compare S. Eilenberg, *op. cit.*, p. 103.

In other words, *a set F is the common boundary of $n+1$ regions $(n \geqslant 1)$ if and only if there exist n functions $f_1, \ldots, f_n$ linearly independent on F* $\operatorname{mod} \Psi(F)$ *and such that $f_i \sim 1$ on every proper closed subset of F (for $i = 1, \ldots, n$).*

Proof. By Theorem 1 of § 49, V, the set F irreducibly cuts $\mathcal{S}_2$ between p and q if it is the common boundary of the components of $\mathcal{S}_2 - F$ which contain p and q respectively.

Hence we have $\operatorname{rank}[\Omega(F)/\Psi(F)] \geqslant m$ by Theorem 2.

On the other hand, let $f_1, \ldots, f_n$ be a system satisfying the second part of Theorem 3. Let $p_j \in R_j$, where $j = 0, \ldots, m$. By Theorem 1 of Section III we have

$$f_i(x) \sim \left(\frac{x-p_1}{x-p_0}\right)^{k_{i,1}} \cdot \ldots \cdot \left(\frac{x-p_m}{x-p_0}\right)^{k_{i,m}} \quad \text{on } F \quad (i = 1, \ldots, n), \quad (2)$$

because, (compare III (3)) if R is a component of $\mathcal{S}_2 - F$ such that $\operatorname{Fr}(R) \neq F$, it follows by hypothesis that $f_i | \operatorname{Fr}(R) \sim 1$ for $i = 1, \ldots, n$.

The system (2) of n homotopies immediately implies that $n \leqslant m$. Identity (1) follows directly.

THEOREM 4 (of invariance)(¹). *The property of being the common boundary of n regions is an intrinsic invariant.*

Proof. This is a direct consequence of Theorem 3.

THEOREM 5 (of decomposition). *Let $A \subset \mathcal{S}_2 - (p_0, \ldots, p_n)$. If A irreducibly cuts $\mathcal{S}_2$ between every pair p_i, p_j $(i \neq j)$ and if A is decomposable, then A is the union of two connected sets A_0 and A_1 such that*

$$A_1 = A \cap \overline{A - A_0} \quad \textit{and} \quad A_0 = A \cap \overline{A - A_1}$$

and $A_0 \cap A_1$ is the union of $n+1$ non-empty sets $F_0, \ldots, F_n$ closed in $A_0 \cap A_1$ and such that each of the sets A_0 and A_1 is irreducibly connected between F_k and $A_0 \cap A_1 - F_k$ for $k = 0, 1, \ldots, n$.

Proof. This is a consequence of Theorem 5 of § 56, VII combined with Theorem 2, which implies that $\operatorname{rank}[\Omega(A)/\Psi(A)] \geqslant n$.

THEOREM 6 (of the union). *Let $A = A_0 \cup A_1$ where A_0 and A_1 are two continua such that $A_0 \cap A_1 = F_0 \cup \ldots \cup F_n$ where $F_0, \ldots, F_n$ are $n+1$ $(\geqslant 2)$ non-empty, disjoint, closed sets. If each of the sets*

(¹) Compare Alexandroff–Hopf, *Topologie* I, p. 392, and S. Eilenberg, *op. cit.*, p. 104.

A_0 and A_1 is irreducibly connected between F_k and $A_0 \cap A_1 - F_k$, for $k = 0, 1, \ldots, n$, then there exists in $\mathscr{S}_2 - A$ a system of points $p_0, \ldots, p_n$ such that the set A irreducibly cuts $\mathscr{S}_2$ between every pair p_i, p_j where $i \neq j$.

In other words, *A is the common boundary of $n+1$ regions.*

Proof. This is a consequence of Theorem 3 combined with Theorem 5 of § 56, VII, which implies that $\operatorname{rank}[\Omega(F)/\Psi(F)] \geqslant n$.

THEOREM 7. *Every set A, which irreducibly cuts $\mathscr{S}_2$ between p and q, is connected and discoherent.*

If, moreover, A is locally connected, it is a simple closed curve.

Proof. This is a consequence of Theorem 1 combined with Theorems 1 and 2 of § 56, VII.

THEOREM 8. *Every set, which irreducibly cuts $\mathscr{S}_2$ between p and q, is either indecomposable or is the union of two sets irreducible between the same pair of points*(1).

Proof. This is a direct consequence of Theorem 5.

THEOREM 9. *If a set A irreducible between a and b irreducibly cuts $\mathscr{S}_2$ between p and q, then A either is indecomposable or is the union of two indecomposable sets which are closed in A*(2).

Proof. This is a consequence of Theorem 3 of § 56, VII (compare also § 48, VII, Theorem 7).

THEOREM 10. *If A irreducibly cuts $\mathscr{S}_2$ between p_0 and p_1, between p_1 and p_2 and between p_2 and p_0, then A is either indecomposable or is the union of two indecomposable sets closed in A*(3).

If, moreover, A is compact, then A is a continuum irreducible between two points.

Proof. This is a consequence of Theorem 2 and of Theorem 6 of § 56, VII.

The following statement follows from Theorem 10 and Theorem 1 of § 49, V.

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(1) Compare my papers *Sur les coupures irréductibles du plan*, Fund. Math. 6 (1924), p. 137, and *Sur la structure des frontières communes à deux régions*, *ibid.* 12 (1928), p. 21.

(2) See my paper *Über geschlossene Kurven und unzerlegbare Kontinua*, Math. Ann. 98 (1927), p. 404. Compare P. Alexandrov, *ibid.* 96 (1926), p. 537.

(3) S. Eilenberg, *op. cit.*, p. 82.

THEOREM 11. *A common boundary of three regions is either an indecomposable continuum or is the union of two indecomposable continua*(1).

The figures 21 and 22 represent common boundaries of three regions, one is indecomposable and the other is the union of two indecomposable continua (2).

Let us add that slightly modifying the construction of the above continua we can replace the term three by n, or even by ∞(3).

Remark 1. Theorem 11 does not hold in $\mathcal{S}_3$. In fact there exist in $\mathcal{S}_3$ *absolute neighbourhood retracts* which are common boundaries of three regions (4).

Remark 2. Let us quote without proof (5) the following theorem.

Let A be a locally connected subset of $\mathcal{S}_2$ and $p_0, \ldots, p_n$ a system of points in $\mathcal{S}_2 - A$ such that A cuts $\mathcal{S}_2$ between every pair p_i, p_j for $i \neq j$.

If the rational function

$$r(z) = (z-p_0)^{k_0} \cdot \ldots \cdot (z-p_n)^{k_n},$$

where $k_0 + \ldots + k_n = 0$ and $k_j \neq 0$ for $0 \leqslant j \leqslant n$, is homotopic on A to a homeomorphism $f\colon A \to \mathcal{P}$, then

$$|k_0| + \ldots + |k_n| \leqslant 2n.$$

VI. Groups $\mathcal{P}^A$ and $\mathfrak{B}_1(A)$ for locally connected A(6). Many properties of plane locally connected continua which have been established in § 61, II, now will be generalized to locally connected sets (closed or not).

Theorems 1 of Section II and 4 of § 56, X, directly imply the following one, which is a generalization of Theorem 5 of § 61, II.

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(1) See my papers cited above, Fund. Math. 6, p. 138, and *ibid.* 12, p. 36.

(2) For the exact definitions, see B. Knaster, *Quelques coupures singulières du plan,* Fund. Math. 7 (1925), pp. 277 and 280.

(3) *Ibid.*, p. 28.

(4) See M. Lubański, Fund. Math. 50 (1953), pp. 29–38.

(5) For the proof see my paper, *Fonctions rationnelles qui sont homotopes à des fonctions biunivoques sur certains sous-ensembles du plan,* Fund. Math. 41 (1954), pp. 107–121.

For a converse theorem see *ibid.*, p. 108, and also A. Pliś, *Rational functions univalent on sets separating the plane,* Bull. Acad. Pol. Sci. 2 (1954), p. 255.

(6) See S. Eilenberg, *op. cit.* pp. 83 and 105.

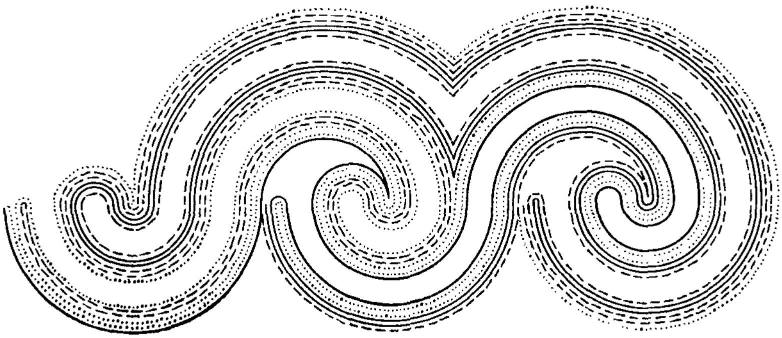

Fig. 21

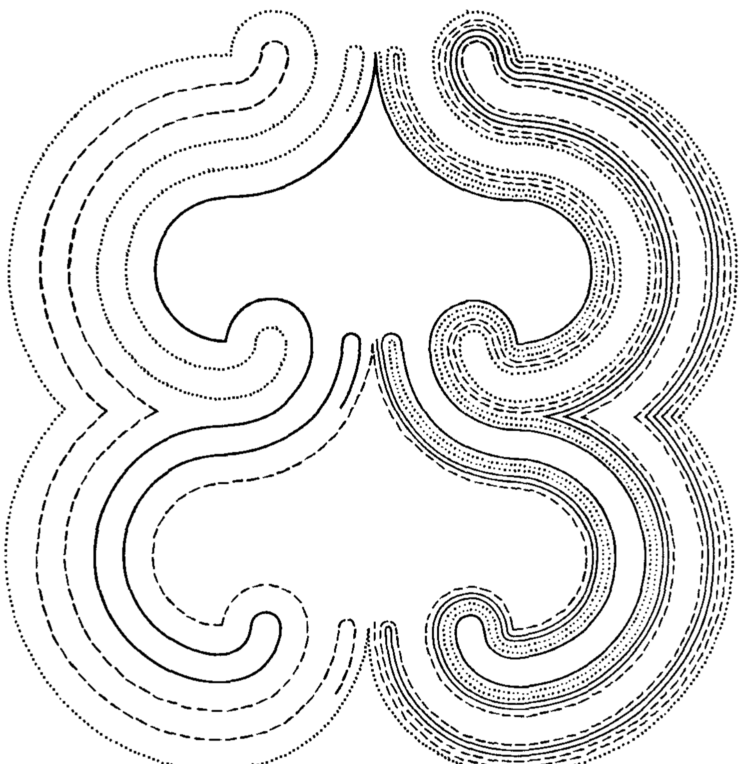

Fig. 22

THEOREM 1. *Every locally arcwise connected set* $A \subset \mathcal{S}_2 - p - q$, *which cuts* $\mathcal{S}_2$ *between* p *and* q, *contains a simple closed curve which does it.*

THEOREM 2. *If* A *is a locally arcwise connected set, every component of* $\mathcal{S}_2 - A$ *is a semi-continuum* (*and hence a constituant of* $\mathcal{S}_2 - A$).

Proof. If p and q are two points of $\mathcal{S}_2 - A$ which cannot be joined in $\mathcal{S}_2 - A$ by a continuum, then by Theorem 1 they are separated by a closed subset of A (even by a simple closed curve). But then they do not belong to the same component of $\mathcal{S}_2 - A$.

THEOREM 3. *If the set* A *does not cut* $\mathcal{S}_2$, *then no set* B *such that*

$$A \subset B \subset A \cup L(A)$$

does it either.

More precisely, *if* A *does not cut* $\mathcal{S}_2$ *between* p *and* q, *then the set* $[A \cup L(A)] - p - q$ *does it neither.*

Proof. It follows by Theorem 1 of Section II that

$$\frac{x-p}{x-q} \sim 1 \quad \text{on } A, \quad \text{hence on} \quad [A \cup L(A)] - p - q$$

according to Theorem 3 of § 56, X.

THEOREM 4. *In order that a locally connected set* A *does not cut* $\mathcal{S}_2$, *it is necessary and sufficient that it be contractible with respect to* $\mathcal{S}$ (*and hence with respect to* $\mathcal{P}$).

Proof. 1. Let A be a locally connected set which does not cut $\mathcal{S}_2$. Let $f\colon A \to \mathcal{S}$ be a continuous function. We must show that $f \sim 1$. Following Theorem 23 of § 49, II, let B be a locally connected $\boldsymbol{G}_\delta$ such that $A \subset B \subset L(A)$, and let $g\colon B \to \mathcal{S}$ be a (continuous) extension of f.

Suppose that f *non* ~ 1, therefore that g *non* ~ 1.

Since B is arcwise connected (by § 50, II, Theorem 1), there exists according to Theorem 4 of § 56, X, a simple closed curve C such that

$$C \subset B, \tag{1}$$

$$g|C \text{ non} \sim 1. \tag{2}$$

Let D_0 and D_1 be the components of $\mathcal{S}_2 - C$. Since B does not cut $\mathcal{S}_2$ (by Theorem 3), then one of these components, say the disk

D_0, satisfies the inclusion

$$D_0 \subset B, \quad \text{hence} \quad \bar{D}_0 \subset B \tag{3}$$

by condition (1).

Since the set $\mathscr{S}_2 - \bar{D}_0 = D_1$ is connected, it follows by Theorem 10 of § 59, II, that

$$g|\bar{D}_0 \sim 1, \quad \text{hence} \quad g|C \sim 1, \tag{4}$$

contradicting (2).

2. Conversely, if A cuts $\mathscr{S}_2$ between p and q, then

$$\frac{x-p}{x-q}\ \textit{non} \sim 1 \quad \text{on } A,$$

by Theorem 1 of Section II. Therefore A is not contractible with respect to $\mathscr{P}$ and hence, neither with respect to $\mathscr{S}$.

THEOREM 5. *Every connected and locally connected set A, which does not cut $\mathscr{S}_2$, is a semi-continuum* (¹).

Proof. Let $p, q \in A$. We have to show that the set $B = \mathscr{S}_2 - A$ is not a cut between p and q. For the sake of simplicity, assume that $p = 0, q = \infty$ and $1 \in B$.

Define

$$f(x, y) = \frac{xy - y}{x - y} \quad \text{where} \quad x \in A \text{ and } y \in B.$$

It is easy to see that

$$f\colon A \times B \to \mathscr{P} \text{ is continuous,} \tag{5}$$

$$f(0, y) = 1, \tag{5'}$$

$$f(\infty, y) = y. \tag{5''}$$

Since A is contractible with respect to $\mathscr{P}$ (by Theorem 4), it follows from (5) and (5′) by Theorem 2 of § 56, X (setting $\mathscr{X} = A$, $\mathscr{Y} = B$ and $a = 0$), that

$$f \sim 1 \quad \text{on } A \times B, \quad \text{which implies} \quad y \sim 1 \quad \text{on } B,$$

by (5″).

(¹) Theorem of S. Eilenberg, Fund. Math. 29 (1937), p. 159. Recall that in $\mathscr{S}_2$ there exist connected, locally connected and totally imperfect sets (which cut $\mathscr{S}_2$, however); compare § 49, I, Example (viii).

Then it follows, according to Theorem 1 of Section II, that B does not cut $\mathscr{S}_2$ between the points 0 and ∞.

Compare the following theorem with Theorem 11 of § 59, II.

THEOREM 6. *If A is locally connected and C is a constituent of $\mathscr{S}_2-A$, every point p of $A\cap\overline{C}$ is accessible from C.*

Proof. Let $c\in C$. We have to show that there is a continuum K such that

$$c,p\in K\subset C\cup p. \tag{6}$$

Let F be a closed set such that

$$c,p\in F\subset C\cup p \quad \text{and} \quad p\in\overline{F-p} \tag{7}$$

(for instance, F may be assumed to consist of c, of p and of a sequence $\subset C$ and converging to p). By (7) every point x of $F-p$ can be joined with c by a continuum disjoint from A, therefore (compare Theorem 3) by a continuum disjoint from $L(A)-x-c$, and hence from $L(A)-F$. In other words, there exists a constituent D of $\mathscr{S}_2-[L(A)-F]$ such that $F-p\subset D$. It follows by (7) that $p\in\overline{D}$, so that the set $D\cup(p)$ is connected.

Since $L(A)$ is a locally connected $\boldsymbol{G}_\delta$ (by § 49, II, Theorems 20 and 22), then so is $L(A)-F$. Therefore, this set is locally arcwise connected (compare 50, II, Theorem 1), and by Theorem 2 the set D is a component of $\mathscr{S}_2-[L(A)-F]$. The set $D\cup(p)$ is its connected subset (since $p\in F$) and therefore $p\in D$. Since D is a semi-continuum, it contains a continuum K irreducible between c and p. By Theorem 3 of § 48, II, the set $K-p$ is connected. Let H be the component of $\mathscr{S}_2-[L(A)-F\cup p]$ which contains $K-p$, and hence contains c. The inclusions (compare (7))

$$A\subset L(A)-F\cup p\subset L(A) \tag{7'}$$

show that the set $L(A)-F\cup p$ is a locally connected $\boldsymbol{G}_\delta$ (compare § 49, II, Theorem 20 through 22), and therefore is locally arcwise connected.

Consequently (see Theorem 2), the set H is a semi-continuum and $H\subset\mathscr{S}_2-A$ by (7').

Since C is a constituent of the point c in $\mathscr{S}_2-A$, we have $H\subset C$, and hence $K-p\subset C$ and condition (6) follows.

THEOREM 7. *Let A be a locally connected set whose complement $\mathcal{S}_2 - A$ consists of a finite number of constituents*

$$\mathcal{S}_2 - A = B_0 \cup \ldots \cup B_n.$$

Let $p_j \in B_j$ for $j = 0, \ldots, n$. Then every continuous $f\colon A \to \mathcal{P}$ is homotopic to a rational function,

$$f(x) \sim (x-p_0)^{k_0} \cdot \ldots \cdot (x-p_n)^{k_n}, \quad \textit{where} \quad k_0 + \ldots + k_n = 0. \tag{8}$$

Consequently (compare Theorem 6 of Section II), *the functions*

$$\frac{x-p_1}{x-p_0}, \ \ldots, \ \frac{x-p_n}{x-p_0}$$

constitute a basis of $\mathcal{P}^A \bmod \Psi(A)$.

Proof. First, consider the case of A arcwise connected. Let, according to Theorem 1, C_{ij} be a simple closed curve contained in A and cutting $\mathcal{S}_2$ between p_i and p_j. Let C be the union of the curves C_{ij} (where $i \neq j$) and R_j the component of $\mathcal{S}_2 - C$ such that $p_j \in R_j$. Since $C \subset A$, then $B_j \subset R_j$. Define

$$F = \mathcal{S}_2 - (R_0 \cup \ldots \cup R_n), \tag{9}$$

hence

$$F \subset A. \tag{10}$$

We can assume (by Theorem 1 of Section III) that conditions (8) are fulfilled on F, for otherwise we set

$$g(x) = f(x) \cdot (x-p_0)^{-k_0} \cdot \ldots \cdot (x-p_n)^{-k_n} \tag{11}$$

and then g *non* ~ 1 on A.

According to Theorem 4 of § 56, X, let K be a simple closed curve such that

$$K \subset A, \tag{12}$$

$$g \text{ non} \sim 1 \quad \text{on } K. \tag{13}$$

Let Q_j be the component of $\mathcal{S}_2 - (F \cup K)$ such that $p_j \in Q_j$. Since $F \cup K \subset A$ (by (10) and (12)), then $B_j \subset Q_j$. Define

$$H = \mathcal{S}_2 - (Q_0 \cup \ldots \cup Q_n), \tag{14}$$

hence

$$F \cup K \subset H \subset A. \tag{15}$$

By (14) and Theorem 1 of Section III, it follows that on H

$$g(x) \sim (x-p_0)^{m_0}\cdot\ldots\cdot(x-p_n)^{m_n}, \tag{16}$$

$$m_0+\ldots+m_n = 0. \tag{17}$$

It follows by (11) and (15) that on F

$$(x-p_0)^{k_0+m_0}\cdot\ldots\cdot(x-p_n)^{k_n+m_n} \sim f(x),$$

which proves that

$$(x-p_0)^{m_0}\cdot\ldots\cdot(x-p_n)^{m_n} \sim 1 \quad \text{on } F, \tag{18}$$

because the homotopy (8) holds on F.

Since the set F cuts $\mathscr{S}_2$ between each pair p_i, p_j $(i \neq j)$, it follows from (17), (18) and Theorem 6 of Section II that $m_0 = \ldots = m_n = 0$. But then $g \sim 1$ on H (by (16)), hence on K (compare (15)), contradicting (13).

Thus, the theorem has been established in the case where A is a G_δ-set. Now the general case will be reduced to this particular case.

Let f: $A \to \mathscr{P}$ be a continuous function. There exists a continuous extension f_1 of f to a G_δ-set, say A_1, containing A, thus $f \subset f_1$: $A_1 \to \mathscr{P}$. Define

$$A^* = A_1 \cap L(A)-(p_0, \ldots, p_n) \quad \text{and} \quad f^* = f_1|A^*.$$

By Theorem 3, A^* is a locally connected G_δ which cuts $\mathscr{S}_2$ into $n+1$ constituents containing the points $p_0, \ldots, p_n$ respectively. As just has been proved, conditions (8) are fulfilled on A^* (replacing f by f^*). The required conclusion follows, because $A \subset A^*$ and $f(x) = f^*(x)$ for $x \in A$.

Let $c(\mathscr{X})+1$ be the number of constituents of $\mathscr{X}$ (provided this number is finite). As in the case of closed sets, the next theorem follows from Theorem 7.

THEOREM 8 (of invariance and of duality). *If A is a locally connected set such that $c(\mathscr{S}_2-A) = n < \infty$, then*

$$\mathfrak{B}_1(A) \underset{\text{gr}}{=} \mathscr{G}^n.$$

Let us mention that many theorems of Section IV can be applied to locally connected sets replacing $b_1(A)$ by $c(\mathscr{S}_2-A)$.

VII. Groups $\mathscr{P}^G$ and $\mathfrak{B}_1(G)$ for open G.

THEOREM 1. *Let G be an open set whose complement $\mathscr{S}_2-G$ consists of a finite number of components $C_0, \ldots, C_n$. Every continuous function $f\colon G \to \mathscr{P}$ is homotopic to a rational function.*

More precisely, *if $p_j \in C_j$, then we have on G*

$$f(x) \sim (x-p_0)^{k_0} \cdot \ldots \cdot (x-p_n)^{k_n} \quad \textit{where} \quad k_0+\ldots+k_n = 0. \tag{1}$$

Proof. This is a direct consequence of Theorem 7 of Section VI.

Remarks. (i) Theorem 1 can be derived from Theorem 1 of Section III in the following way. Let A_j be a closed separator between C_j and $\mathscr{S}_2-G-C_j$ and let, according to Theorem 2 of § 50, III, $F_1, F_2, \ldots$ be a sequence of closed sets such that

$$A_0 \cup \ldots \cup A_n \subset F_1$$

and that

$$G = F_1 \cup F_2 \cup \ldots, \tag{2}$$

$$F_m \subset \operatorname{Int}(F_{m+1}), \tag{3}$$

$$\mathscr{S}_2 - F_m = R_{m,0} \cup \ldots \cup R_{m,n} \quad \text{and}$$

$$C_j \subset R_{m,j} \quad \text{for} \quad j = 0, \ldots, n, \tag{4}$$

where $R_{m,0}, \ldots, R_{m,n}$ are the components of $\mathscr{S}_2 - F_m$.

By (4) and (1), (2) of Section III, we have on F_m

$$f(x) \sim (x-p_0)^{k_{m,0}} \cdot \ldots \cdot (x-p_n)^{k_{m,n}} \quad \text{and} \quad k_{m,0}+\ldots+k_{m,n} = 0. \tag{5}$$

Since $F_1 \subset F_m$, the homotopy (5) holds on F_1. Therefore by Theorem 2 of Section III, $k_{m,j} = k_{1,j}$ for $j = 0, \ldots, n$. Let $k_{1,j} = k_j$. Consequently, the homotopy (1) holds on F_m, and hence on G by (2), (3) and Theorem 8 of § 56, X (setting $\mathscr{X} = G$ and $G_m = \operatorname{Int}(F_m)$).

(ii) *The exponents $k_0, \ldots, k_n$ are uniquely determined.*

This is an easy consequence of Theorem 2 of Section III.

The following theorem shows an analogy to the *theorem of Runge* concerning the representation of holomorphic functions as limits of rational functions.

THEOREM 2. *If G is an open subset of $\mathscr{S}_2$, every continuous function $f: G \to \mathscr{P}$ has the following form*

$$f(x) = \lim_{m=\infty} r_m(x) e^{u_m(x)} \quad \text{where} \quad u_m: \mathscr{S}_2 \to \mathscr{E}^2 \tag{6}$$

is continuous and r_m rational.

The convergence is uniform on every closed subset F of G.

Moreover, *for every F there exists a number m_0 such that for $m \geqslant m_0$ and $x \in F$*

$$f(x) = r_m(x) e^{u_m(x)}. \tag{7}$$

Finally, *if $\{C_j\}$ is a sequence of components of $\mathscr{S}_2 - G$ such that $C \cap \overline{C_0 \cup C_1 \cup \ldots} \neq 0$ whatever is the component C of $\mathscr{S}_2 - G$, and if $p_j \in C_j$, then the zeros and poles of the functions $r_1, r_2, \ldots$ belong to the sequence $\{p_j\}$.*

Proof. Assume conditions (2) and (3), and let $R_{m,0}, R_{m,1}, \ldots$ be the sequence of components of $\mathscr{S}_2 - F_m$; it is admissible to suppose that $R_{m,j}$ contains a component of $\mathscr{S}_2 - G$ (compare § 50, III, (iv)). Since the set $R_{m,j}$ is open, there exists a subscript $t = t(m,j)$ such that

$$C_t \cap R_{m,j} \neq 0, \text{ hence } C_t \subset R_{m,j}, \text{ and therefore } p_t \in R_{m,j}.$$

It follows by Theorem 1 of Section III that the identity (7) holds on F_m for

$$r_m(x) = (x - p_{t(m,0)})^{k_{m,0}} \cdot \ldots \cdot (x - p_{t(m,n_m)})^{k_{m,n_m}}.$$

Let $F = \overline{F}$. By (2) and (3), we have $F \subset F_{m_0}$ for sufficiently large m_0, and hence identity (7) is satisfied for every $x \in F$.

THEOREM 3. *If G is an open set such that the set $\mathscr{S}_2 - G$ consists of an infinite sequence of components, all of which except one, say C_0, are open in $\mathscr{S}_2 - G$, then each continuous function $f: G \to \mathscr{P}$ has the following form*

$$f(x) = \prod_{n=1}^{\infty} \left(\frac{x - p_n}{x - p_0}\right)^{k_n} e^{u_n(x)} \tag{8}$$

where $p_0 \in C_0$, $p_n \in \mathscr{S}_2 - G$ and $u_n: \mathscr{S}_2 \to \mathscr{E}^2$ is continuous.

The convergence is uniform on every closed subset F of G.

Moreover, for every F there exists a number n_0 such that for $n > n_0$ and $x \in F$

$$f(x) = \left(\frac{x-p_1}{x-p_0}\right)^{k_1} \cdot \ldots \cdot \left(\frac{x-p_n}{x-p_0}\right)^{k_n} e^{v_n(x)}, \tag{9}$$

$$v_n(x) = u_1(x) + \ldots + u_n(x). \tag{9'}$$

If $\infty \in C_0$, the term $x-p_0$ can be omitted in (8) *and* (9) (*setting $p_0 = \infty$*).

Proof. By Theorem 4 of § 50, III, the components of $\mathscr{S}_2 - G$ can be ordered into an infinite sequence $C_0, C_1, C_2, \ldots$ in such a manner that

$$\mathscr{S}_2 - F_n = R_{n,0} \cup \ldots \cup R_{n,n}, \tag{10}$$

$$C_j \subset R_{n,j} \quad \text{for} \quad 1 \leqslant j \leqslant n, \tag{11}$$

$$C_0 \cup C_{n+1} \cup C_{n+2} \cup \ldots \subset R_{n,0}, \tag{12}$$

where the closed sets F_n satisfy conditions (2) and (3), and $R_{n,0}, \ldots$ $\ldots, R_{n,n}$ are the components of $\mathscr{S}_2 - F_n$.

Let $p_n \in C_n$. Therefore $p_0 \in R_{n,0}, \ldots, p_n \in R_{n,n}$ by (11) and (12). By (10) and by conditions (1) and (2) of Section III, it follows that on F_n

$$f(x) = \left(\frac{x-p_1}{x-p_0}\right)^{k_{n,1}} \cdot \ldots \cdot \left(\frac{x-p_n}{x-p_0}\right)^{k_{n,n}} e^{v_n(x)} \tag{13}$$

where v_n: $\mathscr{S}_2 \to \mathscr{E}^2$ is a continuous function. Consequently

$$f(x) \sim \left(\frac{x-p_1}{x-p_0}\right)^{k_{n+1,1}} \cdot \ldots \cdot \left(\frac{x-p_{n+1}}{x-p_0}\right)^{k_{n+1,n+1}} \quad \text{on } F_n, \tag{14}$$

since $F_n \subset F_{n+1}$.

By (12) the set F_n does not cut $\mathscr{S}_2$ between p_0 and p_{n+1}. Therefore (by Theorem 1 of Section II)

$$\frac{x-p_{n+1}}{x-p_0} \sim 1 \quad \text{on } F_n, \tag{15}$$

and by (14)

$$f(x) \sim \left(\frac{x-p_1}{x-p_0}\right)^{k_{n+1,1}} \cdot \ldots \cdot \left(\frac{x-p_n}{x-p_0}\right)^{k_{n+1,n}} \quad \text{on } F_n. \tag{16}$$

Conditions (13) and (16) imply, by Theorem 2 of Section III, that

$$k_{n,1} = k_{n+1,1}, \quad \ldots, \quad k_{n,n} = k_{n+1,n}.$$

Set $k_{n,n} = k_n$ for $n = 1, 2, \ldots$. Then condition (13) implies that (9) holds on F_n.

Let $F = \overline{F} \subset G$. By (2) and (3) there exists a number n_0 such that $F \subset F_n$ for $n \geqslant n_0$. Therefore identity (9) is fulfilled on F.

Finally, set

$$u_1(x) = v_1(x) \quad \text{and} \quad u_n(x) = v_n(x) - v_{n-1}(x).$$

Then identity (9′) is fulfilled.

Remarks. (i) *The convergence of the infinite product* (8) *is absolute. Thus, it does not depend upon the order of factors.*

Consequently, *one can assume that $p_0, p_1, \ldots$ is a sequence given in advance of points chosen one by one from all components of the set $\mathscr{S}_2 - G$ (with $p_0 \in C_0$).*

Because for every x there exists only a finite number of factors different from unity.

(ii)[1]. *If* $\operatorname*{Lim}_{n=\infty} C_n = \infty$, *every continuous function $f\colon G \to \mathscr{P}$ is homotopic to a function meromorphic on $\mathscr{E}^2$.*

(iii) Theorem 3 shows a remarkable analogy to the *Weierstrass theorem on the decomposition of an entire function into prime factors.*

If g is an entire function and if G denotes the plane $\mathscr{E}^2$ from which the zeros of g have been removed, then obviously the hypotheses of Theorem 3 are fulfilled.

(iv) *Let G be an open set in $\mathscr{E}^2$. For every continuous function $f\colon G \to \mathscr{P}$ there exists a holomorphic function $h\colon G \to \mathscr{P}$ such that $h \sim f$*[2].

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[1] For the proof, see my above-quoted paper in Fund. Math. 33, p. 340.

[2] See H. Heilbronn, *On the representation of homotopic classes by regular functions*, Bull. Pol. Acad. Sci. 6 (1958), pp. 181–184.

For an analogous theorem concerning the transformations of analytic spaces into Lie groups, see H. Grauert, *Généralisation d'un théorème de Runge et applications à la théorie des espaces fibrés analytiques*, C. R. Paris 242 (1956), pp. 603–605; there are references also to papers by H. Cartan, H. Behnke and others in that connection.

In other words,

$$f(x) = h(x)\, e^{u(x)} \tag{1}$$

where $u\colon G \to \mathscr{E}^2$ is a continuous function properly chosen.

Proof. Let

$$G = F_1 \cup F_2 \cup \ldots, \quad \text{where} \quad F_n \subset \operatorname{Int}(F_{n+1}), \tag{2}$$

with F_n being a compact set (a polygonal domain) such that

$$\text{no component of } \mathscr{S}_2 - F_n \text{ is contained in } G. \tag{3}$$

By Theorem 1 of Section III, there exist for every n a rational function r_n whose zeros and poles belong to $\mathscr{S}_2 - G$, and a continuous function $u_n\colon \mathscr{S}_2 \to \mathscr{E}^2$ such that

$$f(x) = r_n(x)\, e^{u_n(x)} \quad \text{for} \quad x \in F_n. \tag{4}$$

It follows that

$$r_{n+1}(x) : r_n(x) = e^{u_n(x) - u_{n+1}(x)} \quad \text{for} \quad x \in F_n, \tag{5}$$

and therefore the difference $u_n(x) - u_{n+1}(x)$ is holomorphic in $\operatorname{Int}(F_n)$.

We are going to define by induction a sequence of rational functions $s_1, s_2, \ldots$, whose poles belong to $\mathscr{S}_2 - G$ and such that the holomorphic functions (in $\operatorname{Int}(F_n)$)

$$v_n(x) = s_{n+1}(x) - s_n(x) + u_n(x) - u_{n+1}(x) \tag{6}$$

satisfy the condition

$$|v_n(x)| < 1/2^n \quad \text{for} \quad x \in F_{n-1}. \tag{7}$$

Set $s_1(x) = 0$ for all x. Assume that s_n is a rational function whose poles belong to $\mathscr{S}_2 - G$; then $u_{n+1}(x) - u_n(x) + s_n(x)$ is holomorphic in $\operatorname{Int}(F_n)$. By the theorem of Runge(¹) (and by (3)) this function can be uniformly approximated on F_{n-1} by rational functions with poles in $\mathscr{S}_2 - G$. So, there exists among them a function, say s_{n+1}, which satisfies condition (7).

Thus, the infinite sequence $s_1, s_2, \ldots$ is defined. Now we set

$$g_1(x) = r_1(x) \quad \text{and} \quad g_{n+1}(x) = \frac{r_{n+1}(x) \cdot e^{s_{n+1}(x)}}{r_n(x) \cdot e^{s_n(x)}}. \tag{8}$$

(¹) See e.g. S. Saks and A. Zygmund, *Analytic functions*, Monogr. Mat. 28, Second edition, Warszawa 1965, p. 176.

It follows by (4) and (6) that

$$g_{n+1}(x) = e^{v_n(x)} \quad \text{for} \quad x \in F_n, \tag{9}$$

hence

$$|g_{n+1}(x) - 1| < 1/2^{n-1} \quad \text{for} \quad x \in F_{n-1}, \tag{10}$$

by (7) and by the obvious inequality

$$|e^z - 1| < 2|z| \quad \text{for} \quad |z| \leqslant 1.$$

It follows from (10) that the infinite product

$$h(x) = \prod_{n=1}^{\infty} g_n(x) \tag{11}$$

is uniformly convergent on every set F_n and therefore (by (2)) on every compact subset of G. Since, moreover, the functions g_n are holomorphic on G (and do not vanish), then so is h; thus the holomorphic function h maps G into $\mathscr{P}$. It remains to prove that h satisfies condition (1).

Define

$$u(x) = \lim_{n=\infty} [u_n(x) - s_n(x)] \quad \text{for} \quad x \in G. \tag{12}$$

This limit exists and is continuous since by (6) and (7), the sequence $u_n(x) - s_n(x)$ is uniformly convergent on every compact subset F of G.

Set $h_n(x) = g_1(x) \cdot \ldots \cdot g_n(x)$; then by (8) and (4)

$$h_n(x) = r_n(x)\, e^{s_n(x)} = f(x)\, e^{s_n(x) - u_n(x)} \quad \text{for} \quad x \in F_n.$$

Finally, (1) is obtained in the limit (referring to (11) and (12)).

VIII. Multiplicity of a set with respect to a continuous function $f: F \to \mathscr{P}$ where F is closed. Let

$$F = \overline{F} \subset \mathscr{S}_2 \quad \text{and} \quad \mathscr{S}_2 - F = R_0 \cup R_1 \cup \ldots \tag{1}$$

where $R_0, R_1, \ldots$ are the components of $\mathscr{S}_2 - F$.

Let

$$G = R_{l_1} \cup R_{l_2} \cup \ldots. \tag{2}$$

By the multiplicity of G with respect to the continuous function $f: F \to \mathscr{P}$, in symbols $\mu_G f$, we shall understand *the algebraic number* *of zeros and poles belonging to G of an arbitrary rational function homotopic to f* (compare § 60, VIII).

In other terms, setting in view of Theorem 1 of Section III,

$$f(x) \sim (x-p_0)^{k_0}\cdot(x-p_1)^{k_1}\cdot\ldots\,, \quad k_0+k_1+\ldots = 0, \quad p_j \in R_j,$$

we have

$$\mu_G f = k_{l_1}+k_{l_2}+\ldots. \tag{3}$$

According to Theorem 2 of Section III, the number $\mu_G f$ *does not depend on the choice of the rational function.* In particular

$$\mu_{R_n} f = k_n. \tag{4}$$

The following six properties of $\mu_G f$ can easily be established.

THEOREM 1. (*Norm*) $\mu_0 f = 0 = \mu_{\mathscr{S}_2-F} f$.

THEOREM 2. (*Additivity*) $\mu_{G_1\cup G_2} f = \mu_{G_1} f + \mu_{G_2} f$ *if* $G_1 \cap G_2 = 0$.

THEOREM 3. (*Homomorphism*) $\mu_G(f_1\cdot f_2) = \mu_G f_1 + \mu_G f_2$.

THEOREM 4. (*Invariance*) *If* $f_1 \sim f_2$, *then* $\mu_G f_1 = \mu_G f_2$, *in particular*

$$f \sim 1 \quad \textit{implies} \quad \mu_G f = 0 \quad \textit{for any } G.$$

THEOREM 5. (*Characterization of homotopy*) *If* $\mu_{R_j} f_1 = \mu_{R_j} f_2$ *for* $j = 0, 1, \ldots,$ *then* $f_1 \sim f_2$.

Consequently (compare Theorem 4)

$$(f_1 \sim f_2) \equiv (\mu_G f_1 = \mu_G f_2 \textit{ for any } G).$$

In particular,

$$(f \sim 1) \equiv (\mu_G f = 0 \textit{ for any } G).$$

THEOREM 6. (*Continuity*) *If* $\lim\limits_{n=\infty} f_n = f$ (*the convergence being uniform*), *then* $\mu_G f_n = \mu_G f$ *for sufficiently large* n.

Proof. This is a consequence of Theorem 4 and of (1) of § 54, II.

THEOREM 7. *A continuous function* $f\colon F \to \mathscr{P}$ *admits a* (*continuous*) *extension* $f^*\colon F \cup R_j \to \mathscr{P}$ *if and only if* $\mu_{R_j} f = 0$.

Proof. This is a consequence of (4) and of Theorem 2′ of Section III.

THEOREM 8. *If* $f_j = f\,|\,\mathrm{Fr}(R_j)$, *then* $\mu_{R_j} f_j = \mu_{R_j} f$.

Because R_j is a component of the set

$$\mathscr{S}_2 - \mathrm{Fr}(R_j) = R_j \cup (\mathscr{S}_2 - \bar{R}_j).$$

IX. Multiplicity with respect to a continuous function $f: G \to \mathscr{P}$ **where** G **is open.** Let F be a closed-open set in $\mathscr{S}_2 - G$. Thus

$$F = \overline{F} \quad \text{and} \quad \mathscr{S}_2 - G - F = \overline{\mathscr{S}_2 - G - F}.$$

By Theorem 2 of Section VII, there exists a sequence of rational functions $r_1, r_2, \ldots$ whose zeros and poles belong to $\mathscr{S}_2 - G$ and which are such that

$$f(x) = \lim_{n=\infty} r_n(x) \cdot e^{u_n(x)}, \qquad u_n\colon G \to \mathscr{E}^2, \tag{1}$$

where u_n is continuous and the convergence is uniform on every set $F^* = \overline{F^*} \subset G$.

Define

$$\mu_F f = \lim_{n=\infty} \mu_F(r_n | G) \tag{2}$$

where $\mu_F(r_n|G)$ denotes the algebraic number of zeros and poles of r_n which belong to F.

We have to prove that this limit exists and that it does not depend on the choice of the functions r_n.

Let F^* be a closed set which separates the sets F and $\mathscr{S}_2 - G - F$. So, let G^* be closed-open in $\mathscr{S}_2 - F$, contain F and is disjoint from $\mathscr{S}_2 - G - F$; hence $F \subset G^* \subset G \cup F$, therefore $F = G^* - \dot{G}$. Let $f^* = f | F^*$. By (1) and by § 54, II (1), $f^* \sim r_n | F^*$ for $n \geqslant n_0$. Therefore $\mu_{G^*} f^* = \mu_{G^*}(r_n | F^*)$ by Theorem 4 of Section VIII. But $\mu_{G^*}(r_n|F^*) = \mu_F(r_n|G)$, because $F = G^* - G$ and neither zeros nor poles of r_n belong to G. Thus, for $n \geqslant n_0$, the multiplicity $\mu_F(r_n|G)$ has a constant value $\mu_{G^*} f^*$, and therefore a value independent of the choice of the functions r_n.

Remarks. (i) It has been proved simultaneously that:

if F^* is a closed subset of G, and G^* is a closed-open set in $\mathscr{S}_2 - F^*$ such that $F = G^* - G$, then

$$\mu_F f = \mu_{G^*} f^* \quad \textit{where} \quad f^* = f | F^*. \tag{3}$$

This statement reduces the definition of the multiplicity with respect to a function defined on an open set to the definition of the multiplicity of a function defined on a closed set.

(ii) Assume that in formula (1)

$$r_n(x) = c_n (x - p_{n,0})^{k_{n,0}} \cdot \ldots \cdot (x - p_{n,l_n})^{k_{n,l_n}} \tag{4}$$

where $p_{n,0} = \infty$ if $\infty \in \mathscr{S}_2 - G$ and where $k_{n,0} + \ldots + k_{n,l_n} = 0$.

Then it follows by (2) that

$$\mu_F f = \lim_{n=\infty}(k_{n,j_1} + k_{n,j_2} + \ldots) \tag{5}$$

where $p_{n,j_1}, p_{n,j_2}, \ldots$ belong to F.

(iii) In the particular case where $\mathscr{S}_2 - G = C_0 \cup \ldots \cup C_m$ (finite number of components), it follows by Theorem 1 of Section VII, that

$$f(x) = (x-p_0)^{k_0} \cdot \ldots \cdot (x-p_m)^{k_m} e^{u(x)},$$

$$k_0 + \ldots + k_m = 0, \qquad u\colon G \to \mathscr{E}^2 \text{ continuous}, \qquad p_j \in C_j.$$

Assume in formula (1) that $r_n(x) = (x-p_0)^{k_0} \cdot \ldots \cdot (x-p_m)^{k_m}$; then it follows by (5) that

$$\mu_F f = k_{j_1} + \ldots + k_{j_s} \text{ where } p_{j_1}, \ldots, p_{j_s} \in F,$$
$$\text{and in particular } \mu_{C_j} f = k_j. \tag{6}$$

Replace G by F and F by G in Theorems 1 through 4 of Section VIII. Then the following statement is obtained.

THEOREMS 1-4. *If $f\colon G \to \mathscr{P}$ is continuous and G open, the multiplicity has the properties of norm, additivity, homomorphism and of invariance.*

The theorem on the *characterization of homotopy* holds also.

THEOREM 5. *If $\mu_F f_1 = \mu_F f_2$ for every F, then $f_1 \sim f_2$. Therefore these two conditions are equivalent* (by Theorem 4).

In particular, *the condition $f \sim 1$ is equivalent to the following one, $\mu_F f = 0$ for any F.*

Proof. Let $f = f_1 : f_2$. Suppose that f *non* ~ 1. By Theorem 4 of § 56, X, it follows that there exists a closed set $F^* \subset G$ such that setting $f^* = f | F^*$ we have f^* *non* ~ 1. Therefore there exists by Theorem 5 of Section VIII a set G^* closed-open in $\mathscr{S}_2 - F^*$ and such that $\mu_{G^*} f^* \neq 0$. Put $F = G^* - G$. This set is obviously open in $\mathscr{S}_2 - G$ and is also closed there, because

$$G^* = \overline{G^*} - F^*, \qquad \text{hence} \qquad F = \overline{G^*} - G^* - F = \overline{G^*} - G.$$

It follows by (3) that $\mu_F f \neq 0$, hence $\mu_F f_1 \neq \mu_F f_2$.

Relations to some theorems on analytic functions[1].

THEOREM 6. *The multiplicity is a local concept; that is, the value of $\mu_F f$ is not affected by restricting the domain of f to any open set whose union with F is a neighbourhood of F.*

Generally speaking,

Let G and G_0 be two open sets such that $G_0 \subset G \subset \mathscr{S}_2$, and let F and F_0 be two sets closed-open in $\mathscr{S}_2 - G$ and $\mathscr{S}_2 - G_0$ respectively such that $F = F_0 - G$. Let $f\colon G \to \mathscr{P}$ be continuous. Then

$$\mu_{F_0} f_0 = \mu_F f \quad \textit{where} \quad f_0 = f\,|\,G_0.$$

THEOREM 7. *If g is a function holomorphic on an open set and if p is a zero of order k of this function, then*

$$\mu_p g^* = k,$$

where g^* is the restriction of g to the set of x such that $g(x) \neq 0$.

The same theorem holds for *meromorphic* functions.

Remark. Thus, we have $\mu_p g^* > 0$ under the hypotheses of Theorem 7. However, for arbitrary continuous functions an (isolated) zero may be of multiplicity 0, or even of negative multiplicity. This is seen on the following examples.

(i) $g(x) = e^{-1/|x|}$, $g(0) = 0$. Then $\mu_p g^* = 0$ for $p = 0$.

(ii) $g(x) = x^{-n} e^{-1/|x|}$, $g(0) = 0$. Then $\mu_p g^* = -n$ for $p = 0$.

THEOREM 8 (Generalized Rouché theorem). *Let M be a closed subset of $\mathscr{S}_2$ and let $g, g_1\colon M \to \mathscr{E}^2$ be two continuous functions such that $|g_1(x)| < |g(x)|$ for $x \in \mathrm{Fr}(M)$. If F and F^* denote the sets of zeros of g and of $g^* = g + g_1$ respectively, then*

$$\mu_F f = \mu_{F^*} f^*,$$

where $f = g\,|\,G$, $f^* = g^*\,|\,G^*$, $G = \mathrm{Int}(M) - F$ and $G^* = \mathrm{Int}(M) - F^*$.

Remark. It is easy to show, referring to Theorem 7, that if the functions g and g_1 are *holomorphic* on $\mathrm{Int}(M)$, the multiplicities $\mu_F f$ and $\mu_{F^*} f^*$ are the algebraic numbers of zeros of the functions g and g^* respectively.

THEOREM 9. *Let $M = \bar{M} \subset \mathscr{S}_2$ and $g_n\colon M \to \mathscr{E}^2$ $(n = 1, 2, \ldots)$ a sequence of continuous functions uniformly convergent to a function g which does not vanish at any point of* $\mathrm{Fr}(M)$. *Let F_n and F be the*

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[1] For the proofs, see my quoted paper, Sections XII and XX.

sets of zeros of g_n *and* g *respectively. Then, for sufficiently large* n,

$\mu_F f = \mu_{F_n} f_n$ *where* $f = g|[\mathrm{Int}(M) - F]$ *and* $f_n = g_n|[\mathrm{Int}(M) - F_n]$.

THEOREM 10. *If* A *is an elementary continuum or a region, every homeomorphism* $f\colon A \to \mathscr{P}$ *is homotopic to a homographic function.*

More generally, we have

THEOREM 11. *Let* G *be an open set and* $f\colon G \to \mathscr{P}$ *be a continuous function. If the multiplicity* $\mu_F f$ *admits only three values* $k, 0$ *and* $-k$ *(for variable* F*), then*

$$f(x) \sim \left(\frac{x-p}{x-q}\right)^k.$$

X. Characterization of the group $\mathfrak{B}_1(G)$. Recall (see § 58, III) that, given a (compact) $\mathscr{X}$, $\mathfrak{N}(\mathscr{X})$ denotes the totality of normed measures (finitely additive) defined on the family of all closed-open subsets of $\mathscr{X}$.

THEOREM 1. *Let* $G \subset \mathscr{S}_2$ *be open and* $f \in \mathscr{P}^G$. *Write* $\nu_f(F) = \mu_F(f)$ *for each closed-open* $F \subset \mathscr{S}_2 - G$. *Then* $\nu_f \in \mathfrak{N}(\mathscr{S}_2 - G)$.

Moreover, $(f_1 \sim f_2) \equiv (\nu_{f_1} = \nu_{f_2})$. *Thus* ν *associates an element of* $\mathfrak{N}(\mathscr{S}_2 - G)$ *with each member of* $\mathfrak{B}_1(G)$.

Furthermore, ν *is a homomorphism.*

This theorem follows immediately from Theorems 1-5 of Section IX.

We are going to show that ν is an isomorphism.

THEOREM 2 (of duality). $\mathfrak{B}_1(G) \underset{\mathrm{gr}}{=} \mathfrak{N}(\mathscr{S}_2 - G)$.

Proof. It remains to show that for every $\varrho \in \mathfrak{N}(\mathscr{S}_2 - G)$ there is $f \in \mathscr{P}^G$ such that $\nu_f(F) = \varrho(F)$, i.e. that

$$\mu_F f = \varrho(F) \quad \text{for every set } F \text{ closed-open in } \mathscr{S}_2 - G. \tag{1}$$

Referring to Theorem 2 of § 50, III, let

$$G = F_1^* \cup F_2^* \cup \ldots, \qquad F_n^* = \overline{F_n^*} \subset \mathrm{Int}(F_{n+1}^*), \tag{2}$$

where $\mathscr{S}_2 - F_n^*$ has only a finite number of components and where $R - G \neq 0$ for each component R of $\mathscr{S}_2 - F_n^*$.

Since the components of $\mathscr{S}_2 - F_n^*$ are contained in the components of $\mathscr{S}_2 - F_{n-1}^*$, they can be labeled by a finite number of systems $i_1 \dots i_n$, consisting of n non-negative integers, in such a manner that

$$\mathscr{S}_2 - F_n^* = \bigcup R_{i_1 \dots i_n}, \tag{3}$$

$$R_{i_1 \dots i_{n+1}} \subset R_{i_1 \dots i_n}, \tag{4}$$

where the summation runs over all systems $i_1 \dots i_n$ (with fixed n) for which $R_{i_1 \dots i_n}$ is defined.

Define

$$F_{i_1 \dots i_n} = R_{i_1 \dots i_n} - G. \tag{5}$$

It follows for every n (compare (3) through (5)) that

$$F_{i_1 \dots i_n} = \bigcup_j F_{i_1 \dots i_n j}, \tag{6}$$

$$F_{i_1 \dots i_n} \cap F_{l_1 \dots l_n} = 0 \quad \text{if} \quad (i_1 \dots i_n) \neq (l_1 \dots l_n), \tag{7}$$

$$\mathscr{S}_2 - G = \bigcup F_{i_1 \dots i_n}, \tag{8}$$

because by (2) and (3)

$$\mathscr{S}_2 - G = \mathscr{S}_2 - F_n^* - G = \bigcup R_{i_1 \dots i_n} - G.$$

We have $R_{i_1 \dots i_n} - G \neq 0$ by definition of F_n^*. So let (compare (5))

$$p_{i_1 \dots i_n} \in F_{i_1 \dots i_n}. \tag{9}$$

Since $F_{i_1 \dots i_n}$ is closed-open in $\mathscr{S}_2 - G$ by (5), (7) and (8), so let

$$k_{i_1 \dots i_n} = \varrho(F_{i_1 \dots i_n}). \tag{10}$$

By Theorem 2 of Section IX, (6) and (7), it follows that

$$k_{i_1 \dots i_n} = \sum_j k_{i_1 \dots i_n j} \tag{11}$$

and by (i) and (8) we have for every n

$$\sum k_{i_1 \dots i_n} = 0. \tag{12}$$

Consider the rational function

$$r_n(x) = \prod (x - p_{i_1 \dots i_n})^{k_{i_1 \dots i_n}}. \tag{13}$$

We will show that

$$r_n(x) \sim r_{n+1}(x) \quad \text{on } F_n^*. \tag{14}$$

By (9), (6) and (5), we have $p_{i_1 \ldots i_n j} \in R_{i_1 \ldots i_n}$. But, since $R_{i_1 \ldots i_n}$ is a component of $\mathscr{S}_2 - F_n^*$ containing the points $p_{i_1 \ldots i_n}$ and $p_{i_1 \ldots i_n j}$, it follows that on F_n^* (compare II, Theorem 1)

$$\frac{x - p_{i_1 \ldots i_n}}{x - p_{i_1 \ldots i_n j}} \sim 1, \quad \text{hence} \quad \frac{(x - p_{i_1 \ldots i_n})^{k_{i_1 \ldots i_n j}}}{(x - p_{i_1 \ldots i_n j})^{k_{i_1 \ldots i_n j}}} \sim 1,$$

and by (11)

$$\frac{(x - p_{i_1 \ldots i_n})^{k_{i_1 \ldots i_n}}}{\prod_j (x - p_{i_1 \ldots i_n j})^{k_{i_1 \ldots i_n j}}} \sim 1,$$

which implies (14). According to this condition, let

$$r_n(x) = r_{n+1}(x)\, e^{u_{n+1}(x)} \quad \text{for} \quad x \in F_n^*, \tag{15}$$

where

$$u_{n+1} \colon \mathscr{S}_2 \to \mathscr{E}^2 \quad \text{is continuous and} \quad u_1(x) = 0.$$

Define

$$f_n(x) = r_n(x)\, e^{u_1(x) + \ldots + u_n(x)}. \tag{16}$$

It follows according to (15) and (16) that

$$f_n(x) = f_{n+1}(x) = \ldots \quad \text{for} \quad x \in F_n^*. \tag{17}$$

Define

$$f(x) = f_n(x) \quad \text{for} \quad x \in F_n^*. \tag{18}$$

Thus, the function f is uniquely defined on G. According to (2) $f \in \mathscr{P}^G$.

Next we shall show that

$$\mu_{F_{i_1 \ldots i_n}} f = k_{i_1 \ldots i_n} \tag{19}$$

By (5), IX (3) and (18), we have

$$\mu_{F_{i_1 \ldots i_n}} f = \mu_{R_{i_1 \ldots i_n}} f \,|\, F_n^* = \mu_{R_{i_1 \ldots i_n}} r_n \,|\, F_n^*,$$

because $f = f_n \sim r_n$ on F_n^* (compare (16)). Finally $\mu_{R_{i_1 \ldots i_n}} r_n \,|\, F_n^* = k_{i_1 \ldots i_n}$ by (13), (9) and (5).

Thus $\mu_F f = \varrho(F)$ (compare (10)) if F has the form $F = F_{i_1 \dots i_n}$.

If F is an arbitrary closed-open set in $\mathscr{S}_2 - G$, there exists a closed set F^* which separates the (disjoint and closed) sets F and $\mathscr{S}_2 - G - F$. By (2) there exists n such that $F^* \subset F_n^*$. Therefore, we may assume that $F^* = F_n^*$.

Since $R_{i_1 \dots i_n}$ is connected and disjoint from F_n^*, it follows that

$$F \cap R_{i_1 \dots i_n} \neq 0 \quad \text{implies that} \quad (\mathscr{S}_2 - G - F) \cap R_{i_1 \dots i_n} = 0,$$

so that (compare (5)) $F \cap F_{i_1 \dots i_n} \neq 0$ implies the inclusion $F_{i_1 \dots i_n} \subset F$.

In virtue of (13) it follows that F is the union of sets $F_{i_1 \dots i_n}$ such that $F \cap F_{i_1 \dots i_n} \neq 0$. This implies identity (1) by virtue of Theorem 2 of Section IX, (7) and (19), and the proof is completed.

Combining Theorem 2 with Theorem 4 of § 58, III, the following characterization of the group $\mathfrak{B}_1(G)$ is obtained.

THEOREM 3. *If G is an open subset of $\mathscr{S}_2$, then*

$$\textit{either} \quad \mathfrak{B}_1(G) \underset{\text{gr}}{=} \mathscr{G}^n \quad \textit{or} \quad \mathfrak{B}_1(G) \underset{\text{gr}}{=} \mathscr{G}^{\aleph_0},$$

according to whether the set $\mathscr{S}_2 - G$ has a finite number, say $n+1$, or infinitely many components.

Remark 1. It follows from Theorem 3 that, if the open set G separates $\mathscr{S}_2$ in n continua (*n finite*), then n is an *intrinsic invariant* of G. This does not remain true if G separates $\mathscr{S}_2$ in *infinitely* many continua; in the following example(1) two open homeomorphic sets G and H are given in $\mathscr{E}$ such that $\mathscr{E} - G$ has a countable infinity of components, while the family of components of $\mathscr{E} - H$ is uncountable.

Namely $\mathscr{E} - G = \mathscr{G}$ and $\mathscr{E} - H = \mathscr{C}$.

Multiplying these sets by $\mathscr{E}$ (or by $\mathscr{E}^{n-1}$) one gets simple examples of sets in $\mathscr{E}^2$ (or in $\mathscr{E}^n$) whith the same singularity.

Let us recall that this singularity can occur only if G is not connected (compare § 57, IV, Theorem 3).

Remark 2. The function f in formula (6) can be required to be holomorphic (see VII, Remark iv).

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(1) See S. Eilenberg, *An invariance theorem for subsets of S^n*, Bull. Amer. Math. Soc. 47 (1941), p. 75.

XI. Increment of the logarithm. Index. Let $\zeta\colon \mathscr{I} \to \mathscr{E}^2$ be a continuous function such that $\zeta(0) = \zeta(1)$.

Put $C = \zeta(\mathscr{I})$. Let $f\colon C \to \mathscr{P}$ be a continuous function. Therefore $f\zeta\colon \mathscr{I} \to \mathscr{P}$, which implies $f\zeta \sim 1$ (compare § 56, III, Theorem 3), hence

$$f\zeta(t) = e^{u(t)} \quad \text{for} \quad 0 \leqslant t \leqslant 1, \quad \text{and} \quad u \in (\mathscr{E}^2)^{\mathscr{I}}. \tag{0}$$

Since $\zeta(0) = \zeta(1)$, it follows that $u(1) - u(0) = 2n\pi i$.

The integer

$$\Delta_\zeta f = \frac{1}{2\pi i}[u(1) - u(0)]$$

is said to be *the increment of the logarithm of f with respect to ζ*.

It is easy to see that the number $\Delta_\zeta f$ does not depend on the choice of the function u which satisfies condition (0).

Recall that by Theorem 4 of § 56, III, if $f\colon \mathscr{S} \to \mathscr{P}$ is a continuous function, then $f(x) \sim x^n$ where $n = \Delta_\zeta f$ and $\zeta(t) = e^{2\pi i t}$.

Let us mention here that under some regularity hypotheses made on f and C,

$$\Delta_\zeta f = \frac{1}{2\pi i}\int\limits_C \frac{f'(x)}{f(x)}\,dx.$$

Let $p \in \mathscr{S}_2 - C$. Define

$$\operatorname{ind}_\zeta p_\zeta = \Delta_\zeta(x - p). \tag{1}$$

In other words, setting $\zeta(t) - p = e^{u(t)}$ where $u \in (\mathscr{E}^2)^{\mathscr{I}}$, we have

$$\operatorname{ind}_\zeta p = \frac{1}{2\pi i}[u(1) - u(0)]. \tag{2}$$

In particular (compare I, Remark)

$$\operatorname{ind}_\zeta \infty = 0. \tag{2'}$$

Conversely, the increment can be defined by means of the index, namely

$$\Delta_\zeta f = \Delta_{f\zeta} x = \operatorname{ind}_{f\zeta} 0. \tag{3}$$

In a more general way, if $g\colon C \to \mathscr{E}^2$ is a continuous function and $p \in \mathscr{S}_2 - g(C)$, then setting $f(x) = g(x) - p$, we have

$$\Delta_\zeta f = \operatorname{ind}_{g\zeta} p. \tag{4}$$

The transformation ζ of the interval $\mathscr{I}$ determines a transformation ζ^0 of the circle $\mathscr{S}$.

Namely, if $2\pi t$ is the argument of x (where $0 \leqslant t \leqslant 1$), let

$$\zeta^0(x) = \zeta(t).$$

It is easy to see that the function ζ^0 is continuous,

$$\zeta^0\colon \mathscr{S} \to \mathscr{E}^2, \ \zeta^0(\mathscr{S}) = C \text{ and } \zeta^0(e^{2\pi i t}) = \zeta(t). \tag{5}$$

Conversely, if $g\colon \mathscr{S} \to \mathscr{E}^2$ is continuous, then

$$\zeta^0(x) = g(x) \quad \text{setting} \quad \zeta(t) = g(e^{2\pi i t}). \tag{6}$$

It is easy to establish the following Theorems 1 and 2 (in what concerns Theorem 1, compare § 56, III, 4 (ii) and I, 4).

THEOREM 1. $[\operatorname{ind}_\zeta p = n] \equiv [\zeta^0(x) - p \sim x^n$ *(on* $\mathscr{S}$*)*$]$.

THEOREM 2. *The function ζ^0 is a homeomorphism if and only if $\zeta(t) \neq \zeta(t')$ for every pair $t \neq t'$, except when $|t-t'| = 1$.*

THEOREM 3. *If p and q belong to the same component of $\mathscr{S}_2 - C$, where $C = \zeta(\mathscr{I}) \subset \mathscr{E}^2$, then* $\operatorname{ind}_\zeta p = \operatorname{ind}_\zeta q$.

If p belongs to the non-bounded component, then $\operatorname{ind}_\zeta p = 0$ (compare (2′)).

If ζ^0 is a homeomorphism and p belongs to the bounded component of $\mathscr{S}_2 - C$, then $\operatorname{ind}_\zeta p \neq 0$.

Proof. Let us set in Theorems 8 and 9 of Section II, $g = \zeta^0$ and $\mathscr{X} = \mathscr{S}$. Hence $\zeta^0(x) - p \sim \zeta^0(x) - q$ on $\mathscr{S}$, and $\operatorname{ind}_\zeta p = \operatorname{ind}_\zeta q$ by Theorem 1.

If ζ^0 is a homeomorphism, then $\zeta^0(x) - p$ *non*$\sim \zeta^0(x) - \infty \sim 1$ by Theorem 9 of Section II. Therefore $\operatorname{ind}_\zeta p \neq 0$ by Theorem 1.

The last part of Theorem 3 will be strengthened by using Theorems below.

THEOREM 4(1). *Let ζ^0 be a homeomorphism, $f\colon C \to \mathscr{P}$ be a continuous function and $p \in \mathscr{S}_2 - C$. If* $\operatorname{ind}_\zeta p = 1$ *and* $\Delta_\zeta f = n$, *then* $f(x) \sim (x-p)^n$.

Proof. Let

$$\zeta(t) - p = e^{u(t)}, \quad u(1) - u(0) = 2\pi i,$$
$$f\zeta(t) = e^{v(t)}, \quad v(1) - v(0) = 2n\pi i.$$

(1) This is a generalization of Theorem 4 of Section I.

Define the function $w\colon C \to \mathscr{E}^2$ as follows

$$w(x) = v(t) - nu(t) \quad \text{where} \quad x = \zeta(t).$$

Although two values of t correspond to $x = \zeta(0) = \zeta(1)$, the function w is uniquely defined, because

$$v(1) - nu(1) = \big(v(0) + 2n\pi i\big) - \big(2n\pi i + nu(0)\big) = v(0) - nu(0).$$

Thus w is a continuous function and $w\zeta(t) = v(t) - nu(t)$. It follows that

$$\big(\zeta(t) - p\big)^n e^{w\zeta(t)} = e^{nu(t)} \cdot e^{v(t) - nu(t)} = e^{v(t)} = f\zeta(t),$$

which implies $(x-p)^n e^{w(x)} = f(x)$ by setting $x = \zeta(t)$.

THEOREM 5. *Let A be the circle $|x - p| = r$ and $g\colon A \to \mathscr{E}^2$ a homeomorphism of A onto the simple closed curve C. If q is the point of the bounded component D of $\mathscr{S}_2 - C$, then $g(x) - q \sim (x-p)^{\pm 1}$.*

Proof. First, consider the case where C is the circle with the center q and the radius s. Define $\zeta(t) = p + re^{2\pi i t}$ where $0 \leqslant t \leqslant 1$, so that $A = \zeta(\mathscr{I})$. It follows that

$$g\zeta(t) - q = |g\zeta(t) - q|\, e^{2\pi i \varphi(t)} = se^{2\pi i \varphi(t)}, \qquad \text{where } \varphi \in \mathscr{E}^{\mathscr{I}}.$$

Suppose that $|\varphi(1) - \varphi(0)| > 1$. Then there would exist a t_0 such that $|\varphi(t_0) - \varphi(0)| = 1$, so that

$$e^{2\pi i \varphi(t_0)} = e^{2\pi i \varphi(0)}, \qquad \text{hence} \qquad g\zeta(t_0) = g\zeta(0), \qquad \text{thus} \qquad \zeta(t_0) = \zeta(0).$$

Then it would follow by definition of ζ that $e^{2\pi i t_0} = 1$; but this is impossible because $0 < t_0 < 1$.

Therefore $|\varphi(1) - \varphi(0)| \leqslant 1$, and $|\mathrm{ind}_{g\zeta} q| \leqslant 1$.

On the other hand, $\mathrm{ind}_{g\zeta} q \neq 0$. This follows from the last part of Theorem 3, because $g\zeta(t) \neq g\zeta(t')$ for $t \neq t'$ and $|t - t'| < 1$ (compare also Theorem 2).

Thus $\mathrm{ind}_{g\zeta} q = \pm 1$. Assuming in Theorem 4 that $C = A$ and $f(x) = g(x) - q$, we infer from (4) that $g(x) - q \sim (x-p)^{\pm 1}$.

In the general case, let A_0 be the circle $|x - p| = r/2$ and C_0 a circle contained in D with the center q. By Theorem 3 of § 61, V, the homeomorphism $g\colon A \to C$ can be extended to a homeomorphism g^* of two circular annuli contained between A_0 and A and between C_0 and C respectively; so that $g^*(A_0) = C_0$.

Let for $0 \leqslant t \leqslant 1$ and $x \in A$

$$h(x, t) = g^*[p+(x-p)(1-\tfrac{1}{2}t)]-q \quad \text{and} \quad g_0(x) = g^*\left(\frac{p+x}{2}\right).$$

It follows that h is a continuous function such that

$$h\colon A \times \mathscr{I} \to \mathscr{P}, \quad h(x, 0) = g(x)-q \quad \text{and} \quad h(x, 1) = g_0(x)-q.$$

Therefore $g(x)-q \sim g_0(x)-q$ and, since $g_0\colon A \to C_0$ is a homeomorphism, it follows that, as just has been proved,

$$g_0(x)-q \sim (x-p)^{\pm 1}, \quad \text{hence} \quad g(x)-q \sim (x-p)^{\pm 1}.$$

Definition. If ζ^0 is a homeomorphism and if $\operatorname{ind}_\zeta p = 1$ for every point p belonging to the bounded component D of $\mathscr{S}_2 - C$, then ζ is said to be a *positive path of the curve* C.

If $\operatorname{ind}_\zeta p = -1$ for each p, the path is said to be *negative*.

Theorem 6. *If ζ^0 is a homeomorphism, then ζ is either a positive or a negative path of the curve $C = \zeta(\mathscr{I})$.*

Proof. Replace A by $\mathscr{S}$ and g by ζ^0 in Theorem 5. It follows that

$$\zeta^0(x)-p \sim x^{\pm 1}, \quad \text{hence} \quad \operatorname{ind}_\zeta p = \pm 1$$

by Theorem 1. Moreover, $\operatorname{ind}_\zeta p$ has a constant value (for $p \in D$) by Theorem 3.

Remark. If $y = \zeta(t)$ describes a positive path, $y = \zeta(1-t)$ describes a negative one. Therefore each simple closed curve admits two paths, a positive and a negative one.

Theorem 4 implies the following

Theorem 7. *Let ζ be a positive path of the curve C, $f\colon C \to \mathscr{P}$ a continuous function and $p \in D$. If $\Delta_\zeta f = n$, then*

$$f(x) \sim (x-p)^n.$$

If $g\colon C \to \mathscr{E}^2$ is a continuous function and $q \in \mathscr{S}_2 - g(C)$, then

$$g(x)-q \sim (x-p)^{\operatorname{ind}_{g\zeta} q}.$$

XII. Relation to the multiplicity. Kronecker characteristic.

Theorem 1 of Section XI immediately implies the following

Theorem 1. $\operatorname{ind}_\zeta p = \mu_Q[\zeta^0(x)-p]$ *where Q is the disk $|x| < 1$.*

THEOREM 2. *Let ζ be a positive path of a simple closed curve $C \subset \mathscr{E}^2$, D the bounded component of $\mathscr{S}_2 - C$ and $f\colon C \to \mathscr{P}$ a continuous function. Then*

$$\mu_D f = \Delta_\zeta f = \mathrm{ind}_{f\zeta} 0. \tag{1}$$

Consequently *if $g\colon C \to \mathscr{E}^2 - q$ is continuous and $f(x) = g(x) - q$, we have*

$$\mu_D f = \mathrm{ind}_{g\zeta} q.$$

Proof. Let $p \in D$. According to Theorem 1 of Section III, let $f(x) \sim (x-p)^n$. Therefore $\mu_D f = n$ by (4) of Section VIII. On the other hand, $\Delta_\zeta f = n$ according to Theorem 7 of Section XI.

THEOREM 2′. *If D_1 is the unbounded component of $\mathscr{S}_2 - C$ and ζ_1 is a negative path of C, then $\mu_{D_1} f = \Delta_{\zeta_1} f = \mathrm{ind}_{f\zeta_1} 0$.*

Proof. Let $\zeta(t) = \zeta_1(1-t)$. Since ζ is a positive path, the formula (1) is valid. Since $\mu_D f + \mu_{D_1} f = 0$ (compare VIII, Theorems 1 and 2) and since $\Delta_\zeta f = -\Delta_{\zeta_1} f$, the required relation follows.

Let $G \subset \mathscr{S}_2$ be an open set. Assume that p is an *isolated point of the set $\mathscr{S}_2 - G$*. Thus, there exists a disk D with center p and such that $\bar{D} - p \subset G$. Let C be the boundary of D; we can assume that $\infty \in \mathscr{S}_2 - C$. We are going to prove the following statement.

THEOREM 3. *If $f\colon G \to \mathscr{P}$ is a continuous function, then $\mu_p f = \Delta_\zeta f^* = \mathrm{ind}_{f\zeta} 0$, where $f^* = f|C$ and where ζ is a positive or a negative path of C according to whether $p \neq \infty$ or $p = \infty$*(1).

Proof. By (3) of Section IX, $\mu_p f = \mu_D f^*$, and $\mu_D f^* = \Delta_\zeta f^*$ according to Theorems 2 and 2′.

Now, let R be a region whose complement consists of a finite number of components: $\mathscr{S}_2 - R = C_0 \cup \ldots \cup C_n$. Each C_j is a continuum which does not cut $\mathscr{S}_2$ (compare § 46, III, Theorem 5), thus it can be separated from all other C_l's by a simple closed curve (compare § 61, II, Theorem 6). Repeating the above argument we obtain the following theorem.

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(1) Thus, if $g \in (\mathscr{E}^2)^G$ and p is an isolated zero of g and if f is the restriction of g to the set of x's such that $g(x) \neq 0$, then the multiplicity $\mu_p f$ is identical with the "*index*" of the point p in the sense of Alexandroff–Hopf, *op. cit.*, p. 470 ("Index" or "Vielfachheit einer 0-Stelle").

Therefore, if the set F of zeros of the function g is finite, the multiplicity $\mu_F f$ *coincides with the algebraic number of the zeros of* g.

THEOREM 4. *Let $f\colon R \to \mathscr{P}$ be a continuous function. If the simple closed curve $K \subset \mathscr{E}^2$ separates the continuum C_j from all C_l with $l \neq j$, and if D is the component of $\mathscr{S}_2 - K$ which contains C_j, then*

$$\mu_{C_j} f = \Delta_\zeta f^* = \operatorname{ind}_{f\zeta} 0 \quad \textit{where} \quad f^* = f|K,$$

provided that ζ is a positive or a negative path of K according to whether D is bounded or unbounded.

Theorem 3 allows to calculate $\mu_R f$ also in the case where f is defined on a *closed set.*

THEOREM 5. *Let $F = \overline{F} \subset \mathscr{S}_2$, R a component of $\mathscr{S}_2 - F$, $p \in R$ and $f\colon F \to \mathscr{P}$ a continuous function. Let f_1 be a continuous extension of f to $\mathscr{S}_2 - Z$, where Z is finite and $Z \cap R = (p)$* (f_1 may be identified with the right-hand side of III (6), for instance). *Then $\mu_R f = \mu_p f_1$.*

Proof. This is a direct consequence of formula (3) of Section IX, replacing G^* by R, F^* by F, F by p, f by f_1 and G by $\mathscr{S}_2 - Z$.

DEFINITION. Let A be an elementary continuum $\subset \mathscr{E}^2$. Let

$$\mathscr{S}_2 - A = D_0 \cup \ldots \cup D_n, \tag{2}$$

where D_j is a component of $\mathscr{S}_2 - A$ ($j = 0, \ldots, n$). Let the path ζ_j of $\operatorname{Fr}(D_j)$ be negative or positive according to whether D_j is or is not bounded. Then the number

$$\operatorname{car}_A f = \operatorname{ind}_{f\zeta_0} 0 + \ldots + \operatorname{ind}_{f\zeta_n} 0 \tag{3}$$

is said to be the *Kronecker characteristic* of the continuous function $f\colon \operatorname{Fr}(A) \to \mathscr{P}$.

More generally, if A is an elementary closed set

$$A = A_1 \cup \ldots \cup A_m, \tag{4}$$

where A_k is a component of A, then

$$\operatorname{car}_A f = \operatorname{car}_{A_1} f_1 + \ldots + \operatorname{car}_{A_m} f_m \quad \text{where} \quad f_k = f|\operatorname{Fr}(A_k). \tag{5}$$

THEOREM 6. *Let A be an elementary closed set and $f\colon \operatorname{Fr}(A) \to \mathscr{P}$ a continuous function. Setting $I = \operatorname{Int}(A)$, we have*

$$\mu_I f = \operatorname{car}_A f. \tag{6}$$

Proof. First, let A be an elementary continuum. By (2)

$$\mathscr{S}_2 - \mathrm{Fr}(A) = I \cup D_0 \cup \ldots \cup D_n,$$

so that (compare VIII, Theorems 1 and 2)

$$\mu_I f + \mu_{D_0} f + \ldots + \mu_{D_n} f = 0. \tag{7}$$

Let $f_j = f|\mathrm{Fr}(D_j)$. Therefore (see Theorem 8 of Section VIII)

$$\mu_{D_j} f = \mu_{D_j} f_j \quad \text{and} \quad \mu_{D_j} f_j = -\mathrm{ind}_{f\zeta_j} 0 \tag{8}$$

by Theorems 2 and 2′.

Formulas (7), (8) and (3) imply (6).

Now, let A be an elementary closed set satisfying condition (4). Let $I_k = \mathrm{Int}(A_k)$. By Theorem 4 of § 61, III, the set I_k is a component of $\mathscr{S}_2 - \mathrm{Fr}(A_k)$ and also of $\mathscr{S}_2 - \mathrm{Fr}(A)$.

Let $f_k = f|\mathrm{Fr}(A_k)$; then (see Theorem 8 of Section VIII) $\mu_{I_k} f = \mu_{I_k} f_k$. Since (compare § 61, II, Theorem 4) $I = I_1 \cup \ldots \cup I_m$, we infer (compare VIII, Theorem 2) that

$$\begin{aligned}\mu_I f &= \mu_{I_1} f + \ldots + \mu_{I_m} f = \mu_{I_1} f_1 + \ldots + \mu_{I_m} f_m \\ &= \mathrm{car}_{A_1} f_1 + \ldots + \mathrm{car}_{A_m} f_m = \mathrm{car}_A f.\end{aligned}$$

THEOREM 7. *Let G be open, $f: G \to \mathscr{P}$ a continuous function, F a set closed-open in $\mathscr{S}_2 - G$ and A an elementary closed set such that*

$$F \subset \mathrm{Int}(A) \quad \textit{and} \quad A \subset G \cup F. \tag{9}$$

Then

$$\mu_F f = \mathrm{car}_A [f|\mathrm{Fr}(A)]. \tag{10}$$

Proof. According to (9),

$$F \cap \mathrm{Fr}(A) = 0 \quad \text{and} \quad \mathrm{Fr}(A) \subset G \cup F, \quad \text{so that} \quad \mathrm{Fr}(A) \subset G.$$

Thus, the function f is defined on $\mathrm{Fr}(A)$. Since the set $I = \mathrm{Int}(A)$ is closed-open in $\mathscr{S}_2 - \mathrm{Fr}(A)$, it follows by (3) of Section IX (replacing F^* by $\mathrm{Fr}(A)$ and G^* by I) that

$$\mu_F f = \mu_I [f|\mathrm{Fr}(A)],$$

which yields formula (10) by (6).

Theorem 6 combined with Theorem 7 of Section VIII implies (substituting $\mathrm{Fr}(A)$ for F and I for R_j) the two following theorems.

THEOREM 8. *If* $f\colon A \to \mathscr{P}$ *is a continuous function, then*

$$\mathrm{car}_A[f|\mathrm{Fr}(A)] = 0.$$

THEOREM 9. *If* $f\colon \mathrm{Fr}(A) \to \mathscr{P}$ *is a continuous function and* $\mathrm{car}_A f = 0$, *then* $f \subset f^* \in \mathscr{P}^A$.

Remarks. (i) Under some regularity hypotheses on f and A, Theorem 8 follows from the classic Cauchy theorem of the theory of analytic functions.

(ii) Theorem 7 allows to define the multiplicity with the aid of characteristic (and hence of the index), because the existence of an elementary closed set satisfying conditions (9) follows from Theorem 10 of § 61, III (since the set $\mathscr{S}_2 - G - F$ is closed, so $G \cup F$ is an open neighbourhood of F).

THEOREM 10. *Let A be an elementary set and* $g\colon A \to \mathscr{E}^2$ *a continuous function. Assume that* $g(x) \neq 0$ *for* $x \in \mathrm{Fr}(A)$ *and put* $f = g|\mathrm{Fr}(A)$. *If* $\mathrm{car}_A f \neq 0$, *there exists a point* x_0 *such that* $g(x_0) = 0$ (1).

Proof. This is a direct consequence of Theorem 8.

More precisely, *if the set of zeros of g is finite, their algebraic number is equal to* $\mathrm{car}_A f$, *hence to* $\mu_I f$ (2).

The last statement is a particular case of the following theorem (3).

Let $M = \overline{M} \subset \mathscr{S}_2$ and $h\colon M \to \mathscr{S}_2$ a continuous function. Let F be the set of x's such that $h(x) = 0$ or ∞. Assume that $F \cap \mathrm{Fr}(M) = 0$. *Then, setting* $f = h|\mathrm{Fr}(M)$, $I = \mathrm{Int}(M)$ *and* $G = I - F$, *we have* $\mu_I f = \mu_F(h|G)$.

Applications to computing the algebraic number of fixed points (4).

Let E be a compact subset of $\mathscr{E}^2$ and $g\colon E \to E$ a continuous function. Assume that

$$g(x) \neq x \quad \text{for} \quad x \in \mathrm{Fr}(E),$$

and put

$$f(x) = g(x) - x \quad \text{and} \quad f^* = f|[\mathrm{Fr}(E)].$$

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(1) Compare the *existence theorem of Kronecker* (case $n = 2$), Alexandroff–Hopf, *op. cit.*, p. 467 and 470.

(2) Compare *ibidem*, p. 472, Theorem II.

(3) Compare my quoted paper, p. 358.

(4) Compare my paper in Fund. Math. 34 (1947), pp. 261–271.

Thus, $f^*\colon \mathrm{Fr}(E) \to \mathscr{P}$ is a continuous function.

In other words, if Z denotes the set of fixed points of the function g, then $Z \subset I$, where $I = \mathrm{Int}(E)$.

If p is an isolated point of Z, *the order of the fixed point* p is understood to be the index $\mathrm{ind}_{f\zeta} 0$ where ζ is a positive path of the boundary C of a disk D such that

$$p \in D, \quad D \subset \mathrm{Int}(E) \quad \text{and} \quad (p) = D \cap Z.$$

The following statements can be proved(¹).

If Z is finite, the algebraic number of fixed points (the sum of order of fixed points) is equal to $\mu_I f^*$.

If E is a continuum and an absolute neighbourhood retract, the trace of the automorphism of the group $\mathfrak{B}_1(E)$ induced by the function g is equal to $1 - \mu_I f^*$.

In other terms, let $p_1, \ldots, p_n$ be a system of points belonging to (different) bounded components of $\mathscr{E}^2 - E$ and let

$$g(x) - p_j \sim (x-p_1)^{k_{j1}} \cdot \ldots \cdot (x-p_n)^{k_{jn}} \quad \text{on } E;$$

then

$$\mu_I f^* = 1 - (k_{11} + \ldots + k_{nn}).$$

XIII. Positive and negative homeomorphisms. Oriented topology.

THEOREM 1. *Let $C \subset \mathscr{E}^2$ be a simple closed curve, D a component of $\mathscr{S}_2 - C$ and $f\colon C \to \mathscr{P}$ a homeomorphism. If the point 0 belongs to the unbounded component of $\mathscr{S}_2 - f(C)$, then $\mu_D f = 0$.*

If it belongs to the bounded component, then $\mu_D f = \pm 1$, according to whether the path $f\zeta$ of $f(C)$ is positive or negative (ζ denotes a positive or negative path of C according to whether D is or is not bounded).

Consequently, *if D is bounded, then $f(x) \sim (x-p)^{\pm 1}$ for $p \in D$.*

Proof. Since $\mu_{\mathscr{S}_2 - \bar{D}} f = -\mu_D f$, it is sufficient to consider the case where D is bounded.

Put $\eta(t) = f\zeta(t)$. Then the transformation η^0 defined on $\mathscr{S}$ (compare XI (5)) is a homeomorphism. For, if $2\pi t$ is the argument of z, then $\eta^0(z) = \eta(t) = f\zeta(t)$ by the definition of η^0; and therefore the hypothesis $\eta^0(z) = \eta^0(z')$ implies that

$$f\zeta(t) = f\zeta(t'), \text{ hence } \zeta(t) = \zeta(t'), \text{ thus } t = t' \text{ or } |t - t'| = 1.$$

(¹) *Op. cit.*, p. 264 and 267. Compare also S. Lefschetz, *Topology*, p. 385.

Thus, η^0 is a homeomorphism by Theorem 2 of Section XI.

If the point 0 belongs to the unbounded component of $\mathcal{S}_2 - f(C)$, then $\mu_D f = \mathrm{ind}_{f\zeta}\, 0 = 0$ by Theorems 2 and 3 of Sections XII and XI respectively. If it belongs to the bounded component, it follows (replacing ζ by $f\zeta$ and p by 0) that $\mathrm{ind}_{f\zeta}\, 0 = \pm 1$, according to whether the path $f\zeta$ of $f(C)$ is positive or negative; therefore, by Theorem 2 of Section XII, it follows that $\mu_D f = 1$ or $\mu_D f = -1$ respectively.

THEOREM 2. *If $G \subset \mathcal{S}_2$ is open and $g: G \to \mathcal{E}^2$ is a homeomorphism, then, setting $h(x) = g(x) - g(p)$, where $x \in G - p$, we have*

$$\mu_p h = \pm 1 \quad \textit{for every } p \in G. \tag{1}$$

Moreover, *if G is a region, $\mu_p h$ has a constant value*[1].

Proof. Let p be a given point of G. Since $h(x) = g(x) - g(p)$, the function h vanishes only at the point p. Therefore, setting $H = G - p$ and $h^* = h|H$, we infer that $h^*: H \to \mathcal{P}$ is continuous and that p is an isolated point of $\mathcal{S}_2 - H$. Let D be a disk such that $p \in D$ and $\bar{D} \subset G$. Let $C = \mathrm{Fr}(D)$. Since h is a homeomorphism, so the set $h(D)$ is bounded, $h(C)$ is a simple closed curve and the disk $h(D)$ is (by the invariance of the concept of the interior point, compare § 59, IV (0)) the bounded component of $\mathcal{S}_2 - h(C)$. As $0 = h(p) \in h(D)$, then $\mu_D(h|C) = \pm 1$ by Theorem 1, which implies identity (1) (compare IX (3)).

Now, assume that G is a region. Let $p_0 \in G$, $p_1 \in G$, D a disk such that $p_0, p_1 \in D$, $\bar{D} \subset G$ and $C = \mathrm{Fr}(D)$. Define $h_j(x) = g(x) - g(p_j)$ for $j = 0, 1$. It follows, as before, that $\mu_{p_j} h_j = \mu_D(h_j|C)$. Since $g(p_0)$ and $g(p_1)$ belong to $g(D)$, which is a component of $\mathcal{S}_2 - g(C)$, it follows by Theorem 8 of Section II that

$$g(x) - g(p_0) \sim g(x) - g(p_1) \quad \text{on } C, \quad \text{i.e.} \quad (h_0|C) \sim (h_1|C),$$

therefore (compare Theorem 4 of Section VIII)

$$\mu_D(h_0|C) = \mu_D(h_1|C) \quad \text{and hence} \quad \mu_{p_0} h_0 = \mu_{p_1} h_1.$$

DEFINITION. If $R \subset \mathcal{S}_2$ is a region and $g: R \to \mathcal{S}_2$ a homeomorphism, then g is said to be a *positive homeomorphism* if $\mu_p h = 1$ (where $h(x) = g(x) - g(p)$) for every point p such that $g(p) \neq \infty$.

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(1) See Alexandroff–Hopf, *op. cit.*, p. 475.

If $\mu_p h = -1$, the homeomorphism g is said to be *negative.*

Since no point separates R, *every homeomorphism* $g\colon R \to \mathscr{S}_2$ *is* (by Theorem 2) *either positive or negative.*

It is easy to see that, in order to show that the homeomorphism g is positive or that it is negative, it is sufficient to know the value of $\mu_p f$ for one single point p (such that $g(p) \neq \infty$).

The next statement follows by Theorem 3 of Section XII and by formula (4) of Section XI.

THEOREM 3. *If D is a bounded disk with center p, $\bar{D} \subset R$ and* $\operatorname{ind}_{g\zeta} g(p) = 1$ (*where ζ is a positive path of the boundary of D*), *the homeomorphism g is positive.*

Thus, the positive homeomorphisms are just those transformations which do not change the *orientation* of the considered region.

Now consider some particular cases.

THEOREM 4. *If $R \subset \mathscr{E}^2$ is a region such that $\mathscr{E}^2 - R$ is connected and if $g\colon R \to \mathscr{E}^2$ is a homeomorphism, then g is a positive homeomorphism if and only if for each $p \in R$*

$$g(x)-g(p) \sim x-p \quad \text{on } R-p. \tag{2}$$

Proof. Put $h(x) = g(x)-g(p)$. By Theorem 1 of Section VII (setting $p_0 = \infty$), it follows that $h(x) \sim (x-p)^k$, and since $\mu_p h = k$, we have $k = \pm 1$, according to whether the homeomorphism is positive or negative.

A similar argument proves the following statement.

THEOREM 5. *A homeomorphism $g\colon \mathscr{S}_2 \to \mathscr{S}_2$ is positive if and only if*

$$g(x)-g(p) \sim \frac{x-p}{x-p_0} \quad \text{on } \mathscr{S}_2-p-p_0, \quad \text{where} \quad g(p_0) = \infty.$$

In particular, *if $p_0 = \infty$, the relation* (2) *holds.*

It follows that *every homographic transformation is a positive homeomorphism.*

Because

$$\frac{ax-b}{cx-d} - \frac{ap-b}{cp-d} = \lambda \frac{x-p}{x-p_0}, \quad \text{where} \quad p_0 = \frac{d}{c}$$

and where λ is a constant (it is assumed that $ad-bc \neq 0$).

Remark. As an example of a negative homeomorphism of $\mathscr{S}_2$, consider the function $g(\alpha+i\beta) = \alpha-i\beta$, i.e. $g(x) = |x|^2:x$. It is easy to show that $\mathrm{ind}_{g\zeta}0 = -1$ for $\zeta(t) = e^{2\pi it}$, which yields the required conclusion by virtue of Theorem 3.

Theorem 3 immediately implies the following

THEOREM 6. *Let be given two regions $R_1 \subset R$ and a homeomorphism $g\colon R \to \mathscr{S}_2$. If $g|R_1$ is a positive homeomorphism, then so is g.*

THEOREM 7. *The inverse transformation to a positive homeomorphism is a positive homeomorphism.*

In other words, *let R be a region and $g\colon R \to Q$ a positive homeomorphism onto Q; then the inverse mapping $h = g^{-1}$ is a positive homeomorphism.*

Proof. Let D be a bounded disk with center $p \neq 0$, $\overline{D} \subset R$ and $\infty \in \mathscr{S}_2 - g(D)$. Let $q = g(p)$, $H = g(D)$, $f(x) = g(x)-g(p)$. By hypothesis, $\mu_p f = 1$. The same identity holds if the range of x's is restricted to D (compare Theorem 6). Then by Theorem 4

$$g(x)-g(p) \sim x-p$$

on $D-p$, which means that $g(x)-g(p) = (x-p)\cdot e^{u(x)}$.

Substituting $h(y)$ for x, we have $y-q \sim h(y)-h(q)$ on $H-q$, and therefore, setting $h^*(y) = h(y)-h(q)$, we have $\mu_q h^* = 1$ by Theorem 4.

THEOREM 8. *Let $G \subset \mathscr{S}_2$ be an open or a closed set, F a set closed-open in $\mathscr{S}_2 - G$ and $f\colon G \to \mathscr{P}$ a continuous function. The multiplicity $\mu_F f$ is invariant under positive homeomorphisms g of $\mathscr{S}_2$.*

In other terms,

$$\mu_{g^{-1}(F)} fg = \mu_F f. \tag{3}$$

In particular, *the multiplicity is an invariant under homographic transformations.*

Proof. First, consider the case of f rational:

$$f(y) = \lambda(y-q_1)^{k_1}\cdot\ldots\cdot(y-q_n)^{k_n}, \qquad y \in G, \tag{4}$$

$$k_1+\ldots+k_n = 0. \tag{5}$$

Let

$$g(p_0) = \infty, \qquad g(p_1) = q_1, \qquad g(p_2) = q_2, \ldots, g(p_n) = q_n.$$

It follows by Theorem 5 (since g is a positive homeomorphism) that

$$\begin{aligned} fg(x) &= \lambda[g(x)-g(p_1)]^{k_1}\cdot\ldots\cdot[g(x)-g(p_n)]^{k_n} \\ &\sim (x-p_1)^{k_1}\cdot\ldots\cdot(x-p_n)^{k_n}\cdot(x-p_0)^{-(k_1+\ldots+k_n)} \\ &= (x-p_1)^{k_1}\cdot\ldots\cdot(x-p_n)^{k_n} \end{aligned} \tag{6}$$

by (5).

Let $q_{j_1}, \ldots, q_{j_m}$ be the system of points q_j which belong to F. By the definition of multiplicity

$$\mu_F f = k_{j_1}+\ldots+k_{j_m}. \tag{7}$$

Now, since the conditions $q_j \epsilon F$ and $p_j \epsilon g^{-1}(F)$ are equivalent, it follows that, setting $r(x) = (x-p_1)^{k_1}\cdot\ldots\cdot(x-p_n)^{k_n}$, we have

$$\mu_{g^{-1}(F)} r = k_{j_1}+\ldots+k_{j_m}. \tag{8}$$

On the other hand, by (6), Theorem 4 of Section VIII and Theorem 4 of Section IX, we have

$$\mu_{g^{-1}(F)} fg = \mu_{g^{-1}(F)} r,$$

and identity (3) follows from (7) and (8).

Thus, the case where f is a rational function is settled. Let us consider now the general case. If G is closed, there exists by Theorem 1 of Section III, a rational function r such that $f \sim r$, and hence $fg \sim rg$. Consequently (compare Theorem 4 of Section VIII),

$$\mu_F f = \mu_F r \quad \text{and} \quad \mu_{g^{-1}(F)} fg = \mu_{g^{-1}(F)} rg,$$

and so this case is reduced to the preceding one.

Finally, if G is open, there exists a closed set F^* which separates the sets F and $\mathscr{S}_2-G-F$. Therefore its complement consists of two disjoint open sets one of which, say G^*, contains F and the other contains $\mathscr{S}_2-G-F$. Consequently, the set $g^{-1}(G^*)$ contains $g^{-1}(F)$ and is closed-open in $\mathscr{S}_2-g^{-1}(F^*)$. By (3) of Section IX, it follows that

$$\mu_F f = \mu_{G^*}(f\,|\,F^*) \quad \text{and} \quad \mu_{g^{-1}(F)} fg = \mu_{g^{-1}(G^*)}[fg\,|\,g^{-1}(F^*)].$$

Since the right-hand terms of these identities are equal, as we have just proved, formula (3) follows.

THEOREM 9. *The index is invariant under positive homeomorphisms* $g\colon \mathscr{E}^2 \to \mathscr{E}^2$; *i.e.* $\operatorname{ind}_{g\zeta} g(p) = \operatorname{ind}_\zeta p$ (1).

Proof. Put $f(x) = \zeta^0(x) - p$ and $h(x) = g\zeta^0(x) - g(p)$. By Theorem 1 of Section XII,

$$\operatorname{ind}_\zeta p = \mu_Q f \quad \text{and} \quad \operatorname{ind}_{g\zeta} g(p) = \mu_Q h. \tag{9}$$

By hypothesis (compare Theorem 4), $g(y) - g(p) \sim y - p$ on $\mathscr{E}^2 - p$. Hence $g\zeta^0(x) - g(p) \sim \zeta^0(x) - p$ on $\mathscr{S}$ (since the curve $\zeta^0(\mathscr{S})$ lies in $\mathscr{E}^2 - p$). This homotopy implies, by Theorem 4 of Section VIII, that the right-hand terms of the identities (9) are equal; this completes the proof.

The invariants of the positive homeomorphisms (of $\mathscr{S}_2$ or of $\mathscr{E}^2$) can be denoted as the *invariants of the oriented Topology*. As we have seen, they include the multiplicity of a set and the index of a point.

Consequently the absolute values of these invariants are invariants of arbitrary homeomorphisms; thus they are topological concepts (although their definitions use non-topological notions).

Also the increment of the logarithm is such an invariant. For, it follows by condition (3) of Section XI, that

$$\Delta_{g\zeta} f g^{-1} = \operatorname{ind}_{fg^{-1}g\zeta} 0 = \operatorname{ind}_{f\zeta} 0 = \Delta_\zeta f.$$

(1) See Alexandroff–Hopf, *op. cit.*, p. 476.

LIST OF IMPORTANT SYMBOLS

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