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Toward Stronger Robustness of Network Controllability: A Snapback Network Model *

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Abstract

A new complex network model, called q-snapback network, is introduced. Basic topological characteristics of the network, such as degree distribution, average path length, clustering coefficient and Pearson correlation coefficient, are evaluated. The typical 4-motifs of the network are simulated. The robustness of both state and structural controllabilities of the network against targeted and random node- and edge-removal attacks, with comparisons to the multiplex congruence network and the generic scale-free network, are presented. It is shown that the q-snapback network has the strongest robustness of controllabilities due to its advantageous inherent structure with many chain- and loop-motifs.

1 Introduction

The subject of complex networks has gained popularity after two decades of research pursuits with great efforts from various scientific and engineering communities through intensive and extensive studies, and has literally become a self-contained discipline interconnecting network science, systems engineering, statistical physics, applied mathematics and social sciences [1, 2, 3].

This interdisciplinary research area, the interaction between network science and control systems theory in particular, has seen very rapid growth since year 2002 [4, 5, 6, 7]. In fact, it has created a corpus of new opportunities and yet also great challenges for classical control and systems theories and technologies, since a complex dynamical network typically has large numbers of nodes and edges, with higher-dimensional dynamical node-systems interconnected in a complicated structure such as random, small-world or scale-free topology. For a complex dynamical network, to achieve an optimal objective, practically one can only control a small fraction of nodes and/or edges via external inputs. These observation and demand had motivated the long-term endeavor and development of the so-called "pinning control" strategy [7], as a practical control approach to addressing the fundamental questions of how many and which nodes to pin (to control), aiming to design effective control algorithms that could "pull one hair to move the whole body".

For a single-input/single-output (SISO) connected and directed framework of linear time-invariant (LTI) node-systems, the minimum number of external inputs (controllers) required for the network to be structurally controllable is determined by a criterion based on maximum matching. The "minimum inputs theorem" [8] states that, if a network of size N has a perfect matching

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then the number of external controllers is $N_D = 1$ and the controller can be pinned at any node; otherwise, $N_D = N - |E^*|$, where $|E^*|$ is the number of elements in a maximum matching E^*, and the controllers should be pinned at the unmatched nodes.

In the multi-input/multi-output (MIMO) setting, the state controllability of a connected, directed and weighted network of LTI node-systems were studied in [9, 10], where it shows how the network topology, node dynamics, external control inputs, and inner interactions altogether affect the state controllability of the network, with necessary and sufficient conditions derived for the controllability of a connected, directed and weighted network in a general topology. In [9, 10], precise necessary and sufficient conditions are given in terms of the node-system matrices, control gains and the network connectivity matrices. Then, some easily-verified formulas were derived in [11] for verifying the necessary and sufficient controllability conditions for networked MIMO LTI node-systems. Moreover, both state and structural controllabilities for temporal networks, namely the about MIMO LTI setting with a certain time-varying network structure, were studied in [12], establishing some necessary and/or sufficient conditions for the network controllability. Furthermore, in [13], conditions and methods were developed for designing the control input matrices of pinning controllers to guarantee the network state and structural controllabilities.

In retrospect, there had been significant progress in the studies of network controllability in the past decade [4]-[27] These studies were concerned with, for example, pinning small unmatched nodes [14], optimizing the network controllability [15], identifying critical nodes for controllability [16], investigating the exact controllability [17], finding the importance of in- and out-degrees for control [18], and designing targeted control [19]. More recently, studies have evolved to considering, for instance, mathematical and computational approaches to controlling nonlinear networks [20], control properties of complex networks [21], control energy issue [22], sensor-actuator placements for network controllability [23], human protein-protein interaction network [24], structural controllability of temporal networks [25], turning physically uncontrollable networks to become controllable ones [26], and so on. Other related works on network controllability can be found from the recent review articles [4], [28].

On the other hand, the issue of network robustness has been extensively investigated by different means under different criteria in different settings, and there is a vast volume of literature on the subject. Concerning the robustness of the controllability of a complex network against node and/or edge removals, which typically cause cascading failures [29, 30], so that the network could retain its connectivity and functionality (here, the network controllability), relevant research includes the following. In [31], the normalized average edge betweenness is used as a measure of the network vulnerability; in [32], a hybrid method combining similarity-based index and edge-betweenness centrality is proposed for identifying and removing spurious interactions, keeping the network connectivity and preserving the network functionality; in [33], it investigates the vulnerability of complex networks subject to path-based attacks, showing that the more homogeneous the degree distribution is, the more fragile the network will be. Particularly related to the present concern on the network controllability, in [34] the vulnerability of network controllability is considered, where the attacks are based on node degrees or edge betweenness, with simulations showing that the node-based attacks are more harmful to the network controllability than the edge-based attacks and that heterogeneous networks are more vulnerable than homogeneous ones; it was found, however, that for many real-world networks the betweenness-based attacks are actually most harmful to the network controllability.

A piece of recent theoretical work on network controllability and the corresponding robustness is the initiation of a mathematical number-theoretic framework of complex network modeling and analysis [35]. A congruence network is generated as follows. A link (edge) in the congruence network is defined according to the congruence relation $j \equiv r \pmod{i}$, where r is the reminder of j divided by i, and they are all integers (here, natural numbers). For every fixed r, an infinite set of natural number pairs (i, j) can be generated. For each pair of such integers, a directed link from i to j $(i < j)$ characterizes the congruence relation between them. For each r, this process yields a congruence network associated with the reminder r, denote by $G(r, N)$, where N is the largest natural number in the present construction of the network. Then, for different values of r $(r \leq N)$, one obtains various such networks, referred to as multiplex congruence networks (MCN). It was found that every MCN is precisely a scale-free network with a power-law distribution for out-degrees [35].

From the construction of an MCN, one can see that it contains many chains and loops, where as usual every chain has a root-node. Therefore, for each chain, using one external linear selfstate feedback controller is sufficient to guarantees the controllability of the chain. Since typically $r \ll N$, the controllability of the entire network is excellent, in the sense that a very small number of controllers can guarantee the controllability of a large network. This is quite opposite to the common view that scale-free networks are generally not good in controllability by requiring large numbers of controllers because many small nodes need to be individually controlled in general. Moreover, when a chain is being attacked, randomly or intentionally, with one node or one edge removed, in the worst situation it is broken into two sub-chains. In this case, at most one new controller would need to be added at the new chain-root in order to retain the controllability of the network, so it is very robust against attacks. This is also quite opposite to the common view that a scale-free network is fragile against intentional attacks.

The above interesting findings have stimulated our curiosity about the network controllability and its robustness against attacks, urging us to find out why, how, and what the key factors are behind the surprising phenomena regarding the controllability and its robustness against malicious attacks for general complex dynamical networks. In this paper, we attempt to modify and extend the multi-chain structure of the MCN to a multi-ring structure, thereby proposing a new q-snapback network model, which will be shown to be superb in the robustness of network controllability. Extensive simulation results indeed demonstrate that q-snapback networks and MCN outperform general scale-free networks in resisting both targeted and random attacks, and also demonstrate that the q-snapback network is prominently more robust than the MCN against targeted attacks on the nodes with largest betweenness, and also against random attacks. The q-snapback network has similar robustness as the MCN when the targeted attack aims at removing highest degree nodes.

The main contribution of this paper is the introduction of a new network model based on the novel idea of using snapback connections, which turned out to be a good model with the strongest robustness of network controllability against both targeted and random node/edge removal attacks. One technical challenge was to reveal and confirm the key network sub-structures that affect the controllability robustness of the new network model, which were found to be the relatively large numbers of chains and loops existing in the new model, which are not prominent in other well-known network models.

The rest of the paper is organized as follows. Section 2 describes the q-snapback network model. Section 3 presents some analysis on the degree distribution of the new model. Section 4 shows simulation results on various topological features especially degree distributions and motifs. Section 5 discusses both state and structural controllabilities and compare their robustness for three types of networks, *i.e.*, MCN, scale-free and q-snapback networks. Section 6 concludes the investigation.

2 The q-snapback Network Model

Although both the MCN and generic scale-free networks have power-law degree distributions, they behave oppositely against targeted and random attacks; namely, as is well known, generic scale-free networks are robust against random attacks but fragile against targeted attacks, but in contrast MCNs are robust against targeted but fragile against random attacks [35]. This suggests that the power-law degree distribution is not the essential reason supporting the network robustness against attacks and failures.

It is observed that an MCN contains many chains. Since each chain has one root, a subgraph with r chains has at most r roots. According to the matching theory [8], to control a chain only one controller is needed to pin at the root. The chain structure is robust against targeted attack regarding the network controllability, since after one node-removal at most one more new controller is needed to retain its controllability.

It is also observed that, although the structure of chains offers a good controllability to MCN, these chains have feedforward loop connections. Practically, feedback loops are more common and more useful than feedforward ones. For example, the industrial assembly-line illustrated by Fig. 1 is very common in manufacturing processes. On the other hand, the number-theoretic congruence relation has no patterns and no analytic formulas to use for design and analysis considerations. Therefore, a model with feedback connections (called *snapback links*) based on a uniform probability distribution is proposed.

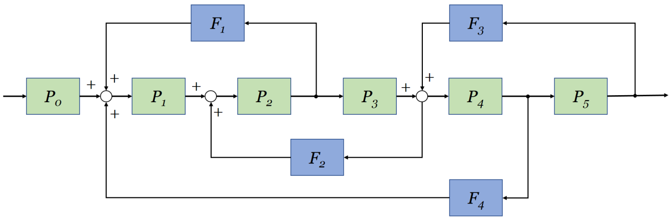

Figure 1: An example of assembly-line automation with a snapback connection structure, where P_i represents the ith plant and F_j represents the jth feedback controller.

In the snapback network model: 1) the feedforward links are replaced by feedback links; 2) the congruence relations are replaced by uniform random connections. Similarly to the MCN, the basic structure of the new model is a backbone chain, which is a maximum matching with all matched nodes except the root. The main difference of the new model compared with the MCN model lies in its large number of loops, which turns out to be advantageous for the controllability robustness as further discussed later below.

More precisely, the q-snapback network consists of multiple layers, generated as follows. Let $q \in [0, 1]$ be a probability parameter. Each layer starts with a directed chain, which will be the backbone of the connected layer. Then, following some rules a number of snapback links are generated connecting to the chain with probability q. Finally, all layers are stacked together as a whole network, where the same nodes will merge into a single node and the same links will also merge into a single link, so as to avoid multiple nodes and multiple links.

Specifically, start from a directed chain with nodes $1, 2, ..., N$.

For $r = 1$, process as follows to generate the first layer network. For every node $i = 2, 3, 4, ..., N$, it connects backward to all previously-appeared nodes $i - 1, i - 2, ..., 2, 1$, all with a (same and small) probability $q \in [0, 1]$. As a result, some will be backward connected but some will not, which happens at random uniformly.

For $r = 2$, continue to process in the same way but it connects backward to only some (not all) previously-appeared nodes, *i.e.*, for nodes $i = 3, 4, ..., N$, they backward connect to nodes $i-2, i-4, ..., i-2\lfloor \frac{i}{2} \rfloor$, with the same probability $q \in [0, 1]$ uniformly. In notation, if $i - 2\lfloor \frac{i}{2} \rfloor = 0$, then the link $(i, i - 2\lfloor \frac{i}{2} \rfloor)$ will not exist.

Next, the construction continues similarly. For the rth layer of the network, with $r = 3, 4, 5, ..., N$, the nodes $i = r + 1, r + 2, ..., N$ will backward connect to nodes $i - r, i - 2r, ..., i - r\lfloor \frac{i}{r} \rfloor$, with the same probability $q \in [0, 1]$ uniformly. Denoted it as $G_r(q, N)$.

The above procedure continues, until it cannot be processed any further.

Finally, stack all so-generated layers together into one, thus establishing the final multiplex network, denoted as $G(q, N) = \bigcup_{r=1}^{N-1} G_r(q, N)$, called the q-snapback multiplex network of size N.

Fig. 2 shows the pseudo codes for generating a q-snapback network. The input parameters include the network size N and the probability of adding snapback links, $q \in [0, 1]$. Note that, with $q = 0$, it is the original chain without any backward connection; with $q = 1$, it is a maximum-size snapback network having the largest number of backward connections.

For each layer, there is a directed chain with some numbers of backward connections. As described above, with $r = 1$, the backward connections on layer G_1 are generated as follows: For every node $i = 2, 3, 4, ..., N$, connect it backward to previously-existing nodes $i-1, i-2, i-3, ..., 2, 1$, all with the same probability q. For this case of $r = 1$, the generated layer is the densest one, with the largest number of links. On the contrary, with $r = N - 1$, there will be only one possible backward connection on the chain, *i.e.*, only node $i = N$ could possibly be connected backward to the first node, which is the sparsest layer (the $(N-1)$st layer).

Each layer can work separately and independently, since it is built on the backbone. Also, all the layers can be stacked together so as to form a multiplex network. Note that all the repeated nodes and links are removed when the layers are put together as one whole network, avoiding multiple nodes and links. Fig. 3 shows an example of stacking three layers together, which has two types of repeated links: 1) the backbone directed links from i to $i + 1$, for $i = 1, 2, ..., N - 1$, and 2) the repeated backward links. As can be seen from the figure, the resultant q-snapback network is an ensemble of links from the three layers without multiple nodes and links.

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Input: 1) network size $N$;  2) probability of adding a snapback link $q$

Start from a directed chain $G_0$ with $N$ nodes and $N-1$ directed links,
directing from $i$ to $i+1$ ($i = 1, 2, \dots, N-1$).
    For $r := 1$ to $N$
        /* This step generates the $r$-th layer $G_r(q, N)$ */
        $G_r := G_0$    /* The backbone of each layer */
        Generate backward connections on $G_r$ as follows: For every
        node $i = r+1, r+2, \dots, N$, connect it backward to nodes
        $i - r \cdot l$, ($l = 1, 2, \dots, \lfloor \frac{i}{r} \rfloor$), with the same probability $q$;
    End for
    Put all the $N-1$ layers together, and remove repeated nodes
    and links.
Output: 1) $q$-snapback network $G(q, N)$; 2) single layers $G_r(q, N)$
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Figure 2: Pseudo codes for generating a q-snapback multiplex network.

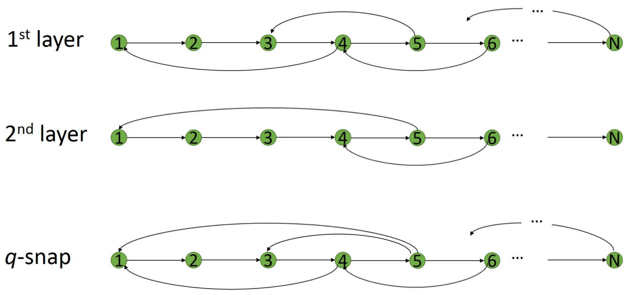


Figure 3: A two-layer example of the q-snapback multiplex network. Each layer is a connected subnetwork and their backbone structures are the same directed chain. On both the 1st and 2nd layers, there is a link connecting from node 6 back to node 4. When stacking them together in the integrated multiplex network, only one link from node 6 to node 4 is kept. The situation for the other nodes is similar.

3 Analysis on Degree Distributions

In this section, degree distribution of the q-snapback network is derived analytically. Here, the out-degrees and in-degrees of a directed network are both considered. First, the degree distribution of each single layer is discussed, followed by the multiplex network.

3.1 Single layers

For the rth layer, $r = 1, \cdots, N-1$, the out-degree of the ith node $d_O(i)$, $i = 1, 2, ..., N$, is calculated by

$$d_O(i) = \begin{cases} 1, & \text{for } i = 1, 2, ..., r \\ 1 + \lfloor \frac{i-1}{r} \rfloor \cdot q, & \text{for } i = r+1, ..., N-1 \\ \lfloor \frac{i-1}{r} \rfloor \cdot q, & \text{for } i = N \end{cases} \tag{1}$$

where $\lfloor x \rfloor$ is the floor function that returns the greatest integer less than or equal to x.

Similarly, the in-degree of the ith node $d_I(i)$, $i = 1, 2, ..., N$, is

$$d_I(i) = \begin{cases} \lfloor \frac{N-1}{r} \rfloor \cdot q, & \text{for } i = 1 \\ 1 + \lfloor \frac{N-i}{r} \rfloor \cdot q, & \text{for } i = 2, ..., N-r \\ 1, & \text{for } i = N-r+1, ..., N \end{cases} \tag{2}$$

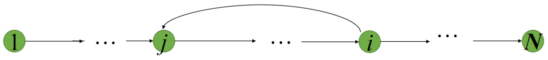


Figure 4: Illustration of a snapback link from any node i to any node j ($j < i$) in a q-snapback network.

where $\lfloor x \rfloor$ is the floor function.

3.2 The multiplex q-snapback network

When a number of layers are stacked together, the multiplex network is formed. As shown in Fig. 4, for any node i, its out-degree is

$$d_O^M(i) = \begin{cases} 1, & \text{for } i = 1 \\ 1 + \sum_{j=1}^{i-1} I_j q, & \text{for } i = 2, 3, ..., N-1 \\ \sum_{j=1}^{N-1} I_j q, & \text{for } i = N \end{cases} \tag{3}$$

where

$$I_j = \begin{cases} 1, & \text{if there exists an edge } (i, j) \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Similarly, the in-degree of the ith node is

$$d_I^M(i) = \begin{cases} \sum_{j=2}^{N} I_j q, & \text{for } i = 1 \\ 1 + \sum_{j=i+1}^{N} I_j q, & \text{for } i = 2, 3, ..., N-1 \\ 1 & \text{for } i = N \end{cases} \tag{5}$$

where I_j is defined in (4).

An edge (i, j) could appear on different layers, as illustrated in Fig. 4. The probability that the edge (i, j) exists on at least one layer, *i.e.*, the probability of existence of edge (i, j), is given by

$$P_{(i,j)} = 1 - (1-q)^{\prod_{i=1}^{m}(x_i+1)} \tag{6}$$

where $i - j = a_1^{x_1} \cdot a_2^{x_2} \ldots a_m^{x_m}$, with $a_1, a_2, ..., a_m$ being prime numbers, and $(1-q)^{\prod_{i=1}^{m}(x_i+1)}$ represents the probability that edge (i, j) does not exist on any layer of the multiplex network. Here, 1 means 1^0.

4 Simulations

Extensive simulations had been performed on the q-snapback network of size $N = 10^4$, with $q = 0.1$ unless otherwise indicated. The following statistical results are averages over 50 independent runs.

It was found that:

1) the average path length is 1667.6, with a standard deviation 0.0;

2) the clustering coefficient is 0.4679, with a standard deviation 6.4×10^{-5};

3) the Pearson correlation coefficient (*i.e.*, assortativity) is -0.4979, with a standard deviation 1.1×10^{-4}.

Next, the out-degree distributions of some single layers $G_r(q = 0.1,\ N = 10^4)$, and of the multiplex network $G(q = 0.1,\ N = 10^4)$, are simulated. Here, only the simulation results on the out-degree distributions are shown, for brevity, since the in-degree distributions (2) and (5) have similar forms as the out-degree ones (1) and (3). Moreover, the influence of the parameter q on the degree distribution of the multiplex network is shown and analyzed. Finally, the distribution of the 4-motifs on the multiplex network is simulated and discussed.

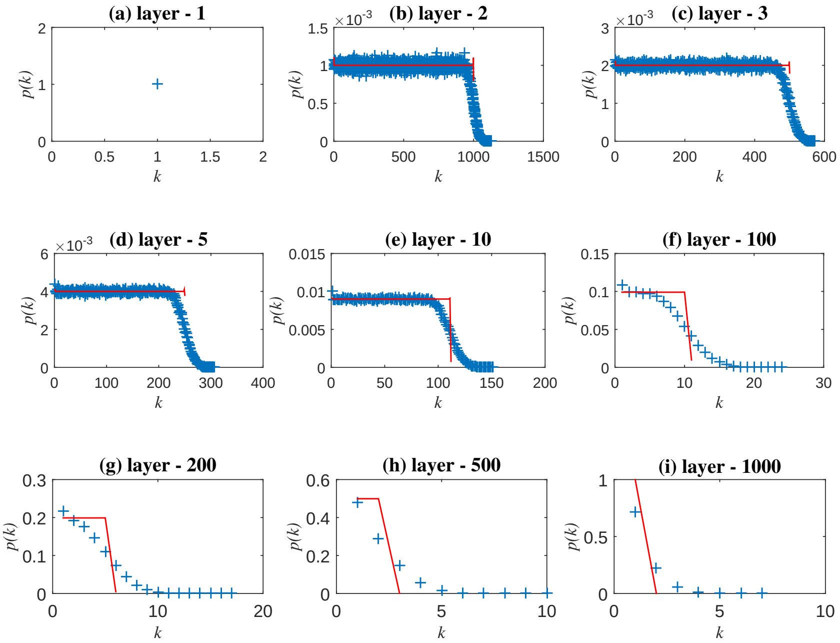


Figure 5: Degree distributions of different single layers. The blue pluses $(+)$ are simulation results and the red lines are calculated using equation (1): (a) layer $r = 1$; (b) layer $r = 2$; (c) layer $r = 3$; (d) layer $r = 5$; (e) layer $r = 10$; (f) layer $r = 100$; (g) layer $r = 200$; (h) layer $r = 500$; (i) layer $r = 1,000$.

4.1 Out-degree distributions of single layers

Fig. 5 shows the degree distribution of 9 single layers of a multiplex q-snapback network. For each single layer, the degree distribution is uniform. The tails of the distribution curves in the figures are due to the well-known finite-size effects. Both empirical and analytical results are presented in the figures. The empirical simulation results (presented by blue pluses in the figures) are averages over 50 independent runs, while the red lines are the analytical solutions calculated by equation (1) for reference.

4.2 Out-degree distribution of the multiplex network

Fig. 6 shows the degree distribution of the multiplex q-snapback network. As can be seen from the figure, the degree distribution is uniform, just like all the single layers, as expected. The analytical degree distribution calculated by equation (3) is also plotted in Fig. 6, for reference. Note that equation (3) gives the expectation of the out-degrees, which is a real number. When plotting Fig. 6, for better visualization the real numbers are rounded to their nearest integers, thus the analytical degree distribution curve appears to be three parallel lines.

4.3 Influence of probability q on degree distributions

Fig. 7 shows the influence of the probability parameter q on the degree distribution of the multiplex network. As can be seen from the figure, the curve becomes more widely distributed as q increases, but it still remains being uniform constantly. This means that uniform distribution is a scale-free property (bigger value of q generates more edges) of the new model.

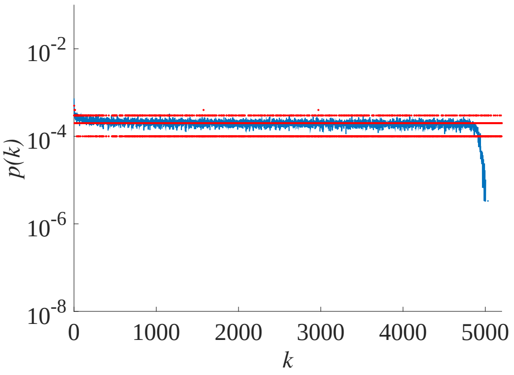


Figure 6: The degree distribution of the multiplex q-snapback network. The blue pluses (+) are simulation results and the red dots are calculated using equation (3), where real numbers are rounded to their nearest integers thus the analytic curve appears to be three parallel lines.

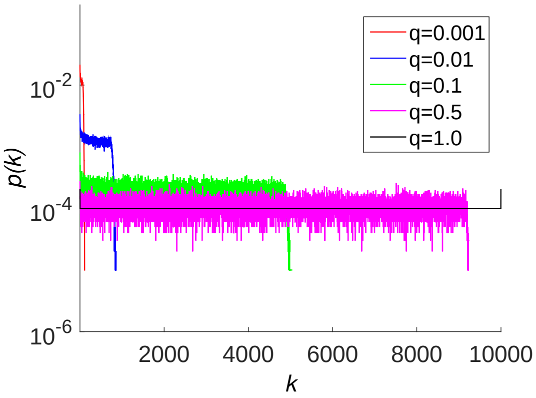


Figure 7: Degree distributions when q is 0.001, 0.01, 0.1, 0.5 and 1.0, respectively.

4.4 Distribution of 4-motifs

Motifs contribute to and even determine many basic properties of a network, thereby becoming an important object for investigation. The distribution of the 4-motifs in the multiplex q-snapback network is shown in Fig. 8. There are 8 of 4-motifs as shown in Fig. 8 (a), labeled from A to H, respectively. Motifs in other sizes are either trivial or too complicated therefore are not discussed here. Fig. 8 (b) shows the average number of each motif on the network $G(q = 0.1, N = 10,000)$. As can be seen from the bar chat, the chains (motif type A) are the most frequently appearing motif, followed by the loops (motif type D). In the next section, it will be shown that these two particular 4-motifs play key roles in the robustness of the network controllability.

5 Controllability

The controllability of the q-snapback network is studied through extensive simulations.

Recall [9] that a system or network described by $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$, where A and B are constant matrices of compatible dimensions, is *state controllable* if and only if the controllability matrix $[B\ AB\ A^2B\ \cdots, A^{n-1}B]$ has a full row-rank, where n is the dimension of A. The concept of *structural controllability* is a slight generalization, dealing with two parameterized matrices A and B, in which the parameters characterize the structure of the underlying system or network. If there are specific parameter values that can make the two parameterized matrices become state controllable, then the underlying system or network is structurally stable.

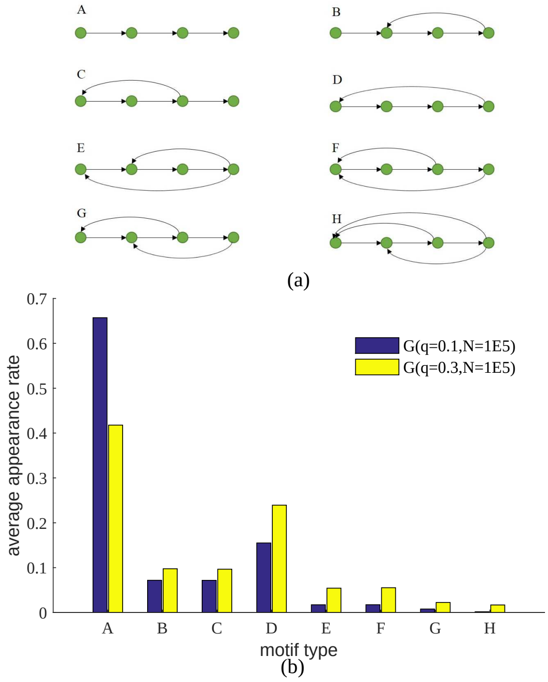


Figure 8: Distribution of the 4-motifs: (a) notation of the eight 4-motifs; (b) comparison of distributions of the eight 4-motifs on $G(q = 0.1, N = 10,000)$ and $G(q = 0.3, N = 10,000)$.

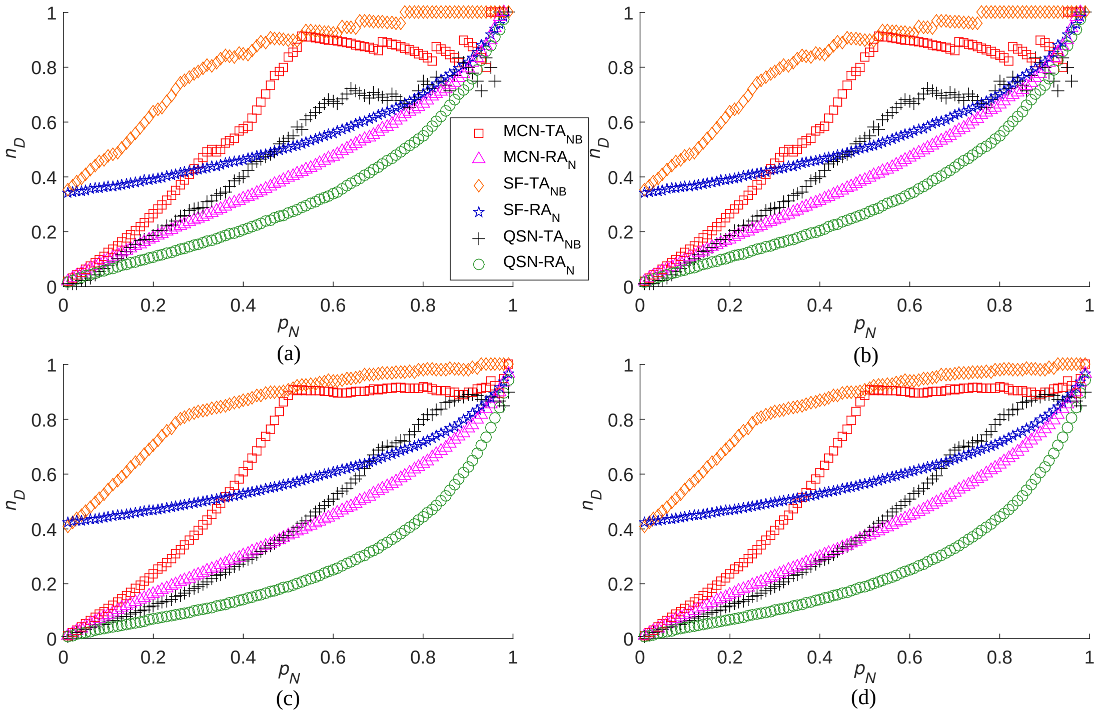
Table 1: Attack methods for simulation.

	Node-removal	**Edge-removal**
Targeted	$\mathrm{TA_{NB}}$: to remove the node with the largest betweenness $\mathrm{TA_{ND}}$: to remove the node with the largest degree	$\mathrm{TA_E}$
Random	$\mathrm{RA_N}$	$\mathrm{RA_E}$

The network controllability is measured by the density of the control-nodes n_D, where $n_D \equiv N_D/N$ and N_D is the number of external controllers (also called driver nodes) needed to retain the network controllability after the network had been attacked, and N is the network size. The smaller the n_D is, the more robust the network controllability will be.

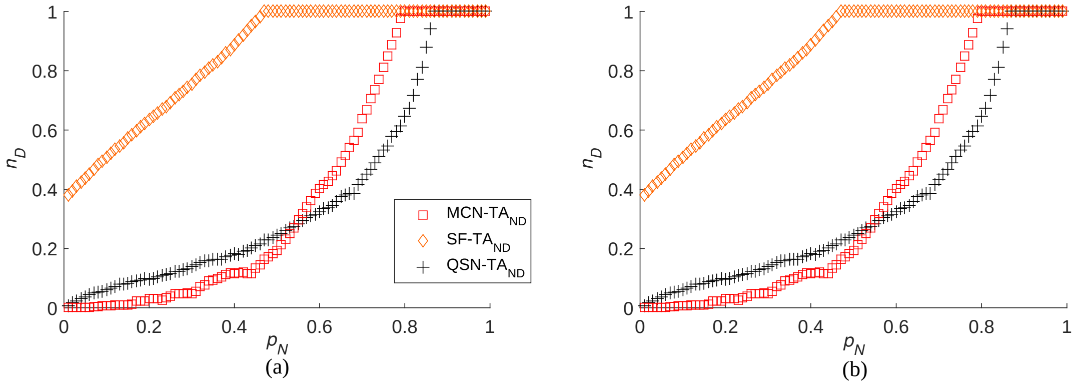
The simulation design here is an extension of the multiplex congruence networks (MCN) simulations reported in [35]. Both MCN and scale-free (SF) networks are taken to compare with the q-snapback network. Scaling property is examined by using two network sizes, with 100 nodes and 1,000 nodes respectively. Five types of attacks, as shown in Table 1, are implemented, *i.e.*, node-betweenness-based targeted attacks ($\mathrm{TA_{NB}}$), node-degree-based targeted attacks ($\mathrm{TA_{ND}}$), node-based random attacks ($\mathrm{RA_N}$), edge-based targeted attacks ($\mathrm{TA_E}$), and edge-based random attacks ($\mathrm{RA_E}$). Here, $\mathrm{TA_E}$ aims at removing the edges with the largest edge-betweenness. To reduce the effect of randomness, the results of node-based RA are averaged over 100 independent runs, and that of edge-based RA are averaged over 30 independent runs. The detailed simulation results are shown in Figs. 9, 10 and 11.

When the network size is set to $N = 100$, as did in [35], for comparison, the results are shown in Figs. 9 (a) and (b). Because the degree distribution is deterministic for the MCN, in the case of 100 nodes it is $\langle k \rangle = 3.82$ [35], the average degrees of the other two types of (SF and q-snapback) networks are set to $\langle k \rangle \approx 3.82$ for a fair comparison, which means that they all have about the same number of links. For SF networks, the average degree cannot be precisely controlled due to the randomness in their generating processes, therefore fine-tunings are performed by adding or deleting a few links, so as to slightly change the average degree such that the difference of average



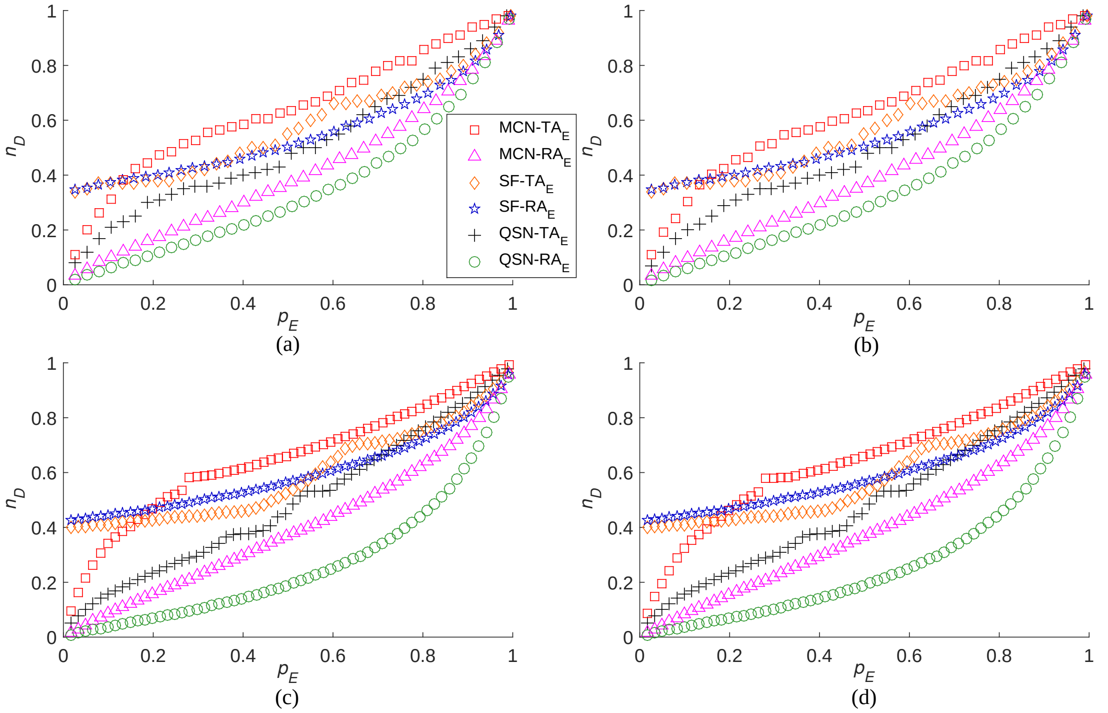
(a) (b) (c) (d)

Figure 9: Density of control-nodes n_D as a function of the proportion p_N of removed nodes, where n_D represents the proportion of needed control-nodes over all nodes of the current network. TA$_{\mathrm{NB}}$ represents the targeted attacks that aim at removing the node with the largest betweenness on the current network: (a) Network size 100, state controllability; (b) network size 100, structural controllability; (c) network size 1,000, state controllability; and (d) Network size 1,000, structural controllability.



(a) (b)

Figure 10: Density of control-nodes n_D as a function of the proportion p_N of removed nodes. TA$_{\mathrm{ND}}$ represents the targeted attacks that aim at removing the node with the largest out-degree in the current network, where for same-degree nodes it randomly removes one. RA$_{\mathrm{N}}$ represents the random attacks that randomly remove a node from the current network: (a) Network size 1,000, state controllability; and (b) network size 1,000, structural controllability.



(a) (b) (c) (d)

Figure 11: Density of control-nodes n_D as a function of the proportion p_E of removed edges. The subscript E represents that both the targeted and random attacks aim at removing edges: (a) Network size 100, state controllability; (b) network size 100, structural controllability; (c) network size 1,000, state controllability; and (d) Network size 1,000, structural controllability.

degrees between SF and MCN networks becomes negligible. As for the q-snapback network, it turns out that $q = 0.06$ is the best value to use, so $G_2(q = 0.06, N = 100)$ is generated, which yields an average degree $\langle k \rangle = 3.78$. As a result, q-snapback network is slightly sparsely linked as compared to MCN and SF.

When the network size is set to $N = 1,000$, the average degree for the three types of networks is set to $\langle k \rangle \approx 6.06$. Similarly, MCN has $\langle k \rangle = 6.06$, SF network has $\langle k \rangle \approx 6.06$ and the q-snapback has $\langle k \rangle = 6.055$; again, the q-snapback network is slightly sparser in terms of edge density in this case.

Both state controllability and structural controllability of these networks are examined. Figs. 9 (a) and (c) show a comparison on the state controllability under TA$_{\mathrm{NB}}$ and RA$_{\mathrm{N}}$, respectively. Both figures show the same phenomenon regardless of the changes of the network sizes. The SF network is the most vulnerable to both TA$_{\mathrm{NB}}$ and RA$_{\mathrm{N}}$. The q-snapback network is the most robust against these two types of attacks. Likewise, in the structural controllability comparison as shown in Figs. 9 (b) and (d), the q-snapback network is the most robust against both TA$_{\mathrm{NB}}$ and RA$_{\mathrm{N}}$.

When the target of the targeted-attacks is shifted, from node-betweenness (TA$_{\mathrm{NB}}$) to node-degree (TA$_{\mathrm{ND}}$), the q-snapback network performs similarly to MCN. More precisely, as shown in Fig. 10, MCN performs more robustly than q-snapback when $p_N < 0.544$ ($p_N = 0.544$ is the intersection of the curves MCN-TA$_{\mathrm{ND}}$ and QSN-TA$_{\mathrm{ND}}$ in Fig. 10). This is because, by the network construction and node-removal mechanisms, the node-degrees of the MCN are arranged in decreasing order and the earlier node-based removals would not affect its connectivity, but after some time its connectivity crashes rapidly. On the contrary, the node-degrees of the q-snapback network are uniformly distributed, so its connectivity remains about the same against removals even after the MCN became disconnected.

Both MCN and the q-snapback network outperform the SF network against the three types of node-based attacks, essentially due to their inherent chain- and loop-motif structures.

Furthermore, simulation on an attack was performed on edge-removals, with results shown in Fig. 11. This kind of attack removes edges from the network, one after another, either in the targeting order or at random. Note that, when the network size is 1,000, there are more than six thousand edges in each network ($\langle k\rangle \approx 6.06$), thus the results of random edge-removal are averaged over 30 independent runs. In this comparison, again, the q-snapback outperforms MCN and SF prominently.

6 Conclusions

A new complex network model, named q-snapback network, has been introduced. Some basic topological characteristics of the network have been calculated, including the degree distribution, average path length, clustering coefficient and Pearson correlation coefficient. The typical 4-motifs of the network have also been evaluated. Most importantly, the robustness of both state controllability and structural controllability of the q-snapback network against five types of attacks (*i.e.*, targeted betweenness-based node-removal, targeted degree-based node-removal, random node-removal, targeted edge-removal, and random edge-removal) have been simulated with comparisons to the multiple congruence network and the generic scale-free network, showing that the multiplex q-snapback network has the strongest robustness of both controllabilities due to its rich inherent chain- and loop-motif structure. The finding reveals that, to build a network with strong robustness of controllabilities against node- and/or -edge removal attacks, it is advantageous to embed more chain- especially loop-microstructures. Whether or not such networks are also good in data traffic management, multi-agent systems and industrial assembly-line automation, as well as other networking performances, remains an important topic for future investigation.

Like the MCN, the q-snapback network has a backbone chain, which has advantage in robustness but also has disadvantage as being less structurally flexible in modeling diverse real-world networks. Looking forward, it is foreseeable that the new q-snapback network model not only have potential applications in industrial assembly-line automation (Fig. 1), but also in biological systems [24] and brain science [27] regarding its feedback control, controllability and information transmission.

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