CALCULUS 1 TUTORIAL EXERCISES IV

In this tutorial you will work with the limit, continuous functions, the intermediate value theorem, limits involving infinity, and the algebra of limits.

Tutor's Example[1]

Find the value of k such that

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x < 2, \\ (x+1)^2 + k, & x \geq 2 \end{cases}$$

is a continuous function.

27. For each function below, identify the value of k that makes the function continuous:
    (a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} \dfrac{x^2+2x-15}{x-3}, & x < 3, \\ x^2 - kx + 1, & x \geq 3. \end{cases}$
    (b) $f : [-3,3] \to \mathbb{R}$ given by $f(x) = \begin{cases} \dfrac{3x-6}{k(x^3+2x^2-11x+6)}, & x < 2, \\ \dfrac{7x}{12} - 1, & x \geq 2. \end{cases}$
    (c) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} 2e^{3x}, & x \leq 0 \\ \dfrac{\sin(kx)}{x}, & x > 0. \end{cases}$
28. If they exist, evaluate the following asymptotic limits:
    (a) $\displaystyle\lim_{x\to\infty} \left[\frac{27x^{26}+24x^{17}-5}{9x^{26}-27x^{17}-5}\right]$,
    (b) $\displaystyle\lim_{x\to-\infty} \left[-x\sin\left(\frac{1}{5x}\right)\right]$,
    (c) $\displaystyle\lim_{y\to\infty} \left[e^{-2yx}\tanh(4y)\right]$ for $x > 0$.
29. Let $n \in \mathbb{N}$, and calculate the limit $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

    [Hint: substitute $n^{-1} = x$ and try to use limits calculated in the lectures.]

[1] To be shown by the tutor at the start of the tutorial.

30. Calculate the following limits, if they exist, without using power series:

(a) $\displaystyle\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$

(b) $\displaystyle\lim_{x\to 0} \frac{\sin(7x)}{\tan(2x)}$

(c) $\displaystyle\lim_{x\to 0} \frac{\sin(x^2)}{x}$

(d) $\displaystyle\lim_{x\to 0} \cosh(x)$

31. Calculate the following limits, if they exist, using the power series representations of the hyperbolic functions:

(a) $\displaystyle\lim_{x\to 0} \frac{\sinh(x)}{x}$

(b) $\displaystyle\lim_{x\to 0} \frac{1-\cosh(x)}{x}$

(c) $\displaystyle\lim_{x\to 0} \frac{1-\cosh(x)}{x^2}$

(d) $\displaystyle\lim_{x\to 0} \frac{\tanh(x)}{x}$

32. Calculate the following limits by making clever substitutions:

(a) $\displaystyle\lim_{x\to \pi} \frac{\sin(x)}{\pi - x}$

(b) $\displaystyle\lim_{x\to 0} \frac{\arcsin(x)}{x}$

(c) $\displaystyle\lim_{x\to 0} \frac{\arcsin(3x)}{\tan(5x)}$

(d) $\displaystyle\lim_{x\to 0} \frac{\operatorname{arctanh}(x)}{x}$

33. This question requires you to think about the definition of a limit in terms of ε and δ.

(a) Let $f(x) = 2x - 1$, which has the following limits

$$\lim_{x\to 0} f(x) = -1 \quad \text{and} \quad \lim_{x\to 5} f(x) = 9.$$

i. For $\varepsilon = 1$, find a $\delta > 0$ such that if $|x| \in (0, \delta)$ then $|f(x) - (-1)| < \varepsilon$. Find a $\delta > 0$ which works for $\varepsilon = 1/10$.

ii. For $\varepsilon = 1$, find a $\delta > 0$ such that if $|x - 5| \in (0, \delta)$ then $|f(x) - 9| < \varepsilon$. Find a $\delta > 0$ which works for $\varepsilon = 1/10$.

iii. What do you notice about the last two parts?

(b) Let $g(x) = x^2 + 3$, which has the following limits

$$\lim_{x\to 0} g(x) = 3 \quad \text{and} \quad \lim_{x\to -2} g(x) = 7.$$

i. For $\varepsilon = 1$, find a $\delta > 0$ such that if $|x| \in (0, \delta)$ then $|g(x) - 3| < \varepsilon$. Find a $\delta > 0$ which works for $\varepsilon = 1/2$.

ii. For $\varepsilon = 1$, find a $\delta > 0$ such that if $|x - (-2)| \in (0, \delta)$ then $|g(x) - 7| < \varepsilon$. Find a $\delta > 0$ which works for $\varepsilon = 1/2$.

iii. What is the difference between $f(x)$ and $g(x)$ which means the pattern in part (a) doesn't hold here?

34. Let $h(x)$ be the Heaviside step function from 1(c) on the first exercise sheet, that is

$$h(x) = \begin{cases} 0 & x < 0 \\ 1/2 & x = 0 \\ 1 & x > 0. \end{cases}$$

(i) What is the *negation* of the definition of a limit:

for all $\varepsilon > 0$*, there exists some* $\delta > 0$ *for which* $|x - x_0| \in (0, \delta)$ *implies* $|h(x) - L| < \varepsilon$.

(ii) For the Heaviside function, the limit as $x \to 0$ does not exists. Give an example of an $\varepsilon > 0$ such that no $\delta > 0$ exists for the definition of the limit to work for any value of L. Explain your answer.

35. **Exam-style question**[2] In this question, we will explore the set of rational numbers $\mathbb{Q}$ as a subset of the reals $\mathbb{R}$ from the point of view of calculus.

(i) Let $p < q \in \mathbb{Q}$ be rational numbers and $r < s \in \mathbb{R}\backslash\mathbb{Q}$ be irrational numbers.

(a) Show that there is an irrational number $a \in \mathbb{R}\backslash\mathbb{Q}$ such that $p < a < q$.
[Hint: What happens when you add a small irrational number to p?]

(b) Show that there is a rational number $b \in \mathbb{Q}$ such that $r < b < s$.
[Hint: Consider the decimal expansions of r and s, how can you change them to get a rational number?]

(ii) Using the first part, argue that the function[3] $d\colon \mathbb{R} \to \mathbb{R}$ given by

$$d(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

is *not* continuous at any point in its domain.

(iii) Show, using the ε-δ definition of a limit, that the function $xd(x)$ *is* continuous at $x = 0$.

36. **Advanced Python**[4] The Python package `sympy` includes the function `limit`. Use this to verify your answers to questions 28, 30, 31, and 32. What happens when you try to use it

[2] Some questions on the final exam will be longer with several parts which delve into a topic and build on each other. To help you prepare, each sheet will include such a question.

[3] This function is sometimes called the *Dirichlet function.*

[4] Each sheet will contain a Python problem which should complement the programming aspect of the course. In some cases, like this one, you may have to look up unfamiliar functions and use your knowledge of Python to apply them.

to compute the example from lectures:

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right).$$

37. **A challenging problem**[5][6] Consider a fixed circle C_1 defined by the equation

$$(x-1)^2 + y^2 = 1$$

and a shrinking circle C_2 with radius r and centre the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?

[5]All challenging problems in the tutorial problem sets are beyond the scope of the course, and are not examinable, but they are, hopefully interesting and inspiring.

[6]This problem is perhaps not so challenging, but the result is interesting - it comes from the blog `https://mrchasemath.wordpress.com/2010/02/23/really-fun-limit-problem/`, where you can find access to a Java applet which will help you understand the solution.