Classical Dynamics

5ccm231

Paul P. Cook* and Neil Lambert†
revised by Chiara Cammarota‡ and Christopher P. Herzog§

Department of Mathematics, King's College London
The Strand, London WC2R 2LS, UK

N.B This module was previously called Intermediate Dynamics but the content is the same

N.B. These notes will be updated as the module progresses.

*email: paul.cook@kcl.ac.uk
†email: neil.lambert@kcl.ac.uk
‡email: chiara.cammarota@kcl.ac.uk
§email: christopher.herzog@kcl.ac.uk

Contents

Chapter 1

Review

1.1 Scalars, Vectors and all That

This chapter is meant to be a review of basic concepts in the geometry of three-dimensional Euclidean space, $\mathbb{R}^3$. Therefore we will be somewhat brief and not as precise as a mathematician should be.

The simplest thing one can imagine is a scalar. Simply put a scalar is a single number, the value of which everyone agrees on. One example is the temperature of a given point in this room, *e.g.*

$$T = 20^\circ C \ . \tag{1.1}$$

(It is a different question whether or not you think that is warm or cold.)

Vectors are more interesting physical quantities. In three dimensions they are given by a triplet of numbers. For example the position of a point in this room is given by a vector:

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ . \tag{1.2}$$

We denote vectors by an underline. The three numerical values on the right hand side give the coordinate values of the point in some coordinate system. Why is this so different from three scalars? Because people using different coordinate systems will not use the same values of x, y and z to describe the same point.

We will often use an index notation r^a, $a = 1, 2, 3$ for the components of a vector $\underline{r}$. In particular

$$r^1 = x \qquad r^2 = y \qquad r^3 = z \ . \tag{1.3}$$

Please note that r^2 in this case does not mean r-squared. There is no meaning to the square of a vector (although below we will consider the length-squared of a vector which will be denoted by $|\underline{r}|^2$).

Vectors live in a vector space. A vector space is equipped with the following two actions that map vectors to vectors:

- multiplication by a scalar a:

$$a\underline{r} = \begin{pmatrix} ax \\ ay \\ az \end{pmatrix} \ . \tag{1.4}$$

- addition of two vectors:

$$\underline{r}_1 + \underline{r}_2 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} . \tag{1.5}$$

Note that the subscript on $\underline{r}_i$ does not denote the components r^a!

For this course three-dimensional space is a vector space: $\mathbb{R}^3$. The three means that it is 3-dimensional. This in turn means that one can pick a basis of three vectors, $\underline{e}_1, \underline{e}_2, \underline{e}_3$ so that any other vector can be written uniquely in terms of these three:

$$\underline{r} = a\underline{e}_1 + b\underline{e}_2 + c\underline{e}_3 . \tag{1.6}$$

Of course for $\mathbb{R}^3$ the most natural choice is

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad \underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} , \tag{1.7}$$

so that, quite simply,

$$\underline{r} = x\underline{e}_1 + y\underline{e}_2 + z\underline{e}_3 . \tag{1.8}$$

But there are many different choices of basis.

More generally a basis for an n-dimensional vector space is a choice of n vectors $\underline{e}_1, ..., \underline{e}_n$ that are linearly independent. This means that the equation

$$a_1\underline{e}_1 + ... + a_n\underline{e}_n = \underline{0} \tag{1.9}$$

only has the trivial solution $a_1 = ... = a_n = 0$.

The **Scalar Product** (*a.k.a.* dot product and inner product) is a map that takes two vectors into a number and has the following properties:

- symmetric: $\underline{r}_1 \cdot \underline{r}_2 = \underline{r}_2 \cdot \underline{r}_1$

- distributive: $\underline{r}_1 \cdot (\underline{r}_2 + \underline{r}_3) = \underline{r}_1 \cdot \underline{r}_2 + \underline{r}_1 \cdot \underline{r}_3$

Sometimes a third property is useful (but not always[1]):

- positive definite: $\underline{r} \cdot \underline{r} \geq 0$ with equality iff $\underline{r} = \underline{0}$.

If this last property holds true, then we can define the length of a vector to be

$$|\underline{r}| = \sqrt{\underline{r} \cdot \underline{r}} , \tag{1.10}$$

which allows us to do geometry.

Although one can consider more general possibilities we will simply take

$$\underline{r}_1 \cdot \underline{r}_2 = \sum_{a=1}^{3} r_1^a r_2^a = x_1 x_2 + y_1 y_2 + z_1 z_2 . \tag{1.11}$$

[1] For example, in special relativity the dot product $(t\underline{e}_0 + x\underline{e}_1 + y\underline{e}_2 + z\underline{e}_3) \cdot (t'\underline{e}_0 + x'\underline{e}_1 + y'\underline{e}_2 + z'\underline{e}_3) = -tt' + xx' + yy' + zz'$ plays a very special role.

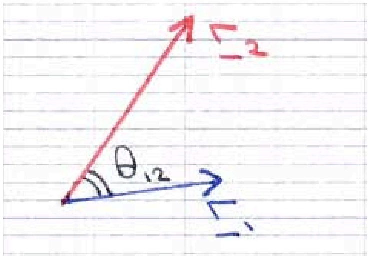

Figure 1.1: Vectors

You can check yourself that this satisfies the symmetry, distributive and positive definite properties above. In this case, the length of a vector is

$$|\underline{r}| = \sqrt{x^2 + y^2 + z^2} \ , \tag{1.12}$$

which is, of course, just the Pythagorian theorem (in 3D). For example

$$\text{if} \quad \underline{r}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \underline{r}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \qquad \text{then} \quad \underline{r}_1 \cdot \underline{r}_2 = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32 \ . \tag{1.13}$$

The interpretation of the scalar product is

$$\underline{r}_1 \cdot \underline{r}_2 = |\underline{r}_1||\underline{r}_2| \cos\theta_{12} \tag{1.14}$$

where θ_{12} is the angle between the two vectors in the plane through the origin defined by $\underline{r}_1$ and $\underline{r}_2$. This is easily seen in two dimensions. Let

$$\underline{r}_1 = \begin{pmatrix} r_1 \cos\theta_1 \\ r_1 \sin\theta_1 \end{pmatrix} \qquad \underline{r}_2 = \begin{pmatrix} r_2 \cos\theta_2 \\ r_2 \sin\theta_2 \end{pmatrix} \ . \tag{1.15}$$

Then it is easy to see that $|\underline{r}_1| = r_1$ and $|\underline{r}_2| = r_2$ and also

$$\underline{r}_1 \cdot \underline{r}_2 = r_1 r_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) = r_1 r_2 \cos(\theta_1 - \theta_2) \tag{1.16}$$

(do you remember your trig identities?!) and so indeed we find $\theta_{12} = \theta_1 - \theta_2$. In three dimensions one can simply rotate the basis until both vectors take the form

$$\underline{r}_1 = \begin{pmatrix} r_1 \cos\theta_1 \\ r_1 \sin\theta_1 \\ 0 \end{pmatrix} \qquad \underline{r}_2 = \begin{pmatrix} r_2 \cos\theta_2 \\ r_2 \sin\theta_2 \\ 0 \end{pmatrix} \ , \tag{1.17}$$

and the result follows again.

Thus, not surprisingly, the basis chosen in (1.7) satisfies:

$$\underline{e}_a \cdot \underline{e}_b = \delta_{ab} = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases} \tag{1.18}$$

i.e. the basis elements are all unit length and orthogonal to each other. Such a basis is called **orthonormal**.

1.2 Vector Product

It is easy to see that the scalar product can be extended to any dimension; we've already used it in two and three dimensions. However in three dimensions there is another product as we will now see.

In three dimensions any two, non-parallel vectors $\underline{r}_1$ and $\underline{r}_2$ define a plane through the origin. A plane in three-dimensions has a **normal vector**, *i.e.* a vector which is orthogonal to every vector in the plane. This allows us to define the vector product which takes two vectors and gives a third that is orthogonal to the original two. Explicitly we define

$$(\underline{v} \times \underline{w})^a = \sum_{bc=1}^{3} \epsilon_{abc} v^b w^c \ . \tag{1.19}$$

Here ϵ_{abc} has the following properties:

$$\epsilon_{abc} = \begin{cases} +1 & \text{if}(a,b,c) = (1,2,3),(2,3,1),(3,1,2) \\ -1 & \text{if}(a,b,c) = (3,2,1),(2,1,3),(1,3,2) \\ 0 & \text{otherwise} \end{cases} \tag{1.20}$$

What does this mean? It means that ϵ_{abc} vanishes unless a, b, c are all distinct. In that case $\epsilon_{abc} = +1$ if (a,b,c) is in 'clockwise'[2] (or cyclic) order or $\epsilon_{abc} = -1$ if (a,b,c) is in 'anti-clockwise' (or anti-cyclic) order.

If this seems tricky then it may be easier to see what is going on by writing out the components:

$$\begin{aligned} (\underline{v} \times \underline{w})^1 &= v^2 w^3 - v^3 w^2 \ , \\ (\underline{v} \times \underline{w})^2 &= v^3 w^1 - v^1 w^3 \ , \\ (\underline{v} \times \underline{w})^3 &= v^1 w^2 - v^2 w^1 \ . \end{aligned} \tag{1.21}$$

Notice again the the sign is determined by whether or not 1,2,3 appear in clockwise or anti-clockwise order.

There are two fundamental properties of the vector product:

- anti-symmetry: $(\underline{v} \times \underline{w}) = -(\underline{w} \times \underline{v})$
- orthogonality: $\underline{v} \cdot (\underline{v} \times \underline{w}) = \underline{w} \cdot (\underline{v} \times \underline{w}) = 0$

Let us check these. In fact anti-symmetry is clear from the definition. Swapping $v^a \leftrightarrow w^a$ in (1.19) or (1.21) changes the overall sign of the right hand sides.

Let us check that the vector product $(\underline{v} \times \underline{w})$ is indeed orthogonal to both $\underline{v}$ and $\underline{w}$. First the fast way:

$$\underline{v} \cdot (\underline{v} \times \underline{w}) = \sum_a v^a (\underline{v} \times \underline{w})^a = \sum_{abc=1}^{3} \epsilon_{abc} v^a v^b w^c = 0 \ . \tag{1.22}$$

Why? because the sum involves $\epsilon_{abc} v^a v^b w^c$ which contains terms of the form

$$\epsilon_{123} v^1 v^2 w^3 + \epsilon_{213} v^2 v^1 w^3 = (+1) v^1 v^2 w^3 + (-1) v^2 v^1 w^3 = 0 \ . \tag{1.23}$$

[2] Imagine a clock face corresponding to a day with just 3 hours.

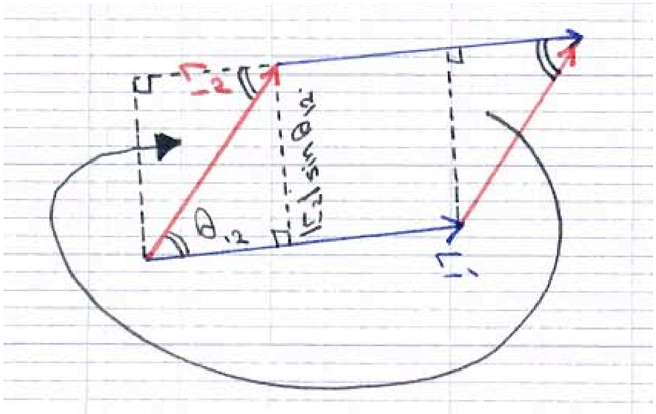

Figure 1.2: Area

Or we can follow the slow way:

$$
\begin{aligned}
\underline{v}\cdot(\underline{v}\times\underline{w}) &= \sum_a v^a(\underline{v}\times\underline{w})^a \\
&= (\epsilon_{123}v^1v^2 + \epsilon_{213}v^2v^1)w^3 + (\epsilon_{312}v^3v^1 + \epsilon_{132}v^1v^3)w^2 + (\epsilon_{231}v^2v^3 + \epsilon_{321}v^3v^2)w^1 \\
&= (v^1v^2 - v^2v^1)w^3 + (v^3v^1 - v^1v^3)w^2 + (v^2v^3 - v^3v^2)w^1 \\
&= 0+0+0 \\
&= 0\ .
\end{aligned}
\tag{1.24}
$$

Lastly we see that

$$
\underline{w}\cdot(\underline{v}\times\underline{w}) = -\underline{w}\cdot(\underline{w}\times\underline{v}) = 0\ , \tag{1.25}
$$

which simply follows from anti-symmetry and switching the names $\underline{v}\leftrightarrow\underline{w}$.

Next we show that

$$
|\underline{r}_1\times\underline{r}_2| = |\underline{r}_1||\underline{r}_2|\sin\theta_{12}\ . \tag{1.26}
$$

You can prove this by writing out all the terms (it helps to observe $|\underline{r}_1|^2|\underline{r}_2|^2\sin^2\theta_{12} = |\underline{r}_1|^2|\underline{r}_2|^2(1-\cos^2\theta_{12}) = |\underline{r}_1|^2|\underline{r}_2|^2 - |\underline{r}_1\cdot\underline{r}_2|^2$). Or we can go back to or choice before (1.17), where we adapted the basis to be convenient for the plane defined by $\underline{r}_1$ and $\underline{r}_2$, Here we see that

$$
\underline{r}_1\times\underline{r}_2 = \begin{pmatrix} r_1\cos\theta_1 \\ r_2\sin\theta_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} r_2\cos\theta_2 \\ r_2\sin\theta_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r_1r_2(\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2) \end{pmatrix} \tag{1.27}
$$

and so

$$
|\underline{r}_1\times\underline{r}_2|^2 = r_1^2r_2^2(\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2)^2 = |\underline{r}_1|^2|\underline{r}_2|^2\sin^2(\theta_1-\theta_2)\ , \tag{1.28}
$$

as promised.

There is also an interpretation of $|\underline{r}_1\times\underline{r}_2|$ as the area of the parallelogram defined by $\underline{r}_1$ and $\underline{r}_2$: From Figure 1.2 one can see that the area consists of two triangles with hight $h_t = |\underline{r}_2|\sin\theta_{12}$ and base $b_t = |\underline{r}_2|\cos\theta_{12}$ as well as rectangle with base $b_r = |\underline{r}_1| - b_t$ and hight $h_r = h_t$. Thus

$$
Area = 2\cdot\frac{1}{2}b_th_t + b_rh_r = h_t(b_t+b_r) = |\underline{r}_1||\underline{r}_2|\sin\theta_{12}\ . \tag{1.29}
$$

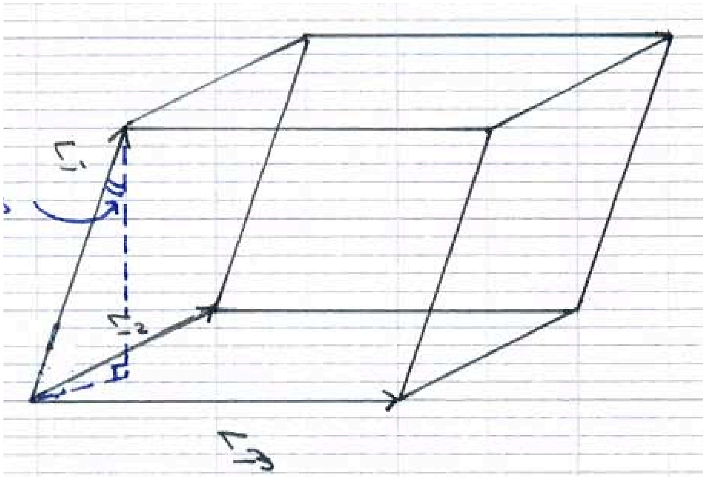

Figure 1.3: Volume

1.3 Triple Products

One can construct a **scalar triple product** of three vectors:

$$\underline{r}_1 \cdot (\underline{r}_2 \times \underline{r}_3) \tag{1.30}$$

which gives a scalar. Geometrically this gives the volume (or more accurately the absolute value gives the volume) of a parallelepiped with sides defined by $\underline{r}_1$, $\underline{r}_2$ and $\underline{r}_3$.

One could also consider the **vector triple product**

$$\underline{r}_1 \times (\underline{r}_2 \times \underline{r}_3) \ . \tag{1.31}$$

But it is not really independent of what we have already constructed. Indeed note that $\underline{r}_2 \times \underline{r}_3$ is orthogonal to both $\underline{r}_2$ and $\underline{r}_3$. Similarly $\underline{r}_1 \times (\underline{r}_2 \times \underline{r}_3)$ is orthogonal to $(\underline{r}_2 \times \underline{r}_3)$, which is orthogonal to the plane defined by $\underline{r}_2$ and $\underline{r}_3$. So it must be that $\underline{r}_1 \times (\underline{r}_2 \times \underline{r}_3)$ lies in the plane defined by $\underline{r}_2$ and $\underline{r}_3$. Therefore we have

$$\underline{r}_1 \times (\underline{r}_2 \times \underline{r}_3) = A\underline{r}_2 + B\underline{r}_3 \ . \tag{1.32}$$

For some scalars A and B.

To compute A and B we can use the definition:

$$\begin{aligned}
(\underline{r}_1 \times (\underline{r}_2 \times \underline{r}_3))^a &= \sum_{b,c} \varepsilon_{abc} r_1^b (\underline{r}_2 \times \underline{r}_3)^c \\
&= \sum_{b,c,d,e} \varepsilon_{abc}\varepsilon_{cde} r_1^b r_2^d r_3^e \ .
\end{aligned} \tag{1.33}$$

Next we observe that

$$\sum_c \varepsilon_{abc}\varepsilon_{cde} = \delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd} \ . \tag{1.34}$$

Why? It's actually just a matter thinking it through: Clearly to be non-zero a must be different from b. Let us fix $a = 1, b = 2$. Then the left hand side is only non-zero if $c = 3$ and $(d, e) = (1, 2)$ or $(d, e) = (2, 1)$. Thus there are only two choices for a non-vanishing answer: $a = d$ and $b = e$ or $a = e$ and $b = d$. Similarly for the other choices of a, b. This is what the right hand side says. The only issue is the minus sign but this arises as $\varepsilon_{abc}\varepsilon_{cab} = (\varepsilon_{abc})^2 = 1$ but $\varepsilon_{abc}\varepsilon_{cba} = -(\varepsilon_{abc})^2 = -1$ (assuming a, b, c are all different).

We can now compute

$$\begin{aligned}(\underline{r}_1 \times (\underline{r}_2 \times \underline{r}_3))^a &= \sum_{b,d,e} (\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}) r_1^b r_2^d r_3^e \\ &= (\underline{r}_1 \cdot \underline{r}_3) r_2^a - (\underline{r}_1 \cdot \underline{r}_2) \underline{r}_3^a \ . \end{aligned} \tag{1.35}$$

which is just

$$\underline{r}_1 \times (\underline{r}_2 \times \underline{r}_3) = (\underline{r}_1 \cdot \underline{r}_3)\underline{r}_2 - (\underline{r}_1 \cdot \underline{r}_2)\underline{r}_3 \ , \tag{1.36}$$

i.e. $A = (\underline{r}_1 \cdot \underline{r}_3)$, $B = -(\underline{r}_1 \cdot \underline{r}_2)$.

1.4 Matrices

.

Lastly we look at matrices which act on vectors as linear maps $M : \mathbb{R}^3 \to \mathbb{R}^3$. Linear here means that

$$\mathbf{M}(A\underline{r}_1 + \underline{r}_2) = A\mathbf{M}(\underline{r}_1) + \mathbf{M}(\underline{r}_2) \tag{1.37}$$

where a is a scalar. We will usually drop the parenthesis and denote matrix multiplication by $\mathbf{M}\underline{r}$.

In terms of the components notation, we can write[3]

$$(\mathbf{M}\underline{r})^a = \sum_{b=1}^{3} M^a{}_b \underline{r}^b \ . \tag{1.38}$$

This can be read as follows: the a^{th} component of $\mathbf{M}\underline{r}$ is given by the scalar product of the a^{th}-row of $\mathbf{M}$ with $\underline{r}$. Thus a matrix has two indices and can be written as an array: *e.g.* the identity matrix $\mathbf{I}$, which doesn't change a vector after multiplication, is

$$(\mathbf{I})^a{}_b = \delta^a{}_b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \tag{1.39}$$

here $\delta^a{}_b$ is known as the Kronecker-delta.

It is clear that we can multiply a matrix by a scalar or add to matrices to get a new matrix in the obvious way:

$$(A\mathbf{M} + \mathbf{N})^a{}_b = AM^a{}_b + N^a{}_b \ . \tag{1.40}$$

In addition we can also define the product of two matrices (we will always be looking at 3×3 matrices) in what may not seem like the obvious way:

$$(\mathbf{MN})^a{}_b = \sum_{c=1}^{3} M^a{}_c N^c{}_b \ . \tag{1.41}$$

[3]In this course you need not worry about why one index is up and the other down. In more general cases, such as special relativity this is important, and there is a rule that you only ever sum over an index that appears in an expression exactly once up and once down.

This can be read as follows: the ab^{th} component of $\mathbf{MN}$ is given by the scalar product of the a^{th}-row $\mathbf{M}$ with the b^{th}-column of $\mathbf{N}$. The reason this definition is useful, as opposed to what might be the obvious way of simply multiplying the individual components together, is that this is what you'd get if you first acted on a vector by $\mathbf{N}$ and then acted again by $\mathbf{M}$.

An important set of matrices are those that leave the scalar product between two vectors invariant:

$$(\mathbf{O}\underline{r}_1) \cdot (\mathbf{O}\underline{r}_2) = \underline{r}_1 \cdot \underline{r}_2 \ , \tag{1.42}$$

for any pair of vectors $\underline{r}_1$ and $\underline{r}_2$. Substituting (1.38) into the left-hand side gives

$$(\mathbf{O}\underline{r}_1) \cdot (\mathbf{O}\underline{r}_2) = \sum_{abc=1}^{3} \mathbf{O}^c{}_a r_1^a \mathbf{O}^c{}_b r_2^b \ , \tag{1.43}$$

while the right hand side is

$$\underline{r}_1 \cdot \underline{r}_2 = \sum_{c=1}^{3} \underline{r}_1^c \underline{r}_2^c \ . \tag{1.44}$$

Thus we must have

$$\sum_{abc=1}^{3} \mathbf{O}^c{}_a r_1^a \mathbf{O}^c{}_b r_2^b = \sum_{c=1}^{3} r_1^c r_2^c \ , \tag{1.45}$$

for any $\underline{r}_1^a$, $\underline{r}_2^a$. This in turn requires that

$$\sum_{c=1}^{3} \mathbf{O}^c{}_a \mathbf{O}^c{}_b = \delta^a{}_b \ . \tag{1.46}$$

In other words

$$\mathbf{O}^T \mathbf{O} = \mathbf{I} \ , \tag{1.47}$$

where

$$(\mathbf{O}^T)^a{}_b = \mathbf{O}^b{}_a \ , \tag{1.48}$$

is called the **transpose**. Linear transformations that don't change angles are more usually referred to as **rotations**[4]. For example you can check that

$$\mathbf{O} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \tag{1.49}$$

is a rotation in the x, y plane. But more complicated examples exist (although they can always be viewed as a two-dimensional rotation in some plane - meaning that there is always one non-zero vector, the normal to the plane, that is left invariant under the action of a rotation in three-dimensions).

[4]Technically, for those who know, one usually also requires that a rotation has determinant 1, which excludes reflections about one axis (but not two).

1.5 Derivatives vs. Partial Derivatives

In this course, there will be a lot of calculus applied to a functions of many variables. It will be important to clearly understand the difference between the following:

$$\frac{df}{dt}\ , \qquad \frac{dF}{dt}\ , \qquad \frac{\partial F}{\partial u_i}\ , \qquad \frac{\partial F}{\partial t}\ . \tag{1.50}$$

Here $f(t)$ is a function of one variable t and $F(u_1, ..., u_n, t)$ is a function of n variables labelled u_i and possibly of t as well (we will often take t to be time but it could be anything). Let us recall some definitions:

$$\begin{aligned}\frac{df}{dt} &= \lim_{\epsilon\to 0}\frac{f(t+\epsilon)-f(t)}{\epsilon}\\ \frac{\partial F}{\partial u_i} &= \lim_{\epsilon\to 0}\frac{f(u_1,...,u_i+\epsilon,...u_n,t)-f(u_1,...,u_i,...,u_n,t)}{\epsilon}\\ \frac{\partial F}{\partial t} &= \lim_{\epsilon\to 0}\frac{f(u_1,...u_n,t+\epsilon)-f(u_1,...,u_n,t)}{\epsilon}\ .\end{aligned} \tag{1.51}$$

The first is the ordinary derivative and measures the rate of change of f with respect to its argument t. The second is a partial derivative and measures the rate of change of F with respect to one of its arguments u_* while holding all the others and t fixed. The third is the rate of change of F with respect to its argument t holding all the u_i fixed.

In this course we will often encounter functions $F(u_1, ..., u_n, t)$ where the variables u_i themselves depend on time t. Thus we might look at a function of the form

$$f(t) = F(u_1(t), ..., u_n(t), t)\ . \tag{1.52}$$

If we want to know the rate of change of f with respect to t, then we use the chain rule:

$$\frac{df}{dt} = \frac{dF}{dt} = \sum_{i=1}^{n}\frac{\partial F}{\partial u_i}\frac{du_i}{dt} + \frac{\partial F}{\partial t}\ . \tag{1.53}$$

Here the first terms give the change in f that arises from the fact that the variables u_i change with t and F depends on u_i. The last term arises if F has an explicit dependence on t.

On a practical level, this means that if we consider a small but finite variation, $u_i \to u_i + \delta u_i$ and $t \to t + \delta t$, then to first order in δu_i, δt we can approximate

$$\delta F = \sum_{i=1}^{n}\frac{\partial F}{\partial u_i}\delta u_i + \frac{\partial F}{\partial t}\delta t + \ldots \tag{1.54}$$

where

$$\delta F = F(u_i + \delta u_i, t + \delta t) - f(u_i, t)\ , \tag{1.55}$$

where from now on we will use a shortcut to refer to the whole collection of variables u_i: $F(u_1(t), ..., u_n(t), t) = F(u_i(t), t)$. The δF is simply the first term in a Taylor expansion. The ellipsis denotes higher order terms. We will often write $\delta u_i = \epsilon T_i$, $\delta t = \epsilon T$ where ϵ is a small parameter, which we can take to be as small as we wish, and T_i, T are some expressions that are not small. Thus the Taylor expansion is

$$\delta F = \epsilon\left(\sum_{i=1}^{n}\frac{\partial F}{\partial u_i}T_i + \frac{\partial F}{\partial t}T\right) + \mathcal{O}(\epsilon^2)\ . \tag{1.56}$$

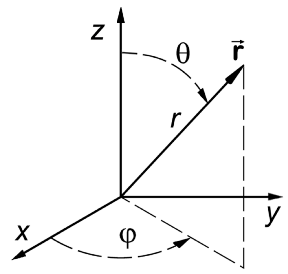

Figure 1.4: Spherical polar coordinates

By taking ϵ suitably small we can neglect the higher order terms as much as we wish.

On a more abstract level we might express this as

$$dF = \sum_{i=1}^{n} \frac{\partial F}{du_i} du_i + \frac{\partial F}{\partial t} dt \ . \tag{1.57}$$

Note that we have not defined dF, du_i or dt. One can think of them as meaning δF, δu_i and δt in the limit that $\epsilon \to 0$. Alternatively one can simply think of (1.57) as a substitute for (1.53) in the sense that if we allow the u_i to depend on any parameter, such as t, then (1.53) holds. This is just like how you are not supposed to think of df/dt as a fraction but in practice it is often helpful to do so. In a sense (1.57) is just a formal expression[5] which encodes the statement that F depends on $u_1, ..., u_n$ and t.

1.6 Co-ordinate systems

When computing derivatives and scalar and vector products, we have been working in the Cartesian co-ordinate system $\{x, y, z\}$. We often use spherical co-ordinates to describe systems with angular momentum or in the case of a central potential (see next chapter). In order to compute vector products we need to be able to convert from one system to another.

In the diagram 1.4 we have coordinates $\{r,\ \theta,\ \phi\}$.

$$r \in (0,\ \infty), \quad \theta \in (0,\ \pi), \quad \phi \in (0,\ 2\pi). \tag{1.58}$$

In order to express a vector $\mathbf{r} = \{r,\ \theta,\ \phi\}$ in cartesian coordinates (so we can perform vector products) we must make the transformations,

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta \ . \tag{1.59}$$

Exercise: What is the inverse transformation? *i.e.* what are $r(x, y, z)$, $\theta(x, y, z)$ and $\phi(x, y, z)$?

[5]Formal because we are not supposed to give a numerical value to dF, du_i dt. Rather we must always understand it as applying within the context of (1.53) or (1.56).

Care should be taken with any example as different conventions are often in use. Sometimes the 'azimuthal' angle θ is measured from below the $z = 0$ plane. In this case, the only transformation to change is that for the z coordinate, which picks up a minus sign, $z = -r\cos\theta$. Some books may even swap the names for the azimuthal angle and the longitudinal angle. It is usually easy to establish the convention in use.

Also note that when $r = 0$, then θ and ϕ are redundant, and when $\theta = 0$ or π, ϕ becomes redundant.

Chapter 2

Newton and His Three Laws

So now it is time to do Physics. Newton's (1642-1727) laws are the basis of classical physics. These laws govern both the motion of objects in our everday lives and the movement of the planets and stars above our heads. This idea of unification is a key feature of progress in theoretical physics, that phenomena we once thought were qualitatively different can be governed by a simple and unified underlying framework. While more modern theories such as special relativity, general relativity and quantum mechanics have to some extent superseded Newton's Laws, classical physics remains essential in our understanding of the world. Classical physics describes almost all of the phenomena in our day-to-day experience and is baked into our intuitive understanding of what happens around us. A key test of these more modern theories is their reproduction of Newton's Laws in a classical limit, and indeed without our classical intuition for the world around us as a jumping off point, it would probably have been impossible to come to any kind of understanding of these more modern approaches.

Newton's greatest work, which lays out his laws, is his (in)famous *Principia Mathematica* in 1687. Although he had already developed calculus, he did not use it in the *Principia*, forcing him to use some very elaborate reasoning. "The calculus" was a new approach and might be viewed with suspicion. His readers, unfamiliar with it, might be more comfortable with a more old fashioned approach. Whatever his reasons, we will use calculus because we assume you know it and it makes things so much easier.

Amusingly his book was to be published by the Royal Society (near Picadilly circus, not far from King's) but they had spent all their publishing budget that year on a book called *The History of Fishes*, which is now only famous for not being the *Principia*.[1] The cost of Newton's book was paid out of pocket by Edmund Halley who was their clerk at the time. If that wasn't bad enough, *The History of Fishes* was a commercial failure and such a drain on the financial resources of the Royal Society that they could only pay Halley his salary by giving him the unsold copies. So that will make you think twice about accepting an FRS.

Newton's laws are incredibly powerful and they basically put physics in pole position as the most predictive science. We still use them all the time even though they have, in a formal sense, been superseded somewhat by relativity but more so by quantum

[1] It's also not clear what history it talks about since it predates evolution - it must have missed that boat too.

theory. Indeed perhaps the first major new ideas beyond Newton came from Maxwell's theory of electromagnetism, which he published while he was a Professor at King's in 1861. You will learn about Maxwell's laws in another course but they ushered in two new concepts: dynamical fields which permeate space and time and the Lorentz transformations of special relativity. Maxwell, who was one of the greatest physicists by any standard, came to King's because he had been down-sized in a merger at the University of Aberdeen in Scotland (a worse *faux pas* than not signing the Beatles[2] or letting Taylor Swift leave your label to record her own songs[3]). So there is hope for everyone except, perhaps, administrators.

2.1 Newton's Laws

In his *Principia*, Newton proposed three laws that govern all motion. We will state them here for the case of a point particle, that is to say a particle small enough that any structure it has does not affect its motion. One can also discuss 'rigid bodies' which do have structure that is important for their motion but that structure itself doesn't change (hence the term rigid). Examples include spinning disney characters `<http://www.youtube.com/watch?v=qquek0c5bt4>`. Of course we think of rigid bodies as being made of particles which are subjected to Newton's Laws.

- [NI] A particle will stay at rest or move with a constant velocity (along a straight line) unless acted on by an external force.[4]

- [NII] The rate of change of momentum of a particle is equal to, and in the direction of, the net force acting on it.

- [NIII] Every action has an opposite and equal reaction.

You've probably heard these before. However, before we continue there are some important comments to make.

First, the law [NI] defines an inertial frame. Not all frames are inertial. Picture yourself on a roller coaster. There is a frame, that is a choice of coordinates, where you are at rest and the amusement park is flying all around you. That frame is *not* inertial. The law [NI] does not hold. Rather you would have to invent all sorts of fictitious forces (centrifugal force is one) to explain the motions of everything that was not strapped in traveling around with you and the roller coaster (including your stomach and your lunch). Instead, Newton is identifying inertial frames as those which are either at rest or moving at constant speed in some fixed direction. To define an inertial frame, we simply say it is a frame where [NI] holds true.

If we have one inertial frame, we can boost it to obtain a new inertial frame. By boost we mean give it a constant velocity relative to what it had before. Rotations and translations also take one inertial frame to another. Mathematically this means that we

[2] *c.f.* Richard Paul Rowe

[3] *c.f.* Big Machine Records and Shamrock Holdings

[4] It is useful to distinguish velocity, which is a vector and hence has both direction and magnitude, and speed, which is the magnitude of a velocity and hence is just a scalar quantity.

can change coordinates according to

$$\begin{aligned} \text{translation}: &\quad \underline{r} \to \underline{r} + \underline{a} \\ \text{rotation}: &\quad \underline{r} \to \mathbf{R}\underline{r} \\ \text{boost}: &\quad \underline{r} \to \underline{r} + \underline{v}t \end{aligned} \tag{2.1}$$

and still be in a frame where [NI] is true. Here $\underline{a}$ and $\underline{v}$ are a fixed vectors and $\mathbf{R}$ is a rotation matrix, that is a matrix such that $\mathbf{R}^T\mathbf{R} = 1$. These transformations form the Galilean group (a group is an important mathematical topic that you might know about but if not should do soon). Physical laws invariant under the Galilean group give rise to Galilean relativity.

Special relativity takes these ideas further and 1) declares that physical laws must be the same in all inertial frames and 2) uses a different notion of boost which leaves the speed of light constant in all frames (which in turn requires that time changes when we go between inertial frames - leading one to replace the Galilean group by the Poincaré group (which is the Lorentz group along with translations)). That was a bit of name dropping not to show off but to try to put classical mechanics in a larger context.

Secondly note that [NII] does not say $\underline{F} = m\ddot{\underline{r}}$, *i.e.* $\underline{F} = m\underline{a}$, where a dot denotes a time derivative. This is true in the simplest cases but not all. Rather, as we have stated it, [NII] is

$$\underline{F} = \dot{\underline{p}} \ , \tag{2.2}$$

where p is called the momentum. The more familiar $\underline{F} = m\ddot{\underline{r}}$ then arises when m is constant, where

$$\underline{p} = m\dot{\underline{r}} \tag{2.3}$$

and

$$\dot{\underline{p}} = m\ddot{\underline{r}} \ . \tag{2.4}$$

For a simple counter example (that we will look at later), consider a rocket ship where m decreases as fuel burns so that

$$\underline{F} = m\ddot{\underline{r}} + \dot{m}\dot{\underline{r}} \ . \tag{2.5}$$

In full generality one also can allow for $\underline{F}$ to depend on $\underline{r}$, $\dot{\underline{r}}$ and t. In these cases [NII] sets up a second order differential equation for $\underline{r}$ which can, in principle if not practice, be solved for all t by knowing the values of $\underline{r}$ and $\dot{\underline{r}}$ at some initial time t.

Finally [NIII] is the famous statement that if you push against something, say the wall, then the wall pushes back on you with equal force but in the opposite direction; after all you can feel it. This is important since if it were not the case, then there would be a net force and, by [NII], the wall would move. There is an intuitive aspect to these laws, familiar to us from many physical situations. A rocket moves forward by throwing its fuel backwards. Another example is a gun where the shooter feels a recoiling force when they pull the trigger.

Note that in the absence of any force, Newton's law tell us that a particle moves in a straight line. Here one simply has

$$\underline{F} = \frac{d}{dt}\underline{p} = \underline{0} \ . \tag{2.6}$$

In other words momentum is conserved, and $\underline{p}$ is time independent.

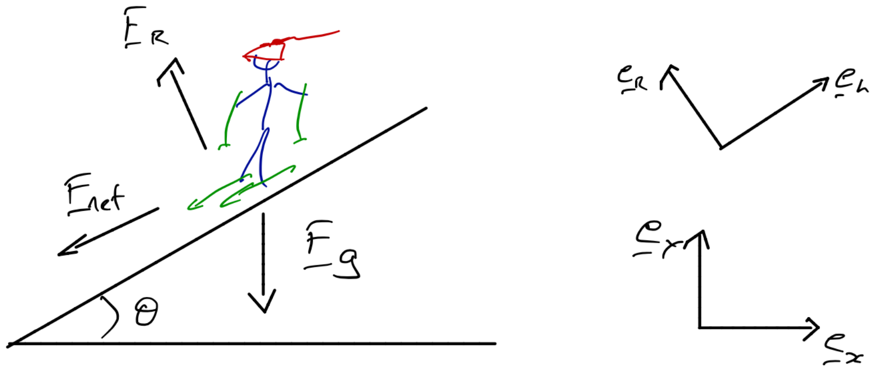

Figure 2.1: The skiier

2.2 Skiing: A Simple Example of Linear Motion

Skiing: Let us look at a skier who descends a slope which makes an angle θ with the horizon.

The force of gravity is constant and acts in the downward direction: $\underline{F}_g = -mg\underline{e}_y$. However it is useful to think of $\underline{e}_y$ in terms of two components

$$\underline{e}_y = \sin\theta\underline{e}_h + \cos\theta\underline{e}_R \ . \tag{2.7}$$

Here $\underline{e}_h$ is the unit vector pointing up the hill against the direction of the skier's motion and $\underline{e}_R$ points up perpendicularly from the slope. Accordingly we can write

$$\begin{aligned} \underline{F}_g &= -mg\underline{e}_y \\ &= -mg\sin\theta\underline{e}_h - mg\cos\theta\underline{e}_R \\ &= \underline{F}_h + \underline{F}_R \ , \end{aligned} \tag{2.8}$$

corresponding to the part which pushes the skier down the hill $\underline{F}_h$ and the part $\underline{F}_R$ that pulls the skier into the slope.

The component $\underline{F}_R = -mg\cos\theta\underline{e}_R$ is canceled by an opposite and equal reactive force of the hill pushing back on the skier. This is intuitively obvious but fundamentally due to the electromagnetic forces inside the atoms of snow and ski-boots. Now in this case $\underline{p} = m\dot{\underline{r}}$ where m is constant so we are simply left with

$$m\ddot{\underline{r}} = \underline{F}_h = -mg\sin\theta\underline{e}_h \ . \tag{2.9}$$

Let us write $\underline{r} = r\underline{e}_h$. Then we find

$$\ddot{r} = -g\sin\theta \ . \tag{2.10}$$

To solve this we simply integrate both sides:

$$\dot{r}(t) = -gt\sin\theta + \dot{r}(0) \ , \tag{2.11}$$

where $\dot{r}(0)$ is a constant of integration. And then integrate again

$$r(t) = -\frac{1}{2}gt^2\sin\theta + \dot{r}(0)t + r(0) \ , \tag{2.12}$$

where $r(0)$ is a another constant of integration. Now clearly the arbitrary constants $\dot{r}(0)$ and $r(0)$ correspond to the speed and position of the skier at $t = 0$. Given these as inputs, we can then compute how the skier will continue to go down the hill. The answer is faster and faster as any novice skier can tell you. Less obvious is that the dependence on the mass of the skiier has dropped out.

To be more explicit, assume that at $t = 0$ the skier is at $r = 100$ m from the bottom and traveling at -1 m s^{-1} down the hill. Then

$$r(t) = -(5\,\mathrm{m\,s^{-2}})t^2\sin\theta - (1\,\mathrm{m\,s^{-1}})t + 100\,\mathrm{m} \tag{2.13}$$

where we have used $g = 10$ m s^{-2} (it is actually closer to 9.8 m s^{-2}). (It can be a useful error checking method to leave the units in your equation, to make sure all the factors of meters and second cancel appropriately.) For a green run, one might have $\theta = \pi/6 = 30^\circ$ so $\sin\theta = 1/2$ leading to

$$r(t)_{\mathrm{green}} = -(2.5\ \mathrm{m\,s^{-2}})t^2 - (1\,\mathrm{m\,s^{-1}})t + 100\,\mathrm{m} \ . \tag{2.14}$$

In the first second, the skier arrives at $r(1)_{\mathrm{green}} = 96.5$ m, just a few meters further, but after 5 seconds $r(10)_{\mathrm{green}} = 67.5$ m, already quite far: 2/3 of the way down the hill. For a red run maybe one has $\theta = \pi/4 = 45^\circ$, $\sin\theta \sim 0.7$ and hence

$$r(t)_{\mathrm{red}} \sim -(3.5\,\mathrm{m\,s^{-2}})t^2 - (1\,\mathrm{m\,s^{-1}})t + 100\,\mathrm{m} \ , \tag{2.15}$$

so that after 1 second $r(1)_{\mathrm{red}} = 95.5$ m, also just a few meters, but now after 5 seconds $r(5)_{red} = 7.5$ m which is almost all the way down.

2.3 Friction

The above discussion is not so realistic since the skier will eventually reach a terminal velocity due to the presence of friction. How do we include friction? Friction is a complicated concept, and there are many different kinds – static, kinetic, rolling, drag, etc. It corresponds to adding another force

$$\underline{F} = \underline{F}_R + \underline{F}_h + \underline{F}_F \tag{2.16}$$

that opposes the acceleration of the object in question. The complication is in how to model $\underline{F}_F$. Here we will assume that the dominant effect comes from a drag force, either from the snow on the skis or the wind on the skiier, that is proportional to the skiier's velocity:

$$\underline{F}_F = -\nu\dot{\underline{r}} \ . \tag{2.17}$$

Here $\nu > 0$ is the friction coefficient. Note the minus sign which means that friction acts in the opposite direction to the velocity.

The effect of friction is therefore to change the equation to

$$m\ddot{\underline{r}} = \underline{F}_h + \underline{F}_F = -mg\sin\theta\underline{e}_h - \nu\dot{\underline{r}}. \tag{2.18}$$

Putting $\underline{r} = r\underline{e}_h$ gives us the equation

$$\ddot{r} = -g\sin\theta - \frac{\nu}{m}\dot{r}. \tag{2.19}$$

To solve this we multiply both sides by $e^{\nu t/m}$ and write it as

$$\frac{d}{dt}(\dot{r}e^{\nu t/m}) = -ge^{\nu t/m}\sin\theta \ . \tag{2.20}$$

We can now integrate both sides once:

$$\dot{r}e^{\nu t/m} = -\frac{mg}{\nu}\sin\theta e^{\nu t/m} + A \ , \tag{2.21}$$

where A is an integration constant. Multiplying both sides by $e^{-\nu t/m}$ gives

$$\dot{r} = -\frac{mg}{\nu}\sin\theta + Ae^{-\nu t/m} \ , \tag{2.22}$$

which can again be integrated to

$$r(t) = -\frac{mg}{\nu}t\sin\theta - \frac{mA}{\nu}e^{-\nu t/m} + B \ , \tag{2.23}$$

Here B is another integration constant. It is easy to see that A and B can be related to the initial position and momentum but not in such a simple way as before. However we can see one universal feature that doesn't depend on A and B. At late times, $t \to \infty$, the exponential becomes negligible and

$$r(t)_{t\to\infty} = -\frac{mg}{\nu}t\sin\theta + B \ , \tag{2.24}$$

meaning that the skier will no longer speed up but will travel with a constant terminal velocity $v_\infty = -\frac{mg}{\nu}\sin\theta$ down the hill. Of course the fun of skiing is that ν is small so v_∞ is big! The same equation also applies if you try skiing without snow; it's just that ν is very large. We have of course assumed that $\nu > 0$, so that friction opposes the acclerating force. For $\nu < 0$ we would find the opposite: the skier speeds up exponentially! Signs are important.

Friction also explains why rain drops don't hurt (usually). Imagine a 1g rain drop that falls from 1km. Without wind resistance its final velocity is

$$r(t) = -5t^2 + 1000 \tag{2.25}$$

where we have used the same equation as the skier but taken the initial value of r to be 1km = 1000m, the initial $\dot{r}$ to be zero and set $\theta = \pi/2$. So it hits your head at $t = 10\sqrt{2}$ s. The speed it hits you with is

$$\dot{r} = (-10\,\mathrm{m\,s^{-2}})t = -100\sqrt{2}\,\mathrm{m\,s^{-1}} \ , \tag{2.26}$$

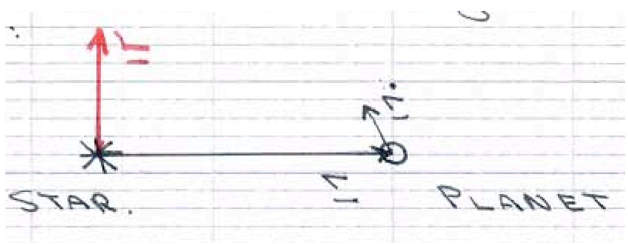

Figure 2.2: Planet going around the sun

and carries momentum

$$p = m\dot{r} = -0.1\sqrt{2}\ \text{kg m s}^{-1} \sim -0.14\ \text{kg m s}^{-1}\ . \tag{2.27}$$

How painful would that be? Consider a 1 kg brick dropped on you from a height of 1 m. Again use the same equations:

$$r(t) = (-5\,\text{m}\,\text{s}^{-2})t^2 + 1 \tag{2.28}$$

so $t = \sqrt{5}/5$ s, the final velocity is $\dot{r} = (-10\,\text{m}\,\text{s}^{-2})t = -2\sqrt{5}$ m s^{-1} and the final momentum is

$$p = m\dot{r} = -2\sqrt{5}\,\text{kg}\,\text{m}\,\text{s}^{-1} \sim -4.47\ \text{kg m s}^{-1}\ . \tag{2.29}$$

So one rain drop would give about 1/40 the punch of such a brick. Maybe not too bad but you can expect more than one rain drop to fall on your head. Indeed you can expect 40 per second! Of course life isn't so cruel. The rain drops reach a modest terminal velocity and only carry a small amount of momentum when they hit your head.

2.4 Angular Motion

A more complicated but still quite trackable example of motion arises when particles move in angular (*e.g.* circular or elliptical) orbits. For example we could be studying planets as they moved around the sun

Since they are not moving in straight lines, linear momentum $\underline{p}$ is not conserved. However in many cases an analogous quantity, angular momentum, is conserved. One can also introduce a suitable notion of force, known as torque, that is better adapted to angular motion.

We start with the definition of angular momentum about the origin:

$$\underline{L} = \underline{r} \times \underline{p}\ . \tag{2.30}$$

Note that $\underline{L}$ points in a direction orthogonal to both $\underline{r}$ and $\underline{p}$.

Next we want to define the analogue of force for angular motion: Torque $\underline{N}$. This is important in sports cars as it tell you how much the engine can turn the wheels around, which then leads to forward motion. In particular

$$\underline{N} = \underline{r} \times \underline{F}\ . \tag{2.31}$$

We observe that, from (NII), if we have a point particle with fixed mass m and momentum $\underline{p} = m\dot{\underline{r}}$ then

$$\begin{aligned} \underline{N} &= \underline{r} \times \underline{F} \\ &= \underline{r} \times \dot{\underline{p}} \\ &= \frac{d}{dt}(\underline{r} \times \underline{p}) \\ &= \dot{\underline{L}} \end{aligned} \tag{2.32}$$

Here we have used the fact that

$$\begin{aligned} \frac{d}{dt}(\underline{r} \times \underline{p}) &= \dot{\underline{r}} \times \underline{p} + \underline{r} \times \dot{\underline{p}} \\ &= m\dot{\underline{r}} \times \dot{\underline{r}} + \underline{r} \times \dot{\underline{p}} \\ &= \underline{0} + \underline{r} \times \dot{\underline{p}} \; . \end{aligned} \tag{2.33}$$

So indeed torque and angular momentum play analogous roles to force and momentum. Let us see how this works in a simple example.

Imagine a particle that moves in a circle. Thus we take

$$\underline{r} = \begin{pmatrix} r\cos\theta(t) \\ r\sin\theta(t) \\ 0 \end{pmatrix} \tag{2.34}$$

Circular motion means that $r = |\underline{r}|$ is constant but not $\theta(t)$! In particular

$$\frac{d}{dt}|\underline{r}| = 0 \; , \qquad i.e. \qquad \dot{r} = 0 \qquad \text{but} \qquad \frac{d}{dt}\underline{r} \neq \underline{0} \qquad \text{because} \qquad \frac{d\theta}{dt} \neq 0 \; . \tag{2.35}$$

As you keep this in mind, we here drop the dependence on t everywhere to simplify the notation. As anticipated we have

$$\dot{\underline{r}} = \begin{pmatrix} -r\dot{\theta}\sin\theta \\ r\dot{\theta}\cos\theta \\ 0 \end{pmatrix} \tag{2.36}$$

and hence

$$|\dot{\underline{r}}|^2 = r^2\dot{\theta}^2(\sin^2\theta + \cos^2\theta) = r^2\dot{\theta}^2 \; . \tag{2.37}$$

Note that $\dot{\theta}$ is called the angular velocity[5] and that (2.37) will be useful in the evaluation of the kinetic energy, to be defined in the following. Moreover we have

$$\ddot{\underline{r}} = \begin{pmatrix} -r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta \\ r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta \\ 0 \end{pmatrix} \; . \tag{2.38}$$

It is important to realise that, even for constant angular velocity, this particle is accelerating. If we do restrict to $\dot{\theta} = \omega$ a constant, then we see that there must be a force:

$$m\ddot{\underline{r}} = \begin{pmatrix} -r\dot{\theta}^2\cos\theta \\ -r\dot{\theta}^2\sin\theta \\ 0 \end{pmatrix} = -m\omega^2\underline{r} \; . \tag{2.39}$$

[5]For those either old enough or cool enough one might know of 33 and 45 rpm records (even 78 for the truly old); rpm stands for revolutions per minute and is a measure of angular velocity.

Thus there is a force pointing inwards whose strength is linear in r:

$$\underline{F} = -m\omega^2\underline{r} \ , \tag{2.40}$$

such as a spring. This is called a centripetal force (as opposed to centrifugal force which is a fictitious force pointing outwards that one feels if one is the particle).

In the general case, let us calculate also the angular momentum. From the expression for $\dot{\underline{r}}$ above we have

$$\begin{aligned}
\underline{L} &= \underline{r} \times \underline{p} \\
&= m \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -r\dot{\theta}\sin\theta \\ r\dot{\theta}\cos\theta \\ 0 \end{pmatrix} \\
&= m \begin{pmatrix} 0 \\ 0 \\ r^2\dot{\theta}(\cos^2\theta + \sin^2\theta) \end{pmatrix} \\
&= mr^2\dot{\theta} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned} \tag{2.41}$$

Note that this points up out of the plane of circular motion. Next we compute the torque for constant angular velocity:

$$\begin{aligned}
\underline{N} &= \underline{r} \times \underline{F} \\
&= -m\omega^2\underline{r} \times \underline{r} \\
&= \underline{0} \ .
\end{aligned} \tag{2.42}$$

We see from the expression for $\ddot{\underline{r}}$ that if there is angular acceleration (but still take r constant):

$$\begin{aligned}
\underline{N} &= \underline{r} \times \underline{F} \\
&= m\underline{r} \times \ddot{\underline{r}} \\
&= m \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta \\ r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta \\ 0 \end{pmatrix} \\
&= m \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -r\ddot{\theta}\sin\theta \\ r\ddot{\theta}\cos\theta \\ 0 \end{pmatrix} \\
&= mr^2\ddot{\theta} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned} \tag{2.43}$$

Note that the second term in the second line is proportional to $\underline{r}$ and hence vanishes in the cross product with $\underline{r}$. This kind of torque is why you buy a BMW M-series and not simply a 316i. It is the ability of a car to speed up the angular velocity of its wheels.

Example: We would like to figure out how to bank a circular track so that a car can travel in a circle without sliding up or down the track. The track has radius R and is idealized to be frictionless. The car travels at constant speed v and has mass m. We want to figure out the angle θ such that the radial acceleration inward is precisely of the form to keep the car moving at constant speed in a circle. We use $\underline{F} = m\underline{a}$. Choosing the same coordinate system that we did for the skier, the left hand side is precisely the same as before.

$$\begin{aligned} F_x &= mg\sin\theta \ , \\ F_y &= F_R - mg\cos\theta \ . \end{aligned}$$

For the right hand side, we resolve the acceleration, which must have magnitude $\omega^2 R$, into components along and perpendicular to the slope of the track. Note that $v = \omega R$ for circular motion. We have then

$$\begin{aligned} ma_x &= m\frac{v^2}{R}\cos\theta \ , \\ ma_y &= m\frac{v^2}{R}\sin\theta \ . \end{aligned}$$

We find then our answer that

$$\tan\theta = \frac{v^2}{gR} \ ,$$

which is independent of the mass of the car. We did not even need to use the y-component of the vector equation although if we wanted to determine F_R, it would have been useful. The reader is invited to try a more complicated version of the problem where there is also friction that opposes the sliding of the car but can never exceed μF_R in magnitude. In thise case, the challenge is to find the range of velocities for which the car can safely negotiate the track without sliding up or down.

2.5 Work, Conservative Forces and Conserved Quantities

We have seen that in the cases of linear and angular motion, if there is no force or torque then momentum $\underline{p}$ and angular $\underline{L}$ are constant in time, or conserved. In particular

$$\underline{F} = \underline{0} \Rightarrow \underline{p} \text{ is conserved}$$

$$\underline{N} = \underline{0} \Rightarrow \underline{L} \text{ is conserved}$$

Conserved quantities play a critical role in the understanding of dynamical system because they can be used to solve a problem, or at least reduce its complexity.

But before we move on to conserved quantities more generally, we need to introduce the notion of work, or more precisely the work done on a system (one doesn't have an absolute notion of work). In an infinitesimal step, the work done on a particle by a force F is the scalar product of the force and of the consequent particle's displacement

$$dW = \underline{F} \cdot d\underline{r} \ . \tag{2.44}$$

For a complete path, the work done is defined as a line integral so that as a particle moves from $\underline{r}_1$ to $\underline{r}_2$ we have

$$\Delta W = \int_{\underline{r}_1}^{\underline{r}_2} \underline{F} \cdot d\underline{r} \ . \tag{2.45}$$

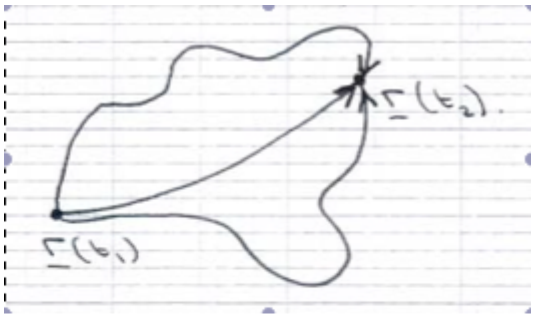

Figure 2.3: Different paths from one point to another.

You should think of this as follows. The particle takes some path described by $\underline{r}(t)$ and goes from $\underline{r}_1 = \underline{r}(t_1)$ to $\underline{r}_2 = \underline{r}(t_2)$ as t goes from t_1 to t_2. Therefore

$$d\underline{r} = \dot{\underline{r}} dt \ , \tag{2.46}$$

and

$$\Delta W = \int_{t_1}^{t_2} \underline{F} \cdot \dot{\underline{r}} dt \ . \tag{2.47}$$

Now if we write $\underline{F} = m\ddot{\underline{r}}$ we have

$$\begin{aligned} \Delta W &= m \int_{t_1}^{t_2} \ddot{\underline{r}} \cdot \dot{\underline{r}} dt \\ &= \frac{m}{2} \int_{t_1}^{t_2} \frac{d}{dt} \left(\dot{\underline{r}} \cdot \dot{\underline{r}} \right) dt \\ &= \frac{1}{2} m |\dot{\underline{r}}(t_2)|^2 - \frac{1}{2} m |\dot{\underline{r}}(t_1)|^2 \ . \end{aligned} \tag{2.48}$$

Here we have introduced the **kinetic** energy

$$T = \frac{1}{2} m |\dot{\underline{r}}|^2 \ . \tag{2.49}$$

The work done is therefore given by the change in kinetic energy over the path of the particle:

$$\Delta W = \Delta T \ . \tag{2.50}$$

It is generally path-dependent: taking different paths will lead to different changes in kinetic energy.

For example if one has only the force of friction, with $\underline{F} = -\nu \dot{\underline{r}}$ then

$$\Delta W_{friction} = -\nu \int_{t_1}^{t_2} \dot{\underline{r}} \cdot \dot{\underline{r}} dt < 0 \ . \tag{2.51}$$

This is negative since, unless there is a counteracting force pushing the particle along, the particle will slow down. The friction will have done negative work (positive work is an achievement: such as speeding up!).

For example if you push your shopping trolley around the supermarket, you must do work to keep the trolley moving at a constant speed because you are constantly fighting agains the friction. Furthermore the more you walk around the isles, the more work you must do. All trips to the supermarket start at the front door and end at the check-out.

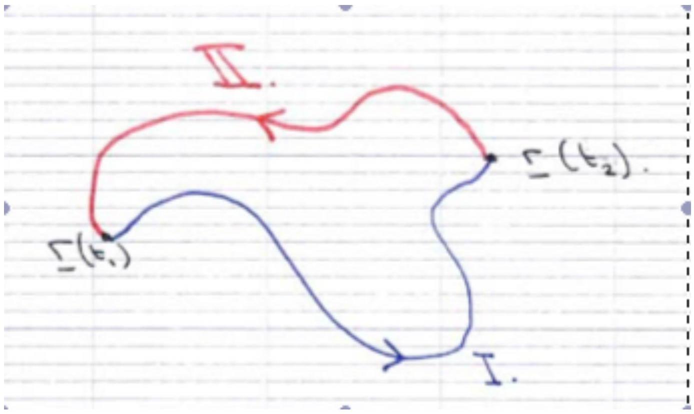

Figure 2.4: Two different paths from one point to another.

But the amount of work you must do to overcome the negative work of friction depends on how long a path around the isles you take to find what you want.

In other words, in order to fight against friction you must supply a force that does positive work. In particular to ensure that your final speed is the same as your initial velocity, you must provide a force $\underline{F}_{you}$ so that

$$\Delta W = \int \underline{F}_{you} \cdot d\underline{r} + \Delta W_{friction} = 0 \ . \tag{2.52}$$

and hence you must do work:

$$\Delta W_{you} = \int \underline{F}_{you} \cdot d\underline{r} > 0 \ . \tag{2.53}$$

But there is a class of forces for which the work done is independent of the path taken. Such a force is said to be **conservative**. Roughly speaking the fundamental forces we observe in Nature (gravity, electromagnetism,...) when acting in empty space are conservative. Non-conservative forces typically arise from some kind of friction force that is due to the microscopic details of many particles bouncing around hitting each other in a disorderly way.

An important class of conservative forces arises if there exists a function $V(\underline{r})$ such that

$$\underline{F} = -\underline{\nabla} V \ . \tag{2.54}$$

Let us check that such an $\underline{F}$ does lead to a definition of work that is path independent. To do this we observe that

$$\begin{aligned}
W &= \int_{\underline{r}_1}^{\underline{r}_2} \underline{F} \cdot d\underline{r} \\
&= -\int_{\underline{r}_1}^{\underline{r}_2} \underline{\nabla} V \cdot d\underline{r} \\
&= -\int_{t_1}^{t_2} \underline{\nabla} V \cdot \dot{\underline{r}} \ dt \\
&= -\int_{t_1}^{t_2} \frac{d}{dt} V(\underline{r}(t)) \ dt \\
&= V(\underline{r}(t_1)) - V(\underline{r}(t_2)) \\
&= V(\underline{r}_1) - V(\underline{r}_2) \ .
\end{aligned} \tag{2.55}$$

As promised, this only depends on the end points and not the path taken.

In fact if $\underline{F}$ only depends on $\underline{r}$ and not, for example $\dot{\underline{r}}$, then such a V always exists (at least locally). We won't prove this here but to see why we note that path independence is equivalent to the statement that the total work done around a closed path vanishes.

For instance, let us consider a closed path, from $\underline{r}_1$ to $\underline{r}_2$ and then back again (along a different path) (see figure 6). In this case the work done between the first and second legs will cancel so that

$$\oint \underline{F} \cdot d\underline{r} = \int_I \underline{F} \cdot d\underline{r} + \int_{II} \underline{F} \cdot d\underline{r} = 0 \tag{2.56}$$

since path I and path II have the same endpoints but in reverse order.

Therefore we have

$$0 = \oint_\gamma \underline{F} \cdot d\underline{r} = \int_{\{B|\partial B = \gamma\}} (\underline{\nabla} \times \underline{F}) \cdot d\underline{A} \tag{2.57}$$

where we have used a corollary of the Stokes theorem applied to a region B whose boundary is the curve γ, $d\underline{r}$ is tangent to the curve and $d\underline{A}$ is perpendicular to the surface B. Since this must be true for any curve γ, we deduce that

$$\underline{\nabla} \times \underline{F} = \underline{0} \tag{2.58}$$

This means (and we won't prove it here) that $\underline{F} = -\underline{\nabla} V$ for some function V.[6]

The notion of a potential function V allows us to introduce the most fundamental of conserved quantities, the total energy:

$$\begin{aligned} E &= T + V \\ &= \frac{1}{2} m |\dot{\underline{r}}|^2 + V(\underline{r}) \ . \end{aligned} \tag{2.59}$$

Claim: The total energy of a conserved system is conserved, *i.e.* constant. To see this we simply differentiate:

$$\begin{aligned} \frac{d}{dt} E &= m\dot{\underline{r}} \cdot \ddot{\underline{r}} + \underline{\nabla} V \cdot \dot{\underline{r}} \\ &= \dot{\underline{r}} \cdot \underline{F} + \underline{\nabla} V \cdot \dot{\underline{r}} \\ &= (\underline{F} + \underline{\nabla} V) \cdot \dot{\underline{r}} \\ &= 0 \ . \end{aligned} \tag{2.60}$$

N.B. There exist more general definitions of energy, of which $T + V$ is a special case. In section 3.5.4, we will encounter other systems where there is a conserved energy, but it takes a different form.

2.6 Solving One-dimensional Dynamics

Conservation of energy is already enough to essentially solve for the motion of a particle in one-dimension with a conservative force. In this case $\underline{r}$ is just a number $r \in \mathbb{R}$. Let the potential be $V(r)$ so that the energy is

$$E = \frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + V(r) \ . \tag{2.61}$$

[6] This is an example of the Poincaré lemma. More interestingly the number of solutions to this equation which are not of this form count the number of "holes" in space in a certain topological sense.

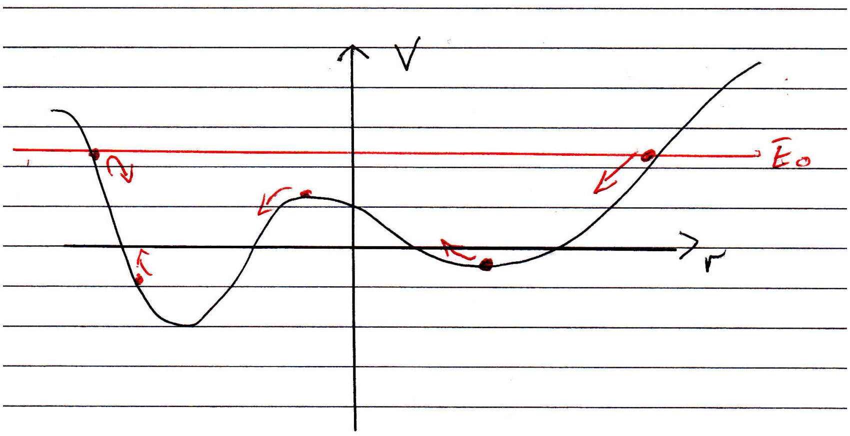

Figure 2.5: Motion in a one dimensional potential.

Note that $E \geq V(r)$ with equality only if $\dot{r} = 0$, *i.e.* the particle is at rest. Conceptually, regardless of the origin of this system, we can think of it as a particle moving in a potential $V(r)$ and, more explicitly, imagine that $V(r)$ is the height of a hill. Then the particle's motion will simply be the same as a skier, moving without friction, along the hill. (Note earlier we used a distance r along the slope to parametrize the position of the skiier instead of the horizontal distance, but regardless of the parametrization, the underlying physics must be the same.)

Given that the energy is conserved along the motion, let's fix it to E_0. E_0 is a constant but one that can be changed from solution to solution. For each solution, one can formally solve for $r(t)$. In fact one solves for t as a function of r and then inverts. To do this we rewrite the conservation energy equation with $E = E_0$ as

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}(E_0 - V(r))}\ , \tag{2.62}$$

and as

$$\frac{dr}{\sqrt{\frac{2}{m}(E_0 - V(r))}} = \pm dt \tag{2.63}$$

which we can integrate to get

$$t - t_0 = \pm\int_{r(t_0)}^{r(t)} \frac{dr'}{\sqrt{\frac{2}{m}(E_0 - V(r'))}}\ . \tag{2.64}$$

The right hand side is, for a given potential $V(r)$, just some integral that can in principle be evaluated.

Note that there is a choice of sign. For a given choice, we solve the system until we reach a **turning point**. At the turning point, $E_0 = V(r)$ and the expression inside the square root vanishes. Here all the energy is potential energy, *i.e.* no kinetic energy. This means that the particle is at rest at that point. The situation is illustrated by the left-most and right-most points in Figure 7. What typically happens is that the particle goes up the hill as far as it can and will now turn around and come back. Thus at such a point $V(r)$ was increasing, has stopped and will now start decreasing. To find the corresponding solution one must match solutions at the turning point with one choice of sign to solutions with the other choice of sign (since time always runs forwards whereas the particle can go up the hill, turn around, and come back).

We can illustrate this with the skier again! This is solvable for the simple case of a constant slope. Here the potential is just $V = mgr\sin\theta$ (why?). Therefore the solution we find from (2.64) is

$$\begin{aligned} t - t_0 &= \pm \int_{r(t_0)}^{r(t)} \frac{dr'}{\sqrt{\frac{2}{m}(E_0 - mgr'\sin\theta)}} \\ &= \mp \frac{1}{g\sin\theta}\sqrt{\frac{2}{m}(E_0 - mgr'\sin\theta)}\Bigg|_{r(t_0)}^{r(t)} \\ &= \mp \frac{1}{g\sin\theta}\sqrt{\frac{2}{m}}\left(\sqrt{E_0 - mgr(t)\sin\theta} - \sqrt{E_0 - mgr(t_0)\sin\theta}\right) \end{aligned} \tag{2.65}$$

As the second term in parentheses is a constant, let us simplify the algebra a bit by calling it c:

$$t - t_0 = \mp \frac{1}{g\sin\theta}\sqrt{\frac{2}{m}}\left(\sqrt{E_0 - mgr(t)\sin\theta} - c\right) . \tag{2.66}$$

We can then solve for $r(t)$. We find

$$\mp\sqrt{\frac{m}{2}}g\sin\theta(t - t_0) + c = \sqrt{E_0 - mgr(t)\sin\theta} . \tag{2.67}$$

And then squaring both sides

$$\frac{mg^2\sin^2\theta}{2}(t - t_0)^2 \mp \sqrt{2m}\, gc(t - t_0)\sin\theta + c^2 = E_0 - mgr(t)\sin\theta . \tag{2.68}$$

Some rearranging leads to

$$r(t) = -\frac{g\sin\theta}{2}(t - t_0)^2 \pm \sqrt{\frac{2}{m}}c(t - t_0) + \frac{E_0 - c^2}{mg\sin\theta} . \tag{2.69}$$

Replacing c with its expression in terms of $r(t_0)$ and E_0, we finally obtain

$$r(t) = -\frac{g\sin\theta}{2}(t - t_0)^2 \pm \sqrt{\frac{2}{m}}\sqrt{E_0 - mgr(t_0)\sin\theta}\,(t - t_0) + r(t_0) . \tag{2.70}$$

To compare with our previous expression for the position of the skiier on the hill, let us start our clock at $t_0 = 0$, in which case, we can write

$$r(t) = -\frac{g\sin\theta}{2}t^2 \pm \sqrt{\frac{2}{m}}\sqrt{E_0 - mgr(0)\sin\theta}\, t + r(0) . \tag{2.71}$$

Our old result was

$$r(t) = -\frac{g\sin\theta}{2}t^2 + v(0)t + r(0) . \tag{2.72}$$

The choice of sign is the choice of sign of the the initial speed $v(0)$. In fact matching the coefficient of the term linear in t we identify

$$v(0) = \pm\sqrt{\frac{2}{m}}\sqrt{E_0 - mgr(0)\sin\theta} \Longleftrightarrow E_0 = \frac{1}{2}mv^2(0) + mgr(0)\sin\theta \tag{2.73}$$

which agrees with what we know. In particular the choice of sign simply corresponds to the fact that the energy is the same for a particle with velocity v or $-v$. Thus there are two branches of solutions depending on this choice.

Let us return to the case of turning points. In particular let us look at the solution we found above (again with $t_0 = 0$):

$$
\begin{aligned}
t &= \mp\frac{1}{g\sin\theta}\sqrt{\frac{2}{m}}\left(\sqrt{E_0 - mgr\sin\theta} - \sqrt{E_0 - mgr(t_0)\sin\theta}\right) \\
&= \mp\frac{1}{g\sin\theta}\sqrt{\frac{2}{m}}\sqrt{E_0 - mgr\sin\theta} + \frac{v(0)}{g\sin\theta} \ .
\end{aligned}
\tag{2.74}
$$

Let us suppose that the skier starts by going up hill with some initial $v(0) > 0$; then $r(t)$ increases until she slows down to a stop. Thus both r and t are increasing (time is always increasing) which means that we must take the minus sign in (2.74) (and correspondingly the plus sign in (2.73)). When does she stop? She stopes when there is no kinetic energy so that $E = V(r(t))$, *i.e.* $E_0 - mgr\sin\theta = 0$, where the term inside the square root vanishes. It follows from (2.74) that this happens at

$$
t = t_{turning} = \frac{v(0)}{g\sin\theta} \ . \tag{2.75}
$$

It's called a turning point because after stopping, the skier will then start to go down the hill with increasing speed. Thus after the turning point, r is decreasing but t is increasing. This corresponds to the plus sign in (2.74). Thus the full solution is

$$
t = \begin{cases} -\frac{1}{g\sin\theta}\sqrt{\frac{2}{m}}\sqrt{E_0 - mg\sin\theta r(t)} + \frac{v(0)}{g\sin\theta} & t \le t_{turning} \\ +\frac{1}{g\sin\theta}\sqrt{\frac{2}{m}}\sqrt{E_0 - mg\sin\theta r(t)} + \frac{v(0)}{g\sin\theta} & t \ge t_{turning} \end{cases} \tag{2.76}
$$

But of course at all times we find that

$$
E_0 = \frac{1}{2}m\dot{r}^2 + mg\sin\theta r \ , \tag{2.77}
$$

regardless of the choice of sign.

2.7 Angular Momentum Revisited

There are other important examples of conserved quantities. Perhaps the next most important one is angular momentum which we have already briefly seen.

A force is called central if $\underline{F} \propto \underline{r}$, *i.e.* $\underline{F} = f(\underline{r})\underline{r}$.

For a conservative force that is derived from a potential $V(\underline{r})$, then central implies that V is only a function of $|\underline{r}|$ (for some choice of the origin). Such forces include the one we used above for circular motion but also, and most importantly, gravity:

$$
V_g = -\frac{G_N M m}{|\underline{r}|} \tag{2.78}
$$

where G_N is Newton's constant and M and m are the masses of the two particles. To see this we note that (here we think of V as a function of $|\underline{r}|$)

$$
\underline{\nabla}V = \left(\frac{dV}{d|\underline{r}|}\right)\underline{\nabla}(|\underline{r}|) \tag{2.79}
$$

and

$$\underline{\nabla}(|\underline{r}|) = \frac{1}{2|\underline{r}|}\underline{\nabla}(|\underline{r}|^2) = \frac{1}{2|\underline{r}|}\underline{\nabla}(\underline{r}\cdot\underline{r}) = \frac{\underline{r}}{|\underline{r}|} \ . \tag{2.80}$$

Thus

$$\underline{F} = -\underline{\nabla}V = -\left(\frac{dV}{d|\underline{r}|}\right)\frac{\underline{r}}{|\underline{r}|} \ , \tag{2.81}$$

i.e.

$$\underline{F} = f\underline{r} \ , \qquad f = -\frac{1}{|\underline{r}|}\frac{dV}{d|\underline{r}|} \ . \tag{2.82}$$

Claim: The angular momentum $\underline{L} = \underline{r}\times\underline{p}$ is conserved for a central force. In other words the torque vanishes. To see this we simply note that

$$\begin{aligned}\frac{d}{dt}\underline{L} &= \frac{d}{dt}\left(\underline{r}\times\underline{p}\right)\\ &= \dot{\underline{r}}\times\underline{p} + \underline{r}\times\dot{\underline{p}}\\ &= m\dot{\underline{r}}\times\dot{\underline{r}} + \underline{r}\times\underline{F}\\ &= \underline{0} + \underline{r}\times f\underline{r}\\ &= \underline{0}\end{aligned} \tag{2.83}$$

2.8 Solving Three-Dimensional Motion in a Central Potential with Effective Potentials

Conservation of angular momentum, along with conservation of energy, is powerful enough to essentially solve the dynamics of a particle in $\mathbb{R}^3$, just like we did for one-dimensional motion. To this end we note that since $\underline{L} = \underline{r}\times\underline{p}$, conservation of $\underline{L}$ implies that $\underline{L}$ is orthogonal to $\underline{r}$. Thus the motion is restricted to a plane: the plane orthogonal to $\underline{L}$. Let us choose coordinates where the plane is

$$\underline{r} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0\end{pmatrix} \qquad \Rightarrow \qquad \dot{\underline{r}} = \begin{pmatrix}\dot r\cos\theta - r\dot\theta\sin\theta \\ \dot r\sin\theta + r\dot\theta\cos\theta \\ 0\end{pmatrix} \tag{2.84}$$

From this we find

$$\begin{aligned}\underline{L} &= m\underline{r}\times\dot{\underline{r}}\\ &= m\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0\end{pmatrix}\times\begin{pmatrix}\dot r\cos\theta - r\dot\theta\sin\theta \\ \dot r\sin\theta + r\dot\theta\cos\theta \\ 0\end{pmatrix}\\ &= \begin{pmatrix}0\\0\\mr^2\dot\theta\end{pmatrix} \ ,\end{aligned} \tag{2.85}$$

and

$$\begin{aligned}|\dot{\underline{r}}|^2 &= (\dot r\cos\theta - r\dot\theta\sin\theta)^2 + (\dot r\sin\theta + r\dot\theta\cos\theta)^2\\ &= \dot r^2 + r^2\dot\theta^2 \ .\end{aligned} \tag{2.86}$$

Because $\underline{L}$ is conserved, we can fix

$$l = mr^2\dot{\theta} \qquad \Rightarrow \qquad \dot{\theta} = \frac{l}{mr^2} \tag{2.87}$$

where l is a constant. Thus we can write the conserved energy as

$$\begin{aligned} E_0 &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \\ &= \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r) \\ &= \frac{1}{2}m\dot{r}^2 + V_{eff}(r) \ . \end{aligned} \tag{2.88}$$

Here

$$V_{eff}(r) = \frac{l^2}{2mr^2} + V(r) \tag{2.89}$$

is an effective potential that incorporates the effect of angular motion. We have now reduced the problem to the same as the one-dimensional case only with the effective potential $V_{eff}(r)$ in place of the original $V(r)$ (and r is now restricted to $r \geq 0$).

Note that one "effect" of angular momentum is to add a so-called angular momentum barrier that stops particles from going to $r = 0$ if $l \neq 0$. In particular because $E \geq V$ and E is constant, the particle cannot get to $r = 0$ since V_{eff} becomes arbitrarily large and hence, for small enough r, $V_{eff} > E$ which is forbidden. (We are assuming that there isn't a negative term in the original potential $V(r)$ that is more dominant than $l^2/2mr^2$ at small r.)

2.9 Celestial Motion about the Sun

Let us now put everything we have learned together and consider the classic case of a planet orbiting the sun with a potential

$$V = -\frac{G_N Mm}{r} \tag{2.90}$$

so that

$$V_{eff} = \frac{l^2}{2mr^2} - \frac{G_N Mm}{r} \ . \tag{2.91}$$

Thus here there is an angular momentum barrier since the first term dominates at small r.

We can qualitatively see how the system behaves in the different cases under the force of gravity. Looking at the plot of V_{eff}, we can identify two types of trajectories. If $E_0 < 0$ then the object is bound and cannot escape to $r \to \infty$. Rather it will oscillate around the minimum of V_{eff}. These are the closer objects in our sky – the moon orbiting around the Earth, the planets orbiting around the sun. (There is also the very special case of an exactly circular orbit where r is held fixed at the minimum of V_{eff}. The satellites we send up to orbit the Earth we often design to follow such paths.) On the other hand if $E_0 > 0$, then the object can escape to $r \to \infty$. Every once in a while, our solar system gets a visitor from far away. An object comes in from $r \to \infty$, reaches a minimal value of r and then goes back out to ∞. This is in fact a hyperbolic orbit

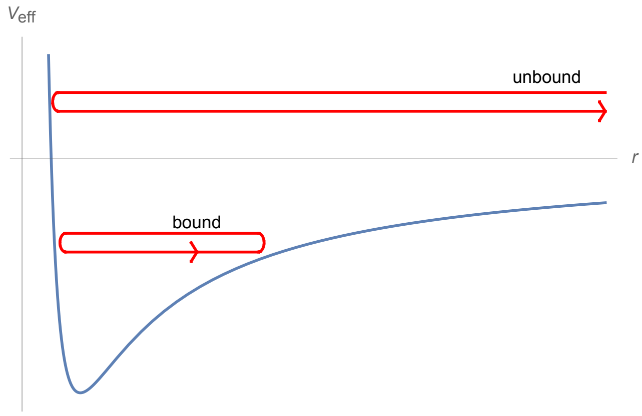

Figure 2.6: Bound and unbound motion in a gravitational potential.

(with again a special case – the parabolic orbit). A recent such interstellar visitor was Oumuamua.[7]

In this case we can also analytically solve for the motion. To see that the solutions are ellipses, hyperbolae, circles, and parabolae (*i.e.* conic sections), it is more helpful to think of r as a function of θ so that

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{l}{mr^2}\frac{dr}{d\theta} \ . \tag{2.92}$$

The energy conservation equation is now

$$E_0 = \left(\frac{dr}{d\theta}\right)^2 \frac{l^2}{2mr^4} + \frac{l^2}{2mr^2} - \frac{G_N M m}{r} \ . \tag{2.93}$$

The smart idea is to note that we can rewrite this as

$$E_0 = \frac{l^2}{2m}\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 + \frac{l^2}{2m}\frac{1}{r^2} - G_N M m\frac{1}{r} \ . \tag{2.94}$$

So let us introduce $u = 1/r$ to find

$$\begin{aligned} E_0 &= \frac{l^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{l^2}{2m}u^2 - G_N M m u \\ &= \frac{l^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{l^2}{2m}\left(u - \frac{G_N M m^2}{l^2}\right)^2 - \frac{G_N^2 M^2 m^3}{2l^2} \end{aligned} \tag{2.95}$$

Next we make a shift

$$u = v + G_N M m^2/l^2 \tag{2.96}$$

so that

$$\begin{aligned} E_0 + \frac{G_N^2 M^2 m^3}{2l^2} &= \frac{l^2}{2m}\left(\frac{dv}{d\theta}\right)^2 + \frac{l^2}{2m}v^2 \\ &= \frac{l^2}{2m}\left[\left(\frac{dv}{d\theta}\right)^2 + v^2\right] \end{aligned} \tag{2.97}$$

[7]While most astronomers believe the object to be naturally occurring, one notable exception is the Harvard astrophysicist Avi Loeb who speculated it could be the product of alien technology!

The solution to this is

$$v = A\cos(\theta - \theta_0)\ , \tag{2.98}$$

where θ_0 is arbitrary. In fact without loss of generality we can choose coordinates such that $\theta_0 = 0$. We see that A satisfies

$$E_0 + \frac{G_N^2 M^2 m^3}{2l^2} = \frac{l^2 A^2}{2m} \qquad \Rightarrow \qquad A = \sqrt{\frac{2mE_0}{l^2} + \frac{G_N^2 M^2 m^4}{l^4}}\ . \tag{2.99}$$

Returning to the original variables we have

$$\begin{aligned} \frac{1}{r(\theta)} &= v + G_N M m^2/l^2 \\ &= A\cos(\theta - \theta_0) + B\ , \end{aligned} \tag{2.100}$$

where $B = G_N M m^2/l^2$. Or in terms of B we have

$$A^2 = B^2 + \frac{2mE_0}{l^2}\ . \tag{2.101}$$

2.10 Conic Sections

The solutions (2.100) are known as conic sections. They played a key role in dynamics since they give the motion of the planets, moons, and comets. (After Galileo and Copernicus (but before Hubble), the stars were treated as fixed.) These solutions literally had cosmic significance, and hence we should look at them more closely. The curves we find (ellipses, parabolas and hyperbolas) are called conic sections because that is what you get by intersecting a cone with a plane in three-dimensions. But they had almost mystic significance for 300 years until the larger universe, with a wider variety of motions, was understood.

Let us choose a coordinate system where $\theta_0 = 0$ and set

$$x = r\cos\theta \qquad y = r\sin\theta\ . \tag{2.102}$$

We can rewrite (2.100) as

$$\begin{aligned} 1 &= Ar\cos\theta + Br \\ &= Ax + B\sqrt{x^2 + y^2} \end{aligned} \tag{2.103}$$

Rearranging this and squaring both sides gives

$$(1 - Ax)^2 = B^2 x^2 + B^2 y^2 \tag{2.104}$$

Some more rearranging gives

$$1 = (B^2 - A^2)x^2 + 2Ax + B^2 y^2 \tag{2.105}$$

which, for $A^2 \neq B^2$, can be written as

$$(B^2 - A^2)\left(x + \frac{A}{B^2 - A^2}\right)^2 + B^2 y^2 = 1 + \frac{A^2}{B^2 - A^2}\ . \tag{2.106}$$

or with a little more rearranging

$$\frac{(B^2 - A^2)^2}{B^2}\left(x + \frac{A}{B^2 - A^2}\right)^2 + (B^2 - A^2)y^2 = 1 \ . \tag{2.107}$$

Let us first look at the $E_0 \geq 0$ possibilities. For $E_0 > 0$, $A^2 - B^2 > 0$ and we have

$$\frac{(A^2 - B^2)^2}{B^2}\left(x - \frac{A}{A^2 - B^2}\right)^2 - (A^2 - B^2)y^2 = 1 \ . \tag{2.108}$$

which is a hyperbolic trajectory. Such an interstellar object has come in from infinity with a non-zero initial velocity so that $E_0 > 0$. At some point it will reach a minimum value of r where it turns around and goes back out. The minimum value of r is easily seen from (2.100) to be $1/(B + A)$.

For $B^2 = A^2$ we find a parabolic trajectory. Here it is easiest to return back to the form (2.105) to find

$$1 - 2Bx = B^2y^2 \qquad \Leftrightarrow \qquad x = \frac{1}{2B} - \frac{1}{2}By^2 \ . \tag{2.109}$$

It is as if the object is being dropped into the sun with a vanishing initial velocity. Again there it makes a closest approach before returning to infinity. The case $E_0 < 0$ is left as an exercise.

2.11 Kepler's Laws

Now we consider the case of planets. These are "bound" to the sun and cannot escape to infinity. Therefore they correspond to solutions with $E_0 < 0$. Before Newton, Kepler was led by observations to propose three laws of planetary motion:

- [KI] The planets move in an ellipse with the sun at one focus.
- [KII] The line joining a given planet to the sun sweeps out equal areas over equal times.
- [KIII] The square of a planet's orbital period is proportional to the cube of the semi-major axis.

Newton was able to derive these three laws from his universal Law of gravitation. So let's do that.

Looking at the conic sections above, it is clear that the planets move in ellipses. It remains to see that the sun, located at $(0, 0)$ is one focus. By definition an ellipse is the set of points on the plane such that the line joining them to one focus plus the line joining them to a second focus has fixed length $2d$. Let us show that what we found corresponds to an ellipse with foci at the origin and at $(-2a, 0)$ (we will have to deduce a and d). The distances to the point (x, y) are

$$d_1 = \sqrt{x^2 + y^2} \ , \qquad d_2 = \sqrt{(x + 2a)^2 + y^2} \ . \tag{2.110}$$

Thus the equation of an ellipse is

$$2d = d_1 + d_2 = \sqrt{x^2 + y^2} + \sqrt{(x + 2a)^2 + y^2} \ . \tag{2.111}$$

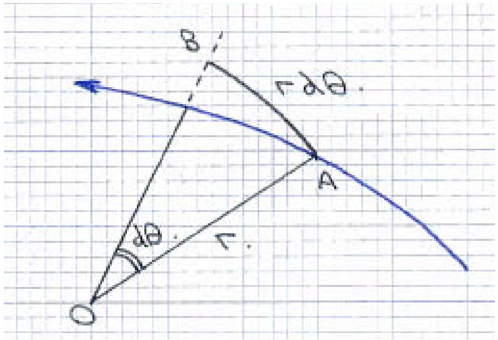

Figure 2.7: A figure for proving Kepler's second law.

We can rearrange this as

$$
\begin{aligned}
(2d - \sqrt{x^2+y^2})^2 &= (x+2a)^2 + y^2 \\
\iff 4d^2 - 4d\sqrt{x^2+y^2} &= 4ax + 4a^2 \\
\iff d^2(x^2+y^2) &= (ax + a^2 - d^2)^2 \\
\iff (d^2-a^2)x^2 + 2a(d^2-a^2)x + d^2y^2 &= (d^2-a^2)^2 \\
\iff (d^2-a^2)\,(x+a)^2 + d^2y^2 &= (d^2-a^2)^2 + (d^2-a^2)a^2 \\
\iff (d^2-a^2)\,(x+a)^2 + d^2y^2 &= (d^2-a^2)d^2 \\
\iff \frac{1}{d^2}\,(x+a)^2 + \frac{1}{d^2-a^2}y^2 &= 1 \qquad (2.112)
\end{aligned}
$$

This agrees with (2.107) and we learn that

$$a = \frac{A}{B^2 - A^2}\ , \qquad d = \frac{B}{B^2 - A^2}\ . \qquad (2.113)$$

Thus we have proven KI.

To prove KII, we note that for an infinitesimal change in θ, the infinitesimal area swept out by the line joining the planet to $(0,0)$ is just given by a triangle with base r and height $rd\theta$ (see figure 2.7). Thus

$$dArea = \frac{1}{2}r^2 d\theta \qquad (2.114)$$

for infinitesimal $d\theta$. Thus the rate of change in time is

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = l/2m \qquad (2.115)$$

where l is the conserved angular momentum. This proves KII.

Finally we look at KIII). To do this we compute the area of the ellipse in two ways. First we note that since the planet sweeps out the same area at equal times we have

$$\text{Area}(E) = \int_0^T \frac{dArea}{dt}dt = \int_0^T \frac{l}{2m}dt = \frac{lT}{2m} \qquad (2.116)$$

where T is the period of the orbit, *i.e.* how long it takes to go around.

On the other hand for an ellipse of the form

$$\frac{(x+a)^2}{d^2} + \frac{y^2}{d^2-a^2} = 1\ , \qquad (2.117)$$

the area is formally

$$\text{Area}(E) = \int\int_E dx\,dy \tag{2.118}$$

where E is the set of points bounded by the curve (2.117). We can make this integral a bit easier with a change of variables, $u = (x+a)/d$ and $v = y/\sqrt{d^2 - a^2}$. Using that $du = dx/d$ and $dv = dy/\sqrt{d^2-a^2}$, we find a new expression for the area

$$\text{Area}(E) = d\sqrt{d^2-a^2}\int\int_{E'} du\,dv = d\sqrt{d^2-a^2}\text{Area}(E') \tag{2.119}$$

where E' is the set of points bounded by the circle $u^2 + v^2 = 1$. But that's easy, $\text{Area}(E') = \pi$. So we learn

$$\text{Area}(E) = \pi d\sqrt{d^2-a^2}\ . \tag{2.120}$$

We already saw that $d = B/(B^2 - A^2)$ and $a = A/(B^2 - A^2)$ so

$$d^2 - a^2 = \frac{B^2}{(B^2-A^2)^2} - \frac{A^2}{(B^2-A^2)^2} = \frac{1}{B^2-A^2} \tag{2.121}$$

and setting the areas equal we find

$$\frac{lT}{2m} = \pi\frac{B}{(B^2-A^2)^{3/2}}\ . \tag{2.122}$$

Finally we note that the semi-major axis is defined as half the widest distance across an ellipse:

$$\begin{aligned} R_{smaj} &= \frac{1}{2}(r(0) + r(\pi)) \\ &= \frac{1}{2}\left(\frac{1}{A+B} + \frac{1}{-A+B}\right) \\ &= \frac{B}{B^2-A^2} \end{aligned} \tag{2.123}$$

Thus

$$\frac{lT}{2m} = \pi B\left(\frac{R_{smaj}}{B}\right)^{3/2} = \pi B^{-1/2}R_{smaj}^{3/2} \tag{2.124}$$

and hence

$$T = \frac{2\pi m}{l}B^{-1/2}R_{smaj}^{3/2} = 2\pi(G_N M)^{-1/2}R_{smaj}^{3/2} \tag{2.125}$$

and we have proven KIII, including a calculation of the constant of proportionality.

2.12 Weighing Planets

We see from the above that by observing the planets, we could deduce $G_N M_{sun}$. Similar equations apply for the case of the moon orbiting the earth. In this case we could measure $G_N M_{earth}$. We can also deduce $g = 9.8m/s^2$ by simply expanding the force of gravity near the surface of the the earth. In particular if we lift a particle up a height h above the earth's surface then

$$\begin{aligned} V(h) &= -\frac{G_N M_{earth} m}{r} \\ &= -\frac{G_N M_{earth} m}{R_{earth} + h} \\ &= -\frac{G_N M_{earth} m}{R_{earth}} \left(\frac{1}{1 + h/R_{earth}} \right) \\ &= -\frac{G_N M_{earth} m}{R_{earth}} + \frac{G_N M m}{R_{earth}^2} h + \dots \end{aligned} \tag{2.126}$$

The first term is a constant that can be ignored. Comparing the second term to the formula $V = mgh$ we deduce that

$$g = \frac{G_N M_{earth}}{R_{earth}^2} \ . \tag{2.127}$$

Since one can measure R_{earth} and g, we can again deduce $G_N M_{earth}$.

However we would really like to know G_N as a universal constant. This was first done by Lord Cavendish up near Russell Square. He was the first to perform an experiment to measure the force of gravitational attraction between two massive balls. Since he knew their mass, he could deduce G_N. It is a subtle experiment based on measuring the oscillations of the two balls suspended in a so-called torsion balance configuration, but he was able to do it. In fact he was very close to the current value of

$$G_N \sim 6.67 \times 10^{-11} \mathrm{m}^3 \mathrm{kg}^{-1} \mathrm{s}^{-2} \ . \tag{2.128}$$

He obtained $6.75 \times 10^{-11} \mathrm{m}^3 \mathrm{kg}^{-1} \mathrm{s}^{-2}$. Although this determined G_N, it is more commonly known as "weighing the earth" since one could then deduce M_{earth} (6×10^{24} kg) and M_{sun} (2×10^{30} kg) and this was his primary objective.

2.13 The Runge-Lenz Vector (Optional)

Lastly it is worth mentioning the famous Runge-Lenz vector although it is generally acknowledged that it was known to Euler, Lagrange and others before Runge and Lenz. Runge apparently wrote about it in a text book, which Lenz referenced in a paper, and ever since then, it has had its current name. In any case the Runge-Lenz vector is

$$\underline{A} = \underline{p} \times \underline{L} + mV(\underline{r})\underline{r} \ . \tag{2.129}$$

To see that it is conserved, we compute:

$$\begin{aligned} \dot{\underline{A}} &= \dot{\underline{p}} \times \underline{L} + \underline{p} \times \dot{\underline{L}} + m(\underline{\nabla} V \cdot \dot{\underline{r}})\underline{r} + mV(|\underline{r}|)\dot{\underline{r}} \\ &= -\underline{\nabla} V \times \underline{L} + \underline{p} \times \underline{0} + m(\underline{\nabla} V \cdot \dot{\underline{r}})\underline{r} + mV(|\underline{r}|)\dot{\underline{r}} \end{aligned} \tag{2.130}$$

where we have used NI with $\underline{F} = -\underline{\nabla} V$ and $\dot{\underline{L}} = \underline{0}$ because $\underline{F}$ is central. Next we note that $\underline{L} = m\underline{r} \times \dot{\underline{r}}$ so that

$$\begin{aligned} \dot{\underline{A}} &= -m\underline{\nabla} V \times (r \times \dot{\underline{r}}) + m(\underline{\nabla} V \cdot \dot{\underline{r}})\underline{r} + mV(|\underline{r}|)\dot{\underline{r}} \\ &= -m(\underline{\nabla} V \cdot \dot{\underline{r}})\underline{r} + m(\underline{\nabla} V \cdot \underline{r})\dot{\underline{r}} + m(\underline{\nabla} V \cdot \dot{\underline{r}})\underline{r} + mV(|\underline{r}|)\dot{\underline{r}} \ . \end{aligned} \tag{2.131}$$

Here we used the triple product identity (1.36). Clearly the first and third terms cancel. To show that the second and fourth also cancel we note that $V \propto |\underline{r}|^{-1}$ so that

$$\begin{aligned} \underline{\nabla} V &= \frac{dV}{d|\underline{r}|}\underline{\nabla}|\underline{r}| \\ &= -\frac{V}{|\underline{r}|}\frac{1}{2|\underline{r}|}\underline{\nabla}(\underline{r}\cdot\underline{r}) \\ &= -\frac{V}{|\underline{r}|^2}\underline{r}\ . \end{aligned} \tag{2.132}$$

Thus $\underline{\nabla} V \cdot \underline{r} = -V$, and indeed the second and fourth terms cancel. The existence of this extra conserved vector is in fact rather miraculous. It arises due to a so-called hidden symmetry that we will return to later. Note also that the only important feature is that $V \sim 1/r$ so it also works for an electric charge orbiting an opposite charge as in the hydrogen atom.

To see how useful it is, we evaluate

$$\begin{aligned} \underline{A}\cdot\underline{r} &= \underline{r}\cdot(\underline{p}\times\underline{L}) + mV(\underline{r})|\underline{r}|^2 \\ &= (\underline{r}\times\underline{p})\cdot\underline{L} + mV(\underline{r})|\underline{r}|^2 \\ &= \underline{L}\cdot\underline{L} + mV(\underline{r})|\underline{r}|^2 \\ &= l^2 - G_N M m^2 r\ , \end{aligned} \tag{2.133}$$

where $r = |\underline{r}|$. In the second line we used the fact that

$$\begin{aligned} \underline{u}\cdot(\underline{v}\times\underline{w}) &= \sum_{abc}\epsilon_{abc}u^a v^b w^c \\ &= \sum_{abc}\epsilon_{cab}u^a v^b w^c \\ &= (\underline{u}\times\underline{v})\cdot\underline{w}\ . \end{aligned} \tag{2.134}$$

On the other hand the right hand side is just $|\underline{A}|r\cos\theta$, where θ is the angle between $\underline{r}$ and the fixed vector $\underline{A}$. Thus we have

$$|\underline{A}|r\cos\theta = l^2 - G_N M m^2 r \tag{2.135}$$

which is the same equation for $r(\theta)$ that we derived if we identify $|\underline{A}| = Al^2$. Thus we have been able to derive the equation for $r(\theta)$ without ever solving a differential equation!

Chapter 3

Multi-Particle Systems and Rigid Body Motion

3.1 Multi-Particle Systems

So far we have mainly concerned ourselves with the motion of a single particle in some external force. Our next step is to consider many particles which we label by i. In this case we can distinguish between two types of forces acting on the ith particle:

- external forces $\underline{F}_i^{ext}$
- inter particle forces $\underline{F}_{ij}^{int}$ between the ith and jth particle.

Note that (NIII) implies that $\underline{F}_{ij}^{int} = -\underline{F}_{ji}^{int}$. And hence also that $\underline{F}_{ii}^{int} = \underline{0}$. Thus (NII) can be written as

$$\frac{d}{dt}\underline{p}^i = \underline{F}_i^{ext} + \sum_j \underline{F}_{ij}^{int} \tag{3.1}$$

It may not be necessary to know exactly what each particle is doing and one might just be interested in the average. To study this we can sum (3.1) over i:

$$\begin{aligned}
\frac{d}{dt}\sum_i \underline{p}^i &= \sum_i \underline{F}_i^{ext} + \sum_i \sum_j \underline{F}_{ij}^{int} \\
&= \sum_i \underline{F}_i^{ext} - \sum_i \sum_j \underline{F}_{ji}^{int} \quad \text{(using } \underline{F}_{ij} = -\underline{F}_{ji} \text{ from NIII)} \\
&= \sum_i \underline{F}_i^{ext} - \sum_j \sum_i \underline{F}_{ij}^{int} \quad \text{(relabeling the variables } i \to j \text{ and } j \to i\text{)} \\
&= \sum_i \underline{F}_i^{ext} - \sum_i \sum_j \underline{F}_{ij}^{int} \quad \text{(permuting the sums)} \\
&= \sum_i \underline{F}_i^{ext}
\end{aligned} \tag{3.2}$$

If we define the total momentum $\underline{P} \equiv \sum_i \underline{p}^i$ and the total external force $\underline{F}^{ext} \equiv \sum_i \underline{F}_i^{ext}$, we have shown that

$$\underline{F}^{ext} = \frac{d}{dt}\underline{P} \ . \tag{3.3}$$

It is tempting the push the single particle analogy further and ask if there is a way of writing also

$$\underline{F}_{ext} = M\underline{\ddot{R}} \tag{3.4}$$

where $M = \sum_i m_i$ is the total mass of the system. Indeed, since $\underline{p}^i = m_i \underline{\dot{r}}_i$ and hence $\underline{P} = \sum_i m_i \underline{\dot{r}}_i$, we can write

$$M\underline{\ddot{R}} = \frac{d}{dt}\underline{P} = \sum_j m_j \underline{\ddot{r}}_j, \tag{3.5}$$

which is true if

$$\underline{R} = \frac{1}{M}\sum_j m_j \underline{r}_j \ . \tag{3.6}$$

This $\underline{R}$ is a vector from the origin to a point called the center of mass. Ignoring rotations which we come to in a moment, the system, from the point of view of the external forces, behaves as a point particle with all its mass concentrated at this center of mass.

It is also easy to see that the work is just the sum of the individual work done by each particle:

$$\begin{aligned}
\Delta W &= \sum_i \int_{t_1}^{t_2} \underline{F}_i \cdot d\underline{r}_i \\
&= \sum_i \int_{t_1}^{t_2} m_i \underline{\ddot{r}}_i \cdot \underline{\dot{r}}_i dt \\
&= \sum_i \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{1}{2} m_i |\underline{\dot{r}}_i|^2\right) dt \\
&= \sum_i (T_i(t_2) - T_i(t_1))
\end{aligned} \tag{3.7}$$

is the sum over the changes of the individual kinetic energies. As before this will be path independent if all the forces are conservative.

How is it that in a world with only fundamental forces, friction arises? Where does the excess work and energy go? Well as you are pushing a shopping trolley around the supermarket, you are heating up the wheels and floor. This leads to an increase in the kinetic energy of the molecules in the wheels and floor. But it is disorganized and not useful as the molecules are pushed in all sorts of directions. So from a macroscopic perspective, it is just wasted. This is essentially the definition of heat: wasted energy (although you might not think the heat of a fire is wasted on a cold day or a romantic evening).

If the internal and external forces can be obtained through potentials:

$$\underline{F}_i^{ext} = -\underline{\nabla}_i V^{ext} \qquad F_{ij}^{int} = -\underline{\nabla}_i V_{ij}^{int} \tag{3.8}$$

where $\underline{\nabla}_i$ is the gradient with respect to $\underline{r}_i$, then we can construct the total energy[1]

$$E = \sum_i \frac{1}{2} m_i |\underline{\dot{r}}_i|^2 + V_{ext} + \sum_j \sum_{i<j} V_{ij}^{int} \ . \tag{3.9}$$

[1]Note we assume that $V_{ij}^{int} = V_{ji}^{int}$. For a two particle system, we would just need a single term, $\sum_j \sum_{i<j} V_{ij}^{int} = V_{12}^{int}$, while for a three particle system, the sum on the interparticle potentials should be $\sum_j \sum_{i<j} V_{ij}^{int} = V_{12}^{int} + V_{23}^{int} + V_{13}^{int}$.

A short calculation shows that this is conserved:

$$\begin{aligned}
\dot{E} &= \sum_i m_i \dot{\underline{r}} \cdot \ddot{\underline{r}} + \sum_i \underline{\nabla}_i V^{ext} \cdot \dot{\underline{r}}_i + \sum_i \sum_{j \neq i} \underline{\nabla}_i V_{ij}^{int} \cdot \dot{\underline{r}}_i \\
&= \sum_i \dot{\underline{r}}_i \cdot \left(m_i \ddot{\underline{r}} + \underline{\nabla}_i V^{ext} + \sum_{j \neq i} \underline{\nabla}_i V_{ij}^{int} \right) \\
&= \sum_i \dot{\underline{r}}_i \cdot \left(m_i \ddot{\underline{r}}_i - \underline{F}_i^{ext} - \sum_{j \neq i} \underline{F}_{ij}^{int} \right) \\
&= 0 \; .
\end{aligned} \tag{3.10}$$

3.2 Multi-particle Systems and Rotation

To continue with our generalization to many bodies, we define the angular momentum of the system to be the sum of the individual angular momenta:

$$\begin{aligned}
\underline{L} &= \sum_i \underline{r}_i \times \underline{p}_i \\
&= \sum_i m_i \underline{r}_i \times \dot{\underline{r}}_i
\end{aligned} \tag{3.11}$$

where the second line holds in the case that $\underline{p}^i = m_i \dot{\underline{r}}_i$ with m_i constant. In particular we don't expect that the individual angular momenta will be conserved for many bodies. We have that the torque is

$$\begin{aligned}
\underline{N} &= \dot{\underline{L}} \\
&= \sum_i \left(\dot{\underline{r}}_i \times \underline{p}_i + \underline{r}_i \times \dot{\underline{p}}_i \right) \; .
\end{aligned} \tag{3.12}$$

Let us assume that $\underline{p}^i = m_i \dot{\underline{r}}_i$ then $\dot{\underline{r}}_i \times \underline{p}_i = \underline{0}$ and hence

$$\begin{aligned}
\underline{N} &= \sum_i \underline{r}_i \times \dot{\underline{p}}_i \\
&= \sum_i \underline{r}_i \times \left(\underline{F}_i^{ext} + \sum_j \underline{F}_{ij}^{int} \right) \; .
\end{aligned} \tag{3.13}$$

Still the remaining expression does not vanish in any obvious way.

Let us focus one a simpler situation at first, one with two particles and no external forces. In this case, the total torque will be

$$\underline{N} = \underline{r}_1 \times \underline{F}_{12}^{int} + \underline{r}_2 \times \underline{F}_{21}^{int} \; . \tag{3.14}$$

By Newton's third law, we must have $\underline{F}_{12}^{int} = -\underline{F}_{21}^{int}$ and hence

$$\underline{N} = (\underline{r}_1 - \underline{r}_2) \times \underline{F}_{12}^{int} \; . \tag{3.15}$$

From this expression, it is clear that if $\underline{F}_{12}^{int}$ is parallel to the line joining the two particles, *i.e.* the vector $\underline{r}_1 - \underline{r}_2$, the torque will vanish. Such a force is called a *central force*. One way of writing such a force is as $\underline{F}_{12}^{int} = f_{12}(|\underline{r}_1 - \underline{r}_2|)(\underline{r}_1 - \underline{r}_2)$, where f_{12} is a scalar function of the distance between the particles.

We have shown then that for an isolated system of two particles and internal forces of this central force type, the total torque must vanish. Or in other words, the angular momentum is conserved. Isolated here means there are no external forces on either of the particles. Indeed, the result is more general. If we have a system of n particles where all the external forces vanish, we can consider all these pairwise contributions to the torque, and for a central force, the total torque will again vanish! Isolated systems subject to internal central forces have a conserved angular momentum!

Such a result bears some reflection. Clearly central forces are a special thing. Angular momentum conservation is not something that comes out of Newton's Laws on its own. While two very important forces are central – gravity and Coulomb forces – there are in fact many forces in our experience which are not central, especially when the particles in question have some substructure or velocity. When electrons move, they generate magnetic fields which do not act centrally. Further, electrons have a spin, and there are spin-spin interactions which do not act centrally either. In fact these forces typically act orthogonally to the line between the particles. In such cases, it is still possible to define a conserved angular momentum, but one must broaden one's definition to include the internal angular momentum of the particles (the spin of the electron) and more exotically angular momentum contained in field strengths (the electric and magnetic fields generated by the electrons and their motions).

Restoring the external forces, we have in full generality for a central force that

$$\begin{aligned}
\underline{N} &= \sum_i \underline{r}_i \times \underline{F}_i^{ext} + \sum_i \sum_j \underline{r}_i \times \underline{F}_{ij}^{int} \\
&= \sum_i \underline{r}_i \times \underline{F}_i^{ext} + \sum_{\substack{\text{pairs}(i,j) \\ i \neq j}} (\underline{r}_i \times \underline{F}_{ij}^{int} + \underline{r}_j \times \underline{F}_{ji}^{int}) \\
&= \sum_i \underline{r}_i \times \underline{F}_i^{ext} + \sum_{\text{pairs}} (\underline{r}_i - \underline{r}_j) \times \underline{F}_{ij}^{int} \ ,
\end{aligned} \tag{3.16}$$

where in the third line we used that by NIII $\underline{F}_{ij}^{int} = -\underline{F}_{ji}^{int}$. By our centrality assumption, $\underline{F}_{ij}^{int} \sim \underline{r}_i - \underline{r}_j$ and the sum over pairs in the third line must vanish. Thus

$$\underline{N} = \dot{\underline{L}} = \sum_i \underline{r}_i \times \underline{F}_i^{ext} \ . \tag{3.17}$$

We saw already the role of the centre-of-mass in describing the change in linear momentum of a system of particles when subject to external forces. The centre-of-mass plays a similar role for angular momentum. By separating out the centre-of-mass motion, the rotation of a system of particles can be decomposed in a nice fashion:

$$\underline{r}_i = \underline{r}'_i + \underline{R} \tag{3.18}$$

The total angular momentum is

$$\begin{aligned}
\underline{L} &= \sum_i m_i \underline{r}_i \times \dot{\underline{r}}_i \\
&= \sum_i m_i (\underline{r}'_i + \underline{R}) \times (\dot{\underline{r}}'_i + \dot{\underline{R}}) \\
&= \sum_i m_i \underline{r}'_i \times \dot{\underline{r}}'_i + \sum_i m_i \underline{r}'_i \times \dot{\underline{R}} + \sum_i m_i \underline{R} \times \dot{\underline{r}}'_i + \sum_i m_i \underline{R} \times \dot{\underline{R}} \ .
\end{aligned} \tag{3.19}$$

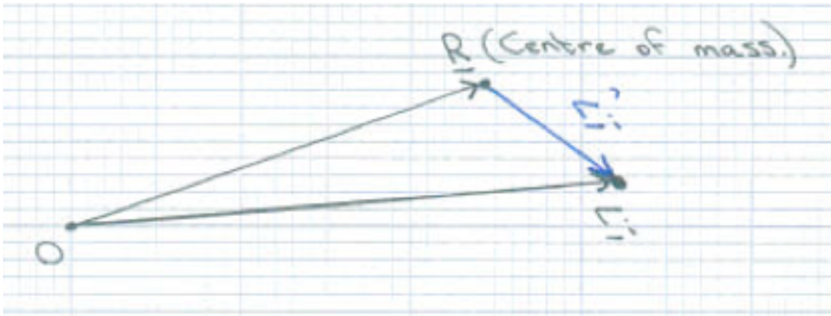

Figure 3.1: Breaking down motion into center of mass motion and relative motion about the center of mass.

However we note that

$$\begin{aligned} M\underline{R} &= \sum_i m_i \underline{r}_i \\ &= \sum_i m_i(\underline{r}'_i + \underline{R}) \\ &= \sum_i m_i \underline{r}' + M\underline{R} \ , \end{aligned} \tag{3.20}$$

and therefore

$$\sum_i m_i \underline{r}' = 0 \qquad \Rightarrow \qquad \sum_i m_i \dot{\underline{r}}' = 0 \ . \tag{3.21}$$

Thus the two terms in the middle of (3.19) vanish, and we are left with

$$\underline{L} = \sum_i m_i \underline{r}'_i \times \dot{\underline{r}}'_i + \sum_i m_i \underline{R} \times \dot{\underline{R}} \ . \tag{3.22}$$

In other words, the total angular momentum of a system of particles can be decomposed into the sum of the individual angular momenta about the centre of mass and the centre of mass angular momentum. Sometimes people use the words "internal angular momentum" or "spin angular momentum" for the first contribution and "orbital angular momentum" for the second.

If there is no external force, or at least one such that $\sum_i \underline{r}_i \times \underline{F}_i^{ext} = \underline{0}$, then by (3.17) we also have a conserved total angular momentum:

$$\begin{aligned} \underline{L} &= \sum_i m_i \underline{r}'_i \times \dot{\underline{r}}'_i + M\underline{R} \times \dot{\underline{R}} \ , \\ &= \underline{L}_{spin} + \underline{L}_{orbital} \ . \end{aligned} \tag{3.23}$$

However unlike in the single particle case, these are not enough to solve for the system in general. There are simply too many variables, too many degrees of freedom. An exception is the two-body case that we will consider a bit later on.

First though, we want to think about rotational motion of rigid systems more carefully.

3.3 Rotation of Rigid Bodies

We want to characterize the motion and in particular the rotation of rigid bodies. We can think of a rigid body as a collection of point masses whose relative distances from each other are fixed. The interesting behavior then is not the relative motion of the point masses, which is either nonexistent or negligible, but the motion of the system as a whole. The analysis that we did on multiparticle systems leading up to this point is not wasted. External forces on a rigid body act like external forces on an arbitrary collection of masses: $\underline{F} = M\ddot{\underline{R}}$ where M is the total mass and $\underline{R}$ is the center of mass. The angular momentum can still be decomposed into an orbital part $\sum_i m_i \underline{R} \times \dot{\underline{R}}$ and an internal portion $\sum_i m_i \underline{r}'_i \times \dot{\underline{r}}'_i$. We will assume that in the absence of external forces, the total angular momentum is conserved.

The first task is to understand how to parametrize the location of a rigid body. We begin by locating the center of mass $\underline{R}$ and its velocity $\dot{\underline{R}}$. We then further identify how the object is spinning about its center of mass. To characterize the spin, we need a vector $\underline{\omega}$, whose direction indicates an axis about which the object is spinning, and whose magnitude indicates a rate, for example in radians per second. Finally, we need to indicate how the object is oriented with respect to $\underline{\omega}$. This last task is where a lot of the mental pain in understanding rigid body motion is often hidden. Our approach in this class will be to take each example on a case-by-case basis, instead of trying to produce a general formalism. (One such formalism goes by the name of Euler angles, but again we will not discuss its details.)

Using the notation from before, if $\underline{r}'_i$ is the distance of one of the constituents of the rigid body from the center of mass, then we can identify $\underline{v}'_i = \underline{\omega} \times \underline{r}'_i$ as the velocity of this constituent relative to the center of mass. Why is this? Let the angle between $\underline{\omega}$ and $\underline{r}'_i$ be θ. Then the distance of this constituent from the axis of rotation is $|\underline{r}'_i| \sin\theta$. If the object is rotating at $|\underline{\omega}|$ radians per second, then this particular constituent must be be traveling at a speed $|\underline{r}'_i||\underline{\omega}| \sin\theta$ in a direction orthogonal to to the plane defined by $\underline{\omega}$ and $\underline{r}'_i$. In other words, $\underline{v}'_i = \underline{\omega} \times \underline{r}'_i$, where the choice of sign is a matter of convention. The figure 3.2 should help better visualize the situation

Let us repackage the formula for the total angular momentum for a rigid body, using this concept of angular velocity $\underline{\omega}$. For the moment, I want to focus on the contribution to the angular momentum that comes purely from rotation about the center of mass:

$$\underline{L}_{spin} = \sum_i \underline{r}'_i \times m_i \dot{\underline{r}}'_i \ . \tag{3.24}$$

To simplify notation, we shall drop the prime. Using angular velocity, the angular momentum can be written in the form

$$\underline{L}_{spin} = \sum_i m_i \underline{r}_i \times (\underline{\omega} \times \underline{r}_i) \ . \tag{3.25}$$

Using cross product identity (1.36), this becomes

$$\underline{L}_{spin} = \sum_i m_i \left[(\underline{r}_i \cdot \underline{r}_i)\underline{\omega} - (\underline{r}_i \cdot \underline{\omega})\underline{r}_i \right] \ . \tag{3.26}$$

Let's look just at the x-component of this vector, temporarily dropping the i-index:

$$m\left[(x^2+y^2+z^2)\omega_x - (x\omega_x + y\omega_y + z\omega_z)x \right] = m(y^2+z^2)\omega_x - mxy\omega_y - mxz\omega_z \ . \tag{3.27}$$

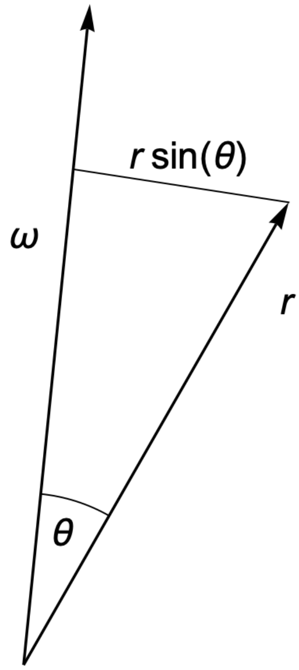

Figure 3.2: Visualizing the relationship between $\underline{\omega}$ and $\underline{v}$.

So we find

$$L_x = \sum_i m_i(y_i^2 + z_i^2)\omega_x - \sum_i m_i x_i y_i \omega_y - \sum_i m_i x_i z_i \omega_z \ . \tag{3.28}$$

To simplify and establish notation, we introduce the following symbols

$$I_{xx} = \sum_i m_i(y_i^2 + z_i^2) \ ,$$

$$I_{xy} = -\sum_i m_i x_i y_i \ , \tag{3.29}$$

$$I_{xz} = -\sum_i m_i x_i z_i \ . \tag{3.30}$$

These are components of the moment of inertia tensor. To find L_y and L_z, we can repeat the discussion above, replacing $x \to y$ and $y \to z$ and $z \to x$. We obtain finally

$$\begin{aligned} L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \ , \\ L_y &= I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \ , \\ L_z &= I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z \ , \end{aligned} \tag{3.31}$$

or more compactly in terms of a matrix $\mathbf{I}$ and a vector $\underline{\omega}$:

$$\underline{L}_{spin} = \mathbf{I} \cdot \underline{\omega} \ . \tag{3.32}$$

The moment of inertia tensor $\mathbf{I}$ is a 3×3 symmetric matrix.

Symmetric matrices are special; they can be diagonalized. Indeed, by appropriately choosing a coordinate system for the rigid body, we can ensure that

$$\mathbf{I} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \ , \tag{3.33}$$

where I_{xx}, I_{yy}, and I_{zz} are all real. In fact something stronger turns out to be true here, which we will not prove. The matrix can be diagonalized by simply rotating the original coordinate system, such that in the new basis, the axes are still mutually orthogonal. For objects with symmetries, these so-called principal axes correspond to axes about which the object has some symmetry. For example, for a box, the moment of inertia tensor is diagonalized by taking axes that are perpendicular and in the center of the faces and sides of the box. The moment of inertia tensor is a source of endless good problems, and I will let you entertain yourself by computing the moment of inertia for some simple objects – a sphere, a cylinder, a rod, a ring, a rectangular plate, etc.

Note that we have used a discretized formalism for discussing a rigid body, employing the index i to list its different constituents. Often it is more convenient to think about a continuous mass distribution. In this case, we have a density function $\rho(x, y, z)$ and an infinitesimal mass element $\rho(x, y, z)dxdydz$ that replaces the point mass m_i we have used up to to this point. Our emphasis in this class will be on rigid bodies where ρ is a constant function although occasionally it is interesting to consider more exotic mass distributions.

Problem: For a box with sides of lengths ℓ_i, $i = x, y, z$, and constant mass density ρ, compute the moment of inertia tensor in a diagonal basis.

Solution: Let us start with I_{xx}, in which case we need to evaluate the three dimensional integral

$$\begin{aligned}
I_{xx} &= \int_{\text{box}} \rho(y^2+z^2)dx\,dy\,dz \\
&= \rho \int_{-\ell_z/2}^{\ell_z/2} \int_{-\ell_y/2}^{\ell_y/2} \int_{-\ell_x/2}^{\ell_x/2} (y^2+z^2)dx\,dy\,dz \\
&= \rho \int_{-\ell_x/2}^{\ell_x/2} dx \int_{-\ell_z/2}^{\ell_z/2} \int_{-\ell_y/2}^{\ell_y/2} (y^2+z^2)dy\,dz \\
&= \rho\ell_x \left(\int_{-\ell_z/2}^{\ell_z/2} dz \int_{-\ell_y/2}^{\ell_y/2} y^2 dy + \int_{-\ell_z/2}^{\ell_z/2} z^2 dz \int_{-\ell_y/2}^{\ell_y/2} dy \right) \\
&= \rho\ell_x \left(\ell_z \frac{y^3}{3}\bigg|_{y=-\ell_y/2}^{y=\ell_y/2} + \ell_y \frac{z^3}{3}\bigg|_{z=-\ell_z/2}^{z=\ell_z/2} \right) \\
&= \frac{\rho\ell_x\ell_y\ell_z}{12}(\ell_y^2+\ell_z^2) = \frac{M}{12}(\ell_y^2+\ell_z^2)
\end{aligned}$$

where in the last line, we introduced the mass of the box $M = \rho\ell_x\ell_y\ell_z$. From this computation, we can deduce also that $I_{yy} = \frac{M}{12}(\ell_x^2 + \ell_z^2)$ and $I_{zz} = \frac{M}{12}(\ell_x^2 + \ell_y^2)$ by permutating the coordinates. The remaining components of the moment of inertia tensor vanish by symmetry. Let's look at I_{xy} to see how that works in more detail

$$\begin{aligned}
I_{xy} &= -\int_{\text{box}} \rho xy\,dx\,dy\,dz \\
&= -\rho \int_{-\ell_z/2}^{\ell_z/2} \int_{-\ell_y/2}^{\ell_y/2} \int_{-\ell_x/2}^{\ell_x/2} xy\,dx\,dy\,dz \\
&= -\rho \int_{-\ell_z/2}^{\ell_z/2} dz \int_{-\ell_y/2}^{\ell_y/2} y\,dy \int_{-\ell_x/2}^{\ell_x/2} x\,dx\ .
\end{aligned}$$

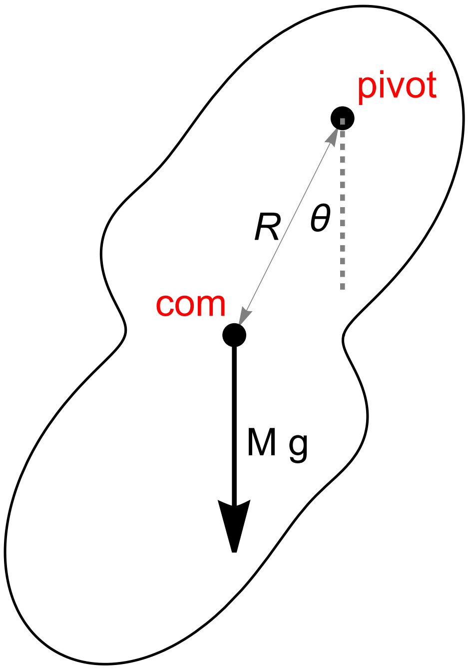

Figure 3.3: The physical pendulum forced to oscillate about a pivot that is a distance R from the center of mass.

Both of the last two integrals vanish because we are integrating odd functions (x and y) over a symmetric domain about the origin. Thus the total integral vanishes as well, as do all the other off-diagonal components of the moment of inertia tensor. In other words, we have picked a simple basis to compute the moment of inertia tensor, a basis in which the tensor is diagonal. Note in general if the sides of the box have different lengths $\ell_x < \ell_y < \ell_z$, then we find as well that $I_{zz} < I_{yy} < I_{xx}$. This ordering will be important for the rotational dynamics of the box, as we will see later.

One last point before moving on. When we work in a diagonal basis for the moment of inertia tensor and then spin the rigid body about one these axes, we can sometimes get away with ignoring the underlying tensor structure. The vector equation $\underline{L} = \mathbf{I} \cdot \underline{\omega}$ boils down to a contribution from a a single component, $|\underline{L}| = I_{xx}\omega_x$, for example if $\underline{\omega} = \underline{e}_x\omega_x$.

Parallel Axis Theorem

We saw above that the angular momentum of an object can be decomposed into a spin and angular momentum contribution (3.23). In some cases, these two components can be completely independent of each other, like the spin of the Earth and the orbital rotation of the Earth around the sun.[2] However, in other cases, the two are closely coupled. The example we have in mind is a physical pendulum, where the object is forced to oscillate about a pivot point that is not the center of mass of the system (see fig. 3.3). In this case, the object rotates about its center of mass once each time it also rotates about the

[2] We will discuss later a weak coupling between these two angular momenta that is leading to a slow increase in the length of the day.

pivot point; in other words the angular velocity of the object about its center of mass is the same as the angular velocity about the pivot point. We take the origin of the coordinate system to be the pivot point and we claim then that $\underline{\omega}_{spin} = \underline{\omega}_{orbital}$ which we can just call $\underline{\omega}$ moving forward.

To be more specific, consider a rigid body with principal moments of inertia I_1, I_2 and I_3. We arrange for the object to spin about an axis parallel to the first principal axis (the one associated with I_1) but displaced from the center of mass a distance R. The angular velocity has magnitude ω. We would like to find the total angular momentum of this system.

Consider the point in the 23-plane intercepted by this new axis of rotation. Call this point the origin of our coordinate system. As we did in decomposing the total angular momentum into an orbital and spin part,

$$\underline{L} = \underline{L}_{orbital} + \underline{L}_{spin} \ ,$$

we can write $\underline{r}_i = \underline{R} + \underline{r}'_i$. Every time the object rotates once about the new origin, any point on the object also rotates once about its center of mass. Therefore, the angular velocity of the object about its center of mass is also ω. From the discussion of the moment of inertia tensor, we have then $\underline{L}_{spin} = I_1\omega$.

The orbital part can be calculated from the formula $\underline{L}_{orbital} = M\underline{R} \times \dot{\underline{R}}$ (3.23). In this coordinate system, the center of mass will be moving in a circle parametrized by

$$\begin{aligned} \underline{R} &= R\cos(\omega t)\underline{e}_2 + R\sin(\omega t)\underline{e}_3 \ , \\ \dot{\underline{R}} &= -R\omega\sin(\omega t)\underline{e}_2 + R\omega\cos(\omega t)\underline{e}_3 \ . \end{aligned}$$

The cross product is

$$\underline{R} \times \dot{\underline{R}} = \omega R^2 \begin{pmatrix} 0 \\ \cos\omega t \\ \sin\omega t \end{pmatrix} \times \begin{pmatrix} 0 \\ -\sin\omega t \\ \cos\omega t \end{pmatrix} = \omega R^2 \underline{e}_1 \ .$$

We conclude that

$$\underline{L} = (I_1 + mR^2)\omega\underline{e}_1 \ . \tag{3.34}$$

It looks like we could reframe this result in terms of our earlier result that $\underline{L}_{spin} = \mathbf{I}\cdot\underline{\omega}$ if we are allowed to redefine the moment of inertia after this shift of the axis of rotation. Indeed, this redefinition is precisely what people often do. We can consider rotation about more general points, but we need to recompute the moment of inertia tensor $\mathbf{I}$. In the case of a parallel shift by a distance ℓ of one of the principal axes, the result takes this simple form, $I_1 \to I_1 + m\ell^2$.

Example: Physical Pendulum

We take an object with mass M. Moreover, it has a moment of inertia component I_{com} about its center of mass, in the directon of one of its principal axes. We then arrange for the object to rotate about a parallel axis a distance ℓ from the center of mass. See fig. 3.3). This parallel axis is also taken to be parallel to the ground. The object is free to oscillate, like a pendulum, under the influence of the gravitational field. Find the frequency of small oscillations.

Solution: Let the line between the center of mass and the pivot point make an angle θ. The magnitude of the torque is $|\underline{N}| = MgR\sin\theta$. The magnitude of the angular momentum about the pivot point is $|\underline{L}| = I_{pivot}\dot{\theta}$. By the parallel axis theorem, $I_{pivot} = I_{com} + MR^2$. Therefore $|\underline{L}| = (I_{com} + MR^2)\dot{\theta}$. We then use $\underline{N} = \dot{\underline{L}}$ and the fact that $\underline{N}$ and $\underline{L}$ point in opposite directions, giving us the differential equation

$$MgR\sin\theta = -(I_{com} + MR^2)\ddot{\theta} \ .$$

For small oscillations, we can replace $\sin\theta \approx \theta$ and we find the differential equation for harmonic motion

$$\ddot{\theta} = -\Omega^2\theta \quad \text{where} \quad \Omega = \sqrt{\frac{MgR}{I_{com} + MR^2}} \ .$$

For a point mass at the end of a massless string, we recover the familiar result that $\Omega = \sqrt{g/R}$.

Kinetic energy of a rotating rigid body

There is an interesting interplay between energy and angular momentum that can explain some peculiar phenomena in the natural world. Often it turns out that angular momentum is a more reliable conserved quantity while the kinetic energy of a rotating system can often leak out into other forms, such as heat or sound. The punchline of this section is two interesting stories, one about satellites and one about the moon, but first we need to understand how to compute the kinetic energy associated with the rotation of a rigid body.

Consider the kinetic energy of a rigid body:

$$T = \frac{1}{2}\sum_j m_j|\dot{\underline{r}}_j|^2 \ . \tag{3.35}$$

As before, we will try to separate the contributions into center of mass motion and rotation about the center of mass, introducing $\underline{r}_j = \underline{R} + \underline{r}'_j$ and its derivative $\dot{\underline{r}}_j = \underline{v}_j = \dot{\underline{R}} + \underline{v}'_j$:

$$T = \frac{1}{2}\sum_j m_j(\dot{\underline{R}} + \underline{v}'_j)^2 = \frac{1}{2}\sum_j (m_j|\dot{\underline{R}}|^2 + 2m_j\underline{v}'_j \cdot \underline{R} + m_j {v'_j}^2) \tag{3.36}$$

By definition of the center of mass, $\sum_j m_j\underline{r}'_j = 0$ and hence $\sum_j m_j\dot{\underline{r}}'_j = \sum_j m_j\underline{v}'_j = 0$. Thus we can divide the kinetic energy up into a center of mass portion and a rotational portion about the center of mass

$$T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\sum_j m_j {v'_j}^2 \ . \tag{3.37}$$

Focusing on the rotational part, we have

$$T_{rot} = \frac{1}{2}\sum_j m_j {v'_j}^2 = \frac{1}{2}\sum_j m_j(\underline{\omega} \times \underline{r}'_j) \cdot (\underline{\omega} \times \underline{r}'_j) \ . \tag{3.38}$$

We now use the vector identity $(\underline{A} \times \underline{B}) \cdot \underline{C} = \underline{A} \cdot (\underline{B} \times \underline{C})$ in the form where $\underline{A} = \underline{\omega}$, $\underline{B} = \underline{r}'_j$ and $\underline{C} = \underline{\omega} \times \underline{r}'_j$. We obtain

$$\begin{aligned} T_{rot} &= \frac{1}{2} \sum_j m_j \underline{\omega} \cdot [\underline{r}'_j \times (\underline{\omega} \times \underline{r}'_j)] \\ &= \frac{1}{2} \underline{\omega} \cdot \sum_j m_j [\underline{r}'_j \times (\underline{\omega} \times \underline{r}'_j)] \ . \end{aligned} \tag{3.39}$$

But this sum is precisely what we identified as the angular momentum about the center of mass (3.25),

$$T_{rot} = \frac{1}{2} \underline{\omega} \cdot \underline{L}_{spin} \ . \tag{3.40}$$

Using our previous result that $\underline{L}_{spin} = I \cdot \underline{\omega}$, we reach the final expressions

$$T_{rot} = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2 \ , \tag{3.41}$$

where we have chosen our coordinates along the principal axes. Alternately, the expression

$$T_{rot} = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3} \ , \tag{3.42}$$

can be more useful because the angular momentum is conserved, not the angular velocity.

Indeed, there is a curious historical incident associated with this equation and the design of satellites in Earth orbit. An early designer of satellites thought it would be a good idea to have a cylindrical satellite spinning about its long axis of rotation. This is the axis with the smallest I_j. Even though there were no external torques on the satellite and its rotational angular momentum should be conserved, the satellite began to wobble after a short time. The problem is that even though the angular momentum is conserved, the kinetic energy is not. Internal friction and movement of the parts of the satellite can begin to dissipate T. The only way to dissipate energy while conserving angular momentum is to reorient the satellite with respect to the direction of its angular momentum such that it is spinning also along axes with larger moments of inertia. Eventually, the satellite should reach a minimum energy configuration when it is spinning only around the axis with the largest moment of inertia. The moral of the story is that flying saucers make better spacecraft than flying cigars.

Another interesting application of these ideas is to understand why there is a side of the moon that we never see from Earth. Naively, moons should both go around their planets and also spin on their axes. The total kinetic energy has contributions both from orbital and spin motion:

$$T_{total} = \frac{1}{2} I \omega_{spin}^2 + \frac{1}{2} M v^2 \ ,$$

where M is the mass of the moon and v is the speed of the moon as it orbits the planet. Likewise, the angular momentum has contributions from both:

$$|\underline{L}| = I \omega_{spin} + MRv \ ,$$

where R is the distance from the planet to the moon. The force of gravity on a moon varies as a function of the distance from the planet, which means the moon experiences an

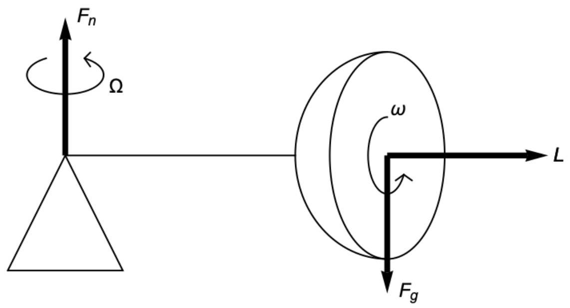

Figure 3.4: The gyroscope

internal tidal force that distorts it longitudinally along the planet-moon axis. (Similarly the planet experiences these tidal forces as well, leading for example to the tides on the Earth.) If the moon were spinning, this tidal force acts as frictional force to slow down its spin rotation. However, the total angular momentum must be conserved and R will increase to compensate. The endpoint of this process is phase locking, where the moon essentially stops spinning, which is exactly what has happened to our moon, and will eventually happen to the Earth as well.

3.4 Gyroscopes

While we have so far emphasized rigid bodies which are free to move through space (with the exception of the physical pendulum), often it is useful to discuss a rigid body where one point is anchored and immovable. Because this point is effectively infinitely massive, the center of mass will be fixed to this immovable point. We can take over wholesale the results above, taking into account that because the fixed point is infinitely massive, there will be no translational motion and taking into account also that we need to recalculate the moment of inertia for the system about this new fixed point. Note that any mass located at this fixed point will not contribute to the moment of inertia about that point.

An important example in the theory of rigid body motion is the gyroscope. An idealized gyroscope consists of a massless rod, fixed at one end to a post. On this rod, at some distance from the post, is a rapidly spinning, massive cylinder. See figure 3.4. There are two external forces acting on the rod and cylinder. Gravity pushes down on the cylinder, and there is a restoring upward normal force from the post at the end of the rod. Intuition tells us the gyroscope should fall down, but conservation of angular momentum has a different story to tell. The full story of the motion is complicated, and we leave it to the optional material at the end of this chapter. However, there is a particular solution to Newton's equations that is interesting and illustrative of the principles we have so far discussed.

Let us imagine a situation where the downward force of gravity $\underline{F}_g$ is exactly balanced by the upward normal force from the rod $\underline{F}_n$. In other words, $\underline{F}_g + \underline{F}_n = 0$. While the forces balance, there is still a torque about the fixed point. The normal force $\underline{F}_n$ cannot exert any torque, because it is located at the fixed point. The gravitational force $\underline{F}_g$

on the other hand will exert a torque. If the center of mass of the cylinder is located a distance $\underline{R}$ out along the rod, then the torque about the pivot will be $\underline{N} = \underline{R} \times \underline{F}_g$. This torque will point orthogonal to the plane established by $\underline{R}$ and $\underline{F}_g$, and will act to change the angular momentum.

What is the angular momentum? We have this heavy, rapidly spinning cylinder spinning about the rod, and hence we have an angular momentum $\underline{L}$ oriented outward (by choice), parallel to the rod. The torque, because it is orthogonal to $\underline{L}$, cannot change the magnitude of $\underline{L}$. However, the torque will change the direction of the angular momentum. The net effect is that the gyroscope will slowly spin around the post, a process called *precession*.

Let us try to be more precise. Assuming the gyroscope is spinning about a principal axis, we can write $|\underline{L}| = I\omega$ where I is one of the principal components of the moment of inertia tensor. Furthermore, the magnitude of the torque is $|\underline{N}| = mgR$. Our ansatz is that the gyroscope should move in a circle and that the torque always points orthogonally to the angular momentum. So let us write then

$$\underline{L} = I\omega\left(\hat{x}\cos(\Omega t) + \hat{y}\sin(\Omega t)\right) \ , \tag{3.43}$$

$$\underline{N} = mgR\left(-\hat{x}\sin(\Omega t) + \hat{y}\cos(\Omega t)\right) \ . \tag{3.44}$$

We have then a solution to $\underline{N} = \frac{d}{dt}\underline{L}$, provided that

$$I\omega\Omega = mgR \ . \tag{3.45}$$

In other words, the precession frequency is

$$\Omega = \frac{mgR}{I\omega} \ . \tag{3.46}$$

This smooth, slow rotation about the post is not the most general solution. More generally, the gyroscope nutates as well. Nutation corresponds to slow up and down oscillations in addition to the precession behavior. So for example, if you release a spinning horizontal gyroscope from rest, it will start to fall briefly before beginning its precession. It will then bob up and down slowly as it moves in a circle. Friction may eventually cause the nutation to damp out, leading to the smooth circular precession motion we described above.

Problem: Consider a gyroscope oriented not perpendicular to the post but at an arbitrary angle. Show that the precession frequency is independent of this angle.

One important example of precession is the Earth. If we think about the plane of rotation in which the Earth moves around the sun, we are familiar with the fact that the Earth spins at an angle of about 22° off from the perpendicular of this plane. Moreover, this angle is responsible for the seasons. It's summer in the hemisphere tilted closer to the sun and winter in the hemisphere oriented away. Less familiar is the fact that, due to tidal forces from the moon and the sun, the direction around which the Earth is spinning is gradually precessing about the perpendicular to its plane of rotation about the sun. This precession takes about 26,000 years. As a result, Polaris will not always be the North Star! Taking into account that the Earth's orbit around the sun is slightly elliptical, there is a further effect that seasons become a bit more extreme in the

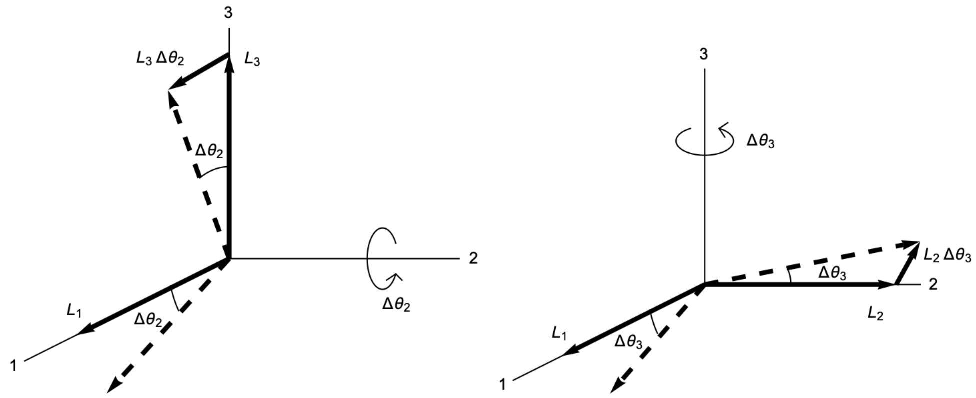

Figure 3.5: Two pictures useful for understanding the derivation of Euler's Equations.

northern hemisphere when the precession cycle orients the northern hemisphere toward the sun at the Earth's point of closest approach to the sun. At the moment, the opposite is true. The northern hemisphere is oriented away from the sun at perihelion, a fact which currently moderates our winters a bit. A fellow named Milankovitch incorporated precession, orbital eccentricity and other effects in a climate model for the Earth in order to predict the ice ages. (Now the dominant effect changing the climate scientists believe is our addition of carbon dioxide to the atmosphere.)

3.5 Euler's Equations and Stability of Rotational Motion (Optional)

We would like to derive the exact equations of motion for a rigid body. To calculate $\dot{\underline{L}}$, we will use a small angle approximation, investigating how the object moves during a small time interval Δt. While the results are correct only to first order, they become exact in the limit $\Delta t \to 0$ and will allow us derive a set of differential equations whose solution describes the motion in question. These equations were first derived by Euler hundreds of years ago and bear his name.

We will work in a frame adapted to the principal axes of the rigid body, labeling these axes 1, 2, and 3. While this frame makes the equations very simple, it hides a different problem – trying to figure out how these axes are oriented in the lab frame. Let us consider how the component L_1 of the angular momentum changes in a time Δt. There will be three contributions. The first comes from a change in ω_1 and is equal to $I_1\Delta\omega_1$. The second comes from the fact that a rotation about the 2-axis will push the L_3 component of the angular momentum slightly into the L_1 direction. This contribution is given by $L_3\Delta\theta_2 = I_3\omega_3\Delta\theta_2$. The third and last contribution comes from the fact that a rotation about the 3-axis will push the L_2 component of the angular momentum slightly into the L_1 direction. This contribution comes with a minus sign and is equal to $-L_2\Delta\theta_3 = -I_2\omega_2\Delta\theta_3$. To understand these last two contributions along with the relative signs, it is useful to stare at figure 3.5. Putting the three contributions together, we find

$$\Delta L_1 = I_1 \Delta\omega_1 + I_3\omega_3\Delta\theta_2 - I_2\omega_2\Delta\theta_3 \ . \tag{3.47}$$

Dividing by Δt and taking the $\Delta t \to 0$ limit gives then

$$\dot{L}_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 \ . \tag{3.48}$$

Repeating the derivation for the other two component and replacing $\dot{L}_i$ with the components of the torque N_i yields Euler's equations:

$$N_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_1\omega_2 \ , \tag{3.49}$$

$$N_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 \ , \tag{3.50}$$

$$N_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 \ . \tag{3.51}$$

If there are no external torques, then of course

$$0 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_1\omega_2 \ , \tag{3.52}$$

$$0 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 \ , \tag{3.53}$$

$$0 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 \ . \tag{3.54}$$

In order to sort out the issue of going between the principal axis frame and lab frame, Euler introduced a set of angles, which also bear his name. The resulting combination of these angles and three equations above is a mess, and we won't repeat it here, but it's good to know there is a general solution. In the cases we need these equations in this class, there will be some other, more geometric way of solving this frame problem, if we need to solve it at all, which we do not in the following example.

Consider the case $\underline{N} = 0$ and also where ω_2 and ω_3 are very small at first. Then we can neglect the second term in (3.52) and write $I_1\dot{\omega}_1 = 0$, or in other words ω_1 is constant. Differentiating the second equation with respect to time and substituting the third equation, we find

$$0 = I_2\ddot{\omega}_2 + (I_3 - I_1)\omega_1\dot{\omega}_3 \ , \tag{3.55}$$

$$= I_2\ddot{\omega}_2 + \frac{(I_3 - I_1)(I_2 - I_1)}{I_3}\omega_1^2\omega_2 \ . \tag{3.56}$$

The expression in the second line is the equation for harmonic motion,

$$\ddot{\omega}_2 = -k\omega_2 \ \text{ where } \ k = \frac{(I_3 - I_1)(I_2 - I_1)}{I_2 I_3}\omega_1^2. \tag{3.57}$$

If $k > 0$, *i.e.* I_1 is either the largest or smallest moment of inertia, then the motion is oscillatory with frequency $\sqrt{k}$, and hence stable. However, if I_1 is neither the biggest nor smallest, then $k < 0$ and the behavior is exponential, indicating an instability. You can see this instability if you have the courage to spin a book in the air about its three principal axes.

[[torque free precession?]]

Chapter 4

Solving the two-body problem

The simplest example of a many-body problem is the 2-body problem. This is meant to refer to two particles undergoing a mutual interaction along with a possible external force. In fact we can solve the 2 body problem for a wide class of forces.

Let us suppose that we have two particles with positions $\underline{r}_1$ and $\underline{r}_2$ which move subject to an external force $\underline{F}_{ext}$ as well as an internal force $\underline{F}_{12} = -\underline{F}_{21}$. We have seen that the center-of-mass $\underline{R}$ only sees $\underline{F}_{ext}$. So let us change variables to

$$\underline{R} = \frac{m_1\underline{r}_1 + m_2\underline{r}_2}{m_1 + m_2} \qquad \underline{r}_{12} = \underline{r}_1 - \underline{r}_2 \ . \tag{4.1}$$

where $M = m_1 + m_2$ is the total mass. We need to invert this to find $\underline{r}_1$ and $\underline{r}_2$ as functions of $\underline{R}$ and $\underline{r}_{12}$

$$M\underline{R} = m_1\underline{r}_1 + m_2(\underline{r}_1 - \underline{r}_{12}) \ . \tag{4.2}$$

Rearranging gives

$$\underline{r}_1 = \frac{M\underline{R} + m_2\underline{r}_{12}}{M} \tag{4.3}$$

and

$$\begin{aligned} \underline{r}_2 &= \underline{r}_1 - \underline{r}_{12} \\ &= \frac{M\underline{R} - m_1\underline{r}_{12}}{M} \end{aligned} \tag{4.4}$$

The conserved energy is

$$\begin{aligned} E &= \frac{1}{2}m_1|\dot{\underline{r}}_1|^2 + \frac{1}{2}m_2|\dot{\underline{r}}_2|^2 + V^{12} + V^{ext} \\ &= \frac{1}{2}M|\dot{\underline{R}}|^2 + \frac{1}{2}\mu|\dot{\underline{r}}_{12}|^2 + V_{12} + V^{ext} \end{aligned} \tag{4.5}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \ , \tag{4.6}$$

is called the reduced mass. Note that in the limit that one mass is much larger than the other, say $m_1 >> m_2$ we simply have

$$\mu = \frac{m_2}{1 + m_2/m_1} = m_2\left(1 - m_2/m_1 + \ldots\right) = m_2 - .. \ , \tag{4.7}$$

corresponding to the lighter particle (but slightly reduced - hence the name) and

$$M = m_1 + m_2 = m_1 (1 + m_2/m_1) = m_1 + ... , \tag{4.8}$$

corresponding to the heavy particle. In addition we find

$$\begin{aligned}
\underline{R} &= \frac{m_1}{m_1+m_2}\underline{r}_1 + \frac{m_2}{m_1+m_2}\underline{r}_2 \\
&= \frac{1}{1+m_2/m_1}\underline{r}_1 + \frac{m_2/m_1}{1+m_2/m_1}\underline{r}_2 \\
&= \underline{r}_1 - (m_2/m_1)\,\underline{r}_{12} + \dots \\
&= \underline{r}_1 + \dots \ ,
\end{aligned} \tag{4.9}$$

so that the centre of mass is essentially just where the heavy particle sits. On the other hand if both particles have the same mass then

$$\mu = \frac{m}{2} \qquad M = 2m \ . \tag{4.10}$$

and

$$\underline{R} = \frac{\underline{r}_1 + \underline{r}_2}{2} \ , \tag{4.11}$$

is the average position.

4.1 From the 2-Body problem to two 1-body problems

A key point of the two body problem is that, under certain conditions, namely if V^{ext} only depends on $\underline{R}$ and V_{12}^{int} only depends on $\underline{r}_{12}$, then we can reduce it to two one body problems: one for $\underline{R}$ and one for $\underline{r}_{12}$. To see this we note that the equations for $\underline{r}_1$ and $\underline{r}_2$ are

$$\begin{aligned}
m_1\ddot{\underline{r}}_1 &= -\underline{\nabla}_1 V^{ext} - \underline{\nabla}_1 V_{12} \\
m_2\ddot{\underline{r}}_2 &= -\underline{\nabla}_2 V^{ext} - \underline{\nabla}_2 V_{12} \ .
\end{aligned} \tag{4.12}$$

The equation for $\underline{R}$ is obtained by summing the two equations in (4.12)

$$\begin{aligned}
M\ddot{\underline{R}} &= -\underline{\nabla}_1 V^{ext} - \underline{\nabla}_2 V^{ext} \\
&= -\frac{m_1}{M}\underline{\nabla}_{\underline{R}} V^{ext} - \frac{m_2}{M}\underline{\nabla}_{\underline{R}} V^{ext} \\
&= -\underline{\nabla}_{\underline{R}} V^{ext} \\
&= \sum_i \underline{F}_i^{ext}
\end{aligned} \tag{4.13}$$

which agrees with our general result above. Here we used the NIII condition that $\underline{\nabla}_1 V_{12} = -\underline{\nabla}_2 V_{12}$ as well as the chain rule to write

$$\underline{\nabla}_1 V^{ext} = \frac{m_1}{M}\underline{\nabla}_{\underline{R}} V^{ext} \ , \qquad \underline{\nabla}_2 V^{ext} = \frac{m_2}{M}\underline{\nabla}_{\underline{R}} V^{ext} \tag{4.14}$$

assuming that V^{ext} only depends on $\underline{R}$. Therefore the equation for $M\ddot{\underline{R}}$ only involves $\underline{R}$ and again leads to the conservation of

$$E_{cm} = \frac{1}{2}M|\dot{\underline{R}}|^2 + V_{ext} \ . \tag{4.15}$$

To obtain an equation for the relative position we multiply the first equation in (4.12) by m_2 and the second by m_1 and then subtract them to find

$$M\mu\ddot{\underline{r}}_{12} = -m_2\underline{\nabla}_1 V^{ext} + m_1\underline{\nabla}_2 V^{ext} - m_2\underline{\nabla}_1 V_{12} + m_1\underline{\nabla}_2 V_{12} \tag{4.16}$$

If V^{ext} is only a function of $\underline{R}$ then

$$-m_2\underline{\nabla}_1 V^{ext} + m_1\underline{\nabla}_2 V^{ext} = -\frac{m_2 m_1}{M}\underline{\nabla}_{\underline{R}} V^{ext} + \frac{m_1 m_2}{M}\underline{\nabla}_{\underline{R}} V^{ext} = \underline{0}\ . \tag{4.17}$$

Also if V_{12} is just a function of $\underline{r}_{12}$ then

$$\underline{\nabla}_2 V_{12} = -\underline{\nabla}_1 V_{12} = -\underline{\nabla}_{12} V_{12}\ . \tag{4.18}$$

Thus (4.16) becomes

$$\mu\ddot{\underline{r}}_{12} = -\underline{\nabla}_{12} V_{12}\ . \tag{4.19}$$

This equation only depends on $\underline{r}_{12}$ and not $\underline{R}$. As before this equation tells us that

$$E_{12} = \frac{1}{2}\mu|\dot{\underline{r}}_{12}|^2 + V_{12} \tag{4.20}$$

is conserved on its own.

Thus we have reduced the two-body problem to two one-body problems. In particular the total energy $E = E_{12} + E_{cm}$ is actually the sum of two conserved energies.

Finally if V_{12} only depends on $r_{12} = |\underline{r}_{12}|$ then energy and angular momentum of the relative system will be conserved leading to a single one-dimensional problem:

$$E_{12} = \frac{1}{2}\mu\dot{r}_{12}^2 + V_{eff}^{int} \qquad V_{eff}^{int} = V_{12}(r_{12}) + \frac{l_{12}^2}{2\mu r_{12}^2} \tag{4.21}$$

and similarly if V^{ext} only depends on $R = |\underline{R}|$ then

$$E_{cm} = \frac{1}{2}M\dot{R}^2 + V_{eff}^{ext} \qquad V_{eff}^{ext} = V^{ext}(R) + \frac{l_{cm}^2}{2MR^2}\ . \tag{4.22}$$

Next you might try the three-body problem, but that is unsolvable in general. But some things can be said in special cases or limits (such as when one mass is much heavier than the others).

Example

Let us consider N electrons moving in the presence of electric and magnetic fields $\underline{E}$, $\underline{B}$ which we assume to be constant and pointing in the same direction. This gives an external force

$$\underline{F}_i^{ext} = e\underline{E} + e\dot{\underline{r}}_i \times \underline{B}\ . \tag{4.23}$$

that acts on the ith electron. However in addition there will be inter-electron repulsive forces due to the fact that like charges repel. These give rise to the internal forces

$$\underline{F}_{ij}^{int} = \frac{e^2}{4\pi}\frac{1}{|\underline{r}_i - \underline{r}_j|^2}\frac{\underline{r}_i - \underline{r}_j}{|\underline{r}_i - \underline{r}_j|}\ . \tag{4.24}$$

This force is similar in form to gravity and one can easily check that it arises from the potential

$$V_{ij} = \frac{e^2}{4\pi}\frac{1}{|\underline{r}_i - \underline{r}_j|} \tag{4.25}$$

Thus the equation for a single electron is

$$m\ddot{\underline{r}}_i = e\underline{E} + e\dot{\underline{r}}_i \times \underline{B} + \sum_{j\neq i} \frac{e^2}{4\pi} \frac{1}{|\underline{r}_i - \underline{r}_j|^2} \frac{\underline{r}_i - \underline{r}_j}{|\underline{r}_i - \underline{r}_j|} \tag{4.26}$$

This is clearly a tricky thing to solve in general (impossible might be a better term, and we have not even included the magnetic interactions between the electrons). However if the electrons are well separated, then the final term is small compared to the $\underline{E}$ and $\underline{B}$ terms. In this case the problem of each electron is separated and individually solvable. Roughly speaking this is true if

$$\frac{e^2}{|\underline{r}_i - \underline{r}_j|^2} << e\sqrt{|\underline{E}|^2 + v^2|\underline{B}|^2} \tag{4.27}$$

where v is the speed of the electrons. This can be achieved by making the background electric and magnetic fields to be large.

Alternatively, we can essentially solve the system with inter-electron force if we consider just two electrons. There isn't a potential V^{ext} for the magnetic field (though the magnetic force is conservative!) so let us set that to zero and only look at a constant electric field $\underline{E}$. For the centre of mass we find (recall $M = 2m$)

$$M\ddot{\underline{R}} = \underline{F}_1^{ext} + \underline{F}_2^{ext} = 2e\underline{E} \tag{4.28}$$

which corresponds to the potential

$$V^{ext} = -e\underline{E}\cdot(\underline{r}_1 + \underline{r}_2) = -2e\underline{E}\cdot\underline{R} \tag{4.29}$$

This is of the form we needed above in that it only depends on $\underline{R}$. The electric field is just a constant force like gravity was for the skier going down the hill (with down now being determined by the direction in which $\underline{E}$ points). So the solution will be similar. Thus we can solve for the centre-of-mass by writing

$$\underline{R} = \frac{1}{2}at^2\underline{E} + \underline{v}t + \underline{R}(0) \tag{4.30}$$

where $\underline{v}$ and $\underline{R}(0)$ are the initial centre of mass velocity and position. To determine a we substitute into (4.28) and find

$$Ma = 2e \qquad \Longrightarrow \qquad a = \frac{2e}{M} = \frac{e}{m}\ . \tag{4.31}$$

Thus the background electric field causes a constant acceleration of the electrons along the direction of $\underline{E}$.

Next we look at $\underline{r}_{12}$. From the formulae above we find the relation (recall $\mu = m/2$)

$$E = \frac{1}{2}\mu\dot{r}_{12}^2 + V_{eff} \qquad V_{eff}(r_{12}) = \frac{e^2}{4\pi}\frac{1}{r_{12}} + \frac{l^2}{2\mu r_{12}^2}\ . \tag{4.32}$$

This is similar to to the gravitational case we saw before when studying planetary motion except that all terms are positive and hence V_{eff} is monotonically decreasing from infinity at $r_{12} = 0$ to zero as $r_{12} \to \infty$. Therefore there are no bound state solutions and r_{12} always ends up growing arbitrarily large. In other words if the two electrons are sufficiently far away from each other there is little attractive force but once they come close there is a repulsive force that sends them far away again from each other. Thus the pair of electrons will constantly accelerate along the direction of $\underline{E}$ and scatter off each other whenever they come too close.

Problem Show that if we ignore the internal forces and set $\underline{E} = \underline{0}$, then a constant background magnetic field causes the electrons to move in circles in the plane orthogonal to $\underline{B}$.

Chapter 5

Lagrangian Mechanics

So when a rain drop falls do you think that it is trying to solve differential equations arising from Newtons laws to figure out what to do on the way down? Surely not. So how does it know? Here we need to introduce a new level of abstraction. And an apparent miracle.

5.1 The Principle of Least Action

You may have noticed in the treatment of the planets, that we never really used the force at all. Although we used Newtons Laws to introduce the notion of a potential which we then used to derive a conserved energy. Let us return to the case of a single particle. In particular we had kinetic energy

$$T = \frac{1}{2}m|\dot{\underline{r}}|^2 \tag{5.1}$$

and if the force was conservative, a potential energy $V(\underline{r})$. This lead to the total energy

$$E = T + V \ . \tag{5.2}$$

This is conserved: for a given path of a particle, E remains constant.

There is something else that we can consider:

$$L = T - V = \frac{1}{2}m|\dot{\underline{r}}|^2 - V(\underline{r}) \ . \tag{5.3}$$

This is not conserved, it changes in time as the particle moves. But we can consider instead the functional

$$S[\underline{r}] = \int_{t_1}^{t_2} L(\underline{r}, \dot{\underline{r}})dt \ . \tag{5.4}$$

S is called the **action** and L is the **Lagrangian**. The action is a functional in the sense that it is a function of a function: given a function, namely the entire path $\underline{r}(t)$ of a particle from t_1 to t_2, then $S[\underline{r}]$ gives a number. This number depends on the whole path not just any given point on it. This is indicated by the square brackets $S = S[\underline{r}]$. Note that the Lagrangian is not a functional because it depends on $\underline{r}$ and $\dot{\underline{r}}$ at a single time. Rather one tends to think of it as a formal function of $\underline{r}$ and $\dot{\underline{r}}$ as independent variables, without thinking of the fact that $\underline{r}$ and $\dot{\underline{r}}$ are also themselves functions of time.

Finally the action is somewhat analogous to the work which is also a functional (*i.e.* a function of a function):

$$W[\underline{r}] = \int_{t_1}^{t_2} \underline{F} \cdot \dot{\underline{r}} dt \tag{5.5}$$

except that whereas work is path-independent for conservative forces, the action S is very much path-dependent also in this case. Although there is the following crude analogy. When you are pushing a shopping cart around the supermarket looking for the marmite, if you know what you are doing you will not wander around everywhere but rather take the shortest path to the marmite and then the shortest path to the check-out. You do this to minimize the amount of work that you must do to push the shopping trolley. As we will now see particles make a similar calculation. We now state the

Principle of Least Action: Particles move so as to extremize the action S as a functional of all possible paths between $\underline{r}(t_1)$ and $\underline{r}(t_2)$. That is to say Newton's Laws of motion are equivalent to the statement that

$$\delta S = 0 \ , \tag{5.6}$$

where δS is the first order variation of the action obtained by shifting the path $\underline{r} \to \underline{r}+\delta\underline{r}$ (and $\dot{\underline{r}} \to \dot{\underline{r}} + \delta\dot{\underline{r}}$), subject to the condition that the end points of the path are fixed: $\delta\underline{r}(t_1) = \delta\underline{r}(t_2) = \underline{0}$.

Let us prove this. To do this we compute

$$\begin{aligned} S[\underline{r}+\delta\underline{r}] &= \int_{t_1}^{t_2} \left[\frac{1}{2}m(\dot{\underline{r}}+\delta\dot{\underline{r}})\cdot(\dot{\underline{r}}+\delta\dot{\underline{r}}) - V(\underline{r}+\delta\underline{r})\right] dt \\ &= \int_{t_1}^{t_2} \left[\frac{1}{2}m\dot{\underline{r}}\cdot\dot{\underline{r}} + \frac{1}{2}m\delta\dot{\underline{r}}\cdot\dot{\underline{r}} + \frac{1}{2}m\dot{\underline{r}}\cdot\delta\dot{\underline{r}} - V(\underline{r}+\delta\underline{r})\right] dt + \ldots \\ &= \int_{t_1}^{t_2} \left[\frac{1}{2}m\dot{\underline{r}}\cdot\dot{\underline{r}} + m\dot{\underline{r}}\cdot\delta\dot{\underline{r}} - V(\underline{r}+\delta\underline{r})\right] dt \end{aligned} \tag{5.7}$$

where the dots denote higher order terms in $\delta\underline{r}$ which we have dropped in the last line as we will only be interested in the first order variation. Next we need to taylor expand $V(\underline{r}+\delta\underline{r})$:

$$V(\underline{r}+\delta\underline{r}) = V(\underline{r}) + \underline{\nabla}V\cdot\delta\underline{r} + \ldots \tag{5.8}$$

where again the dots denote higher order terms in $\delta\underline{r}$. Putting this back in we find

$$\begin{aligned} S[\underline{r}+\delta\underline{r}] &= \int_{t_1}^{t_2} \left[\frac{1}{2}m\dot{\underline{r}}\cdot\dot{\underline{r}} - V(\underline{r}) + m\dot{\underline{r}}\cdot\delta\dot{\underline{r}} - \underline{\nabla}V\cdot\delta\underline{r}\right] dt \\ &= S[\underline{r}] + \int_{t_1}^{t_2} \left[m\dot{\underline{r}}\cdot\delta\dot{\underline{r}} - \underline{\nabla}V\cdot\delta\underline{r}\right] dt + \ldots \end{aligned} \tag{5.9}$$

Our next step is to note that the third term can be manipulated using integration by parts:

$$m\dot{\underline{r}}\cdot\delta\dot{\underline{r}} = \frac{d}{dt}\left(m\dot{\underline{r}}\cdot\delta\underline{r}\right) - m\ddot{\underline{r}}\cdot\delta\underline{r} \ . \tag{5.10}$$

Thus we have

$$\begin{aligned} S[\underline{r}+\delta\underline{r}] &= S[\underline{r}] + \int_{t_1}^{t_2} \frac{d}{dt}\left(m\dot{\underline{r}}\cdot\delta\underline{r}\right) dt - \int_{t_1}^{t_2} \left[m\ddot{\underline{r}}\cdot\delta\underline{r} + \underline{\nabla}V\cdot\delta\underline{r}\right] dt + \ldots \\ &= S[\underline{r}] + m\dot{\underline{r}}\cdot\delta\underline{r}\,\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[m\ddot{\underline{r}} + \underline{\nabla}V\right]\cdot\delta\underline{r} dt + \ldots \end{aligned} \tag{5.11}$$

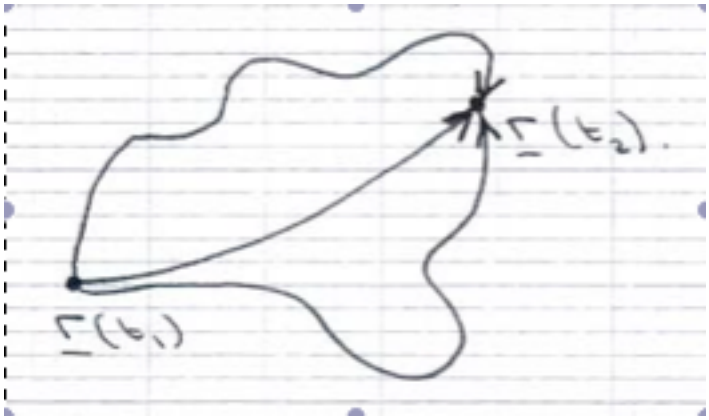

Figure 5.1: More paths

Now we assume that we vary the action over all paths which begin at $\underline{r}(t_1)$ and end at $\underline{r}(t_2)$. Thus we impose that $\delta\underline{r}(t_1) = \delta\underline{r}(t_2) = \underline{0}$. Therefore we find

$$S[\underline{r} + \delta\underline{r}] = S[\underline{r}] - \int_{t_1}^{t_2} [m\ddot{\underline{r}} + \underline{\nabla} V] \cdot \delta\underline{r}\, dt + \dots \tag{5.12}$$

from which we read off the first order variation:

$$\delta S[\underline{r}] = S[\underline{r} + \delta\underline{r}] - S[\underline{r}] = - \int_{t_1}^{t_2} [m\ddot{\underline{r}} + \underline{\nabla} V] \cdot \delta\underline{r}\, dt \ . \tag{5.13}$$

The claim is that for arbitrary variations in the path, the particle will extremise S: $\delta S = 0$. The path which extremises S against an arbitrary variation is therefore the one for which:

$$m\ddot{\underline{r}} + \underline{\nabla} V = \underline{0} \ . \tag{5.14}$$

This is just NII with $\underline{F} = -\underline{\nabla} V$ and $\underline{p} = m\dot{\underline{r}}$!

5.2 Generalized Coordinates and Lagrangians

The power of the Lagrangian method lies in the fact that we can relatively simply construct Lagrangians for different physical systems by determining the kinetic and potential energies as a function of the dynamical variables. We get to use scalar quantities – the kinetic and the potential energy – instead of vectorial quantities – force, momentum, and angular momentum – in setting up the system and trying to understand it. The Lagrangian method will give you a very straightforward way, for example, of using an angle θ as a coordinate without having to at the same time think about which way an angular momentum is pointing, which way a torque is pointing, etc. It's a streamlined process for setting up $\underline{F} = \dot{\underline{p}}$ without having to think in three dimensions.

Indeed, as we have already seen the Cartesian coordinates (x, y, z) may not be the best way of setting up a particular problem. Instead, it is often useful to develop **generalized coordinates**. Instead of thinking of the dynamical variables as the positions $\underline{r}_i$ of N particles, we consider a generic system with "generalized" coordinates that we denote by q_i. Here each q_i is treated as a scalar, not a vector, and we will take the index i to be generic and range over all the generalized coordinates in the problem at hand.

A related concept is the **degree of freedom**. A degree of freedom is a generalized coordinate that is allowed to evolve in time without constraint. If we have a system of N free particles with positions $\underline{r}_1, ..., \underline{r}_N$ then the generalized coordinates q_i are just the $3N$ components of the positions so that $i = 1, ..., 3N$. This has $3N$ degrees of freedom.

We will want to look at constrained systems where the various generalized coordinate are related to each other leading to a reduction in the number of degrees of freedom. Another example is the skier we considered early on. The skier is constrained to lie on a hill slope and furthermore we assume that they went straight down the hill. Thus even though a skier is described by a three-vector $\underline{r}$, in the end we only used the distance that the skier was up the slope from the bottom, which we denoted by r. Consider a pendulum where the weight is constrained to sit at the end of a rod of fixed length. In this case the generalized coordinate is simply the angle of the pendulum from the equilibrium position.

Thus we wish to know how to evaluate the principle of least action $\delta S = 0$ for a general Lagrangian. We will assume that the Lagrangian is a function of $q_i(t)$ and $\dot{q}_i(t)$ but not higher order derivatives (although this can also be considered). It may also have an explicit dependence on t. Thus we start with

$$S[q_i] = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t), t)dt \; . \tag{5.15}$$

To evaluate δS we expand $q_i \to q_i + \delta q_i$, $\dot{q}_i \to \dot{q}_i + \delta\dot{q}_i$. We treat q_i and $\dot{q}_i$ as independent. Although these are functions, at a fixed value of t we could just think of them as ordinary variables. Therefore we can expand L using the familiar rules of calculus:[1]

$$\begin{aligned} S[q_i + \delta q_i] &= \int_{t_1}^{t_2} L(q_i(t) + \delta q_i(t), \dot{q}_i(t) + \delta\dot{q}_i(t), t)dt \\ &= \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) + \sum_i \frac{\partial L}{\partial q_i}\delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i + \ldots dt \end{aligned} \tag{5.16}$$

where the dots denote higher order terms in δq_i and $\delta\dot{q}_i$. We do not need these terms to evaluate δS which is, by definition, the first order term in the variation:

$$\delta S = S[q_i + \delta q_i] \mid_{\mathcal{O}(\delta)} - S[q_i] \; . \tag{5.17}$$

Therefore we find

$$\delta S = \sum_i \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i \right) dt \; . \tag{5.18}$$

Next we want to write the second term in terms of δq_i, not $\delta\dot{q}_i$. To do this we rewrite the second term as a total derivative plus something else:

$$\frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \; . \tag{5.19}$$

Substituting this into δS we find

$$\begin{aligned} \delta S &= \sum_i \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i + \frac{\partial L}{\partial q_i}\delta q_i dt \\ &= \left. \frac{\partial L}{\partial \dot{q}_i}\delta q_i \right|_{t_1}^{t_2} - \sum_i \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} \right] \delta q_i dt \; . \end{aligned} \tag{5.20}$$

[1] In the literature one sometimes sees $\delta L/\delta q_i$ instead of $\partial L/\partial q_i$.

To deal with the boundary term, we note that we want to make an arbitrary variation of the path

$$q_i(t) \to q_i(t) + \delta q_i(t) \ . \tag{5.21}$$

However we only want to consider paths that start at a fixed starting point $q_i(t_1)$ and end at a fixed end point. Thus we keep $\delta q_i(t)$ arbitrary except that

$$\delta q_i(t_1) = \delta q_i(t_2) = 0 \ . \tag{5.22}$$

And therefore

$$\frac{\partial L}{\partial \dot{q}_i}\delta q_i\bigg|_{t_1}^{t_2} = 0 \ , \tag{5.23}$$

so that finally we find

$$\delta S = -\sum_i \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right] \delta q_i dt \ . \tag{5.24}$$

Now the principle of least action asserts that the dynamical path of the system is the one for which $\delta S = 0$ for any choice of $\delta q_i(t)$ (subject to (5.22)). This will only be the case if

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \ . \tag{5.25}$$

These are known as the Euler-Lagrange equations.

Finally we note that two Lagrangians that differ by a total derivative will give the same Euler-Lagrange equations and hence correspond to the same physical system. In particular if

$$L'(q_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \frac{d}{dt}\Omega(q_i, t) \ , \tag{5.26}$$

then the associated actions differ by boundary terms

$$S'[q_i] = S[q_i] + \Omega(q_i(t_2), t_2) - \Omega(q_i(t_1), t_1) \ . \tag{5.27}$$

Since we do not vary the boundary values $\delta q_i(t_1) = \delta q_i(t_2) = 0$, it follows that

$$\delta S' = \delta S \ , \tag{5.28}$$

and hence we would find the same Euler-Lagrange equation.

5.2.1 Simple Examples

This is all quite abstract. Let us see how it works in several examples.

Example 1: a particle in 3D

Let us go back to the single particle that we originally studied:

$$\begin{aligned} L &= \frac{1}{2}m|\dot{\underline{r}}|^2 - V(\underline{r}) \\ &= \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - V(q_1, q_2, q_3) \end{aligned} \tag{5.29}$$

where in the second line we have rewritten the Lagrangian in terms of the "generalized" coordinates q_i, $i = 1, 2, 3$ which are simply the components of $\underline{r} = (q^1, q^2, q^3)$, *i.e.* $q_i = r^a$. From here we see that (recall that we think of q_i and $\dot{q}_i$ as independent):

$$\begin{aligned}\frac{\partial L}{\partial q_i} &= -\frac{\partial V}{\partial q_i} \\ \frac{\partial L}{\partial \dot{q}_i} &= m\dot{q}_i \end{aligned} \tag{5.30}$$

Thus the Euler-Lagrange equation (5.25) is

$$m\ddot{q}_i + \frac{\partial V}{\partial q_i} = 0 \ , \tag{5.31}$$

which is the component version of (5.14). In particular we have simply rediscovered NII with $F_i = -\partial V/\partial q_i$.

Example 2: The skier

Let us look at our skier again. Here $\underline{r} = r\underline{e}_D$, where $\underline{e}_D$ was the constant unit vector pointing down the hill. Thus

$$\begin{aligned} T &= \frac{1}{2}m|\dot{\underline{r}}|^2 \\ &= \frac{1}{2}m(\dot{r}\underline{e}_D)\cdot(\dot{r}\underline{e}_D) \\ &= \frac{1}{2}m\dot{r}^2 \ . \end{aligned} \tag{5.32}$$

The potential energy is just proportional to the height $h = r\sin\theta$:

$$V = mgr\sin\theta \ . \tag{5.33}$$

Thus the Lagrangian is

$$L = \frac{1}{2}m\dot{r}^2 - mgr\sin\theta \ . \tag{5.34}$$

Here there is just one generalized coordinate $q = r$. We can easily evaluate

$$\begin{aligned}\frac{\partial L}{\partial r} &= -mg\sin\theta \\ \frac{\partial L}{\partial \dot{r}} &= m\dot{r} \end{aligned} \tag{5.35}$$

and hence the Euler-Lagrange equation is

$$m\ddot{r} + mg\sin\theta = 0 \ , \tag{5.36}$$

which is what we found before. Note that here we did not have to worry about resolving the forces into the part $\underline{F}_D$ along the direction of motion and the part $\underline{F}_V$ that is cancelled by the upwards force of the hill pushing back on the skier. All we needed was to identify the potential, which is the height.

In fact we can easily allow for a more interesting ski slope where the ski slope has a non-trivial profile $y = h(x)$. Here x is the direction along the horizon and y is the height above the ground.

In this case the potential is

$$V = mgh(x) \tag{5.37}$$

The kinetic energy is (we only consider motion straight up and down the hill and not side-to-side across the slope)

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) \\ &= \frac{1}{2}m\left[\dot{x}^2+\left(\frac{dh}{dx}\dot{x}\right)^2\right] \\ &= \frac{1}{2}m\left[1+\left(\frac{dh}{dx}\right)^2\right]\dot{x}^2 \ . \end{aligned} \tag{5.38}$$

Thus the Lagrangian is

$$L = \frac{1}{2}m\left[1+\left(\frac{dh}{dx}\right)^2\right]\dot{x}^2 - mgh(x) \tag{5.39}$$

From here we find

$$\begin{aligned} \frac{\partial L}{\partial x} &= m\frac{dh}{dx}\frac{d^2h}{dx^2}\dot{x}^2 - mg\frac{dh}{dx} \\ \frac{\partial L}{\partial \dot{x}} &= m\left[1+\left(\frac{dh}{dx}\right)^2\right]\dot{x} \end{aligned} \tag{5.40}$$

and hence the Euler-Lagrange equation is

$$\begin{aligned} & \frac{d}{dt}\left(m\left[1+\left(\frac{dh}{dx}\right)^2\right]\dot{x}\right) - m\frac{dh}{dx}\frac{d^2h}{dx^2}\dot{x}^2 + mg\frac{dh}{dx} = 0 \\ \Longleftrightarrow \quad & m\left[1+\left(\frac{dh}{dx}\right)^2\right]\ddot{x} + m\frac{dh}{dx}\frac{d^2h}{dx^2}\dot{x}^2 + mg\frac{dh}{dx} = 0 \end{aligned} \tag{5.41}$$

This equation would have been pretty hard to determine based directly on Newton's Laws. Although in this case it is simply equivalent to the conservation of energy

$$E = T + V = \frac{1}{2}m\left[1+\left(\frac{dh}{dx}\right)^2\right]\dot{x}^2 + mgh \tag{5.42}$$

as can be seen by evaluating $dE/dt = 0$.

We find the simple skier we had before if we identify $x = r\cos\theta$ and $h = r\sin\theta = x\tan\theta$.

Example 3: Circular motion and centrifugal force

Let us look at a free particle $V = 0$, lying in a plane but using polar coordinates. We looked at this problem before and took:

$$\underline{r} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix} \ . \tag{5.43}$$

To compute the Lagrangian we note that

$$\dot{\underline{r}} = \begin{pmatrix} \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ 0 \end{pmatrix} \ . \tag{5.44}$$

and hence

$$\begin{aligned} L &= \frac{1}{2}m|\underline{\dot{r}}|^2 \\ &= \frac{1}{2}m((\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2) \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \ . \end{aligned} \tag{5.45}$$

The generalized coordinates are $q_1 = r$ and $q_2 = \theta$. Let us look at the Euler-Lagrange equations; there will be one for r and one for θ. First r:

$$\begin{aligned} \frac{\partial L}{\partial r} &= mr\dot{\theta}^2 \\ \frac{\partial L}{\partial \dot{r}_i} &= m\dot{r} \ , \end{aligned} \tag{5.46}$$

so

$$m\ddot{r} - mr\dot{\theta}^2 = 0 \ . \tag{5.47}$$

The second term is the centrifugal force term. Recall this was a fictitious force and indeed it arises here because we have not used Cartesian coordinates. Indeed in the Lagrangian we have set $V = 0$ so that there is no 'real' force.

For θ we find

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= 0 \\ \frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta} \ , \end{aligned} \tag{5.48}$$

so

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = 0 \ . \tag{5.49}$$

This equation immediately gives us the conservation of angular momentum:

$$|\underline{L}| = mr^2\dot{\theta} = constant. \tag{5.50}$$

We could also consider adding a potential term $V(r,\theta)$. This would then correct the Euler-Lagrange equations to

$$\begin{aligned} m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} &= 0 \ , \\ mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + \frac{\partial V}{\partial \theta} &= 0 \ , \end{aligned} \tag{5.51}$$

corresponding to a force with components $F_r = -\partial V/\partial r$, $F_\theta = -\partial V/\partial\theta$. We also see that the F_θ component generates torque, corresponding to non-conservation of angular momentum.

5.3 Constraints

The last two examples were actually cases where there was a constraint on the system. In particular, rather than looking at an object that was free to move in all three-dimensions there was always some kind of restriction: The skier was required to stay on the slope, and the particle undergoing circular motion was required to lie in the plane. These constraints were so simple that we hardly noticed them at all. Nevertheless they are examples of **holonomic** constraints:

Definition: A holonomic constraint on a system is a function of the form

$$C(q_i, t) = 0 \tag{5.52}$$

that is imposed on the coordinates, but not their derivatives, at all times for some function C.

There can of course be more than one such constraint on a system. Constraints that are not of this type are called, no surprise here, **non-holonomic**.

In general each holonomic constraint reduces the number degrees of freedom of the system by one. In particular one uses each constraint to solve for one of the generalized coordinates in terms of the others.

One of the great powers of the Lagrangian formulation is that it can handle holonomic constraints quite easily (in principle - one can always come up with examples that are tough to solve in practice). This is because we are only required to determine the kinetic and potential energies of the system. We do not need to analyze each of the forces and counter-forces to determine the net force on each particle. For the examples above this isn't so hard to do. We did it for some of them. Let us now look at some more difficult problems that involve constraints. Although possible, you would find it very tricky to solve them by analysing all the forces.

(One set of problems that Lagrangians cannot deal with involve non-conservative forces. If friction is an important effect in your problem, you need to go back to Newton's Laws.)

5.3.1 The Pendulum and Double Pendulum in the Plane

A pendulum is a weight of mass m that is attached to a rigid rod of length l which is itself held fixed at the other end. There is a story that Galileo was in church and saw the chandeliers swinging and by the end of the evidently less than riveting sermon, he had deduced the form of their motion with the classic result that the period of oscillation is independent of the mass of the chandelier.

To construct the Lagrangian we note that the position of the chandelier is

$$\underline{r} = \begin{pmatrix} l\sin\theta \\ -l\cos\theta \\ 0 \end{pmatrix} . \tag{5.53}$$

Since l is fixed we have

$$\dot{\underline{r}} = \begin{pmatrix} l\dot{\theta}\cos\theta \\ l\dot{\theta}\sin\theta \\ 0 \end{pmatrix} . \tag{5.54}$$

and hence the kinetic energy is

$$T = \frac{1}{2}m|\dot{\underline{r}}|^2 = \frac{1}{2}ml^2\dot{\theta}^2 \tag{5.55}$$

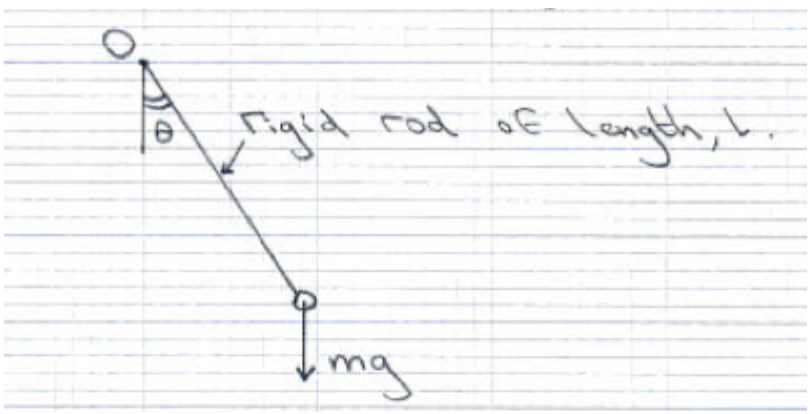

Figure 5.2: The pendulum

Indeed this is the same in example 3 just with $r \to l$ a constant (as well as $\theta \to \theta - \pi/2$). Unlike example 3 there is a potential due to gravity which is just the height (much like the skier):

$$V = -mgl\cos\theta \ . \tag{5.56}$$

Therefore the Lagrangian is

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \ . \tag{5.57}$$

We evaluate

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= -mgl\sin\theta \ , \\ \frac{\partial L}{\partial \dot{\theta}} &= ml^2\dot{\theta} \ , \end{aligned} \tag{5.58}$$

so that the Euler-Lagrange equation for θ is

$$ml^2\ddot{\theta} + mgl\sin\theta = 0 \ . \tag{5.59}$$

Here we see that the factors of m cancel, corresponding to Galileo's observation:

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \ . \tag{5.60}$$

The full solution involves a special function. However for small oscilations, $\sin\theta = \theta + \ldots$ and one can simply take

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \ . \tag{5.61}$$

The solution to this is

$$\theta = A\sin\left(\sqrt{\frac{g}{l}}t\right) + B\cos\left(\sqrt{\frac{g}{l}}t\right) \tag{5.62}$$

for arbitrary constants A and B. The independence of the period of oscillation on the amplitude of oscillation is precisely why pendulums make good clocks. Another amusing observation: if you know the length of the cord holding a chandelier and you have a good time keeping device, you can measure the gravitational constant g.

Let us next consider a pendulum that is attached to a second pendulum (see Figure 5.3). The rods connecting then both have length l and they both have the same mass

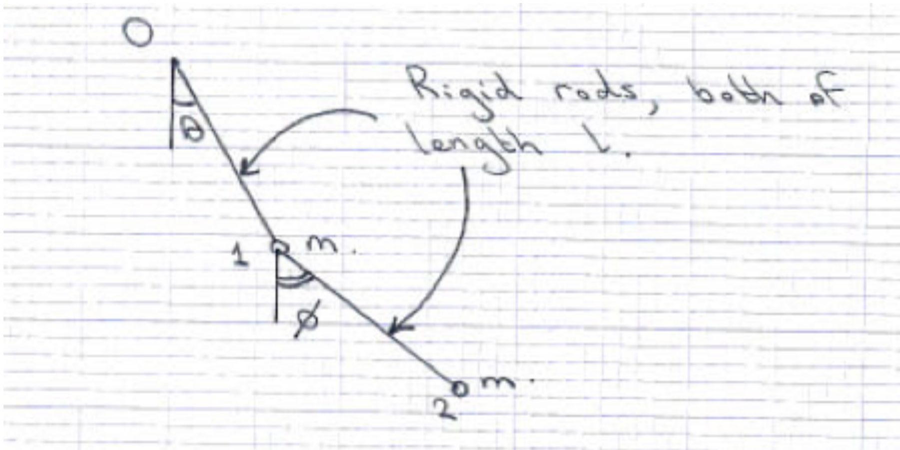

Figure 5.3: The double pendulum

m. The details of this are left as a problem so let us just give some sketch of what to do.

Let their positions be (x_1, y_1, z_1) and (x_2, y_2, z_2). Pendulum one has the same form as the single pendulum above in terms of θ. In particular it satisfies the constraints

$$\begin{aligned} C_1(x_1, y_1, z_1, x_2, y_2, z_2, t) &= x_1^2 + y_1^2 - l^2 = 0 \\ C_2(x_1, y_1, z_1, x_2, y_2, z_2, t) &= z_1 = 0 \end{aligned} \tag{5.63}$$

We are taking the z direction to point out of the page in Figure 13. These reduce the 3 degrees of freedom of the first pendulum to a single degree of freedom θ that was seen above.

The second pendulum satisfies the constraints

$$\begin{aligned} C_3(x_1, y_1, z_1, x_2, y_2, z_2, t) &= (x_2 - x_1)^2 + (y_2 - y_1)^2 - l^2 = 0 \\ C_4(x_1, y_1, z_1, x_2, y_2, z_2, t) &= z_2 = 0 \ . \end{aligned} \tag{5.64}$$

Again this reduces the 3 degrees of freedom down to one

$$x_2 - x_1 = l \sin\phi \ , \qquad y_2 - y_1 = -l \cos\phi \ . \tag{5.65}$$

This allows one to compute the kinetic energy T in terms of $\dot\theta$ and $\dot\phi$. The potential is the sum of two terms, one for each weight,

$$\begin{aligned} V_1 &= mgy_1 \ , \\ V_2 &= mgy_2 \ . \end{aligned} \tag{5.66}$$

This allows you to write down the Lagrangian as a function of the generalized coordinates θ, ϕ and their time derivatives. In particular we have

$$\begin{aligned} x_1 &= l \sin\theta & y_1 &= -l\cos\theta \\ x_2 &= l\sin\theta + l\sin\phi & y_2 &= -l\cos\theta - l\cos\phi \end{aligned}$$

So the kinetic term is

$$
\begin{aligned}
T &= \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) \\
&= \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}(\cos\theta\cos\phi + \sin\theta\sin\phi)) \\
&= \frac{1}{2}ml^2(2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi))
\end{aligned}
\tag{5.67}
$$

and hence

$$
L = ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\phi}^2 + ml^2\dot{\theta}\dot{\phi}\cos(\theta - \phi) + mgl(2\cos\theta + \cos\phi)\ . \tag{5.68}
$$

From here we can read off the θ equation of motion

$$
\begin{aligned}
\frac{d}{dt}\left(2ml^2\dot{\theta} + ml^2\dot{\phi}\cos(\theta - \phi)\right) - \left(-ml^2\dot{\theta}\dot{\phi}\sin(\theta - \phi) - 2mgl\sin\theta\right) &= 0 \\
&\Longleftrightarrow \\
2ml^2\ddot{\theta} + ml^2\ddot{\phi}\cos(\theta - \phi) - ml^2\dot{\phi}(\dot{\theta} - \dot{\phi})\sin(\theta - \phi) + ml^2\dot{\theta}\dot{\phi}\sin(\theta - \phi) + 2mgl\sin\theta &= 0 \\
&\Longleftrightarrow \\
\ddot{\theta} + \frac{1}{2}\ddot{\phi}\cos(\theta - \phi) + \frac{1}{2}\dot{\phi}^2\sin(\theta - \phi) + \frac{g}{l}\sin\theta &= 0\ ,
\end{aligned}
\tag{5.69}
$$

and the ϕ equation of motion:

$$
\begin{aligned}
\frac{d}{dt}\left(ml^2\dot{\phi} + ml^2\dot{\theta}\cos(\theta - \phi)\right) - \left(ml^2\dot{\theta}\dot{\phi}\sin(\theta - \phi) - mgl\sin\phi\right) &= 0 \\
&\Longleftrightarrow \\
ml^2\ddot{\phi} + ml^2\ddot{\theta}\cos(\theta - \phi) - ml^2\dot{\theta}(\dot{\theta} - \dot{\phi})\sin(\theta - \phi) - ml^2\dot{\theta}\dot{\phi}\sin(\theta - \phi) + mgl\sin\phi &= 0 \\
&\Longleftrightarrow \\
\ddot{\phi} + \ddot{\theta}\cos(\theta - \phi) - \dot{\theta}^2\sin(\theta - \phi) + \frac{g}{l}\sin\phi &= 0\ .
\end{aligned}
\tag{5.70}
$$

Of course it is altogether a different problem to solve these equations! In fact they are known to exhibit chaotic behaviour. You can take a look at some cool pictures and movies about it here: http://en.wikipedia.org/wiki/Double_pendulum.

We can look at these equations in the limit where (θ, ϕ) are small, as we did for the single pendulum - although here we also need to assume that $(\dot{\theta}, \dot{\phi})$ are small to obtain linear differential equations. In this case we approximate $\sin(\theta - \phi) \sim \theta - \phi$, $\cos(\theta - \phi) \sim 1$ and $\sin\phi \sim \phi$, and neglect terms of higher order in θ and ϕ. In this case we find

$$
\begin{aligned}
\ddot{\theta} + \frac{1}{2}\ddot{\phi}\ + \frac{g}{l}\theta &= 0 \\
\ddot{\phi} + \ddot{\theta} + \frac{g}{l}\phi &= 0
\end{aligned}
\tag{5.71}
$$

To solve this we write our system in terms of matrices:

$$
K\ddot{\Theta} + \Omega\Theta = 0 \tag{5.72}
$$

$$\Theta = \begin{pmatrix} \theta \\ \phi \end{pmatrix} \ , \qquad K = \begin{pmatrix} 1 & 1/2 \\ 1 & 1 \end{pmatrix} , \qquad \Omega = \frac{g}{l}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.73}$$

We then invert K and write our equation as

$$\ddot{\Theta} + K^{-1}\Omega\Theta = 0 \tag{5.74}$$

Note that K^{-1} always exists otherwise it would mean that some linear combination of θ and ϕ did not have kinetic energy in the Lagrangian. Next we construct the eigenvalues $\omega_{1,2}^2$ of $K^{-1}\Omega$ and their eigenvectors Θ_1 and Θ_2 respectively. The solution to the equation $\ddot{\Theta} + K^{-1}\Omega\Theta = 0$ is then

$$\Theta = \mathrm{Re}\left(A_1 e^{i\omega_1 t}\Theta_1 + A_2 e^{i\omega_2 t}\Theta_2\right) \tag{5.75}$$

where A_1 and A_2 are arbitrary complex numbers. Here we use the linearity of the equation to take the real part (or we could take the imaginary part) to obtain a real solution.

The $\Theta_{1,2}$ and $\omega_{1,2}$ are called the **normal modes** and **normal frequencies** respectively. In our case the normal frequencies are real but in general they could be complex. This is okay as the equations are linear and so one just takes the real (or imaginary) part to obtain a physically acceptable solution. However an imaginary part to the frequency ω indicates an instability as the solution will have an exponential dependence of the form $e^{i\omega t} \sim e^{-\mathrm{Im}(\omega)t}$ which diverges in the past or future.

In our case we find

$$K^{-1}\Omega = \frac{1}{1-1/2}\begin{pmatrix} 1 & -1/2 \\ -1 & 1 \end{pmatrix}\frac{g}{l}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{g}{l}\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \tag{5.76}$$

The eigenvalue equation is therefore

$$\begin{aligned} 0 &= (2g/l - \omega_{1,2}^2)^2 - 2g^2/l^2 \\ &\Longleftrightarrow \\ \omega_{1,2}^2 &= \frac{g}{l}(2 \mp \sqrt{2}) \ . \end{aligned} \tag{5.77}$$

Next we find the eigenvectors. To this end we write

$$\Theta = \begin{pmatrix} 1 \\ b \end{pmatrix} \tag{5.78}$$

and substitute into $K^{-1}\Omega\Theta = \omega_{1,2}^2\Theta$ which leads to the condition

$$\frac{g}{l}(2-b) = \frac{g}{l}(2 \mp \sqrt{2}) \tag{5.79}$$

Thus $b = \pm\sqrt{2}$ and our eigenvectors are

$$\Theta_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \ , \qquad \Theta_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \tag{5.80}$$

Thus our solution is

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\left(A_1 e^{i\omega_1 t} + A_2 e^{i\omega_2 t}\right) \\ \sqrt{2}\mathrm{Re}\left(A_1 e^{i\omega_1 t} - A_2 e^{i\omega_2 t}\right) \end{pmatrix} \tag{5.81}$$

For example taking $A_1 = A_2 = A$ a real constant gives:

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} A\cos\omega_1 t + A\cos\omega_2 t \\ \sqrt{2}A\cos\omega_1 t - \sqrt{2}A\cos\omega_2 t \end{pmatrix} \tag{5.82}$$

Lastly we note that it is curious to see irrational values showing up which gives a hint of the complicated and chaotic motion of the full system. In particular the ratio

$$\frac{\omega_1}{\omega_2} = \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} \tag{5.83}$$

is irrational so that the motion is not periodic, *i.e.* there is no time $t \neq 0$ for which both $\omega_1 t$ and $\omega_2 t$ are integer multiples of 2π.

5.3.2 A Marble in a Bowl

Let us derive the equations of motion of a marble rolling about without friction in a bowl under the force of gravity. In particular suppose that the bowl is defined by the curve, for $z \geq 0$,

$$z = x^2 + y^2 \qquad \Longleftrightarrow \qquad C(x,y,z) = z - x^2 - y^2 = 0 \ . \tag{5.84}$$

Solving this constraint reduces us from three degrees of freedom to two. In particular let us switch to polar coordinates for x and y:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{5.85}$$

so that the constraint is simply solved by taking

$$z = r^2 \ . \tag{5.86}$$

The kinetic energy is then

$$\begin{aligned} T &= \frac{1}{2}m((\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2 + 4r^2\dot{r}^2) \\ &= \frac{1}{2}m(1+4r^2)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 \ . \end{aligned} \tag{5.87}$$

The potential energy is again just the height:

$$V = mgr^2 \ , \tag{5.88}$$

so that

$$L = \frac{1}{2}m(1+4r^2)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr^2 \ . \tag{5.89}$$

More generally one could consider a bowl with a shape given by a function $z = f(x^2+y^2)$ so that the constraint is satisfied by $z = f(r^2)$. Therefore $\dot{z} = 2r\dot{r}f'(r^2)$ and hence

$$L = \frac{1}{2}m(1+4r^2(f'(r^2))^2)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgf(r^2) \ . \tag{5.90}$$

To find the Euler-Lagrange equations we first evaluate

$$\begin{aligned}\frac{\partial L}{\partial \dot r} &= m(1+4r^2)\dot r \\ \frac{\partial L}{\partial r} &= 4mr\dot r^2 + mr\dot\theta^2 - 2mgr \ ,\end{aligned} \tag{5.91}$$

so that the r Euler-Lagrange equation is

$$m(1+4r^2)\ddot r + 8mr\dot r^2 - 4mr\dot r^2 - mr\dot\theta^2 + 2mgr = 0 \ . \tag{5.92}$$

For the θ equation we again notice that since $\partial L/\partial\theta = 0$ there is a conservation law:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = \frac{d}{dt}(mr^2\dot\theta) = 0 \ , \tag{5.93}$$

which is equivalent to the conservation of angular momentum $l = mr^2\dot\theta$.

We can now use the conservation of l to obtain a reduced dynamical system that only involves r and an effective potential as we did before. To this end we note that the energy

$$\begin{aligned}E &= T + V \\ &= \frac{1}{2}m(1+4r^2)\dot r^2 + \frac{1}{2}mr^2\dot\theta^2 + mgr^2 \\ &= \frac{1}{2}m(1+4r^2)\dot r^2 + \frac{l^2}{2mr^2} + mgr^2\end{aligned} \tag{5.94}$$

is conserved.

This can also be seen by starting from the r-equation of motion and substituting in for $\dot\theta$:

$$m(1+4r^2)\ddot r + 4mr\dot r^2 - \frac{l^2}{mr^3} + 2mgr = 0 \tag{5.95}$$

Next we multiply by $\dot r$ and integrate up with respect to time:

$$\begin{aligned}m(1+4r^2)\ddot r\dot r + 4mr\dot r^3 - \frac{l^2}{mr^3} + 2mgr\dot r &= 0 \\ &\Longleftrightarrow \\ \frac{d}{dt}\left(\frac{1}{2}m(1+4r^2)\dot r^2 + \frac{l^2}{2mr^2} + mgr^2\right) &= 0\end{aligned} \tag{5.96}$$

Thus E is indeed constant.

Just as before we can rewrite the conservation of energy as

$$\frac{1}{2}m\dot r^2 + \frac{l^2}{2mr^2}\frac{1}{1+4r^2} + \frac{mgr^2}{1+4r^2} - \frac{E}{1+4r^2} = 0 \tag{5.97}$$

This is of the form

$$\frac{1}{2}m\dot r^2 + V_{eff} = 0 \tag{5.98}$$

but with

$$V_{eff} = \frac{l^2}{2mr^2}\frac{1}{1+4r^2} + \frac{mgr^2}{1+4r^2} - \frac{E}{1+4r^2} \ . \tag{5.99}$$

This can be quantitatively and qualitatively analysed as we did before for 3D problems with conserved angular momentum. Note that E now appears as part of V_{eff} rather than as a line which the particle must stay below.

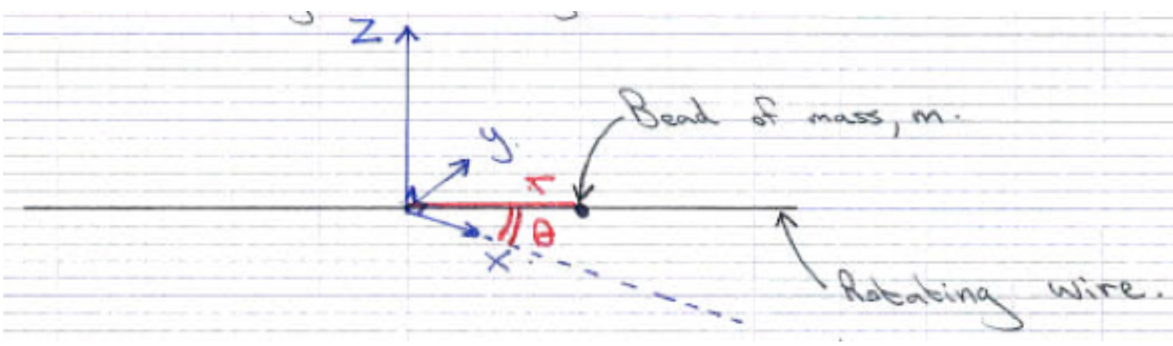

Figure 5.4: Bead on a rotating wire

We could also have simply written the energy as

$$E = \frac{1}{2}m(1+4r^2)\dot{r}^2 + \hat{V}_{eff} \qquad \hat{V}_{eff} = \frac{l^2}{2mr^2} + mgr^2 \tag{5.100}$$

In this case there is a non-standard kinetic term but this won't have much of a qualitative effect on the dynamics since the non-constant coefficient $1+4r^2$ never vanishes. Its effect is to modify the relation between kinetic energy and velocity depending on the value of r. But qualitatively one still has that the kinetic energy is an increasing positive function of velocity. It will of course have quantitative effects.

5.3.3 A Bead on a Rotating Wire

Let us now look at something with a time-dependent constraint. We consider a straight wire that is lying in the $x-y$ plane and rotating about its midpoint at the origin with constant angular velocity $\omega = \dot{\theta}$. Let us imagine a bead moves on the wire without friction. We also assume that the wire is infinitely long so the bead never falls off the end.

Let us write the position of the bead in cylindrical coordinates

$$\underline{r} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix} . \tag{5.101}$$

The constraint that the bead is on the wire, and the wire is rotating, can be written as

$$\begin{aligned} C_1(r,\theta,z,t) &= \theta - \omega t = 0 \\ C_2(r,\theta,z,t) &= z = 0 \ , \end{aligned} \tag{5.102}$$

for a fixed ω. These are solved by taking

$$\theta = \omega t \qquad z = 0 \ . \tag{5.103}$$

Thus there are initially three degrees of freedom but there are two constraints leading to just one generalized coordinate or degree of freedom r. To continue we just compute:

$$\dot{\underline{r}} = \begin{pmatrix} \dot{r}\cos(\omega t) - \omega r\sin(\omega t) \\ \dot{r}\sin(\omega t) + \omega r\cos(\omega t) \\ 0 \end{pmatrix} . \tag{5.104}$$

There is no potential energy so the Lagrangian is just the kinetic energy

$$\begin{aligned} L &= \frac{1}{2}m|\dot{\underline{r}}|^2 \\ &= \frac{1}{2}m((\dot{r}\cos(\omega t) - \omega r\sin(\omega t))^2 + (\dot{r}\sin(\omega t) + \omega r\cos(\omega t))^2) \\ &= \frac{1}{2}m(\dot{r}^2 + \omega^2 r^2) \ . \end{aligned} \tag{5.105}$$

This is just like an unconstrained particle in a potential $V = -m\omega^2 r/2$ corresponding to a force $F = m\omega^2 r$ that points radially outwards. This is a **centrifugal** force and is again fictitious, in the sense that there is no force or potential term in the original Lagrangian.

Let us look at the equation of motion:

$$\ddot{r} - \omega^2 r = 0 \ . \tag{5.106}$$

Rather than finding sine and cosine as solutions the minus sign in second term indicates an instability. The solutions are given by

$$r = Ae^{\omega t} + Be^{-\omega t} \ . \tag{5.107}$$

At late times only the first term is important and the bead flies off to $r \to \infty$ getting ever faster and faster due to the centrifugal force. In particular if at $t = 0$ we assume $\dot{r} = 0$ then we require

$$A\omega - B\omega = 0 \qquad \Longrightarrow A = B \tag{5.108}$$

so that $r = 2A\cosh(\omega t)$.

5.4 The Coriolis Effect

Next we consider a more involved and famous example: the Coriolis effect which is important for the weather. This is not related to constraints but rather relates to what (fictitious) forces arise when one switches between different coordinate systems where there is an explicit time dependence.

In particular consider a mass of air above the Earth with coordinates $\underline{r}' = (x', y', z')$ where the z' coordinate runs north-south. Since the Earth is rotating $\underline{r}'$ is not an inertial frame. Therefore it makes sense to switch to a inertial coordinate system $\underline{r} = (x, y, z)$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x'\cos(\omega t) - y'\sin(\omega t) \\ y'\cos(\omega t) + x'\sin(\omega t) \\ z' \end{pmatrix} \ , \tag{5.109}$$

where $\omega = 2\pi/60/60/24 \sim 0.00007$ is the angular velocity of the Earth per second. For $\omega > 0$ this means that the (x', y') plane is rotating with angular velocity ω with respect to the (x, y) plane. We can denote this as

$$\underline{r} = \mathbf{R}\underline{r}' \qquad \mathbf{R} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{5.110}$$

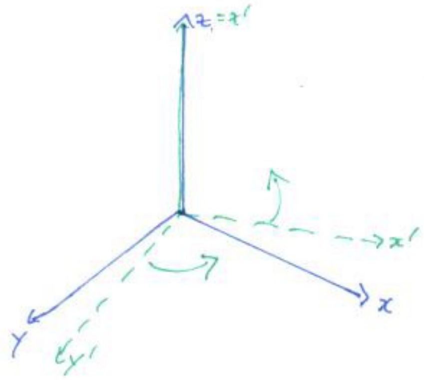

Figure 5.5: The Coriolis force

We don't need a potential. The Atmosphere is of course subjected to Earth's gravity but the pressure of the lower air levels keeps the higher air from falling. So in effect the gravitational force is cancelled, at least in the approximation that we will make. Thus the Lagrangian of an air molecule is the kinetic energy

$$\begin{aligned} L &= T \\ &= \frac{1}{2}m|\dot{\underline{r}}|^2 \\ &= \frac{1}{2}m\dot{\underline{r}}^T\dot{\underline{r}} \end{aligned} \tag{5.111}$$

where we are thinking of the positions as 1×3 matrices. Now

$$\dot{\underline{r}} = \mathbf{R}\dot{\underline{r}}' + \dot{\mathbf{R}}\underline{r}' \tag{5.112}$$

so that

$$\begin{aligned} L &= \frac{1}{2}m(\mathbf{R}\dot{\underline{r}}' + \dot{\mathbf{R}}\underline{r}')^T(\mathbf{R}\dot{\underline{r}}' + \dot{\mathbf{R}}\underline{r}') \\ &= \frac{1}{2}m(\dot{\underline{r}}'^T\mathbf{R}^T + \underline{r}'^T\dot{\mathbf{R}}^T)(\mathbf{R}\dot{\underline{r}}' + \dot{\mathbf{R}}\underline{r}') \\ &= \frac{1}{2}m(\dot{\underline{r}}'^T\mathbf{R}^T\mathbf{R}\dot{\underline{r}}' + \underline{r}'^T\dot{\mathbf{R}}^T\mathbf{R}\dot{\underline{r}}' + \dot{\underline{r}}'^T\mathbf{R}^T\dot{\mathbf{R}}\underline{r}' + \underline{r}'^T\dot{\mathbf{R}}^T\dot{\mathbf{R}}\underline{r}')\ . \end{aligned} \tag{5.113}$$

The middle two terms are actually equal:

$$\begin{aligned} \underline{r}'^T\dot{\mathbf{R}}^T\mathbf{R}\dot{\underline{r}}' &= \left(\underline{r}'^T\dot{\mathbf{R}}^T\mathbf{R}\dot{\underline{r}}'\right)^T \\ &= \dot{\underline{r}}'^T\mathbf{R}^T\dot{\mathbf{R}}\underline{r}'\ . \end{aligned} \tag{5.114}$$

Further since $\mathbf{R}$ is a rotation we have $\mathbf{R}^T\mathbf{R} = \mathbf{I}$. Thus we see that

$$L = \frac{1}{2}m|\dot{\underline{r}}'|^2 + m\dot{\underline{r}}'^T\mathbf{R}^T\dot{\mathbf{R}}\underline{r}' + \frac{1}{2}m\underline{r}'^T\dot{\mathbf{R}}^T\dot{\mathbf{R}}\underline{r}'\ . \tag{5.115}$$

Next we need to compute

$$\dot{\mathbf{R}} = \omega\begin{pmatrix} -\sin(\omega t) & -\cos(\omega t) & 0 \\ \cos(\omega t) & -\sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.116}$$

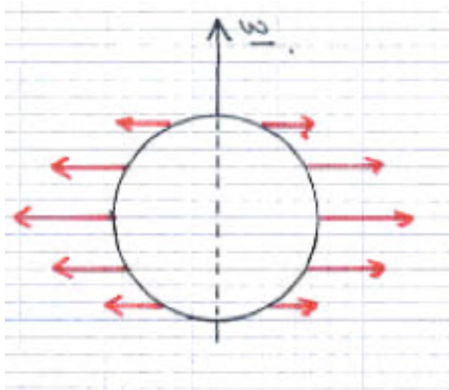

Figure 5.6: Centrifugal force

so that

$$
\begin{aligned}
\mathbf{R}^T\dot{\mathbf{R}} &= \omega \begin{pmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin(\omega t) & -\cos(\omega t) & 0 \\ \cos(\omega t) & -\sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \omega \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (5.117) \\
\dot{\mathbf{R}}^T\dot{\mathbf{R}} &= \omega^2 \begin{pmatrix} -\sin(\omega t) & \cos(\omega t) & 0 \\ -\cos(\omega t) & -\sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\sin(\omega t) & -\cos(\omega t) & 0 \\ \cos(\omega t) & -\sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \omega^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \qquad (5.118)
\end{aligned}
$$

Putting this all together we find

$$L = \frac{1}{2}m((\dot{x}')^2 + (\dot{y}')^2 + (\dot{z}')^2) + m\omega(x'\dot{y}' - y'\dot{x}') + \frac{1}{2}m\omega^2((x')^2 + (y')^2) \ . \qquad (5.119)$$

This can be rewritten as

$$L = \frac{1}{2}m|\underline{\dot{r}}'|^2 + m\omega(\underline{r}' \times \underline{\dot{r}}') \cdot \underline{e}_z + \frac{1}{2}m\omega^2|\underline{r}' \times \underline{e}_z|^2. \qquad (5.120)$$

where $\underline{e}_z$ is the unit vector point north.

What are these terms? The first is the familiar kinetic term and is independent of the rotation of the earth. The third is a centrifugal force term. In particular the third term can written as a potential

$$V = -\frac{1}{2}m\omega^2|\underline{r}' \times \underline{e}_z|^2 = -\frac{1}{2}m\omega^2|\underline{r}'|^2 \sin^2\theta \ , \qquad (5.121)$$

where θ is the angle between $\underline{r}'$ and $\underline{e}_z$ - so the minimum at $\theta = \pi/2$ is the equator. So you are lighter, by about 0.3% at the equator.

The second term gives a velocity dependent force and is known as the Coriolis effect. Let us look at the effect of this term on the equations of motion (Euler-Lagrange equation). To do this we consider the x' equation:

$$\begin{aligned}\frac{d}{dt}\left(m\dot{x}' - m\omega y'\right) - m\omega\dot{y}' + \mathcal{O}(\omega^2) &= 0\\ \ddot{x}' - 2\omega\dot{y}' + \mathcal{O}(\omega^2) &= 0 \ . \end{aligned} \tag{5.122}$$

Here we do not want to worry about the effects of the centrifugal force which is higher order in ω. The y' equation is:

$$\begin{aligned}\frac{d}{dt}\left(m\dot{y}' + m\omega x'\right) + m\omega\dot{x}' + \mathcal{O}(\omega^2) &= 0\\ \ddot{y}' + 2\omega\dot{x}' + \mathcal{O}(\omega^2) &= 0 \ . \end{aligned} \tag{5.123}$$

On the other hand the z' equation is unaffected:

$$\ddot{z}' = 0 \ . \tag{5.124}$$

Thus the Coriolis term gives and extra velocity dependent force. We can integrate these equations:

$$\begin{aligned}\dot{x}' - 2\omega y' &= 2\omega A\\ \dot{y}' + 2\omega x' &= 2\omega B\\ \dot{z}' &= C\end{aligned} \tag{5.125}$$

where A, B, C are constants. Clearly we can integrate up the z' equation again to find

$$z' = Ct + D \ . \tag{5.126}$$

To solve for x' and y' we first substitute $\dot{y}' = 2\omega B - 2\omega x'$ into the $\ddot{x}'$ equation:

$$\ddot{x}' + 4\omega^2 x' = 4\omega^2 B \ . \tag{5.127}$$

Again we can solve this by writing

$$x' = B + x_0' \ , \tag{5.128}$$

where x_0' satisfies

$$\ddot{x}_0' + 4\omega^2 x_0' = 0 \ . \tag{5.129}$$

Thus x' is oscillating about B with frequency ω:

$$x' = B + \alpha\sin(2\omega t) + \beta\cos(2\omega t) \ . \tag{5.130}$$

We can now solve for y' by writing:

$$\begin{aligned}\dot{y}' &= 2\omega B - 2\omega x'\\ &= -2\omega\alpha\sin(2\omega t) - 2\omega\beta\cos(2\omega t)\\ y' &= E + \alpha\cos(2\omega t) - \beta\sin(2\omega t) \ . \end{aligned} \tag{5.131}$$

Here E is just a constant (not related to the energy). We see that

$$(x' - B)^2 + (y' - E)^2 = \alpha^2 + \beta^2 \ . \tag{5.132}$$

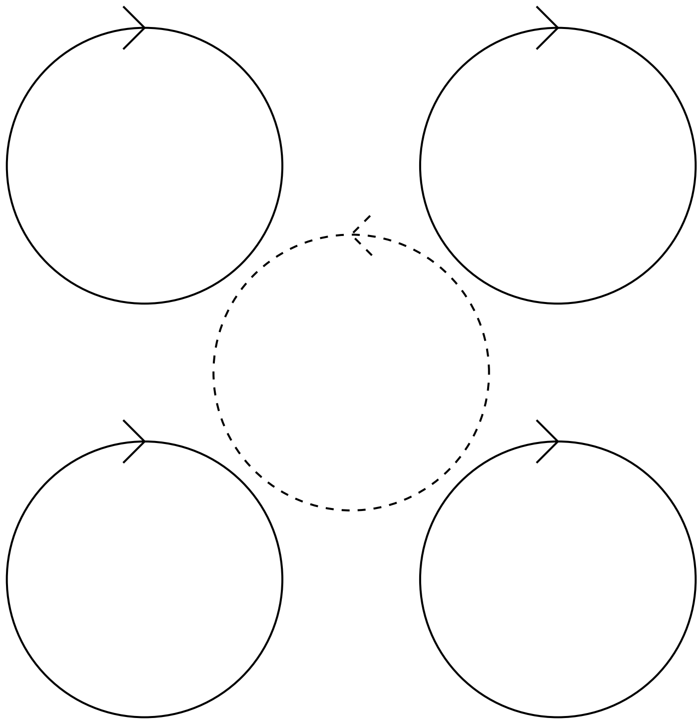

Figure 5.7: Four high pressure areas with clockwise circulation surrounding a low pressure area with anti-clockwise circulation

Thus a particle will move in circles in the $x' - y'$ plane.

This effect is important in meteorology, but we need to add an additional epicycle onto the interpretation of ω in the circular solutions found above. On the surface of the Earth, the force of gravity points radially inward while the Earth itself is spinning about a vertical axis through the poles. As mentioned already above, we want to ignore the motion of air in the radial direction, and focus instead on the motion of air tangent to the surface. Thus, we need to resolve the angular rotation into this tangential plane. If ϕ is the latitude of our position on the Earth, then the component of angular frequency in our tangential plane is $\omega \sin \phi$: at the north pole, the plane makes one rotation per day, while at the equator, there is no rotation at all about these tangential directions. For the circle solutions we found above, in the context of meteorology, we need to make the replacement $\omega \to \omega \sin \phi$. In the northern hemisphere, we then find circles of air moving in a clockwise fashion, while in the southern hemisphere, the circles move anti-clockwise.

This effect helps to explain the direction of wind currents in a hurricane or cyclone although there is yet one more confusing sign issue, for cyclones and hurricanes in the northern hemisphere rotate anti-clockwise. In the center of a hurricane, there is a low pressure region which draws the air from nearby. The nearby air is rotating in a clockwise fashion. As this air gets sucked into the low pressure region, it naturally rotates anticlockwise, because it is surrounded by nearby neighboring cells rotating in a clockwise fashion. Figure 5.7 hopefully makes the situation clearer.

5.5 Symmetries, Conservation Laws and Noether's Theorem

So far we have been using the principle of least action to obtain the equations of motion of a system from the Euler-Lagrange Equations. But we haven't usually been trying to solve them. In order to proceed we need to study conserved quantities again, but this time in the Lagrangian formulation.

If we define the **conjugate momentum** as

$$p^i = \frac{\partial L}{\partial \dot{q}_i} \ , \tag{5.133}$$

and

$$F_i = \frac{\partial L}{\partial q_i} \ , \tag{5.134}$$

then the Euler-Lagrange equation reads as

$$\frac{d}{dt}p^i = F_i \ . \tag{5.135}$$

which is in the form of NII.

In the examples above we saw several times that if the Lagrangian was independent of a particular coordinate q (quite often it was the angle θ), but not $\dot{q}$, then we could immediately identify a conserved quantity (when the Lagrangian is independent of the angle θ, the conserved quantity was the angular momentum):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \qquad \Longrightarrow \qquad Q = \frac{\partial L}{\partial \dot{q}} \text{ is conserved} \ . \tag{5.136}$$

Such a coordinate is said to be **ignorable**.

If L is independent of a particular coordinate, say $q_{i'}$, then there is a symmetry: $L[q_i + \delta q_i, \dot{q}_i + \delta\dot{q}_i, t] = L[q_i, \dot{q}_i, t]$ where

$$\delta q_{i'} = \epsilon \ , \qquad \delta q_i = 0 \text{ otherwise} \ , \qquad \delta\dot{q}_i = 0 \ , \tag{5.137}$$

where ϵ is a constant. This can be made more general as follows. A (continuous or infinitesimal) symmetry of the Lagrangian is a transformation

$$q_i \to q_i + \epsilon T_i \qquad \dot{q}_i \to \dot{q}_i + \epsilon\dot{T}_i \ , \tag{5.138}$$

where ϵ is an infinitesimal parameter and T_i is a function of the q_i's and t, under which L is invariant:

$$L[q_i + \epsilon T_i, \dot{q}_i + \epsilon\dot{T}_i, t] = L[q_i, \dot{q}_i, t] \ . \tag{5.139}$$

This implies that, expanding to first order in ϵ,

$$\delta L = \sum_i \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i = 0 \ . \tag{5.140}$$

It is important to emphasize that this is a specific variation of the coordinates and their derivatives that must be true for any configuration q_i, $\dot{q}_i$. This is quite different, in a sense opposite, to when we evaluated $\delta S = 0$. In that case we required that the action was invariant under all variations of the coordinates and their derivatives and this led to the Euler-Lagrange equation that selects a particular path q_i, $\dot{q}_i$.

We can now state **Noether's Theorem**: For every symmetry of the Lagrangian there is a conserved quantity:

$$Q = \sum_i \frac{\partial L}{\partial \dot{q}_i} T_i \ . \tag{5.141}$$

Let us prove this. To do so we simply compute

$$\begin{aligned} \frac{dQ}{dt} &= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) T_i + \frac{\partial L}{\partial \dot{q}_i}\frac{dT_i}{dt} \\ &= \sum_i \frac{\partial L}{\partial q_i} T_i + \frac{\partial L}{\partial \dot{q}_i}\dot{T}_i \ , \end{aligned}$$

where in the second line we used the Euler-Lagrange equation. Next we note that this can be written as

$$\begin{aligned} \frac{dQ}{dt} &= \frac{1}{\epsilon}\sum_i \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \\ &= \frac{1}{\epsilon}\delta L \\ &= 0 \ , \end{aligned} \tag{5.142}$$

since δq_i is a symmetry of L.

Finally we note that actually all we require is a symmetry of the action S. Thus it is enough if the Lagrangian is invariant up to a total derivative:

$$\delta L = \epsilon \frac{dJ}{dt} \ . \tag{5.143}$$

In this case we must shift the definition of Q to

$$Q = \sum_i \frac{\partial L}{\partial \dot{q}_i} T_i - J \ , \tag{5.144}$$

so that the second to last line in (5.142) is cancelled by the variation of the second term in (5.144). Even more generally one might also allow for t to change under the transformation. In this case one must also be careful to include the change in t when evaluating the action as in integral over time, as we will see below.

This is a deep connection between symmetry and conservation laws. There are a few important symmetries that many physical systems have and these lead to well known conserved charges. Let us look at some.

5.5.1 Elementary examples of symmetries

Let us look at a single particle in a plane with Cartesian coordinates (x, y):

$$L_{cartesian} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \ . \tag{5.145}$$

Here we see that both x and y are ignorable. This leads to two symmetries:

$$\begin{aligned} x &\to x + \epsilon_1 \qquad y \to y \\ y &\to y + \epsilon_2 \qquad x \to x \end{aligned} \tag{5.146}$$

parameterized by ϵ_1 and ϵ_2.

Let us look at the same system but in polar coordinates:

$$L_{polar} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{5.147}$$

where

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \arctan(y/x) \ . \end{aligned} \tag{5.148}$$

Here θ is an ignorable coordinate and hence one has the symmetry

$$\theta \to \theta + \epsilon_3 \qquad r \to r \tag{5.149}$$

However these represent the same system. So both Lagrangians must have all three symmetries. To see that this is indeed the case we can first compute the change in r, θ coming from ϵ_1 and ϵ_2:

$$\begin{aligned} \delta r &= \frac{\partial r}{\partial x}\delta x + \frac{\partial r}{\partial y}\delta y \\ &= \frac{x}{r}\epsilon_1 + \frac{y}{r}\epsilon_2 \\ &= \cos\theta\epsilon_1 + \sin\theta\epsilon_2 \end{aligned} \tag{5.150}$$

and

$$\begin{aligned} \delta\theta &= \frac{\partial\theta}{\partial x}\delta x + \frac{\partial\theta}{\partial y}\delta y \\ &= \frac{1}{1+y^2/x^2}\frac{-y}{x^2}\epsilon_1 + \frac{1}{1+y^2/x^2}\frac{1}{x}\epsilon_2 \\ &= -\frac{y}{x^2+y^2}\epsilon_1 + \frac{x}{x^2+y^2}\epsilon_2 \\ &= -\frac{\sin\theta}{r}\epsilon_1 + \frac{\cos\theta}{r}\epsilon_2 \ . \end{aligned} \tag{5.151}$$

From these we can compute

$$\begin{aligned} \delta\dot{r} &= -\dot{\theta}\sin\theta\epsilon_1 + \dot{\theta}\cos\theta\epsilon_2 \\ \delta\dot{\theta} &= -\dot{\theta}\frac{\cos\theta}{r}\epsilon_1 - \dot{\theta}\frac{\sin\theta}{r}\epsilon_2 + \dot{r}\frac{\sin\theta}{r^2}\epsilon_1 - \dot{r}\frac{\cos\theta}{r^2}\epsilon_2 \end{aligned} \tag{5.152}$$

From these we can compute

$$\begin{aligned} \delta L_{polar} &= \frac{\partial L_{polar}}{\partial r}\delta r + \frac{\partial L_{polar}}{\partial \theta}\delta\theta + \frac{\partial L_{polar}}{\partial \dot{r}}\delta\dot{r} + \frac{\partial L_{polar}}{\partial \dot{\theta}}\delta\dot{\theta} \\ &= m\dot{r}\delta\dot{r} + mr\delta r\dot{\theta}^2 + mr^2\dot{\theta}\delta\dot{\theta} \\ &= m\dot{r}(-\dot{\theta}\sin\theta\epsilon_1 + \dot{\theta}\cos\theta\epsilon_2) + mr(\cos\theta\epsilon_1 + \sin\theta\epsilon_2)\dot{\theta}^2 \\ &\quad + mr^2\dot{\theta}\left(-\dot{\theta}\frac{\cos\theta}{r}\epsilon_1 - \dot{\theta}\frac{\sin\theta}{r}\epsilon_2 + \dot{r}\frac{\sin\theta}{r^2}\epsilon_1 - \dot{r}\frac{\cos\theta}{r^2}\epsilon_2\right) \\ &= 0 \end{aligned} \tag{5.153}$$

Thus the symmetries generated by ϵ_1 and ϵ_2 also extend to symmetries of L_{polar}. We can compute the Noether charges. For the symmetry generated by ϵ_1 we set $\epsilon_2 = 0$ and find

$$\begin{aligned} Q_1 &= \frac{1}{\epsilon_1}\frac{\partial L}{\partial \dot{r}}\delta r + \frac{1}{\epsilon_1}\frac{\partial L}{\partial \dot{\theta}}\delta\theta \\ &= m\dot{r}(\cos\theta) + mr^2\dot{\theta}\left(-\frac{\sin\theta}{r}\right) \\ &= m\frac{d}{dt}(r\cos\theta) \\ &= m\dot{x}\ . \end{aligned} \tag{5.154}$$

While for the symmetry generated by ϵ_2 we set $\epsilon_1 = 0$ and find

$$\begin{aligned} Q_2 &= \frac{1}{\epsilon_2}\frac{\partial L}{\partial \dot{r}}\delta r + \frac{1}{\epsilon_2}\frac{\partial L}{\partial \dot{\theta}}\delta\theta \\ &= m\dot{r}(\sin\theta) + mr^2\dot{\theta}\left(\frac{\cos\theta}{r}\right) \\ &= m\frac{d}{dt}(r\sin\theta) \\ &= m\dot{y}\ . \end{aligned} \tag{5.155}$$

On the other hand we can also compute the ϵ_3 symmetry in cartesian coordinates

$$\begin{aligned} \delta x &= \frac{\partial x}{\partial r}\delta r + \frac{\partial x}{\partial \theta}\delta\theta \\ &= -r\sin\theta\epsilon_3 \\ &= -y\epsilon_3 \end{aligned} \tag{5.156}$$

and

$$\begin{aligned} \delta y &= \frac{\partial y}{\partial r}\delta r + \frac{\partial y}{\partial \theta}\delta\theta \\ &= r\cos\theta\epsilon_3 \\ &= x\epsilon_3\ . \end{aligned} \tag{5.157}$$

Therefore $\delta\dot{x} = -\dot{y}\epsilon_3$ and $\delta\dot{y} = \dot{x}\epsilon_3$.

$$\begin{aligned} \delta L_{cartesian} &= \frac{\partial L_{cartesian}}{\partial x}\delta x + \frac{\partial L_{cartesian}}{\partial y}\delta y + \frac{\partial L_{cartesian}}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L_{cartesian}}{\partial \dot{y}}\delta\dot{y} \\ &= m\dot{x}\delta\dot{x} + m\dot{y}\delta\dot{y} \\ &= -m\dot{x}\dot{y}\epsilon + m\dot{y}\dot{x}\epsilon \\ &= 0 \end{aligned} \tag{5.158}$$

Again we can compute compute the conserved charge due to the symmetry generated by ϵ_3:

$$\begin{aligned} Q_3 &= \frac{1}{\epsilon_3}\frac{\partial L}{\partial \dot{x}}\delta x + \frac{1}{\epsilon_3}\frac{\partial L}{\partial \dot{y}}\delta y \\ &= m\dot{x}(-y) + m\dot{y}(x) \\ &= m(\dot{y}x - y\dot{x}). \end{aligned} \tag{5.159}$$

If we substitute $x = r\cos\theta$ and $y = r\sin\theta$ then

$$\begin{aligned} Q_3 &= m(\dot{r}\sin\theta + r\dot{\theta}\cos\theta)r\cos\theta - mr\cos\theta(\dot{r}\cos\theta - r\dot{\theta}\sin\theta) \\ &= mr^2\dot{\theta}(\cos^2\theta + \sin^2\theta) \\ &= mr^2\dot{\theta} \\ &= l\ . \end{aligned} \tag{5.160}$$

The point of this is to show that although some symmetries may be realised rather trivially, such as those generated by ϵ_1 and ϵ_2 in cartesian coordinates, there may still be other symmetries which have a non-trivial realization. In this case the there is a third symmetry generated by ϵ_3. Similarly in polar coordinates where ϵ_3 is rather simple there are in fact still two more symmetries generated by ϵ_1 and ϵ_2. Later in the course we will see that there can be still more symmetries that are not al all apparent in the Lagrangian formulation.

5.5.2 Invariance under spatial translations gives conserved momentum

This is the simplest example of a symmetry. Let us suppose that the generalized coordinates are positions in space and that the potential and kinetic terms only depend on the separation between any two pairs of particles, *e.g.* $V = V(\underline{r}_i - \underline{r}_j)$ then we have an overall translational symmetry:

$$\underline{r}_i \to \underline{r}_i + \epsilon\underline{a} \qquad \dot{\underline{r}}_i \to \dot{\underline{r}}_i\ , \tag{5.161}$$

where $\underline{a}$ is a fixed vector. Therefore $\underline{r}_i - \underline{r}_j$ is invariant. This corresponds to picking up every particle in your system and moving it over a tiny bit in the direction of $\underline{a}$. Since we assume $\dot{\underline{a}} = \underline{0}$ the Lagrangian will be invariant and we find

$$Q = \sum_i \frac{\partial L}{\partial \dot{\underline{r}}_i} \cdot \underline{a} = \sum_i \underline{p}_i \cdot \underline{a}_i\ . \tag{5.162}$$

This is just the total momentum along the direction $\underline{a}$. This symmetry reflects the **homogeneity** of space, namely that there is no preferred location.

5.5.3 Invariance under rotations gives conserved angular momentum

Let us make the same assumption as for spatial translations but then also assume that the potential and kinetic terms only depend on the distance $|\underline{r}_i - \underline{r}_j|$ between any pair of particles (and not the direction). This is the case for all known fundamental forces. Then we can consider a rotation of all the particles:

$$\underline{r}_i \to \underline{r}_i + \epsilon\mathbf{T}\underline{r}_i \qquad \dot{\underline{r}}_i \to \dot{\underline{r}}_i + \epsilon\mathbf{T}\dot{\underline{r}}_i\ , \tag{5.163}$$

where $\mathbf{T}$ is a constant anti-symmetric matrix: $\mathbf{T}^T = -\mathbf{T}$. To show that the Lagrangian is invariant we must show that $|\underline{r}_i - \underline{r}_j|$ is invariant:

$$\begin{aligned} \delta(|\underline{r}_i - \underline{r}_j|^2) &= 2(\underline{r}_i - \underline{r}_j) \cdot \delta(\underline{r}_i - \underline{r}_j) \\ &= 2\epsilon(\underline{r}_i - \underline{r}_j) \cdot \mathbf{T}(\underline{r}_i - \underline{r}_j)\ , \end{aligned} \tag{5.164}$$

Now $\mathbf{T}$ is an anti-symmetric 3×3 matrix so we can write is as

$$\mathbf{T} = \begin{pmatrix} 0 & -T^3 & T^2 \\ T^3 & 0 & -T^1 \\ -T^2 & T^1 & 0 \end{pmatrix} . \tag{5.165}$$

If we think in terms of components we have

$$\mathbf{T}^a{}_b = -\sum_{c=1}^{3} \epsilon_{abc} T^c . \tag{5.166}$$

and therefore

$$\begin{aligned} \delta(|\underline{r}_i - \underline{r}_j|^2) &= 2\epsilon(\underline{r}_i - \underline{r}_j) \cdot \mathbf{T}(\underline{r}_i - \underline{r}_j) \\ &= -2\epsilon \sum_{abc} \epsilon_{abc}(r_i^a - r_j^a)(r_i^b - r_j^b)T^c \\ &= 0 . \end{aligned} \tag{5.167}$$

The expression vanishes because $\epsilon_{abc} = -\epsilon_{bac}$ but $(r_i^a - r_j^a)(r_i^b - r_j^b)$ is symmetric in $a \leftrightarrow b$. Thus the Lagrangian will be invariant.

The Noether charge is

$$\begin{aligned} Q &= \sum_i \frac{\partial L}{\partial \dot{\underline{r}}_i} \cdot \mathbf{T}\underline{r}_i \\ &= \sum_i \underline{p}_i \cdot \mathbf{T}\underline{r}_i \\ &= \sum_i \sum_{ab} \underline{p}_i^a \mathbf{T}^a{}_b \underline{r}_i^b \\ &= -\sum_i \sum_{abc} p_i^a \epsilon_{abc} r_i^b T^c \\ &= \underline{T} \cdot \sum_i \underline{r}_i \times \underline{p}_i . \end{aligned} \tag{5.168}$$

This is just the component of the total angular momentum along the direction $\underline{T} = (T^1, T^2, T^3)$.

This symmetry reflects the **isotropy** of space, namely that it looks the same in all directions.

Note that putting in a massive body such as the sun and holding it fixed will break homogeneity but not isotropy about the sun. Of course in reality the sun isn't fixed, just heavy compared to the planets, and so space is really homogeneous as the sun is free to move.

5.5.4 Invariance under time translations gives conserved energy

Lastly, but most importantly, let us show that conservation of energy arises from invariance under time translations. So let us assume that the Lagrangian does not have any explicit time dependence:

$$\frac{\partial L}{\partial t} = 0 . \tag{5.169}$$

First note that the action S depends only on t_1 and t_2 (and not on the integration variable t which has been integrated over). Thus a translation in time means a shift

$t_1 \to t_1 + \epsilon$, $t_2 \to t_2 + \epsilon$. In this case we must be a little more subtle with the variation of the action which is now

$$\begin{aligned}
\delta S &= \int_{t_1+\epsilon}^{t_2+\epsilon} L(q_i(t), \dot{q}_i(t))dt - \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t))dt \\
&= \int_{t_1}^{t_2} L(q_i(t+\epsilon), \dot{q}_i(t+\epsilon)) - L(q_i(t), \dot{q}_i(t))dt \ , \qquad (5.170)
\end{aligned}$$

where in the first term of the second line we use a change of variables $t' = t + \epsilon$ in the integral. In this case the coordinates will transform as

$$q_i(t+\epsilon) = q_i(t) + \epsilon\dot{q}_i(t) \qquad \dot{q}(t+\epsilon) = \dot{q}_i(t) + \epsilon\ddot{q}_i(t) \ . \qquad (5.171)$$

In other words

$$\delta q_i = \epsilon\dot{q}_i \qquad \delta\dot{q}_i = \epsilon\ddot{q}_i \qquad (5.172)$$

Since the Lagrangian has no explicit t dependence it too simply transforms as

$$L(q_i(t+\epsilon), \dot{q}_i(t+\epsilon)) = L(q_i(t), \dot{q}_i(t)) + \epsilon\frac{dL}{dt} \ , \qquad (5.173)$$

where

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \ . \qquad (5.174)$$

Thus

$$\delta S = \epsilon \int_{t_1}^{t_2} \frac{dL}{dt}dt \ , \qquad (5.175)$$

and we need to use the modified form for the Noether charge (5.144):

$$\begin{aligned}
Q &= \frac{1}{\epsilon}\sum_i \frac{\partial L}{\partial \dot{q}_i}\delta q_i - L \\
&= \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L \\
&= \sum_i p^i\dot{q}_i - L
\end{aligned}$$

If we evaluate this for a simple Lagrangian of the form

$$L = \frac{1}{2}m|\dot{\underline{r}}|^2 - V(\underline{r}_1) \ , \qquad (5.176)$$

then

$$\underline{p} = m\dot{\underline{r}} \ , \qquad (5.177)$$

and hence the conserved charge is indeed the energy E:

$$\begin{aligned}
Q &= \underline{p}\cdot\dot{\underline{r}} - \frac{1}{2}m|\dot{\underline{r}}|^2 + V(\underline{r}) \\
&= \frac{1}{2}m|\dot{\underline{r}}|^2 + V(\underline{r}) \\
&= E \qquad (5.178)
\end{aligned}$$

as previously defined. Here we see how to extend it to a general Lagrangian. Note that for a general Lagrangian, one that isn't of the form (5.176), this definition of energy is not simply of the form $E = \frac{1}{2}m|\dot{\underline{r}}|^2 + V$.

Let us check that this is indeed conserved. To do so we first compute:

$$\begin{aligned}
\frac{dL}{dt} &= \frac{\partial L}{\partial t} + \sum_i \left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \right) \\
&= \frac{\partial L}{\partial t} + \sum_i \left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right] - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right]\dot{q}_i \right) \\
&= \frac{\partial L}{\partial t} + \frac{d}{dt}\sum_i \left[\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right] + \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] \right)\dot{q}_i \\
&= \frac{\partial L}{\partial t} + \frac{d}{dt}\sum_i p_i \dot{q}_i \ , \qquad (5.179)
\end{aligned}$$

where we have used the Euler-Lagrange equation. Combining this with

$$E = \sum_i p^i \dot{q}_i - L \qquad (5.180)$$

tells us that

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t} \ . \qquad (5.181)$$

Thus, provided that L does not explicitly depend on time, then E will be conserved.

Example: A spherical pendulum

Let us return to the simple pendulum but now we not longer constrained it to lie in a plane but can move in all three dimensions. However it still has one constraint: it must be a fixed distance l from the origin:

$$C_1(\underline{r}, t) = |\underline{r}| - l = 0 \ . \qquad (5.182)$$

To solve this constraint we introduce spherical coordinates, $\underline{r} = (x, y, z)$ with

$$\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= -r\cos\theta \qquad (5.183)
\end{aligned}$$

Note that the range of θ and ϕ are $[0, \pi]$ and $[0, 2\pi)$ respectively, therefore the constraint is

$$x^2 + y^2 + z^2 - l^2 = r^2 - l^2 = 0 \qquad (5.184)$$

solved by taking $r = l$.

To continue we compute the kinetic energy

$$\dot{\underline{r}} = l \begin{pmatrix} \dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi \\ \dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi \\ \dot{\theta}\sin\theta \end{pmatrix} \qquad (5.185)$$

and hence

$$\begin{aligned}
T &= \frac{1}{2}m|\dot{\underline{r}}|^2 \\
&= \frac{1}{2}ml^2\left((\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi)^2 + (\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi)^2 + \dot{\theta}^2\sin^2\theta \right) \\
&= \frac{1}{2}ml^2\left(\dot{\theta}^2\cos^2\theta\cos^2\phi + \dot{\phi}^2\sin^2\theta\sin^2\phi + \dot{\theta}^2\cos^2\theta\sin^2\phi + \dot{\phi}^2\sin^2\theta\cos^2\phi + \dot{\theta}^2\sin^2\theta \right) \\
&= \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2 \right) \ . \qquad (5.186)
\end{aligned}$$

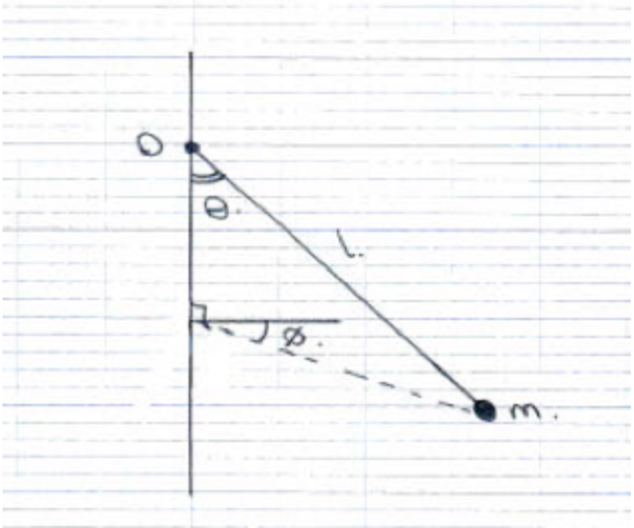

Figure 5.8: Spherical pendulum

The potential is just $V = mgz$:

$$V = -mgl\cos\theta \ . \tag{5.187}$$

Hence we arrive at the Lagrangian

$$L = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mgl\cos\theta \ . \tag{5.188}$$

Here we see that ϕ is an ignorable coordinate.

What are the conserved charges? Since L does not depend explicitly on t we have a conserved energy:

$$\begin{aligned} E &= \frac{\partial L}{\partial\dot{\theta}}\dot{\theta} + \frac{\partial L}{\partial\dot{\phi}}\dot{\phi} - L \\ &= ml^2\dot{\theta}^2 + ml^2\sin^2\theta\dot{\phi}^2 - \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - mgl\cos\theta \\ &= \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - mgl\cos\theta \ . \end{aligned} \tag{5.189}$$

From the symmetry $\phi \to \phi + \epsilon$ we have conserved angular momentum about the z-axis

$$\begin{aligned} L_z &= \frac{\partial L}{\partial\dot{\phi}} \\ &= ml^2\sin^2\theta\dot{\phi} \ . \end{aligned} \tag{5.190}$$

As before this is enough to reduce the system to two a single first order differential equation. In particular we write

$$\dot{\phi} = \frac{L_z}{ml^2\sin^2\theta} \ , \tag{5.191}$$

so that the energy is

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{L_z^2}{2ml^2\sin^2\theta} - mgl\cos\theta \ . \tag{5.192}$$

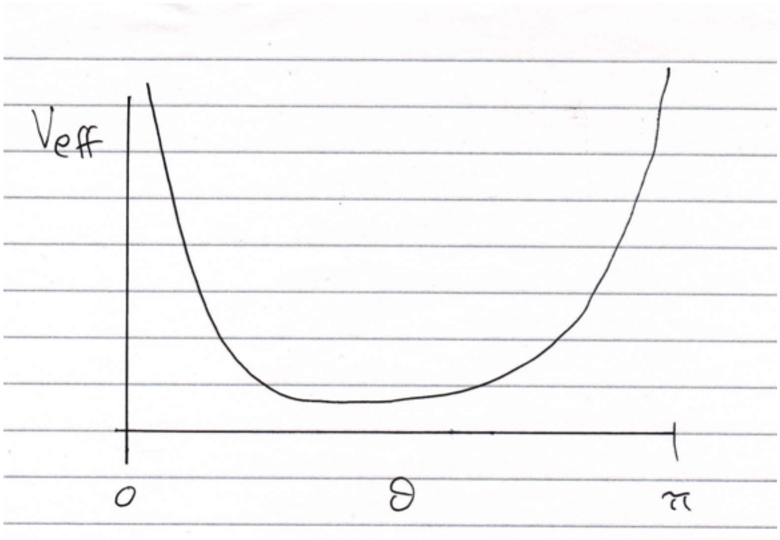

Figure 5.9: Effective potential

This gives the effective potential

$$V_{eff} = \frac{L_z^2}{2ml^2 \sin^2\theta} - mgl\cos\theta \ , \tag{5.193}$$

so that the solution for $\theta(t)$ comes from the integral

$$t - t_0 = \sqrt{\frac{m}{2}} l \int_{\theta_0}^{\theta} \frac{d\theta'}{\sqrt{(E - V_{eff}(\theta'))}} \ . \tag{5.194}$$

Once one knows $\theta(t)$ one can integrate (5.191) to obtain $\phi(t)$. Clearly finding $\theta(t)$ is not easy but it can be done numerically with computers.

However qualitatively one can understand the dynamics by looking at a sketch of V_{eff} for $L_z \neq 0$ (see figure 17). We see that there is a minimum of V_{eff} at

$$-\frac{L_z^2}{ml^2\sin^3\theta}\cos\theta + mgl\sin\theta = 0 \qquad \Longleftrightarrow \qquad \frac{\sin^4\theta}{\cos\theta} = \frac{L_z^2}{m^2gl^3} \tag{5.195}$$

At this value of, $\theta = \theta_0$, we obtain circular orbits where $\theta = \theta_0$ is constant and

$$\phi = \frac{L_z t}{ml^2 \sin^2\theta_0} \tag{5.196}$$

Notice that small values of L_z lead to small values of θ_0 but large values lead to $\theta_0 \sim \pi/2$.

More typically θ will oscillate around this minimum, while ϕ winds around, leading to paths depicted in figure 5.10. Of course we could also choose initial conditions where $\dot{\phi} = 0$, *i.e.* $L_z = 0$. In which case we recover the simple pendulum in a plane.

5.6 Calculus of Variations in Other Contexts: Catenary and Brachistochrone

The techniques we have learned to deal with Lagrangians can be adapted to find other functions which maximize or minimize some desired quantity. In this section, we will

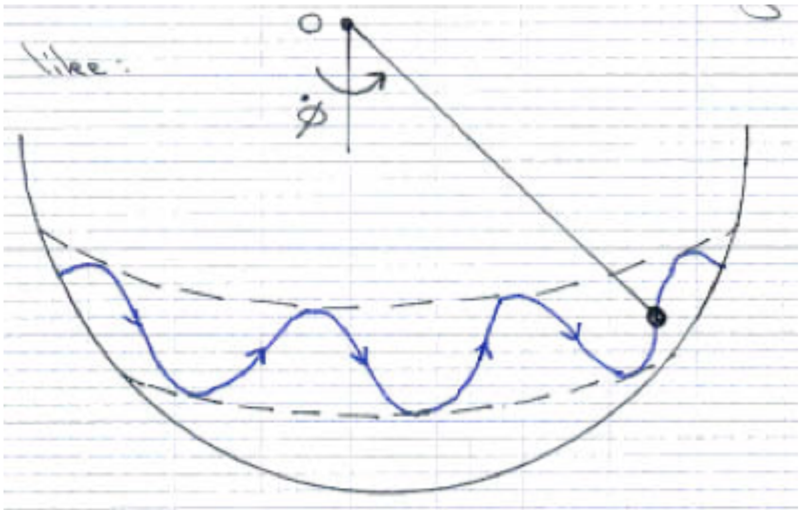

Figure 5.10: Spherical pendulum motion

look at two famous examples, the catenary and the brachistochrone. The catenary is the curve made by a wire hung between two posts. The brachistochrone is the (frictionless) track joining two points that allows a massive particle under the influence of gravity to travel the most quickly.

Catenary

Let us start with the catenary as the integrals are more straightforward. Imagine we have a wire of linear mass density ρ and length s. We have two posts a distance ℓ apart, and we will hang the ends of the wire from the same height on each post. Moreover, we assume that the posts are high enough that the wire does not drag along the ground in between. What quantity should we minimize to find the path? One candidate is the potential energy. If the potential energy were not at least at a local minimum, then the wire could "fall" a little bit and rearrange its path to further reduce its potential energy.

We divide the wire up into many small segments, each of length ds. Let the path the wire takes be a height function $y(x)$ of its horizontal position along the ground. We can rewrite the path length in terms of the horizontal distance dx as

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \ . \tag{5.197}$$

The weight of this short length segment is $\rho g\, ds$ and its potential energy hence $\rho g y(x)\, ds$. The total potential energy is then the integral

$$V = \int_{x_i}^{x_f} \rho g y \sqrt{1 + (y')^2}\, dx \ , \tag{5.198}$$

where we have defined $y' \equiv \frac{dy}{dx}$.

We can treat the integrand as a Lagrangian, replacing the time coordinate t for a spatial coordinate x:

$$L(y', y, x) = \rho g y \sqrt{1 + (y')^2} \ . \tag{5.199}$$

Moreover, the independence of $L(y', y)$ from x means that there will be a conserved "energy". As the ignorable coordinate is x and not t, let us call this quantity a momentum P:

$$P = \frac{\partial L}{\partial y'} y' - L = \frac{\rho g y (y')^2}{\sqrt{1+(y')^2}} - \rho g y \sqrt{1+(y')^2} = \frac{\rho g y}{\sqrt{1+(y')^2}} \ . \tag{5.200}$$

This first order differential equation for y we can solve; we need to isolate y'. We square both sides,

$$1 + (y')^2 = \left(\frac{\rho g y}{P}\right)^2 \ , \tag{5.201}$$

and then separate variables,

$$\int \frac{dy}{\sqrt{\left(\frac{\rho g y}{P}\right)^2 - 1}} = \pm \int dx \ . \tag{5.202}$$

Introducing a new variable $\rho g y / P = u$, we find

$$\frac{P}{\rho g} \int \frac{du}{\sqrt{u^2 - 1}} = \pm \int dx \ . \tag{5.203}$$

The integral is an inverse hyperbolic cosine function,

$$\cosh^{-1} u = \pm \frac{\rho g}{P}(x - x_0) \ , \tag{5.204}$$

where x_0 is an integration constant. Note the $\pm$ disappears because cosh is an even function of its argument. Restoring the original variable y, we obtain the general solution

$$y = \frac{P}{\rho g} \cosh\left(\frac{\rho g}{P}(x - x_0)\right) \ . \tag{5.205}$$

As a last step, we can try to fix the constants x_0 and P in terms of the data of the original question. By arranging the posts at $x = \pm \ell/2$, we can set x_0 to zero, as physically we then expect a minimum in $y(x)$ at the midpoint, where the sinh function vanishes. To fix P in terms of s, we need to do one last integral

$$\begin{aligned} s &= \int_{-\ell/2}^{\ell/2} \sqrt{1+(y')^2}\, dx = \int_{-\ell/2}^{\ell/2} \sqrt{1 + \sinh^2\left(\frac{\rho g x}{P}\right)}\, dx \\ &= \int_{-\ell/2}^{\ell/2} \cosh\left(\frac{\rho g x}{P}\right) dx = \frac{P}{\rho g} \sinh\left(\frac{\rho g x}{P}\right)\Big|_{x=-\ell/2}^{x=\ell/2} \\ &= \frac{2P}{\rho g} \sinh\left(\frac{\rho g \ell}{2P}\right) \ . \end{aligned} \tag{5.206}$$

The transcendental nature of the equation makes it difficult to invert it, but at least we now have an expression for the length of the wire s as a function of P.

Brachistochrone

For this problem, we need to minimize a time. Similar to the catenary problem, our unknown will be the height function of the path $y(x)$. Locally, for a particle traveling a distance ds at speed v, we expect the time taken to be $dt = ds/v$. We can write the infinitesimal path length again in the form

$$ds = \sqrt{1+(y')^2}\, dx \ .$$

As we saw in the case of the skiier, the velocity is fixed by energy conservation, $E = \frac{1}{2}mv^2 + mgy(x)$. In other words

$$v^2 = \frac{2}{m}\left(E - mgy\right) \ . \tag{5.207}$$

Let us pick our coordinate system such that the beginning of the track is at the origin and our initial condition such that the particle is released from rest. In this case, we find $E = 0$ and $v^2 = -2gy$; the height function y will be negative. To avoid carrying around this annoying negative sign, I will define a positive quantity $h(x) \equiv -y(x)$.

The quantity we need to minimize is the total time

$$T[h] = \int dt = \int \frac{ds}{v} = \int_{x_i}^{x_f} \frac{1}{\sqrt{2gh}}\sqrt{1+(h')^2}\,dx \ . \tag{5.208}$$

Our effective Lagrangian is

$$L(h, h', x) = \frac{1}{\sqrt{2gh}}\sqrt{1+(h')^2} \ , \tag{5.209}$$

which similar to the catenary problem is evidently independent of x. We can thus define a conserved quantity

$$P = \frac{\partial L}{\partial h'}h' - L = \frac{(h')^2}{\sqrt{2gh}\sqrt{1+(h')^2}} - \frac{1}{\sqrt{2gh}}\sqrt{1+(h')^2} = -\frac{1}{\sqrt{2gh}\sqrt{1+(h')^2}} \tag{5.210}$$

Squaring both sides yields

$$1 + (h')^2 = \frac{1}{2ghP^2} \ , \tag{5.211}$$

after some rearrangement. Separating variables, we find the first integral expressoin

$$\int \frac{dh}{\sqrt{\frac{1}{2ghP^2} - 1}} = \pm \int dx \tag{5.212}$$

Let us introduce dimensionless lengths $v = 2ghP^2$ and $u2gxP^2$: The integral takes the slightly simpler form

$$\int \frac{dv}{\sqrt{\frac{1}{v} - 1}} = \pm \int du \tag{5.213}$$

The left hand side can be looked up in a table of integrals, but the expression cannot be inverted to give h as a function of x (equivalently $v(u)$):

$$f(v) = \pm u + \text{const} \ \text{ where } \ f(v) \equiv -v\sqrt{\frac{1}{v} - 1} - \tan^{-1}\sqrt{\frac{1}{v} - 1} \ . \tag{5.214}$$

The function $f(v)$ is defined in the range $0 < v < 1$. Moreover, we find $f(0) = -\frac{\pi}{2}$ and $f(1) = 0$. The consequence of these facts is that the range u is seemingly limited to $|u_{\max} - u_{\min}| < \frac{\pi}{2}$. However, by taking advantage of the sign choice $\pm$, we can glue two solutions together and extend the range.

We assume that the particle is released from rest at the origin $u = v = 0$ and travels along a path constructed for $u > 0$. Then initially, the particle will follow

$$f(v) + \frac{\pi}{2} = u \ . \tag{5.215}$$

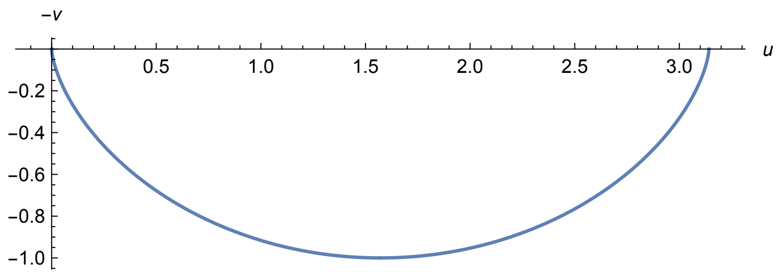

Figure 5.11: The path followed by a falling particle to minimize the time between two points.

However, if the endpoint of the track is located at $\frac{\pi}{2} < u_{\mathrm{max}} < \pi$, we can glue a second solution on

$$-f(v) + \frac{\pi}{2} = u \ . \tag{5.216}$$

The path in uv-space it plotted as figure 5.11.

It's not entirely obvious but the fact that P is arbitrary allows us to choose any pair (x, y) in the lower right hand quadrant of the plane as the final point. Note the ratio $\frac{y}{x} = \frac{v}{u}$ is equal to $\frac{2}{\pi}$ where the two solutions join. For $\frac{y}{x} > \frac{2}{\pi}$, we can use just the first solution $f(v) + \frac{\pi}{2} = u$. For points that are separated more horizontally than vertically, $\frac{y}{x} < \frac{2}{\pi}$, we have to use the second solution as well.

Chapter 6

Hamiltonian Mechanics

In our discussion of Lagrangians we already introduced the notion of the conjugate momentum:

$$p^i = \frac{\partial L}{\partial \dot{q}_i} \ . \tag{6.1}$$

Once all the (holonomic only) constraints have been solved for, leading to a reduced number of degrees of freedom there is one conjugate momentum for each generalized coordinate q_i. The Euler-Lagrange equation then gives a second order differential equation for the time evolution of the system. Since it is second order one must specify the initial values of q_i and $\dot{q}_i$. That is the initial positions and velocities of the particles.

We now consider an equivalent description of dynamical systems known as the Hamiltonian formulation. Here one essentially swaps $\dot{q}_i$ for the conjugate momentum p^i and doubles the number of variables. Since both positions and velocities are needed to describe a system the Hamiltonian formulation puts both of q_i and p^i on an equal footing. The upside of this is that the second order Euler-Lagrange equations are replaced by first order evolution equations known as Hamilton's equations. In effect we wish to go from thinking in terms of the Lagrangian $L(q_i, \dot{q}_i, t)$ to a new function $H(q_i, p^i, t)$ known as the Hamiltonian which encodes the same information. In particular in the Hamiltonian view $\dot{q}_i$ never appears, only q_i and p^i. To emphasize this point we will not use a dot as a short-hand for a time derivative once we are in the Hamiltonian formulation (although we will use it when we discuss Lagrangians). Thus, once in the Hamiltonian formulation $\dot{q}$ never appears.

The general procedure is called a **Legendre transformation**. Let us consider a function $F(x, y)$ which we want to swap for a new function $\tilde{F}(x, u)$ without losing any information. To do this we note that the total differential of F is

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy \ . \tag{6.2}$$

Let us introduce a new function

$$\tilde{F}(x, y, u) = uy - F(x, y) \ , \tag{6.3}$$

which is initially a function of (x, y, u) so that

$$d\tilde{F} = ydu + udy - \frac{\partial F}{\partial x} dx - \frac{\partial F}{\partial y} dy \ . \tag{6.4}$$

However we see that the dy term in $d\tilde{F}$ drops out if we take

$$u = \frac{\partial F}{\partial y} \ . \tag{6.5}$$

In this case $\tilde{F}$ is only a function of (x, u):

$$\tilde{F}(x,u) = uy(x,u) - F(x, y(x,u)) \tag{6.6}$$

where we use (6.5) to find $y(u, x)$. This is the **Legendre transformation** and it preserves all the information of the system since we can undo it by a further Legendre transformation. To see this consider

$$\tilde{\tilde{F}}(x,z) = \frac{\partial \tilde{F}}{\partial u} u(x,z) - \tilde{F}(x, u(x,z)) \ , \tag{6.7}$$

where $z = \partial\tilde{F}/\partial u$ so that $\tilde{\tilde{F}}$ is only a function of (x, z) . But from (6.6)

$$\begin{aligned} \frac{\partial \tilde{F}}{\partial u} &= y + u\frac{\partial y}{\partial u} - \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} \\ &= y + u\frac{\partial y}{\partial u} - u\frac{\partial y}{\partial u} \\ &= y \ , \end{aligned} \tag{6.8}$$

where we have used (6.5) in the second line. Therefore

$$\begin{aligned} \tilde{\tilde{F}}(x,z) &= yu(x,z) - \tilde{F}(x, u(x,z)) \\ &= F(x,z) \ , \end{aligned} \tag{6.9}$$

and we have the original function back.

N.B. This isn't quite true. In computing the Legendre transform we have assumed that we can invert the expression $u = \partial F/\partial y$ to find y as a function of u (and x). If F is only linear in y then u is y-independent and hence we can not invert to find y as a function of u. We will largely ignore this special case since y will be taken to be the velocity and Lagrangians are typically quadratic in velocity.

So what? We have already encountered the conjugate momentum:

$$p^i = \frac{\partial L}{\partial \dot{q}_i} \ . \tag{6.10}$$

This can be viewed as part of a Legendre transform which produces the Hamiltonian from the Lagrangian:

$$H(q_i, p^i, t) = \sum_i p^i \dot{q}_i - L(q_i, \dot{q}_i, t) \ . \tag{6.11}$$

We have seen this before, where H was the conserved energy E (when there is no explicit time dependence). Thus the physical significance of the Hamiltonian is that it is the energy of the system.

Example: To make things concrete let us look at a Lagrangian of the form

$$L(q_i, \dot{q}_i) = \sum_i \frac{1}{2} m_i \dot{q}_i^2 - V(q_i) \ . \tag{6.12}$$

For example the q_i's could be the $3N$ position variables of N particles in $\mathbb{R}^3$. Here we see that the conjugate momenta are

$$p^i = m_i\dot{q}_i \qquad \Longleftrightarrow \qquad \dot{q}_i = p^i/m_i \ . \tag{6.13}$$

The Hamiltonian is then

$$\begin{aligned} H(q_i, p^i) &= \sum_i p^i\dot{q}_i - L(q_i, \dot{q}_i(q_i, p^i)) \\ &= \sum_i p^i\dot{q}_i - \sum \frac{1}{2}m_i\dot{q}_i^2 + V(q_i) \\ &= \sum_i \frac{(p^i)^2}{2m_i} + V(q_i) \ . \end{aligned} \tag{6.14}$$

6.1 Hamilton's Equations

As $H \equiv H(q_i, p^i; t)$ then,

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p^i}dp^i + \frac{\partial H}{\partial t}dt \right). \tag{6.15}$$

While as $H = \sum_i \dot{q}_i p^i - L$ we also have

$$\begin{aligned} dH &= \sum_i \left(d\dot{q}_i p^i + \dot{q}_i dp^i - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i - \frac{\partial L}{\partial t}dt \right) \\ &= \sum_i \left(\dot{q}_i dp^i - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial t}dt \right) \end{aligned} \tag{6.16}$$

where we have used the definition of the conjugate momentum $p^i = \frac{\partial L}{\partial \dot{q}_i}$ to eliminate two terms in the final line. By comparing the coefficients of dq_i, $d\dot{q}_i$ and dt in the two expressions for dH we find

$$\dot{q}_i = \frac{\partial H}{\partial p^i}, \qquad -\frac{\partial L}{\partial q_i} = \frac{\partial H}{\partial q_i}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{6.17}$$

Next we use the Euler-Lagrange equation to observe that $\dot{p}_i = \frac{\partial L}{\partial q_i}$ so that the first two equations give

$$\frac{d}{dt}q_i = \frac{\partial H}{\partial p^i}, \qquad \frac{d}{dt}p^i = -\frac{\partial H}{\partial q_i} \tag{6.18}$$

These are referred to as Hamilton's equations of motion. Notice that these are $2n$ first order differential equations compared to Lagrange's equations which are n second-order differential equations. The space of (q_i, p^i) is known as **phase space** and typically for N unconstrained particles it is $\mathbb{R}^{6N}$ since each particle has three position variables and three momentum variables. In classical mechanics the state of a system is given by a point in phase space.

Example 1: Let us look at a free particle on $\mathbb{R}^3$ The Lagrangian is

$$L = \frac{1}{2}m|\dot{\underline{r}}|^2 \tag{6.19}$$

Clearly the Euler-Lagrange equations are just

$$m\ddot{\underline{r}} = 0 \tag{6.20}$$

which have linear solutions $\underline{r}(t) = \underline{v}(0)t + \underline{r}(0)$. To construct the Hamiltonian we note that

$$\underline{p} = \frac{\partial L}{\partial \dot{\underline{r}}} = m\dot{\underline{r}} \tag{6.21}$$

and hence

$$\begin{aligned} H &= \underline{p} \cdot \dot{\underline{r}} - L \\ &= \frac{1}{2m}|\underline{p}|^2 \ . \end{aligned} \tag{6.22}$$

What are Hamiltons equations? Just:

$$\begin{aligned} \frac{d}{dt}\underline{r} &= \frac{\partial H}{\partial \underline{p}} = \frac{\underline{p}}{m} \\ \frac{d}{dt}\underline{p} &= -\frac{\partial H}{\partial \underline{r}} = \underline{0} \end{aligned} \tag{6.23}$$

Thus the solution is

$$\begin{aligned} \underline{p}(t) &= \underline{p}(0) \\ \underline{r}(t) &= \frac{\underline{p}(0)}{m}t + \underline{r}(0) \end{aligned} \tag{6.24}$$

In this case the Hamiltonian flow simply consists of straight lines in phases space with a constant value of $\underline{p} \neq \underline{0}$. In particular the flow goes to the left for lines above the $\underline{r}$ axis but to the right for lines below the $\underline{r}$ axis. The $\underline{p} = \underline{0}$ axis itself is special as points that start on it remain on it. Therefore it simply consists of an infinite collection of disjoint points (this case is a bit singular).

Example 2: Let us look at the simplest next possible system in detail. Its called the harmonic oscillator and consists of a single degree of freedom q with mass m moving in a potential $V(q) = \frac{1}{2}kq^2$. Thus the force is linear $F = -kq$. Such a system could be a spring displaced by an amount q (using Hook's law with spring constant k) or the small angle approximation to a pendulum with $q = \theta << 1$, since in that case we had $V = -mgl\cos\theta \sim -mgl + \frac{1}{2}mgl\theta^2$ so that $k^2 = mgl$ and the constant term $-mgl$ is irrelevant.

First let us solve this problem using the Lagrangian approach. Here we construct

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 \tag{6.25}$$

from which we obtain the Euler-Lagrange equation

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} \\ &= \frac{d}{dt}(m\dot{q}) + kq \\ &= m\ddot{q} + kq \ . \end{aligned} \tag{6.26}$$

This can be readily solved by taking

$$q(t) = A\sin(\omega t) + B\cos(\omega t) \qquad \omega = \sqrt{k/m} \ . \tag{6.27}$$

Here we identify $B = q(0)$ and $\omega A = \dot{q}(0)$ so that we could write

$$q(t) = \omega^{-1}\dot{q}(0)\sin(\omega t) + q(0)\cos(\omega t) \ . \tag{6.28}$$

Let us look at this in the Hamiltonian formalism. Here the phase space is $\mathbb{R}^2$ parameterized by (q, p). First we note that

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \tag{6.29}$$

so that

$$\dot{q} = p/m \ . \tag{6.30}$$

Thus the Hamiltonian is

$$\begin{aligned} H &= p\dot{q} - L \\ &= \frac{q^2}{m} + \frac{1}{2}\left(\frac{p}{m}\right)^2 + \frac{1}{2}kq^2 \\ &= \frac{1}{2m}p^2 + \frac{1}{2}kq^2 \ . \end{aligned} \tag{6.31}$$

From here we read off Hamilton's equations:

$$\begin{aligned} \frac{d}{dt}q &= \frac{\partial H}{\partial p} = p/m \\ \frac{d}{dt}p &= -\frac{\partial H}{\partial q} = -kq \ . \end{aligned} \tag{6.32}$$

Note that we can substitute the first equation into the second equation to find

$$\frac{d}{dt}p = m\frac{d^2}{dt^2}q = -qk \tag{6.33}$$

which is just the same equation as in the Lagrangian formulation. However this is not how we want to think of the problem. Rather we want to solve for $q(t)$ and $p(t)$. The simplest way to do this is take the previous solution and reinterpret it:

$$\begin{aligned} q(t) &= A\sin(\omega t) + B\cos(\omega t) \\ p(t) &= m\dot{q} \\ &= mA\omega\cos(\omega t) - mB\omega\sin(\omega t) \end{aligned} \tag{6.34}$$

From here we see that

$$B = q(0) \qquad A = \frac{p(0)}{m\omega} \tag{6.35}$$

so we have

$$\begin{aligned} q(t) &= \frac{p(0)}{m\omega}\sin(\omega t) + q(0)\cos(\omega t) \\ p(t) &= p(0)\cos(\omega t) - mq(0)\omega\sin(\omega t) \ . \end{aligned} \tag{6.36}$$

Notice that these parameterize an ellipse:

$$(q(t))^2 + \frac{1}{m^2\omega^2}(p(t))^2 = +(q(0))^2 + \frac{1}{m^2\omega^2}(p(0))^2 \ . \tag{6.37}$$

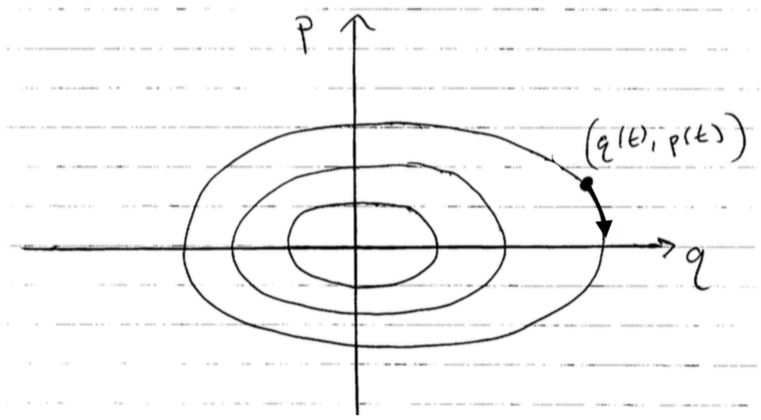

Figure 6.1: Phase space

In particular the right hand side is simply proportional to the Hamiltonian which is the energy of the system:

$$\begin{aligned} H &= \frac{1}{2m}p^2 + \frac{1}{2}kq^2 \\ &\frac{k}{2}\left(q^2 + \frac{p^2}{mk}\right) \\ &= \frac{m\omega^2}{2}\left(q^2 + \frac{p^2}{mk}\right) \end{aligned} \tag{6.38}$$

so that

$$(q(t))^2 + \frac{1}{m^2\omega^2}(p(t))^2 = \frac{2E}{m\omega^2} \ . \tag{6.39}$$

Thus from the Hamiltonian point of view the dynamical motion consists of concentric ellipses in phase space. Note that any given point in phase space lies on just one ellipse.

This is a general feature: the ellipses are known as **Hamiltonian flows** and as a consequence of the first order dynamical differential equations a given point in phase space lies on just one curve of the Hamiltonian flow since the solution to Hamilton's equations only depends on the initial value of (q, p).

Let us observe that there appears to be a symmetry of this Hamiltonian:

$$q' = \frac{1}{\sqrt{mk}}p \qquad p' = -\sqrt{mk}q \tag{6.40}$$

then one has

$$\begin{aligned} H(q,p) &= \frac{1}{2m}\left(\sqrt{mk}q'\right)^2 + \frac{k}{2}\left(-\frac{p'}{\sqrt{mk}}\right)^2 \\ &= \frac{1}{2m}p'^2 + \frac{1}{2}kq'^2 \\ &= H(q', p') \ . \end{aligned} \tag{6.41}$$

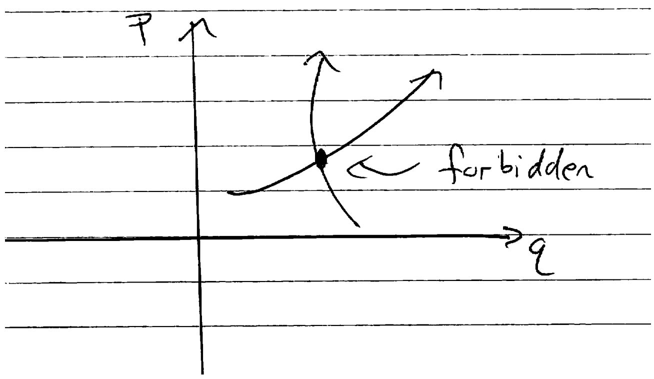

Figure 6.2: Flows in phase space that cross are forbidden.

From the point of view of the phase space diagrams this corresponds to swapping m with $1/k$ (which effectively swaps the longer and shorter radii of the ellipses), and then rotating (p, q) by $90°$. This maps the ellipses of phase space back to themselves and, as we have just seen, this is a symmetry of the Hamiltonian.

However from the Lagrangian point of view this is weird: how can we swap positions with momenta? Furthermore the Lagrangian has no such symmetry to swap q with $\dot{q}$ and m with $1/k$. However we have just seen that the set of solutions is left invariant. So something is up. To make sense of this in the Hamiltonian formulation we need to develop the concept of Poisson brackets and canonical transformations.

6.2 Poisson Brackets

Phase space is always even-dimensional (at least as we've constructed it here). As a result there is a useful skew-symmetric structure known as a **symplectic structure**. It is determined by the **Poisson bracket** which is defined by

$$\{f, g\} \equiv \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p^i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p^i} \right) , \tag{6.42}$$

where $f = f(q_i, p^i)$ and $g = g(q_i, p^i)$ are arbitrary functions on phase space.

The Poisson bracket has several properties:

- $\{f, g\} = -\{g, f\}$.
- $\{f, g + \lambda h\} = \{f, g\} + \lambda\{f, h\}$, for $\lambda \in \mathbb{R}$
- $\{f, gh\} = \{f, g\}h + g\{f, h\}$

The first two should be obvious from the definition. The third requires a little calculation (and is given as a problem).

One can write the equations of motion using the Poisson bracket as

$$\frac{d}{dt} q = \{q_i, H\} = \frac{\partial H}{\partial p^i} \qquad \text{and} \qquad \frac{d}{dt} p = \{p^i, H\} = -\frac{\partial H}{\partial q_i}. \tag{6.43}$$

In fact for any function $f(q_i, p^i)$ on phase space we have that $\dot{f} = \{f, H\}$. To prove this we note that

$$
\begin{aligned}
\{f, H\} &= \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p^i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p^i} \right) \qquad (6.44)\\
&= \sum_i \left(\frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \frac{dp^i}{dt}\frac{\partial f}{\partial p^i} \right)\\
&= \frac{df}{dt} ,
\end{aligned}
$$

if $f = f(q_i, p^i)$.

6.3 Canonical Transformations And Symmetries

The set of Poisson brackets acting on simply q_i and p^j are known as the fundamental or canonical Poisson brackets. They have a simple form:

$$
\begin{aligned}
\{q_i, p^j\} &= \delta_{ij} \qquad (6.45)\\
\{q_i, q_j\} &= 0\\
\{p^i, p^j\} &= 0 ,
\end{aligned}
$$

which one may easily confirm by direct computation.

Definition: **Canonical transformations** $q_i \to q_i'$, $p^i \to p_i'$ are transformations which preserve fundamental Poisson brackets, *i.e.*

$$\{q_i', p_j'\} = \delta_{ij}, \quad \{q_i', q_j'\} = 0 \quad \text{and} \qquad \{p_i', p_j'\} = 0. \qquad (6.46)$$

Note that q_i' and p_i' are both allowed to be functions of q_i and p^i. From the Lagrangian point of view this would be weird: it would mean redefining q_i' as a function of q_i and $\dot{q}_i$. But in the Hamiltonian formulation q_i and p^i are on equal footings. We will see that this leads to a larger class of symmetries than in the Lagrangian formulation.

The key thing about canonical transformations is that Hamilton's equations remain valid

$$\frac{d}{dt}q_i' = \frac{\partial H(q_i', p_i')}{\partial p_i'} \qquad \text{and} \qquad \frac{d}{dt}p_i' = -\frac{\partial H(q_i', p_i')}{\partial q_i'}. \qquad (6.47)$$

To see this we write (6.44) for q' instead of f and we note that

$$
\begin{aligned}
\frac{d}{dt}q_i' &= \{q_i', H\}\\
&= \sum_j \left(\frac{\partial q_i'}{\partial q_j}\frac{\partial H}{\partial p^j} - \frac{\partial q_i'}{\partial p^j}\frac{\partial H}{\partial q_j} \right)\\
&= \sum_{j,k} \left(\frac{\partial q_i'}{\partial q_j} \left(\frac{\partial p_k'}{\partial p^j}\frac{\partial H}{\partial p_k'} + \frac{\partial q_k'}{\partial p^j}\frac{\partial H}{\partial q_k'} \right) - \frac{\partial q_i'}{\partial p^j} \left(\frac{\partial p_k'}{\partial q_j}\frac{\partial H}{\partial p_k'} + \frac{\partial q_k'}{\partial q_j}\frac{\partial H}{\partial q_k'} \right) \right)\\
&= \sum_{j,k} \left(\left(\frac{\partial q_i'}{\partial q_j}\frac{\partial p_k'}{\partial p^j} - \frac{\partial q_i'}{\partial p^j}\frac{\partial p_k'}{\partial q_j} \right) \frac{\partial H}{\partial p_k'} + \left(\frac{\partial q_i'}{\partial q_j}\frac{\partial q_k'}{\partial p^j} - \frac{\partial q_i'}{\partial p^j}\frac{\partial q_k'}{\partial q_j} \right) \frac{\partial H}{\partial q_k'} \right)\\
&= \sum_k \left(\{q_i', p'^k\}\frac{\partial H}{\partial p_k'} + \{q_i', q_k'\}\frac{\partial H}{\partial q_k'} \right)\\
&= \frac{\partial H}{\partial p'^i} . \qquad (6.48)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\frac{d}{dt}p'^i &= \{p'^i, H\} \\
&= \sum_j \left(\frac{\partial p'^i}{\partial q_j}\frac{\partial H}{\partial p^j} - \frac{\partial p'^i}{\partial p^j}\frac{\partial H}{\partial q_j} \right) \\
&= \sum_{j,k} \left(\frac{\partial p'^i}{\partial q_j} \left(\frac{\partial p'^k}{\partial p^j}\frac{\partial H}{\partial p'^k} + \frac{\partial q'_k}{\partial p^j}\frac{\partial H}{\partial q'_k} \right) - \frac{\partial p'_i}{\partial p^j} \left(\frac{\partial p'_k}{\partial q_j}\frac{\partial H}{\partial p'_k} + \frac{\partial q'_k}{\partial q_j}\frac{\partial H}{\partial q'_k} \right) \right) \\
&= \sum_{j,k} \left(\left(\frac{\partial p'^i}{\partial q_j}\frac{\partial p'^k}{\partial p^j} - \frac{\partial p'^i}{\partial p^j}\frac{\partial p'_k}{\partial q_j} \right) \frac{\partial H}{\partial p'_k} + \left(\frac{\partial p'_i}{\partial q_j}\frac{\partial q'_k}{\partial p^j} - \frac{\partial p'_i}{\partial p^j}\frac{\partial q'_k}{\partial q_j} \right) \frac{\partial H}{\partial q'_k} \right) \\
&= \sum_k \left(\{p'^i, p'^k\}\frac{\partial H}{\partial p'_k} + \{p'^i, q'_k\}\frac{\partial H}{\partial q'_k} \right) \\
&= -\frac{\partial H}{\partial q'_i} \; .
\end{aligned}
\tag{6.49}
$$

This allows us to define the notion of a **symmetry** in the Hamiltonian formulation: A symmetry is a canonical transformation $q \to q'_i(q_i, p^i)$, $p^i \to p'^i(q_i, p^i)$ such that

$$H(q'_i, p'^i) = H(q_i, p^i)$$

Example: Harmonic Oscilator If we return to the Harmonic oscilator before we see that

$$q' = \frac{1}{\sqrt{mk}}p \qquad p' = -\sqrt{mk}q \tag{6.50}$$

is indeed a canonical transformation:

$$\{q', p'\} = \left\{ \frac{1}{\sqrt{mk}}p, -\sqrt{mk}q \right\} = -\{p, q\} = \{q, p\} = 1 \tag{6.51}$$

Furthermore $\{q', q'\} = \{p', p'\} = 0$ automatically as the Poisson bracket is anti-symmetric. Furthermore we have already noted that

$$H(q', p') = H(q, p)$$

So we can call this a symmetry.

More generally an infinitesimal canonical transformation may be generated by any arbitrary function $f(q_i, p^i)$ (called generating function) on phase space via

$$
\begin{aligned}
q_i \to q'_i &= q_i + \epsilon\{q_i, f\} \equiv q_i + \delta q_i \\
p^i \to p'_i &= p^i + \epsilon\{p^i, f\} \equiv p^i + \delta p^i \; ,
\end{aligned}
\tag{6.52}
$$

where

$$
\begin{aligned}
\delta q_i &= \epsilon\{q_i, f\} = \epsilon\frac{\partial f}{\partial p^i} \\
\delta p^i &= \epsilon\{p^i, f\} = -\epsilon\frac{\partial f}{\partial q_i} \; .
\end{aligned}
\tag{6.53}
$$

Next we show that, expanding to first order in $\epsilon \ll 1$, the transformation is an infinitesimal canonical transformation. It is easy to check that this preserves the fundamental

Poisson brackets up to terms of order $\mathcal{O}(\epsilon^2)$, *e.g.*

$$\begin{aligned}
\{q_i', p_j'\} &= \{q_i + \epsilon\{q_i, f\}, p^j + \epsilon\{p^j, f\}\} \\
&= \{q_i, p^j\} + \epsilon\{\{q_i, f\}, p^j\} + \epsilon\{q_i, \{p^j, f\}\} + \mathcal{O}(\epsilon^2) \\
&= \{q_i, p^j\} + \epsilon(\{\frac{\partial f}{\partial p^i}, p^j\} + \{q_i, -\frac{\partial f}{\partial q_j}\}) + O(\epsilon^2) \\
&= \delta_{ij} + \epsilon\left(\frac{\partial^2 f}{\partial q_j \partial p^i} - \frac{\partial^2 f}{\partial p^i \partial q_j}\right) + \mathcal{O}(\epsilon^2) \\
&= \delta_{ij} + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{6.54}$$

We must also check that

$$\begin{aligned}
\{q_i', q_j'\} &= \{q_i + \epsilon\{q_i, f\}, q_j + \epsilon\{q_j, f\}\} \\
&= \{q_i, q_j\} + \epsilon\{\{q_i, f\}, q_j\} + \epsilon\{q_i, \{q_j, f\}\} + \mathcal{O}(\epsilon^2) \\
&= \{q_i, q_j\} + \epsilon(\{\frac{\partial f}{\partial p^i}, q_j\} + \{q_i, \frac{\partial f}{\partial p^j}\}) + O(\epsilon^2) \\
&= 0 + \epsilon\left(-\frac{\partial^2 f}{\partial p^j \partial p^i} + \frac{\partial^2 f}{\partial p^i \partial p^j}\right) + \mathcal{O}(\epsilon^2) \\
&= 0 + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{6.55}$$

and

$$\begin{aligned}
\{p_i', p_j'\} &= \{p^i + \epsilon\{p^i, f\}, p^j + \epsilon\{p^j, f\}\} \\
&= \{p^i, p^j\} + \epsilon\{\{p^i, f\}, p^j\} + \epsilon\{p^i, \{p^j, f\}\} + \mathcal{O}(\epsilon^2) \\
&= \{p^i, p^j\} + \epsilon(-\{\frac{\partial f}{\partial q_i}, p^j\} + \{p^i, -\frac{\partial f}{\partial q_j}\}) + O(\epsilon^2) \\
&= 0 + \epsilon\left(-\frac{\partial^2 f}{\partial q_j \partial q_i} + \frac{\partial^2 f}{\partial q_i \partial q_j}\right) + \mathcal{O}(\epsilon^2) \\
&= 0 + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{6.56}$$

Canonical transformations therefore generalise simple coordinate transformations of the form $q_i' = q_i'(q_i)$ to also allow for transformations that define q_i' in terms of q_i and p^i. In particular when we looked at Noether's theorem we considered infinitesimal change of variables of the form

$$q_i' = q_i + \epsilon T_i(q) \qquad i.e. \qquad \delta q_i = \epsilon T_i \tag{6.57}$$

where $\epsilon << 1$. In the Lagrangian formulation we can use this to compute the change in p^i:

$$\begin{aligned}
p^{i\prime} &= \frac{\partial L}{\partial \dot{q}_i'} \\
&= \sum_j \frac{\partial \dot{q}_j}{\partial \dot{q}_i'}\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial q_j}{\partial \dot{q}_i'}\frac{\partial L}{\partial q_j} \\
&= \sum_j \frac{\partial \dot{q}_j}{\partial \dot{q}_i'}\frac{\partial L}{\partial \dot{q}_j}
\end{aligned} \tag{6.58}$$

where we have used the fact that q_i and $\dot{q}_i$ are independent in the Lagrangian formulation. To proceed we note that, to first order in ϵ, we can write

$$\begin{aligned} \dot{q}_j &= \dot{q}'_j - \epsilon\dot{T}_j(q') \\ &= \dot{q}'_j - \epsilon\sum_k \frac{\partial T_j(q')}{\partial q_k}\dot{q}_k \end{aligned} \tag{6.59}$$

so that

$$\begin{aligned} \frac{\partial \dot{q}_j}{\partial \dot{q}'_i} &= \delta_{ij} - \epsilon\sum_k \frac{\partial T_j}{\partial q'_k}\delta_{ki} - \epsilon\sum_k \frac{\partial^2 T_j}{\partial q_k \partial \dot{q}'_i}\dot{q}_k \\ &= \delta_{ij} - \epsilon\frac{\partial T_j}{\partial q'_i} \ , \end{aligned} \tag{6.60}$$

where we used the fact that T does not depend on $\dot{q}_i$ so that the second derivative term vanishes. Therefore

$$\begin{aligned} p'^i &= \frac{\partial L}{\partial \dot{q}'_i} \\ &= \sum_j \frac{\partial \dot{q}_j}{\partial \dot{q}'_i}\frac{\partial L}{\partial \dot{q}_j} \\ &= \sum_j \left(\delta_{ij} - \epsilon\frac{\partial T_j}{\partial q_i}\right)\frac{\partial L}{\partial \dot{q}_j} \\ &= p^i - \epsilon\sum_j \frac{\partial T_j(q)}{\partial q_i}p^j \ , \end{aligned} \tag{6.61}$$

where we have dropped all terms that are higher order in ϵ.

To see that these transformations are canonical we can just take the generating function f to be

$$f = \sum_j p^j T_j(q) \tag{6.62}$$

(note that this is just the Noether charge (5.141) written in terms of p^j) so that

$$\begin{aligned} \delta q_i &= \epsilon\sum_j \{q_i, p^j T_j(q)\} \\ &= \epsilon\sum_j \{q_i, p^j\}T_j(q) + \epsilon\sum_j \{q_i, T_j(q)\}p^j \\ &= \epsilon\sum_j \delta_{ij}T_j(q) \\ &= \epsilon T_i(q) \ , \end{aligned} \tag{6.63}$$

whereas for δp^i we find

$$\begin{aligned} \delta p^i &= \epsilon\sum_j \{p^i, p^j T_j(q)\} \\ &= \epsilon\sum_j \{p^i, p^j\}T_j(q) + \epsilon\sum_j p^j\{p^i, T_j(q)\} \\ &= -\epsilon\sum_j p^j\frac{\partial T_j(q)}{\partial q_i} \ . \end{aligned} \tag{6.64}$$

Which agrees with what we found using the Lagrangian formulation. Thus an infinitesimal coordinate transformation in the Lagrangian formulation leads to an infinitesimal canonical transformation in the Hamiltonian formulation where the generating function f is linear in momenta.

However clearly there are many more types of canonical transformations in the Hamiltonian formulation since f can be an arbitrary function of both q_i and p^i. However the Lagrangian formulation only sees canonical transformations that are generated by a function f that is linear in the momentum variables.

Example: Harmonic Oscilator Let us return again to the harmonic oscilator. In fact any rotation (along with an appropriate rescaling of coordinates) is a symmetry. In particular if

$$\begin{aligned} q' &= \cos\alpha q + \frac{1}{\sqrt{mk}} p \sin\alpha \\ p' &= \cos\alpha p - \sqrt{mk} q \sin\alpha \end{aligned} \tag{6.65}$$

then

$$\begin{aligned} H(q', p') &= \frac{1}{2m}(\cos\alpha p - \sqrt{mk}\sin\alpha q)^2 + \frac{k}{2}(\cos\alpha q + \frac{1}{\sqrt{mk}} p \sin\alpha)^2 \\ &= \frac{1}{2m}(\cos^2\alpha + \sin^2\alpha)p^2 + \frac{k}{2}(\cos^2\alpha + \sin^2\alpha)q^2 \\ &= H(q, p) \end{aligned} \tag{6.66}$$

To see whether or not it is canonical we again compute ($\{q', q'\} = \{p', p'\} = 0$ automatically)

$$\begin{aligned} \{q', p'\} &= \{\cos\alpha q + \frac{1}{\sqrt{mk}} p \sin\alpha, \cos\alpha p - \sqrt{mk} q \sin\alpha\} \\ &= \cos^2\alpha\{q, p\} - \sin^2\alpha\{p, q\} \\ &= \{q, p\} \end{aligned} \tag{6.67}$$

So this is indeed a canonical transformation. In fact if we start with a the solutions we found before:

$$\begin{aligned} q(t) &= \frac{p(0)}{m\omega}\sin(\omega t) + q(0)\cos(\omega t) \\ p(t) &= p(0)\cos(\omega t) - mq(0)\omega\sin(\omega t) \ . \end{aligned} \tag{6.68}$$

then the new solutions are (recall $\omega = \sqrt{k/m}$ and some trig identities)

$$\begin{aligned} q'(t) &= \frac{p(0)}{m\omega}\sin(\omega t)\cos\alpha + q(0)\cos(\omega t)\cos\alpha + \frac{p(0)}{\sqrt{mk}}\cos(\omega t)\sin\alpha - \frac{mq(0)}{\sqrt{mk}}\omega\sin(\omega t)\sin\alpha \\ &= \frac{p(0)}{m\omega}\left(\sin(\omega t)\cos\alpha + \cos(\omega t)\sin\alpha\right) + q(0)\left(\cos(\omega t)\cos\alpha - \sin(\omega t)\sin\alpha\right) \\ &= \frac{p(0)}{m\omega}\sin(\omega t + \alpha) + q(0)\cos(\omega t + \alpha) \\ p'(t) &= p(0)\cos(\omega t)\cos\alpha - mq(0)\omega\sin(\omega t)\sin\alpha - \sqrt{mk}\frac{p(0)}{m\omega}\sin(\omega t)\sin\alpha - \sqrt{mk}q(0)\cos(\omega t)\sin\alpha \\ &= p(0)\left(\cos(\omega t)\cos\alpha - \sin(\omega t)\sin\alpha\right) - m\omega q(0)\left(\sin(\omega t)\sin\alpha + \cos(\omega t)\sin\alpha\right) \\ &= p(0)\cos(\omega t + \alpha) - mq(0)\omega\sin(\omega t + \alpha) \ . \end{aligned} \tag{6.69}$$

Thus we see that the canonical transformations corresponding to rotations are simply time translations. Furthermore the Hamiltonian itself is the function that generates these transformations infinitesimally (up to a factor of ω that can be absorbed into α):

$$\begin{aligned}\delta q &= \frac{\alpha}{\omega}\{q,H\} = \frac{\alpha}{2m\omega}\{q,p^2\} = \frac{\alpha}{\sqrt{mk}}p \\ \delta p &= \frac{\alpha}{\omega}\{p,H\} = \frac{k\alpha}{2\omega}\{p,q^2\} = -\alpha\sqrt{mk}q\end{aligned}$$

which agrees with (6.65) to first order in α.

In fact this is a general result. That is to say if at time t a Hamiltonian system is at $q_i(t), p^i(t)$ then at time $t+\epsilon$ the system is in

$$\begin{aligned}q_i(t+\epsilon) &= q_i(t) + \epsilon\frac{d}{dt}q_i(t) & p^i(t+\epsilon) &= p^i(t) + \epsilon\frac{d}{dt}p^i(t) \\ &= q_i(t) + \epsilon\{q_i, H\} & &= p^i(t) + \epsilon\{p^i, H\}\end{aligned} \tag{6.70}$$

Thus time evolution is just a series of infinitesimal canonical transformations on phase space.

6.3.1 Noether's Theorem

As we have said, symmetries in the Hamiltonian formulation are canonical transformations that leave the Hamiltonian invariant:

$$\begin{aligned}&H(q_i', p'^i) = H(q_i, p^i) \\ &\{q'_i, p'^j\} = \delta_i^j\ , \qquad \{q'_i, q'_j\} = \{p'^i, p'^j\} = 0\end{aligned}$$

In particular if the infinitesimal canonical transformation generated by a function f on phase space is a symmetry of the Hamiltonian then $\delta H = 0$ under the transformation. Now,

$$\begin{aligned}\delta H &= \sum_i \left(\frac{\partial H}{\partial q_i}\delta q_i + \frac{\partial H}{\partial p^i}\delta p^i\right) \\ &= \epsilon\sum_i \left(\frac{\partial H}{\partial q_i}\{q_i, f\} + \frac{\partial H}{\partial p^i}\{p^i, f\}\right) \\ &= \epsilon\sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p^i} - \frac{\partial H}{\partial p^i}\frac{\partial f}{\partial q_i}\right) \\ &= \epsilon\{H, f\} \\ &= -\epsilon\frac{df}{dt}\ ,\end{aligned} \tag{6.71}$$

where we have assumed that f is an explicit function of the phase space variables and not time, *i.e.* $\frac{\partial f}{\partial t} = 0$. Hence if the transformation is a symmetry $\delta H = 0$ then $f(q_i, p^i)$ is a conserved quantity. Thus Noether's theorem is manifest in the Hamiltonian formulation: every conserved quantity generates a canonical transformation that is a symmetry and vice-versa: every (infinitesimal) canonical transformation is generated by a function f and if this is a symmetry then f is conserved. And indeed we will see that there are more symmetries that are manifest in the Hamiltonian formulation.

If the Hamiltonian has no explicit time dependence then clearly it generates a canonical transformation that is a symmetry since

$$\delta H = \epsilon\{H, H\} = 0 \ . \tag{6.72}$$

So that H itself is the conserved quantity: the total energy $E = H$.[1]

6.3.2 Kepler Revisited

Let us construct the Hamiltonian for the Kepler problem. It is simply taken from the previous examples with a potential $V(\underline{r}) = -G_N Mm/|\underline{r}|$:

$$H = \frac{1}{2m}|\underline{p}|^2 - \frac{G_N Mm}{|\underline{r}|} \ , \tag{6.73}$$

where $\underline{q} = \underline{r}$ is the position of the planet. The phase space of this system is six-dimensional (3 from $\underline{p}$ and 3 $\underline{r}$). So its not easy to draw a phase diagram. However we can consider the effective one-dimensional theory for the radius $r = |\underline{r}|$. We saw before that that was (switching to $p = m\dot{r}$)

$$E = \frac{1}{2m}p^2 + \frac{l^2}{2mr^2} - \frac{G_N Mm}{r} \tag{6.74}$$

So in this reduced theory we have the Hamiltonian

$$\begin{aligned} H(r,p) &= \frac{1}{2m}\left(p^2 + \frac{l^2}{r^2} - \frac{2G_N Mm^2}{r}\right) \\ &= \frac{1}{2m}\left(p^2 + \left(\frac{l}{r} - \frac{G_N Mm^2}{l}\right)^2\right) - \frac{G_N^2 M^2 m^3}{2l^2} \ . \end{aligned} \tag{6.75}$$

We want to draw the phase diagram for this system. Recall that the phase flow consists of curves with constant $E = H$. At large r we have $E = p^2/2m$ which is independent of r, so lines of constant E have constant p. At small r (note that $r > 0$) we have $E = p^2/2m + l^2/2mr^2$ so $p = \pm\sqrt{2mE - l^2/r^2}$ which implies there is a minimum value of r: $r > l\sqrt{1/2mE}$ and then $|p|$ increases as r increases. On the other hand near $r = l^2/G_N Mm^2$ we can expand $r = l^2/G_N Mm^2 + \rho$ for small ρ to find

$$E = \frac{p^2}{2m} + \frac{G_N^4 M^4 m^7}{2l^6}\rho^2 - \frac{G_N^2 M^2 m^3}{2l^2} \tag{6.76}$$

which is like the harmonic oscillator (since we are near a minimum of the potential). A little bit of thought shows that the phase diagram looks like the figure below. In particular the closed orbits are the planets moving in ellipses with $E < 0$ and the open orbits that extend to $r \to \infty$ are asteroids with $E \geq 0$.

Although we won't necessarily need them we should, for completeness, compute Hamilton's equations:

$$\begin{aligned} \frac{d\underline{r}}{dt} &= \frac{\partial H}{\partial \underline{p}} = \frac{1}{m}\underline{p} \\ \frac{dr^a}{dt} &= \frac{1}{m}p^a \end{aligned} \tag{6.77}$$

[1] Note that by this equation we mean that H is a function of q_i and p^i and can be evaluated for any path $(q_i(t), p^i(t))$. If it is evaluated on a specific flow that solves Hamilton's equations then it is a constant that we identify with the energy E of that flow.

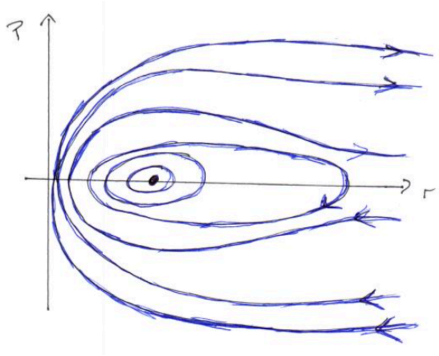

Figure 6.3: Phase flow for the Kepler problem

and

$$\begin{aligned}\frac{d\underline{p}}{dt} &= -\frac{\partial H}{\partial \underline{r}} = -\frac{G_N M m}{|\underline{r}|^3}\underline{r}\\ \frac{dp^a}{dt} &= -\frac{G_N M m}{r^3}r^a\end{aligned} \tag{6.78}$$

where in the second lines we have written the equations in terms of the components r^a, p^a of $\underline{r}, \underline{p}$ with $r = \sqrt{(r^1)^2 + (r^2)^2 + (r^3)^2}$. If we substitute the first equation into the second then we obtain the equation of motion that we would get from NII:

$$\begin{aligned}m\frac{d^2\underline{r}}{dt^2} &= -\frac{G_N M m}{|\underline{r}|^3}\underline{r}\\ m\frac{d^2 r^a}{dt^2} &= -\frac{G_N M m}{r^3}r^a \ .\end{aligned} \tag{6.79}$$

Let us look at the conserved charges. Since the Hamiltonian doesn't depend explicitly on time, $H = E$ will be conserved:

$$\dot{E} = \{H, H\} = 0 \ . \tag{6.80}$$

As we have seen the canonical transformation generated by $H = E$ is just time-translation: $t \to t + \epsilon$.

We also saw above that there was conservation of angular momentum $\underline{L} = \underline{r} \times \underline{p}$, or in components:

$$L_b = \sum_{c,d} \epsilon_{bcd} r_c p_d \ . \tag{6.81}$$

First we should check that $\{H, L_a\} = 0$. Indeed this will be true for any H of the form:

$$H = \frac{1}{2m}\underline{p} \cdot \underline{p} + V(|\underline{r}|^2) \ . \tag{6.82}$$

We need to evaluate

$$
\begin{aligned}
\{H, L_a\} &= \frac{1}{2m}\sum_{c,d} \epsilon_{bcd}\{|\underline{p}|^2, r_c p_d\} + \sum_{c,d} \epsilon_{bcd}\{V(|\underline{r}|^2), r_c p_d\} \\
&= \frac{1}{2m}\sum_{c,d} \epsilon_{bcd}\{|\underline{p}|^2, r_c\} p_d + \sum_{c,d} \epsilon_{bcd} r_c\{V(|\underline{r}|^2), p_d\} \\
&= \frac{1}{2m}\sum_{c,d} \epsilon_{bcd}\{|\underline{p}|^2, r_c\} p_d + \frac{dV}{d|\underline{r}|^2}\sum_{c,d} \epsilon_{bcd} r_c\{|\underline{r}|^2, p_d\}
\end{aligned}
$$

Next we simply note that

$$
\{|\underline{p}|^2, r_c\} = -\sum_d \{r_c, p_d p_d\} = -2p_c \tag{6.83}
$$

and similarly $\{|\underline{r}|^2, p_d\} = 2r_d$. Thus

$$
\{H, L_a\} = -\frac{1}{m}\sum_{c,d} \epsilon_{bcd} p_c p_d + 2\frac{dV}{d|\underline{r}|^2}\sum_{c,d} \epsilon_{bcd} r_c r_d = 0\ . \tag{6.84}
$$

So this is indeed a conserved quantity.

Let us look at the canonical transformation generated by L_b:

$$
\begin{aligned}
\delta_b r_a &= \epsilon\{r_a, L_b\} \\
&= \epsilon\sum_{c,d} \epsilon_{bcd}\{r_a, r_c p_d\} \\
&= \epsilon\sum_{c,d} \epsilon_{bcd}(\{r_a, r_c\} p_d + r_c\{r_a, p_d\}) \\
&= \epsilon\sum_{c} \epsilon_{cba} r_c \\
&= \epsilon\sum_{c} \epsilon_{abc} r_c\ ,
\end{aligned} \tag{6.85}
$$

and

$$
\begin{aligned}
\delta_b p_a &= \epsilon\{p_a, L_b\} \\
&= \epsilon\sum_{c,d} \epsilon_{bcd}\{p_a, r_c p_d\} \\
&= \epsilon\sum_{c,d} \epsilon_{bcd}(\{p_a, r_c\} p_d + r_c\{p_a, p_d\}) \\
&= -\epsilon\sum_{d} \epsilon_{bad} p_d \\
&= \epsilon\sum_{c} \epsilon_{abc} p_c
\end{aligned} \tag{6.86}
$$

Hence for every choice of T_b a general canonical transformation generated by $L = \sum_b L_b T^b$ is just a rotation

$$
\begin{aligned}
\underline{r} &\to \underline{r} + \epsilon\underline{T} \times \underline{r} & \underline{p} &\to \underline{p} + \epsilon\underline{T} \times \underline{p} \\
&= \underline{r} + \epsilon\mathbf{T}\underline{r} & &= \underline{p} + \epsilon\mathbf{T}\underline{p}
\end{aligned} \tag{6.87}
$$

where the matrix $\mathbf{T}$ has components $\mathbf{T}_{ac} = \sum_b \epsilon_{abc}T^b = -\sum_b \epsilon_{acb}T^b$. It is easy to see that as before these generate a symmetry of H since in particular both $|\underline{p}|^2$ and $|\underline{r}|^2$ are invariant under rotations:

$$\begin{aligned}\delta|\underline{r}|^2 &= 2\underline{r}\cdot\delta\underline{r} \\ &= 2\sum_a r_a \delta r_a \\ &= 2\epsilon\sum_{ac} r_a\epsilon_{abc}T^b r_c \\ &= 2\epsilon\ \underline{r}\cdot(\underline{T}\times\underline{r}) \\ &= 0\ , \end{aligned} \tag{6.88}$$

since $a_a r_c = r_c r_a$ but $\epsilon_{abc} = -\epsilon_{cba}$. Similarly one sees that $\delta|\underline{p}|^2 = 0$.

Let us also recall the Runge-Lenz vector

$$\begin{aligned}\underline{A} &= \underline{p}\times\underline{L} - \frac{G_N M m^2}{|\underline{r}|}\underline{r} \\ &= \underline{p}\times(\underline{r}\times\underline{p}) - \frac{G_N M m^2}{|\underline{r}|}\underline{r} \\ &= (\underline{p}\cdot\underline{p})\underline{r} - (\underline{r}\cdot\underline{p})\underline{p} - \frac{G_N M m^2}{|\underline{r}|}\underline{r}\end{aligned} \tag{6.89}$$

This was conserved but it does not arise from a symmetry of the Lagrangian. But by construction it generates a canonical transformation that is a symmetry of the Hamiltonian. What is it? First we write $\underline{A}$ in components:

$$A_b = r^b\sum_c p_c p_c - p^b\sum_c p_c r_c - \frac{G_N M m^2}{|\underline{r}|} r_b\ . \tag{6.90}$$

Note that this is not linear in p_a and so the associated canonical transformation is not simply a coordinate transformation of r_a. Next we evaluate

$$\begin{aligned}\delta_b r_a &= \epsilon\{r_a, A_b\} \\ &= \epsilon\{r_a, r^b\sum_c p_c p_c\} - \epsilon\{r_a, p^b\sum_c p_c r_c\} \\ &= \epsilon\left(\sum_c r^b p_c\{r_a,p_c\} + \sum_c\{r_a,p_c\}r^b p_c - \sum_c\{r_a,p^b\}p_c r_c - \sum_c p^b\{r_a,p_c\}r_c\right) \\ &= 2\epsilon r_b p_a - \epsilon\delta_{ab}(\underline{p}\cdot\underline{r}) - p^b r_a\ ,\end{aligned} \tag{6.91}$$

where we have dropped terms which involve $\{r_a, r_c\}$.

For the momenta we find

$$\begin{aligned}\delta_b p_a &= \epsilon\{p_a, A_b\} \\ &= \epsilon\{p_a, r^b\}\sum_c p_c p_c - \epsilon p^b\sum_c\{p_a, r_c\}p_c - \epsilon G_N M m\{p_a, r_b/|\underline{r}|\} \\ &= -\epsilon\delta_{ab}|\underline{p}|^2 + \epsilon p_a p_b - \epsilon\frac{G_N M m}{|\underline{r}|}\{p_a, r_b\} - \epsilon G_N M m^2 r_b\{p_a, |\underline{r}|^{-1}\}\ .\end{aligned} \tag{6.92}$$

To evaluate the last term we can use the chain rule

$$\begin{aligned}
\{p_a, |\underline{r}|^{-1}\} &= \sum_d \left(\frac{\partial p_a}{\partial r_d} \frac{\partial |\underline{r}|^{-1}}{\partial p_d} - \frac{\partial p_a}{\partial p_d} \frac{\partial |\underline{r}|^{-1}}{\partial r_d} \right) \\
&= -\frac{\partial |\underline{r}|^{-1}}{\partial r_a} \\
&= \frac{r_a}{|\underline{r}|^3} \ ,
\end{aligned} \tag{6.93}$$

so that

$$\delta_b p_a = -\epsilon \delta_{ab} |\underline{p}|^2 + \epsilon p_a p_b + \epsilon \frac{G_N M m^2}{|\underline{r}|} \delta_{ab} - \epsilon \frac{G_N M m^2}{|\underline{r}|^3} r_b r_a \ . \tag{6.94}$$

If we consider a general canonical transformation generated by $A = \sum_b A_b U^b$ we have

$$\begin{aligned}
\delta \underline{r} &= 2\epsilon (\underline{U} \cdot \underline{r}) \underline{p} - \epsilon (\underline{p} \cdot \underline{r}) \underline{U} - \epsilon (\underline{U} \cdot \underline{p}) \underline{r} \\
\delta \underline{p} &= \epsilon \underline{U} \left(-|\underline{p}|^2 + \frac{G_M m^2}{|\underline{r}|} \right) + \epsilon (\underline{U} \cdot \underline{p}) \underline{p} - \epsilon \frac{G_N M m^2}{|\underline{r}|^3} (\underline{U} \cdot \underline{r}) \underline{r} \ .
\end{aligned} \tag{6.95}$$

To see that this is indeed a symmetry of H we first note that

$$\delta H = \frac{1}{m} \underline{p} \cdot \delta \underline{p} + \frac{G_N M m}{|\underline{r}|^3} \underline{r} \cdot \delta \underline{r} \ . \tag{6.96}$$

Next we compute

$$\begin{aligned}
\underline{p} \cdot \delta \underline{p} &= \epsilon (\underline{U} \cdot \underline{p}) \left(-|\underline{p}|^2 + \frac{G_N M m^2}{|\underline{r}|} \right) + \epsilon (\underline{U} \cdot \underline{p}) |\underline{p}|^2 - \epsilon \frac{G_N M m^2}{|\underline{r}|^3} (\underline{U} \cdot \underline{r})(\underline{r} \cdot \underline{p}) \\
&= \epsilon \frac{G_N M m^2}{|\underline{r}|} (\underline{U} \cdot \underline{p}) - \epsilon \frac{G_N M m^2}{|\underline{r}|^3} (\underline{U} \cdot \underline{r})(\underline{r} \cdot \underline{p}) \ ,
\end{aligned} \tag{6.97}$$

so

$$\frac{1}{m} \underline{p} \cdot \delta \underline{p} = \epsilon \frac{G_N M m}{|\underline{r}|} (\underline{U} \cdot \underline{p}) - \epsilon \frac{G_N M m}{|\underline{r}|^3} (\underline{U} \cdot \underline{r})(\underline{r} \cdot \underline{p}) \ , \tag{6.98}$$

On the other hand

$$\begin{aligned}
\frac{G_N M m}{|\underline{r}|^3} \underline{r} \cdot \delta \underline{r} &= 2\epsilon \frac{G_N M m}{|\underline{r}|^3} (\underline{U} \cdot \underline{r})(\underline{p} \cdot \underline{r}) - \epsilon \frac{G_N M m}{|\underline{r}|^3} (\underline{U} \cdot \underline{r})(\underline{p} \cdot \underline{r}) \\
&\quad - \epsilon \frac{G_N M m}{|\underline{r}|^3} (\underline{U} \cdot \underline{p})(\underline{r} \cdot \underline{r}) \\
&= \epsilon \frac{G_N M m}{|\underline{r}|^3} (\underline{U} \cdot \underline{r})(\underline{p} \cdot \underline{r}) - \epsilon \frac{G_N M m}{|\underline{r}|} (\underline{U} \cdot \underline{p}) \\
&= -\frac{1}{m} \underline{p} \cdot \delta \underline{p}
\end{aligned} \tag{6.99}$$

Therefore these two cancel so that $\delta H = 0$ as required. You would be hard-pressed to guess this symmetry! Note that it mixes $\underline{p}$ into $\underline{r}$ and vice-versa. Therefore one would not see it as a symmetry of the Lagrangian. But it explains why we find the conserved Runge-Lenz vector.

For completeness we should also show that $\{H, A_b\} = 0$. In fact this follows from the fact that $\delta H = 0$ since

$$\begin{aligned}
\delta H &= \sum_a \frac{\partial H}{\partial r_a}\delta r_a + \sum_a \frac{\partial H}{\partial p_a}\delta p_a \\
&= \epsilon \sum_a \frac{\partial H}{\partial r_a}\{r_a, \underline{U}\cdot\underline{A}\} + \sum_a \frac{\partial H}{\partial p_a}\{p_a, \underline{U}\cdot\underline{A}\} \\
&= \epsilon \sum_a \left(\frac{\partial H}{\partial r_a}\frac{\partial(\underline{U}\cdot\underline{A})}{\partial p_a} - \frac{\partial H}{\partial p_a}\frac{\partial(\underline{U}\cdot\underline{A})}{\partial q_a}\right) \\
&= \epsilon\{H, \underline{U}\cdot\underline{A}\} \\
&= \epsilon\underline{U}\cdot\{H, \underline{A}\}\ .
\end{aligned} \tag{6.100}$$

Since $\delta H = 0$ for arbitrary $\underline{U}$ we see that $\{H, \underline{A}\} = \underline{0}$.

However for those interested we can also do the full computation:

$$\{A_b, H\} = \{r^b \sum_c p_c p_c, H\} - \{p^b \sum_c p_c r_c, H\} - \{\frac{G_N M m^2}{|\underline{r}|} r_b, H\}\ . \tag{6.101}$$

Expanding this out will give six terms, so this is going to be complicated. To break up our calculation into smaller pieces we note that this must be true without imposing any equations of motion. Therefore it must be separately true for the terms that are proportional to various different powers of G_N: G_N^0, G_N^1 and G_N^2. The terms that are independent of G_N arise from taking $H = \frac{1}{2m}|\underline{p}|^2$ as well as dropping the last term:

$$\begin{aligned}
\{A_b, H\}_{G_N^0} &= \frac{1}{2m}\{r^b \sum_c p_c p_c, \sum_d p_d p_d\} - \frac{1}{2m}\{p^b \sum_c p_c r_c, \sum_d p_d p_d\} \\
&= \frac{1}{2m}\sum_{c,d} p_c p_c \{r^b, p_d p_d\} - \frac{1}{2m} p^b \sum_{c,d} p_c\{r_c, p_d p_d\} \\
&= \frac{1}{m}\sum_{c,d} p_c p_c \{r^b, p_d\} p_d - \frac{1}{m} p^b \sum_{c,d} p_c\{r_c, p_d\} p_d \\
&= \frac{1}{m}\sum_{c,d} p_c p_c \delta_{bd} p_d - \frac{1}{m} p^b \sum_{c,d} p_c \delta_{cd} p_d \\
&= \frac{1}{m}\sum_c p_c p_c p_b - \frac{1}{m} p^b \sum_c p_c p_c \\
&= 0\ .
\end{aligned} \tag{6.102}$$

Next we look at the terms that are proportional to G_N (but not G_N^2). In the first two terms these come from taking $H = -G_N M m/|\underline{r}|$ but we must take $H = \frac{1}{2m}|\underline{p}|^2$ in the third term:

$$
\begin{aligned}
\{A_b, H\}_{G_N^1} &= -\{r^b \sum_c p_c p_c, \frac{G_N M m}{|\underline{r}|}\} + \{p^b \sum_c p_c r_c, \frac{G_N M m}{|\underline{r}|}\} - \sum_c \{\frac{G_N M m^2}{|\underline{r}|} r_b, \frac{1}{2m} p_c p_c\} \\
&= G_N M m \left(-r^b \sum_c \{p_c p_c, |\underline{r}|^{-1}\} + \sum_c r_c \{p^b p_c, |\underline{r}|^{-1}\} - \frac{1}{2} \sum_c \{|\underline{r}|^{-1} r_b, p_c p_c\} \right) \\
&= G_N M m \left(-2r^b \sum_c p_c \{p_c, |\underline{r}|^{-1}\} + \sum_c r_c \{p^b p_c, |\underline{r}|^{-1}\} - \sum_c \{|\underline{r}|^{-1} r_b, p_c\} p_c \right) \\
&= G_N M m \left(-2r^b \sum_c p_c \{p_c, |\underline{r}|^{-1}\} + \sum_c r_c p_c \{p^b, |\underline{r}|^{-1}\} + \sum_c r_c p^b \{p_c, |\underline{r}|^{-1}\} \right. \\
&\qquad \left. - \sum_c r_b \{|\underline{r}|^{-1}, p_c\} p_c - |\underline{r}|^{-1} \sum_c \{r_b, p_c\} p_c \right) \qquad (6.103)
\end{aligned}
$$

Next we recall that

$$
\{p_d, |\underline{r}|^{-1}\} = -\frac{\partial |\underline{r}|^{-1}}{\partial r_d} = |\underline{r}|^{-3} r_d \qquad (6.104)
$$

so that

$$
\begin{aligned}
\{A_b, H\}_{G_N^1} &= \frac{G_N M m}{|\underline{r}|^3} \left(-2r^b \sum_c p_c r_c + \sum_c r_c p_c r^b + \sum_c r_c p^b r_c + \sum_c r_b r_c p_c - |\underline{r}|^2 \sum_c \delta_{bc} p_c \right) \\
&= \frac{G_N M m}{|\underline{r}|^3} \left(-2r^b \sum_c p_c r_c + 2 \sum_c r_c p_c r^b + |\underline{r}|^2 p_b - |\underline{r}|^2 p_b \right) \\
&= 0 \qquad (6.105)
\end{aligned}
$$

Finally there is a term that is proportional to G_N^2 that comes from taking $H = -G_N M m / |\underline{r}|$ in the last term:

$$
\begin{aligned}
\{A_b, H\}_{G_N^2} &= \{\frac{G_N M m^2}{|\underline{r}|} r_b, \frac{G_N M m}{|\underline{r}|}\} \\
&= 0 \qquad (6.106)
\end{aligned}
$$

But this is clearly zero as no p_a appears in the Poisson bracket. Thus indeed $\{A_b, H\} = 0$.

It is worth commenting that conservation of angular momentum arises from rotational symmetry which is generated by 3×3 special orthogonal matrices known as $SO(3)$. The conservation of the Runge-Lenz also gives rise to a separate "rotational" symmetry. Here we use quotes as the canonical transformations generated by the Runge-Lenz vector are not simply rotations that preserve the lengths of $\underline{r}$ and $\underline{p}$ however they are still associated with an $SO(3)$ (viewed as a group - if you know what that means). So the Kepler Hamiltonian (or the Hamiltonian for a Hydrogen atom which is structurally the same just with different constants) has an $SO(3) \times SO(3)$ symmetry. It is amusing to note that (ignoring some subtlties) $SO(3) \times SO(3) \cong SO(4)$ so it is as if there were an a hidden extra dimension of space.

6.4 Liouville's Theorem and Poincare Recurrence

Hamiltonian's equations define time evolution as a flow in phase space, known as the Hamiltonian flow. One of the most important features is that since Hamilton's equations

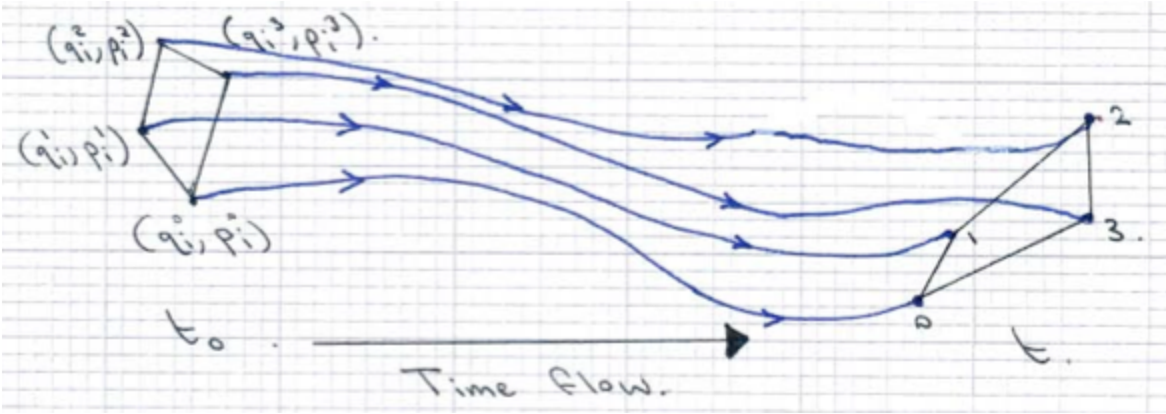

Figure 6.4: Phase flow in time

are first order in time, given an initial point (p^i, q_i) there is a unique flow that passes through that point at time $t = 0$. If we start at a nearby point at $t = 0$ then we will flow along a different path. If two paths ever intersect at a point then they must be the same path everywhere (here we are assuming that there is no explicit time dependence in the Hamiltonian). To see this we stop the motion at the point were the two paths meet. Then restart the time evolution. Since it is first order the subsequent time evolution of each path must be the same. In addition we can run the Hamiltonian flow backwards to deduce that the flows must have been the same in the past.

On the other hand we could instead consider region of phase space, not just a single point. This region will then also evolve smoothly in time since no two points in it will ever intersect. Why might we be interested in this? Well since one can never measure anything with 100% precision one never really quite knows which point in phase space a system is in. By looking at regions of phase space we allow for a certain amount of experimental error. It also touches on the subject of chaos which mathematically means sensitivity to initial conditions. In particular one can ask what happens to points in phase space that start off near each other, will they stay nearby?

Here we encounter a theorem:

Liouville's Theorem: The volume of a region of phase space is constant along a Hamiltonian flow. Here volume means

$$Volume(R) = \int_R dq_1...dq_N dp_1...dp_N = \int_R dV \ , \tag{6.107}$$

where R is the region under consideration. So in general it is a high-dimensional integral in $2N$ dimensions and not simply a 'volume' in as we encounter in three-dimensions that is say measured in litres.

Proof: We wish to show that dV is time-independent. Under time evolution

$$\begin{aligned} q_i(t) \to q_i(t+\epsilon) &= q_i + \epsilon\frac{dq_i}{dt} = q_i + \epsilon\frac{\partial H}{\partial p^i} = q_i' \\ p^i(t) \to p^i(t+\epsilon) &= p^i + \epsilon\frac{dp_i}{dt} = p^i - \epsilon\frac{\partial H}{\partial q_i} = p_i' \ . \end{aligned} \tag{6.108}$$

Thus the volume at time $t + \epsilon$ is

$$\begin{aligned} dV' &= dq_1'...dq_N'dp_1'...dp_N' \\ &= \det(\mathbf{J})dq_1...dq_N dp_1...dp_N \ , \end{aligned} \tag{6.109}$$

where

$$\begin{aligned} \mathbf{J} &= \begin{pmatrix} \frac{\partial q_i'}{\partial q_j} & \frac{\partial q_i'}{\partial p^j} \\ \frac{\partial p_i'}{\partial q_j} & \frac{\partial p_i'}{\partial p^j} \end{pmatrix} \\ &= \begin{pmatrix} \delta^i_j + \epsilon\frac{\partial^2 H}{\partial p^i \partial q_j} & \epsilon\frac{\partial^2 H}{\partial p^i \partial p^j} \\ -\epsilon\frac{\partial^2 H}{\partial q_i \partial q_j} & \delta^i_j - \epsilon\frac{\partial^2 H}{\partial q_i \partial p^j} \end{pmatrix} \ . \end{aligned} \tag{6.110}$$

Let us write this as

$$\mathbf{J} = \mathbf{1} + \epsilon\mathbf{X} \qquad \mathbf{X} = \begin{pmatrix} \frac{\partial^2 H}{\partial p^i \partial q_j} & \frac{\partial^2 H}{\partial p^i \partial p^j} \\ -\frac{\partial^2 H}{\partial q_i \partial q_j} & -\frac{\partial^2 H}{\partial q_i \partial p^j} \end{pmatrix} \ . \tag{6.111}$$

To continue we need to use the relation, valid to first order in ϵ,

$$\det(\mathbf{J}) = \mathbf{1} + \epsilon \mathrm{tr}(\mathbf{X}) \ . \tag{6.112}$$

In fact this is the first order in ϵ form of the relation:

$$\det e^{\mathbf{A}} = e^{\mathrm{tr}\mathbf{A}} \ , \tag{6.113}$$

with $A = \epsilon\mathbf{X}$ One quick way to see this relation, or at least convince oneself it's true, is to note that it is a basis independent. Let us assume that there is a basis where $\mathbf{A}$ is diagonal with eigenvalues $a_1, ..., a_n$ then

$$e^{\mathbf{A}} = \begin{pmatrix} e^{a_1} & 0 & & \\ 0 & e^{a_2} & & \\ & & \ddots & \\ & & & e^{a_n} \end{pmatrix} \tag{6.114}$$

Thus

$$\det e^{\mathbf{A}} = e^{a_1} e^{a_2} \dots e^{a_n} = e^{a_1 + a_2 + \dots + a_n} = e^{\mathrm{tr}\mathbf{A}} \ . \tag{6.115}$$

Let us prove this relation at order ϵ for a generic matrix $\mathbf{A} = \epsilon\mathbf{X}$ by induction. It is clearly true for a 1×1 matrix and let us assume it is true for an $n \times n$ matrix. Consider an $(n+1) \times (n+1)$ matrix then

$$\begin{aligned} \det \mathbf{J} &= J_{11} \det \hat{\mathbf{J}}_{11} - J_{12} \det \hat{\mathbf{J}}_{12} + ... + (-1)^n J_{1n+1} \det \hat{\mathbf{J}}_{1n+1} \\ &= (1 + \epsilon X_{11}) \det \hat{\mathbf{J}}_{11} - \epsilon X_{12} \det \hat{\mathbf{J}}_{12} + ... + (-1)^n \epsilon X_{1n+1} \det \hat{\mathbf{J}}_{1n+1} \ , \end{aligned} \tag{6.116}$$

where $\hat{\mathbf{J}}_{ij}$ is the reduced matrix with the ith row and jth column deleted whereas J_{ij} is the ij entry of $\mathbf{J}$. Next we note that we only need to go to first order in ϵ so that in all but the first term we can replace $\hat{\mathbf{J}}_{ij}$ with $\hat{\mathbf{1}}_{ij}$:

$$\begin{aligned} \det \mathbf{J} &= (1 + \epsilon X_{11}) \det \hat{\mathbf{J}}_{11} - \epsilon X_{12} \det \hat{\mathbf{I}}_{12} + ...(-1)^n \epsilon X_{1n+1} \det \hat{\mathbf{I}}_{1n+1} \\ &= (1 + \epsilon X_{11}) \det \hat{\mathbf{J}}_{11} \ , \end{aligned} \tag{6.117}$$

the second line follows because $\det \hat{\mathbf{I}}_{ij} = 0$ if $i \neq j$ (there will always be a row of zeros somewhere in $\hat{\mathbf{I}}_{ij}$ if $i \neq j$). Thus using the induction hypothesis on $\hat{\mathbf{J}}_{11} = \hat{\mathbf{1}}_{11} + \epsilon\hat{\mathbf{X}}_{11}$ and dropping terms that are higher order in ϵ, we find

$$\begin{aligned}\det \mathbf{J} &= (1+\epsilon X_{11})(1+\epsilon \mathrm{tr}(\hat{\mathbf{X}}_{11}))\\ &= (1+\epsilon X_{11})(1+\epsilon(X_{22}+X_{33}+...+X_{n+1n+1}))\\ &= (1+\epsilon X_{11})+\epsilon(X_{22}+X_{33}+...+X_{n+1n+1})\\ &= 1+\epsilon \mathrm{tr}\mathbf{X} \ . \end{aligned} \tag{6.118}$$

Evaluating our expressions we obtain

$$\begin{aligned}\det \mathbf{J} &= 1+\epsilon\sum_i \frac{\partial^2 H}{\partial q_i \partial p^i} - \epsilon\sum_i \frac{\partial^2 H}{\partial p^i \partial q_i}\\ &= 1 \ . \end{aligned} \tag{6.119}$$

Thus $dV' = dV$ so the volume is preserved by the flow. Note that the Hamiltonian appeared here because it generates time translations but was otherwise arbitrary. Thus we see that more generally any infinitesimal canonical transformation preserves the phase space volume: $dV' = dV$. As an exercise you should convince yourself of this.

Poincare Recurrence Theorem: If the phase space of a system is bounded then given any open neighbourhood B of a point (q_i, p^i) then a system that starts at (q_i, p^i) will return to B in a finite time.

Proof. Let $V(t)$ be the volume of phase space that is swept out by the initial region B in time t. Since the volume is preserved we have

$$\frac{dV}{dt} = C_B \ , \tag{6.120}$$

where $C_B \geq 0$ is constant and hence the volume at time t is of the form

$$V(t) = V(0) + C_B t \ . \tag{6.121}$$

If no point in B ever returns to B then the volume swept out by B will grow linearly in time. However since the total phase space volume is finite, $V(t)$ cannot become arbitrarily large. This is a contradiction which means that at least a finite volume part of B must return to B in finite time.

Note that this refers to late times. Given a finite region B of phase space at $t = 0$ then in a small enough time step $B(t)$ will naturally still intersect $B(0)$ as all the points in B will only have shifted a small amount and hence some must still remain in B. However after a finite time, call it t_1 $B(t_1)$ will generically[2] no longer intersect $B(0)$. The important part of this claim is that there must be a later time $t_{recurrence} >> t_1$ where $B(t_{recurrence})$ again intersects $B(0)$.

Next we have to ask about any remaining part of B that does not return, call it B'. If B' has finite volume then we repeat the argument above applied to B' and reach another contradiction: at least a finite volume part of B' must return to B', otherwise

[2]Generically because one could chose B so large that it always self-intersects and then the theorem is trivially true.

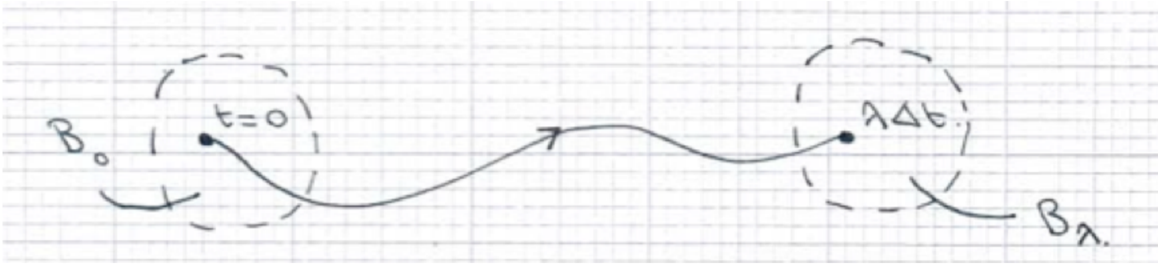

Figure 6.5: More flows

it would go on to sweep out an infinite volume of phase space. Thus there is at most a zero-volume set in B that does not return to B eventually.

Put another way given even the smallest region of phase space, so long as it has non-zero volume, it will sweep out an arbitrarily large region of phase space unless it eventually intersects itself again. Since there isn't an infinite amount of phase space to cover, it follows that it must intersect itself again. Thus a system either repeats its motion in phase space or evolves to fill up a subregion of the phase space, or all of it. In the last case the system is said to be **ergodic**.

Note that it is sufficient that the subsets of phase space corresponding to constant H are bounded as the system is constrained (assuming no explicit time dependence) to evolve such that H is fixed.

We can illustrate this with the simple pendulum. In the case of a single pendulum we saw that the motion was periodic (at least for small oscillations) with frequency $\omega = \sqrt{g/l}$. For example one solution is

$$\theta = A\sin\omega t \qquad\qquad p_\theta = mA\omega\cos\omega t \tag{6.122}$$

which has $\theta = 0$ and $p_\theta = Aml^2\omega$ at $t = 0$. Clearly every single point in phase space returns to itself after a time $2\pi/\omega$. Note that in phase space we also require that the momenta return to themselves which means $t = 2\pi n/\omega$ and not just $t = \pi n/\omega$.

For the double pendulum we saw that a single point never returns to itself as the ratio of the periods of the normal modes was irrational. For example one solution is

$$\begin{aligned}\begin{pmatrix}\theta\\ \phi\end{pmatrix} &= \begin{pmatrix}A\cos\omega_1 t + A\cos\omega_2 t\\ \sqrt{2}A\cos\omega_1 t - \sqrt{2}A\cos\omega_2 t\end{pmatrix}\\ \begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} &= \begin{pmatrix}-ml^2A\omega_1\sin\omega_1 t - ml^2A\omega_2\sin\omega_2 t\\ \sqrt{2}ml^2A\omega_1\sin\omega_1 t + \sqrt{2}ml^2A\omega_2\sin\omega_2 t\end{pmatrix}\end{aligned} \tag{6.123}$$

At $t = 0$ we have

$$\begin{pmatrix}\theta\\ \phi\end{pmatrix} = \begin{pmatrix}2A\\ 0\end{pmatrix} \qquad \begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \tag{6.124}$$

But this never happens again since this requires $\cos\omega_1 t = \cos\omega_2 t = 1$. However this means that

$$t = 2\pi n_1/\omega_1 = 2\pi n_2/\omega_2 \tag{6.125}$$

for integers n_1, n_2 which implies that $\omega_1/\omega_2 = n_1/n_2$ is rational, but it isn't. However given any rational approximation N_1/N_2 to ω_1/ω_2 we can take $t = 2\pi N_1/\omega_1$ so that $\cos\omega_1 t = 1$ and

$$\cos\omega_2 t = \cos\left(2\pi\frac{\omega_2}{\omega_1}\frac{N_1}{N_2}N_2\right) . \tag{6.126}$$

By taking better and better rational approximations to ω_1/ω_2 the argument can be made arbitrarily close to $2\pi N_2$. Hence $\cos\omega_2 t$ will be arbitrarily close to, but always less than, 1. Thus $\theta \leq 2A$ with equality only at $t = 0$ but it can get as close to $2A$ as one wants by waiting a sufficiently long enough time. Thus given any small region around the initial starting point the system will eventually come back to it, even if it never exactly comes back to where it started. Clearly the smaller we make the region, the better rational approximation to ω_1/ω_2 we must use and hence the value of N_1, N_2 must be large, corresponding to a long recurrence time.

This is an amazing theorem and the cause for much pub-chatter[3]. Why? One example is the long-run stability of the solar system. Let us consider a system of (say) 8 planets in orbit around the sun. Planets can't escape to infinity because they have negative energy. And they can't go faster that the speed of light. So they effectively live in a bounded phase space. Therefore the solar system is either completely stable and periodic or it will eventually explore all available phase space. It seems unlikely that the planets orbits are so "fine-tuned" that the motion will be periodic (in particular why should the orbital periods of the planets be rational multiples of an earth year). So we are led to suspect that the solar system will eventually fill up all of its phase space, which could include having at least one planet go very far away (or perhaps move very very fast). Thus one is led to expect that the solar system is, ultimately, unstable (meaning that it will eventually look very different to how it looks now).

Another example is to consider a finite sized room that is empty of air. Suppose you then place some air into a corner of the room in such a way that each air molecules have vanishing or small initial velocity. The air will subsequently naturally disperse and fill the whole room. In this case the phase space is bounded (for constant energy) because the room is bounded and the momentum of a given air molecule cannot be so large that its kinetic energy is greater that than the total energy. So the Poincare Recurrence Theorem applies. That means that in a finite time all the air will be back in the corner of the room with small velocities and the people who subsequently entered will spontaneously suffocate.

This seems to contract the notion that entropy, the amount of disorder, never decreases since the initial and final configurations are highly ordered whereas a generic room of air is a highly disordered state. So whats wrong? The cheap answer is that a finite time can still be an incredibly long time. In this case the time scale for the recurrence is longer than the age of the Universe.

But this answer is cheap because a theorem is a theorem and it can't depend on how long is a long time. Which theorem is wrong? This is much debated[4]. Arguably the theorem of the increase in entropy is more suspect. Its proof is far less simple and

[3]If you are lucky(...) enough to be in the pub with a physicist.

[4]Typically also in the aforementioned pub.

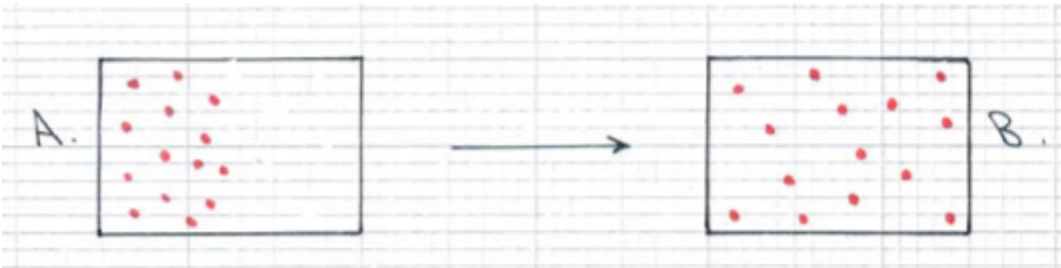

Figure 6.6: Motion of gas in a box when the volume of the box is instananeously doubled.

any known proof of it includes an assumption such as “repeated interactions between particles can be treated as independent”. Whereas in essence the Poincare recurrence theorem shows that they are not truly independent.

Acknowledgments

This veracity of these notes owes a debt to the sharp eyes of students who have pointed out various typos. First in our list to thank are Joshua Haholongan, Guoqinxin Huang, Audrey Herman, Trisha Sadhwani and Zhiping (Frank) She.