Exercises to Section 4

Exercises in red are from the list of the typical exercises for the exam.
Exercises marked with a star * are for submission to your tutor.

Derivative

1. For a function f defined on $\mathbb{R}$ and for $h > 0$, define a new function

$$\Delta_h f(x) = f(x+h) - f(x).$$

Prove the "sum rule" and "product rule"

$$\begin{aligned} \Delta_h(f+g)(x) &= \Delta_h f(x) + \Delta_h g(x) \\ \Delta_h(fg)(x) &= g(x+h)\Delta_h f(x) + f(x)\Delta_h g(x). \end{aligned}$$

Determine $\Delta_h f(x)$, if

(a) $f(x) = ax + b$;
(b) $f(x) = ax^2 + bx + c$;
(c) $f(x) = e^x$.

Explain how formulas for the derivative of f arise from here.

Differentiability

2. For each of the following functions, determine the set of $x \in \mathbb{R}$ where the derivative fails to exist:

(a) $f(x) = |(x-1)(x-2)^2(x-3)^3|$;
(b) $f(x) = |\pi^2 - x^2| \sin x$;
(c) $f(x) = \begin{cases} \frac{x-1}{4}(x+1)^2, & |x| \leqslant 1, \\ |x| - 1, & |x| > 1; \end{cases}$
(d) $f(x) = \lfloor x \rfloor \sin \pi x$;
(e) $f(x) = \begin{cases} x/(1+e^{1/x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$
(f)* $f(x) = |\log|x||$.

3. Let $n \geqslant 0$ be an integer, and let $f(x) = x^n \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Determine the range of n for which

(a) f is continuous at $x = 0$;
(b) f is differentiable at $x = 0$;
(c) f' is continuous at $x = 0$.

4. Let φ be a function continuous at $x = a$.

(a) Let $f(x) = (x-a)\varphi(x)$; show that f is differentiable at $x = a$ and find $f'(a)$.

(b) Let $f(x) = |x-a|\varphi(x)$; show that f is not differentiable at $x = a$ unless $\varphi(a) = 0$.

The algebra of differentiation

5. What can you say about the differentiability of $f(x) + g(x)$ and $f(x)g(x)$ at $x = 0$, if

   (a) $f(x)$ is differentiable at 0, and $g(x)$ is not differentiable at $x = 0$;

   (b)* neither $f(x)$ nor $g(x)$ are differentiable at $x = 0$.

The Mean Value Theorem

6. Using the Mean Value Theorem, prove the following inequalities:

   (a) $|\tan^{-1} x - \tan^{-1} y| \leqslant |x - y|$;

   (b) $|x^p - y^p| \leqslant p \max\{|x|^{p-1}, |y|^{p-1}\}|x-y|$, $p \geqslant 1$;

   (c) $|\log(x/y)| \leqslant |x-y|/\min\{x, y\}$ for $x > 0$, $y > 0$.

Taylor's formula

7. For each of the following functions $f(x)$, write down the first several terms of the Taylor expansion near $x = 0$, up to and including the term of the given power:

   (a) $f(x) = \dfrac{1 + x + x^2}{1 - x + x^2}$ up to the term with x^4;

   (b) $f(x) = \dfrac{(1+x)^{100}}{(1-2x)^{40}(1+2x)^{60}}$ up to the term with x^2;

   (c) $f(x) = (a^m + x)^{1/m}$ up to the term with x^2 ($a > 0$ and $m \in \mathbb{N}$);

   (d)* $f(x) = \sqrt{1-2x} - \sqrt[3]{1-3x}$ up to the term with x^2;

   (e) $f(x) = \log \frac{\sin x}{x}$ up to the term with x^4.

8. Using Taylor's formula (with the remainder term), give an upper estimate for the error in the following approximations:

   (a) $e^x \approx 1 + x + \cdots + \frac{x^n}{n!}$ for $0 \leqslant x \leqslant 1$

   (b) $\sin x \approx x - \frac{x^3}{6}$ for $|x| \leqslant 1$

   (c) $\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8}$ for $0 \leqslant x \leqslant 1$

9. Using Taylor's expansions, compute the following limits:

   (a) $\displaystyle\lim_{x\to 0} \frac{e^x \sin x - x(1+x)}{x^3}$

   (b) $\displaystyle\lim_{x\to 0} \frac{a^x + a^{-x} - 2}{x^2}$ (where $a > 0$)

   (c) $\displaystyle\lim_{x\to\infty} \left(x - x^2 \log(1 + 1/x)\right)$

10. Let $n \in \mathbb{N}$, and let f be a function on $\mathbb{R}$ such that $f^{(n)}(x) = 0$ everywhere. What can you say about this function?

Challenging exercises

11. Let f be a function on $\mathbb{R}$ satisfying

$$|f(x) - f(y)| \leqslant C|x - y|^2$$

for all $x, y \in \mathbb{R}$ and some $C > 0$; what can you say about this function?

12. Prove that if f is differentiable on $(0, \infty)$ and $f'(x) = o(1)$ as $x \to \infty$, then $f(x) = o(x)$ as $x \to \infty$.

    Hint: use the Mean Value Theorem.

13. Let p be a polynomial of degree n with real coefficients such that p has n distinct real roots. Prove that all roots of p' are also distinct and real. What can you say about the relative location of the roots of p and p'?

14. Prove that for the *Legendre polynomial*

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \Big((x^2 - 1)^n \Big)$$

all roots are real and located between -1 and 1.

Hint: use Rolle's theorem n times.