Classical Dynamics – Problem Sheet 1

to be discussed in tutorial the week of 2 October 2023

1. In this question we will show that the volume of a parallelepiped whose parallel edges are defined by $\underline{r}_1$, $\underline{r}_2$ and $\underline{r}_3$ is equal to the scalar triple product, $\underline{r}_1 \cdot (\underline{r}_2 \times \underline{r}_3)$.

   a) Consider the parallelogram constituting the base of a parallelepiped. Let its defining edges be $\underline{r}_2$ and $\underline{r}_3$. Show, with the aid of a diagram, that its area is given by $|\underline{r}_2 \times \underline{r}_3|$.

   b) Write an expression for a vector $\vec{v}$ perpendicular to both $\underline{r}_2$ and $\underline{r}_3$.

   c) The vector $\vec{v}$ is orthogonal to the base of the parallelepiped. The third edge of the parallelepiped is given by $\underline{r}_1$. Write the scalar product of $\underline{r}_1$ and $\vec{v}$, letting θ denote the angle between the two vectors. Draw a sketch of the parallelepiped showing θ.

   d) Calculate the height of the parallelepiped and use this together with the result of part (a) to find the volume of the parallelepiped (HEIGHT $\times$ AREA = VOLUME).[1]

   e) If we had interchanged $\underline{r}_1$, $\underline{r}_2$ and $\underline{r}_3$ the orientation of the parallelepiped will have changed but its volume will have stayed the same up to a sign. Consequently find two more expressions for $\underline{r}_1 \cdot (\underline{r}_2 \times \underline{r}_3)$ all having the same value and sign for the parallelepiped volume.

2. (for submission) Consider the function

$$F(x, y, t) = x^2 + 2xy + y^2 - 3t$$

   a) Compute $\partial F/\partial x$, $\partial F/\partial y$ and $\partial F/\partial t$

   b) Suppose that x and y evolve as $x = \sin 3t$, $y = t - \sin 3t$. Compute $f(t) = F(x(t), y(t), t)$ and df/dt.

   c) Compare this expression for df/dt to that obtained from the chain rule:

$$\frac{df}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial t}$$

3. Given

$$\begin{aligned} z &= r\cos\theta \\ x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \end{aligned}$$

   find $r(z, y, x)$, $\theta(x, y, z)$ and $\phi(x, y, z)$.

4. Show that in spherical coordinates

$$\underline{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \end{pmatrix}$$

   Deduce that if f is only a function of r then

$$\underline{\nabla} \cdot \underline{\nabla} f = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$$

[1] Note that it is the same as the scalar triple product, $\underline{r}_1 \cdot (\underline{r}_2 \times \underline{r}_3)$, which you should have calculated in part (c).

5. (challenge problem useful for rigid body motion) Consider a symmetric 3×3 matrix $\mathbf{I}$ with real entries. Assume that it has three eigenvectors $\{\underline{r}_1, \underline{r}_2, \underline{r}_3\}$ with distinct nonzero real eigenvalues. Recall that $\underline{r}$ is an eigenvector with eigenvalue $\lambda \in \mathbb{C}$ if $\mathbf{I}\underline{r} = \lambda \underline{r}$. Show that the the $\underline{r}_i$ are mutually orthogonal.