Exercises to Section 8

Exercises in red are from the list of the typical exercises for the exam.
Exercises marked with a star * are for submission to your tutor.

Theory

1. Assume that the integral $\int_{-\infty}^{\infty} f(x)dx$ converges. Prove that

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{b\to\infty}\int_c^b f(x)dx + \lim_{a\to-\infty}\int_a^c f(x)dx$$

for any choice of the point $c \in \mathbb{R}$.

2. Prove the "only if" part of the Lemma in the section on the absolute and conditional convergence of integrals: for a bounded f on $[a,\infty)$, if the limit $\lim\limits_{x\to\infty} f(x)$ exists, then for any $\varepsilon > 0$ there exists $R > 0$ such that for any $x_1 \geqslant R$ and $x_2 \geqslant R$ we have $|f(x_1) - f(x_2)| < \varepsilon$.

3. Prove the integral convergence test for series (see lecture notes). Proceed as follows.

   (a) Check that for any $k \in \mathbb{N}$,

$$f(k+1) \leqslant \int_k^{k+1} f(t)dt \leqslant f(k).$$

   (b) Denote

$$A_n = \sum_{k=1}^{n} f(k), \quad B_n = \int_0^n f(t)dt, \quad n \in \mathbb{N}.$$

   Using the previous step, prove that A_n is a Cauchy sequence if and only if B_n is a Cauchy sequence.

   (c) Show that B_n converges as $n \to \infty$ if and only if $\int_0^x f(t)dt$ converges as $x \to \infty$. Conclude the proof.

Improper integrals

4. Determine whether the following improper integrals converge:

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

   (b) $\displaystyle\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$

   (c) $\displaystyle\int_{0}^{\infty} \frac{x^2+1}{x^4+1}dx$

   (d)* $\displaystyle\int_{0}^{\infty} \frac{x\log x}{(1+x^2)^2}dx$

(e) $\int_0^2 \frac{dx}{\log x}$

(f) $\int_0^\infty x^{p-1} e^{-x} dx$, $p \in \mathbb{R}$

(g) $\int_0^\infty \frac{\log(1+x)}{x^n} dx$, $n \in \mathbb{N}$

(h) $\int_0^1 \frac{x^n dx}{\sqrt{1-x^4}}$, $n \in \mathbb{N}$

(i) $\int_0^{\pi/2} \frac{\log(\sin x)}{\sqrt{x}} dx$

(j) $\int_0^\infty \frac{(\sin x)^2}{x} dx$

5. Using the integral test, determine whether the following series converge:

(a) $\sum_{n=2}^\infty \frac{1}{n(\log n)^p}$, $p > 0$

(b) $\sum_{n=3}^\infty \frac{1}{n(\log n)(\log\log n)^p}$, $p > 0$

Absolute and conditional convergence

6. Do the following improper integrals converge absolutely? Conditionally? You can use Theorem 8.3 from the lecture notes.

(a) $\int_0^\infty \frac{\sin x}{\sqrt[4]{x+1}} dx$

(b) $\int_0^\infty \frac{\sqrt{x}\cos x}{x+100} dx$

(c)* $\int_{-\infty}^\infty \frac{\sin x \tan^{-1} x}{\sqrt{x^4+1}} dx$

Challenging exercises

7. Using integration by parts, prove Theorem 8.3 from the lecture notes. The statement is repeated below.

Theorem. *Let f and g be continuous functions on $[a, \infty)$, such that (i) the integral*

$$\int_a^x f(t)dt$$

is bounded and (ii) $g(x)$ is continuously differentiable, goes to zero as $x \to \infty$ and is monotone. Then the integral

$$\int_a^\infty f(x)g(x)dx$$

converges.

8. Do the following improper integrals converge absolutely? Conditionally?

   (a) $\int_0^\infty \sin x^2 dx$ *Hint:* change of variable

   (b) $\int_0^\infty x^p \cos x \, dx$, $p < 0$.

9. Let $n < m$ be natural numbers, and let $f \in C^1[n, m]$. Prove the first *Euler-Maclaurin formula*

$$\sum_{i=n}^{m} f(i) - \int_n^m f(x)dx = \frac{1}{2}(f(n) + f(m)) + \int_n^m f'(x)(x - \lfloor x \rfloor - \tfrac{1}{2})dx.$$

10. In the Euler-Maclaurin formula, take $f(i) = i^{-s}$, $s > 1$, $n = 1$ and let $m \to \infty$. Conclude that

$$\zeta(s) = \frac{1}{s-1} + O(1), \quad s \to 1_+,$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function.