Exercises to Section 2

Exercises in red are from the list of the typical exercises for the exam.
Exercises marked with a star * are for submission to your tutor.

Continuity

1. A factory produces square metal plates with the side length L_0 =10cm. What is the maximum accepted deviation δ of the side length from L_0, if the area of the plate must be within the bounds 100cm$^2 \pm \varepsilon$, where the tolerance ε for the area is (a) $\varepsilon = 1$cm^2; (b) $\varepsilon = 0.1$cm^2; (c) $\varepsilon = 0.01$cm^2?

2. Let $f(x) = 1/x$ for $x > 0$, let $\varepsilon = 0.001$ and let (a) $x_0 = 0.1$; (b) $x_0 = 0.01$; (c) $x_0 = 0.001$. Find the maximal possible number $\delta = \delta(\varepsilon, x_0)$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.

3. Using the trigonometric identity
$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$
and the inequality $|\sin x| \leqslant |x|$, prove the continuity of the function $f(x) = \sin x$. Follow the "ε-δ definition".

4. For what value of the parameter a (if any) are the following functions continuous?
   (a) $f(x) = \frac{\sin x}{|x|}$ if $x \neq 0$ and $f(0) = a$;
   (b) $f(x) = \sin x \sin(1/x)$ if $x \neq 0$ and $f(0) = a$;
   (c) $f(x) = e^{-\frac{1}{x}}$ of $x \neq 0$ and $f(0) = a$;
   (d) $f(x) = x\log(x^2)$ if $x \neq 0$ and $f(0) = a$;
   (e) $f(x) = x^x$ if $x > 0$ and $f(0) = a$.

Types of discontinuity

5. For each of the following functions, find the points of discontinuity and determine their nature (i.e. removable, jump, infinite or oscillatory).
   (a) $f(x) = \dfrac{x}{(1+x)^2}$;
   (b) $f(x) = \dfrac{1+x}{1+x^3}$;
   (c)* $f(x) = \dfrac{\frac{1}{x} - \frac{1}{x+1}}{\frac{1}{x-1} - \frac{1}{x}}$;
   (d) $f(x) = \operatorname{sign}(\sin(\pi/x))$;
   (e) $f(x) = \tan^{-1}(1/x)$;
   (f) $f(x) = \sqrt{\dfrac{1-\cos(\pi x)}{4-x^2}}$.

The algebra of continuous functions

6. Determine whether $f \circ g$ and $g \circ f$ are continuous on $\mathbb{R}$, where

   (a) $f(x) = \operatorname{sign}(x)$ and $g(x) = 1 + x^2$

   (b)* $f(x) = \operatorname{sign}(x)$ and $g(x) = x(1 - x^2)$

   (c) $f(x) = \operatorname{sign}(x)$ and $g(x) = 1 + x - \lfloor x \rfloor$

7. For each of the following statements, determine whether they are true or false. If they are true, give a brief argument to support your claim. If they are false, give a counterexample.

   (a) If f is continuous and g is discontinuous at x_0, then $f(x) + g(x)$ is discontinuous at x_0.

   (b) If both f and g are discontinuous at x_0, then $f(x) + g(x)$ is discontinuous at x_0.

   (c)* If f is continuous and g is discontinuous at x_0, then $f(x)g(x)$ is discontinuous at x_0.

   (d) If both f and g are discontinuous at x_0, then $f(x)g(x)$ is discontinuous at x_0.

   (e) If f has a jump discontinuity at x_0, and g is continuous on $\mathbb{R}$, then $g \circ f$ has a jump discontinuity at x_0.

Challenging exercises

8. Prove that if the functions f and g are continuous on (a, b), then the functions

$$\varphi(x) = \min\{f(x), g(x)\} \quad \text{and} \quad \psi(x) = \max\{f(x), g(x)\}$$

   are also continuous on (a, b).

   Hint: first prove that if $|u - u'| < \varepsilon$ and $|v - v'| < \varepsilon$, then

$$|\min\{u, v\} - \min\{u', v'\}| < \varepsilon,$$
$$|\max\{u, v\} - \max\{u', v'\}| < \varepsilon.$$

9. Complete the proof of the theorem from the lecture notes *All discontinuities of a monotonic function are jump discontinuities.* Proceed as follows. Let f be a non-decreasing function and let x_0 be a point in the domain of f. The set $f((-\infty, x_0))$ is bounded above, and let M be its supremum. Prove that $\lim_{x \to x_0-} f(x) = M$ by using the "$\varepsilon - \delta$ definition" of the limit. Repeat this for the right limit $\lim_{x \to x_0+} f(x)$.

10. Let $E \subset (0, 1)$ be a countable set. Construct a function on $[0, 1]$ which has a discontinuity at every point of E.

11. A function f is called left continuous (resp. right continuous) at x, if $f(x) = \lim_{x' \to x-} f(x)$ (resp. $f(x) = \lim_{x' \to x+} f(x)$). Let f be a bounded function on $[a, b]$. Prove that the functions

$$m(x) = \inf_{a \leqslant \xi < x} f(\xi) \quad \text{and} \quad M(x) = \sup_{a \leqslant \xi < x} f(\xi)$$

    are left continuous on $[a, b]$.