Linear Algebra and Geometry I 2023 **Problem Sheet 5**

You are encouraged to work with other students on the module. If you are having difficulty with any of the questions or want feedback on specific answers you should ask your tutor in your next tutorial or attend the lecturer's office hours.

1. Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 7 \\ -5 \\ -7 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}.$$

Determine which of the following sets are a basis for $\mathbb{R}^3$. You must justify your answers. Which sets are linearly independent? Which sets span $\mathbb{R}^3$?

(a) $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$

(b) $\mathbf{v}_3$ and $\mathbf{v}_4$

(c) $\mathbf{v}_1, \mathbf{v}_3$ and $\mathbf{v}_4$

(d) $\mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$

2. (a) Find a basis for the following subspaces of $\mathbb{R}^3$. In each case, you should demonstrate how you know that it is a basis.

   (i) The set of vectors (x, y, z) with $x + y + z = 0$.

   (ii) The set of vectors (x, y, z) with $x = 2y = 4z$.

   (iii) The set of vectors (x, y, z) that are perpendicular to $(1, 1, 0)$ and $(2, 0, -1)$.

   (b) Complete each of the bases that you found in Part (a) to a basis of $\mathbb{R}^3$.

3. Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v}_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

   (a) Find the largest possible number of linearly independent vectors among $\mathbf{v}_1, \ldots, \mathbf{v}_6$.

   (b) Hence determine the dimension of $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_6\}$.

4. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^4$ be *distinct* vectors. What are the possible values for the dimension of $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Give an example of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to show each possibility.

5. Let V be vector space. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in V$ be a set of linearly independent vectors which *do not* span V and let $\mathbf{u} \in V$ be any vector which is not in the linear span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$. We want to prove that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m, \mathbf{u}$ are still linearly independent.

   Suppose that there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta$ such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots \alpha_m \mathbf{v}_m + \beta \mathbf{u} = \mathbf{0}.$$

   (a) Show that if $\beta = 0$ then $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$.

   (b) Explain why β must be 0.

   (*Hint: Assume that $\beta \neq 0$ and show that this leads to a contradiction. You may want to look at the proof of Proposition 2.4.14 in the notes.*)

   (c) Use Parts (a) and (b) to write a proof of the fact that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m, \mathbf{u}$ are linearly independent.

6. Let V be a vector space and U a subspace of V. A linear algebra student wishes to prove that $\dim U \leq \dim V$.

   The student provides the following proof:

   Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis of V so that $\dim V = n$. Since U is a subspace of V, any basis of V can be reduced to a basis of U, by removing vectors if necessary. It follows that some subset of $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ must be a basis of U. Therefore U has a basis consisting of at most n elements. Hence $\dim U \leq n = \dim V$.

   Is this proof correct? If not, what is the error and how can the argument be changed to make it correct?