Linear Algebra and Geometry I 2023 — Problem Sheet 10

You are encouraged to work with other students on the module. If you are having difficulty with any of the questions or want feedback on specific answers you should ask your tutor in your next tutorial or attend the lecturer's office hours.

1. (a) Compute the determinant of each of the following matrices by using a cofactor expansion along an appropriate row or column. Hence determine which of these matrices are invertible.

(i) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & -5 \\ 6 & -4 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & -1 \end{pmatrix}$ (iii) $\begin{pmatrix} 4 & 2 & 0 & 2 \\ -6 & 1 & -3 & 2 \\ 4 & 0 & 1 & -3 \\ 4 & 0 & 3 & -5 \end{pmatrix}$

(b) Use row operations to simplify then compute the determinants of the following matrices.

(i) $\begin{pmatrix} 1 & -4 & 3 & 2 \\ 2 & -7 & 5 & 1 \\ 1 & 2 & 6 & 0 \\ 2 & -10 & 14 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{pmatrix}$

2. Let A be a 3×3 matrix such that $\det(A) = 5$. Compute the determinants of the following matrices:

(a) $3A$ (b) A^2 (c) $2A^{-1}$ (d) $A^{-1}A^T$

3. Let A be the matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

How are the determinants of A and B related in the following cases:

(a) $B = \begin{pmatrix} 2a_1 & 3a_2 & 5a_3 \\ 2b_1 & 3b_2 & 5b_3 \\ 2c_1 & 3c_2 & 5c_3 \end{pmatrix}$, (b) $B = \begin{pmatrix} 3a_1 & 4a_2 + 5a_1 & 5a_3 \\ 3b_1 & 4b_2 + 5b_1 & 5b_3 \\ 3c_1 & 4c_2 + 5c_1 & 5c_3 \end{pmatrix}$.

4. Determine whether each of the following statements is true or false. In each case you should either give a short proof or a counterexample to justify your answer.

(a) The determinant of $I + A$ is $1 + \det(A)$.

(b) The determinant of ABC is $\det(A)\det(B)\det(C)$.

(c) The determinant of $4A$ is $4\det(A)$.

(d) The determinant of $AB - BA$ is zero.

5. Let J_n be the $n \times n$ "reverse identity" matrix:

$$J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \ldots$$

(a) Compute the determinants of J_2, J_3, J_4 and J_5.

(b) Show that if either n or $n-1$ is divisible by 4 then $\det(J_n) = +1$, otherwise $\det(J_n) = -1$.
(Hint: think about what row exchanges you need to transform J_n to the identity matrix.)

6. Let A be an $n \times n$ matrix.

(a) A is called *nilpotent* if $A^k = \mathbf{0}$ for some positive integer k. Show that if A is nilpotent then $\det(A) = 0$.

(b) A is called *orthogonal* if A is real and $A^T A = I$. Show that if A is orthogonal then $\det(A) = \pm 1$.

(c) A is called *skew-symmetric* if $A^T = -A$. Show that if A is skew-symmetric and n is odd then $\det(A) = 0$. Is this still true for n even?