Linear Algebra and Geometry I 2023 **Problem Sheet 2**

You are encouraged to work with other students on the module. If you are having difficulty with any of the questions or want feedback on specific answers you should ask your tutor in your next tutorial or attend the lecturer's office hours.

1. (a) Find the modulus and principal argument of the following complex numbers.

   (i) $e^{3\pi i}$ (ii) $2e^{i9\pi/4}$ (iii) $e^{2+\pi i/2}$

   (b) Write each of the following complex numbers in polar form $re^{i\theta}$, with $r > 0$ and $-\pi < \theta \leq \pi$.

   (i) $15i$ (ii) $1 - i$ (iii) $-1 - i\sqrt{3}$

2. (a) Write down all sixth roots of unity in the form $a + ib$.

   (b) Show that if $|z| = 1$ and $\mathrm{Re}\,(z) = -1/2$ then $z^3 = 1$.

3. Find all solutions $z \in \mathbb{C}$ of the following equations and sketch them on the complex plane.

   (a) $2z^2 - 4z + 12 = 0$ (b) $(z - i)^6 = 64$ (c) $z/\overline{z} = i$

4. (a) Let $z = \pi/6 - i\log 2$ ($\log 2 = \ln 2$ is the natural logarithm of 2). Write e^{iz} in the form $a + ib$, where $a, b \in \mathbb{R}$.

   (b) Let a and b be real numbers. Find the real and imaginary parts of the following complex numbers:

   (i) $\left(e^{a+ib}\right)^2$ (ii) $\dfrac{e^a e^{ib}}{a + ib}$ (iii) $e^{e^{ia}}$

5. Use De Moivre's theorem to prove the *triple angle identities*:

$$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta);$$
$$\sin(3\theta) = -4\sin^3(\theta) + 3\sin(\theta).$$

6. Let $n \geq 2$ be an integer. We say that $\omega \in \mathbb{C}$ is a *primitive* n^{th} root of unity if $\omega^n = 1$ and $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are all distinct (so that these are the complete set of n^{th} roots of unity).

(a) Write down the primitive n^{th} roots of unity for $n = 4, 5, 6$. You can write them in any form you wish, polar form is probably easiest.

(b) Let $\omega = e^{2\pi i k/n}$ with $0 \leq k < n$. Prove that if ω is primitive, then the greatest common divisor of k and n is 1.

Hint: If d is a common divisor of k and n, what is $\omega^{n/d}$?

(c) Hence show that if every n^{th} root of unity except 1 is primitive, then n is prime.

Make sure you write in sentences, with punctuation, and you explain your reasoning at each step. Look at the proofs in the notes for an idea of how you should try to write.