CCM115 (2023-24): Assignment Sheet 1

You are strongly encouraged to try as many of the problems on this sheet as possible (*if you are short of time*, then you may wish to focus on the problems marked with the symbol '†'). A selection of the problems (usually without the symbol †) will also be discussed in your tutorials.

At the beginning of the course, let me make an important general point: you may find that some of the problems on this, and later, assignment sheets require you to think hard and you do not immediately know how to proceed. However, the key thing is that you keep trying to solve such problems since this is a really essential part of the process of properly learning mathematics!

Set Theory

1. List all elements of the set $\{x \in \mathbb{N} \mid -3 < x \leq 5\}$.

2. Give a more direct description of the following sets:

   (a)† $(-1, 1) \cap [-1, 2)$;
   (b) $(-\infty, 0) \cap (0, \infty)$;
   (c)† $(-\infty, 0] \cap [0, \infty)$;
   (d) $[1, \infty) \setminus (1, 2)$;
   (e)† $[-1, 1] \setminus (-1, 1)$;
   (f) $(-\infty, 0) \cup (-1, \infty)$;
   (g)† $(-\infty, 1) \cap (-3, \infty) \cap [0, 5]$.

3. Give a more direct description of the following sets:

   (a)† $A = \{x \in \mathbb{R} \mid x^2 - 8x + 15 = 0\}$
   (b)† $B = \{x \in \mathbb{N} \mid -11 < x \leq -7\}$
   (c)† $C = \{x \in \mathbb{Z} \mid -11 < x \leq -7\}$

4. Give a more direct description of the following sets:

   (a)† $\bigcap_{j=1}^{\infty} [j, \infty)$
   (b) $\bigcup_{j=1}^{\infty} [j, \infty)$
   (c)† $\bigcup_{j=1}^{\infty} (0, 1/j)$
   (d) $\bigcup_{n=1}^{\infty} [1 + \frac{1}{n}, 1 + n]$
   (e)† $\bigcap_{n=1}^{\infty} [1 + \frac{1}{n}, 1 + n]$

Logic

5. Mark each of the following statements true or false:

   (a)† $\forall k \in \mathbb{N}$, $\exists n \in \mathbb{N}$ such that $k^2 = n$

   (b) $\forall k \in \mathbb{Z}$, $\exists n \in \mathbb{N}$ such that $k^2 = n$

   (c)† $\forall n \in \mathbb{Z}$, $\exists k \in \mathbb{Z}$ such that $k^2 = n$

   (d) $\forall n \in \mathbb{Z}$, $\exists k \in \mathbb{Z}$ such that $n^2 = k$

   (e)† $\exists a \in \mathbb{R}$ such that $a^2 = 3$

   (f) $\exists a \in \mathbb{Z}$ such that $a^2 = 3$

   (g)† $\forall \varepsilon > 0$ $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$

   (h) $\exists \varepsilon > 0$ such that $\forall n \in \mathbb{N}$ one has $\frac{1}{n} < \varepsilon$

   (i)† $\forall \varepsilon > 0$ $\exists N \in \mathbb{R}$ such that $\forall n \geq N$ one has $\frac{1}{n} < \varepsilon$

6. Negate the following propositions. You **don't** need to decide whether they are true or false.

   (a) x is rational and greater than 1.

   (b)† $0 < x \leq 1$.

   (c) There is a real number whose square is negative

   (d)† $\exists x \in \mathbb{R}$ such that $\forall n \in \mathbb{Z}$ one has $n \leq x$

   (e)† $\forall a \in \mathbb{N}$ one has $a > 0$

   (f) $\forall a \in \mathbb{N}$ $\exists x \in \mathbb{R}$ such that $a = x^2$

   (g)† $\forall \varepsilon > 0$ $\exists N \in \mathbb{R}$ such that $\forall n \geq N$ one has $\frac{1}{n} < \varepsilon$.

   (h) $\forall a > 0$ one has $(a \geq 1)$ OR $(1/a \geq 1)$.

   (i)† $\forall n \in \mathbb{N}$ $\exists k \in \mathbb{Z}$ such that (k^n is even) AND ($(k+1)^n$ is odd).

7. For each of the following implications, give the converse and the contrapositive. Decide whether the original implication is true and whether the converse and the contrapositive is true.

   (a)† $\forall n \in \mathbb{N}$: n is odd $\Rightarrow$ n^2 is odd.

   (b) $\forall x \in \mathbb{R}$: $(x > 0$ and $x < 10) \Rightarrow x^2 < 100$.

   (c)† $\forall n \in \mathbb{Z}$: $n^2 = 100 \Rightarrow (n = 10$ or $n = -10)$.