Introduction to Abstract Algebra: Sheet 6

For discussion in week 6 skills sessions and week 7 tutorials

You should write up your solutions for the participation mark problems and submit them using gradescope on the KEATS page for the course. The problems marked with (SS) will be discussed in the Skills Session for Week 6. Please attempt these *before* your Skills Session. Some of the other problems will be discussed in your tutorial for Week 7. Please attempt all core problems *before* your tutorial.

Participation mark problems

Exercise 1. Let G be a group with identity element e, and let $g \in G$. Suppose $g^2 = g$. Prove that $g = e$.

Exercise 2. Let G be an abelian group, let $g, h \in G$, and let $n \in \mathbb{N}$. Use induction to show that $(gh)^n = g^n h^n$.

Exercise 3. Find the order of the following elements of S_5:

(i) $(1\ 2)(1\ 3\ 4)$

(ii) $(1\ 2)(3\ 4)$

(iii) $(1\ 2)(3\ 4\ 5)$

Core problems

Exercise 4 (SS)**.** Let G be a group and let $g, h, k \in G$. Show that if $gh = gk$, then $h = k$.

Exercise 5 (SS)**.** Let G_1 and G_2 be groups. Assume G_1 is nonabelian. Show that $G_1 \times G_2$ is nonabelian.

Exercise 6 (SS)**.** Let G be a group, let $g \in G$, and let $n \in \mathbb{N}$. Show that $(g^n)^{-1} = (g^{-1})^n$.

Exercise 7. Let G be a group, and let $g \in G$. Prove that $(g^m)^n = g^{mn}$ for all $m, n \in \mathbb{Z}$ as follows:

(i) First prove the formula holds for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. (Use induction on n and the fact proved in lecture that $g^a g^b = g^{a+b}$ for all $a, b \in \mathbb{Z}$.)

(ii) Deduce that the formula holds for all $m, n \in \mathbb{Z}$.

Exercise 8. Find the order of every element of each of the following groups:

(i) $\mathbb{Z}_8$;

(ii) $\mathbb{Z}_{15}^\times$;

(iii) D_4.

Additional practice

Exercise 9. Suppose that G is a group with identity element e, and that g and h are elements of G. Prove that there is a unique element $x \in G$ such that $g(xh) = e$.

Exercise 10. Suppose that G is a group such that $g^2 = e$ (the identity element) for all $g \in G$. Prove that G is abelian.

Exercise 11. Let G be a group, and let $g \in G$. Show that the function $f : G \to G$ defined by $f(x) = gx$ for all $x \in G$ is bijective.

Exercise 12. Let G be a group, let $g, h \in G$, and let $n \in \mathbb{Z}$.

(i) Show that if G is abelian, then $(gh)^n = g^n h^n$.

(ii) Give an example to show that the statement in (i) is not necessarily true if G is not abelian.

Exercise 13. Let G be a group, and let $g \in G$. Prove that g and g^{-1} have the same order.

Exercise 14. Let G_1 and G_2 be groups, let $g_1 \in G_1$, and let $g_2 \in G_2$. Assume $\mathrm{ord}(g_1) = n_1$ and $\mathrm{ord}(g_2) = n_2$. Find a formula for the order of $(g_1, g_2) \in G_1 \times G_2$.