INTRODUCTION TO NUMBER THEORY HW 1

Solve the given problems and show ALL of your work, each answer should be justified by a sound mathematical argument. The ones tagged with $(*)$ should be submitted on Gradescope by 11:59 on October 5, following the link on the KEATs page.

Questions tagged with (*) denote a longer (or harder) exercise for those who are interested — it is not examinable material.

Exercise 1. Calculate $d = \gcd(a, b)$ and find integers u, v such that $d = au + bv$ in the following cases:

(1) $a = 359$, $b = 133$
(2) $a = 1771$, $b = 179$
(3) $a = 2437$, $b = 875$

Exercise 2 $(*)$**.** Let a, b be integers, not both zero.

(1) Let m be a non-zero integer. Show that $\gcd(ma, mb) = |m| \gcd(a, b)$.
(2) Show that $\frac{a}{\gcd(a,b)}$ and $\frac{b}{\gcd(a,b)}$ are coprime integers.

Exercise 3. If a and b are positive integers and $\gcd(a, b) = \mathrm{lcm}(a, b)$, show that $a = b$.

Exercise 4 $(*)$**.** Find all pairs of integers (x, y) which are solutions to the equation

$$1485x + 1745y = 15$$

Exercise 5. Let $a > b > 1$ be integers.

(1) Consider the first two steps of the Euclidean algorithm for computing $\gcd(a, b)$:

$$a = q_1 b + r_1$$
$$b = q_2 r_1 + r_2$$

Show that $r_2 < \frac{b}{2}$.

(2) (*) Let $\lambda(a, b)$ be the number of steps taken by the Euclidean algorithm for computing $\gcd(a, b)$ — more precisely we let $\lambda(a, b) = n$ where r_n is the first zero remainder in the Euclidean algorithm.

Show that $\lambda(a, b) \leq 2\lceil \frac{\log b}{\log 2} \rceil$ (for a real number x the *ceiling* $\lceil x \rceil$ is the smallest integer $\geq x$).

Exercise 6 (*)**.** The *Fibonacci numbers* $f_n = 1, 1, 2, 3, 5, \ldots$ are defined by $f_1 = f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for $n \geq 1$.

(1) What is $\lambda(f_{n+2}, f_{n+1})$? ($\lambda(a, b)$ is defined as in Exercise 5)
(2) Suppose $a > b > 0$ and $\lambda(a, b) = n$. Moreover, suppose that if we have another pair of integers $a' > b' > 0$ and $\lambda(a', b') = n$ then $a' \geq a$ (so a is the *smallest* integer for which the Euclidean algorithm takes n steps to work out its gcd with another integer $b < a$).

Show that a and b are consecutive Fibonacci numbers.