Computation of equivalent ellipsoid coefficients from moment matrix

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August 28, 2019

Abstract *A short technical note to summarize the methods used for computing the equivalent ellipsoid with same principal axes than a given 3D region.*

Contents

1 Introduction

Region analysis of 3D images involves computation of geometric properties from 3D binary or label regions. In 2D, equivalent ellipses are commonly used for describing 2D regions (Russ, 2006; Gonzales and Woods, 2018; Burger and Burge, 2008). A similar approach can be used for 3D images, but the notations do not seem as standard as for the 2D case.

This document attempts to summarize the conventions and the methods I used for computing the parameters of 3D equivalent ellipsoid from 3D binary or label images. The general principle is to compute the second order centered moments, perform singular value decomposition of the resulting matrix, and identify the parameters of the 3D ellipsoid with same moments.

2 Notations

We consider euclidean spaces of dimension 3. Let X be the a 3D set representing the structure of interest.

2.1 Moments

In 3D, the **moments** m_{pqr} of order (p,q,r) are defined as:

$$m_{pqr} = \int\int\int I_X(x,y,z)x^p y^q z^r \cdot dx \cdot dy \cdot dz \tag{1}$$

where $I_X(x,y,z)$ is the indicator function of the set X, taking value 1 if the specified point is within the set X, and 0 otherwise. The moment m_{000} corresponds to the **volume** of the structure. The **centered moments** are expressed as:

$$\mu_{pqr} = \int\int\int I_X(x,y,z)\,(x-x_c)^p\,(y-y_c)^q\,(z-z_c)^r \cdot dx \cdot dy \cdot dz \tag{2}$$

where $(x_c,y_c,z_c) = (\frac{m_{100}}{m_{000}}, \frac{m_{010}}{m_{000}}, \frac{m_{001}}{m_{000}})$ is the **centroid** of X.

The matrix of second order moments is the symmetric 3×3 matrix that combines all the second order centered moments:

$$\boldsymbol{M} = \begin{bmatrix} \mu_{200} & \mu_{110} & \mu_{101} \\ \mu_{110} & \mu_{020} & \mu_{011} \\ \mu_{101} & \mu_{011} & \mu_{002} \end{bmatrix} \tag{3}$$

Note that this is NOT the same matrix as the inertia matrix (or inertia tensor).

2.2 Eigen values decomposition

The matrix $\boldsymbol{M}$ can be factorized using eigen value decomposition:

$$\boldsymbol{M} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^t \tag{4}$$

where

- $\boldsymbol{Q}$ is a 3×3 orthogonal matrix whose i-th column is the eigenvector q_i of $\boldsymbol{M}$
- $\boldsymbol{\Lambda}$ is a diagonal 3×3 matrix whose diagonal elements are the corresponding eigen values, $\Lambda_{ii} = \lambda_i$.

Eigen values are ordered in decreasing order. The matrix $\boldsymbol{Q}$ can be used to extract the orientation of the principal axes of the structure, whereas the eigen values λ_i can be related to the dimensions along these axes.

2.3 Ellipsoid

It is convenient to consider the equivalent ellipsoid with the same orientation and the same eigen values as the structure of interest. The three radiusses of the ellipsoid are noted a, b, c, with $a > b > c$. The orientation of the ellipsoid can be represented using three Euler angles.

The determination of the coefficients of the equivalent ellipsoid requires to compute the radiusses from the eigen values λ_i, and to find Euler angles from the rotation matrix $\boldsymbol{Q}$.

3 Computation of radiusses

Let us suppose that the ellipsoid is centered and aligned with the main axes. From centering, we have $m_{pqr} = \mu_{pqr}$. For integration, it is more convenient to use spherical coordinates (ρ, θ, φ), corresponding to the radius, inclination with the vertical, and azimuth (these notations are different from the ones used for Euler angles). This corresponds to:

$$\begin{aligned} x &= \rho a \cos\varphi \sin\theta \\ y &= \rho b \sin\varphi \sin\theta \\ z &= \rho c \cos\theta \end{aligned} \tag{5}$$

with $\rho \in [0; +\infty[$, $\varphi \in [0; 2\pi]$ and $\theta \in [0; \pi]$.

The Jacobian matrix of the transformation is as follow (see example 3 in "Jacobian matrix and determinant" on Wikipedia[1]):

$$J_\Phi(x, y, z) = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} a\cos\varphi\sin\theta & \rho a \cos\varphi\cos\theta & -\rho a \sin\varphi \sin\theta \\ b\sin\varphi\sin\theta & \rho b \sin\varphi\cos\theta & \rho b\cos\varphi\sin\theta \\ c\cos\theta & -\rho c \sin\theta & 0 \end{bmatrix} \tag{6}$$

The determinant equals $\rho^2 \sin\theta$ if $a = b = c = 1$. Introducing a, b and c gives a determinant equal to $abc\rho^2\sin\theta$. Then the moment integral is given by:

$$\mu_{pqr} = abc \int_0^{2\pi} \int_0^{\pi} \int_0^1 (\rho\cos\varphi\sin\theta)^p (\rho\sin\varphi\sin\theta)^q (\rho\cos\theta)^r \cdot \rho^2 \sin\theta \cdot d\rho \cdot d\theta \cdot d\varphi \tag{7}$$

[1] https://en.wikipedia.org/wiki/Jacobian_matrix_and_determinant#Example_3:_spherical-Cartesian_transformation

3.1 Volume moment

It is easy to check that the moment m_{000} corresponds to the volume of the ellipsoid (equal to $\frac{4\pi}{3}abc$):

$$\begin{aligned} m_{000} &= abc\int_0^{2\pi}\int_0^{\pi}\int_0^{1}\rho^2\sin\theta\cdot d\rho\cdot d\theta\cdot d\varphi \\ &= \frac{abc}{3}\int_0^{2\pi}\int_0^{\pi}\sin\theta\cdot d\theta\cdot d\varphi = \frac{abc}{3}\int_0^{2\pi}(-\cos\pi+\cos 0)\cdot d\varphi \\ &= \frac{2abc}{3}\int_0^{2\pi}d\varphi = \frac{4\pi}{3}abc \end{aligned} \tag{8}$$

3.2 Moment m200

When considering centered ellipsoid aligned with principal axes, the eigen values λ_i simply correspond to the moments μ_{200}, μ_{020} and μ_{002}. Let us focus on the moment μ_{200}:

$$\begin{aligned} \mu_{200} &= abc\int_0^{2\pi}\int_0^{\pi}\int_0^{1}(a\rho\cos\varphi\sin\theta)^2\rho^2\sin\theta\cdot d\rho\cdot d\theta\cdot d\varphi \\ &= a^3bc\int_0^{2\pi}\int_0^{\pi}\int_0^{1}\rho^4 d\rho\cdot\cos^2\varphi\sin^3\theta\cdot d\theta\cdot d\varphi \\ &= \frac{a^3bc}{5}\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\cdot d\theta\cdot\cos^2\varphi\cdot d\varphi \end{aligned} \tag{9}$$

The integral over θ can be developed using the linearisation of $\sin^3\theta$:

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta) \tag{10}$$

Incorporation into the integral gives:

$$\begin{aligned} \int_0^{\pi}\sin^3\theta d\theta &= \int_0^{\pi}\left(\frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta)\right)d\theta \\ &= -\frac{3}{4}(\cos\pi - \cos 0) - \frac{-1}{12}(\cos 3\pi - \cos 0) \\ &= \frac{3}{2} - \frac{1}{6} = \frac{4}{3} \end{aligned} \tag{11}$$

Coming back to moment integral:

$$\mu_{200} = \frac{a^3bc}{5}\frac{4}{3}\int_0^{2\pi}\cos^2\varphi \cdot d\varphi == \frac{a^3bc}{5}\frac{4}{3}\int_0^{2\pi}\frac{1}{2}+\frac{1}{2}\cos(2\varphi)\,d\varphi$$
$$= \frac{a^3bc}{5}\frac{4}{3}\frac{1}{2}(2\pi - 0) = \frac{4\pi}{3}\frac{a^3bc}{5} = \frac{a^2}{5}m_{000} \tag{12}$$

Then, the central moments are expressed as:

$$\begin{aligned}\lambda_1 &= \mu_{200} = \frac{a^2}{5}m_{000}\\ \lambda_2 &= \mu_{020} = \frac{b^2}{5}m_{000}\\ \lambda_3 &= \mu_{002} = \frac{c^2}{5}m_{000}\end{aligned} \tag{13}$$

We therefore have $\lambda_i = \frac{r_i^2}{5}m_{000}$, then $r_i = \sqrt{5\lambda_i/m_{000}}$.

4 Computation of angles

This section largely follows the document of Greg Slabaugh for obtaining Euler angles from a rotation matrix (Slabaugh, 1999). We follow his notations. Note that in this section, ψ, φ and θ correspond to angle values of 3D rotations.

4.1 Rotation matrices

A rotation of ψ radians about the x-axis is noted by:

$$\boldsymbol{R}_x(\psi) = \begin{bmatrix} 1 & 0 & 0\\ 0 & \cos\psi & -\sin\psi\\ 0 & \sin\psi & \cos\psi \end{bmatrix} \tag{14}$$

Similarly, a rotation of θ radians about the y-axis is noted by:

$$\boldsymbol{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{15}$$

Finally, a rotation of φ radians about the z-axis is noted by:

$$\boldsymbol{R}_z(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

A general rotation matrix may have the following form:

$$\boldsymbol{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \tag{17}$$

Such a matrix can be represented by a sequence of three successive rotations around the main axes. As matrix multiplication does not commute, the order of the axes one rotates about will affect the result. I follow the choice of G. Slabaugh and consider rotation first about the x-axis, then about the y-axis, and finally about the z-axis. Then, the angles ψ, θ and φ correspond to Euler angles. Angle ψ corresponds to the "roll", angle θ to the "pitch", and angle φ to the "yaw".

The global rotation matrix can be written as follow:

$$\begin{aligned} \boldsymbol{R} &= \boldsymbol{R}_z(\varphi)\boldsymbol{R}_y(\theta)\boldsymbol{R}_x(\psi) \\ &= \begin{bmatrix} \cos\theta\cos\varphi & \sin\psi\sin\theta\cos\varphi - \cos\psi\sin\varphi & \cos\psi\sin\theta\cos\varphi + \sin\psi\sin\varphi \\ \cos\theta\sin\varphi & \sin\psi\sin\theta\sin\varphi + \cos\psi\cos\varphi & \cos\psi\sin\theta\sin\varphi - \sin\psi\cos\varphi \\ -\sin\theta & \sin\psi\cos\theta & \cos\psi\cos\theta \end{bmatrix} \end{aligned} \tag{18}$$

The problem is now to identify the three Euler angles ψ, θ and φ from the matrix coefficients. This results in nine equations.

4.2 Elevation

We start considering the elevation θ, or "pitch". From element R_{31} of the matrix, one finds

$$R_{31} = -\sin\theta$$

One identifies θ with the following

$$\theta = -\sin^{-1}(R_{31}) \tag{19}$$

by keeping the value of $\theta \in [-\frac{\pi}{2}; \frac{\pi}{2}]$ to remove ambiguity. The case $R_{31} = \pm 1$ will be considered later. In practice, we replace the $\sin^{-1}$ function by atan2 function for better numerical stability.

Compared to the document of Slabaugh, keeping $\theta \in [-\frac{\pi}{2}; \frac{\pi}{2}]$ removes the sign ambiguity of $\cos\theta$, and therefore simplifies the remaining computations.

4.3 Roll

The possible values of the roll ψ around the x-axis can be found by

$$\frac{R_{32}}{R_{33}} = \tan\psi$$

This lead to the value of ψ as

$$\psi = \operatorname{atan2}(R_{32}, R_{33}) \tag{20}$$

where $\operatorname{atan2}(y, x)$ is the arc tangent function of the two variables y and x that extends the atan function to the all four quadrants. The function atan2 is available in most programming languages.

4.4 Azimut

The azimut φ, or "yaw", can be obtained from

$$\begin{aligned} \frac{R_{21}}{R_{11}} &= \tan\varphi \\ \varphi &= \operatorname{atan2}(R_{21}, R_{11}) \end{aligned} \tag{21}$$

4.5 Special case of $\cos\theta = 0$

When the matrix element $R_{31} = \pm 1$, this corresponds to an elevation angle of either $+\pi/2$ or $-\pi/2$: The main axis of the ellipsoid is aligned with the z-axis.

If $\theta = \pi/2$ then

$$\begin{aligned} R_{12} &= \sin\psi\cos\varphi - \cos\psi\sin\varphi = \sin(\psi - \varphi) \\ R_{13} &= \cos\psi\cos\varphi - \sin\psi\sin\varphi = \cos(\psi - \varphi) \\ R_{22} &= \sin\psi\sin\varphi - \cos\psi\cos\varphi = \cos(\psi - \varphi) = R_{13} \\ R_{23} &= \cos\psi\sin\varphi - \sin\psi\cos\varphi = -\sin(\psi - \varphi) = -R_{12} \end{aligned} \tag{22}$$

The value of ψ may determined by setting the value φ for the azimut to 0 arbitrarily.

$$\psi = \operatorname{atan2}(-R_{23}, R_{22}) \tag{23}$$

5 Implementations

The method described in the document has been implemented as a Matlab function: "imEquivalentEllipsoid". The results are similar to those obtained with the "regionprops3" function in Matlab, with a difference in the constant used for scalingthe principal axes (Matlab uses a factor of 2, I use a factor of $\sqrt{5}$).

References

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