CALCULUS 1 TUTORIAL EXERCISES VI

In this tutorial you will revise and work with the derivative.
Questions 45(c) and (g), 46(c), 47(h), and 48(c) should be submitted to GradeScope via the link under week 6 of the KEATs page for marking and feedback. The deadline is **4pm on Friday 17th November 2023**.

Tutor's Example[1]

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sqrt{1 + \sin x}$. Compute $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ and sketch the graph of $f(x)$ for $x \in [-\pi, \pi]$. Is the function f differentiable for $x \in [-\pi, \pi]$?

45. Calculate the following derivatives:

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, e^{-x^2}$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, \ln(\tan x)$

(c) **To be submitted:** $\dfrac{\mathrm{d}}{\mathrm{d}x}\, (x \ln x - x)$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, \arcsin(x^2)$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, \arctan(e^x)$

(f) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, e^{\sin x}$

(g) **To be submitted:** $\dfrac{\mathrm{d}}{\mathrm{d}x}\, \left(x e^{\arctan x}\right)$

(h) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, \ln|\arcsin x|$

46. **To be submitted (part (c)):** For parts (a)–(d) in question 45, use your answers to help help sketch each of these functions together with their derivatives on the same graph, noting the position of any local maxima/minima, and the slope of the graphs.

47. Calculate the following derivatives:

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, \arcsin\left(\dfrac{1-x}{1+x}\right)$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, 3^{\sin x}$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, \ln|x + \sqrt{x^2 - a^2}|$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, \dfrac{\sqrt{1+x}}{x}$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, \left(\dfrac{1}{x} \arccos\sqrt{1-x^2}\right)$

(f) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, x^{x \sin x}$

(g) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, \ln(\cosh x + \sinh x)$

(h) **To be submitted:** $\dfrac{\mathrm{d}}{\mathrm{d}x}\, \ln\sqrt{\dfrac{1+x^2}{1-x^2}}$

[1] To be shown by the tutor at the start of the tutorial.

48. Let $f^{(n)}(x)$ denote the n^{th} derivative with respect to x of the function $f : \mathbb{R} \to \mathbb{R}$, where $n \in \mathbb{N}$. Calculate $f^{(n)}(x)$ at $x = 0$ when

    (a) $f(x) = e^x$.

    (b) $f(x) = \sin x$.

    (c) **To be submitted:** $f(x) = x^n e^x$.

49. Differentiate $\sin(e^{\arcsin x})$ with respect to x.

50. Use the formula $\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$ to find the derivative of x^x with respect to x for $x > 0$.

51. Consider the hyperbola defined as the set of points satisfying $x^2 - y^2 = 16$. Let $A = (-5, y_A)$, $B = (5, y_B)$ and C the point where the tangent to the hyperbola at A meets the tangent to the hyperbola at B. Compute the area of the triangle ABC.

52. **Exam-style question**[2] Consider the function

$$f(x) = \frac{4x^2 - 1}{x^3 - x^2 - 2x}.$$

    (i) [**5 marks**] By completely factorising the denominator as $(x-a)(x-b)(x-c)$ for some a, b, and c in $\mathbb{R}$, rewrite $f(x)$ as[3]

$$f(x) = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}.$$

    (ii) [**5 marks**] Sketch $f(x)$, marking axis intercepts, asymptotes, and determine the behaviour as $x \to \infty$ and as $x \to -\infty$.

    (iii) [**6 marks**] Using part (i) or otherwise, compute the 2023^{rd} derivative of $f(x)$.

53. **Advanced Python**[4] Use Python to check your answers to questions 47 and 49.

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[2] Some questions on the final exam will be longer with several parts which delve into a topic and build on each other. To help you prepare, each sheet will include such a question.

[3] This is called a *partial fraction decomposition* of $f(x)$, and it will be a key tool later in the course for computing certain types of integrals.

[4] Each sheet will contain a Python problem which should complement the programming aspect of the course. In some cases, like this one, you may have to look up unfamiliar functions and use your knowledge of Python to apply them.

54. **A challenging problem**[5][6] Show that the function

$$f(x) = \begin{cases} x^4 \left(2 + \sin \frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

has an extreme value at a point where the sign of the derivative does not make a simple change.

[5]All challenging problems in the tutorial problem sets are beyond the scope of the course, and are not examinable, but they are, hopefully interesting and inspiring.

[6]This problem is, perhaps, less challenging than it is interesting. The function is taken from the marvellous book "Counterexamples in Analysis" by Bernard R. Gelbaum and John M. H. Olmstead.