Monte Carlo Simulation in Financial Modeling

Koray D. Simsek

Koray D. Simsek
is a professor of finance at Crummer Graduate School of Business at Rollins College in Winter Park, FL.
ksimsek@rollins.edu

KEY FINDINGS

- Monte Carlo simulation is a useful quantitative tool that incorporates consideration of uncertainty in many financial models.
- Random number generation and various other statistical concepts are central to the successful implementation of Monte Carlo simulation.
- Option pricing and portfolio performance/risk models are among the most common application areas for Monte Carlo simulation in portfolio management.

ABSTRACT

Models in asset management require consideration of uncertainty. Monte Carlo simulation is a popular quantitative tool that assigns random values to input variables in order to draw inferences about an uncertain outcome. This article explains and illustrates the main characteristics of Monte Carlo simulation and presents examples for its application in option pricing, portfolio insurance, and portfolio risk management.

Monte Carlo simulation is a quantitative method that uses randomly generated samples to evaluate outcomes and decisions under uncertainty. In this article, I review the main characteristics of Monte Carlo (MC) simulation and present examples of its applications in financial modeling. The conceptual origins of Monte Carlo simulation can be traced back to the "Buffon's needle" experiment in the early 18th century. However, the modern version of the method was developed and first used in the late 1940s during the Manhattan project by prominent scientists at the Los Alamos National Laboratory. The invention is attributed to Stanislaw Ulam and John von Neumann, while the name Monte Carlo was suggested to Ulam by Nicholas Metropolis. The timing naturally coincides with the development of the first electronic programmable computer, as Monte Carlo simulation requires computational tools to produce meaningful results.

This article begins by highlighting the main features of Monte Carlo simulation models as they relate to the common steps of this approach. Next, it demonstrates the use of Monte Carlo simulation in asset management through examples in risk measurement and option pricing. For each area of application, comparisons with other methods are presented.

MAIN CHARACTERISTICS OF MONTE CARLO SIMULATION MODELS

Monte Carlo simulation helps financial decision makers to explicitly model the source of uncertainty. For example, the future return on a stock portfolio depends on those of its constituent stocks. In this context, portfolio return is uncertain since the stock returns are not known in advance. Therefore, the true source of uncertainty would be the stock returns, which should be considered as input variables in a portfolio model, while the portfolio return is the output variable. Similarly, the price of a stock option is linked to its future uncertain payoff and, therefore, depends on the underlying stock's price, which constitutes the true source of uncertainty. From a modeling perspective, the future path of the stock price is an input variable and is mathematically linked to the option price, which is the output variable.

With respect to this relationship between input and output variables, running a Monte Carlo simulation involves several steps. The actual number of steps and their order of execution may depend on the purpose of the simulation; however, a typical setup can be outlined as follows:

- *Step 1*: Generating a random value for each input variable according to a specified distribution or a complex stochastic process,
- *Step 2*: Converting the inputs to one or more output variables using a direct formula or a detailed mathematical model,
- *Step 3*: Repeating Steps 1 and 2 as many times as desired, where each repeat is usually called a trial or a scenario,
- *Step 4*: Summarizing the distribution of output variable(s) across all scenarios.

The following subsections provide insights about various aspects of these steps.

Random Number Generation

As outlined in Step 1, at the heart of each Monte Carlo simulation lies a set of randomly generated numbers. Despite the use of word "random" in the terminology, Monte Carlo simulations typically work with pseudorandom numbers. These numbers are formulated to follow a very long but deterministic sequence according to an algorithm, however, they appear to bear characteristics of a random uniform distribution between 0 and 1.

The roots of these pseudorandom number can be traced back to John Von Neumann, one of the inventors of the modern Monte Carlo methods; however, the interest in developing faster algorithms with better randomness characteristics has been very high since the early 1950s. The most popular pseudorandom number generator currently in use is the MT19937 implementation of the Mersenne Twister algorithm. The RAND function in Microsoft Excel and the *random* module in Python uses this algorithm while it is also the default option in R.

One obvious advantage of pseudorandom number generators is reproducibility. In other words, they enable someone else to replicate a Monte Carlo simulation using the same set of random numbers. This is possible through the specification of a seed value, which points to the same initial random number. Since the algorithm is deterministic a given seed will always provide the same sequence of random numbers guaranteeing perfect reproducibility. Exhibit 1 presents the first 10 random numbers in the sequence that corresponds to seed values 1 and 2, respectively, generated using Microsoft Excel's VBA function Rnd. On any computer with the current version of Excel, the same exact random number sequences will be generated for these seed values.

EXHIBIT 1

First 10 Random Numbers in the Sequence That Correspond to Seed Values 1 and 2 in Microsoft Excel (using the VBA function rnd)

Random Numbers for Seed = 1	Random Numbers for Seed = 2
0.3335753083229060	0.5828428864479060
0.0681638717651367	0.5703611373901360
0.5938293337821960	0.3372375369071960
0.7660394906997680	0.7858148813247680
0.1892893910408020	0.3799632191658020
0.5373985767364500	0.2153770923614500
0.3269943594932560	0.5430588126182550
0.3939369916915890	0.9957436323165890
0.0734191536903381	0.5179992318153380
0.8315424919128410	0.7478022575378410

SOURCE: Author's calculations.

Once uniformly distributed random numbers are generated, they can be easily converted to random variables that follow other popular commonly used distributions. For example, in order to generate standard normally distributed random variables, the inverse transformation method can be used in Microsoft Excel via NORM.S.INV(RAND()). If the objective is to generate random variables for a finite discrete distribution, one of Excel's lookup functions (such as VLOOKUP) can be utilized to perform a search among uniformly distributed random numbers and transform those to the finite outcomes by matching the cumulative probabilities.

Probabilistic Modeling of Inputs

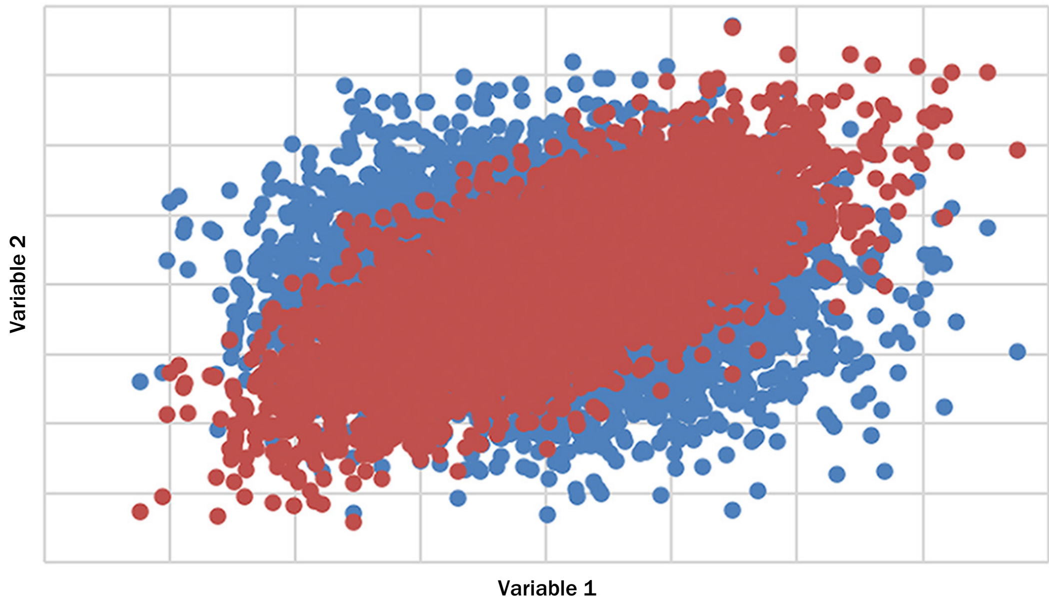
The main purpose of Monte Carlo simulation is to incorporate uncertainty into a financial model. As a result, the decision maker has full control on the choice of variables that would be modeled probabilistically as opposed to deterministically. From a theoretical perspective, there is no limit on the number of variables that could be probabilistic. However, as with any risk analysis methodology, it would make sense to designate the most impactful variables as probabilistic to reduce the computational cost of the model while ensuring that desired benefits are achieved. From this perspective, impact is often aligned with relevance and degree of uncertainty. For example, if the purpose of the simulation is to price an option on a highly volatile stock, it would make sense to simulate the future price of the stock by assuming that its volatility is uncertain. However, if the interest rates are expected to be stable over the life of the option, one may assume constant interest rates instead of simulating paths for them.

A typical choice for modeling uncertainty in Monte Carlo simulation is a continuous-time stochastic process, such as geometric Brownian motion (GBM), which was famously used by the Black–Scholes option pricing model (Black and Scholes 1973). Since GBM has a closed-form analytical solution, its simulation is an exact one. However, for more complex stochastic processes or multifactor models, discrete versions of the continuous-time models should be utilized so that the length of discrete time steps—in other words, how often the variable needs to be simulated over the time horizon—becomes another choice the modelers should make. The use of discretization schemes, such as Euler and Milstein, inevitably introduces errors into the simulation due to their inexact nature (Glasserman 2004). This highly technical topic is beyond the scope of this article.

In many instances, there are multiple probabilistic variables. For example, the payoff of a stock option may depend on the future stock price, which may be affected by the volatility of the stock that may be correlated with the returns on the stock. Similarly, the future value of a portfolio depends on the future prices of its constituents while the returns on these constituents may be correlated with one another. Monte Carlo simulations provide substantial flexibility in modeling correlations. Cholesky factorization is a popular technique that converts uncorrelated random variables from a standard normal distribution to those that are correlated. This is done through a simple equation in the case of two variables while higher orders require use of matrix algebra. Exhibit 2 presents two overlaid scatter plots. The blue one shows the absence of a statistical relationship between two independent normally distributed

EXHIBIT 2

Scatter Plot for a Pair of Independent Variables (in blue) Overlaid with Their Correlated Counterparts (in red) after Transformation via Cholesky Factorization



SOURCE: Hypothetical data simulated by the author.

variables while the red one shows the positive relationship after the second variable is transformed into a +0.7 positively correlated variable through Cholesky factorization.

Parameter values for the stochastic processes or statistical distributions are often chosen to match the time-series or cross-sectional characteristics of the market data using a calibration technique such as maximum-likelihood estimation. Built on top of all the probabilistic and deterministic inputs is a financial model that generates the output variable(s). This financial model itself is practically the same as the one that is built for completely deterministic inputs. The only difference for a Monte Carlo simulation is the fact that some parameters of the financial model are based on randomly generated values, and therefore, the output variable (and the actionable decision associated with it) is likely to change every time a new scenario is generated.

Summarizing Output Variables

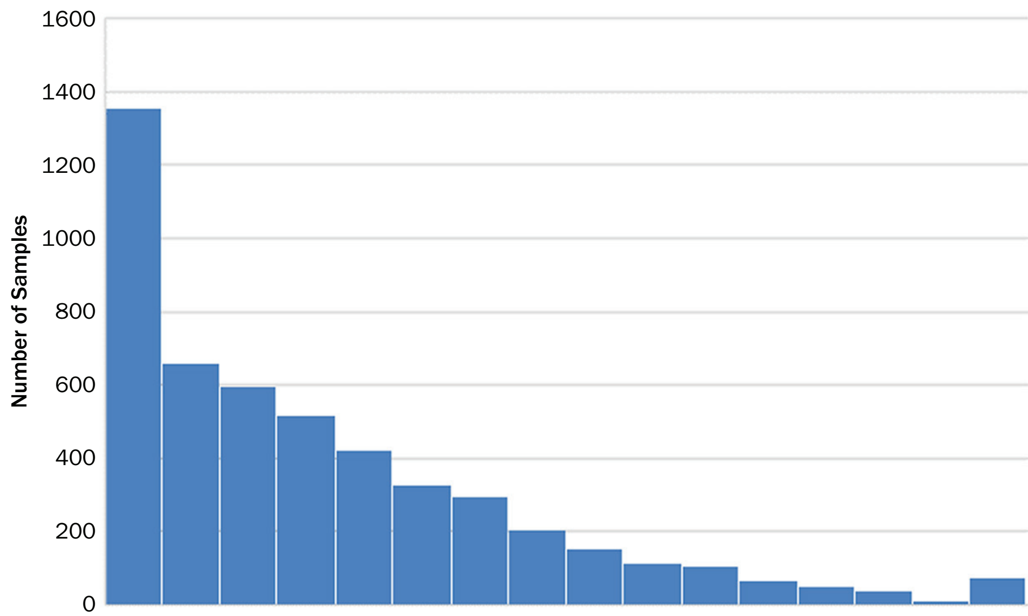
A Monte Carlo simulation effectively produces many realizations of an output variable. In the case of option pricing, it generates an option payoff for each scenario of the underlying asset values. When computing the downside risk of a portfolio, each sample of portfolio risk factors is converted into a possible future portfolio value.

As a result, once many scenarios are created as highlighted by Step 3 above, many realizations of the output variable are produced. These realizations effectively generate an empirical distribution for the output variable. As with all such distributions, one can plot a histogram or produce a frequency table for the output variable. Exhibit 3 displays such a histogram for the payoff of a European call option.

As informative as it is, a histogram provides too much detail and needs to be converted into summary statistics for actionable financial decisions. The mean of the empirical distribution provides an unbiased estimate of the output variable. Since the Monte Carlo simulation assumes that each scenario is independent of one

EXHIBIT 3

Histogram for the Payoff of a European Call Option in a 5,000-Scenario Monte Carlo Simulation



SOURCE: Author's calculations.

EXHIBIT 4

Summary Statistics for an Option Pricing Example with Monte Carlo Simulation

Mean Estimate	Standard Deviation of the Estimate	Standard Error of the Estimate	95% Confidence Interval for the Mean Estimate
10.36330	14.80523	0.20938	9.95293–10.77368

SOURCE: Author's calculations.

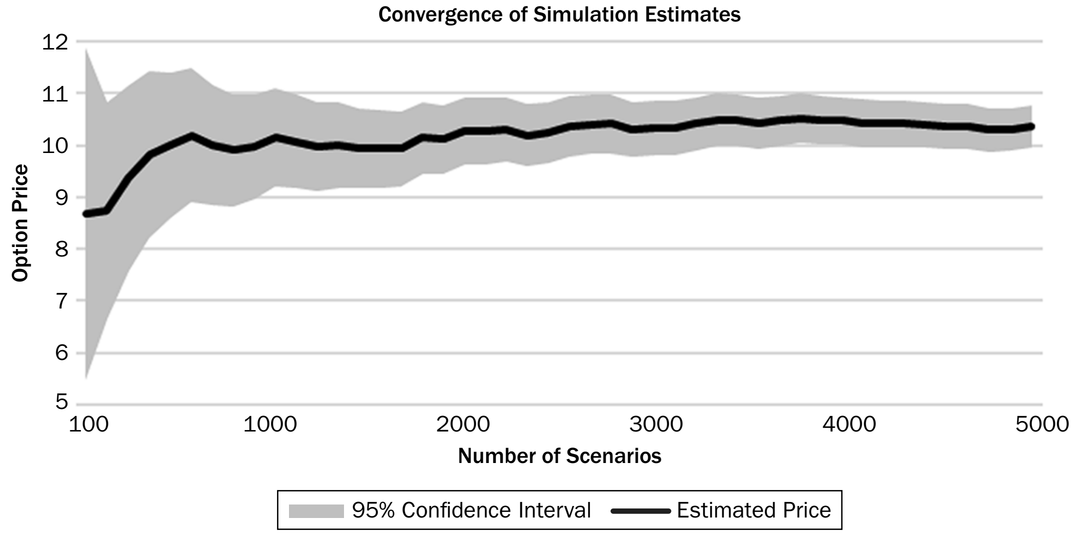
another and has the same chance to be sampled as any other scenario, the mean becomes the average of all realizations of the output variable. It turns out that the average is susceptible to sampling errors. In other words, for two samples of 1,000 scenarios, the average will likely be quite different. The sampling error for the mean can be measured by the standard error of the estimate, which can be computed as the standard deviation of the output variable divided by the square root of the number of scenarios. Since the mean is quite sensitive to the set of scenarios used in a simulation, decision makers are advised to compute a confidence interval for its estimate. Due to the central limit theorem, the mean for the output variable is normally distributed around the average value. More specifically, the 95% confidence interval can be defined to be approximately two (or 1.96) standard errors around the mean. Exhibit 4 presents the 95% confidence interval estimate as well as the mean estimate for the price of a European call option whose payoff histogram is given in Exhibit 3.

Variance Reduction Methods

Despite the relative mathematical simplicity of Monte Carlo simulation, especially when compared with analytical methods for finding exact solutions, one of its obvious disadvantages is the need to generate a significant number of random samples to achieve an acceptable level of precision and accuracy. More specifically, due to the well-known central limit theorem, the number of simulated scenarios needs to be increased by approximately four times to cut the estimation errors by half. Exhibit 5 demonstrates how the estimation errors are reduced by increasing the number of

EXHIBIT 5

Convergence of Monte Carlo Estimates as the Number of Scenarios Increases for an Option Pricing Simulation



SOURCE: Author's calculations.

scenarios for an option pricing simulation. The 95% confidence interval, as measured by the height of the shaded area begins shrinking quickly initially, but this improvement in accuracy slows down as the number of scenarios increase linearly. This slow rate of convergence has led researchers to use various techniques to reduce the variance of the simulation outputs.

One popular and simple variance reduction technique is the *antithetic variates* method, which relies on using the antithetic counterparts of already generated random numbers. In this method, once a uniform random number between 0 and 1 is generated, its antithetic counterpart, which is equal to one minus the random number, is also utilized. For normally distributed random numbers, this is equivalent to generating the positive and the negative of the same value. Both the random number and its antithetic counterpart are used to generate two distinct values for the output variable. The average of the two output values becomes the estimate generated by this scenario. Therefore, for a given number of scenarios, implicitly twice as many random numbers are used, while half of these random numbers are negatively correlated with the other half. This negative correlation creates a reduction in the variance of the overall estimate, which is based on the average across all scenarios. In Exhibit 6, results for two Monte Carlo simulations are compared. The first one is based on one set of random numbers, while the second one applies the antithetic variates technique to the same set of random numbers. This simulation with 5,000 scenarios prices a one-year at-the-money European call option on a stock that follows GBM with a risk-free rate of 5% and volatility of 20% assuming an initial stock price of $100. The standard error is approximately halved through this approach, which is computationally less costly than increasing the number of scenarios by four-fold.

The next section presents three different applications of Monte Carlo simulations in finance.

EXHIBIT 6

Option Pricing Example Illustrating the Benefits of Variance Reduction on the Precision of Simulation Estimates

Variance Reduction Technique	Price Estimate ($)	Standard Error of the Estimate ($)
None	10.3633	0.20938
Antithetic Variates	10.3630	0.10267

SOURCE: Author's calculations.

APPLICATIONS OF MONTE CARLO SIMULATION IN PORTFOLIO MANAGEMENT

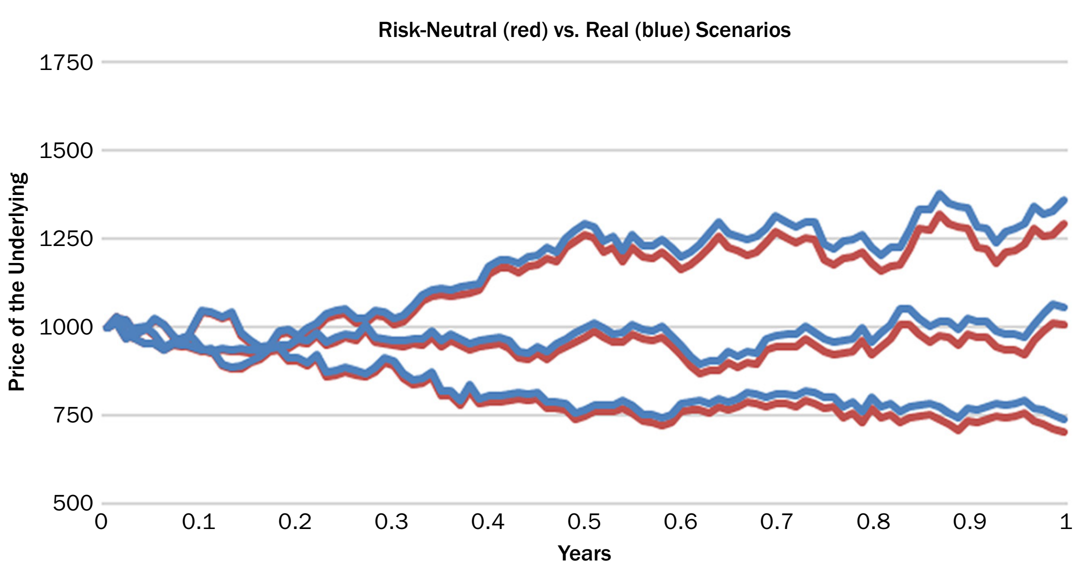
Application areas of Monte Carlo simulation are wide-ranging, spanning from physical sciences and engineering to computational biology, business, and law. In finance, Hertz (1964) introduced a capital budgeting approach that incorporates Monte Carlo simulation into risk analysis. Later, Boyle (1977) proposed the use of Monte Carlo simulation in option pricing. A related application is on option-based portfolio insurance, a concept formalized by Leland (1980). Computation of portfolio risk measures such as value-at-risk (VaR) using Monte Carlo simulation has been made popular by the public release of the RiskMetrics™ methodology (see J.P. Morgan and Reuters 1996).

Option Pricing

Option pricing is based on the concept of risk-neutral pricing, which asserts that the fair value of an option is its expected payoff in a risk-neutral world discounted at the risk-free rate. This does not imply that one must assume risk-neutrality of all investors. However, since it is not trivial to find the appropriate discount rate for a risky and nonlinear investment like an option, a transformation to the risk-neutral world makes valuation more plausible. According to the risk-neutrality assumption, every asset (including the risky underlying asset and the option on it) earns the risk-free rate on average. With this probabilistic assumption, one can model the price of the underlying with a risk-free growth rate, compute the expected option payoff with risk-neutral probabilities, and finally discount the expected option payoff at the risk-free rate. In the case of Monte Carlo simulation, expected payoff is simply the average payoff across all the scenarios. Exhibit 7 illustrates the difference between paths simulated using the risk-neutral assumption and those under the actual growth rate assumption. The blue lines have the same type of uncertainty as the red lines (i.e., with similar directional moves) and their level differences are due to the higher

EXHIBIT 7

Comparison of Simulated Paths Using Risk-Neutral vs Actual Growth Rates



SOURCE: Author's calculations based on hypothetical data.

drift of the risky asset in the real world with respect to the risk-free rate of the risk-neutral world.

When compared with the Black–Scholes formula, which provides an analytical closed-form solution for European option prices under the GBM assumption, or the lattice-based methods such as the binomial option pricing method (see Cox et al. 1979), Monte Carlo simulation is the preferred methodology when either the option of interest is a complex path-dependent option such as lookback options or the underlying process is so complex that a mathematical solution is not tractable. For American options, there are regression-based algorithms that make the use of Monte Carlo simulation feasible (see Carriere 1996); however, lattice-based methods have an edge in this domain due to their computational simplicity.

Asian option payoffs are based on the average price of an underlying asset over the life of the option. Therefore, Monte Carlo simulation can be used to generate price path scenarios for the underlying asset and the average price can be computed for each scenario. Under the GBM assumption, there exists a closed-form solution for the so-called geometric Asian option, where the payoff is computed as a function of the continuous-time geometric average price of the underlying asset. In practice, though, Asian options are of the arithmetic type, where the average price of the underlying is computed across discrete time intervals, such as the average of end-of-month prices. Monte Carlo simulation is clearly suitable for pricing arithmetic Asian options. However, its precision can be significantly improved using a variance reduction method called *control variates*. In this method, a preliminary simulation is conducted to find the regression relationship between the discounted payoffs of the geometric and arithmetic Asian options. Since the analytical solution of the geometric Asian option is known, the simulation pricing error for that option is reflected in the simulation price of the arithmetic Asian option using the regression slope as a multiplier for this error. Exhibit 8, adapted from Pachamanova and Fabozzi (2010), illustrates the improvement in the precision of price estimates for arithmetic Asian options using control variates for the Monte Carlo simulation. The computational success of this method depends on the availability of suitable and highly correlated control variates. When applicable, the enhancement is quite significant, as demonstrated in this example.

Portfolio Insurance

Portfolio insurance strategies aim to protect an investment against dropping below a certain level. Leland (1980) showed that plain option-based portfolio insurance can be achieved by holding the reference portfolio and buying a put option on the portfolio. The level of strike price on the put option determines the level of protection that will be achieved in the portfolio insurance strategy.

Perold and Sharpe (1988) used Monte Carlo simulation to compare several portfolio insurance strategies with other dynamic investment strategies. Since put-option-based portfolio insurance can be created with an investment in the put option and the underlying portfolio, in principle, any method that is suitable for option pricing can be used to analyze portfolio insurance strategies. However, when the underlying portfolio evolves according to a complex process, Monte Carlo simulation is the preferred methodology.

Exhibit 9, based on a model with stochastic interest rates and stochastic volatility used in Goltz et al. (2008), compares the performance of portfolio insurance strategies with all-equity and all-bond investments in a Monte Carlo simulation. This simulation

EXHIBIT 8

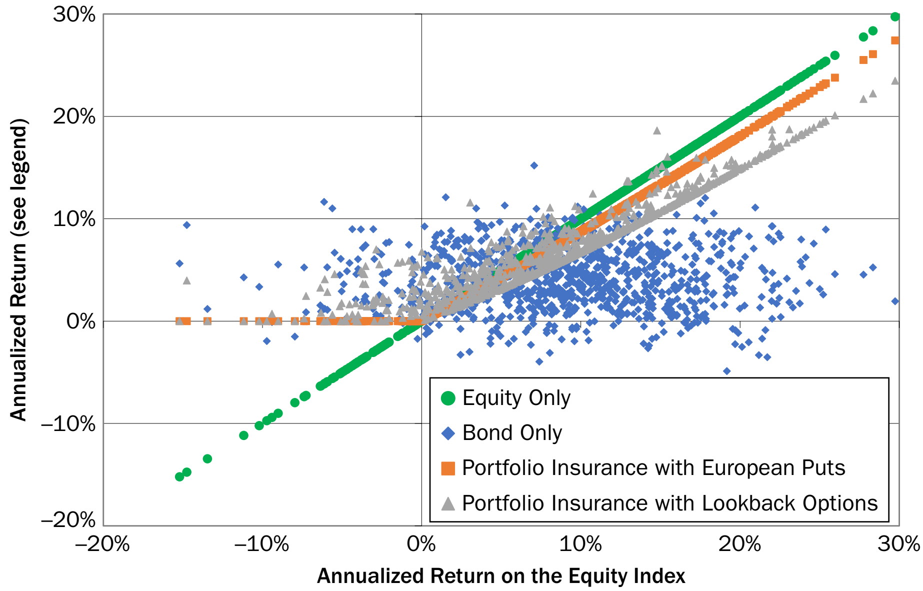
Pricing Asian Options with Monte Carlo Simulation and Improving Precision with Variance Reduction

Variance Reduction Technique	Price Estimate	Standard Error	95% Confidence Interval
None	1.8332	0.1009	1.6355–2.0309
Control Variates	1.8381	0.0026	1.8329–1.8432

SOURCE: Author's calculations based on hypothetical data.

EXHIBIT 9

Monte Carlo Simulation Results of Portfolio Insurance Strategies Using Options



SOURCE: Author's calculations based on hypothetical data.

shows that portfolio insurance using European put options (in orange) protects the downside of the portfolio by sacrificing a small portion of the upside in all positive-return scenarios (compared with all equity investment in green). When lookback options that are more expensive are preferred within the portfolio insurance strategy (in gray), upside potential is more restricted; however, the potential to beat the index even in up markets is attained. Monte Carlo simulation facilitates the presentation of these strategies in a more intuitive way.

Measuring Portfolio Risk

Value-at-risk (VaR) is a downside risk measure used in banking and portfolio risk management that identifies the amount of money such that the portfolio losses will not exceed given a certain confidence level. Despite its relative popularity among practitioners and regulators, its performance in risk management has been problematic during periods of market stress, such as the 2007–2008 Global Financial Crisis. Expected shortfall (ES), which is defined as the expected portfolio loss that exceeds the VaR level, displays better risk measurement characteristics and performs better in such scenarios.

There are three commonly used methods for computing VaR or ES: Model building, historical simulation, and Monte Carlo simulation.

Model building (also called the variance/covariance) approach assumes the presence of a multivariate normal distribution for the portfolio constituents. Based on the estimation of a variance/covariance matrix from the historical data for the constituents, portfolio variance is computed. VaR is then calculated as a percentile of the normal distribution with that variance. ES is computed using the conditional expectation also but with the normal distribution assumption. Even though this is a computationally simple method, it doesn't work well in practice due to the unrealistic normal distribution assumption.

EXHIBIT 10

Value-at-Risk and Expected Shortfall Measures for a Portfolio Invested in 10 S&P 500 GICS Sectors

Risk Measure	Variance-Covariance Approach	Historical Simulation	Monte Carlo Simulation
Value-at-Risk	7.79%	6.27%	5.96%
Expected Shortfall	13.63%	9.46%	10.71%

SOURCE: Author's calculations using S&P 500 sector returns from FactSet.

Historical simulation samples each day randomly from the historical database by assigning either equal probability to each day or varying probabilities. Once a day is sampled, the returns for each portfolio constituent is taken as if that day is going to repeat itself in the future. As in Monte Carlo simulation, as many days from the history are samples as needed for statistical convergence. This method provides a more realistic approach compared to the normal distribution assumption; however, it is still restrictive as it does not allow any new scenarios that has not happened in the past. Moreover, if correlations between assets change to a situation that is not presented in the historical data, this method will be insufficient.

Monte Carlo simulation is the most flexible of these three methods, but it is also the most expensive one in terms of computation costs. The random nature of asset returns can be modeled using any stochastic process desired, and correlations can be easily embedded using Cholesky factorization. Using the main features highlighted in the previous section, a Monte Carlo simulation provides risk managers with the greatest flexibility in measuring the risk of their portfolios.

In Exhibit 10, a risk analysis of a cap-weighted portfolio of S&P 500 is presented using all three methods highlighted above. For the model-building approach, we assume that the returns of the 10 GICS sectors (communication services is excluded due to lack of sufficient historical data) are normally distributed by estimating the annualized monthly variance-covariance matrix between 2016 and 2023. For the historical simulation, annual rolling holding period returns are calculated between October 2016 and February 2013 and sampled once without replacement. For Monte-Carlo simulation, we use the same variance-covariance matrix estimated and generate returns from the lognormal distribution using Cholesky factorization. The market weights for the 10 sectors at the end of February 2023 are used. Since this time period represents a significant bull market with right-skewed returns, the normality assumption is violated. As a result, historical simulation results in the most optimistic estimate.

CONCLUSIONS

This article reviews Monte Carlo simulation as a useful and popular quantitative tool for incorporating uncertainty and risk analysis into asset management. At the core of every Monte Carlo simulation is a pseudorandom number generator. Software used in statistical analysis and spreadsheet modeling as well as programming languages all have built-in algorithms that generate uniformly distributed random numbers. These pseudorandom numbers can be easily transformed into other statistical distributions or fed into complex financial models to generate needed outputs. Once the full distribution of the output variable is determined, summary statistics that reflect the objective of the simulation is produced.

Among many uses of Monte Carlo simulation in finance, the most well-known examples can be found in option pricing, portfolio insurance, and risk measurement. In option pricing and portfolio insurance, when analytical methods such as the Black–Scholes model fails either due to the complexity of the option or the underlying process, Monte Carlo simulation provides an answer. This is especially true for path-dependent options. Even though the lattice models are standard in American option pricing, regression-based algorithms enable the use of Monte Carlo simulation as an alternative in that domain. When measuring risk via value-at-risk or expected shortfall, Monte Carlo simulation presents a flexible alternative to normal distribution–based parametric methods and historical simulation. Especially in market conditions under stress, this flexibility becomes crucial.

As the computational capabilities and algorithmic efficiency are enhanced through technological progress, the importance of Monte Carlo simulation as a financial decision-making tool under uncertainty naturally increases. It remains as an important quantitative method in the toolbox of financial analysts.

REFERENCES

Black, F., and M. Scholes. 1973. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (3): 637–654.

Boyle, P. P. 1977. "Options: A Monte Carlo Approach." *Journal of Financial Economics* 4 (3): 323–338.

Carriere, J. 1996. "Valuation of the Early-Exercise Price for Options Using Simulations and Nonparametric Regression." *Insurance: Mathematics and Economics* 19 (1): 19–30.

Cox, J. C., S. A. Ross, and M. Rubinstein. 1979. "Option Pricing: A Simplified Approach." *Journal of Financial Economics* 7 (3): 229–263.

Glasserman, P. 2004. *Monte Carlo Methods in Financial Engineering*. New York: Springer.

Goltz, F., L. Martellini, and K. D. Simsek. 2008. "Optimal Static Allocation Decisions in the Presence of Portfolio Insurance." *Journal of Investment Management* 6 (2): 37–56.

Hertz, D. B. 1964. "Risk Analysis in Capital Investment." *Harvard Business Review* 42: 95–106.

J.P. Morgan and Reuters. 1996. *RiskMetrics™—Technical Document*. 4th edition. New York, NY.

Leland, H. E. 1980. "Who Should Buy Portfolio Insurance?" *The Journal of Finance* 35 (2): 581–594.

Pachamanova, D. A., and F. J. Fabozzi. 2010. *Simulation and Optimization in Finance: Modeling with MATLAB, @RISK, or VBA*. Hoboken, NJ: John Wiley & Sons.

Perold, A. F., and W. F. Sharpe. 1988. "Dynamic Strategies for Asset Allocation." *Financial Analysts Journal* 44 (1): 16–27.