Exercises to Section 7

Exercises in red are from the list of the typical exercises for the exam.
Exercises marked with a star * are for submission to your tutor.

Identities

1. Let $\lambda > 0$ and $f \in \mathcal{R}[a,b]$; prove that $\lambda f \in \mathcal{R}[a,b]$ and

$$\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx.$$

2. Let $f \in \mathcal{R}[a,b]$; prove that $-f \in \mathcal{R}[a,b]$ and

$$\int_a^b (-f(x))dx = -\int_a^b f(x)dx.$$

Hint: prove first that for any interval Δ,

$$\sup_{\Delta}(-f(x)) = -\inf_{\Delta} f(x).$$

Inequalities

3. Compare the two given integrals – which one is greater?

(a) $\int_0^1 \sin x \, dx$ or $\int_0^1 (\sin x)^2 dx$
(b) $\int_0^4 |\sin x| \, dx$ or $\int_0^4 (\sin x)^2 dx$
(c) $\int_0^1 e^{-x} dx$ or $\int_0^1 e^{-x^2} dx$
(d)* $\int_0^\pi e^{-x^2} (\cos x)^2 dx$ or $\int_\pi^{2\pi} e^{-x^2} (\cos x)^2 dx$

4. For each of the following integrals, determine whether it is positive or negative.

(a) $\int_0^{2\pi} x \sin x \, dx$
(b)* $\int_0^{2\pi} \frac{\sin x}{x} \, dx$
(c) $\int_{-2}^2 x^3 2^x \, dx$
(d) $\int_{1/2}^2 x^2 \log x \, dx$

5. Using Jensen's inequality, compare the two given expressions – which one is greater?

(a) $\left(\int_0^1 e^{-x^2} dx\right)^3$ or $\int_0^1 e^{-3x^2} dx$
(b) $\int_0^\pi x \sin x \, dx$ or $\frac{1}{\pi}\left(\int_0^\pi \sqrt{x \sin x} \, dx\right)^2$
(c) $-\log\left(\int_0^1 e^{-x^2} dx\right)$ or $1/3$

6. Let $p > 1$ and $q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Consider the function $f(t) = t - (t^p/p)$ for $t \geqslant 0$; prove that $f(t) \leqslant f(1)$ for all $t \geqslant 0$.

(b) Substitute $t = |u|/|v|^{q-1}$ for some $u, v \in \mathbb{R}$; rearrange to obtain

$$|uv| \leqslant \frac{1}{p}|u|^p + \frac{1}{q}|v|^q.$$

(*Hint:* remember to use $\frac{1}{p} + \frac{1}{q} = 1$.)

(c) Prove that for $f, g \in \mathcal{R}[a,b]$ and for an additional parameter $\lambda > 0$

$$\int_a^b |f(x)g(x)|dx \leqslant \frac{1}{p}\lambda^p \int_a^b |f(x)|^p dx + \frac{1}{q}\lambda^{-q} \int_a^b |g(x)|^q dx.$$

(d) Choose an appropriate value of λ to obtain the *Hölder's inequality*

$$\int_a^b |f(x)g(x)|dx \leqslant \left(\int_a^b |f(x)|^p dx\right)^{1/p} \left(\int_a^b |g(x)|^q dx\right)^{1/q}.$$

When $p = q = 2$ it is usually called the *Schwarz inequality*.

7. Let $f \in \mathcal{R}[a,b]$ such that $f(x) \geqslant 0$ for all x, and let $\varphi \in C[a,b]$.

(a) Prove that

$$m \int_a^b f(x)dx \leqslant \int_a^b f(x)\varphi(x)dx \leqslant M \int_a^b f(x)dx,$$

where $M = \sup_{[a,b]} \varphi$, $m = \inf_{[a,b]} \varphi$.

(b) Using the previous step and the Intermediate Value Theorem, prove that

$$\int_a^b f(x)\varphi(x)dx = \varphi(c) \int_a^b f(x)dx$$

for some $c \in [a,b]$.

Change of variable

8. Prove the Theorem on the change of variable in integrals from the lecture notes. Proceed as follows.

(a) Define

$$F(y) = \int_A^y f(s)\mathrm{d}s, \quad y \in [A,B],$$

and set $G(x) = F(\varphi(x))$, $x \in [a,b]$. By the Fundamental Theorem of Calculus (part 1), F is differentiable and $F' = f$.

(b) Differentiate G by applying the chain rule.

(c) Apply the Fundamental Theorem of Calculus (part 2) to the identity obtained on the previous step; you should get $G(b) - G(a) = \ldots$

(d) Express the left hand side directly from the definition of G.

9. Prove that if $f \in \mathcal{R}[a,b]$, then

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0_+} \int_{a+\varepsilon}^{b-\varepsilon} f(x)dx.$$

Challenging exercises

10. Let f be a continuous function on $[0,\infty)$ such that the limit $A = \lim_{x\to\infty} f(x)$ exists. Prove that

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(t)dt = A.$$

11. Let f be an infinitely differentiable function on $[a,b]$. Use integration by parts to prove the following asymptotic expansion as $\lambda \to \infty$:

$$\int_a^b e^{i\lambda x} f(x)dx \sim e^{i\lambda a} \sum_{k=0}^{\infty} (-i\lambda)^{-k-1} f^{(k)}(a) - e^{i\lambda b} \sum_{k=0}^{\infty} (-i\lambda)^{-k-1} f^{(k)}(b).$$