For submission in Week 5, please choose one of the first 5 problems to answer. For submission in Week 6, please choose one of the remaining 5 problems to answer. But as usual, I encourage you to attempt all of them.

Problem 1. Let $J = (xy, xz, yz)$. What is $V(J)$? Is it irreducible? If not, what are its irreducible components? Does $J = \text{rad}(J)$?

Problem 2. Let $J = (xy, (x-y)z)$. What is $V(J)$? Is it irreducible? If not, what are its irreducible components? Does $J = \text{rad}(J)$?

Problem 3. Use the Nullstellensatz to deduce that any algebraically closed field must have infinitely many elements.

Problem 4. Show by example that there exists a field k (which is not algebraically closed) and a polynomial f such that $V(f) = k^n$ but $f \neq 0$.

Problem 5. Let k be a field. Show that the only algebraic sets in k are $\emptyset$, finite collections of points, or all of k.

Problem 6. Let k be a field. In this problem we will go through a series of steps to show that two distinct irreducible curves in the plane meet at only finitely many points.

(1) Show that we may freely assume that k is algebraically closed.
(2) Let $C_1 = V(f_1)$ and $C_2 = V(f_2)$ be two irreducible curves in k^2. If $C_1 \neq C_2$, then show that f_1 and f_2 have no common factors.
(3) Let $K = k(x)$ and consider f_1 and f_2 as polynomials in $K[y]$. Show that f_1 and f_2 have no common factors as polynomials in $K[y]$.
(4) Show that there exists polynomials p_1 and p_2 in $K[y]$ such that $p_1 f_1 + p_2 f_2 = 1$ (Hint: use the fact that $K[y]$ is a Euclidean domain- if you haven't heard of this notion before, you can just assume this part before continuing onto the next step.)
(5) Deduce that there exists a polynomial $q(x) \in k[x]$ such that if (x_0, y_0) is a solution to $f_1 = f_2 = 0$ then $q(x_0) = 0$ (Hint: trying clearing the denominator in the equation $p_1 f_1 + p_2 f_2 = 1$).
(6) Conclude that there are only finitely many solutions to $f_1 = f_2 = 0$.

Problem 7 Show that the ideal $(XZ - Y^2, YW - Z^2, XW - YZ)$ cannot be generated by two elements.

Problem 8. Let $X \subset \mathbb{K}^n$ be an algebraic set. Show that there is a one-to-one correspondence between points $x \in X$ and maximal ideals of $I(X) \subset \mathfrak{m} \subset \mathbb{K}[x_1, \ldots, x_n]$. Deduce there is a one-to-one correspondence between the maximal ideals of $\mathbb{K}[X] := \mathbb{K}[x_1, \ldots, x_n]/I(X)$ and points of X.

Problem 9. Let R be a finitely generated $\mathbb{K}$-algebra and suppose that R has no nilpotent elements, i.e., if $r^n = 0$ then $r = 0$. Show that $(0) = \bigcap_{\mathfrak{m} \subset R} \mathfrak{m}$, where the intersection is over all maximal ideal of R.

Problem 10. Show that a local $\mathbb{K}$-algebra R without any nilpotents is a finitely generated $\mathbb{K}$-algebra if and only if R is a field. (Hint: use Problem 9).