Linear Algebra and Geometry II - Sheet 8

Please attempt all of the problems on this sheet. You are welcome to make use of the Lecturers' office hours and the tutorial for asking questions about any of the problems. Solutions will be made available in Week 10.

Participation Mark: Please submit your solution to Problems 3 and 6, marked with a blue triangle (△), by using the submission link on Keats. They are due on Monday 18th March 2024 at 4pm GMT.

1. Show that for a square matrix $A \in M_n(\mathbb{C})$ the equality $\det(A^*) = \overline{\det(A)}$ holds.

2. Let $A \in M_{m,n}(\mathbb{C})$. Show that $\ker(A) = \ker(A^*A)$.

   (*Hint*: To do that you need to prove two inclusions, $\ker(A^*A) \subset \ker(A)$ and $\ker(A) \subset \ker(A^*A)$. One of the inclusions is trivial, for the other one use the fact that $\|A\mathbf{x}\|^2 = \langle A\mathbf{x}, A\mathbf{x}\rangle = \langle A^*A\mathbf{x}, \mathbf{x}\rangle$.)

3. △ Show that a product of unitary (orthogonal) matrices is again unitary (orthogonal).

4. Let V be an inner product space of dimension n over $\mathbb{F}$. Show that there is an isometry
$$T : V \to \mathbb{F}^n$$
   to the standard vector space $\mathbb{F}^n$ with the dot product.

   Hint: you should use the fact that V has an orthonormal basis.

5. True or false? A matrix is unitarily equivalent to a diagonal matrix if and only if it has an orthogonal basis of eigenvectors.

6. △Which of the following pairs of matrices are unitarily equivalent? Give a reason or proof for each answer.

   (a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

   (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;

   (c) $\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$;

   (d) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$;

(e) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

7. [Optional bonus problem] This is related to next week's topic. Let $U : V \to V$ be a unitary operator on a complex inner product space V.

   (a) Let $\mathbf{x} \in V$ be a non-zero eigenvector of U. Let $V_1 = \text{span}(\mathbf{x})^{\perp}$. Show that $U(V_1) \subset V_1$.

   (b) Show that U defines (by restriction) a unitary operator on V_1.

   (c) Prove that unitary operators on complex inner product spaces are diagonalizable. (*Hint: try a proof by induction on the dimension of the space.*)