CALCULUS 1 TUTORIAL EXERCISES II

In this tutorial you will work with inverse functions, exponents, logarithms, polynomial functions and trigonometric functions.

Tutor's Example[1]

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = (x+1)(x-2)(x+3).$$

Prove that the function is NOT a bijection. Identify domains (which together cover $\mathbb{R}$) over which $f(x)$ becomes invertible.

7. Identify domains (which together cover $\mathbb{R}$) over which $f : \mathbb{R} \to \mathbb{R}$ becomes invertible, where
   (a) $f(x) = -x^3 + x^2 + 9x - 9$,
   (b) $f(x) = \tan(x^2)$,
   (c) $f(x) = 2^x + 2^{-x}$.
8. Find the value of x for which
   $$4^{\frac{x}{y}+\frac{y}{x}} = 32$$
   and
   $$\log_3(x-y) + \log_3(x+y) = 1.$$
   [Hat-tip: this problem was set at math10.com[2].]
9. For what value of n does $\log_2(3)\log_3(4)\log_4(5)\ldots\log_n(n+1) = 10$? [Hat-tip: this problem was set at math10.com[3].]
10. Use the definitions of the sine, cosine and tangent functions, to prove the following pair of trigonometric identities:
    (a) $1 + \tan^2 x = \sec^2 x$ and
    (b) $1 + \cot^2 x = \csc^2 x$.

    Both of these identities arise from using Pythagoras' theorem for right-angled triangles. Identify the right-angled triangles for which Pythagoras' theorem gives the identities above.

[1] To be shown by the tutor on the board at the start of the tutorial
[2] See `https://www.math10.com/problems/logarithmic-equation-problems/difficult/`
[3] See `https://www.math10.com/problems/logarithmic-equation-problems/difficult/`

11. We have expressed e^x as the infinite sum $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, but written it notationally as a power of Euler's number e. In this question show that the infinite sum expression of e^x obeys the following properties of exponents:

    (a) $e^x e^y = e^{x+y}$

    (b) $e^{-x} = \frac{1}{e^x}$

    You may wish to use the binomial expansion $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

12. **Exam-style question**[4]

    (i) [5]**[6 marks]** By first deriving the double-angle formula for $\tan(a+b)$, show that if $-\pi/2 < \arctan(x) + \arctan(y) < \pi/2$ then arctan satisfies

    $$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right), \tag{1}$$

    for $x, y \in \mathbb{R}$. You may state without proof, the double-angle formulae for cos and sin.

    (ii) **[5 marks]** Explain carefully why the condition $-\pi/2 < \arctan(x) + \arctan(y) < \pi/2$ is necessary for (1) to hold, and how one can adapt this formula in case the condition is not satisfied.

    (iii) **[4 marks]** Hence, or otherwise, compute the value of

    $$\arctan(1) + \arctan(2) + \arctan(3).$$

13. **Advanced Python**[6] Use Python to plot $f(x) = \sum_{k=0}^{N} \frac{1}{k!} x^k$ for $N = 2$, 5, and 10 and compare your plot to the exponential function. Plot all four graphs on the same set of axes for the

[4] Some questions on the final exam will be longer with several parts which delve into a topic and build on each other. To help you prepare, each sheet will include such a question.

[5] You may notice that sometimes the parts of questions are labelled (a), (b), (c),..., and sometimes they are labelled (i), (ii), (iii),.... While there is no universally applied rule, often the convention is that parts labelled (a), (b), (c),... are independent, you can solve them in any order, and the answer to one does not affect the answer to the others. On the other hand parts labelled (i), (ii), (iii),... build on what went before. Thus in this question, you'll probably need to use what you learn in the first part to solve the second part and then the third part.

[6] Each sheet will contain a Python problem which should complement the programming aspect of the course. In some cases, like this one, you may have to look up unfamiliar functions and use your knowledge of Python to apply them.

domain $x \in [-5, 5]$. N.B. in Python one way to sum values is to put them in a list and use the `sum` function.

```
values = [ 1, 2, 3, 4 ]

total = sum( values )
print( total )
```

which outputs

```
10
```

You can use `exp` from the `numpy` package, and `factorial` from the `math` package which can be called using

```
import numpy as np
import math

e = np.exp( 1 )
fac = math.factorial( 4 )
print( e )
print( fac )
```

which outputs

```
2.718281828459045
24
```

14. **A challenging problem**[7] In the lecture notes we considered differential equations of the form
$$\frac{d^n}{dx^n}(f(x)) = Af(x).$$

- When $A = 1$ and $n = 1$,
$$\frac{d}{dx}(f(x)) = f(x)$$
the solution satisfying $f(0) = 1$ is the exponential function;
- when $A = 1$ and $n = 2$,
$$\frac{d^2}{dx^2}(f(x)) = f(x)$$
the solutions are the hyperbolic functions ($f(x) = \cosh(x)$ when $f(0) = 1$ and $f'(0) = 0$ and $f(x) = \sinh(x)$ when $f(0) = 0$ and $f'(0) = 1$);

[7] All challenging problems in the tutorial problem sets are beyond the scope of the course, and are not examinable, but they are, hopefully interesting and inspiring.

- when $A = -1$ and $n = 2$,
$$\frac{d^2}{dx^2}(f(x)) = -f(x)$$
the solutions are the trigonometric functions ($f(x) = \cos(x)$ when $f(0) = 1$ and $f'(0) = 0$ and $f(x) = \sin(x)$ when $f(0) = 0$ and $f'(0) = 1$).

What are the solutions to the equation when

(a) $A = 1$, $n = 3$ for $f(0) = 1$, $f'(0) = 0$ and $f''(0) = 1$?

(b) $A = -1$, $n = 3$ for $f(0) = 1$, $f'(0) = 0$ and $f''(0) = 1$?

Give power series for each solution.