Introduction to Abstract Algebra: Sheet 2

For discussion in Week 2 Skills Sessions and Week 3 Tutorials

Please complete all core problems. The additional practice problems are not required: try them if you're interested and have time. The problems marked with (SS) will be discussed in the Skills Session for Week 2. Please attempt these *before* your Skills Session. Some of the other problems will be discussed in your tutorial for Week 3. Please attempt these problems *before* your tutorial.

Core problems

Exercise 1 (SS)**.** Let $a, b, c \in \mathbb{Z}$. Assume $a|b$ and $b|c$. Show that $a|c$.

Exercise 2. Fix integers $a, b \in \mathbb{Z}$, not both zero. Define $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $f((x, y)) = ax + by$. Show f is surjective if and only if a and b are relatively prime.

Exercise 3 (SS)**.** Let $a, b \in \mathbb{Z}$, not both zero. Let $d = \gcd(a, b)$. Show that $\frac{a}{d}$ and $\frac{b}{d}$ are relatively prime.

Exercise 4 (SS)**.** Let $a, b, c \in \mathbb{Z}$. Assume a and b are relatively prime.

(i) Assume $a|c$ and $b|c$. Show $ab|c$.

(ii) Is the statement in (i) true if we remove the assumption that a and b are relatively prime? Prove it or give a counterexample.

Exercise 5. Let a, b, c and d be integers, with $a \neq 0$. Suppose that a and b are relatively prime.

(i) Prove that if $d|a$, then b and d are relatively prime.

(ii) Prove that $\gcd(a, c) = \gcd(a, bc)$.

(iii) Prove that a^2 and b^2 are relatively prime.

Exercise 6. The **Fibonacci numbers** are defined by $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$ and in general by the recursive formula

$$F_{n+1} = F_n + F_{n-1}$$

for all $n \geq 2$. Prove that F_n and F_{n+1} are relatively prime for all $n \geq 1$.

Exercise 7. A fancy restaurant only has two options for dinner: a 4-course tasting menu for £126 and a 3-course tasting menu for £108 (these prices include taxes, gratuity, drinks, etc). Show that the bill will add up to a multiple of £18 for every table at this restaurant.

Additional practice

Exercise 8. Let a, b, c be integers, not all zero, and set

$$S = \{ax + by + cz \mid \ x, y, z \in \mathbb{Z}\}.$$

Show that there exists an integer $m \geq 0$ such that $S = \{mu \mid \ u \in \mathbb{Z}\}$.

Exercise 9. Let a, b and c be positive integers.

(i) Prove that if $a|b$ and $b|c$, then $\frac{b}{a} | \frac{c}{a}$.

(ii) Prove that if c is a common divisor of a and b, then $\gcd(\frac{a}{c}, \frac{b}{c}) = \frac{\gcd(a,b)}{c}$.