Exponentially weighted forecasts

FORECASTING IN R

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Simple exponential smoothing

Forecasting Notation:

$$\hat{y}_{t+h|t} = \text{ point forecast of } \hat{y}_{t+h} \text{ given data } y_1, ..., y_t$$

Forecast Equation:

$$\hat{y}_{t+h|t} = \alpha y_t + \alpha(1-\alpha)y_{t-1} + \alpha(1-\alpha)^2 y_{t-2} + ...$$

$$where\ 0 \le \alpha \le 1$$

Observation	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
y_t	0.2	0.4	0.6	0.8
y_{t-1}	0.16	0.24	0.24	0.16
y_{t-2}	0.128	0.144	0.096	0.032
y_{t-3}	0.1024	0.0864	0.0384	0.0064
y_{t-4}	$(0.2)(0.8)^4$	$(0.4)(0.6)^4$	$(0.6)(0.4)^4$	$(0.8)(0.2)^4$
y_{t-5}	$(0.2)(0.8)^5$	$(0.4)(0.6)^5$	$(0.6)(0.4)^5$	$(0.8)(0.2)^5$

Simple exponential smoothing

Component form	
Forecast equation	$\hat{y}_{t+h\|t} = \ell_t$
Smoothing equation	$\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$

- ℓ_t is the level (or the smoothed value) of the series at time t
- We choose α and ℓ_0 by minimizing SSE:

$$SSE = \sum_{t=1}^{T} (y_t - \hat{y}_{t|t-1})^2$$

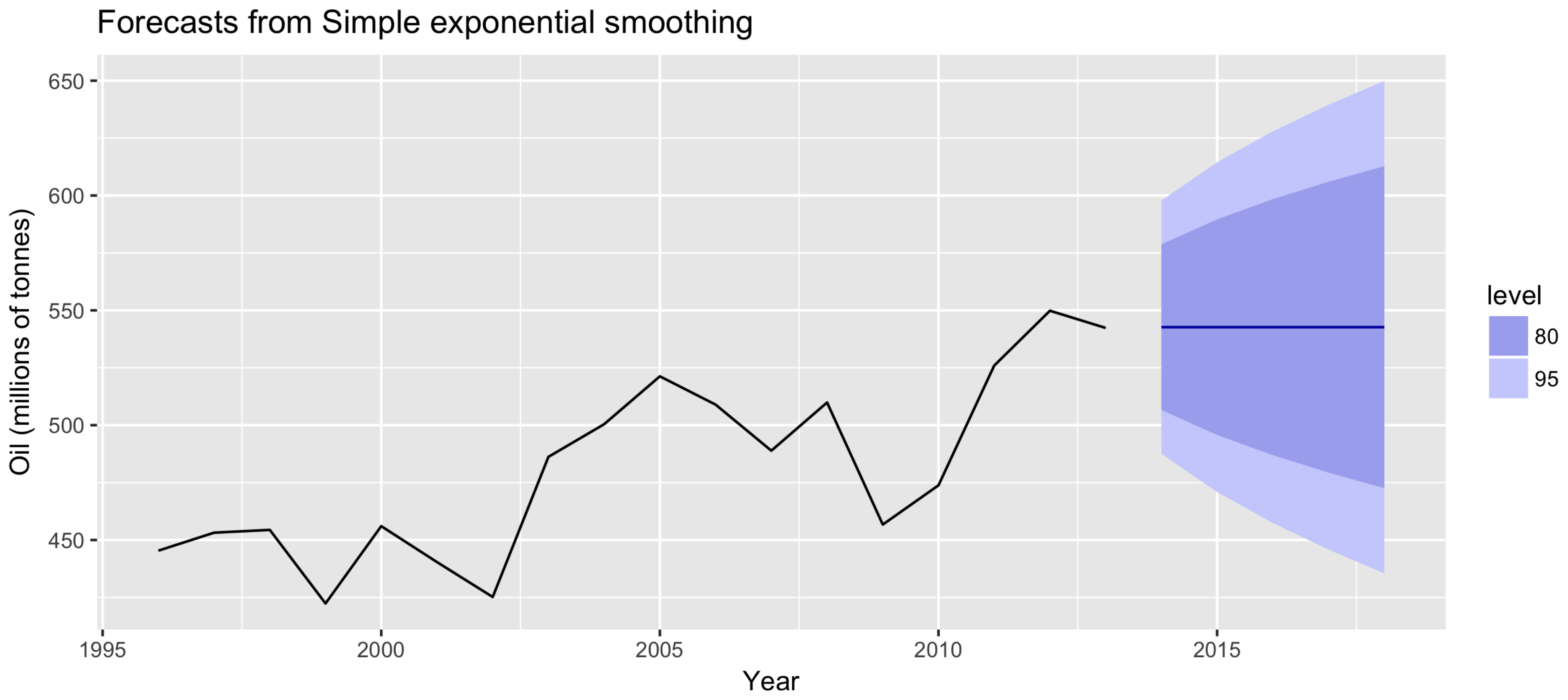
Example: oil production

```
oildata <- window(oil, start = 1996)    # Oil Data
fc <- ses(oildata, h = 5)               # Simple Exponential Smoothing
summary(fc)
```

```
Forecast method: Simple exponential smoothing
Model Information:
Simple exponential smoothing
Call:
 ses(y = oildata, h = 5)
  Smoothing parameters:
    alpha = 0.8339
  Initial states:
    l = 446.5759
  sigma:  28.12
*** Truncated due to space
```

Example: oil production

```
autoplot(fc) +
  ylab("Oil (millions of tonnes)") + xlab("Year")
```



Let's practice!

FORECASTING IN R

Exponential smoothing methods with trend

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Holt's linear trend

Simple exponential smoothing	
Forecast	$\hat{y}_{t+h\|t} = \ell_t$
Level	$\ell_t = \alpha y_t + (1-\alpha)\ell_{t-1}$

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Holt's linear trend	
Forecast	$\hat{y}_{t+h\|t} = \ell_t + hb_t$
Level	$\ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$
Trend	$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$

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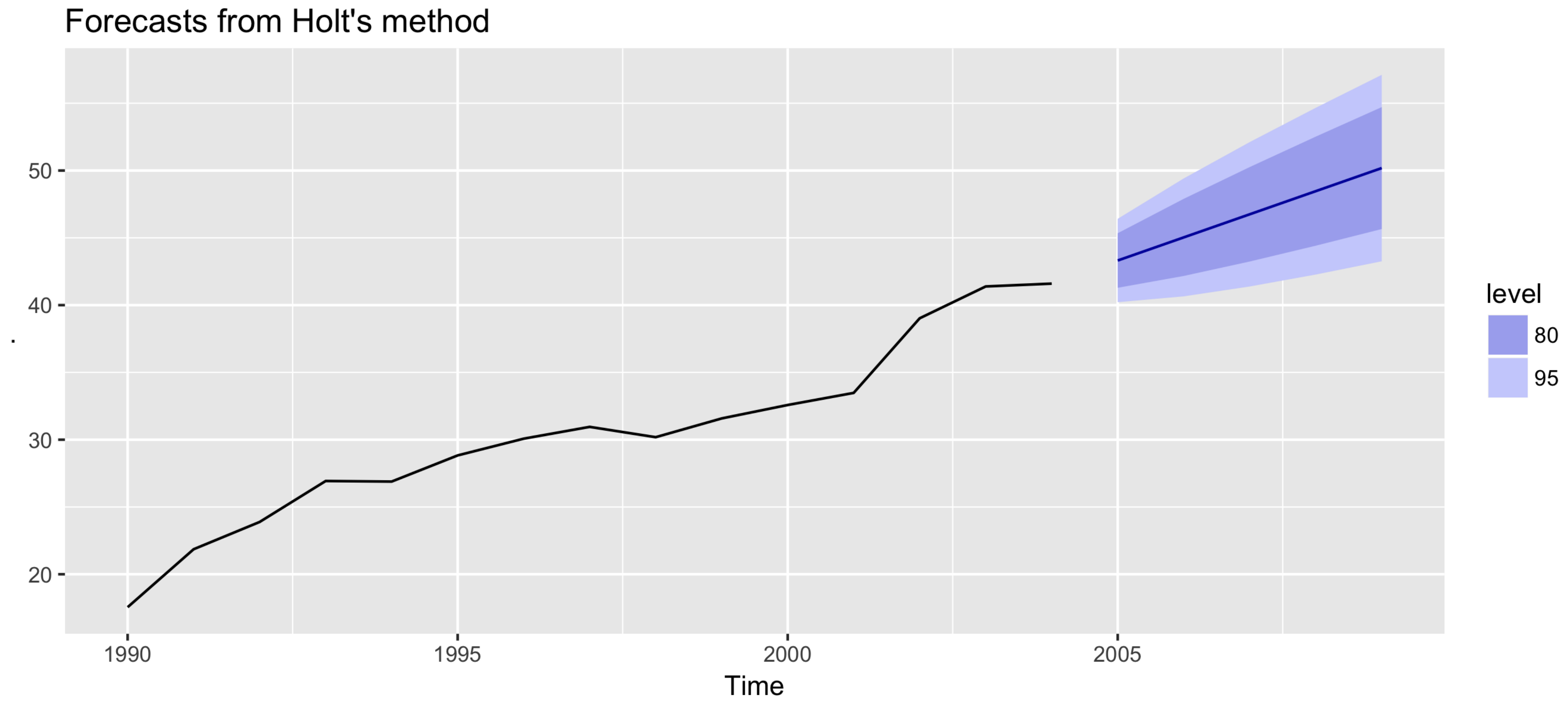
Trend	$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$

- Two smoothing parameters α and $\beta^* (0 \leq \alpha, \beta^* \leq 1)$
- Choose $\alpha, \beta^*, \ell_0, b_0$ to minimize SSE

Holt's method in R

```
airpassengers %>% holt(h = 5) %>% autoplot
```



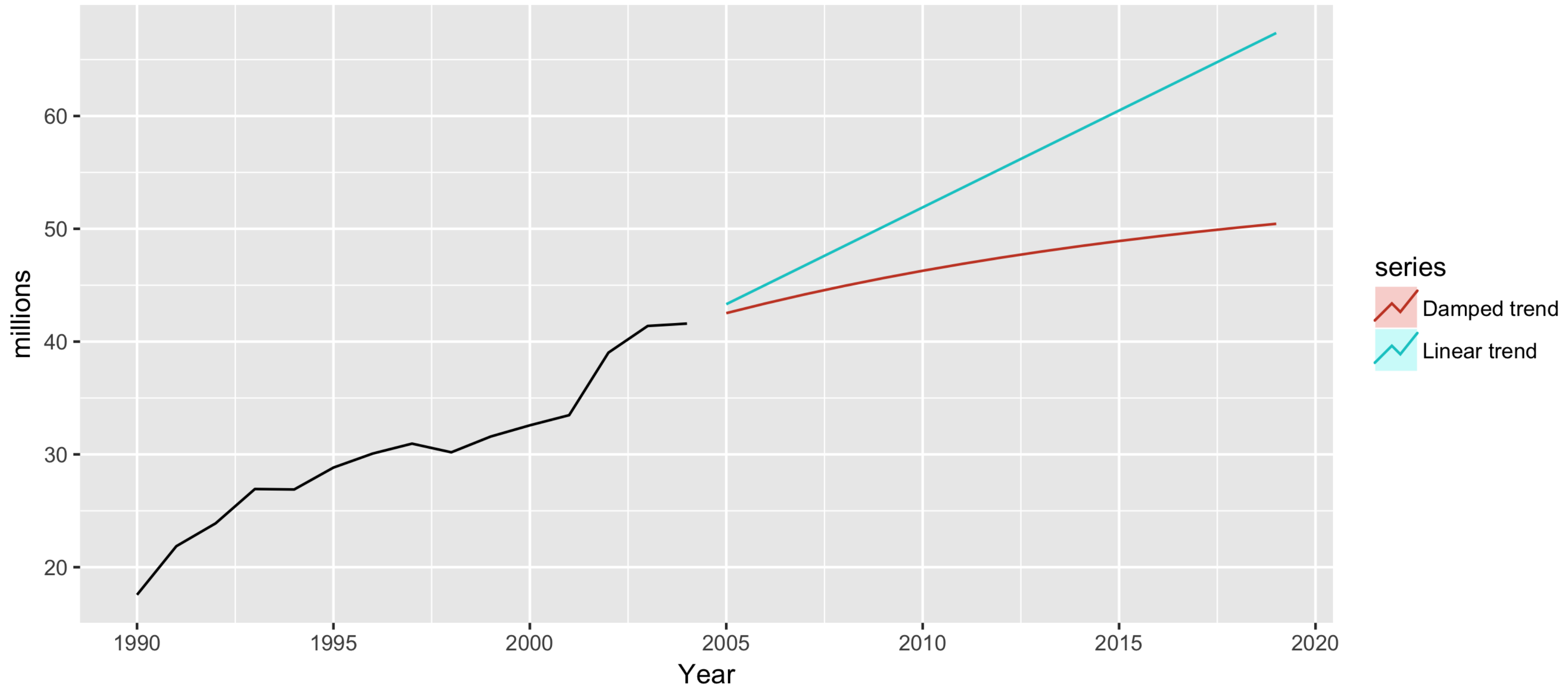
Damped trend method

Component form
$\hat{y}_{t+h\|t} = \ell_t + (\phi + \phi^2 + \cdots + \phi^h) b_t$
$\ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + \phi b_{t-1})$
$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)\phi b_{t-1}$

- Damping parameter $0 < \phi < 1$
- If $\phi = 1$, identical to Holt's linear trend
- Short-run forecasts trended, long-run forecasts constant

Example: Air passengers

```
fc1 <- holt(airpassengers, h = 15, PI = FALSE)
fc2 <- holt(airpassengers, damped = TRUE, h = 15, PI = FALSE)
autoplot(airpassengers) + xlab("Year") + ylab("millions") +
  autolayer(fc1, series="Linear trend") +
  autolayer(fc2, series="Damped trend")
```



Let's practice!

FORECASTING IN R

Exponential smoothing methods with trend and seasonality

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Holt-Winters' additive method

Holt-Winters additive method
$\hat{y}_{t+h\|t} = \ell_t + hb_t + s_{t-m+h_m^+}$
$\ell_t = \alpha(y_t - s_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1})$
$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$
$s_t = \gamma(y_t - \ell_{t-1} - b_{t-1}) + (1-\gamma)s_{t-m}$

- $s_{t-m+h_m^+}$ = seasonal component from final year of data
- Smoothing parameters:
  $0 <\leq \alpha \leq 1,\ 0 \leq \beta^* \leq 1,\ 0 \leq \gamma \leq 1 - \alpha$
- m = period of seasonality (e.g. m = 4 for quarterly data)
- seasonal component averages **zero**

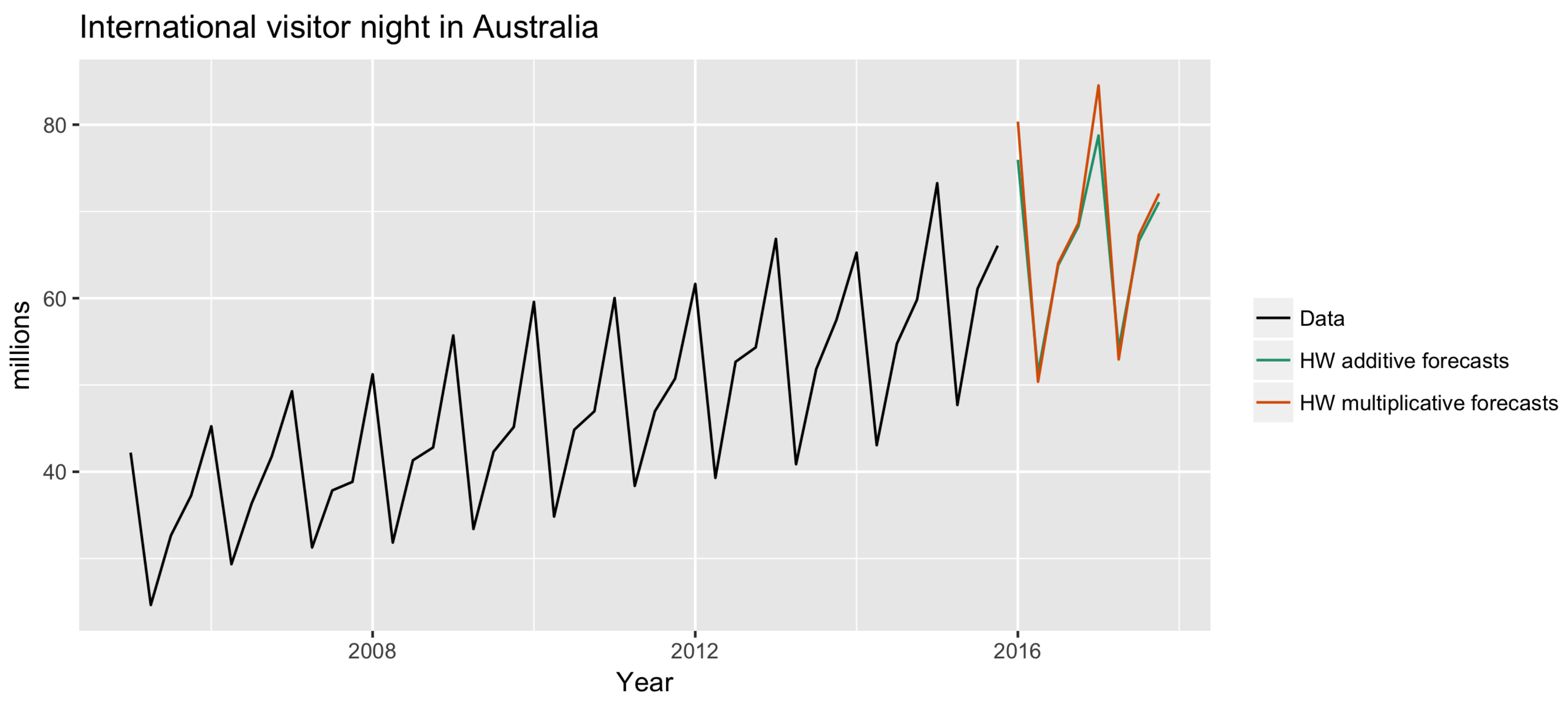
Holt-Winters' multiplicative method

Holt-Winters multiplicative method
$\hat{y}_{t+h\|t} = (\ell_t + hb_t)s_{t-m+h_m^+}$
$\ell_t = \alpha \dfrac{y_t}{s_{t-m}} + (1-\alpha)(\ell_{t-1} + b_{t-1})$
$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$
$s_t = \gamma \dfrac{y_t}{(\ell_{t-1} + b_{t-1})} + (1-\gamma)s_{t-m}$

- $s_{t-m+h_m^+}$ = seasonal component from final year of data
- Smoothing parameters:
  $0 << \alpha \leq 1,\ 0 \leq \beta^* \leq 1,\ 0 \leq \gamma \leq 1 - \alpha$
- m = period of seasonality (e.g. m = 4 for quarterly data)
- seasonal component averages **one**

Example: Visitor Nights

```
aust <- window(austourists, start = 2005)
fc1  <- hw(aust, seasonal = "additive")
fc2  <- hw(aust, seasonal = "multiplicative")
```



Taxonomy of exponential smoothing methods

	Seasonal Component		
Trend Component	N (None)	A (Additive)	M (Multiplicative)
N (None)	(N, N)	(N, A)	(N, M)
A (Additive)	(A, N)	(A, A)	(A, M)
A_d (Additive Damped)	(A_d, N)	(A_d, N)	(A_d, N)

Taxonomy of exponential smoothing methods

	Seasonal Component		
Trend Component	N (None)	A (Additive)	M (Multiplicative)
N (None)	(N, N)	(N, A)	(N, M)
A (Additive)	(A, N)	(A, A)	(A, M)
A_d (Additive Damped)	(A_d, N)	(A_d, N)	(A_d, N)

(N, N)	Simple exponential smoothing	ses()
(A, N)	Holt's linear method	holt()
(A_d, N)	Additive damped trend method	hw()
(A, A)	Additive Holt-Winter's method	hw()
(A, M)	Multiplicative Holt-Winter's method	hw()
(A_d, M)	Damped multiplicative Holt-Winter's	hw()

Let's practice!

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State space models for exponential smoothing

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Innovations state space models

- Each exponential smoothing method can be written as an **"innovations state space model"**
    - Trend = {N, A, A_d}

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  - Trend = $\{N, A, A_d\}$
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3 x 3= 9 possible exponential smoothing methods

Innovations state space models

- Each exponential smoothing method can be written as an **"innovations state space model"**

  - Trend = {N, A, A_d}
  - Seasonal = {N, A, M}
  - Error = {A, M}

3 x 3= 9 possible exponential smoothing methods

Innovations state space models

- Each exponential smoothing method can be written as an "innovations state space model"

    - Trend = {N, A, A_d}
    - Seasonal = {N, A, M}
    - Error = {A, M}

3 x 3= 9 possible exponential smoothing methods

9 x 2= 18 possible state space models

- ETS models: Error, Trend, Seasonal

ETS models

- Parameters: estimated using the **"likelihood"**, the probability of the data arising from the specified model
- For models with additive errors, this is **equivalent to minimizing SSE**
- Choose the best model by minimizing a corrected version of Akaike's Information Criterion (AIC_c)

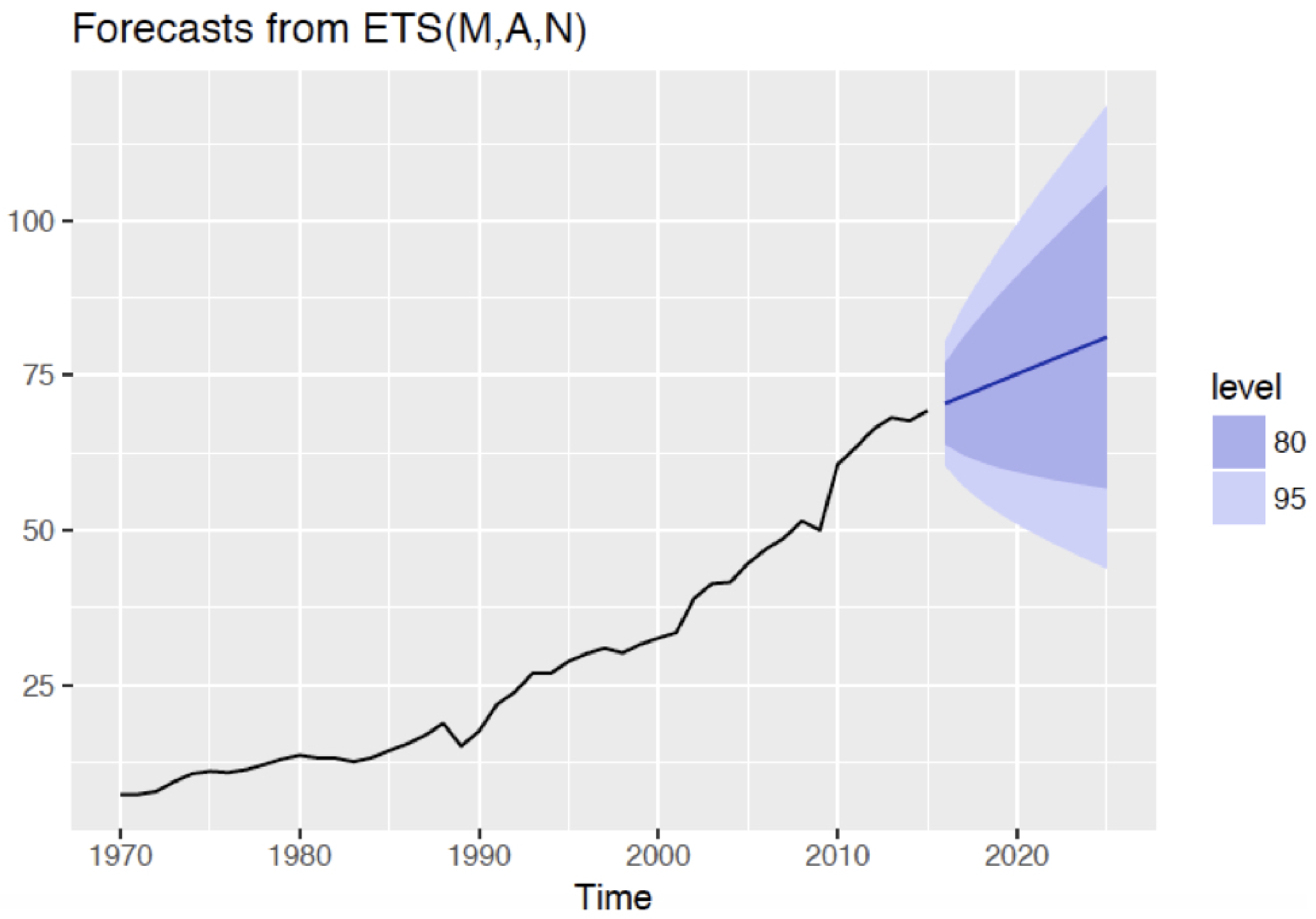
Example: Australian air traffic

```
ets(ausair)
```

```
ETS(M,A,N)
Call:
 ets(y = ausair)
  Smoothing parameters:
    alpha = 0.9999
    beta  = 0.0176
  Initial states:
    l = 6.5242
    b = 0.7584
  sigma:  0.0729
     AIC     AICc      BIC
234.5273 236.0273 243.6705
```

Example: Australian air traffic

```
ausair %>% ets() %>% forecast() %>% autoplot()
```



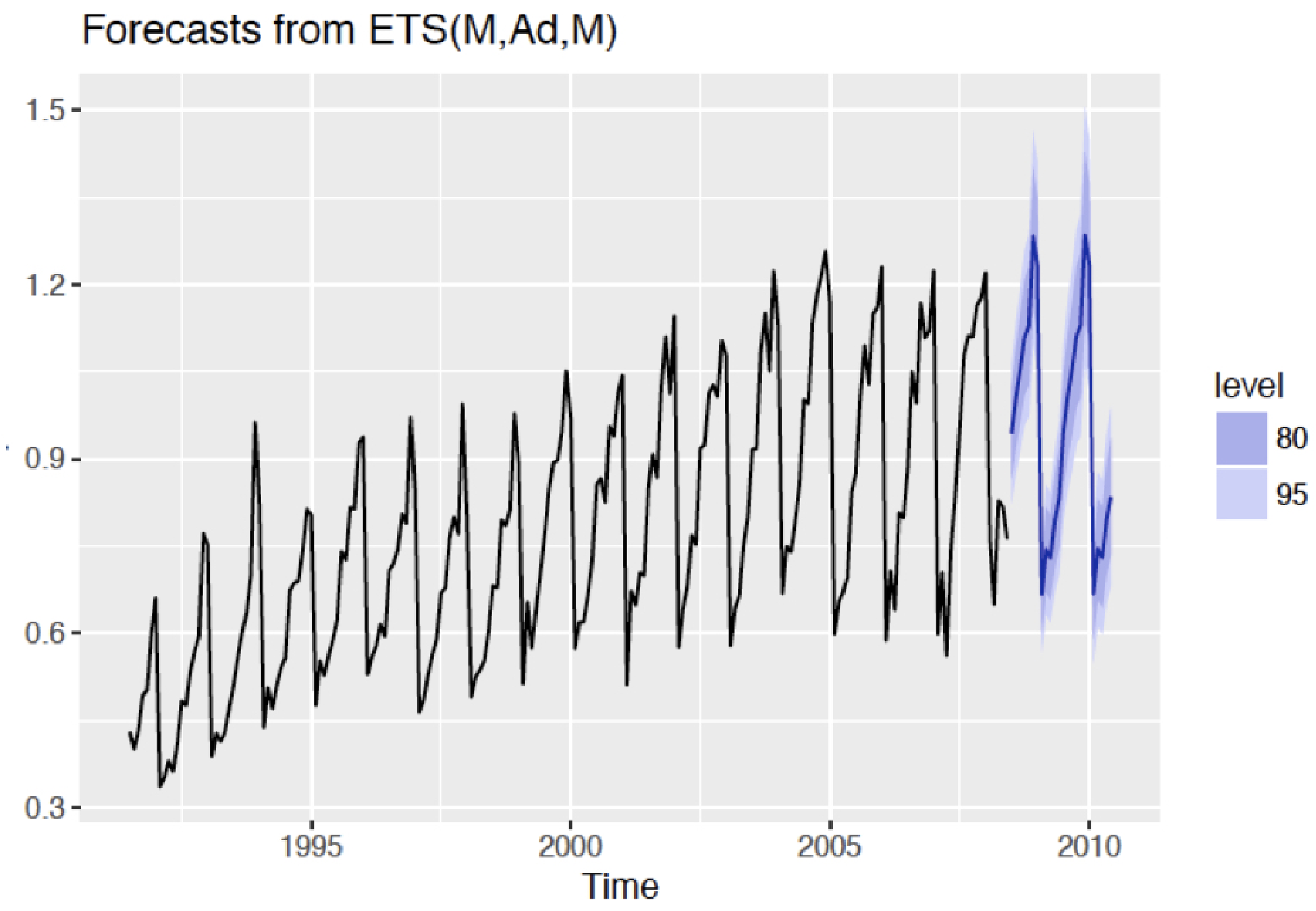
Example: Monthly cortecosteroid drug sales

```
ets(h02)
```

```
ETS(M,Ad,M)
Call:
 ets(y = h02)
  Smoothing parameters:
    alpha = 0.2173
    beta  = 2e-04
    gamma = 1e-04
    phi   = 0.9756
  Initial states:
    l = 0.3996
    b = 0.0098
    s=0.8675 0.8259 0.7591 0.7748 0.6945 1.2838
           1.3366 1.1753 1.1545 1.0968 1.0482 0.983
  sigma:  0.0647
       AIC       AICc        BIC
-123.21905 -119.52175  -63.49289
```

Example: Monthly cortecosteroid drug sales

```
h02 %>% ets() %>% forecast() %>% autoplot()
```



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