Linear Algebra and Geometry I 2023 — Problem Sheet 1

You are encouraged to work with other students on the module. If you are having difficulty with any of the questions or want feedback on specific answers you should ask your tutor in your next tutorial or attend the lecturer's office hours.

1. (a) Evaluate the following sums.

(i) $\displaystyle\sum_{k=5}^{9} 4(k-5)$ (ii) $\displaystyle\sum_{j=0}^{3}\sum_{k=0}^{5}(j-k)$ (iii) $\displaystyle\sum_{k=1}^{n}\sum_{j=1}^{n} 2j$

(b) Fill in the blanks so that the following sums are all equal.

$$\sum_{j=-1}^{12}\sqrt{2j+7} = \sum_{j=\square}^{18}\sqrt{2j+\square} = \sum_{j=\square}^{\square}\sqrt{2j+3}$$

(c) Show that

$$\frac{1}{2}\sum_{k=0}^{n}\sum_{j=0}^{n}(j+k)^2 = (n+1)\sum_{k=0}^{n}k^2 + \left(\sum_{k=0}^{n}k\right)^2.$$

2. (a) Express each of the following numbers in the form $a+ib$ for some $a,b\in\mathbb{R}$, then compute its modulus. Do *not* use decimal approximations (so 0.2 or 1/5 are both okay, but while 1/3 is fine, 0.33 is not).

(i) $(-3+4i)+(6+7i)$

(ii) $(2+5i)(3-2i)$

(iii) $\dfrac{1}{3i-2}$

(iv) $(i+\sqrt{5})^2$

(v) $(1+i)(1+2i)(1+3i)$

(vi) $\left(\dfrac{1+i}{1-i}\right)^2$

(b) Let $z=(1+i)/\sqrt{2}$. Determine the following in the form $a+ib$. (*Do you spot a pattern that would allow you to solve the last part?*)

(i) z^2 (iii) z^4 (v) z^8

(ii) z^3 (iv) z^6 (vi) z^{100}

3. Decide whether each of the following statements are true or false. In each case either explain why it is true or give a counterexample to show that it is false.

   (a) If $z+w$ is real then z and w must be real.

   (b) If zw is real then z and w must be real.

   (c) If $\overline{z}$ is real then z must be real.

   (d) If $z-\overline{z}$ is real then z must be real.

4. Determine the solutions $z \in \mathbb{C}$ of each of the following equations and sketch each set of solutions on the complex plane.

   (a) $|z+1| = 2$ (b) $z = \overline{z} + 2i$ (c) $z^2 = (\overline{z})^2$

5. Let $z, w \in \mathbb{C}$ with $w \neq 0$. Prove the following statements.

   (a) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$ (b) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$