Classical Dynamics – Problem Sheet 3

to be discussed in tutorial the week of 14 October

1. A particle of mass m moves in three dimensions under the action of a conservative force that is obtained from a potential: $\underline{F} = -\underline{\nabla} V$.

   (a) Define the kinetic and potential energy for this particle and prove that the total energy is conserved.

   (b) Show that only one of the two forces

   $$\begin{aligned}(1)\quad \underline{F} &= zx\underline{e}_x + yz\underline{e}_y + z^2\underline{e}_z \\ (2)\quad \underline{F} &= yz^2\underline{e}_x + xz^2\underline{e}_y + 2xyz\underline{e}_z\end{aligned}$$

   is conservative and find a potential for it.

   (c) Define the torque and angular momentum for a particle moving in three dimensions. Suppose the particle is acted on by the force (1) above. Compute the torque $\underline{N}$. Why is this result obvious from the form of $\underline{F}$? If at time $t = 0$ the particle is located at $\underline{r}(0) = 2\underline{e}_x + 2\underline{e}_y$ with velocity $\underline{\dot{r}}(0) = 2\underline{e}_x + 3\underline{e}_y$, find the angular momentum at all times. Consider a particle moving in a straight line given by $\underline{r} = \underline{v}t + \underline{r}(0)$ where $\underline{v}$ and $\underline{r}(0)$ are constant vectors. Show that both momentum and angular momentum are conserved.

2. Consider a particle moving in three-dimensions with no force. Show that, after reducing to just the $r = |\underline{r}|$ variable, the conservation of energy equation

   $$E = \frac{m}{2}\dot{r}^2 + \frac{l^2}{2mr^2}$$

   has straight lines as solutions. What do the solutions $l = 0$ correspond to?

3. Consider a simple harmonic oscillator given by a particle of mass m moving in a potential $V = (k/2)r^2$, with k a positive constant and $r \equiv |\underline{r}|$.

   (a) Find the force $\underline{F} = -\nabla V$ acting on the particle.

   (b) Show that the general solution to Newton's second law of motion can be written in the form $\underline{r}(t) = \underline{a}\cos\omega t + \underline{b}\sin\omega t$, with $\omega = \sqrt{k/m}$.

   (c) For the general solution $\underline{r}(t)$, calculate the angular momentum $\underline{J}$ and the total energy E of the particle as a function of time. You should find that both $\underline{J}$ and E are in fact time-independent; explain from the properties of the force $\underline{F}$ why this is so.

4. This *is* rocket science. Consider a rocket of mass $M(t)$ at time t moving in a straight line which ejects its fuel at a constant rate and at a constant speed, u, relative to the rocket (think about the signs!).

   (a) In a small time interval Δt the rocket ejects $|\Delta M|$ of fuel and gains in velocity by Δv. Use the conservation of momentum for the rocket and fuel system as a whole to show that:

   $$\Delta v = -u\frac{\Delta M}{M}$$

   You may neglect any terms of the form $\Delta M \Delta v$.

   (b) By promoting this equation to a differential equation (i.e. let $\Delta \to d$) show by integrating it that the velocity of the rocket satisfies:

   $$v(t) = -u\ln\frac{M(t)}{M(0)}$$

(c) If $u = 2{,}000\,\mathrm{m\,s^{-1}}$, what is the speed of the rocket when half of its mass has been ejected. Assume the rocket is stationary at $t = 0$.