King's College London

University Of London

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

INSERT this paper with your answer scripts **IN THE ENVELOPE PROVIDED**

Candidate No: ..... **Desk No:** .....

BSc and MSci Examination

4CCM111A Calculus I

Specimen exam

Time Allowed: Two Hours

This paper consists of two sections, Section A and Section B.

Section A contributes 75% and section B contributes 25% of the total marks for the paper.

Answer all questions.

NO calculators are permitted.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

TURN OVER WHEN INSTRUCTED

Section A

In section A, each question has *exactly one* correct answer. There are 5 marks for a correct answer, −1 for a wrong answer, 0 for a blank answer.

Indicate your chosen answer to each multiple-choice question with a tick in the appropriate box.

✔	B	C	D	E	F	

If you want to change an answer, please *fill in* the box that contains the unintended tick and write your final answer in the right-hand column:

■	B	C	D	E	F	B

Only answers recorded using the rules above will be counted.

You can use the space provided under each question for rough work, but this will not be marked.

You may use if you wish, without proof, the following standard limits:

$$\lim_{x\to 0}\frac{\sin x}{x}=1, \qquad \lim_{x\to 0}\frac{e^x-1}{x}=1, \quad \text{and} \quad \lim_{x\to 0}\frac{\ln(1+x)}{x}=1.$$

Answers written on this page will NOT be marked

See Next Page

A 1. Consider the function $f : \mathbb{R} \to \mathbb{R}$. Which, if any, of the following definitions of f is an injection:

A. $f(x) = \sqrt{x^2}$ B. $f(x) = x^2$ C. $f(x) = \cos x$

D. $f(x) = \sin x$ E. $f(x) = \tan x$ F. None of these

Answer: A B C D E F

Rough work:

See Next Page

A 2. Consider the function $f : \mathbb{R} \to B$ given by $f(x) = \frac{1}{2}\sin(3x^2)$. For which of the following sets B is the function a surjection?

A. $B = \mathbb{R}$ B. $B = (-1, 1)$ C. $B = [-1, 1]$

D. $B = [-1/2, 1/2]$ E. $B = [-3, 3]$ F. None of these

Answer: A B C D E F

Rough work:

See Next Page

A 3. The function $f : A \to [-1, 1]$ given by $f(x) = \cos(x^2)$ is invertible when

A. $A = [-\sqrt{\pi}, \sqrt{\pi}]$ B. $A = [\sqrt{\pi}, \sqrt{2\pi}]$ C. $A = [0, \pi]$

D. $A = [-\sqrt{3\pi}, \sqrt{\pi}]$ E. $A = [-\pi, \pi]$ F. $A = [-\pi, 3\pi]$

Answer: A B C D E F

Rough work:

See Next Page

A 4. The value of $\tan(-5\pi/6)$ is equal to

A. $\dfrac{1}{3\sqrt{2}}$ B. $\dfrac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}$ C. $\dfrac{1}{2\sqrt{3}}$

D. $\dfrac{1}{\sqrt{3}}$ E. $\dfrac{1}{\sqrt{2}}$ F. None of these

Answer: | A | B | C | D | E | F |

Rough work:

See Next Page

A 5. The sum $\cosh^2(x) + \sinh^2(x)$ is equal to:

A. $\sinh(2x)$ B. $\cosh(x)$ C. $\tanh(x)$

D. $\cosh(2x)$ E. $\operatorname{sech}(x)$ F. None of these

Answer: | A | B | C | D | E | F |

Rough work:

See Next Page

A 6. For $x \in \mathbb{R}$, the function $\operatorname{arcsinh}(x)$ is equal to:

A. $\ln(x + \sqrt{x^2 - 1})$ B. $\ln(x - \sqrt{x^2 + 1})$ C. $\ln(x + \sqrt{x^2 + 1})$

D. $\ln(-x + \sqrt{x^2 - 1})$ E. $\ln(-x - \sqrt{x^2 + 1})$ F. None of these

Answer: | A | B | C | D | E | F |

Rough work:

See Next Page

A 7. The limit $\lim_{x\to\infty}\left[\dfrac{3x^3-2x^2-1}{4x^3+2x+1}\right]$ is equal to:

A. 1 B. $-1/2$ C. 0

D. $\ln(3/4)$ E. $3/4$ F. None of these

Answer: | A | B | C | D | E | F |

Rough work:

See Next Page

A 8. Identify the value of $k \in \mathbb{R}$ that makes the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \dfrac{x^2+2}{x+1} & x \leq 0 \\ \dfrac{\tan(4x)}{kx} & x > 0 \end{cases}$$

continuous at $x = 0$:

A. 1 B. 0 C. limit does not exist

D. 2 E. 4 F. None of these

Answer: A B C D E F

Rough work:

See Next Page

A 9. Let $f : (0, 1] \to [0, \infty)$ be given by $f(x) = \operatorname{arccosh}(1/x)$, then $f'(x)$ is equal to:

A. $\dfrac{1}{\sqrt{x^2 - 1}}$ B. $-\operatorname{arcsinh}(x)$ C. $\dfrac{1}{\sqrt{x^2 + 1}}$

D. $\dfrac{-1}{\sqrt{x^2 - 1}}$ E. $\dfrac{-1}{\sqrt{x^2 + 1}}$ F. None of these

Answer: | A | B | C | D | E | F |

Rough work:

See Next Page

A 10. Let a curve be defined by $x(t) = \cosh(t)$ and $y(t) = \sinh(t)$ for $t \in (-\infty, 0)$. The derivative $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is equal to to:

A. $\dfrac{x}{\sqrt{1+x^2}}$ B. $\dfrac{x}{\sqrt{1-x^2}}$ C. $\dfrac{x}{\sqrt{x^2-1}}$

D. $\dfrac{-x}{\sqrt{x^2-1}}$ E. $\dfrac{\sqrt{x^2-1}}{x}$ F. None of these

Answer: | A | B | C | D | E | F |

Rough work:

See Next Page

A 11. Let a curve be defined as the set of points (x, y) satisfying the relation $\cosh(y/x) - x = 1$. The derivative $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is equal to

A. $\dfrac{x}{y} - \dfrac{x}{\cosh(y/x)}$ B. $\dfrac{y}{x} - \dfrac{x}{\cosh(y/x)}$ C. $\dfrac{x}{y} - \dfrac{x}{\sinh(y/x)}$

D. $\dfrac{y}{x} + \dfrac{x}{\sinh(y/x)}$ E. $\dfrac{1}{\sinh(y/x)}$ F. None of these

Answer: A B C D E F

Rough work:

See Next Page

A 12. The integral $\int_{3/2}^{\sqrt{3}} \frac{1}{\sqrt{3-x^2}} \mathrm{d}x$ is equal to:

A. $\pi/3$ B. $\pi/2$ C. $\pi/6$

D. π E. $\pi/12$ F. None of these

Answer: | A | B | C | D | E | F |

Rough work:

See Next Page

A 13. The volume enclosed by rotating the segment of the curve $y = e^x$ between $x = 0$ and $x = 2$ about the x-axis is equal to:

A. $\frac{\pi}{4}(e-1)$ B. $2\pi e$ C. $\pi/2$

D. $e^2 - 1$ E. $\frac{\pi}{2}(e^4 - 1)$ F. None of these

Answer: | A | B | C | D | E | F |

Rough work:

See Next Page

A 14. The radius of convergence of the series $\sum_{n=0}^{\infty} \left(2^n x^n/(n+1)\right)$ is equal to:

A. $3/2$ B. e C. $1/e$

D. 1 E. $1/2$ F. None of these

Answer: A B C D E F

Rough work:

See Next Page

A 15. The Taylor expansion of $f(x) = x/e^x$ about $x = 0$ is:

A. $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$ B. $x - x^2 + \frac{1}{2}x^3 - \cdots$ C. $x - \frac{1}{2}x^2 + \frac{1}{3!}x^3 - \cdots$

D. $1 - x + \frac{1}{2}x^2 - \cdots$ E. $1 - 2x + \frac{3}{2}x^2 - \cdots$ F. None of these

Answer: | A | B | C | D | E | F |

Rough work:

See Next Page

Section B

In section B, you must provide FULL DERIVATIONS of your solutions to gain marks.

Your answers to each question should be written in the space provided on pages 20–25. If you require extra space to write your answers, please use the space on pages 26–27.

You may use if you wish, without proof, the following standard limits:

$$\lim_{x\to 0}\frac{\sin x}{x}=1, \qquad \lim_{x\to 0}\frac{e^x-1}{x}=1, \qquad \text{and} \qquad \lim_{x\to 0}\frac{\ln(1+x)}{x}=1.$$

See Next Page

B 16. Consider the function

$$f(x) = \frac{4x^2 - 1}{x^3 - x^2 - 2x}.$$

(i) [5 marks] Sketch $f(x)$, marking axis intercepts, asymptotes, and determine the behaviour as $x \to \infty$ and as $x \to -\infty$.

(ii) [5 marks] Say whether the integral

$$\int_0^{1/2} \frac{4x^2 - 1}{x^3 - x^2 - 2x} \mathrm{d}x,$$

converges. If it does, compute its value, otherwise explain why it does not converge.

[10 marks]

B 17. Consider the ellipse defined parametrically by

$$\begin{cases} x(\theta) = a\sin(\theta) \\ y(\theta) = b\cos(\theta) \end{cases} \qquad \text{for } \theta \in [0, 2\pi],$$

where $a, b > 0$ are constants.

i) [4 marks] Show that the square of the slope of the curve is given by

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{b^2}{a^2}\frac{x^2}{a^2 - x^2}.$$

From now on, set $a = 1$ and $b = 2$.

ii) [2 marks] Let $L(t)$ be the length of the segment of the ellipse parametrised by $x \in [0, t]$, for $-1 \leq t \leq 1$. Show that

$$L(t) = \int_0^t \sqrt{\frac{1 + 3x^2}{1 - x^2}} \mathrm{d}x.$$

iii) [5 marks] Explain why

$$\frac{\mathrm{d}}{\mathrm{d}t} L(t) = \sqrt{\frac{1 + 3t^2}{1 - t^2}}$$

and compute $\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} L(t)$ and $\dfrac{\mathrm{d}^3}{\mathrm{d}t^3} L(t)$.

iv) [4 marks] Hence write down the Taylor expansion of $L(t)$ near $t = 0$ up to terms involving t^3. Use your expression to approximate the value of $L(2/5)$.

[15 marks]

See Next Page

Write your solutions to questions in section B here.
In section B, you must provide FULL DERIVATIONS of your solutions to gain marks.
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See Next Page

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See Next Page

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See Next Page

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