Exercises to Section 3

Exercises in red are from the list of the typical exercises for the exam.
Exercises marked with a star * are for submission to your tutor.

Functions on closed and bounded intervals

1. Let $f \in C[a,b]$; assume that $f(x) \neq 0$ for all $x \in [a,b]$. Deduce that $1/f(x)$ is bounded on $[a,b]$. Show that without the assumption of continuity of f the conclusion is in general false.

2. Use the Intermediate Value Theorem to prove that a solution to the following equations exists on the given interval:

   (a) $e^{-x} + x^3 = 0$ on $[-2, 0]$

   (b) $e^{-x^2} = x$ on $[0, 1]$

3.* Let n be an odd natural number and let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a monic polynomial with real coefficients. Use the intermediate value theorem to prove that $P(x)$ has at least one root on the real line.

4. Let n be an even natural number and let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a monic polynomial with real coefficients such that $a_0 < 0$. Use the intermediate value theorem to prove that $P(x)$ has at least two roots on the real line.

5. Use the Intermediate Value Theorem to prove that a solution to the following equations exists on the given interval:

   (a) $x^3 - \sqrt{x} = 10$ on $[0, \infty)$

   (b) $P(x) = \sin x$ on $\mathbb{R}$, where P is a polynomial of odd degree with real coefficients.

6. Let $f : [0,1] \to [0,1]$ be a continuous function. Prove that there exists $c \in [0,1]$ such that $f(c) = c$ (i.e. c is a *fixed point* of f).

   Hint: consider the function $g(x) = f(x) - x$.

Uniform continuity

7. Which of the following functions are uniformly continuous on the given interval? Justify your answer.

   (a) $f(x) = e^x$ on $(-\infty, 0)$; on $(0, \infty)$

   (b) $f(x) = x^2$ on $(0, 1)$; on $(1, \infty)$

   (c) $f(x) = \sin \pi x^2$ on $\mathbb{R}$

   (d) $f(x) = x + \sin x$ on $\mathbb{R}$

   (e)* $f(x) = \log x$ on $(0, \infty)$

   (f)* $f(x) = \frac{\sin x}{x}$ on $(0, 1)$

   (g) $f(x) = \frac{x}{4-x^2}$ on $(-1, 1)$

   (h) $f(x) = x \sin x$ on $\mathbb{R}$

8. Let f be a uniformly continuous function on a bounded (not necessarily closed) set Δ. Prove that f is bounded.

   Hint: take $\varepsilon = 1$ and cover Δ by intervals of length $< \delta/2$.

   Challenging exercises

9. Prove that if f is continuous on $[0, \infty)$ and a (finite) limit $A = \lim_{x\to\infty} f(x)$ exists, then f is uniformly continuous on $[0, \infty)$.

10. Construct a uniformly continuous function f on $[0, 1]$ such that the derivative of f is unbounded.

11. Let f be a uniformly continuous function on an interval (a, b). Prove that the limits $\lim_{x\to a_+} f(x)$ and $\lim_{x\to b_-} f(x)$ exist and thus f extends to a continuous function on $[a, b]$. Proceed as follows.

    (a) By taking $\varepsilon = 1$ in the definition of uniform continuity, prove that f is bounded.

    (b) Let $x_n \to a_+$ as $n \to \infty$. By using the definition of uniform continuity, prove that $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence. Deduce that it converges, denote the limit by A.

    (c) Let $x'_n \to a_+$, $n \to \infty$, be another sequence. By using the definition of uniform continuity, prove that $\lim_{n\to\infty} f(x'_n) = A$.

    (d) Conclude that f extends to $[a, b)$ as a continuous function.

    (e) Consider the point b in the same fashion.

12. Let $E \subset \mathbb{R}$ be any set; define the distance from $x \in \mathbb{R}$ to E by

    $$d_E(x) = \inf_{z\in E} |x - z|.$$

    Prove that d_E is Lipschitz continuous on $\mathbb{R}$, with the Lipschitz constant 1.

    Hint: we have $d_E(x) \leqslant |x - z| \leqslant |x - y| + |y - z|$.

13. Let $D \subset \mathbb{C}$ be a set in the complex plane such that the conclusion of the Bolzano-Weierstrass theorem for D holds true; i.e. assume that for any sequence of points $\{x_n\}_{n=1}^{\infty}$ there is a convergent subsequence with a limit in D. (Such sets are called *compact* and can be identified with closed and bounded subsets of the complex plane, but this is another story; you will learn about such sets in the *Metric Spaces and Topology* course.) Let $f : D \to \mathbb{R}$ be a continuous function. By mimicking the proofs of the relevant statements from the lecture notes, prove that:

    (a) f is bounded on D;

    (b) f attains its maximal and minimal values on D;

    (c) f is uniformly continuous on D.