CCM115a Sequences and Series: Assignment 4

You are strongly encouraged to try as many of the problems on this sheet as possible (*if you are short of time*, then you may wish to focus on the problems marked with the symbol '†').

A selection of these problems will also be discussed in your tutorials.

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1. Negate the following proposition:

$$\forall\, \delta > 0,\ \exists\, m \in \mathbb{N},\ \forall\, n \geq m,\ -\delta < \frac{(-1)^n}{n} \text{ and } \frac{(-1)^n}{n} < \delta.$$

2. For each given set S, find (i) the set S_- of all lower bounds of S; (ii) $\inf S = \max S_-$; (iii) the set S_+ of all upper bounds of S; (iv) $\sup S = \min S_+$. No proof is required.

   (a) $S = \{x \in \mathbb{R} \mid x^2 - 6x + 5 < 0\}$

   (b)† $S = \{\frac{1}{\sqrt{n}} + (-1)^n \mid n \in \mathbb{N}\}$

   (c) $S = \{(-1)^n - \frac{1}{n^2} \mid n \in \mathbb{N}\}$

3. In the following exercises, for a given set S, you need to determine $\sup S$ or $\inf S$ and *prove* your claim. Proceed as follows. In order to prove that $m = \inf S$, you need to prove that (i) for all $x \in S$, one has $m \leq x$; (ii) for any given $\varepsilon > 0$, there exists an element $x \in S$ such that $x < m + \varepsilon$ (specify x explicitly in terms of ε). In order to prove that $M = \sup S$, you need to prove that (i) for all $x \in S$, one has $x \leq M$; (ii) for any given $\varepsilon > 0$, there exists an element $x \in S$ such that $M - \varepsilon < x$ (specify x explicitly in terms of ε).

   (a)† Let $S = \{\frac{2\sqrt{n}+1}{\sqrt{n}} \mid n \in \mathbb{N}\}$. Determine $m = \inf S$ and prove your claim.

   (b) Let $S = \{\frac{\log(n/2)}{\log(n)} \mid n = 2, 3, 4, \ldots\}$. Determine $m = \sup S$ and prove your claim.

   (c) Let $S = \{\frac{1}{\sqrt{n}} + (-1)^n \mid n \in \mathbb{N}\}$. Determine $m = \inf S$ and prove your claim.

   (d) Let $S = \{(-1)^n - \frac{1}{n^2} \mid n \in \mathbb{N}\}$. Determine $M = \sup S$ and prove your claim.

4.† Let $x, y \in \mathbb{R}$ and suppose that $y \leq x + \varepsilon$ for all real $\varepsilon > 0$. Prove that $y \leq x$; use the method of proof by contradiction.

5. Let $A \subset B \subset \mathbb{R}$.

   (a)† Suppose that B is bounded above; prove that A is also bounded above and $\sup A \leq \sup B$.

   (b) Suppose that B is bounded below; prove that A is also bounded below and $\inf B \leq \inf A$.

6. Using the definition of convergence, check that the following sequences s_n converge to zero as $n \to \infty$. That is, given $\varepsilon > 0$, find explicitly $n_0 \in \mathbb{N}$ (your n_0 should depend on ε) such that $\forall n \geq n_0$ the inequality $|s_n| \leq \varepsilon$ holds true.

   (a) $s_n = \frac{10}{n}$

   (b) $s_n = \frac{1}{10n^3}$

   (c)† $s_n = \frac{1}{10^n}$

   (d) $s_n = \frac{\sin(n)}{\sqrt{n}}$

7. Let $A \subset \mathbb{R}$ be bounded above and let $B = \{x \in \mathbb{R} \mid -x \in A\}$. Prove that B is bounded below and $\inf B = -\sup A$.