Stationary measures and orbit closures for the action of $SL_2(\mathbb{Z})$ on the torus following Y.Benoist & J-F.Quint.

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Abstract

We reproduce the proof of the classification of stationary measures that appears in the seminal paper of Benoist and Quint «Mesures stationnaires et fermes invariants des espaces homogenes » in the simplest case of linear actions on the torus.

Contents

1 Introduction

Let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus. It is endowed with an action of $G = SL_2(\mathbb{Z})$, the famous group of integer matrices.

Theorem 1.1. *Let μ be a compactly supported probability measure on G whose support span a Zariski dense sub-semigroup. Then any atom-free μ-stationary measure on $\mathbb{T}$ is the Lebesgue measure.*

This result is due to Benoist and Quint and is a particular case of a much more genreal result. The goal of those notes is to present the proof as it is conducted in [?], though we may slighty change the presentation from time to time. There is no conceptual shorcut in the proof of this simpler case but we feel some concepts or details became less intimidating or easier to expose in that context. We also added a few details or comments in some proofs that we thought were relevant. For the sake of completeness, we added in subsection 6.2 the proof of the classification of closed sets that is a spectacular application of the classification of stationary measures given by Benoist and Quint.

2 Looking into the past: the backward dynamical system and its renormalisation

2.1 Space of past trajectories

We denote by $\mathcal{G}$ the Borel σ-algebra on G. Let B be the product space $G^{\mathbb{N}}$, let $\mathcal{G}^{\otimes\mathbb{N}}$ be the product algebra, let β be the product measure $\mu^{\otimes\mathbb{N}}$, let $\mathcal{B}$ be the β-completion of $\mathcal{G}^{\otimes\mathbb{N}}$ and let $\mathcal{B}_n$ be the σ-algebra generated by the projection the on the n first coordinates of B. Finally, let $\mathcal{X}$ be the Borel σ-algebra of X. The set $B \times X$ will be referred to as the set of past the past trajectories and will be denoted by B^X. We think of elements of B^X as possible trajectories of the random walk if it were happening since the origin of time up to the present, the projection on the X factor corresponding to where the random walked arrived at the present. An element $(b, x) \in B^X$ corresponds to a trajectory that arrived at x, and that was at $b_1^{-1}x$ one step ago, at $b_2^{-1}b_1^{-1}x$ two steps ago... etc. This heurisitic explains the choice of terminology. There is a natural measure on the space of past trajectories defined as follows. Let $(\nu_b)_{b\in B}$ be the Furstenberg measures associated to the random walk on X induced by the measures μ and ν (see Appendix) and define:

$$\beta^X = \int_B \delta_b \otimes \nu_b \; d\beta(b)$$

We denote by π_B and π_X the projection on the two factors of B^X, of which think as random variables on the probability space (B^X, β^X). The following proposition characterizes the measure β^X.

Proposition 2.1. *The measure β^X is the only measure that satisfies the following two properties:*

1. $(\pi_B)_*\beta^X = \beta$
2. *The conditionnal law $\beta^X(\pi_X \in A \mid \pi_B = b)$ is given by $\nu_b(A)$*

In particular, if f is $\mathcal{B}_n \otimes \mathcal{X}$-measurable:

$$\int_{B^X} f(b,x)\ d\beta^X(b,x) = \int_B \int_X f(b, b_1 \cdots b_n \cdot y)\ d\nu(y) d\beta(b) \tag{1}$$

Proposition 2.1 provides an interpretation of the measure β^X in the language of probability theory. If $E \in \mathcal{B}\otimes\mathcal{X}$ is a measurable set (or event), we think of the quantity $\beta^X(E)$ as the probability the event E is realized as a trajectory of the random walk from the origin of time to the present. The first item of proposition 2.1 simply translates the fact that, by definition of the random walk, any step is chosen independently with law μ. To understand the second item, remember that the starting of the random walk is chosen, by definition, with law ν and thus the probability to end up in A along b is $(b_1 \cdots b_\infty)_*\nu$. Of course, this equality can only be made rigorous with a limit and we should write $\underset{n\to\infty}{lim}(b_1 \cdots b_n)_*\nu$, which is precisely the definition of ν_b.

Remark. The second item of proposition 2.1 can be rephrased in: the disintegration of β^X on the fiber of b is $\delta_B \otimes \nu_b$. See the appendix for more details on disintegration of measures.

2.2 Backward dynamical system

The space of past trajectories is associated with a natural dynamical system that we define now. We denote by $T : B \to B$ the one sided shift that deletes the first letter and we define:

$$T^X : \begin{array}{ccc} B^X & \to & B^X \\ (b,x) & \mapsto & (T(b), b_1^{-1} \cdot x) \end{array}$$

The map T^X corresponds to looking one step into the past. It is natural in the following sense:

Proposition 2.2. *The measure β^X is preserved by the map T^X.*

We can now introduce one of the principal ingredient of the the method developped by Benoist and Quint.

Definition 2.1. *The quadruplet $(B^X, \mathcal{B}\otimes\mathcal{X}, \beta^X, T^X)$ is called the backward dynamical system.*

2.3 Measure of the stable leaves

Definition 2.2. *The support of the measure μ is the smallest closed subgroup Γ_μ such that $\mu(\Gamma_\mu)$ has measure 1.*

Lemma 2.1. *The action of Γ_μ on $\mathbb{R}^2$ satifies the following two properties:*

- *It is stongly irreducible: no finite index subgroup of Γ_μ preserves a line in $\mathbb{R}^2$.*
- *It is proximal: there is a sequence $\gamma_n \in \Gamma_\mu$ and a rank 1 endomorphism π such that:*

$$\frac{\gamma_n}{\|\gamma_n\|} \underset{n\to\infty}{\to} \pi$$

Proof. We start with item 1. Saying that a finite index subgroup of Γ_μ preserves a line amounts to saying that Γ_μ preserves a finite union of lines which is a contradiction with the Zariski density assumption.

For the second item, recall that Γ_μ is discrete as a subgroup of $SL_2(\mathbb{Z})$. This implies that it is unbounded as otherwise it would be finite and could not act strongly irreducibily. Let γ_n be a sequence in Γ_μ such that $\|\gamma_n\| \to \infty$. The sequence of matrices $\|\gamma_n\|^{-1} \cdot \gamma_n$ is bounded and thus, up to extracting, converges to some matrix π. By continuity, the determinent of π is the limit of the determinant of $\|\gamma_n\|^{-1} \cdot \gamma_n$ which is $\|\gamma_n\|^{-1}$ as Γ_μ is contained in $SL_2(\mathbb{R})$. Consequently, π is not invertible and its norm equals 1 as a limit of matrices with norm 1: it is a rank 1 endormorphism. □

Lemma 2.1 has the following important consequence, which results from ? in the appendix.

Proposition 2.3. *For β-almost every $b \in B$, there is a line V_b such that image of any limit point of $\|b_1 \cdots b_n\|^{-1} \cdot b_1 \cdots b_n$ is given by this line V_b. In particular, for β-almost every $b \in B$, the following holds:*

$$V_b = b_1 \cdot V_{T(b)}$$

Proof. This follows from lemma 2.1 and proposition A.2. □

The remainder of this subsection is dedicated to the proof of the following proposition. It (or more precisely its consequence stated in proposition 2.12 below) is a key step in the approach of Benoist and Quint. This is what they need to initiate their exponential drift that is presented in proposition 5.1

Proposition 2.4. *If ν is a non atomic measure, then for β^X-almost every $(b, x) \in B^X$, we have:*

$$\nu_b(x + V_b) = 0$$

If we denote by $\rho_n(b)$ the product $b_n^{-1} \cdots b_1^{-1}$, notice that by definition of the V_b, $y \in x + V_b$ if and only if $d(\rho_n(b) \cdot x, \rho_n(b) \cdot y)$ converges to zero as n gets large. It will be convenient to work with more general compact spaces. If (Y, d) is a compact metric space endowed with a continuous action of G, we define the unstable leaf of (b, x) by:

$$W_b(x) = \{y \in Y \mid \lim_{n\to\infty} d(\rho_n(b) \cdot x, \rho_n(b) \cdot y) = 0\}$$

On such a set Y, we denote

$$A_\mu(v) = (x, y) \mapsto \int_G v(g \cdot x, g \cdot y) d\mu(g)$$

We shall say that μ has the contraction property on Y if there is an unbounded positive function $v : Y \times Y - \Delta_Y \to [0, \infty[$ such that :

1. For any compact K of X, the restriction of v to $K \times K - \Delta_K$ is proper
2. There are two constants $0 < a < 1$ and $C > 0$ such that $A_\mu(v) \leq av + C$

The meaning of the contraction property is that the diagonal is repulsive for the random walk on the product Y^2 at a rate given by a. Indeed, by the properness assumption, v gets large only close to the diagonal. When that happens, the constant C becomes irrelevant and then item 2 says that the expected value of v after one step of the random walk has to uniformly decrease according to the factor a. If ν is a stationary measure on Y that is concentrated on a single point y_0, then almost all the ν_b also are Dirac masses on y_0. The following proposition establishes the converse when the contraction property holds.

Proposition 2.5. *Let (Y, d) be a compact metric space endowed with a continous action of G and assume that μ has the contraction property on Y. If ν is a stationary measure on Y such that for β-almost every b in B, the measure ν_b is a Dirac mass, then the measure ν is a Dirac mass itself.*

Proof. We will consider the product random walk on X^2. We will show that the contraction property forces typical random trajectories that start off the diagonal to stay away from it while the fact that the ν_b are Dirac masses forces the random trajectories to converge to the diagonal. Those two behavior are not compatible unless all the mass of $\nu \otimes \nu$ is concentrated at a single point of the diagonal, that is ν is a dirac mass. More precisely, we define $S_p(x, y)$ to be the time average $\frac{1}{p} \sum_{k=1}^{p} A_\mu^k(v)(x, y)$. We shall first prove that this function is bounded above whenever $x \neq y$ but that if ν is not a dirac mass, there are two points $x \neq y$ such that $S_p(x, y)$ is unbounded.

We start by finding elements x, y for which the time average $S_p(x, y)$ is unbounded. Define $\kappa : B \to X$ to be the map such that ν_b is the Dirac mass at $\kappa(b)$. Notice that by equation (6) in the appendix, we have the follwing:

$$\kappa(gb) = g \cdot \kappa(b) \tag{2}$$

and

$$\nu = \kappa_* \beta \tag{3}$$

By Lusin's theorem, we chose a compact K_0 in B of mass $\beta(K_0) > 1-\varepsilon$ on restriction to which the function κ is absolutely continous. Restricting to such a compact allows to rule out the trajectories along which convergence doesnt hold, such as trajectories of the form $(g, g^{-1}, g, g^{-1}, \cdots)$. By assumption, the function v is proper on $K \times K - \Delta_K$ where $K = \kappa(K_0)$. Using the fact that v is also unbounded, we show that for any $M > 0$ there is a η such that if $d(x, y) \leq \eta$ then $v(x, y) \geq M$. In particular, using the uniform continuity of κ on restriction to K_0, there is a range n_M such that for any $n \geq n_M$, any $b, b' \in B$ and $g_1, \cdots, g_n \in G$ such that $g_1 \cdots g_n b$ and $g_1 \cdots g_n b$ are in K, one has $d(\kappa(g_1 \cdots g_n b), \kappa(g_1 \cdots g_n b')) \leq \eta$ (as words sharing the same first letters are close) and thus:

$$v(\kappa(g_1 \cdots g_n b), \kappa(g_1 \cdots g_n b')) \geq M$$

To use this, we shall exhibit for β-almost every $b \in B$ many $g_1, \cdots, g_n$ such that $g_1 \cdots g_n b$ is in K_0. We define the following operator on $L^1(B, \mu)$:

$$L_\mu(\varphi)(b) = \int_G \varphi(gb) d\mu(g)$$

The Chacon-Ornstein ergodic theorem applied to L_μ and 1_{K_0} gives that there is a null set N such that for $b \in B - N$:

$$\frac{1}{p} \sum_{n=1}^{p} \mu^{\otimes n}\{(g_1, \cdots, g_n) \in G^n \mid g_1 \cdots g_n b \in K_0\} \underset{p \to \infty}{\to} \beta(K_0) \geq 1 - \varepsilon$$

Up to increasing N, according to formula (2), we have that for any b in $B - N$, and for $\mu^{\otimes n}$-almost every $(g_1, \cdots, g_n) \in G^n$:

$$\kappa(g_1 \cdots g_n b) = (g_1 \cdots g_n) \cdot \kappa(b)$$

If ν is not a Dirac mass, or equivalently $\nu \otimes \nu(\Delta_Y) < 1$, then, according to equation (3) there are two points b and b' not in N such that $\kappa(b) \neq \kappa(b')$. Consequently, there is a $p_0 \geq n_M$ such that this sum is bigger than $1 - 2\epsilon$ and thus:

$$\begin{aligned}
S_p(\kappa(b), \kappa(b')) &= \frac{1}{p} \sum_{n=1}^{p} \int_{G^n} v(g_1 \cdots g_n \cdot \kappa(b), g_1 \cdots g_n \kappa(b')) d\mu^{\otimes n}(g_1, \cdots, g_n) \\
&= \frac{1}{p} \sum_{n=1}^{p} \int_{G^n} v(\kappa(g_1 \cdots g_n \cdot b), \kappa(g_1 \cdots g_n \cdot b')) d\mu^{\otimes n}(g_1, \cdots, g_n) \\
&\geq \frac{1}{p} \sum_{n=p_0}^{p} M \mu^{\otimes n}\{(g_1, \cdots, g_n) \in G^n \mid g_1 \cdots g_n b \in K_0 \ \&\ g_1 \cdots g_n b' \in K_0\} \\
&\geq (1 - 4\epsilon - \frac{p_0}{p}) M
\end{aligned}$$

On the other hand, the contraction property gives:

$$A_\mu^n(v) \leq a^n v + (1 + \cdots + a^{n-1})$$

and thus

$$S_p(v)(\kappa(b),\kappa(b')) \leq \frac{1}{p(1-a)} v(\kappa(b),\kappa(b'))) + \frac{1}{1-a} C$$

□

So eventually, taking the limit when p goes to infinity:

$$(1-4\epsilon)M \leq \frac{C}{1-a}$$

As M was arbitrary, that is a contradiction whenever $\epsilon < \frac{1}{4}$.

In order to show that μ has the contraction property on X, we need the following lemma that appears in [?] (proposition 4.2):

Lemma 2.2. *There are $\delta > 0$, $n \geq 1$ and $a < 1$ such that:*

$$\int_G \|g \cdot v\|^{-\delta} d\mu^n(g) \leq a\ \|v\|^{-\delta}$$

Proposition 2.6. *The measure μ has the contraction property on the space* $(\mathbb{T}^2, d_{eucl})$

Proof. Let δ, n and a be as in lemma 2.2, Observe first that if μ^{*n} has the contraction property on X, so has μ. Indeed, let v_0 be as in the contraction property for μ^{*n}. Define:

$$v = \sum_{k=0}^{n-1} a^{-k} A_\mu^k(v_0)$$

It is easy to see that the function v satisfies the contraction property for μ with the same constants a and C. Consequently, up to replacing μ by μ^n, we can assume that $n = 1$ in lemma 2.2. Now, define:

$$v : \begin{array}{ccc} X^2 - \Delta_X & \to & \mathbb{R} \\ (x,y) & \mapsto & d(x,y)^{-\delta} \end{array}$$

It is clear that this function is proper on restriction to product of compacts and unbounded. It remains to show the contraction property. Since the μ is compactly supported, define:

$$R = \sup_{g \in \Gamma_\mu}\ max(\{\|g\|, \|g^{-1}\|\})$$

Let $(x,y) \in X \times X - \Delta_X$. We single out two cases:

1. If $d(x,y) \leq \frac{1}{2R}$, then there is a unique vector of norm smaller than $\frac{1}{2R}$ such that $y = x + u$. Consequently, for any $g \in \Gamma_\mu$, we have $g \cdot y = g \cdot x + g \cdot u$ with $\|g \cdot u\| \leq \frac{1}{2}$ and then $d(g \cdot x, g \cdot y) = \|g \cdot u\|$. Thus, using lemma 2.2, we have:

$$
\begin{aligned}
A_\mu(v)(x,y) &= \int_G d(g\cdot x, g\cdot y)^{-\delta} d\mu(g) \\
&= \int_G \|g\cdot u\|^{-\delta} d\mu(g) \\
&\leq a\|u\|^{-\delta} \\
&\leq av(x,y)
\end{aligned}
$$

2. Now, if $d(x,y) \geq \frac{1}{2R}$, then $d(g\cdot x, g\cdot y) \geq \frac{1}{2R^2}$ and thus:

$$
\begin{aligned}
A_\mu(v)(x,y) &= \int_G d(g\cdot x, g\cdot y)^{-\delta} d\mu(g) \\
&\leq \int_G (2R^2)^\delta d\mu(g) = C
\end{aligned}
$$

In any case, we obtained:

$$
A_\mu(v)(x,y) \leq av(x,y) + C
$$

□

Proposition 2.7. *If the measure ν is non atomic, then for β-almost every $b \in B$, the measure ν_b is non atomic*

Proof. We suppose to a contradiction that the set $D := \{b \in B \mid \nu_b \text{ has atomes}\}$ is given positive measure by β. As for β-almost any $b \in B$, $b_{1*}\nu_{T(b)} = \nu_b$, The set contains a T-invariant conull set D_0. By ergodicity of the shift map, we get that D has full measure. Using the same argument, we can also show that the number of atoms with the biggest mass and the number of such atoms is constant on D_0. We shall denote by N_0 and m_0 those quantities. For any $b \in D_0$, we define ν'_b to be the average measure of the N_0 atoms of ν_b of mass m_0. By contstruction, we get for any such b the equivariance property $b_{1*}\nu'_{T(b)} = \nu'_b$ and thus $\nu' := \int_B \nu'_b d\beta(b)$ is also a μ-stationary measure of mass m_0. By construction, ν' is absolutely continous with respect to ν and thus is also non atomic.

Let $\mathfrak{S}_{N_0}$ be the symmetric group of $\{1, \cdots, N_0\}$ and define $Y = X^{N_0}/\mathfrak{S}_{N_0}$. We denote by p the canonical projection from $X^{N_0} \to Y$ and by η_b the measure $p_*(\nu_b'^{\otimes N_0})$. Notice that those measures are now Dirac masses. They still satisfy the equivariance property and thus $\eta = \int_B \eta_b \, d\beta(b)$ is a μ-stationary measure. There is a natural continuous action of G on Y by taking the quotient of the diagonal action on X^{N_0} and we claim that μ has the contraction property on Y. To see this, let $v : X \times X - \Delta_X$ be the map gien by proposition 2.6 and define:

$$
\begin{array}{rccc}
 & Y & \to & Y \\
w : & (p(x_i), p(x'_i)) & \mapsto & \sum_{\sigma \in \mathfrak{S}_{N_0}} \min_{i \in [\![1, N_0]\!]} v(x_i, x'_{\sigma(i)})
\end{array}
$$

It is easily verified that w is proper and that:

$$A_\mu w \leq aw + CN_0!$$

We can thus apply proposition 2.5 to deduce that η is a Dirac mass. In turn, this implies that the ν_b are β-almost all supported on the same finite set of points and thus ν' has finite support, which is a contradiction. □

Before going on to the last step of the proof of proposition 2.4, we shall first proof the following technical lemma:

Lemma 2.3. *Let (Ω, η) be a probability space together with a measurable map $f : \Omega \to \Omega$ that preserves the measure η and let $\varphi : \Omega \to \mathbb{R}$ be a measurable map such that $\varphi^{-1}\{0\}$ is a null set, then $Z_\varphi = \{\omega \in \Omega \mid \varphi \circ f^p(\omega) \underset{n\to\infty}{\to} 0\}$ is also a null set.*

Proof. Write $\varphi^{-1}(0) = \cup_n \varphi^{-1}([n^{-1}, \infty[)$ and suppose to a contradiction that $\eta(Z) > 0$. Then there exists $n > 0$ such that $\eta(Z \cap \varphi^{-1}[n^{-1}, \infty[) > 0$. Poincaré recurrence theorem applied to this set yields a point ω and a sequence n_p going to infinity such that $\varphi \circ f^{n_p}(\omega)$ both converge to 0 when p goes to infinity and is bounded below by n^{-1}. This is a contradition and Z is a null set. □

Proposition 2.8. *If for β-almost every $b \in B$, the measure ν_b is non atomic then for β^X-almost every $(b, x) \in B^X$, we have $\nu_b(x + V_b) = 0$.*

Proof. The idea is to apply the lemma 2.3 to a well chosen dynamical system. Define $\Omega = B \times X \times X$ that we endow with the measure $\eta = \int_B \delta_b \otimes \nu_b \otimes \nu_b \; d\beta(b)$ and let f be the map $(b, x, y) \mapsto (T(b), b_1^{-1} \cdot x, b_1^{-1} \cdot y)$ and φ be the map $(b, x, y) \mapsto d(x, y)$. The measure η is preserved by the transformation f. Notice that $Z_\varphi = \{(b, x, y) \mid y \in x + V_b\}$ and since the ν_b are non atomic, the set $\varphi^{-1}\{0\}$ has measure 0. Consequently, lemma 2.3 shows:

$$\eta(Z_\varphi) = \int_{B^X} \nu_b(x + V_b) \; d\beta^X(b, x) = 0$$

□

Proof of proposition 2.4. If ν is non atomic then proposition 2.7 shows that the ν_b are non atomic either and then we conclude using proposition 2.8. □

2.4 Renormalization and the horocycle flow

Recall that for β-almost every $b \in B$, the following holds:

$$b_1 \cdot V_{T(b)} = V_b$$

Put otherwise, this relation means that the slices $\{b\} \times V_b$ are permuted by T^X. There is thus a well defined measurable map $s : B \to \mathbb{R}$ such that for β-almost every $b \in B$, the following holds:

$$b_1 \cdot v_{T(b)} = s(b) \cdot v_b$$

This allows to define a flow on B^X by $\varphi_t(b,x) = (b, x + t \cdot v_b)$. The fact that the slices $\{b\} \times V_b$ are permuted is translated into the following:

$$\varphi_t \circ T^X(b,x) = T^X \circ \varphi_{t \cdot s(b)}(b,x)$$

For a reason that will become clear later, it would be interesting to reparametrise the V_b so that the time change in the orbit of φ_t that appears in the previous equation is orientation-preserving and not dependent on $b \in B$. It turns out that this is not always possible but this difficulty can be bypassed with a "suspension" construction that we now detail. Let $\tau : B \to \mathbb{R}$ be a measurable map and define $\tau_p(b) = \sum_{k=0}^{p-1} \tau \circ T^p(b)$. We denote $B^\tau = \{(b,k,\varepsilon) \in B \times \mathbb{R} \times \{\pm 1\} \mid 0 \leq t < \tau(b)\}$ and the corresponding suspended flow over the shift map T. For any $c = (b,k,\varepsilon)$

$$T_l^\tau(c) = (T^{p_c^l}(b), \tau_{p_c^l}(b) - l + k, sgn(s(b)) \cdot \varepsilon)$$

Where we set:

$$p_c^l = min\ \{p \in \mathbb{N} \mid \tau_p(b) - (l-k) \geq 0\}$$

The heuristic of those constructions is that we now attach to any sequence of steps of the random walk $b \in B$ two new quantities. The first one is $\tau(b)$. It corresponds to the duration needed to perfom the last step along b. By extension τ_p is the duration needed to perfom the p last steps. The second information, encoded in the factor $\{\pm 1\}$, records if the last jump matches or not the orientation of the attracting spaces V_b and $V_{T(b)}$ defined by s. With this in mind, the semi-flow T_l^τ corresponds to looking up in past at time $t = -l$. We have the corresponding space of past trajectories $B^{\tau,X} = B^\tau \times X$ and the corresponding semi flow:

$$T_l^{\tau,X} : \begin{array}{ccc} B^{\tau,X} & \to & B^{\tau,X} \\ (c,x) & \mapsto & (T_l^\tau(c), b_{p_l c}^{-1} \cdots b_1^{-1} \cdot x) \end{array}$$

This semi-flow maps past trajectories (c,x) to its past l units of time ago. We now need to chose carefully the map τ to solve our initial problem, this is the purpose of the following proposition. We denote $\theta = log(|s|)$.

Proposition 2.9. *There are $\varepsilon > 0$, a measurable map $\tau : B \to \mathbb{R}$ bounded below by ε and a bounded measurable measurable map $\varphi : B \to \mathbb{R}$ such that:*

$$\theta - \varphi \circ \theta + \varphi = \tau$$

From now on, we chose τ as in proposition 2.9. The set B^τ has finite measure for the measure $\beta \otimes \lambda$ as a subset of $B \times \mathbb{R} \times \{\pm 1\}$, where λ is the Haar measure on $\mathbb{R} \times \{\pm 1\}$. We denote by β^τ the probability measure induced on B^τ by restriction of $\beta \otimes \lambda$. If $c = (b,k,\varepsilon)$, we denote $\nu_c = \nu_b$ and we define a measure $\beta^{\tau,X}$ on $B^{\tau,X}$ by:

$$\beta^{\tau,X} = \int_{B^\tau} \delta_c \otimes \nu_c \ d\beta^\tau(c)$$

Proposition 2.10. *The measure $\beta^{\tau,X}$ is $T_l^{\tau,X}$-invariant.*

We denote by $\mathcal{B}^\tau$ the completion for the measure β^τ of the product σ-algebra $\mathcal{G}^{\otimes N} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\{\pm 1\}))$ restricted to B^τ.

Definition 2.3 (Normalized backward dynamical system). *The space* $(B^{\tau,X}, \mathcal{B}^\tau \otimes \mathcal{X}, \beta^{\tau,X}, T_l^{\tau,X})$ *is called the normalized backward dynamical system.*

Definition 2.4 (Horocycle flow). *We define a flow the space* $B^{\tau,X}$*, called the horocycle flow, as follows:*

$$\Phi_t((b,k,\varepsilon),x) = ((b,k,\varepsilon), x + t \cdot \varepsilon e^{k-\varphi(b)} v_b)$$

This flow is the suspension of the flow φ_t defined at the beginning of this section. The constructions made in this section result in, as advertised, the following crucial result:

Proposition 2.11. *For* $\beta^{\tau,X}$*-almost every* $(c,x) \in B^{\tau,X}$*, we have:*

$$T_l^{\tau,X} \circ \Phi_t(c,x) = \Phi_{e^{-l}t} \circ T_l^{\tau,X}(c,x)$$

Proof. To be added. We refer to lemma 6.10 in [?] □

We can draw from proposition 2.4 a first important fact about the renormalized backward dynamical system. For any $c = (b,k,\varepsilon) \in B^\tau$, we write:

$$W_c = \{v \in \mathbb{R}^2 \mid \sup_{l>0} (e^l \| b_{p_c^l}^{-1} \cdots b_1^{-1} \cdot v \|) < \infty\}$$

Proposition 2.12. *Let* F_0 *be any* $\mathcal{B}^{\tau,X}$*-measurable set of positive measure. There exists a subset* $F \subset F_0$ *of full measure such that for any* $(c,x) \in F$*, there is a sequence* (u_n) *converging to* 0 *in* $\mathbb{R}^2 \setminus W_c$ *such that* $(c, x+u_n) \in F$*.*

Proof. to be added. We refer to corollary 6.15 in [?] □

3 Leaf-wise measures

3.1 Construction

Throughout this section, let $v \in \mathbb{R}$ be a non-zero vector and define for any $x \in X$ and $t \in \mathbb{R}$:

$$\phi_s(x) = x + s \cdot v$$

A flow box for ϕ_s is a homeomorphism $\varphi : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{U} \in X$ with $\mathcal{U}_i$ open sets contained in $\mathbb{R}$ and $\mathcal{U}$ open set contained X such that for any $t \in \mathcal{U}_2$, the slice $\varphi(\mathcal{U}_1 \times \{t\})$ is a piece of orbit of ϕ. This means that, for any $t \in \mathcal{U}_2$, there is a map $\gamma_t : \mathcal{U}_t \to \mathbb{R}$ such that $\varphi(s,t) = \phi_{\gamma_t(s)}(\varphi(0,t))$. Let $v^\perp$ be a non-zero vector in $\mathbb{R}^2$ perpendicular to v and define the following map:

$$a : \begin{array}{ccc} \mathbb{R}^2 & \to & X \\ (s,t) & \mapsto & s \cdot v + t \cdot v^\perp \bmod \mathbb{Z}^2 \end{array}$$

The simplest example of a flowbox is given by $(s,t) \mapsto a(x+s, y+t)$. In that case the map γ_t is the identity map. There is an important construction associated to any flowbox φ that we now explain. Let $t \in \mathcal{U}_1$ and define μ^φ to be the measure on $\mathcal{U}_2$ obtained by pushing forward the restriction of ν to $\mathcal{U}$ by $\pi_2 \circ \varphi^{-1}$. Here, π_2 is the canonical projection $\mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{U}_2$. The disintegration theorem yieds a familly of measure ν_s^φ, uniquely defined up to a conull set, such that for any ν-integrable function $f : \mathcal{U} \to \mathbb{R}$:

$$\int_{\mathcal{U}} f \, d\nu = \int_{\mathcal{U}_2} \left(\int_{\mathcal{U}_1} f \circ \varphi(s,t) \, d\nu_t^\varphi(s) \right) d\mu^\varphi(t)$$

Finally, if λ is a measure, we shall denote by $[\lambda]$ its projective class, i.e its equivalence class modulo multiplication by a positive scalar and by $\tau_c : \mathbb{R} \to \mathbb{R}$ the translation by c.

Proposition 3.1 (Leafwise measures)**.** *There are a measurable map $\sigma : X \to \mathbb{P}\mathcal{M}(\mathbb{R})$ and a conull set E which satisfy the following two properties:*

1. *Let $\varphi : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{U}$ be a flowbox. For ν-almost every $x = \varphi(s_x, t_x) \in \mathcal{U}$, the following holds:*

$$(\tau_{\gamma_{t_x}(s_x)})_* \sigma(x) = (\gamma_{t_x})_* [\nu_t^\varphi]$$

2. *For any $t \in \mathbb{R}$ and $x \in E$ such that $\phi_u(x) \in E$, the following holds:*

$$\sigma(x) = (\tau_u)_* \sigma(\phi_u(x))$$

Furthermore, if σ' is another such map then it coincides with σ on a set of full measure.

remark. Item 1 is an equality of measures on $\gamma_{t_x}(\mathcal{U}_1)$. Using item 1, one can locally recover the $\sigma(x)$ using flowboxes and disintegration of the measure ν along pieces of orbits of ϕ. Item 2 indicates how to patch those pieces up. Notice that in the case of a flowbox $\varphi(u,v) = a(s+u, t+v)$, item 1 rewrites as:

$$(\tau_u)_* \sigma(x) = [\nu_v^\varphi], \text{ where } x = \varphi(u,v)$$

proof of proposition 3.1. First, we construct the map σ. It will be obtained as the quotient of a map defined on $\mathbb{R}^2$. Let $\tilde{\nu}$ be the locally finite measure defined such that for any For any compactly supported continuous function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$\int_{\mathbb{R}^2} f \, d\tilde{\nu} = \int_X \sum_{a^{-1}(x)} f(s,t) \, d\nu(x)$$

Now, let $\tilde{\mu}$ to be a finite measure equivalent to $\tilde{\nu}$. We mean by that that $\tilde{\mu}$ is absolutely continous with respect to $\tilde{\nu}$ and that the corresponding Radon-Nikodym derivative is nowhere vanishing. Such a measure can be constructed by multipliying $\tilde{\nu}$ by a map with mean 1 and that is decreasing quickly at infinity. We denote by μ the pushforward of $\tilde{\mu}$ by the projection on the second factor and disintegration of the measure $\tilde{\nu}$ on the fibers of the projection, we get a familly of measure $\tilde{\nu}_y$ such that for any $\tilde{\nu}$-integrable function, the following holds:

$$\tilde{\nu}(f) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(s,t)\ d\tilde{\nu}_t(s) \right) d\mu(t)$$

We claim that if $a(s,0) = a(0,t)$ then for μ-almost any $y \in \mathbb{R}$, the following holds:

$$\tilde{\nu}_y \propto (\tau_s)_* \tilde{\nu}_{y+t} \tag{4}$$

To prove this claim, notice that, by construction, the measure $\tilde{\nu}$ is invariant by translation by $(s, -t)$ and thus, since $\tilde{\nu}$ and $\tilde{\mu}$ are equivalent, so are $\tau_t * \mu$ and μ. Denote by g the corresponding Radon-Nikodym derivative. For any $\tilde{\nu}$-integrable function $f : \mathcal{U} \to \mathbb{R}$, we have:

$$\begin{aligned}
\int_{\mathbb{R}^2} f(u,v)\ d\tilde{\nu}(u,v) &= \int_{\mathbb{R}^2} f(u+s, v+t)\ d\tilde{\nu}(u,v) \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(s+u, v-t)\ d\tilde{\nu}_v(u) \right) d\mu(v) \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(u,v)\ d(\tau_s)_* \tilde{\nu}_v(u) \right) d(\tau_{-t})_* \mu(v) \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(u,v)\ dg(v) \cdot (\tau_s)_* \tilde{\nu}_v(u) \right) d\mu(v)
\end{aligned}$$

This proves our claim as g is everywhere positive and that the $\tilde{\nu_y}$ are uniquely defined up to a conull set. As they are only coutably many pairs (s,t) as before, let E be a conull set on which the previous equality holds for any such pair. We now define the following measurable map:

$$\tilde{\sigma} : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathcal{M}(\mathbb{R}) \\ (s,t) & \mapsto & (\tau_{-s})_* \tilde{\nu}_t \end{array}$$

This map descends to a well defined measurable map $\sigma : X \to \mathbb{P}\mathcal{M}(\mathbb{R})$. This is a consequence of equation (4) that we proved earlier. We now check that item 1 and 2 are satisfied.

For the first item, let $\tilde{\varphi} : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathbb{R}^2$ be a homeomorphism such that $a \circ \tilde{\varphi} = \varphi$. By assumption, there are two maps $\varphi_1 : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathbb{R}$ and $\varphi_2 : \mathcal{U}_2 \to \mathbb{R}$ such that $\tilde{\varphi} = (\varphi_1, \varphi_2)$. Notice that, since $\tilde{\nu}$ and $\tilde{\mu}$ are equivalent, the measures ν^{φ} is absolutely continuous with respect to the measure $\varphi_2^* \mu$. Let h be the corresponding Radon-Nikodym derivative and denote by $\mathcal{U}_2'$ the subset of $\mathcal{U}_2$ where h does not vanish. Notice that by unicity of the disintegration, $\nu_{\varphi_2(t)}(\varphi_1(\cdot, t)(\mathcal{U}_1))$ vanishes whenever $t \notin \mathcal{U}_2'$. Let f be any ν-integrable function.

$$
\begin{aligned}
\int_{\mathcal{U}} f \, d\nu &= \int_{\tilde\varphi(\mathcal{U}_1\times\mathcal{U}_2)} f\circ a \, d\tilde\nu \\
&= \int_{\varphi_2(\mathcal{U}_2)} \left(\int_{\varphi_1(\mathcal{U}_1\times\{\varphi_2^{-1}(t)\})} f\circ a(s,t) \, d\tilde\nu_t(s) \right) \, d\mu(t) \\
&= \int_{\mathcal{U}_2} \left(\int_{\varphi_1(\mathcal{U}_1\times\{t\})} f\circ a(s,\varphi_2(t)) \, d\tilde\nu_{\varphi_2(t)}(s) \right) \, d\varphi_2^*\mu(t) \\
&= \int_{\mathcal{U}_2} \left(\int_{\mathcal{U}_1} f\circ a(\varphi_1(s,t),\varphi_2(t)) \, d(\varphi_1(\cdot,t))^*\tilde\nu_{\varphi_2(t)}(s) \right) \, d\varphi_2^*\mu(t) \\
&= \int_{\mathcal{U}_2'} \left(\int_{\mathcal{U}_1} f\circ a\circ\tilde\varphi(s,t) \, d(\varphi_1(\cdot,t))^*\tilde\nu_{\varphi_2(t)}(s) \right) \, d\varphi_2^*\mu(t) \\
&= \int_{\mathcal{U}_2'} \left(\int_{\mathcal{U}_1} f\circ \varphi(s,t) \, dh^{-1}(t)\cdot(\varphi_1(\cdot,t))^*\tilde\nu_{\varphi_2(t)}(s) \right) \, d\mu^\varphi(t) \\
&= \int_{\mathcal{U}_2} \left(\int_{\mathcal{U}_1} f\circ \varphi(s,t) \, dh^{-1}(t)\cdot(\varphi_1(\cdot,t))^*\tilde\nu_{\varphi_2(t)}(s) \right) \, d\mu^\varphi(t)
\end{aligned}
$$

From this, we get that for μ^φ-almost every $t\in\mathcal{U}_2$:

$$\nu_t^\varphi = h^{-1}(t)\cdot\varphi_1(\cdot,t)^*\tilde\nu_{\varphi_2(t)} = h^{-1}(t)\cdot\varphi_1(\cdot,t)^*\tilde\nu_{\varphi_2(t)}$$

And since $a\circ\tilde\varphi(s,t)=\varphi(s,t)$, one gets that for μ^φ-almost every $t\in\mathcal{U}_2$ and for any $s\in\mathcal{U}_1$:

$$\tau_{\varphi_1(s,t)_*}\sigma(\varphi(s,t)) = [\tau_{-\varphi_1(s,t)_*}\tilde\nu_{\varphi_2(s,t)}] = [\varphi_1(\cdot,t)_*\nu_t^\varphi]$$

To conclude, one just has to notice that for any $t\in\mathcal{U}_2$, $\varphi_1(\cdot,t)=\varphi(\tilde{0},t)+\gamma_t$.

We now check the second item of the statement. Let $x\in E$ and let $t\in\mathbb{R}$ such that $\phi_t(x)$ belongs to E. Let (u,v) such that $a(u,v)=x$. Notice that $a(u+s,v)=\phi_s(x)$. By construction $\sigma(x)=(\tau_{-u})_*\tilde\nu_v$ and $\sigma(\phi_t(x))=(\tau_{-u-s})_*\tilde\nu_t=(\tau_{-s})_*\sigma(x)$.

Finally, the unicity property follows from the remark made before the proof.

□

3.2 Renormalization of the leafwise measures

We now arrive to a cornerstone of the approach followed by Benoist and Quint. In the previous subsection, we associated to any $v\in\mathbb{R}^2$ a measurable map $\sigma: X\to\mathbb{P}\mathcal{M}(\mathbb{R})$. From now on, we shall denote this map σ_v to stress the dependence on v. We define:

$$\sigma: \begin{array}{ccc} B^{\tau,X} & \to & \mathbb{R} \\ (b,k,\varepsilon,x) & \mapsto & \sigma_{\varepsilon e^{k-\varphi(b)}v_b} \end{array}$$

The proof of Theorem 1.1 is based on an analysis of some invariance property of σ.

Proposition 3.2. *For any $l \geq 0$ am for $\beta^{\tau,X}$-almost every $(c,x) \in B^{\tau,x}$, the following holds:*

$$\sigma \circ T_l^{\tau,X}(c,x) = (e^{-l})_* \sigma(c,x)$$

Proof. To be added. We refer to lemma 6.12 in [?] □

Corollary 3.1. *The map $\sigma : B^{\tau,X} \to \mathcal{M}(\mathbb{R})$ is $\mathcal{Q}_\infty^{\tau,X}$- measurable.*

Proof. To be added. We refer to corollary 6.13 in [?] □

4 Rewritting the near past

4.1 Law of the last jumps

We now introduce some definitions. For any $q \geq 0$ and $a \in B$, we denote by $a[q] = (a_1, \cdots, a_q)$. If $b \in B$, we will also denote by $a[q]b = (a_1, \cdots, a_q, b_1, b_2, \cdots)$. It corresponds to a trajectory of the random walk that went from the origin of time to $t = 0$ following b and then did another q jumps following a. Now, let $c = (b, k, \varepsilon) \in B^\tau$ and define:

$$\tilde{q}_c^l(a) = \sup r\{q \mid \tau_q(a[q]b) - l + k < 0\}$$

If, starting from c, the random walk follows the steps given by a, ie does a_1 then a_2 etc, then the number of steps achieved in l unit of time is precisely $\tilde{q}_c^l(a)$. We also define

$$\tilde{h}_c^l(a) = (a[q]b, k + l - \tau_q(a[q]b), \sigma_q(a[q]b) \cdot \varepsilon) \text{ with } q = \tilde{q}_c^l(a)$$

Where we have denoted by $\sigma_q(b,k,\varepsilon) = sgn(b) \times \cdots \times sgn(T^q(b))$. This is the trajectory of the random walk obtained if we start at c, walk along a and wait l unit of times. In the same fashion, we denote by $\tilde{h}_{c,x}^l(a) = (\tilde{h}_c^l(a), a_{\tilde{q}_c^l(a)} \cdots a_1 \cdot x)$. We start with a technical lemma:

Lemma 4.1. *For any $\beta^{\tau,X}$-integrable fonction φ and $l > 0$, we have:*

$$\int_{B^{\tau,X}} \int_B \varphi(\tilde{h}_{c,x}^l(a)) \; d\beta(a) \; d\beta^{\tau,X}(c,x) = \int_{B^{\tau,X}} \varphi \; d\beta^{\tau,X}(c,x)$$

We now denote by $q_c^l(a) = \tilde{q}_{T_l^\tau(c)}^l(a)$, $h_c^l(a) = \tilde{h}_{T_l^\tau(a)}$ and finally by $h_{c,x}^l(a) = \tilde{h}_{T_l^{\tau,X}(c,x)}(a)$. It corresponds to delete the past of (c,x) between $t = -l$ and $t = 0$ and replacing it by a. In particular, for any $a \in B$, the point (c,x) and $h_{c,x}^l(a)$ have the same image by $T_l^{\tau,X}$. The following proposition is called the law of the last jumps in [?]. It will be crucial in the proof of the main proposition 5.1

Proposition 4.1. *Let $\varphi : B^{\tau,X} \to \mathbb{R}$ be a $\beta^{\tau,X}$-measurable map. For any $l > 0$ and for $\beta^{\tau,X}$-almost every (c,x) in $B^{\tau,X}$ we have:*

$$\mathbb{E}[\varphi \mid \mathcal{Q}_l^{\tau,X}](c,x) = \int_B \varphi(h_{c,x}^l(a)) d\beta(a)$$

Proof. Let ψ be a $\mathcal{Q}_l^{\tau,X}$-measurable function. By definition of the conditional expectation, it is enough to prove:

$$\int_{B^{\tau,X}} \psi(c,x) \int_B \varphi(h^l_{c,x}(a)) d\beta(a) d\beta^{\tau,X}(c,x) = \int_{B^{\tau,X}} \psi(c,x)\varphi(c,x)\ d\beta^{\tau,X}(c,x)$$

Since the function ψ is $\mathcal{Q}_l^{\tau,X}$-measurable, for any $(c,x) \in B^{\tau,X}$ and $a \in B$, we have $\psi(c,x) = \psi(h^l_{c,x}(a))$. Thus:

$$\begin{aligned}\int_{B^{\tau,X}} \psi(c,x) \int_B \varphi(h^l_{c,x}(a) d\beta(a) d\beta^{\tau,X}(c,x) &= \int_{B^{\tau,X}} \int_B \psi(h^l_{c,x}(a))\varphi(h^l_{c,x}(a)) d\beta(a) d\beta^{\tau,X}(c,x) \\ (by\ definition\ of\ h) &= \int_B \int_{B^{\tau,X}} \psi \times \varphi(\tilde{h}^l_{T_l^{\tau,X}(c,x)}(a)) d\beta^{\tau,X}(c,x)\ d\beta(a) \\ (using\ that\ \beta^{\tau,X}\ is\ T_l^{\tau,X} - invariant) &= \int_B \int_{B^{\tau,X}} \psi \times \varphi(\tilde{h}^l_{(c,x)}(a)) d\beta^{\tau,X}(c,x)\ d\beta(a) \\ (using\ lemma\ 4.1) &= \int_{B^{\tau,X}} \psi \times \varphi(c,x)\ d\beta^{\tau,X}(c,x)\end{aligned}$$

□

4.2 Random product of matrices

Proposition 4.2. *For any $\alpha > 0$ and $\eta > 0$, there are $r > 0$ and $q_0 \geq 1$ such that, for any non zero $v \in \mathbb{R}^2$ and $D \in \mathbb{P}^2(\mathbb{R})$, the following holds:*

- $\beta(\{a \in B \mid \forall q \geq q_0\ \|a_q \cdots a_1 \cdot v\| \geq \frac{1}{r}\ \|a_q \cdots a_1\|\|v\|\}) \geq 1 - \alpha$
- $\beta(\{a \in B \mid \forall q \geq q_0\ d(\mathbb{R}a_q \cdots a_1 \cdot v, a_q \cdots a_1 \cdot D) \leq \eta\}) \geq 1 - \alpha$

Proof. to be added. For nowm the reader is refered to corollary 5.5 in [**?**] □

5 Drift and invariance of the stationary probabilities

5.1 The exponential drift

Proposition 5.1. *Let $(Y, \mathcal{Y})$ be a standard Borel set, let $f : B^{\tau,X} \to Y$ be a $\mathcal{Q}_\infty^{\tau,X}$-measurable map and let $E \subset B^X$ be a conull set: $\beta^X({}^cE) = 0$. Then, for $\beta^{\tau,X}$-almost every $(c,x) \in B^{\tau,X}$ and $\varepsilon > 0$, there is a $0 < t < \varepsilon$ and $(c',x') \in E$ such that:*

$$f(b', x' + t \cdot v_b) = f(b', x') = f(b, x)$$

Proof. Before going into the details of the proof, let us say a word on the strategy used by Benoist and Quint. Let (c,x) and (c',x') be two past trajectories of the renormalized random walk that only differ during the last l seconds, ie:

$$T_l^{\tau,X}(c,x) = T_l^{\tau,X}(c',x')$$

A small deformation of x into $y = x+u$ yields a deformation of x' into $y' = x+v$. All the efforts made in the previous section are dedicated to controling the vector v. More precisely, we are going to see that for some choice of u, l and c', this vector v can be chosen of norm smaller than ϵ and arbitrarily close to $V_{c'}$. Limiting thius construction when u converges, we obtain $y'_\infty = x'_\infty + v_\infty$ with $v_\infty \subset V_{c'}$, or equivalently there is a $0 < t \leq \varepsilon$ such that:

$$(c'_\infty, y'_\infty) = \Phi_t(c'_\infty, x'_\infty)$$

But the fact that φ is $\mathcal{Q}_\infty^{\tau,X}$-measurable implies that whenever two past trajectories have the same past before a time $t = -l$, they have the same image by φ and so

$$\varphi(c_\infty, y_\infty) = \varphi(c, y) \text{ and } \varphi(c_\infty, x_\infty) = \varphi(c, x)$$

If x and y belong to a continuity set of φ, then we also have:

$$\varphi(c, x) = \varphi(c, y)$$

Hence:

$$\varphi(c, x) = \varphi(c'_\infty, x'_\infty) = \varphi(\Phi_t(c'_\infty, x'_\infty))$$

Let $\varepsilon > 0$ and let B_0 be a conull set contained in B on which the equivariance relation of the furstenberg measure holds. We denote by $B_0^{\tau,X} = B_0 \times \mathbb{R} \times \tilde{X}$ and let $F \subset B^{\tau,X}$ be the conull set given by proposition 2.12 applied to $B_0^{\tau,X}$.

1. We treat first the case where $(Y, \mathcal{Y})$ is a topological space and f is continuous. The following result is essential.

Proposition 5.2. *For any $(c, x) \in F$, $\eta > 0$, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there are $l > 0$, $u \in \mathbb{R}^2 - \{0\}, v \in \mathbb{R}^2$ and $a \in B$ such that, setting $y = x + u$:*

1. $\|u\| \leq \varepsilon_1$
2. $h^l_{c,y}(a) = h^l_{c,x}(a) + v$
3. $0 < \|v\| \leq e^{-M}\varepsilon_2$
4. $d(\mathbb{R} \cdot v, V_{h_{l,c}(a)}) \leq \eta$

Proof of proposition 5.2. First notice that there is a $\lambda > 0$ such that for any $(c, x) \in F$, $l \geq l'$ and $u \in \mathbb{R}^2$, we have:

$$\frac{\|b^{-1}_{p^l_c} \cdots b^{-1}_1 \cdot u\|}{\|b^{-1}_{p'_l(c)} \cdots b^{-1}_1 \cdot u\|} < e^{\lambda(l-l')} \tag{5}$$

Indeed, $p^l_c - p^{l'}_c$ is the number of steps made between the times $t = -l$ and $t = -l'$ and this quantity is bouded below by $\frac{l-l'}{\min \tau}$. Then the inequality is just a consequence of the fact that μ is compactly supported. Additionally, there is a constant $C > 0$ such that for $\beta^{\tau,X}$-almost every $(c, x) \in B^{\tau,X}$ and $l > 0$, we have:

$$C^{-1} \leq e^{-l} \, \|b_1 \cdots b_{p_c^l} \cdot v_{T_l^\tau(c)}\| \leq C$$

By proposition 2.12, let $u \in \mathbb{R}^2 - V_c$ of norm smaller than ε_1 such that $(c, y) \in B_0^{\tau,X}$ where $y = x + u$ and let r be as in lemma 4.2 for a fixed $\alpha < 1$. Define:

$$l_0 = inf \; \{l > 0 \mid e^l \; \|b_{p_c^l}^{-1} \cdots b_1^{-1} \cdot u\| > \frac{e^{-(1+\lambda+M)}\varepsilon}{Cr}\}$$

Notice that, up to taking a smaller u, which is licit according to proposition 2.12, l_0 can be chosen arbitrarily large. We now apply the inequality (5) with $l = l_0$ and $l' = l_0 - 1$ to obtain:

$$\begin{aligned} e^{l_0} \; \|b_{p_{l_0}(c)}^{-1} \cdots b_1^{-1} \cdot u\| &\leq e^{(1+\lambda)} e^{l_0 - 1} \|b_{p_{l_0-1}(c)}^{-1} \cdots b_1^{-1} \cdot u\| \\ &\leq \frac{e^{1+\lambda} e^{-(1+\lambda+M)}\varepsilon}{Cr} \\ &\leq \frac{e^{-M}\varepsilon}{Cr} \end{aligned}$$

Consequently, denoting $C_0 = e^{1+\lambda}$, we get:

$$\frac{e^{-M}}{CC_0 r}\varepsilon \leq e^{l_0} \; \|b_{p_{l_0}(c)}^{-1} \cdots b_1^{-1} \cdot u\| \leq \frac{e^{-M}}{Cr}\varepsilon$$

Show that we can choose $l = l_0$. Since α was chosen to be smaller than 1, proposition 4.2 provides us with a $a \in B$ satisfying the two inequalities this proposition features. Notice that $h_{l_0,c,y}(a) = h_{l_0,c,x}(a) + v$, where $v = a_{q_l} \cdots a_1 b_{p_c^l}^{-1} \cdots b_1^{-1} \cdot u$. On the one hand:

$$\begin{aligned} \|v\| &\leq \|a_{q_{l_0}} \cdots a_1\| \|b_{p_c^l}^{-1} \cdots b_1^{-1} \cdot u\| \\ &\leq r \|a_{q_l} \cdots a_1 \cdot v_{T_l^{\tau,X}(c,x)}\| \|b_{p_c^l}^{-1} \cdots b_1^{-1} \cdot u\| \\ &\leq rCe^{l_0} \|b_{p_{l_0}(c)}^{-1} \cdots b_1^{-1} \cdot u\| \\ &\leq e^{-M}\varepsilon \end{aligned}$$

And on the other hand:

$$\begin{aligned} \|v\| &\geq \frac{1}{r} \|a_q \cdots a_1\| \|b_p^{-1} \cdots b_1^{-1} \cdot u\| \\ &\geq \frac{1}{r} \; \|a_q \cdots a_1 \cdot v_{T_l^\tau(c')}\| \|b_p^{-1} \cdots b_1^{-1} \cdot u\| \\ &\geq \frac{1}{Cr} e^l \; \|b_p^{-1} \cdots b_1^{-1} \cdot u\| \\ &\geq \frac{1}{Cr} \frac{e^{-M}}{CC_0 r}\varepsilon > 0 \end{aligned}$$

Finally, item 4 results from item 2 of proposition 4.2 applied to $b_{p_l}^{-1} \cdots b_1^{-1} \cdot u$ and $V_{T_l^\tau(c)} = V_{T_l^\tau(c')}$. Indeed, since $a \in B_0$, we have $a_{q_l} \cdots a_1 \cdot V_{T_l^\tau(c)} = V_{h_{l,c}(a)}$

□

Now, Let (c, x) be an element of F and let (η_n) and (ε_n) be two sequences converging to 0 when n goes to infinity. Proposition 5.2 applied to the point (c, x) with η_n and ε_n converginf to 0 and $e^{-M}\varepsilon$ provides us with l_n, u_n, v_n as in the statement. Since f is $\mathcal{Q}_\infty^{\tau,X}$-measurable, we have:

$$f(h_{l_n,c,x}(a_n)) = f(c, x) \text{ and } f(h_{l_n,c,y_n}(a_n)) = f(c, y_n)$$

Since X is compact, up to taking a subsequence we can assume that $h_{l_n,c,y_n}(a_n)$ and $h_{l_n,c,x}(a_n)$ converge to points (c', x') and (c', y'). Using the continuity of f and the fact that $y_n = x_n + u_n$ converges to x, we have:

$$f(c', x') = f(c, x) \text{ and } f(c', y') = f(c, x)$$

We can further extract to have that v_n converges to a non zero vector v_∞, bounded above by $e^M\varepsilon$ and lying in $V_{c'}$. Thus there is a $0 < t_\infty \leq \varepsilon$ such that $v_\infty = t_\infty \cdot v_{c'}$ and thus:

$$(c', y') = \Phi_{t_\infty}(c', x')$$

and then

$$f(\Phi_{t_\infty}(c', x')) = f(c', x') = f(c, x)$$

2. We now treat the general case where we drop the continuity assumption. The strategy is to use Lusin's theorem to work on a compact subset of arbitrarily large measure on restriction to which the function f is continuous and redo the argument we did before. The difficulty is to show that for a given (c, x) is this compact set, one can find sufficiently many points in K whose past coincide at some relevant time with the past of (c, x) ie we need an upgrade of proposition 5.2. This is where the last jumps law comes into play. More precisely, Lusin's theorem provides us with a compact set K of mesure at least $1 - \alpha^2$ on restriction to which all the functions we shall consider are continuous. Now, consider the function $\mathbb{E}[1_K|\mathbb{Q}_\infty^{\tau,X}]$. It is bounded above by 1 and its integral over X is, by definiton of the conditional expectation bounded below by $1 - \alpha^2$. Consequently, this function is bounded below by $1 - \alpha$ on a set of measure at least $1 - \alpha$. Using Lusin's theorem once more, this means that there is a set L of measure at least $1 - \alpha$ on restriction to which f is continuous and such that for any (c, x) in L we have:

$$\mathbb{E}[1_K \mid \mathcal{Q}_\infty^{\tau,X}](c, x) > 1 - \alpha$$

Now, an application of Doob's martingale convergence theorem, shows that for $\beta^{\tau,X}$-almost every $(c, x) \in B^{\tau,X}$:

$$\mathbb{E}[1_K \mid \mathcal{Q}_l^{\tau,X}](c, x) \underset{l\to\infty}{\to} \mathbb{E}[1_K \mid \mathcal{Q}_\infty^{\tau,X}](c, x)$$

Up to taking a smaller α, Egorov's theorem applied to the countable family $(\mathbb{E}[1_K \mid \mathcal{Q}_l^{\tau,X}])_{l\in|\mathbb{Q}}$ shows that there is an l_0 such that for any rational number l such that $l \geq l_0$ and for any $(c,x) \in L$, we have

$$\mathbb{E}[1_K \mid \mathcal{Q}_l^{\tau,X}](c,x) \geq (1-\alpha)$$

Since α is arbitrary, it is enough to show that proposition 5.1 holds on L. Let F be as in proposition 2.12 for the set $L \cap B_0^{\tau,X}$. The following is the upgrade of proposition 5.2 that we need to conclude.

Proposition 5.3. *For any $(c,x) \in F$, $\eta > 0$, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there are $l > 0$, $u \in \mathbb{R}^2 - \{0\}$, $v \in \mathbb{R}^2$ and $a \in B$ such that, setting $y = x + u$:*

1. $\|u\| \leq \varepsilon_1$
2. $(c,y) = (c,x+u)$ *belongs to* L
3. $0 < \|v\| \leq e^{-M}\varepsilon_2$
4. $h^l_{c,y}(a) = h^l_{c,x}(a) + v$
5. $d(\mathbb{R}\cdot v, V_{h_{l,c}(a)}) \leq \eta$
6. $h^l_{c,x}(a)$ *and* $h^l_{c,y}(a)$ *both belong to* K

Proof. The proposition goes exactly the same as in the continous case but a is now chosen to also belong K, which is possible as soon as $\alpha > \frac{1}{3}$ so that the intersection of K and the two sets of proposition 5.5 have a non trivial intersection. □

Since by construction, f continous on L and K, the same argument as before can be reproduced and we are able find for any (c,x) in F, a t in $\mathbb{R}$ with $0 < t \leq \varepsilon$ and (c',x') in $B^{\tau,X}$ such that:

$$f(\Phi_t(c',x')) = f(c',x') = f(c,x)$$

□

5.2 Invariance of the Furstenberg measures

Proposition 5.4. *If ν is non atomic, then for β-almost every $b \in B$, the measure ν_b is V_b-invariant.*

Proof. Let E be as in proposition 3.1. By Corollary 3.1, we know that the map $\sigma : B^{\tau,X} \to \mathbb{P}\mathcal{M}(\mathbb{R})$ is $\mathcal{Q}_\infty^{\tau,X}$-measurable and thus we can apply proposition 5.1. We obtain that for any $\epsilon > 0$ and for $\beta^{\tau,X}$-almost every $(c,x) \in B^{\tau,X}$, there is a $t \leq \varepsilon$ and $(c',x') \in E$ such that:

$$\sigma(\Phi_t(c',x')) = \sigma(c',x') = \sigma(c,x)$$

but since (c',x') lies in E, we know that:

$$\sigma(\Phi_t(c',x')) = (\tau_t)_*\sigma(c',x')$$

As a consequence, one gets:

$$(\tau_t)_*\sigma(c,x) = \sigma(c,x)$$

Notice that, because of item 2 of 3.1 together with the fact that ν is a probability measure, for $\beta^{\tau,X}$-almost any $(c,x) \in B^{\tau,X}$ the class $\sigma(x,x)$ is represented by a σ-finite measure. This allows us to pick a well defined representatives as follows: let $n > 0$ be the smallest integer such that $\sigma(c,x)([-n,n])$ is non-zero and choose the unique measure $\bar{\sigma}(c,x)$in the class $\sigma(c,x)$ such that $\sigma([-\bar{n},n]) = 1$. Now, the invariance by translation of the $\sigma(c,x)$ means that there is a measurable map $\alpha : B^{\tau,X} \to \mathbb{R}$ such that for $\beta^{\tau,X}$-almost every $(c,x) \in B^{\tau,X}$, the following holds:

$$(\tau_t)_*\bar{\sigma}(c,x) = e^{\alpha(c,x)\cdot t}\bar{\sigma}(c,x)$$

Now, using the relation proved in corollary 3.2, we easily get:

$$\alpha(T_l^{\tau,X}(c,x)) = e^l\alpha(c,x)$$

We can now conclude that α vanishes on a conull set using everywhere Poincare recurence theorem. Indeed, otherwise we could assume that, up to replacing α by $-\alpha$, there are two real numbers $0 < a < b$ such that $\alpha^{-1}([a,b])$ has positive measure. Poincare recurence theorem would provide us with an unbounded sequence $(l_n)_{n\in\mathbb{N}}$ and $(c,x) \in \alpha^{-1}([a,b])$ such that for any $n > 0$, $T_{l_n}^{\tau,X}(c,x)$ belongs to $\alpha^{-1}([a,b])$. This is a contradiction as soon as $e^{l_n}a > b$. The consequence of all this is that for $\beta^{\tau,X}$-almost every $(c,x) \in B^{\tau,X}$, the measure $\bar{\sigma}(c,x)$ is invariant by translation and thus is a Lebesgue measure. Recall that Finally, using item 2 of proposition 3.1, we see that for $\beta^{\tau,X}$-almost every $(c,x) \in B^{\tau,X}$, the restriction of ν_c to any flowbox is a sum of measures that are invariant by translation by V_c and thus is itself invariant by translation V_c. This concludes.

□

6 The harvest

6.1 Invariance of the stationary measures

Proposition 6.1. *If ν is non atomic, then it is the Haar measure.*

Proof. Suppose to a contradiction that there is a set $\{b \in B \mid \nu_b \neq Haar\}$ of positive measure. Since this set is invariant by T, it has full measure. By proposition 5.4, the ν_b are then lebesgue on a subtori of dimension 1 that is the image on a line of rational slope. There is thus a well defined measurable map $S : B \to \mathbb{P}(\mathbb{Q}^2)$ that maps a point b to the line that supports ν_b. Let η be the pushforward $S_*\beta$. There is a natural action of G on $\mathbb{P}(\mathbb{Q}^2)$ induced by its action on $\mathbb{Q}^2$. for any $g \in G$, We define $L_g : b \mapsto (g, b_1, b_2, \cdots)$. The following relation is satisfied:

$$g \cdot S(b) = S \circ L_g(b)$$

We claim that the measure $S_*\beta$ is μ-stationary. Indeed,

$$\begin{aligned}\int_G g_*\eta \ d\mu(g) &= \int_G (g\cdot S)_*\beta \ d\mu(g)\\ &= \int_G (S\circ L_g)_*\beta \ d\mu(g)\\ &= S_*(\int_G (L_g)_*\beta \ d\mu(g))\\ &= S_*\beta\\ &= \eta\end{aligned}$$

Now, let A be the set a atoms of maximal mass of η. Since η is a probability measure this set is finite and we claim that this set is invariant by Γ_μ. indeed let $x \in A$. By stationnarity:

$$1 = \eta(\{x\}) = \int_G g_*\eta(\{x\})d\mu(g)$$

This shows that for μ-almost every $g \in G$, $g \cdot A = A$ and thus by definition of Γ_μ, it preserves A and this proves our claim. This yields a finite union of lines that are preserved by Γ_μ, in contradiction with the strong irreducibility. Thus, for almost every $b \in B$, the measure ν_b coincides with the Lebesgue measure on the torus. □

6.2 Classification of closed μ-invariant sets

We will need the following. Compare with proposition 2.6

Proposition 6.2. *Let F be a finite Γ-invariant set. For any ε, there is a compact K_ε of $X - F$ such that for any $x \in X - F$, there is a $M_x \leq 1$ such that for any $n \geq M_x$:*

$$\int_G 1_{K_\varepsilon}(g\cdot x) \ d\mu^{*n}(g) \geq 1-\varepsilon$$

Furthermore, M_x can be chosen to be constant on the compacts of F^c.

Proof. Let $a_0 < 1$, $\delta_0 > 0$ and $n_0 \geq 1$ be as in lemma 2.2. and using the same trick that we used at the beginning of 2.6, we can assume that $n_0 = 1$. Let d_F be smallest distance between to distinct points in F and define:

$$u_F : \begin{array}{ccc} F^c & \to & \mathbb{R} \\ x & \mapsto & d(x,F)\end{array}$$

This map is proper and we claim that there are $a < 1$ and $C > 0$ such that:

$$A_\mu(u_F) \leq a \cdot u_F + C$$

To prove this claim we define as before

$$R = \underset{g\in\Gamma_\mu}{sup} \ max(\{\|g\|, \|g^{-1}\|\})$$

and we consider the following two cases:

1. If $d(x,F)(x) < \frac{d_F}{2R}$, then there is a unique $x' \in F$ and $u \in \mathbb{R}^2$ of norm at most $\frac{d_F}{2R}$ such that $x = x' + u$. Consequently, for any $g \in \Gamma_\mu$, we have that $g \cdot x = g \cdot x' + g \cdot u$ and the norm of $g \cdot u$ is smaller than $\frac{d_F}{2}$. This means that for any $g \in \Gamma_\mu$, one has $u_F(g \cdot x) = \|g \cdot u\|$ and thus according to lemma 2.2:

$$A_\nu(u_F)(x) = \int_G \|g \cdot u\|^{-\delta} \leq a_0 \cdot \|u\|^{-\delta} = a_0 \cdot u_F(x)$$

2. Now, if $d(x,F) > \frac{d_F}{2R}$, then $d(g \cdot x, F) > \frac{d_F}{2R^2}$ and thus:

$$A_\mu(u_F)(x) \leq (\frac{2R^2}{d_F})^\delta =: C$$

Consequently:

$$A_\mu(u_F) \leq a_0 \cdot u_F + C$$

Now, let $\varepsilon > 0$ and define $K = \{z \in F^c \mid u_F(z) \leq \frac{2B}{\varepsilon}\}$. Notice that $1_{K^c} \leq \frac{\varepsilon}{2B} f$ and that for any $n \geq 1$, we have $A^n_\mu(u_F) \leq a_0^n \cdot u_F + \frac{b}{1-a}$. Consequently, for any $y \in F^c$:

$$A^n_\mu(u_F)(y) \leq \frac{\varepsilon}{2B} A^n_\mu(f)(y) \leq \frac{\varepsilon a_0^n}{2B} f(y) + \frac{\varepsilon}{2}$$

Since f is bounded on compact, this proves our claim.

□

Proposition 6.3. *The torus X is the only infinite closed Γ-invariant set.*

Proof. We will first do a simplification. As G is simple and Γ is Zariski dense, there are $g_1, \cdots, g_l \in \Gamma$ such that $\Gamma' = \langle g_1, \cdots, g_l \rangle$ is still Zariski dense and finite index in Γ. It is enough to prove the result for Γ' instead of Γ. Define the following measure:

$$\mu = \frac{1}{l}(\delta_{g_1} + \cdots + \delta_{g_l})$$

The idea is to use the classification of ν-stationary measures that we established to deduce the result. Notice first that the set of finite Γ'-orbits is countable. This comes from the fact Γ' has countably many finite index subgroups. Indeed, any finite orbit yields a finite index subgroups by looking at the stabilizer of a point in the orbit and the fixed points set of such a finite index subgroup are isolated, hence finite as X is compact. The isolation property comes from the fact that Γ' is finite index in Γ together with the fact that the action of Γ is strongly irreducible. Thus, there is a nested familly $(F_i)_{i\in\mathbb{N}}$ of finite Γ'-invariant sets such that any finite and Γ'-invariant subset is contained in one of the F_i. As F is infinite, one can choose a sequence $(x_i)_{i\in\mathbb{N}}$ such that $x_i \notin F_i$.

Now, according to proposition 6.2, for any $i \in \mathbb{N}$, there is a set K_i which is a compact set of F_i^c such that for any $j \geq 1$, there is a $M_j \geq 1$ such that for any $n \geq M_j$ and $i \leq j$:

$$\mu^{*n} * \delta_{x_j}(K_i^c) \leq \frac{1}{i}$$

Set $n_j = jM_j$ and define:

$$\nu_j = \frac{1}{n_j} \sum_{k=1}^{n_j} \mu^{*k} * \delta_{x_j}$$

For any $i \leq j$, one has:

$$\nu_j(K_i^c) \leq \frac{M_j}{n_j} + \frac{n_j - M_j}{n_j} \frac{1}{i} \leq \frac{2}{i}$$

This ensures that the limit points of the sequence $(\nu_j)_{j \in \mathbb{N}}$ for the weak* topology are μ-stationary measures that do not give mass to the F_i. According to Theorem 1.1, this measure is then the haar measure and thus $F = X$. $\square$

A Furstenberg measures

Proposition A.1. *For β-almost every $b \in B$, the sequence of measure $(b_1 \cdots b_n)_* \nu$ converges to a measure ν_b in the weak-$*$ topology. In addition, those measures satisfy the following:*

$$\nu_b = (b_1)_* \nu_{Tb} \tag{6}$$

and

$$\nu = \int_B \nu_b \; d\beta(b) \tag{7}$$

Reciprocally, for any $\mathcal{B}$-measurable $b \mapsto \nu_b \in \mathcal{P}(X)$ satisfying (6), the average measure $\nu = \int_B \nu_b \; d\beta(b)$ is μ-stationary.

Proof. We introduce the filtration $\mathcal{S}_n$ of the σ-algebra $\mathcal{B}$ generated by the projection on the n first coordinates of B. The information contained in $\mathcal{S}_n$ corresponds to the last n steps of the random walk. Let $f \in C_c^0(X)$ and define a sequence of random variables $(X_n)_{n \in \mathbb{N}^*}$ by:

$$X_n(b) = (b_1 \cdots b_n)_* \nu(f)$$

We observe that X_n is $\mathcal{S}_n$-measurable. We claim that $\mathbb{E}[X_{n+1} | \mathbb{S}_n] = X_n$. To prove this claim it is enough to prove that for any $A = A_1 \times \cdots A_n \times B \in \mathcal{S}_n$:

$$\int_A X_{n+1} \; d\beta = \int_A X_n \; d\beta$$

Let us compute:

$$
\begin{aligned}
\int_A X_{n+1}\ d\beta &= \int_{A_1\times\cdots\times A_n} \left(\int_G (b_1\cdots b_n g)_*\nu(f)\ d\mu(g)\right) d\mu^{\otimes n}(b_1,\cdots,b_n)\\
&= \int_{A_1\times\cdots\times A_n} \left(\int_G g_*\nu(f\circ b_1\cdots b_n)\ d\mu(g)\right) d\mu^{\otimes n}(b_1,\cdots,b_n)\\
&= \int_{A_1\times\cdots\times A_n} \nu(f\circ b_1\cdots b_n)\ d\mu^{\otimes n}(b_1,\cdots,b_n)\\
&= \int_A X_n\ d\beta
\end{aligned}
$$

In other words, $(X_n)_n$ is a martingale. Doob's martingale convergence theorem ensures thus that for almost every $b \in B$, the sequence $(b_1\cdots b_n)_*\nu(f)$ converges. Now, let $(f_p)_p$ be a dense famille of function in $C_c^0(X)$. What has be done previously ensures that for almost every $b\in B$, the sequence $((b_1\cdots b_n)_*\nu(f_p))_n$ converges for any p. We denote by $\nu_b(f_p)$ this limit. Observe that:

$$|(b_1\cdots b_n)_*\nu(f_p)| \leq \|f_p\|_\infty$$

This means that if f_{n_p} converges to a function f, the sequence $((b_1\cdots b_n)_*\nu(f_{n_p}))_p$ converges by completeness of $\mathbb{R}$ and the limite does not depend on the choice of the limiting sequence. We denote this limit $\nu_b(f)$. Finally, we observe that $\nu_b(af+bg) = a\nu_b(f)+b\nu(g)$ and thus by Riesz representation theorem, let ν_b be the measure corresponding the the linear form $f\mapsto \nu_b(f)$. □

Proposition A.2. *Let μ be a measure supported on $GL_n(\mathbb{R})$ such that the support of the measure Γ_μ acts strongly irreducibly and proximaly on $\mathbb{R}^n$. Then there is a measurable map $J : B \to \mathbb{R}\mathrm{P}^{n-1}$ such that for β-almost every $b\in B$, the image of any limit point of $\|b_1\cdots b_n\|^{-1}\cdot b_1\cdots b_n$ is given by $J(b)$.*

Proof. To be added. We refer to [**?**] □

Proposition A.3. *Let $\varphi \in L_1(B,\beta)$. There is a conull subset $N\subset B$ such that if $b\in N$, then for β almost every $(g_0,g_2,\cdots)\in B$ we have:*

$$\frac{1}{N}\sum_{n=1}^{N}\varphi(g_{n-1}\cdots g_0\cdot b) \to \int_B \varphi\ d\beta$$

Proof. Recall that the bi infinite shift $S : G^{\mathbb{Z}} \to G^{\mathbb{Z}}; (\omega_n)_n \mapsto (\omega_{n+1})_n$ is ergodic for the measure $\mu^{\otimes\mathbb{Z}}$. Define $\iota : G^{\mathbb{Z}} \to B, (\omega_n)\mapsto(\omega_{-1},\omega_{-2},\cdots)$ and $\tilde\varphi = \varphi\circ\iota$. Notice that $\tilde\varphi\circ S^n(\omega) = \varphi(\omega_{n-1}\cdots\omega_0\cdot\iota(\omega))$ and $\iota_*\mu^{\otimes\mathbb{Z}} = \beta$. The Birkhoff ergodic theorem gives that for $\mu^{\mathbb{Z}}$-almost every ω:

$$\frac{1}{N}\sum_{n=1}^{N}\tilde\varphi\circ S^n(\omega) = \frac{1}{N}\sum_{n=1}^{N}\varphi(\omega_{n-1}\cdots\omega_0\cdot\iota(\omega)) \to \int_{G^{\mathbb{Z}}}\tilde\varphi\ d\mu^{\otimes\mathbb{Z}} = \int_B\varphi\ d\beta$$

To conclude, let $(\beta_b)_b$ be the conditionnal measures for the disintegration of $\mu^{\mathbb{Z}}$ along the fibers of ι and let $N = \{b \in B \mid \beta_b(\iota^{-1}(b)) = 1\}$. By construction this set has full measure and any $b \in N$ satisfies the conclusion of the statement.

$\square$

B Rohlin's theorem for disintegration of measures

Proposition B.1 (Rohlin)**.** *Let (X, μ) and (Y, ν) be two standard Borel spaces. Let $\pi : X \to Y$ be a measurable map such that $\pi_* \mu << \nu$. Then there is a collection of measures $(\mu_y)_{y \in Y}$ on X*

- *For ν-almost any $y \in Y$, $\mu_y(\pi^{-1}(y)) = 1$*
- *For any $f \in L^1(\mu)$, the map $y \mapsto \int_X f \ d\mu_y$ is in $L^1(\nu)$*
- *For any $f \in L^1(\mu)$, $\int_X f \ d\mu = \int_Y (\int_X f \ d\mu_y(x)) \ d\nu(y)$*

Moreover, if $(\mu'_y)_{y \in Y}$ is another such collection, then for ν-almost every $y \in Y$, we have $\mu_y = \mu'_y$.

A proof of this result can be found in [**?**]