Linear Algebra and Geometry I 2023 **Problem Sheet 3**

You are encouraged to work with other students on the module. If you are having difficulty with any of the questions or want feedback on specific answers you should ask your tutor in your next tutorial or attend the lecturer's office hours.

1. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$ and let α be a scalar. State whether each of the following operations are defined. If you think it is not defined, explain why.

   (a) $3\alpha + \mathbf{v}$ (c) $\mathbf{u}/\mathbf{v}$ (e) $\|\mathbf{u}\|\mathbf{v} - \alpha\mathbf{u}$

   (b) $\mathbf{w} + \alpha^2\mathbf{v}$ (d) $\|\mathbf{v}\| - \alpha\mathbf{u}$ (f) $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$

2. Let $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -3 \\ 5 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$, $\mathbf{v}_4 = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix}$.

   (a) Compute the following linear combinations:

   (i) $\frac{1}{3}(3\mathbf{v}_1 - 4\mathbf{v}_2 + 2\mathbf{v}_3)$ (ii) $\frac{1}{2}(\mathbf{v}_2 - \mathbf{v}_1) + 4(\mathbf{v}_4 - \mathbf{v}_1)$

   (b) Find $\mathbf{x} \in \mathbb{R}^3$ such that $2\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{x} = 3(\mathbf{x} + 2\mathbf{v}_4)$.

3. Let

$$\mathbf{u} = \begin{pmatrix} -1+2i \\ 2i \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ i \\ 1-i \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -i \\ 3 \\ 4 \end{pmatrix}.$$

   (a) Evaluate the following expressions.

   (i) $3\mathbf{u} - i\mathbf{v} + 2\mathbf{w}$ (ii) $\dfrac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|}$ (iii) $(\sqrt{2}e^{i\pi/4}\mathbf{u} + \mathbf{v}) \cdot \mathbf{w}$

   (b) Determine whether each of the following vectors is contained in $\operatorname{span}\{\mathbf{v}, \mathbf{w}\}$. If it is, express it as a linear combination of $\mathbf{v}$ and $\mathbf{w}$. (*Recall that* $\operatorname{span}\{\mathbf{v}, \mathbf{w}\}$ *denotes the linear span of* $\mathbf{v}$ *and* $\mathbf{w}$.)

   (i) $\begin{pmatrix} 2-3i \\ 9+6i \\ 12+8i \end{pmatrix}$ (ii) $\begin{pmatrix} 1+7i \\ 2-3i \\ 2+3i \end{pmatrix}$ (iii) $\begin{pmatrix} -3i \\ 12+i \\ 10-4i \end{pmatrix}$

4. Let $\mathbf{u} = (1, -2, 3)$, $\mathbf{v} = (3, 0, 1)$ and $\mathbf{w} = (-2, 2, 1)$.

(a) Determine the cosine of the angle between

(i) $\mathbf{u}$ and $\mathbf{v}$ (ii) $\mathbf{v}$ and $\mathbf{w}$

(b) Find $\alpha \in \mathbb{R}$ such that $\mathbf{u} + \alpha\mathbf{v}$ is perpendicular to $\mathbf{w}$.

5. (a) Find the parametric equation of the line in $\mathbb{R}^3$ passing through the point $(4, 1, 5)$ and parallel to the vector $(1, 0, 1)$.

(b) Find the parametric equation of the line in $\mathbb{R}^3$ passing through the points $(2, -7, 12)$ and $(2, 9, -6)$.

(c) Show that the lines in Parts (a) and (b) intersect, and find their point of intersection.

(d) Let L_1 and L_2 be the lines in $\mathbb{R}^3$ with parametric equations

$$L_1 : \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad L_2 : \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Show that both lines pass through the points $(0, 1, 1)$ and $(1, 2, 2)$. Does that mean L_1 and L_2 are equal? Explain your answer.

6. Let $\mathbf{p} = (-2, 1, -3)$ and $\mathbf{q} = (0, 4, 1)$.

(a) Show that the vector $(13, 2, -8)$ is orthogonal (i.e perpendicular) to both $\mathbf{p}$ and $\mathbf{q}$.

(b) Hence determine the Cartesian equation of the plane with parametric equation

$$\mathbf{v} = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} + t\mathbf{p} + s\mathbf{q}.$$

7. (a) Prove the *triangle inequality* in $\mathbb{R}^n$, i.e. prove that for every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ we have

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

You may use the fact that $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$.
(*Hint: Express the length of* $\mathbf{u} + \mathbf{v}$ *using the dot product.*)

(b) Show that $\mathbf{u}$ and $\mathbf{v}$ satisfy

$$\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$$

if and only if $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|$.

Make sure you write in sentences, with punctuation, and you explain what you are doing. Look at the other proofs in the notes for guidance on how to write a proof.