Linear Algebra and Geometry I 2023 — Problem Sheet 8

You are encouraged to work with other students on the module. If you are having difficulty with any of the questions or want feedback on specific answers you should ask your tutor in your next tutorial or attend the lecturer's office hours.

1. Determine whether each of the following matrices is invertible. If it is, compute the inverse.

(a) $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 7 & 3 \\ 2 & 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 2 & 4 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \\ 6 & 1 & 5 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

2. Consider the matrix
$$A = \begin{pmatrix} \alpha & \beta & \beta \\ \alpha & \alpha & \beta \\ \alpha & \alpha & \alpha \end{pmatrix}.$$
Show that if $\alpha \neq 0$ and $\alpha \neq \beta$ then A is invertible.

3. Let A be the matrix
$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{pmatrix}.$$
   (a) Find three elementary matrices E_1, E_2, E_3 such that $A = E_1 E_2 E_3$.
   (b) Determine the inverse of A (using any method you wish).

4. Let A be a 3×3 matrix. Let $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ denote the first, second and third row of A respectively. Suppose that $\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{r}_3$.
   (a) Explain why the equation $A\mathbf{x} = (1, 0, 0)$ has no solution.
   (b) What condition on $\mathbf{b} = (b_1, b_2, b_3)$ is necessary in order that $A\mathbf{x} = \mathbf{b}$ has a solution?
   (c) Is this condition also sufficient?
   (d) Is it possible that $A\mathbf{x} = \mathbf{b}$ has a unique solution? Justify your answer.

5. In each of the following cases, determine whether there exists a matrix A with the given property. Either construct an example of such a matrix or explain why no such matrix exists.

   (a) The only solution of $A\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

   (b) The only solution of $A\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

6. *Read through Section 4.3.1 in the lecture notes before attempting this question*

   For each of the linear ODEs below do the following:

   (i) Find a particular solution. *Look for a solution of the form suggested.*

   (ii) Use the substitution $y(x) = e^{\lambda x}$ to find the general solution of the associated homogeneous equation.

   You may use the fact that every solution of the homogeneous equations is a linear combination of exponential functions (i.e functions of the form $x \mapsto e^{\lambda x}$).

   (iii) Find the general solution.

   (a) $\dfrac{dy}{dx} + 4y = e^{-x}$. *Try* $y(x) = \alpha e^{-x}$.

   (b) $\dfrac{dy}{dx} - 2y = 3\sin 2x$. *Try* $y(x) = \alpha \sin 2x + \beta \cos 2x$.

   (c) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = \cos x$. *Try* $y(x) = \alpha \sin x + \beta \cos x$.

   (d) $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 6x^2 - 10x + 2$. *Try* $y(x) = \alpha x^2 + \beta x + \gamma$.

7. Let A be a square matrix. Prove that the columns of A are linearly independent if and only if the rows of A are linearly independent. (This is "if and only if" so make sure you prove both directions.)

   (Hint: Think about what having linearly independent columns tells you about the matrix A. What does this tell you about A^T ?)