Problem 1. (N.b. This is quite a challenging problem.) Show that any non-degenerate cubic in $\mathbb{P}^2_{\mathbb{C}}$ can be transformed by a linear transformation into a cubic of the form $\{(x, y) : y^2 = x^3 + ax + b\}$ (in affine coordinates). (Hint: if you get stuck, look up "depressed cubic" or "Weierstrass form of cubic").

Problem 2. Show that if C is the closure in $\mathbb{P}^2$ of the affine cubic $C_0 = \{(x, y) : y^2 = x^3 + ax + b\}$ then $C \setminus C_0 = [0 : 1 : 0]$.

Problem 3. Let $P_0, \dots P_n \in \mathbb{P}^n_k$ be points such that no three lie on a line, no four lie on a plane, etc. Show that there exists a projective linear transformation $A\colon \mathbb{P}^n_{\mathbb{K}} \to \mathbb{P}^n_{\mathbb{K}}$ such that $A(P_i) = [0 : \cdots : 1 : \cdots : 0]$ (1 is in the i-th spot).

Problem 4.

1. Show by example that Bezout's theorem is false without either projectivity assumption or the assumption that k is algebraically closed. (Keep in mind that you need to produce two examples- one for each hypothesis.)
2. Let $\mathbb{K}$ be a field (not necessarily algebraically closed). Let $C_1 = \{f_1 = 0\}$ and $C_2 = \{f_2 = 0\}$ be two projective curves of degree d_1 and d_2, respectively. Suppose that f_1 and f_2 have no common factors. Show that $\#C_1 \cap C_2 \le d_1 d_2$.
3. What happens if we allow f_1 and f_2 to have a common factor?

Problem 5. Let $v = \sum_{i=0}^{n} X_i \frac{\partial}{\partial X_i}$. A polynomial $f \in \mathbb{K}[X_0, \dots, X_n]$ is homogeneous of degree m if and only if $v(f) = mf$.

Problem 6. Let R be a commutative ring with a unit $1 \in R$. Recall that an ideal $I \subset R$ is said to be radical if $f^n \in I$ then $f \in I$. Which of the following ideals in $\mathbb{K}[x, y]$ are radical?

1. $(y, x - y^2)$
2. $(y, y - x^2)$
3. $(x^2 - y^3)$
4. $(x^2 - 2x + 1)$

Problem 7. Show that the space of degree $= d$ projective curves can be naturally identified with $\mathbb{P}^{N(d)}$. Calculate what $N(d)$ is.

Problem 8. Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth cubic and let $K \subset \mathbb{C}$ be a subfield. Suppose that X can be defined by a polynomial with coefficients in K. We define $C(K) = C \cap \mathbb{P}^2_K$, i.e., the points of C with coordinates in K. Show that if $P, Q \in C(K)$, then $P + Q \in C(K)$.