6 - Traiettorie

Friday, 24 June 2022 23:03

#1 - SISTEMA

$\dot{x} = Ax \qquad A = \begin{vmatrix} -1 & 2 \\ p & -2 \end{vmatrix} \quad p \in \mathbb{R}$

EQUILIBRIO

$\det(A) = +2 - 2p = 2(1-p)$

$\underline{p \neq 1} \rightarrow \det A \neq 0 \rightarrow \exists!\ \bar{x} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

$\underline{p = 1} \rightarrow \det A = 0$ SOSTITUENDO OTTENGO

$\begin{cases} \dot{x}_1 = -x_1 + 2x_2 \\ \dot{x}_2 = x_1 - 2x_2 \end{cases} \qquad \bar{x}_1 = 2\bar{x}_2$

$\exists \infty$ EQUILIBRI CON $\bar{x} = \begin{vmatrix} 2\alpha \\ \alpha \end{vmatrix} \quad \alpha \in \mathbb{R}$

STABILITÀ

STUDIO GLI AUTOVALORI DI A

$\det(\lambda I - A) = \det \begin{vmatrix} \lambda+1 & -2 \\ -p & \lambda+2 \end{vmatrix} = \Delta_A(\lambda)$

$\Delta_A(\lambda) = (\lambda+1)(\lambda+2) - 2p = \lambda^2 + 3\lambda + (2-2p)$

$\Delta_A(\lambda) = 0 \rightarrow \lambda_{1,2} = \dfrac{-3 \pm \sqrt{9 - 4(2-2p)}}{2} = \dfrac{-3 \pm \sqrt{1+8p}}{2}$

$\underline{\text{CASO 1}} \quad 1 + 8p < 0 \quad p < -\frac{1}{8}$

$\mathrm{Re}(\lambda_i) < 0\ \forall i \rightarrow \underline{AS}$

$\lambda_D = \max(\mathrm{Re}\,\lambda_i) = -\frac{3}{2}$

$T_D = -\frac{1}{\lambda_D} = \frac{2}{3} \qquad T_R = 5T_D = \frac{10}{3}$

$\underline{\text{CASO 2}} \quad 1 + 8p = 0 \quad p = -\frac{1}{8}$

$\{\lambda\}_A = \{-\frac{3}{2}, -\frac{3}{2}\} \quad \mathrm{Re}(\lambda_i) < 0\ \forall i \rightarrow \underline{AS}$

$T_D = -\frac{1}{\lambda_D} = \frac{2}{3} \qquad T_R = 5T_D = \frac{10}{3}$

$\underline{\text{CASO 3}} \quad 1 + 8p > 0 \quad p > \frac{1}{8}$

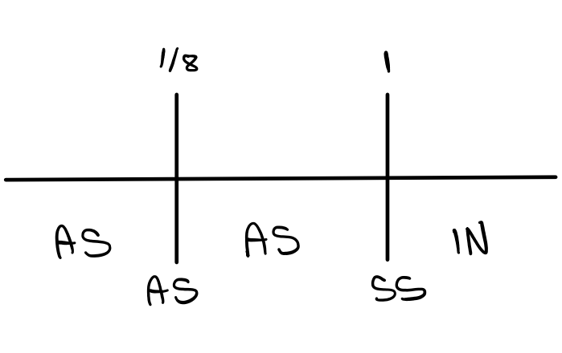
$\{\lambda\}_A = \{\underset{?}{-\frac{3}{2} + \frac{1}{2}\sqrt{1+8p}},\ \underset{\text{SEMPRE NEGATIVO}}{-\frac{3}{2} - \frac{1}{2}\sqrt{1-8p}}\}$

CHE SEGNO HA IL PRIMO AUTOVALORE?

$-3 + \sqrt{1+8p} > 0 \qquad \sqrt{1+8p} > 3 \qquad 1 + 8p > 9$

SE $\underline{p > 1}$ IL PRIMO AUTOVALORE È POSITIVO → INSTABILE

SE $\underline{p = 1}$ UN AUTOVALORE NULLO E UNO NEGATIVO → SS



$p < 1$ STABILE ASINTOTICO
$p = 1$ STABILE SEMPLICE
$p > 1$ INSTABILE

TEMPO PER ANDARE A REGIME

DEFINITO SOLO IN AS

$\underline{\text{CASO 1}} \quad p \leq \frac{1}{8}$

$\{\lambda\}_A = \{\frac{-3 \pm \sqrt{1+8p}}{2}\}$

$\sqrt{1+8p}$ COMPLESSA O AL PIÙ NULLA ($p = 1/8$)

ENTRAMBI GLI AUTOVALORI HANNO $\mathrm{Re}(\lambda_i) = -\frac{3}{2}$

$T_R = 5T_D = 5\,\frac{-1}{-3/2} = +\frac{10}{3}$

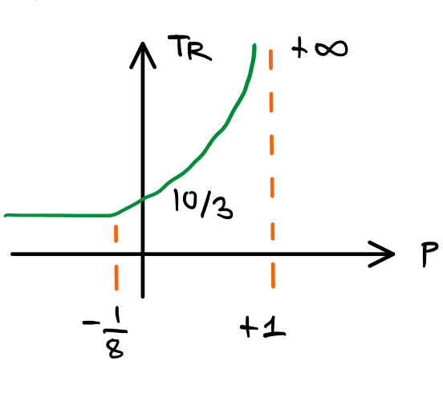
$\underline{\text{CASO 2}} \quad p \in (\frac{1}{8}, 1)$

$\{\lambda\}_A = \{\frac{-3 + \sqrt{1+8p}}{2}, \frac{-3 - \sqrt{1+8p}}{2}\}$

$\sqrt{1-8p}$ REALE POSITIVA $\qquad \lambda_D = \frac{-3 + \sqrt{1+8p}}{2}$

$T_R = 5T_D = 5\,\frac{-1}{\lambda_D} = \frac{10}{-3 + \sqrt{1+8p}}$

GRAFICO DEL TR



$T_R \xrightarrow{p \to 1} \infty$

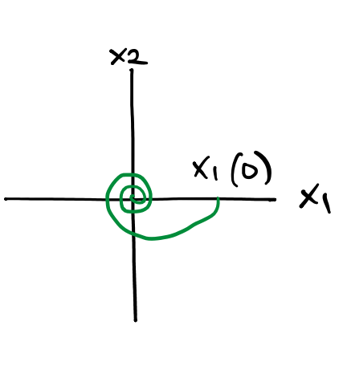
ESISTONO INFINITE OSCILLAZIONI ?

$\exists \infty$ OSCILLAZIONI $\rightarrow \exists \lambda \in \mathbb{C} \rightarrow p < -\frac{1}{8}$

TRAIETTORIA $\underline{p = -1}$

$\left.\begin{array}{l}\lambda_{1,2} \in \mathbb{C}\ + \\ \text{ASINTOTICA STABILITÀ}\end{array}\right\} \rightarrow$ FUOCO STABILE

$\begin{cases} \dot{x}_1 = -x_1 + 2x_2 \\ \dot{x}_2 = -x_1 - 2x_2 \end{cases}$



$x(0) = \begin{vmatrix} x_1(0) \\ 0 \end{vmatrix}$

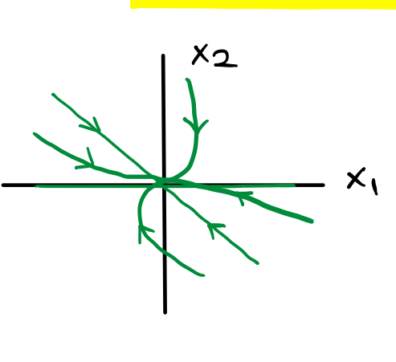
$\dot{x}(0) = \begin{vmatrix} -x_1(0) \\ -x_1(0) \end{vmatrix}$

TRAIETTORIA $p = 0$

$\frac{-3 \pm \sqrt{1+8p}}{2} = \frac{-3 \pm 1}{2} = \begin{array}{l} \rightarrow -2 \\ \rightarrow -1 \end{array} \qquad \{\lambda\}_A = \{-1, -2\}$

$\left.\begin{array}{l}\lambda_{1,2} \in \mathbb{R}\ + \\ \text{ASINTOTICA STABILITÀ}\end{array}\right\} \rightarrow$ NODO STABILE

$\begin{cases} \dot{x}_1 = -x_1 + 2x_2 \\ \dot{x}_2 = -2x_2 \end{cases}$



$\underline{\lambda_1 = -1} \quad Aw = \lambda_1 w \quad \begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \end{vmatrix} = -1 \begin{vmatrix} w_1 \\ w_2 \end{vmatrix}$

$\begin{cases} -w_1 + 2w_1 = -w_1 \\ -2w_2 = -w_2 \end{cases} \rightarrow w_2 = 0 \quad \forall w_1$

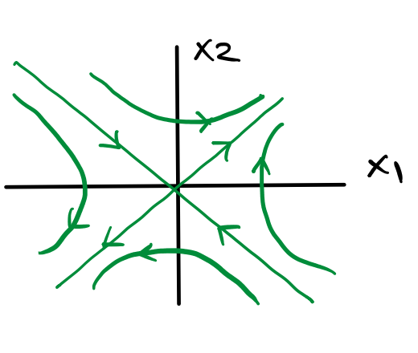
$\underline{\lambda_2 = -2} \quad Aw = \lambda_2 w \quad \begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \end{vmatrix} = -2 \begin{vmatrix} w_1 \\ w_2 \end{vmatrix}$

$\begin{cases} -w_1 + 2w_2 = -2w_1 \\ -2w_2 = -2w_2 \end{cases} \rightarrow w_1 = -2w_2$

TRAIETTORIA $p = +3$

$\lambda_{1,2} = \dfrac{-3 \pm \sqrt{1+8p}}{2} \begin{array}{l} \rightarrow -4 \\ \rightarrow +1 \end{array}$ SELLA INSTABILITÀ

$\begin{cases} \dot{x}_1 = -x_1 + 2x_2 \\ \dot{x}_2 = +3x_1 - 2x_2 \end{cases}$



$\underline{\lambda_1 = -4} \quad Aw = \lambda_1 w$

$\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \end{vmatrix} = -4 \begin{vmatrix} w_1 \\ w_2 \end{vmatrix}$

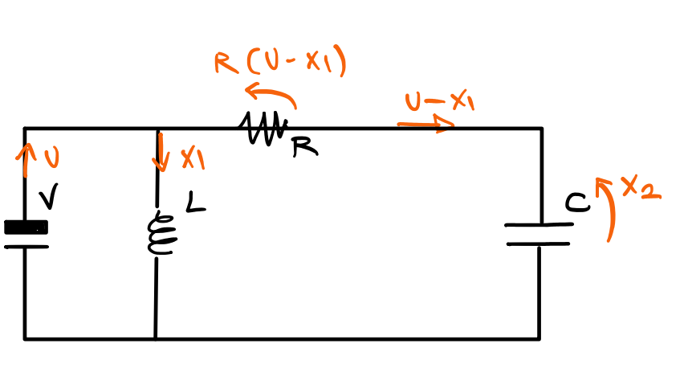
$\begin{cases} -w_1 + 2w_2 = -4w_1 \\ 3w_1 - 2w_2 = -4w_1 \end{cases} \rightarrow w_2 = -\frac{3}{2}w_1$

$\underline{\lambda_2 = +1} \quad Aw = \lambda_2 w$

$\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \end{vmatrix} = \begin{vmatrix} w_1 \\ w_2 \end{vmatrix}$

$\begin{cases} -w_1 + 2w_2 = w_1 \\ 3w_1 + 2w_2 = w_2 \end{cases} \rightarrow w_1 = w_2$

CIRCUITO RLC



RESISTENZA $V = RI$
CONDENSATORE $I = C\dot{V}$
INDUTTANZA $V = L\dot{I}$

DIMOSTRA CHE SE $\bar{U}$ COSTANTE, CORRENTE x_1 IN INDUTTORE PUÒ TENDERE A $\bar{x}_1$ CON ∞ OSCILLAZIONI SMORZATE O SINGOLA SOVRAELONGAZIONE

$U(t)$ CORRENTE IN ENTRATA
$x_1(t)$ CORRENTE IN INDUTTANZA
$x_2(t)$ DDP IN CONDENSATORE

MODELLO FISICO

$\begin{cases} \dot{x}_1 = \frac{1}{L}(x_2 + R(U - x_1)) \\ \dot{x}_2 = \frac{1}{C}(U - x_1) \end{cases}$

$\begin{cases} \dot{x}_1 = -\frac{R}{L}x_1 + \frac{1}{L}x_2 + \frac{R}{L}U \\ \dot{x}_2 = -\frac{1}{C}x_1 + \frac{1}{C}U \end{cases} \qquad A = \begin{vmatrix} -\frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{vmatrix}$

EQUILIBRIO

$\dot{x}_1 = \dot{x}_2 = 0$

$\begin{cases} 0 = -R\bar{x}_1 + \bar{x}_2 + R\bar{U} \\ 0 = -\bar{x}_1 + \bar{U} \end{cases} \qquad \begin{cases} \bar{x}_1 = \bar{U} \\ \bar{x}_2 = 0 \end{cases}$

STABILITÀ

$A = \begin{vmatrix} -\frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{vmatrix} \qquad R, L, C \in \mathbb{R}^+$

STUDIO GLI AUTOVALORI DI A

$\det(\lambda I - A) = \det \begin{vmatrix} \lambda + \frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & \lambda \end{vmatrix} = \Delta_A(\lambda)$

$\Delta_A(\lambda) = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$

$\lambda_{1,2} = \dfrac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2} = \dfrac{-\frac{R}{L} \pm \sqrt{\quad L^2C}}{2}$

$\underline{\text{CASO 1}} \quad R^2C - 4L \leq 0 \qquad C \leq 4\frac{L}{R^2}$

TUTTI AUTOVALORI CON $\mathrm{Re}(\lambda_i) < 0 \rightarrow \underline{AS}$

$\underline{\text{CASO 2}} \quad C > 4\frac{L}{R^2}$

$\{\lambda\}_A = \{\underset{?}{-\frac{R}{2L} + \sqrt{\frac{R^2C - 4L}{4L^2C}}},\ \underset{-}{-\frac{R}{2L} - \sqrt{\frac{R^2C - 4L}{4L^2C}}}\}$

VALUTO IL SEGNO DELL'AUTOVALORE 1

$\sqrt{\frac{R^2C - 4L}{4L^2C}} > \frac{R}{2L} \qquad \frac{R^2C - 4L}{\cancel{4L}\,C} > \frac{R^2}{\cancel{4L}}$

$R^2\cancel{C} - 4L > R^2\cancel{C}$ IMPOSSIBILE

QUESTO VUOLE DIRE CHE $\forall R, L, C$ NON SI AVRÀ MAI UN AUTOVALORE POSITIVO

$\underline{\text{RIASSUNTO}}$ AS $\forall R, L, C$

POTEVAMO RISOLVERLA ANCHE COSÌ:

$\left.\begin{array}{l} \mathrm{TR}(A) = -\frac{R}{L} < 0 \\ \det(A) = \frac{1}{LC} > 0 \end{array}\right. \rightarrow AS \ \forall R, L, C$

DIMOSTRA CHE SE $R < R^* = 2\sqrt{L/C}$ SI HANNO OSCILLAZIONI SMORZATE E SOVRAELONGAZIONE SE $R > R^*$

$\Delta_A(\lambda) = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$

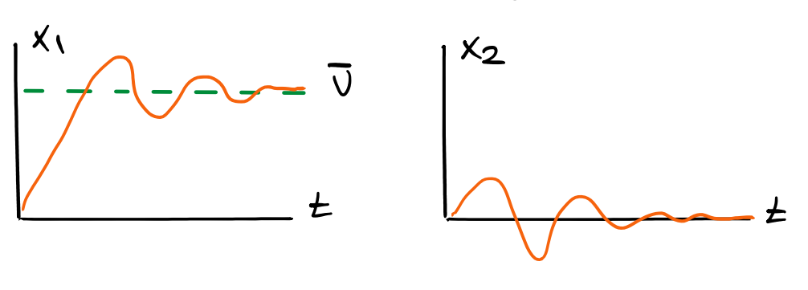
$\{\lambda\}_A = \{\frac{R}{2L} \pm \sqrt{\frac{R^2C - 4L}{4L^2C}}\}$

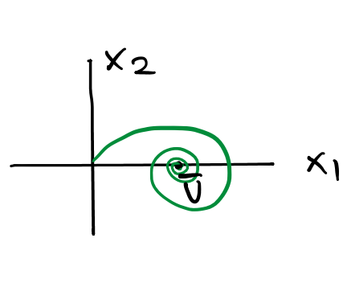
RESISTENZA CRITICA

$\frac{R^2C - 4L}{L^2C} = 0 \qquad R^2C = 4L \qquad R = 2\sqrt{\frac{L}{C}} = R^*$

1) $R < R^*$

$\lambda_{1,2} \in \mathbb{C}$
- FUOCO STABILE
- ASINTOTICAMENTE STABILE
- ∞ OSCILLAZIONI





$x(0) = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \qquad \dot{x}(0) = \begin{vmatrix} \frac{R}{L}\bar{U} \\ \bar{U}/C \end{vmatrix}$

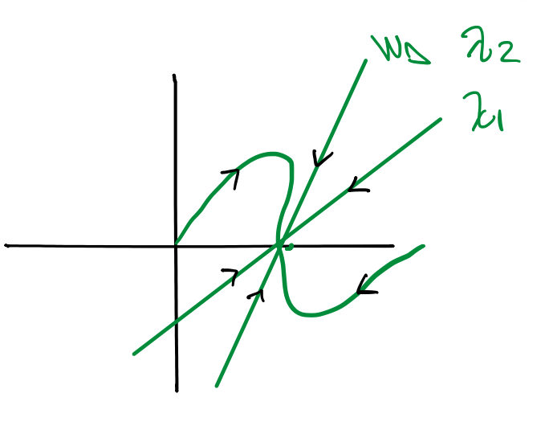
2) $R > R^* \rightarrow \lambda_{1,2} \in \mathbb{R},\ \mathrm{Re}(\lambda_{1,2}) < 0 \rightarrow \underline{AS}$ NODO STABILE

$Aw = \lambda w \quad \begin{vmatrix} -\frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \end{vmatrix} = \lambda \begin{vmatrix} w_1 \\ w_2 \end{vmatrix}$

$-\frac{1}{C}w_1 = \lambda w_2 \rightarrow w_1 = -C\lambda w_2 \qquad -\frac{1}{C\lambda}$ PENDENZA

$\lambda_1 = \dfrac{-\frac{R}{L} - \sqrt{\ }}{2}$

$\lambda_2 = \dfrac{-\frac{R}{L} + \sqrt{\ }}{2} = \lambda_D$



$-\frac{1}{C\lambda_2} > \frac{1}{C\lambda_1}$ PENDENZA

SISTEMA

$\dot{x} = Ax \rightarrow A = \begin{vmatrix} 0 & 1 \\ p & -2 \end{vmatrix} \quad p \in \mathbb{R}$

EQUILIBRIO

$\det(A) = -p$

SE $\det(A) \neq 0 \rightarrow p \neq 0 \rightarrow \exists!\ \bar{x} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

SE $\det(A) = 0 \rightarrow p = 0$

$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_2 \end{cases} \quad x_2 = 0 \rightarrow \exists\infty\ \bar{x} = \begin{vmatrix} \alpha \\ 0 \end{vmatrix} \quad \alpha \in \mathbb{R}$

INFINITI EQUILIBRI LUNGO LA RETTA REALE

STABILITÀ

$\det(A) = -p$

$\mathrm{TR}(A) = -2 < 0$

$p < 0$	$\det(A) > 0$	AS
$p > 0$	$\det(A) < 0$	IN
$p = 0$	$\det(A) = 0$	
$\begin{vmatrix} 0 & 1 \\ 0 & -2 \end{vmatrix}$	TRIANGOLARE $\{\lambda\} = \{0, -2\}$	SS

$\mathrm{Re}(\lambda_i) \leq 0\ \forall i + \lambda = 0$ RADICE SEMPLICE

TEMPO DI RISPOSTA

DEFINITO SOLO PER AS ($p < 0$)

$\underline{\text{AUTOVALORI DI A}}$

$\det(\lambda I - A) = \det \begin{vmatrix} \lambda & -1 \\ -p & \lambda+2 \end{vmatrix} = \Delta_A(\lambda)$

$\Delta_A(\lambda) = \lambda^2 + 2\lambda - p = 0$

$\lambda_{1,2} = -1 \pm \sqrt{1+p} \qquad p < 0$

$\{\lambda\}_A = \{-1 - \sqrt{1+p}, -1 + \sqrt{1+p}\}$

ENTRAMBI SEMPRE NEGATIVI IN $p < 0$

IL SECONDO AUTOVALORE È CC SE $p < -1$

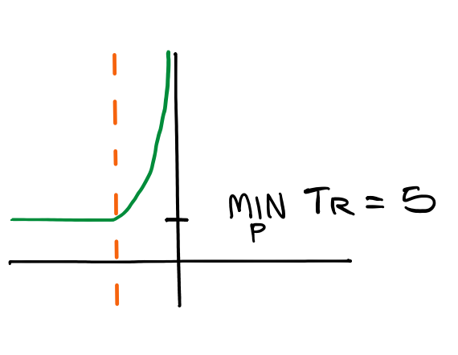
$p < -1 \quad \lambda_D = -1 \quad T_D = +1 \quad T_R = 5$

$-1 < p < 0 \quad \lambda_D = -1 + \sqrt{1+p}$

$T_D = \frac{-1}{\lambda_D} \qquad T_R = 5T_D = \frac{5}{1 - \sqrt{1+p}}$

GRAFICO DI TR

$T_R(p) = \begin{cases} 5 & p < -1 \\ \frac{5}{1 - \sqrt{1+p}} & p \in (0, -1] \end{cases}$



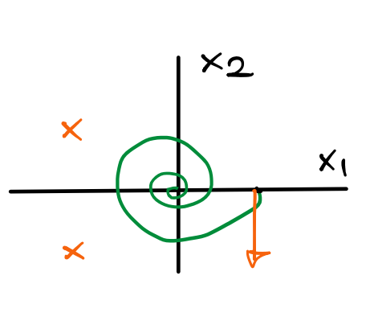
QUADRO DELLE TRAIETTORIE

$p < -1$ AS

$\lambda_{1,2} = -1 \pm i\sqrt{-1-p} \quad \in \mathbb{C},\ \mathrm{Re}(\lambda_i) < 0$

FUOCO STABILE IN $\begin{vmatrix} 0 \\ 0 \end{vmatrix}$

EQUILIBRIO DEFINITO PRIMA PER $\det \neq 0$ ($p \neq 0$)



POSTO $x(0) = \begin{vmatrix} x_1^0 \\ 0 \end{vmatrix}$

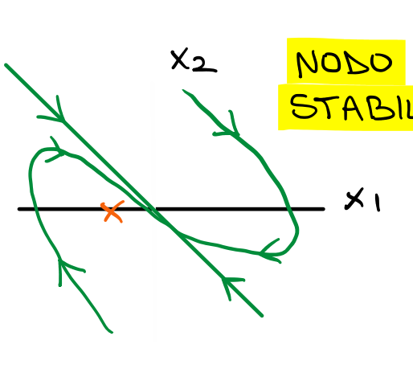
$\begin{vmatrix} 0 & 1 \\ p & -2 \end{vmatrix} \quad \begin{cases} \dot{x}_1(0) = \cancel{x_2(0)} \\ \dot{x}_2(0) = p\,x_1(0) - 2\cancel{x_2(0)} \end{cases} \rightarrow \dot{x}(0) = \begin{vmatrix} 0 \\ p\,x_1^0 \end{vmatrix}$

SE $x_1^0 > 0 \qquad p\,x_1^0 < 0 \qquad \dot{x}(0) < 0$

$p = -1$ AS

$\lambda_1 = \lambda_2 = -1 \in \mathbb{R}^-$ COINCIDENTI NEGATIVI

$\underline{\text{EQUILIBRIO}}$ IN $\begin{vmatrix} 0 \\ 0 \end{vmatrix}$ POICHÉ $p \neq 0$



NODO STABILE

POSTO $x(0) = \begin{vmatrix} x_1^0 \\ 0 \end{vmatrix} \rightarrow \dot{x}(0) = \begin{vmatrix} 0 \\ -x_1^0 \end{vmatrix}$

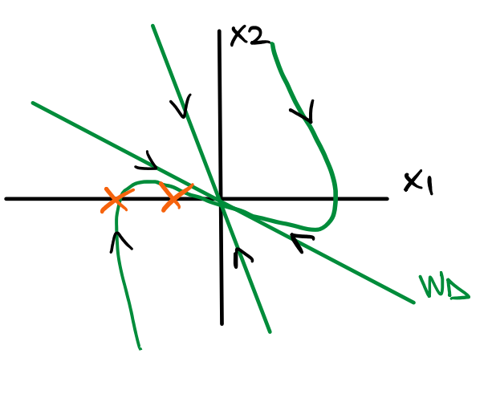
$x_1^0 > 0 \rightarrow \dot{x}(0) < 0$

$-1 < p < 0$ AS

$\lambda_{1,2} = -1 \pm \sqrt{1+p} \in \mathbb{R}^- \rightarrow$ NODO STABILE IN $\begin{vmatrix} 0 \\ 0 \end{vmatrix}$

AUTOVETTORI $Aw = \lambda w$

$\begin{vmatrix} 0 & 1 \\ p & 2 \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \end{vmatrix} = \lambda \begin{vmatrix} w_1 \\ w_2 \end{vmatrix}$



$w_2 = \lambda w_1 = (-1 \pm \sqrt{1+p})\,w_1$

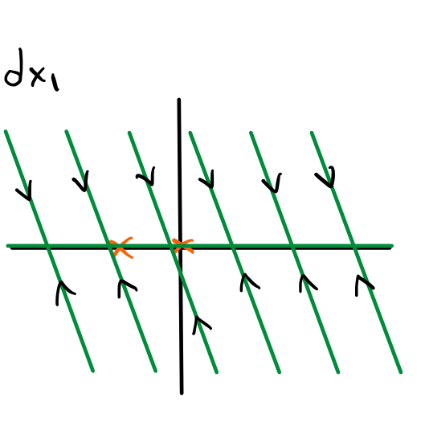
$\lambda_D = -1 + \sqrt{1+p}$

$p = 0 \quad \exists\infty$ EQUILIBRI $\quad$ STABILITÀ SEMPLICE

$\lambda_1 = -2 \quad \lambda_2 = 0$

$\begin{vmatrix} 0 & 1 \\ 0 & -2 \end{vmatrix} \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_2 \end{cases} \ \% \quad \frac{dx_2}{dx_1} = -2$

$dx_2 = -2\,dx_1 \qquad \int_{x_2(0)}^{x_2(t)} dx_2 = -2\int_{x_1(0)}^{x_1(t)} dx_1$



$x_2(t) - x_2(0) = -2(x_1(t) - x_1(0))$

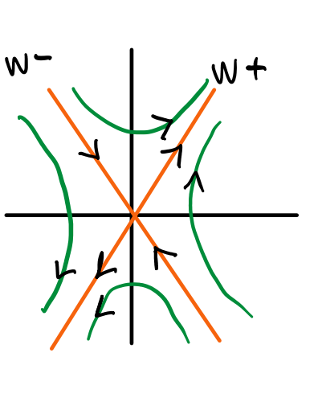
$x_2(t) = x_2(0) - 2(x_1(t) - x_1(0))$

RETTE CON PENDENZA -2

$p > 0$ INSTABILE

$\lambda_{1,2} = -1 \pm \sqrt{1+p} \quad \exists \lambda_i > 0 \rightarrow$ SELLA IN $\begin{vmatrix} 0 \\ 0 \end{vmatrix}$

$Aw = \lambda w \quad \begin{vmatrix} 0 & 1 \\ p & -2 \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \end{vmatrix} = \lambda \begin{vmatrix} w_1 \\ w_2 \end{vmatrix}$



$\begin{cases} w_2 = \lambda w_1 \\ p w_1 - 2w_2 = \lambda w_2 \end{cases} \rightarrow w_2 = \lambda w_1$

$\exists\infty$ OSCILLAZIONI ?

SI, SOLO SE $\lambda \in \mathbb{C} \rightarrow \underline{p < 1}$

CONDIZIONI SU P AFFINCHE NON ESISTANO SOVRAELONGAZIONI PER x_1, x_2 CON $x(0) = (-2, -3)$