Modern Portfolio Theory

2

2.1 Introduction

Investors universally seek ideal investments that offer substantial returns. However, given market volatility, this pursuit is challenged by uncertainty. Instead of a perfect investment, a sound strategy that balances high returns with minimal risk is the more practical goal. In today's investment climate, such a strategy is the closest we can get to an optimal/efficient solution.

While the concept of maximizing returns and minimizing risk might seems intuitive today, the mean-variance strategy didn't become a standard investment approach until the late twentieth century. Before the mean-variance strategy, investors often lacked a systematic approach to managing risk. This groundbreaking approach revolutionized investment theory by offering a structured framework to optimize portfolios based on individual risk tolerance and return objectives. While the ideal investment remains elusive, sophisticated strategies like mean-variance analysis empower investors to navigate the complexities of the financial world.

In 1952, Harry Markowitz revolutionized the field of portfolio management with a groundbreaking discovery. He devised an intuitive formula that enables investors to mathematically balance their risk tolerance against their expectations of reward, ultimately crafting an optimal portfolio. The objective of this chapter is to delve into the principles of mean-variance analysis and explore the optimal allocation advocated by Modern Portfolio Theory (MPT), as proposed by Markowitz in his seminal work (Markowitz, 1952). By employing MPT, investors aim to maximize their overall return while keeping their exposure to risk at an acceptable level.

The MPT presents a stark departure from traditional qualitative stock selection methods. Yet, when wielded with precision, this quantitative portfolio management tool has the potential to yield well-diversified and lucrative investment portfolios. Despite its 70-year history, the core principles of MPT remain as relevant as ever.

M. Donadelli et al., *Essentials of Financial Economics*, Springer Texts in Business and Economics, https://doi.org/10.1007/978-3-031-86189-5_2

Furthermore, owing to its simplicity in application and interpretation, the mean-variance approach continues to serve as a prominent benchmark for asset allocation strategies.

In Sect. 2.2, we briefly review the risk–return trade-off and decision-making under uncertainty, building on the previous chapter. Section 2.3 introduces the *global minimum variance portfolio*, where we allocate a fixed budget (i.e., resources/wealth) across stocks to minimize portfolio variance. We start with a simple two-asset economy (Sect. 2.3.1) and then extend it to a three-asset economy (Sect. 2.3.1). Sections 2.4 and 2.5 present the mathematical foundation of Markowitz's mean-variance optimal allocation framework for N risky assets and for N risky assets with a risk-free asset, respectively. Finally, Sect. 2.6 provides a practical application of the mean-variance framework to a portfolio of five stocks.

2.2 The Risk–Return Trade-Off

In the context of MPT, investors are typically assumed to be risk-averse, meaning they prefer higher expected returns for a given level of risk. Variance of portfolio returns is commonly used as a proxy for risk. Consider an investor faced with a choice between two risky portfolios, Portfolio 1 and Portfolio 2, each consisting of N risky assets.

Consider then a scenario, denoted as **Scenario A**, where the following conditions are met:

1. the expected return of Portfolio 1 is greater than the expected return of Portfolio 2
$$\mathbb{E}(\tilde{r}_1) > \mathbb{E}(\tilde{r}_2)$$
2. the variance of Portfolio 1 is less than or equal to the variance of Portfolio 2, or alternatively, the standard deviation of Portfolio 1 is less than or equal to that of Portfolio 2.
$$\sigma_1^2 \leq \sigma_2^2 \quad \text{or} \quad \sigma_1 \leq \sigma_2$$

In this scenario, according to the mean-variance criterion, Portfolio 1 is the preferred choice over Portfolio 2. This preference arises from Portfolio 1's superior risk–return profile: it offers a higher expected return and lower variance. As risk-averse investors seek to maximize returns while minimizing risk, Portfolio 1 is the more attractive option. In essence, Portfolio 1 exhibits *mean-variance dominance* over Portfolio 2.

Consider a different scenario, denoted as **Scenario B**, where the following conditions hold:

1. the expected return of Portfolio 1 is greater than or equal to the expected return of Portfolio 2.

$$\mathbb{E}(\tilde{r}_1) \geq \mathbb{E}(\tilde{r}_2)$$

2. the variance of Portfolio 1 is greater than the variance of Portfolio 2, or alternatively, the standard deviation of Portfolio 1 is greater than that of Portfolio 2.

$$\sigma_1^2 > \sigma_2^2 \quad \text{or} \quad \sigma_1 > \sigma_2$$

In this scenario, determining a clear preference between Portfolio 1 and Portfolio 2 becomes challenging. Neither portfolio demonstrates mean-variance dominance, indicating that the mean-variance criterion alone may not always suffice in decision-making.

William Sharpe proposes an alternative approach for situations like **Scenario B**, suggesting the comparison of the portfolios' *Sharpe ratios*. The *Sharpe ratio*, defined as the ratio of expected return to standard deviation (or risk), offers a measure of a portfolio's average performance per unit of risk. Although the *Sharpe ratio* is conventionally defined with respect to the excess return over a risk-free rate, for simplicity, we assume a risk-free rate of return (r_f) to be zero in this context. However, solely relying on the *Sharpe ratio* for decision-making assumes equal weightage of return and risk. This implies that investors perceive a doubling of risk (σ_j) as acceptable if accompanied by a doubling of return ($\mathbb{E}(\tilde{r}_j)$). Yet, this approach may not necessarily align with the preferences of risk-averse investors, who are typically unwilling to accept disproportionate increases in risk for incremental returns.[1]

We will discover that a subset of portfolios satisfying the mean-variance criterion are known as *efficient portfolios* (see Sect. 2.4). These portfolios represent asset combinations that offer the highest return for a given level of risk or, conversely, the lowest risk for a given level of return. To illustrate this concept, consider **Scenario C**, where:

1. The expected return of Portfolio 1 is less than or equal to that of Portfolio 2:

$$\mathbb{E}(\tilde{r}_1) \leq \mathbb{E}(\tilde{r}_2)$$

2. The variance of Portfolio 1 is greater than that of Portfolio 2:

$$\sigma_1^2 > \sigma_2^2$$

[1] Indeed, similar to the expected value, the *Sharpe ratio* may not offer a robust criterion for evaluating risky investments. As discussed in Chap. 1, a more nuanced and comprehensive approach is often required when comparing random cash flows, especially for risk-averse investors.

In this case, Portfolio 1 is *inefficient*. A rational, risk-averse investor would never choose Portfolio 1 over Portfolio 2, as the latter offers both higher expected returns and lower risk.

The concept of *efficient* and *inefficient* portfolios is fundamental to MPT. By identifying *efficient portfolios*, investors can optimize their allocations to maximize returns while minimizing risk.

2.3 The Global Minimum Variance Portfolio

Assume for now that the investor aims to distribute their total wealth across various assets to minimize portfolio risk. How can the investor effectively mitigate overall risk? Should the wealth be concentrated in a single asset (e.g., the one with the lowest standard deviation), or is it more prudent to diversify across multiple assets? If diversification is the preferred approach, what should be the optimal allocation for each asset? Section 2.3.1 provides a solution for scenarios involving two risky assets, while Sect. 2.3.2 extends the analysis to cases with three or more risky assets. In both sections, the primary objective is to determine the optimal allocation of wealth among risky assets to minimize portfolio risk.

2.3.1 2-Risky Assets

Let us consider our investor who allocates a fraction w_1 of their wealth to asset 1 and the remaining fraction $w_2 = 1 - w_1$ to asset 2, thus ensuring that $w_1 + w_2 = 1$. The random return of this portfolio can be expressed as:

$$\tilde{r}_p = w_1 \tilde{r}_1 + w_2 \tilde{r}_2. \tag{2.1}$$

Here, we define the expected returns of $\tilde{r}_1$ and $\tilde{r}_2$ as $\mathbb{E}(\tilde{r}_1) = \mu_1$ and $\mathbb{E}(\tilde{r}_2) = \mu_2$, respectively. Consequently, the expected return of the portfolio is:

$$\mathbb{E}(\tilde{r}_p) = w_1 \mathbb{E}(\tilde{r}_1) + w_2 \mathbb{E}(\tilde{r}_2) = w_1 \mu_1 + w_2 \mu_2 = \mu_p, \tag{2.2}$$

where $\mathbb{E}(\tilde{r}_p) = \mu_p$. The variance of the portfolio is determined by:

$$\begin{aligned} \sigma_p^2 &= \mathbb{E}[\tilde{r}_p - \mu_p]^2 = \mathbb{E}[w_1(\tilde{r}_1 - \mu_1) + w_2(\tilde{r}_2 - \mu_2)]^2 \\ &= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_{1,2}, \end{aligned} \tag{2.3}$$

where $\sigma_{1,2}$ represents the covariance between $\tilde{r}_1$ and $\tilde{r}_2$.

It is worth noting that the covariance between two random variables can be expressed as the product of their correlation coefficient and the product of their standard deviations, denoted as $\sigma_{1,2} = \rho_{1,2}\sigma_1\sigma_2$, where $\rho_{1,2}$ represents the correlation between the returns of asset 1 and asset 2. With this understanding, we can reformulate the variance of our portfolio consisting of two risky assets as follows:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{1,2}\sigma_1\sigma_2. \tag{2.4}$$

To determine the *global minimum variance portfolio*, the investor must solve the following optimization problem:

$$\min_{\{w_1,w_2\}} \quad \sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{1,2} \quad \text{subject to} \quad w_1 + w_2 = 1. \tag{2.5}$$

Here, the constraint $w_1 + w_2 = 1$ indicates that the sum of the weights cannot exceed 100%. The Lagrangian for this problem is:

$$\underbrace{\Lambda(w_1, w_2, \lambda)}_{\text{Lagrangian Function}} = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{1,2} + \lambda(1 - w_1 - w_2),$$

where the first-order conditions (FOCs) for a minimum are:

$$\frac{\partial\Lambda(w_1, w_2, \lambda)}{\partial w_1} = 0 = 2w_1\sigma_1^2 + 2w_2\sigma_{1,2} - \lambda \tag{2.6}$$

$$\frac{\partial\Lambda(w_1, w_2, \lambda)}{\partial w_2} = 0 = 2w_2\sigma_2^2 + 2w_1\sigma_{1,2} - \lambda \tag{2.7}$$

$$\frac{\partial\Lambda(w_1, w_2, \lambda)}{\partial \lambda} = 0 = 1 - w_1 - w_2. \tag{2.8}$$

The FOCs (2.6)–(2.8) yield a system of three equations with three unknowns. Solving this system provides the weights w_1 and w_2 for the *global minimum variance portfolio*. Equation (2.8) allows us to express w_2 as $1 - w_1$, which we can substitute into Eqs. (2.6) and (2.7) to obtain:

$$2w_1\sigma_1^2 + 2(1 - w_1)\sigma_{1,2} = \lambda = 2(1 - w_1)\sigma_2^2 + 2w_1\sigma_{1,2} = \lambda.$$

By solving for w_1 and then using $w_2 = 1 - w_1$, we derive the weights that define the *global minimum variance portfolio*, denoted as $w_{1,gmv}$ and $w_{2,gmv}$:

$$w_1 = w_{1,gmv} = \frac{\sigma_2^2 - \sigma_{1,2}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{1,2}}, \quad w_2 = w_{2,gmv} = \frac{\sigma_1^2 - \sigma_{1,2}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{1,2}}. \tag{2.9}$$

An alternative approach to tackling the minimization problem (2.5) involves directly substituting the constraint $w_2 = 1 - w_1$ into the objective function, σ_p^2. This integration enables expressing both the expected return and portfolio variance solely as functions of w_1, thus simplifying the analysis. Formally, we have:

$$\mu_p = f(w_1) = \mu_2 + w_1(\mu_1 - \mu_2)$$
$$\sigma_p^2 = f(w_1) = w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1(1-w_1)\rho_{1,2}\sigma_1\sigma_2$$
$$= \sigma_2^2 + w_1^2\left[\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2\right] + 2w_1\left[\rho_{1,2}\sigma_1\sigma_2 - \sigma_2^2\right].$$

As previously discussed, our objective remains centered on determining the value of w_1 corresponding to the potential minimum variance (referred to as $w_{1,gmv}$). To ascertain the optimal proportions of the two assets required to achieve the lowest risk level, we must solve the equation $f'(w_1) = 0$. If both assets exhibit risk (i.e., $\sigma_1^2 > 0$ and $\sigma_2^2 > 0$), the equation $f'(w_1) = 0$ possesses a unique solution. Furthermore, since $f''(w_1) > 0$ for all w_1, the function f_{w_1} is strictly convex, ensuring that the solution $w_1 \equiv w_{1,gmv}$ represents a unique global minimum. This minimum can be determined as follows:[2]

$$\min_{\{w_1\}} \quad \sigma_p^2 = w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1(1-w_1)\rho_{1,2}\sigma_1\sigma_2$$
$$= \sigma_2^2 + w_1^2\left[\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2\right] + 2w_1\left[\rho_{1,2}\sigma_1\sigma_2 - \sigma_2^2\right]. \tag{2.10}$$

The FOC w.r.t. the unique choice variable w_1 is

$$\frac{d\sigma_p^2}{dw_1} = 0 = 2w_1\left[\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2\right] + 2\left[\rho_{1,2}\sigma_1\sigma_2 - \sigma_2^2\right]. \tag{2.11}$$

Solving (2.11) for w_1 yields the following *global minimum variance portfolio* weights:

$$w_1^* = w_{1,gmv} = \frac{\sigma_2^2 - \rho_{1,2}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2}, \quad (1-w_1^*) = w_{2,gmv} = \frac{\sigma_1^2 - \rho_{1,2}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2}.$$

The minimum value $f(w_{1,gmv})$ is derived by substituting the optimal solution $w_1^* = w_{1,gmv}$ into the definition of σ_p^2. Through algebraic manipulation, we arrive at the following closed-form solution for the risk level of the *global minimum variance portfolio*, $\sigma_{p,gmv}^2$:

$$\sigma_{p,gmv}^2 = f(w_{1,gmv})$$
$$= \sigma_2^2 + \frac{(\sigma_2^2 - \rho_{1,2}\sigma_1\sigma_2)^2}{(\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2)^2}\left[\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2\right]$$

[2] Note that the case where $\sigma_1^2 = \sigma_2^2$ with $\rho_{1,2} = 1$ cannot be considered. This implies $f(w_1) = \sigma_p^2 = \sigma_1^2 = \sigma_2^2$ for all w_1. Therefore, all portfolios have the same variance, and a unique min-variance portfolio cannot exist.

$$
\begin{aligned}
&+2\left(\frac{\sigma_2^2-\rho_{1,2}\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho_{1,2}\sigma_1\sigma_2}\right)\left[\rho_{1,2}\sigma_1\sigma_2-\sigma_2^2\right]\\
&=\sigma_2^2+\frac{(\sigma_2^2-\rho_{1,2}\sigma_1\sigma_2)^2}{(\sigma_1^2+\sigma_2^2-2\rho_{1,2}\sigma_1\sigma_2)}-2\frac{(\sigma_2^2-\rho_{1,2}\sigma_1\sigma_2)^2}{(\sigma_1^2+\sigma_2^2-2\rho_{1,2}\sigma_1\sigma_2)} \qquad (2.12)\\
&=\sigma_2^2-\frac{(\sigma_2^2-\rho_{1,2}\sigma_1\sigma_2)^2}{(\sigma_1^2+\sigma_2^2-2\rho_{1,2}\sigma_1\sigma_2)}\\
&=\frac{\sigma_1^2\sigma_2^2-\rho_{1,2}^2\sigma_1^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2-2\rho_{1,2}\sigma_1\sigma_2)}=\frac{\sigma_1^2\sigma_2^2(1-\rho_{1,2}^2)}{\sigma_1^2+\sigma_2^2-2\rho_{1,2}\sigma_1\sigma_2}.
\end{aligned}
$$

Note that by setting either $\rho_{1,2} = 1$ or $\rho_{1,2} = -1$, we obtain a risk-free minimum variance portfolio, i.e., $\sigma^2_{p,gmv} = 0$. On the other hand, when $\rho_{1,2} = 0$, the portfolio's variance is given by $\sigma^2_{p,gmv} = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}$.

Finally, we can define the expected return of the *global minimum variance portfolio*, $\mu_{p,gmv}$, as follows:

$$
\mu_{p,gmv} = \mu_2 + \underbrace{\left[\frac{\sigma_2^2-\rho_{1,2}\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho_{1,2}\sigma_1\sigma_2}\right]}_{w_{1,gmv}}(\mu_1-\mu_2).
$$

In the context of the risk–return trade-off, understanding how investors can simultaneously enhance returns while reducing risk is essential. The solution lies in a fundamental principle: **diversification**. By spreading investments across a variety of assets, investors can achieve a more favorable balance between risk and return. To better appreciate the advantages of diversification, it is useful to examine specific scenarios that clearly illustrate its benefits. These intuitive cases are explored in greater detail in Box (2.1), providing a practical understanding of how diversification can optimize a portfolio's performance.

Box (2.1)

Correlation and Diversification

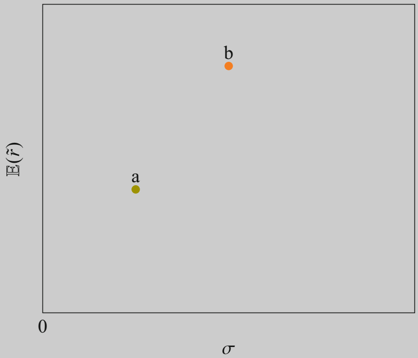
In mean-variance analysis, the interrelationship between assets, quantified by their covariance or correlation, plays a pivotal role in determining the efficacy of portfolio diversification. Consider a scenario where an investor allocates resources between two risky assets, a and b. The portfolio variance for these two assets, with weights, ω_a and $\omega_b = (1 - \omega_a)$, is given by:

(continued)

Box (2.1) (continued)

$$\sigma_p^2 = \omega_a^2\sigma_a^2 + (1-\omega_a)^2\,\sigma_b^2 + 2\omega_a\,(1-\omega_a)\,\sigma_a\sigma_b\rho_{a,b}, \tag{2.13}$$

where $\sigma_a\sigma_b\rho_{a,b} = \sigma_{a,b}$. Equation (2.13) proves particularly valuable for assessing the risk and diversification advantages of combining assets, leveraging their correlation. In the upcoming figure, we illustrate the positions of two assets, denoted as a and b, on the $(\sigma, \mathbb{E}(\tilde{r}))$-plane. This plane represents the relationship between risk (measured by standard deviation, σ) and expected return ($\mathbb{E}(\tilde{r})$). For each asset, its risk is plotted along the horizontal axis, while its expected return is plotted along the vertical axis. Specifically, the figure highlights the expected returns $\mathbb{E}(r_a)$ and $\mathbb{E}(r_b)$ corresponding to assets a and b, respectively, providing a visual representation of their risk–return profiles.



Special Cases

In a simplified economy featuring only two risky assets, the efficient frontier showcases two extreme scenarios: one characterized by perfect negative correlation ($\rho_{a,b} = -1$) and the other by perfect positive correlation ($\rho_{a,b} = 1$).

1. **The case**: $\rho_{a,b} = -1$. *(perfect negative correlation)*
   When $\rho_{a,b} = -1$, the portfolio variance σ_p^2 in Eq. (2.13) can be expressed as:

$$\begin{aligned}\sigma_p^2 &= \omega_a^2\sigma_a^2 + (1-\omega_a)^2\,\sigma_b^2 - 2\omega_a\,(1-\omega_a)\,\sigma_a\sigma_b \\ &= (\omega_a\sigma_a - (1-\omega_a)\sigma_b)^2 = (\omega_a(\sigma_a+\sigma_b) - \sigma_b)^2\,.\end{aligned}$$

   This suggests the feasibility of achieving a *perfect hedge* (ph^{-1}), denoting a portfolio with zero variance:

(continued)

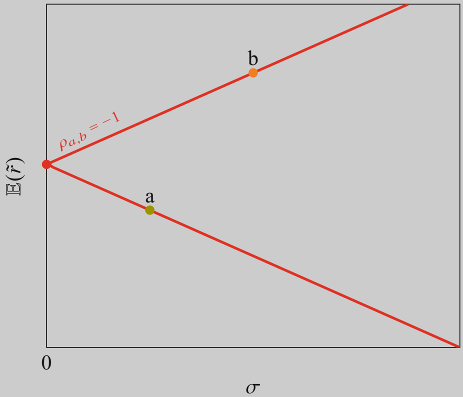
Box (2.1) (continued)

$$\sigma^2_{ph^{-1}} = (\omega_a(\sigma_a + \sigma_b) - \sigma_b)^2 = 0 \rightarrow \omega_a = \frac{\sigma_b}{\sigma_a + \sigma_b}, \quad \omega_b = \frac{\sigma_a}{\sigma_a + \sigma_b}$$

and an expected (deterministic) return equal to

$$\mathbb{E}(r_{ph^{-1}}) = \frac{\sigma_b}{\sigma_a + \sigma_b}\mathbb{E}(\tilde{r}_a) + \frac{\sigma_a}{\sigma_a + \sigma_b}\mathbb{E}(\tilde{r}_b).$$

In this scenario, the presence of a *perfect hedge* against portfolio risk (ph^{-1}) indicates that the *global minimum variance portfolio* possesses zero variance, as depicted in the following figure (see red dot). Alternatively, this can also be deduced by setting the correlation between the two assets equal to -1 in the definition of the variance of the *global minimum variance portfolio*, as outlined in Eq. (2.12).



It is noteworthy that in such circumstances, the efficient frontier (illustrated by the red solid line) demonstrates the highest degree of concavity. This arises from the capability to achieve an optimal portfolio blend of assets a and b, effectively mitigating the overall portfolio risk to zero. A higher concavity indicates that the benefits of diversification are more pronounced at lower risk levels. Investors can significantly reduce risk without sacrificing much return by combining different assets. However, as they move up the frontier, the incremental benefits of diversification decrease.

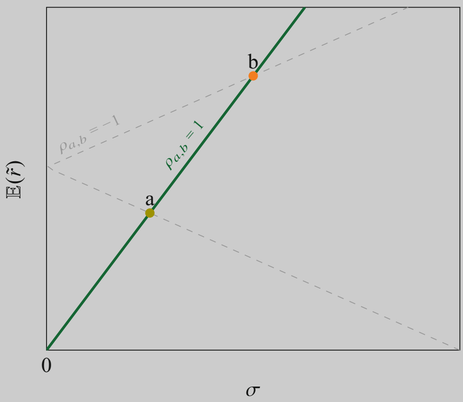
2. **The case**: $\rho_{a,b} = 1$. *(perfect positive correlation)*
   When $\rho_{a,b} = 1$, indicating *perfect positive correlation* between assets a and b, the variance of the portfolio σ_p^2 in Eq. (2.13) can be expressed as:

$$\sigma_p^2 = \omega_a^2\sigma_a^2 + (1-\omega_a)^2\sigma_b^2 + 2\omega_a(1-\omega_a)\sigma_a\sigma_b.$$

(continued)

Box (2.1) (continued)

In this scenario, the portfolio lacks the benefit of risk diversification due to the *perfect positive correlation* between assets a and b. Consequently, the efficient frontier (depicted by the green solid line) is no longer strictly concave. Instead, each conceivable risk combination intersects the straight line connecting assets a and b, resulting in a linear efficient frontier.



To illustrate this transition, the efficient frontier for the scenario of *perfect negative correlation* ($\rho_{a,b} = -1$) is also displayed (represented by a gray dashed line). These two extreme cases serve as critical benchmarks, highlighting the impact of correlation on the efficient frontier's configuration and curvature. They demonstrate the transformation of the frontier from a distinctly concave curve to a linear relationship as the correlation approaches *perfect positive correlation*. It is important to note that all efficient frontiers presented here represent different correlation coefficients between assets a and b, ranging from $-1 \leq \rho_{a,b} \leq 1$. This demonstrates the impact of correlation on portfolio optimization.

However, it is noteworthy that a *perfect hedge* portfolio is still achievable even when the two assets exhibit *perfect positive correlation*. In practice, a portfolio with zero variance can be attained when $\rho_{a,b} = 1$, under the condition that $-\infty \leq \omega_a \leq \infty$ holds true. In such instances, the variance of the portfolio is represented by the following expression:

$$\sigma^2_{ph^{+1}} = (\omega_a(\sigma_a - \sigma_b) + \sigma_b)^2 = 0 \rightarrow \omega_a = -\frac{\sigma_b}{\sigma_a - \sigma_b}, \quad \omega_b = \frac{\sigma_a}{\sigma_a - \sigma_b},$$

and the expected return is equal to

$$\mathbb{E}(r_{ph^{+1}}) = -\frac{\sigma_b}{\sigma_a - \sigma_b}\mathbb{E}(\tilde{r}_a) + \frac{\sigma_a}{\sigma_a - \sigma_b}\mathbb{E}(\tilde{r}_b).$$

(continued)

Box (2.1) (continued)

Theoretically, it is possible to construct a portfolio comprising two risky assets that has zero risk, even when the assets exhibit a *perfect positive correlation*. This can be achieved by taking a short position in one of the assets, specifically asset a. The proportion of this short position can be determined using the formula $\omega_a = -\frac{\sigma_b}{\sigma_a - \sigma_b}$, where σ_a and σ_b represent the standard deviations of assets a and b, respectively.

Independence vs. Positive Correlation:

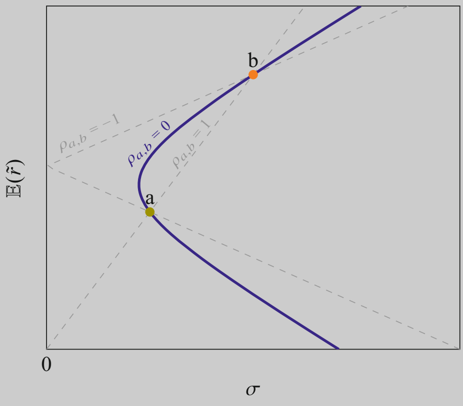
In scenarios of zero or positive correlation ($\rho_{a,b} \geq 0$), investors can still benefit from diversification, as evidenced by the concave shape of the efficient frontiers. As elaborated earlier, these frontiers are confined within the parameter space delineated by $\rho_{a,b} \in (-1, 1)$. With increasing values of $\rho_{a,b}$, the concavity of the efficient frontier diminishes, gradually transitioning toward a linear configuration. This convergence toward a linear efficient frontier underscores the diminishing diversification advantages as correlation strengthens.

1. The case: $\rho_{a,b} = 0$. *(no correlation)*

   When $\rho_{a,b} = 0$, the variance of the portfolio σ_p^2 in Eq. (2.13) simplifies to:

$$\sigma_p^2 = \omega_a^2 \sigma_a^2 + (1 - \omega_a)^2 \sigma_b.$$

   The efficient frontier corresponding to $\rho_{a,b} = 0$ (depicted by the blue solid line) is situated within the parameter space delineated by $\rho_{a,b} \in (-1, 1)$ (indicated by gray dashed lines).



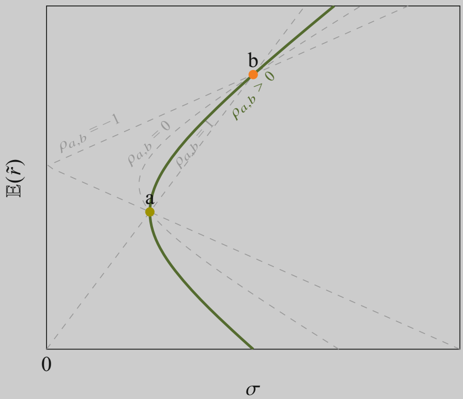
(continued)

Box (2.1) (continued)

2. **The case**: $\rho_{a,b} > 0$. *(positive correlation)*
   Let us consider a scenario where $\rho_{a,b} = 0.5$ as an example. In this case, the portfolio variance is expressed as:

$$\sigma_p^2 = \omega_a^2 \sigma_a^2 + (1-\omega_a)^2 \sigma_b^2 + \omega_a (1-\omega_a) \sigma_a \sigma_b \rho_{a,b}.$$

   Comparing this to the scenario with zero correlation, we observe that positive correlation ($\rho_{a,b} = 0.5$) introduces changes in risk diversification, primarily influenced by the term $\omega_a (1-\omega_a) \sigma_a \sigma_b \rho_{a,b}$. Graphically, the efficient frontier (shown as the green solid line) tends to converge closer to the $\rho_{a,b} = 1$ frontier and moves away from $\rho_{a,b} = -1$. This shift underscores the impact of positive correlation on portfolio risk and diversification strategies. Conversely, when $\rho_{a,b} = -0.5$, the opposite effect is observed. The efficient frontier aligns closer to the $\rho_{a,b} = -1$ frontier and moves away from $\rho_{a,b} = 1$. This dynamic relationship highlights the crucial role of correlation in shaping portfolio risk management.



A simple two-asset portfolio analysis clearly demonstrates the significant impact of asset correlation on the extent of achievable diversification. Positive correlation limits diversification benefits, while negative correlation enhances them.

In Box (2.2), we explore the intriguing concept that an exceptionally large number of assets within a portfolio can potentially result in a variance of zero. This phenomenon is made possible through the effective elimination of idiosyncratic risk. Idiosyncratic risk, also known as unsystematic risk, refers to the inherent variability associated with individual assets within a portfolio. By diversifying across a broad spectrum of assets, investors can effectively mitigate this risk. As the number of assets in the portfolio increases significantly, the impact of any single asset's fluctuations diminishes relative to the overall portfolio. In essence, the unique risks associated with individual assets tend to cancel each other out on a large scale, resulting in a reduction, and potentially complete elimination, of portfolio variance. This concept underscores the importance of diversification in modern portfolio theory. Through thoughtful asset allocation and diversification strategies, investors aim not only to optimize returns but also to minimize risk. Understanding how a vast array of assets can collectively contribute to a portfolio's stability highlights the nuanced dynamics at play in investment management.

Box (2.2)

The Insurance Principle

Let us generalize the variance of the portfolio defined in Eq. (2.3) for the case of N risky assets. This gives

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j$$

$$\text{with} \quad i \neq j.$$

Suppose now that all asset returns are uncorrelated (i.e., $\rho_{i,j} = 0$). In this special case, the portfolio variance reduces to

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 = \left(w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + \cdots + w_N^2 \sigma_N^2 \right).$$

One can further assume that all assets have the same variance (i.e., $\sigma_i^2 = \sigma^2$ for all i) and are held in the same proportion (i.e., $1/N$). Under all these assumptions, we have

$$\sigma_p^2 = \left(\frac{1}{N}\right)^2 \sigma^2 + \left(\frac{1}{N}\right)^2 \sigma^2 + \cdots + \left(\frac{1}{N}\right)^2 \sigma^2 = N \left(\frac{1}{N}\right)^2 \sigma^2$$
$$= \frac{1}{N^2} N \sigma^2 = \frac{1}{N} \sigma^2.$$

The result suggests that as the number of assets approaches infinity (i.e., for $N \to \infty$) the portfolio's variance converges to zero (i.e., $\sigma_p^2 = 0$). This implies that aggregating uncorrelated risks can reduce overall risk, similar

(continued)

Box (2.2) (continued)

to the insurance principle. Insurance companies set premiums based on the assumption of largely uncorrelated events across individuals. For instance, in a large pool of insured individuals, while some may experience accidents, many others will not, balancing the risk. This parallels the diversification of idiosyncratic risk, a financial concept, through a large number of insurance contracts. However, systemic risks, such as natural disasters, can affect all individuals simultaneously. Nevertheless, the insurance principle suggests that investors can mitigate idiosyncratic risk through diversification across a broad portfolio. Bad news affecting some firms can be offset by good news from others. This aligns with the concept, as discussed later in the context of the Capital Asset Pricing Model (CAPM), that stocks should, on average, compensate investors for systemic risk, not individual risk.

2.3.2 3-Risky Assets

Consider a three-asset portfolio problem with assets labeled as 1, 2, and 3. Let $\tilde{r}_i$ (with $i = 1, 2, 3$) represent the random return on asset i. Define w_i as the proportion of wealth invested in asset i, subject to the constraint $w_1 + w_2 + w_3 = 1$. The portfolio return is a random variable given by:

$$\tilde{r}_p = w_1\tilde{r}_1 + w_2\tilde{r}_2 + w_3\tilde{r}_3. \tag{2.14}$$

The expected return on the portfolio is

$$\mathbb{E}(\tilde{r}_p) = \mu_p = w_1\mathbb{E}(\tilde{r}_1) + w_2\mathbb{E}(\tilde{r}_2) + w_3\mathbb{E}(\tilde{r}_3) = w_1\mu_1 + w_2\mu_2 + w_3\mu_3, \tag{2.15}$$

where $\mathbb{E}(\tilde{r}_i) = \mu_i$ is the excepted return (mean) of asset i.

Let $\text{cov}(\tilde{r}_i, \tilde{r}_j) = \sigma_{i,j}$ denote the covariance between assets i and j. The variance of the portfolio's return can thus be expressed as follows:

$$\text{Var}(\tilde{r}_p) = \sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_{1,2} + 2w_1w_3\sigma_{1,3} + 2w_2w_3\sigma_{2,3}. \tag{2.16}$$

The variance of the portfolio return is a function of three variance terms and six covariance terms. Notably, the number of covariance terms is twice that of variance terms, even for a relatively simple three-asset portfolio. While this algebraic complexity can become unwieldy, it can be significantly streamlined through the use of matrix notation.

Let us define the following (3×1) column vectors:

- Asset Returns:

$$\tilde{\boldsymbol{r}} = \begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \tilde{r}_3 \end{bmatrix},$$

where $\tilde{r}_i$ represents the random return on asset i.
- Portfolio Weights:

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix},$$

where w_i represents the weight or proportion of the portfolio invested in asset i.

Utilizing matrix notation, we can consolidate the multiple returns into a single vector denoted as $\tilde{\boldsymbol{r}}$. It is important to note that each element in $\tilde{\boldsymbol{r}}$ represents a random variable; hence, we refer to $\tilde{\boldsymbol{r}}$ as a random vector. The probability distribution of this random vector, $\tilde{\boldsymbol{r}}$, is simply the joint distribution of its elements. All returns are assumed to be jointly normally distributed, and this joint distribution is fully characterized by the means, variances, and covariances of the returns. Now, the (3×1) vector of portfolio expected returns is:

$$\mathbb{E}[\tilde{\boldsymbol{r}}] = \mathbb{E}\begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \tilde{r}_3 \end{bmatrix} = \begin{bmatrix} \mathbb{E}[\tilde{r}_1] \\ \mathbb{E}[\tilde{r}_2] \\ \mathbb{E}[\tilde{r}_3] \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \boldsymbol{\mu},$$

and the (3×3) variance-covariance matrix is:

$$\operatorname{Var}(\tilde{\boldsymbol{r}}) = \begin{bmatrix} \operatorname{Var}(\tilde{r}_1) & \operatorname{cov}(\tilde{r}_1, \tilde{r}_2) & \operatorname{cov}(\tilde{r}_1, \tilde{r}_3) \\ \operatorname{cov}(\tilde{r}_2, \tilde{r}_1) & \operatorname{Var}(\tilde{r}_2) & \operatorname{cov}(\tilde{r}_2, \tilde{r}_3) \\ \operatorname{cov}(\tilde{r}_3, \tilde{r}_1) & \operatorname{cov}(\tilde{r}_3, \tilde{r}_2) & \operatorname{Var}(\tilde{r}_3) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \sigma_{1,3} \\ \sigma_{2,1} & \sigma_2^2 & \sigma_{2,3} \\ \sigma_{3,1} & \sigma_{3,2} & \sigma_3^2 \end{bmatrix} = \mathbf{\Sigma}.$$

It is important to note that the variance–covariance matrix is symmetric, meaning that $\operatorname{cov}(\tilde{r}_2, \tilde{r}_1) = \operatorname{cov}(\tilde{r}_1, \tilde{r}_2)$, $\operatorname{cov}(\tilde{r}_1, \tilde{r}_3) = \operatorname{cov}(\tilde{r}_3, \tilde{r}_1)$, and $\operatorname{cov}(\tilde{r}_2, \tilde{r}_3) = \operatorname{cov}(\tilde{r}_3, \tilde{r}_2)$. In essence, the off-diagonal elements are equal, implying that $\mathbf{\Sigma} = \mathbf{\Sigma}'$, where $\mathbf{\Sigma}'$ represents the transpose of $\mathbf{\Sigma}$.

The random return of the portfolio using matrix notation is

$$\tilde{r}_p = \mathbf{w}'\tilde{\boldsymbol{r}} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \tilde{r}_3 \end{bmatrix} = w_1\tilde{r}_1 + w_2\tilde{r}_2 + w_3\tilde{r}_3.$$

Similarly, the expected return on the portfolio is

$$\boldsymbol{\mu}_p = \mathbb{E}[\mathbf{w}'\tilde{\boldsymbol{r}}] = \mathbf{w}'\mathbb{E}[\tilde{\boldsymbol{r}}] = \mathbf{w}'\boldsymbol{\mu} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = w_1\mu_1 + w_2\mu_2 + w_3\mu_3.$$

The variance of the portfolio's return can be expressed as:[3]

$$\begin{aligned} \sigma_p^2 =& \mathrm{Var}(\mathbf{w}'\tilde{\boldsymbol{r}}) = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \sigma_{1,3} \\ \sigma_{2,1} & \sigma_2^2 & \sigma_{2,3} \\ \sigma_{3,1} & \sigma_{3,2} & \sigma_3^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\ =& w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_{1,2} + 2w_1w_3\sigma_{1,3} + 2w_2w_3\sigma_{2,3}. \end{aligned}$$

[3] Let us stress that the quadratic form $\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$ is always positive. This is so if the variance–covariance matrix $\boldsymbol{\Sigma}$ is positive definite. For $\boldsymbol{\Sigma}$ to be positive definite, it must satisfy the following conditions: (a) all diagonal elements (variances) must be positive; (b) the determinant of the matrix must be positive; and (c) all leading principal minors (submatrices formed by selecting rows and columns from the top left corner) must have positive determinants. Let us explore the scenario with two assets:

$$\begin{aligned} \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} &= \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1\sigma_1^2 + w_2\sigma_{2,1} & w_1\sigma_{1,2} + w_2\sigma_2^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ &= w_1^2\sigma_1^2 + w_1w_2\sigma_{2,1} + w_1w_2\sigma_{1,2} + w_2^2\sigma_2^2. \end{aligned}$$

To show that the portfolio variance is positive $\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} > 0$ for any $\mathbf{w} \neq 0$, one has to show that the aforementioned conditions (a)–(c) are satisfied. Let us analyze each condition:

(a) **Positivity of diagonal elements**: Since variance represents the square of standard deviation, it is always non-negative. Hence, $\sigma_1^2 \geq 0$ and $\sigma_2^2 \geq 0$.

(b) **Determinant of the matrix**: The determinant of our (2×2) variance–covariance matrix is given by $\det(\boldsymbol{\Sigma}) = \sigma_1^2\sigma_2^2 - \sigma_{12}^2$. For the variance–covariance matrix to be positive definite, this determinant must be positive. To prove that this determinant is always positive, we need to show that $\sigma_1^2\sigma_2^2 > \sigma_{12}^2$. Since variance is the square of standard deviation, it is always non-negative. Therefore, $\sigma_1^2 \geq 0$ and $\sigma_2^2 \geq 0$. The covariance term σ_{12} represents the relationship between the returns of assets 1 and 2. By Cauchy–Schwarz Inequality, we have:

$$\sigma_{12}^2 \leq \sigma_1^2\sigma_2^2.$$

This inequality states that the square of the covariance cannot exceed the product of the variances. Hence, σ_{12}^2 is always less than or equal to $\sigma_1^2\sigma_2^2$. Therefore, we can conclude that $\sigma_1^2\sigma_2^2 - \sigma_{12}^2$ is always positive, ensuring that the determinant of the variance–covariance matrix is positive.

(c) **Leading principal minors**: The (1×1) leading principal minor is just the determinant of the top-left element of the matrix, which is σ_1^2. The 2×2 leading principal minor is the determinant of the entire matrix, which is given by: $\det(\boldsymbol{\Sigma}) = \sigma_1^2\sigma_2^2 - \sigma_{12}^2$. The variance of the first asset, σ_1^2, is always positive or non-negative by definition. The $\det(\boldsymbol{\Sigma})$ is also positive unless $\sigma_1^2\sigma_2^2 = \sigma_{12}^2$.

Since all conditions are satisfied, the variance–covariance matrix of returns of a portfolio of 2 risky assets is always positive, indicating that it is positive definite.

We can represent the requirement that the portfolio weights add up to one as follows:

$$\mathbf{w}'\mathbf{1} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = w_1(1) + w_2(1) + w_3(1) = 1,$$

where $\mathbf{1}$ is a (3×1) column vector of ones (i.e., each element of the vector is 1).

To find the *global minimum variance portfolio* in the three-asset case, one has to solve the following constrained minimization problem:

$$\begin{aligned} \min_{\{w_1, w_2, w_3\}} \quad & \sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_{1,2} + 2w_1w_3\sigma_{1,3} + 2w_2w_3\sigma_{2,3} \\ s.t. \quad & w_1 + w_2 + w_3 = 1. \end{aligned} \tag{2.17}$$

The Lagrangian for this problem is:

$$\begin{aligned} \underbrace{\Lambda(w_1, w_2, w_3, \lambda)}_{\text{Lagrangian function}} = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_{1,2} + 2w_1w_3\sigma_{1,3} \\ + 2w_2w_3\sigma_{2,3} + \lambda(1 - w_1 - w_2 - w_3), \end{aligned}$$

where λ denotes the Lagrange multiplier. The FOCs for a minimum are

$$\begin{aligned} 0 &= \frac{\partial \Lambda}{\partial w_1} = 2w_1\sigma_1^2 + 2w_2\sigma_{1,2} + 2w_3\sigma_{1,3} - \lambda, \\ 0 &= \frac{\partial \Lambda}{\partial w_2} = 2w_2\sigma_2^2 + 2w_1\sigma_{1,2} + 2w_3\sigma_{2,3} - \lambda, \\ 0 &= \frac{\partial \Lambda}{\partial w_3} = 2w_3\sigma_3^2 + 2w_1\sigma_{1,3} + 2w_2\sigma_{2,3} - \lambda, \\ 0 &= \frac{\partial \Lambda}{\partial \lambda} = w_1 + w_2 + w_3 - 1. \end{aligned} \tag{2.18}$$

It is worth noting that the FOCs outlined in (2.18) yield a system of four equations in four unknowns, which must be solved to determine the global minimum variance weights. The system of four linear equations can be represented in matrix form as:

$$\begin{bmatrix} 2\sigma_1^2 & 2\sigma_{1,2} & 2\sigma_{1,3} & 1 \\ 2\sigma_{2,1} & 2\sigma_2^2 & 2\sigma_{2,3} & 1 \\ 2\sigma_{3,1} & 2\sigma_{3,2} & 2\sigma_3^2 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

which can be rewritten in a more compact form as follows:

$$\begin{bmatrix} 2\Sigma & \mathbf{1} \\ \mathbf{1}' & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}. \tag{2.19}$$

Let us recall that the system (2.19) is of the classical form

$$\boldsymbol{A_w \nu} = \mathbf{b},$$

where

$$\mathbf{A_w} = \begin{bmatrix} 2\Sigma & \mathbf{1} \\ \mathbf{1}' & 0 \end{bmatrix}, \quad \boldsymbol{\nu} = \begin{bmatrix} \mathbf{w} \\ \lambda \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}.$$

The solution for $\boldsymbol{\nu}$ is then given by

$$\boldsymbol{\nu} = \mathbf{A_w^{-1} b}. \tag{2.20}$$

The first three elements of $\boldsymbol{\nu}$ are the portfolio weights $\mathbf{w} = \begin{bmatrix} w_1 \ w_2 \ w_3 \end{bmatrix}$ for the *global minimum variance portfolio* with expected return $\mu_p = \mathbf{w}'\boldsymbol{\mu}$ and variance $\sigma_p^2 = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$. Put differently, the solution (2.20) elucidates that the optimal weights defining the portfolio with the minimal variance can be derived by computing the inverse of the matrix $\mathbf{A_w}$. Box (2.3) provides a simplified and direct method for calculating the weight vector of the *global minimum variance portfolio*.

Box (2.3)

Alternative Derivation of the Global Minimum Variance Portfolio

Solving the system defined in (2.18) explicitly can be quite cumbersome. However, employing matrix notation, we can express the constrained minimization problem outlined in (2.17) as:

$$\min_{\{\mathbf{w}\}} \quad \sigma_p^2 = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \quad \text{subject to} \quad \mathbf{w}'\mathbf{1} = 1. \tag{2.21}$$

To solve the constrained minimization problem in (2.21), we can employ a standard approach by computing the FOCs with respect to the vector of weights, $\mathbf{w}$, and the Lagrange multiplier, λ. We define the Lagrangian function as:

$$\Lambda(\mathbf{w}, \lambda) = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \lambda(1 - \mathbf{w}'\mathbf{1}) \tag{2.22}$$

The FOCs are derived as follows:

(continued)

Box (2.3) (continued)

$$\underbrace{\mathbf{0}}_{(3\times1)} = \frac{\partial\Lambda(\mathbf{w},\lambda)}{\partial\mathbf{w}} = 2\boldsymbol{\Sigma}\mathbf{w} - \lambda\mathbf{1}, \tag{2.23}$$

$$\underbrace{0}_{(1\times1)} = \frac{\partial\Lambda(\mathbf{w},\lambda)}{\partial\lambda} = \mathbf{w}'\mathbf{1} - 1 \tag{2.24}$$

The aforementioned FOCs provide the foundation for solving the optimization problem. Solving for $\mathbf{w}$ yields:

$$\mathbf{w} = \frac{1}{2}\lambda\boldsymbol{\Sigma}^{-1}\mathbf{1}.$$

By multiplying both sides by $\mathbf{1}'$ and utilizing the first-order condition $\mathbf{w}'\mathbf{1} - 1 = 0$, we can derive a solution for the unknown Lagrange multiplier, λ:

$$1 = \mathbf{1}'\mathbf{w} = \frac{1}{2}\lambda(\mathbf{1}')\boldsymbol{\Sigma}^{-1}\mathbf{1} \quad \Rightarrow \quad \lambda = 2\frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}.$$

Finally, substituting the value for λ back into $\mathbf{w} = \frac{1}{2}\lambda\boldsymbol{\Sigma}^{-1}\mathbf{1}$, we solve for $\mathbf{w}$:

$$\underbrace{\mathbf{w}}_{(3\times1)} = \frac{1}{2}\left((2)\frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\right)\boldsymbol{\Sigma}^{-1}\mathbf{1} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \equiv \underbrace{\boldsymbol{w}_{gmv}}_{(3\times1)}. \tag{2.25}$$

In conclusion, Eq. (2.25) shows that computing the optimal weights for the *global minimum variance portfolio* simplifies to calculating the inverse of the variance–covariance matrix, $\boldsymbol{\Sigma}^{-1}$.

2.4 Mean-Variance Asset Allocation

Markowitz's mean-variance (hereinafter M-V) asset allocation approach is a cornerstone of MPT. The portfolio theory introduced by Markowitz (1952) is founded on a fundamental concept: *investors aim to construct portfolios that offer the most favorable trade-off between expected returns and risk (known as efficient portfolios)*. In his seminal paper, he demonstrates that a collection of *efficient portfolios* can be characterized in two interrelated ways:

1. **Maximizing Expected Return for a Given Risk**: Investors endeavor to identify portfolios that maximize the expected return while maintaining a specified level

of risk, typically measured by the portfolio's variance. This approach enables investors to optimize their portfolio's return potential without exceeding their tolerance for risk.
2. **Minimizing Portfolio Variance at a Target Expected Return**: Alternatively, investors may strive to minimize the variance of their portfolio while targeting a specific expected return level. By minimizing variance, investors aim to mitigate the uncertainty and volatility associated with their investment, while still achieving their desired level of return.

These two perspectives represent complementary approaches to portfolio optimization, offering investors flexibility in achieving their investment objectives within their risk constraints. Markowitz (1952)'s framework laid the foundation for MPT, revolutionizing the way investors conceptualize and construct investment portfolios. For the sake of computational simplicity and the inherent preference of investors, targeting expected returns is often favored over specifying risk levels. As a result, problem 2 is commonly employed in practice. In solving problem 2, it is typical to vary the target return within a specified range. A rigorous and comprehensive derivation of the constrained portfolio variance minimization problem is presented in the next section.

2.4.1 Deriving Optimal Weights

Consider an economy with $N \geq 2$ linearly independent risky assets. This implies that the return of no asset can be perfectly predicted as a linear combination of the returns of other assets. Furthermore, assume the absence of a risk-free asset, forcing investors to allocate their wealth entirely among the risky assets. These N risky assets exhibit different expected returns and variances. These differences in variance and covariance contribute to the elements of the $(N \times N)$ variance–covariance matrix of returns, denoted as $\mathbf{\Sigma}$. Within this matrix, the elements $\left\{\sigma_{i,j}\right\}_{i,j=1}^{N}$ represent the covariance between the returns of assets i and j, while $\sigma_{i,i} = \sigma_i^2 = \text{Var}(\tilde{r}_i)$ captures the variance of asset i. It is worth noting that the matrix $\mathbf{\Sigma}$ is symmetric and positive definite due to the linear independence of the N assets. Moreover, it possesses full rank, rendering it invertible. Formally, the $(N \times N)$ variance–covariance matrix is represented as follows:

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \sigma_{1,3} & \cdots & \cdots & \sigma_{1,N} \\ \sigma_{2,1} & \sigma_2^2 & \sigma_{2,3} & \cdots & \cdots & \sigma_{2,N} \\ \sigma_{3,1} & \sigma_3^2 & \sigma_{3,3} & \cdots & \cdots & \sigma_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ \sigma_{N-1,1} & \sigma_{N-1,2} & \sigma_{N-1,3} & \cdots & \sigma_{N-1}^2 & \sigma_{N-1,N} \\ \sigma_{N,1} & \sigma_{N,2} & \sigma_{N,3} & \cdots & \sigma_{N,N-1} & \sigma_N^2 \end{bmatrix}.$$

As previously noted, the variance–covariance matrix $\boldsymbol{\Sigma}$ is symmetric. This symmetry implies that $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}'$, where $\boldsymbol{\Sigma}'$ denotes the transpose of $\boldsymbol{\Sigma}$.

Remark. The symmetry arises from the equality of the off-diagonal elements, such as $\sigma_{1,2} = \sigma_{2,1}$, $\sigma_{1,3} = \sigma_{3,1}$, $\sigma_{N-1,1} = \sigma_{1,N-1}$, and so on.

Now, let $\boldsymbol{\mu}$ be an $(N \times 1)$ vector representing the expected returns on the N assets. This vector comprises elements $\{\mu_i\}_{i=1}^{N}$, where $\mu_i = \mathbb{E}[\tilde{r}_i]$. Thus, the $(N \times 1)$ vector of portfolio expected returns is:

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_{N-1} \\ \mu_N \end{bmatrix}.$$

The estimation methods for the vector $\boldsymbol{\mu}$ and the matrix $\boldsymbol{\Sigma}$ will be addressed in later discussions. For now, let us assume that these values are commonly known. This means that all investors have full information on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, and they are aware that others possess the same information.

Within the framework of M-V asset allocation theory, it is assumed that *capital markets are perfect*. This assumption encompasses several key features:

- **Frictionless Markets**: Capital markets operate without any transaction costs or constraints on trading activity. Investors can buy or sell assets without incurring any fees or encountering limitations on their transactions.
- **Unlimited Trading**: Investors have the ability to purchase or short-sell unlimited quantities of any asset. Short-selling, in this context, refers to the practice of selling an asset that the investor does not own, typically achieved by borrowing it from another investor or broker. The short-seller is obligated to return the borrowed asset by the end of the borrowing period.
- **Infinitely Divisible Assets**: Assets are considered infinitely divisible, allowing investors to purchase fractions of shares freely. This ensures that investors can adjust their portfolios with precision, even with small amounts of capital.
- **Equal Access to Information**: All investors have equal and unrestricted access to information regarding asset returns. This ensures that the information set available to investors is identical, promoting fairness and transparency in the market.

These assumptions collectively create an idealized environment where investors can make decisions based on perfect information and unrestricted trading conditions.

In light of the aforementioned assumptions, a portfolio of assets is characterized by an $(N \times 1)$ vector denoted as $\mathbf{w}$. This vector comprises the portfolio weights, denoted as $\{w_i\}_{i=1}^{N}$, representing the proportion of an investor's wealth allocated to each of the N assets.

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}.$$

Given the allowance for short-selling, weights in the vector $\mathbf{w}$ can assume negative values. However, investors are constrained by the limitation that they cannot invest more than their total wealth. Consequently, the sum of these weights must equal 1. This condition can be formally expressed as:

$$\sum_{i=1}^{N} w_i = 1 = \mathbf{1}'\mathbf{w} = \mathbf{w}'\mathbf{1}, \tag{2.26}$$

where $\mathbf{1}$ is a $(N \times 1)$ vector of 1s (i.e., every element of the vector is equal to 1). This restriction qualifies the meaning of the vector $\mathbf{w}$. In fact, this vector characterizes the composition of the portfolio, not its overall value. The value of the portfolio corresponds initially to the initial wealth,W_0, of the investor who selects such portfolio. Given such initial wealth, W_0, the vector $\mathbf{w}$ will be important in determining the final value of the portfolio, p. Thus, suppose the investor chooses the portfolio's composition characterized by the vector $\mathbf{w}$, then the return on portfolio p, $\tilde{r}_p$, is defined as

$$\tilde{r}_p = \sum_{i=1}^{N} w_i \tilde{r}_i = \mathbf{w}'\tilde{\mathbf{r}},$$

where $\tilde{\mathbf{r}}$ represents the vector of random assets' returns,

$$\tilde{\mathbf{r}} = \begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \vdots \\ \tilde{r}_N \end{bmatrix}.$$

The aforementioned expression denotes that the portfolio's return is determined by the weighted average of the returns on individual assets, where the weights represent the proportion of wealth invested in each asset. Consequently, we can compute the investor's wealth level over a specified holding period, from $t = 0$ to $t = 1$ (e.g., 1 year), as:

$$\tilde{W}_1 = (1 + \tilde{r}_p)W_0 = (1 + \mathbf{w}'\tilde{\mathbf{r}})W_0.$$

M-V portfolio theory dictates that investors leverage the statistical properties of assets to construct efficient portfolios. Specifically, portfolios are evaluated based on the first and second moments of returns. Leveraging the linearity of the expectation operator and the properties of variance, we determine that, with weights defined in the vector $\mathbf{w}$, the expected return and variance of portfolio p are

$$\mathbb{E}[\tilde{r}_p] = \sum_{i=1}^{N} w_i \mu_i \equiv \mathbf{w}'\boldsymbol{\mu}, \tag{2.27}$$

$$\sigma_p^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{i,j} \equiv \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}. \tag{2.28}$$

The expected return on portfolio p is intriguingly a weighted average of the expected returns on the N assets, with weights determined by the proportion of wealth invested in each individual asset.

In principle, an infinite number of portfolios of risky assets can be considered. However, investors are typically attracted to only a subset of these portfolios. Specifically, a "M-V investor" exhibits a preference for portfolios that offer the lowest possible risk (i.e., minimum standard deviation) while achieving a specified target expected return level. From a practical standpoint, this preference translates into solving the following constrained optimization problem:[4]

$$\begin{aligned} \min_{\{\mathbf{w}\}} \ \left(\frac{1}{2}\right)\sigma_p^2 &\equiv \left(\frac{1}{2}\right)\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}, \\ s.t. \quad \mathbf{w}'\boldsymbol{\mu} &= \mu_p, \\ \mathbf{w}'\mathbf{1} &= 1. \end{aligned} \tag{2.29}$$

[4] It is important to highlight that in the context of the dual equivalent problem, investors strive to maximize the expected return of their portfolio while maintaining a predetermined level of risk. This entails solving a constrained maximization problem to identify an equivalent efficient portfolio. Mathematically, this can be expressed as:

$$\max_{\{\mathbf{w}\}} \ \mathbf{w}'\boldsymbol{\mu} = \mu_p,$$

$$\begin{aligned} s.t. \quad \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} &\equiv \sigma_{p,0}^2, \\ \mathbf{w}'\mathbf{1} &= 1, \end{aligned}$$

where $\sigma_{p,0}^2$ denotes the desired level of risk that the investor is prepared to tolerate.

It is noteworthy that our objective entails the minimization of a quadratic form.[5] Given the positive definiteness of the variance–covariance matrix, we can reasonably deduce that this optimization problem possesses a unique solution. This solution can be obtained by solving the system of its FOCs, which arise from the optimization process. The derivation of these FOCs involves the construction of the Lagrangian associated with the optimization problem. This entails formulating an augmented objective function that incorporates the constraints of the optimization problem. The Lagrangian is defined as the original objective function plus a weighted sum of the constraint functions, where the weights are Lagrange multipliers. Thus, the system of FOCs is derived by applying the necessary conditions for optimality to the Lagrangian. These conditions typically involve setting the partial derivatives of the Lagrangian with respect to the decision variables and the Lagrange multipliers equal to zero. To summarize, the optimization process entails minimizing a quadratic form subject to the imposed constraints. The existence of a unique solution depends on solving the system of FOCs derived from the Lagrangian, which can be expressed mathematically as:

$$\min_{\{\mathbf{w},\lambda,\psi\}} \Lambda \equiv \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \lambda(\mu_p - \mathbf{w}'\boldsymbol{\mu}) + \psi(1 - \mathbf{w}'\mathbf{1}). \tag{2.30}$$

The FOCs read as follows:[6]

$$\frac{\partial \Lambda}{\partial \mathbf{w}} \equiv \boldsymbol{\Sigma}\mathbf{w} - \lambda\boldsymbol{\mu} - \psi\mathbf{1} = 0, \tag{2.31}$$

$$\frac{\partial \Lambda}{\partial \lambda} \equiv \mu_p - \mathbf{w}'\boldsymbol{\mu} = 0, \tag{2.32}$$

[5] Please note that the factor of $\left(\frac{1}{2}\right)$ in front of the objective function has been introduced solely to simplify the algebraic manipulations. In the subsequent sections, we will present derivations that do not use this factor but lead to the same results.

[6] Here below, we present a summary of the derivatives involving vectors and matrices. With the aim of facilitating students' understanding, in the following table we present vector and matrix derivatives juxtaposed with their scalar counterparts. In the ensuing illustrations, k is representative of a constant scalar, while $\mathbf{k}$ and $\mathbf{x}$ denote constant vectors of dimensions $(N \times 1)$, and $\mathbf{K}$ denotes a constant matrix with dimensions $(N \times N)$.

ScalarDerivative			**VectorDerivative**		
$f(x)$	$\rightarrow$	$\frac{df(x)}{dx}$	$f(\mathbf{x})$	$\rightarrow$	$\frac{df(\mathbf{x})}{d\mathbf{x}}$
kx	$\rightarrow$	k	$\mathbf{x}'\mathbf{K}$	$\rightarrow$	$\mathbf{K}$
kx	$\rightarrow$	k	$\mathbf{x}'\mathbf{k}$	$\rightarrow$	$\mathbf{k}$
x^2	$\rightarrow$	$2x$	$\mathbf{x}'\mathbf{x}$	$\rightarrow$	$2\mathbf{x}$
kx^2	$\rightarrow$	$2kx$	$\mathbf{x}'\mathbf{K}\mathbf{x}$	$\rightarrow$	$2\mathbf{K}\mathbf{x}$

$$\frac{\partial \Lambda}{\partial \psi} \equiv 1 - \mathbf{w}'\mathbf{1} = 0. \tag{2.33}$$

It is imperative to emphasize that condition (2.31) corresponds to a system of N equations, each representing a constraint for every weight within the vector $\mathbf{w}$. This system lends itself to straightforward resolution through pre-multiplication of Eq. (2.31) by the inverse of the variance–covariance matrix, denoted as $\mathbf{\Sigma}^{-1}$. This manipulation yields:

$$\mathbf{w} = \lambda(\mathbf{\Sigma}^{-1}\boldsymbol{\mu}) + \psi(\mathbf{\Sigma}^{-1}\mathbf{1}). \tag{2.34}$$

Due to the presence of the undetermined Lagrange multipliers, λ and ψ, Eq. (2.34) does not yield a definitive closed-form solution for the optimal weights vector. In the following, we focus on determining the values of these multipliers. Pre-multiplying both sides of Eq. (2.34) by $\boldsymbol{\mu}'$ and subsequently by $\mathbf{1}'$ yields:

$$\underbrace{\boldsymbol{\mu}'\mathbf{w}}_{\mu_p} = \lambda(\boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu}) + \psi(\boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\mathbf{1}),$$

and

$$\underbrace{\mathbf{1}'\mathbf{w}}_{1} = \lambda(\mathbf{1}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu}) + \psi(\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}).$$

Defining $A = \boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu}$, $B = \boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\mathbf{1}$, and $C = \mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}$, and utilizing the FOCs (2.32) and (2.33), we arrive at the following system of equations:[7]

$$\mu_p = A\lambda + B\psi, \tag{2.35}$$

$$1 = B\lambda + C\psi. \tag{2.36}$$

Given that A, B, and C are scalars, we can effortlessly solve the system of equations above through substitution. This yields the following solutions for λ and ψ:

$$\lambda = \frac{C\mu_p - B}{\Delta}, \qquad \psi = \frac{A - B\mu_p}{\Delta},$$

where $\Delta = AC - B^2$. Alternatively, the system of equations 2.35–2.36 can be expressed more concisely using matrix notation:

[7] Note that substituting the optimal solution (2.34) directly into the FOCs (2.32) and (2.33) yields the same result.

$$\begin{bmatrix} \mu_p \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} A & B \\ B & C \end{bmatrix}}_{M} \begin{bmatrix} \lambda \\ \psi \end{bmatrix}.$$

The vector containing the values of λ and ψ can be readily computed as:

$$\begin{bmatrix} \lambda \\ \psi \end{bmatrix} = \underbrace{\begin{bmatrix} C/\Delta & -B/\Delta \\ -B/\Delta & A/\Delta \end{bmatrix}}_{M^{-1}} \begin{bmatrix} \mu_p \\ 1 \end{bmatrix},$$

where M^{-1} represents the inverse of the (2×2) matrix M, and $\Delta = AC - B^2$ is the determinant of the matrix M. Hence, upon multiplying the matrix M^{-1} by the vector $\begin{bmatrix} \mu_p \\ 1 \end{bmatrix}$, we obtain:

$$\begin{bmatrix} \lambda \\ \psi \end{bmatrix} = \begin{bmatrix} \frac{C\mu_p - B}{\Delta} \\ \frac{A - B\mu_p}{\Delta} \end{bmatrix}.$$

Substituting λ and ψ into (2.34) and isolating the terms involving μ_p yield the optimal solution for $\mathbf{w}$:

$$\mathbf{w} = \mathbf{g} + \mathbf{h}\mu_p, \tag{2.37}$$

where the vectors $\mathbf{g}$ and $\mathbf{h}$ are defined as follows:

$$\mathbf{g} = \frac{1}{\Delta}[A(\boldsymbol{\Sigma}^{-1}\mathbf{1}) - B(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})],$$

$$\mathbf{h} = \frac{1}{\Delta}[C(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) - B(\boldsymbol{\Sigma}^{-1}\mathbf{1})].$$

Although the derivation of this result may involve considerable complexity, the final solution is surprisingly concise. By specifying the desired minimum expected return, μ_p, the optimal weight vector, $\mathbf{w}$, is expressed as a linear combination of two vectors, $\mathbf{g}$ and $\mathbf{h}$. In essence, for any target expected return μ_p chosen by the investor, Eq. (2.37) provides the corresponding optimal portfolio allocation.

2.4.2 The Portfolio Frontier

In summary, the portfolio p, characterized by the optimal vector of weights, $\mathbf{w}$, as defined in Eq. (2.37), serves as a solution to the optimal choice problem (2.29). Within the realm of M-V asset allocation theory, this specific portfolio is referred to as a *frontier portfolio*. Notably, since the optimal solution given by Eq. (2.37) suggests that a distinct *frontier portfolio* is attained for each chosen value of μ_p, we consequently have a collection of optimal portfolios, known as the *portfolio frontier*.

It is worth emphasizing that the vectors $\mathbf{g}$ and $\mathbf{g} + \mathbf{h}$ correspond to two specific portfolios. They represent the solutions to the optimal choice problem (2.29) for $\mu_p = 0$ and $\mu_p = 1$, respectively. These two corner solutions highlight that the set of portfolios addressing the optimal choice problem (2.29) adhere to some crucial properties.

Primarily, it is imperative to acknowledge that the *portfolio frontier* is inherently a convex set. This fundamental property signifies that any convex combination of multiple frontier portfolios maintains its status as a *frontier portfolio*. Establishing this assertion follows a relatively straightforward path. Let us consider two distinct *frontier portfolios*, denoted as k and q, each associated with divergent expected return values, μ_k and μ_q, respectively, where $\mu_k \neq \mu_q$. Representing these portfolios as $\mathbf{w}_k = \mathbf{g} + \mathbf{h}\mu_k$ and $\mathbf{w}_q = \mathbf{g} + \mathbf{h}\mu_q$, we proceed to define a convex combination of these portfolios. For any scalar value α within the interval (0, 1), the convex combination allocates α percent of the investor's wealth to portfolio k and $(1 - \alpha)$ percent to portfolio q. Consequently, the resulting portfolio possesses a weight vector denoted as:

$$\mathbf{w}_p = \alpha \mathbf{w}_k + (1 - \alpha)\mathbf{w}_q.$$

It is evident that this combination represents a legitimate portfolio. Notably, the summation of its weights equates to unity:

$$\mathbf{1}'\mathbf{w}_p = \alpha \mathbf{1}'\mathbf{w}_k + (1 - \alpha)\mathbf{1}'\mathbf{w}_q = \alpha + (1 - \alpha) = 1.$$

In addition, this is a *frontier portfolio*. This conclusion can be drawn from the fact that:

$$\begin{aligned}
\mathbf{w}_p &= \alpha \mathbf{w}_k + (1 - \alpha)\mathbf{w}_q = \\
&= \alpha(\mathbf{g} + \mathbf{h}\mu_k) + (1 - \alpha)(\mathbf{g} + \mathbf{h}\mu_q) \\
&= \mathbf{g} + \mathbf{h}[\alpha\mu_k + (1 - \alpha)\mu_q] \\
&= \mathbf{g} + \mathbf{h}\mu_p,
\end{aligned}$$

where μ_p denotes the weighted average return of portfolios k and q.

2.4.3 A Graphical Representation of the Portfolio Frontier

The portfolio variance associated with the optimal portfolio p can be derived by leveraging the solution (2.34). Specifically, let us pre-multiply both sides of

Eq. (2.34) by $\mathbf{w}'\boldsymbol{\Sigma}$. This results in:[8]

$$\underbrace{\left(\mathbf{w}'\boldsymbol{\Sigma}\right)\mathbf{w}}_{\sigma_p^2} = \lambda\left(\mathbf{w}'\boldsymbol{\Sigma}\right)\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \psi\left(\mathbf{w}'\boldsymbol{\Sigma}\right)\boldsymbol{\Sigma}^{-1}\mathbf{1} = \lambda(\mathbf{w}'\boldsymbol{\mu}) + \psi(\mathbf{w}'\mathbf{1}) = \lambda\mu_p + \psi 1,$$

where we have used $\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1} = 1$, and FOCs (2.31) and (2.32), i.e., $\mathbf{w}'\boldsymbol{\mu} = \mu_p$ and $\mathbf{w}'\mathbf{1} = 1$. Substituting λ and ψ into the equation results in:

$$\sigma_p^2 = \left(\frac{C\mu_p - B}{\Delta}\right)\mu_p + \left(\frac{A - B\mu_p}{\Delta}\right)1 = \frac{1}{\Delta}(C\mu_p^2 - 2B\mu_p + A). \tag{2.38}$$

Equation (2.38) elucidates the relationship between portfolio variance and the target return. It highlights that each level of the investor's desired return corresponds to a specific level of risk. However, this risk is minimized in accordance with the minimization problem. The dynamic interaction between these statistical metrics can be visually represented in the variance-mean (σ^2, μ) space. Conceptually, it materializes as a rotated parabola, encompassing the array of mean and variance combinations across the *frontier portfolios*. Alternatively, we can examine the standard deviation of returns on portfolio p. This yields:

$$\sigma_p = \frac{1}{\sqrt{\Delta}}[C\mu_p^2 - 2B\mu_p + A]^{1/2}. \tag{2.39}$$

The aforementioned expression delineates a hyperbola within the space of standard deviation and mean, denoted as (σ, μ), as depicted in Fig. 2.1. To show this, we can further develop Eq. (2.39) by expressing μ_p as a function of σ_p^2. This process yields:[9]

$$\mu_p = \frac{B}{C} \pm \frac{\sqrt{\Delta(C\sigma_p^2 - 1)}}{C}, \tag{2.40}$$

where $(1/\sqrt{C}, B/C)$ represent the coordinates of the *global minimum variance portfolio*.

The hyperbola depicted in Fig. 2.1 represents the set of feasible combinations of standard deviation and expected return for the *frontier portfolios*. These combinations provide critical insights into the trade-off between risk and return, which is central to portfolio management. It is important to emphasize that the hyperbola specifically characterizes the risk–return profiles of the *frontier portfolios*, reflecting

[8] Alternatively, one can directly substitute (2.34) into $\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$.

[9] To demonstrate this, one simply needs to pre-multiply both sides of Eq. (2.38) first by Δ and then by $(1)^2$. By rearranging terms, a second-order degree equation in μ_p is obtained. Upon solving it, Eq. (2.40) is precisely derived.

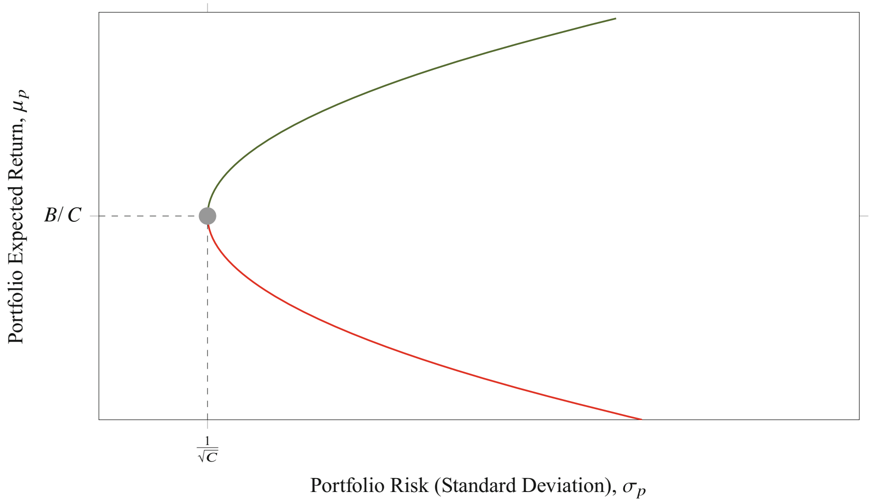


Fig. 2.1 The mean-variance portfolio frontier

the optimal trade-off between risk reduction and return maximization. In contrast, portfolios that do not lie on the *efficient frontier* correspond to points within the (σ_p, μ_p) space but outside the hyperbolic curve. These portfolios are suboptimal, as they do not provide the best possible expected return for a given level of risk. Specifically, any portfolio located to the right of the *efficient frontier* has a higher risk for a given expected return, indicating a less favorable risk–return combination than those on the frontier.

2.4.4 The Global Minimum Variance Portfolio

It is worth noting that the expression for the variance of a *frontier portfolio*, as indicated in (2.38), can alternatively be defined as follows:[10]

$$\sigma_p^2 = \frac{C}{\Delta}\left(\mu_p - \frac{B}{C}\right)^2 + \frac{1}{C}. \tag{2.41}$$

[10] Following are the mathematical details:

$$\sigma_p^2 = \frac{C}{\Delta}\left(\mu_p - \frac{B}{C}\right)^2 + \frac{1}{C} = \mu_p^2\frac{C}{\Delta} - 2\mu_p\frac{B}{C}\frac{C}{\Delta} + \frac{B^2}{C^2}\frac{C}{\Delta} + \left(\frac{1}{C}\right) = \mu_p^2\frac{C}{\Delta} - 2\mu_p\frac{B}{\Delta} + \frac{B^2}{C\Delta} + \frac{1}{C}$$
$$= \frac{\mu_p^2C^2 - 2\mu_pBC + B^2 + AC - B^2}{C\Delta} = \frac{\mu_p^2C^2 - 2\mu_pBC + AC}{C\Delta} = \frac{\mu_p^2C - 2\mu_pB + A}{\Delta}. \blacksquare$$

Examining Fig. 2.1 and leveraging Eq. (2.41), it is evident that the point $(1/\sqrt{C}, B/C)$ denotes the pairing of standard deviation and expected return for the minimum variance portfolio. This portfolio, characterized by attaining the global minimum for the variance of its return, holds paramount significance in portfolio management. If $\mu_p = \frac{B}{C}$ designates the expected return yielding the lowest portfolio variance, the vector of optimal portfolio weights is expressed as:

$$\mathbf{w}_{gmv} = \mathbf{g} + \mathbf{h}\frac{B}{C}. \tag{2.42}$$

These coordinates can be derived by computing the first-order derivative of Eq. (2.38) with respect to μ_p, i.e.,

$$\frac{d\sigma_p^2}{d\mu_p} = 0 \quad \Rightarrow \quad 2C\mu_p - 2B = 0 \Rightarrow \quad C\mu_p = B \Rightarrow \quad \mu_{p,gmv} = \frac{B}{C}.$$

By utilizing $\mu_{p,gmv} = \frac{B}{C}$ and substituting it into Eq. (2.38), we straightforwardly determine the risk level of the *global minimum variance portfolio*:

$$\sigma_{p,gmv}^2 = \frac{1}{\Delta}\left[C\left(\frac{B}{C}\right)^2 - 2A\frac{B}{C} + A\right] = \frac{C\frac{B^2}{C^2} - 2\frac{B^2}{C} + A}{AC - B^2} = \frac{1}{C}.$$

Note that $\mu_{p,gmv} = \frac{B}{C}$ and $\sigma_{p,gmv}^2 = \frac{1}{C}$ can now be utilized to determine the weights of the *global minimum variance portfolio*. Initially, we substitute $\mu_{p,gmv}$ into the definitions for λ and ψ. This yields the values of λ and ψ corresponding to the global minimum variance:

$$\lambda_{gmv} = \frac{\mu_{p,gmv}C - B}{AC - B^2} = \frac{\frac{B}{C}C - B}{AC - B^2} = \frac{\frac{BC - BC}{C}}{AC - B^2} = 0,$$

and

$$\psi_{gmv} = \frac{A - B\mu_{p,gmv}}{AC - B^2} = \frac{A - B\frac{B}{C}}{AC - B^2} = \frac{\frac{AC - B^2}{C}}{AC - B^2} = \frac{1}{C}.$$

Following this, we substitute $\lambda_{gmv} = 0$ and $\psi_{gmv} = \frac{1}{C}$ into Eq. (2.34) to derive the vector of weights defining the *global minimum variance portfolio*:

$$\begin{aligned}\mathbf{w_{gmv}} &= (\lambda_{gmv})\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + (\psi_{gmv})\boldsymbol{\Sigma}^{-1}\mathbf{1} = (0)\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \left(\frac{1}{C}\right)\boldsymbol{\Sigma}^{-1}\mathbf{1} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{C} \\ &= \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}.\end{aligned} \tag{2.43}$$

Box (2.4)

Alternative Derivation of the Global Minimum Variance Portfolio

The determination of the final expression for the vector of weights delineating the *global minimum variance portfolio* can be achieved through algebraic manipulation of Eq. (2.42). Employing the designated definitions for $\mathbf{g}$, $\mathbf{h}$, and $\mu_p = \frac{B}{C}$, the following expression is derived:

$$\mathbf{w_{gmv}} = \frac{1}{\Delta}[A(\boldsymbol{\Sigma}^{-1}\mathbf{1}) - B(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})] + \frac{1}{\Delta}[C(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) - B(\boldsymbol{\Sigma}^{-1}\mathbf{1})]\left(\frac{B}{C}\right)$$

For brevity in notation, let us introduce the shorthand notation $\mathbf{K} = \left(\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)$ and $\mathbf{J} = (\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})$. Hence, the expression simplifies as follows:

$$\begin{aligned}
\mathbf{w_{gmv}} &= \frac{1}{\Delta}[A(\mathbf{K}) - B(\mathbf{J})] + \frac{1}{\Delta}[C(\mathbf{J}) - B(\mathbf{K})]\left(\frac{B}{C}\right) = \\
&= \frac{A\mathbf{K}C - B\mathbf{J}C}{(AC - B^2)}\left(\frac{1}{C}\right) + \frac{C\mathbf{J}B - B^2\mathbf{K}}{(AC - B^2)}\left(\frac{1}{C}\right) = \\
&= \frac{A\mathbf{K}C}{(AC - B^2)C} - \frac{B\mathbf{J}C}{(AC - B^2)C} + \frac{C\mathbf{J}B}{(AC - B^2)C} - \frac{B^2\mathbf{K}}{(AC - B^2)C} = \\
&= \frac{A\mathbf{K}C}{(AC - B^2)C} - \frac{B^2\mathbf{K}}{(AC - B^2)C} = \frac{(AC - B^2)\mathbf{K}}{(AC - B^2)C} = \frac{\mathbf{K}}{C} = \frac{\left(\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}. \blacksquare
\end{aligned}$$

Note: This closed-form solution corresponds to the optimal solutions presented in Eqs. (2.25) and (2.43).

This result aligns with the solution derived in Sect. 2.3.2 (refer to Eq. (2.25) in Box (2.3)). Alternatively, the *global minimum variance portfolio* weight vector can be obtained by substituting $\mu_{p,gmv} = \frac{B}{C}$ directly into the optimal solution outlined in Eq. (2.37). The detailed algebraic steps for this alternative approach are provided in Box (2.4).

2.4.5 The Efficient Frontier

As clearly illustrated in Fig. 2.1, the *portfolio frontier* is divided into two distinct branches, which are connected at the point representing the *global minimum variance portfolio*. As a result, the *portfolio frontier* contains two portfolios that exhibit the same level of risk but differ in their expected returns, with the exception of the *global minimum variance portfolio*, which uniquely minimizes risk without considering return trade-offs. It is unequivocal that investors invariably prioritize portfolios offering superior expected returns. Thus, for a specified risk level, investor preference converges solely toward portfolios situated within the upper branch of the *portfolio frontier* (i.e., portfolios along the green branch in Fig. 2.1). These portfolios consistently offer higher expected returns in proportion to their risk, surpassing those located on the lower branch of the hyperbola (i.e., portfolios along the red branch in Fig. 2.1). Consequently, any portfolio, denoted p, positioned along the upper branch of the hyperbola in Fig. 2.1 is classified as an *efficient portfolio* (EP) if its expected return, μ_p, is greater than or equal to that of the minimum variance portfolio, denoted by $\frac{B}{C}$. The *efficient frontier* (EF) accordingly comprises the collective of EPs, delineating the green branch within Fig. 2.1. Evidently, any portfolio situated outside the EF is surpassed by a portfolio positioned within it. For every portfolio outside the EF, there exists an *efficient portfolio* (EP) that matches its standard deviation while providing a higher expected return. Intuitively, rational investors typically avoid selecting portfolios located outside the EF, as they offer suboptimal risk–return combinations.

The set of EPs constitutes a convex set. Exploiting this fundamental property, we can demonstrate that any convex combination of a group of EPs remains efficient. Consider a scenario where we have S distinct portfolios on the EF, denoted by weight vectors $\mathbf{w}^1, \mathbf{w}^2, ..., \mathbf{w}^S$. Let $\alpha_1, \alpha_2, ..., \alpha_S$ represent real numbers such that $\sum_{s=1}^{S} \alpha_s = 1$. The convex combination of the S portfolios, defined as:

$$\mathbf{w}^c \equiv \sum_{s=1}^{S} \alpha_s \mathbf{w}^s,$$

also qualifies as an EP. To establish this, let us begin by observing that:

$$\mathbf{1}'\mathbf{w}^c = \sum_{i=1}^{N} \sum_{s=1}^{S} \alpha_s w_i^s = \sum_{s=1}^{S} \alpha_s \sum_{i=1}^{N} w_i^s = \sum_{s=1}^{S} \alpha_s = 1,$$

where the notation $\sum_{i=1}^{N} \sum_{s=1}^{S} \alpha_s w_i^s$ signifies the ability to invest in N different risky assets for each portfolio s, and $\sum_{i=1}^{N} w_i^s$ ensures that the sum of weights in each portfolio s equals 1. Subsequently, we establish the following:

$$\mathbf{w}^c \equiv \sum_{s=1}^{S} \alpha_s \mathbf{w}^s = \sum_{s=1}^{S} \alpha_s (\mathbf{g} + \mathbf{h}\mu_s) = \mathbf{g} + \mathbf{h} \sum_{s=1}^{S} \alpha_s \mu_s = \mathbf{g} + \mathbf{h}\mu_c$$

$$\mu_c = \sum_{s=1}^{S} \alpha_s \mu_s \geq \underbrace{\sum_{s=1}^{S} \alpha_s}_{=1} \frac{B}{C} = \frac{B}{C}.$$

It is revealed that $\mathbf{w}^c$ represents a portfolio situated on the frontier and possesses the attribute of efficiency. This finding carries significant implications: in a scenario where all investors exclusively hold EPs, the resulting *market portfolio*, structured as a convex combination of investors' portfolios, with weights proportional to individuals' shares of the total wealth, also achieves efficiency.

2.4.6 The "Two-Fund" Separation Theorem

The convexity property of the EF plays a pivotal role in establishing the so-called *two-fund separation theorem*. This property allows us to assert that any selection of two distinct *efficient portfolios* (EPs) can form a basis to span the entire EF. For instance, let $\mathbf{w}^1$ and $\mathbf{w}^2$ represent the weight vectors associated with two distinct portfolios, labeled as 1 and 2, situated on the EF. Additionally, let $\mathbf{w}_p$ denote the weight vector of any other EP, denoted as p. Furthermore, let μ_1, μ_2, and μ_p denote the respective expected returns of these portfolios. It follows that there exists $\alpha \in \mathbb{R}$ such that:

$$\mu_p = \alpha\mu_1 + (1-\alpha)\mu_2.$$

This equation illustrates the capability to express the expected return of any EP, p, as a convex combination of the expected returns of portfolios 1 and 2. Consider thus the scenario where portfolios 1, 2, and p are all EPs. In such a case, the following relationship holds:

$$\begin{aligned}
\mathbf{w}_p &= \mathbf{g} + \mathbf{h}\mu_p = \mathbf{g} + \mathbf{h}[\alpha\mu_1 + (1-\alpha)\mu_2] \\
&= \alpha(\mathbf{g} + \mathbf{h}\mu_1) + (1-\alpha)(\mathbf{g} + \mathbf{h}\mu_2) \\
&= \alpha\mathbf{w}^1 + (1-\alpha)\mathbf{w}^2,
\end{aligned}$$

It turns out that portfolio p can be replicated through a linear combination of EPs 1 and 2. The two-fund separation principle, consequently, asserts that as long as at least two distinct EPs are publicly traded in financial markets, individual investors will always have the ability to construct any desired *frontier portfolio* by appropriately combining the two traded EPs. Consequently, individual investors can acquire a specific EP p not only by investing in individual assets in desired proportions but also by investing in the two traded EPs.

Now, consider Fig. 2.2. Here, investor a aims to hold a portfolio characterized by weight vector $\mathbf{w}^a$, expected return μ_a, and standard deviation σ_a. Similarly, investor b seeks to hold a portfolio characterized by weight vector $\mathbf{w}^b$, expected return μ_b, and standard deviation σ_b. These portfolios can be acquired either by purchasing quantities $\{w_i^a\}_{i=1}^N$ and $\{w_i^b\}_{i=1}^N$ of individual assets or by investing in different proportions of funds 1 and 2.

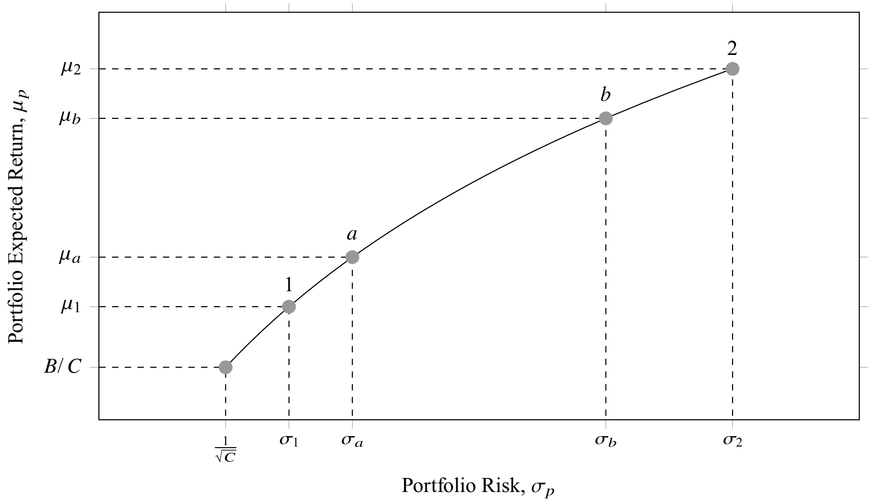


Fig. 2.2 Two fund separation

2.5 Adding a Riskless Asset

Let us consider a scenario wherein investors have the option to invest in a riskless asset. This newly introduced riskless asset offers a deterministic rate of return (i.e., $\sigma^2_{r_f} = 0$), denoted as r_f, with the condition that $r_f < \frac{B}{C}$. Similar to the M-V asset allocation framework outlined in Sect. 2.4, we can characterize a portfolio by specifying the vector $\mathbf{w}$, representing the allocation of initial wealth among N risky assets. However, in this expanded economy comprising $N + 1$ assets, there is no necessity to enforce the constraint that the elements of $\mathbf{w}$ sum up to 1. This relaxation stems from investors distributing their total wealth between a portfolio of risky assets and the riskless asset. Consequently, the share of wealth not allocated to the N risky assets is held in the riskless asset, denoted as $w_f = (1 - \mathbf{w}'\mathbf{1})$. It is implicit that investors utilize all available resources. Formally, this is expressed as:

$$\mathbf{w}'\mathbf{1} + (1 - \mathbf{w}'\mathbf{1}) = \mathbf{w}'\mathbf{1} + w_f = 1.$$

It is worth noting that the allocation of wealth to the riskless asset, w_f, may assume negative values, indicating that investors can *borrow* the riskless asset at the risk-free rate r_f through short-selling. Conversely, when $w_f > 0$, investors can *lend* the riskless asset at the risk-free rate r_f. In essence, the introduction of the riskless asset presupposes that investors can freely *borrow* and *lend* at a predetermined deterministic interest rate. Consequently, the expected return of this $N + 1$ portfolio can be formulated as:

$$\mathbf{w}'\boldsymbol{\mu} + w_f r_f = \mathbf{w}'\boldsymbol{\mu} + (1 - \mathbf{w}'\mathbf{1})r_f = \mu_p, \tag{2.44}$$

where $\boldsymbol{\mu}$ denotes the standard vector of expected returns of the risky assets.

2.5.1 Deriving Optimal Weights

Let us revisit the optimization problem discussed earlier in the context of defining the *portfolio frontier*, as outlined in Sect. 2.4. Given the constraint specified in (2.44), the task of determining the weight vector $\mathbf{w}$ and the risk-free asset weight w_f involves solving the following optimization problem:

$$\min_{\{\mathbf{w}\}} \left(\frac{1}{2}\right) \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \quad \text{s.t.} \quad \mathbf{w}'\boldsymbol{\mu} + (1 - \mathbf{w}'\mathbf{1})r_f = \mu_p. \tag{2.45}$$

The corresponding *Lagrangian* for this optimization problem is defined as

$$\min_{\{\mathbf{w},\lambda\}} \Lambda \equiv \left(\frac{1}{2}\right) \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \lambda[\mu_p - \mathbf{w}'\boldsymbol{\mu} - (1 - \mathbf{w}'\mathbf{1})r_f], \tag{2.46}$$

with the associated FOCs regarding $\mathbf{w}$ and the unique Lagrange multiplier λ being

$$\frac{\partial\Lambda}{\partial\mathbf{w}} = 0 \;\Rightarrow\; \mathbf{w} \;= \lambda\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f), \tag{2.47}$$

$$\frac{\partial\Lambda}{\partial\lambda} = 0 \;\Rightarrow\; \mu_p = r_f + \mathbf{w}'(\boldsymbol{\mu} - \mathbf{1}r_f). \tag{2.48}$$

The vector $(\boldsymbol{\mu} - \mathbf{1}r_f)$ represents the $(N \times 1)$ array of *expected excess returns* of the risky assets over the risk-free asset. It represents the premium that risky assets, on average, provide over the risk-free rate. Substituting (2.47) into (2.48) yields the following expression for λ:

$$\lambda = \frac{\mu_p - r_f}{H} \tag{2.49}$$

where $H = (\boldsymbol{\mu} - \mathbf{1}r_f)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)$. Substituting (2.49) into (2.47) provides the solution for $\mathbf{w}$:

$$\mathbf{w} = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)\frac{\mu_p - r_f}{H}. \tag{2.50}$$

Equation (2.50) provides insight into the allocation of weights for the risky assets that solve the optimization problem. It can be broken down into two components: first, a $(N \times 1)$ vector that depends on the covariance matrix $\boldsymbol{\Sigma}$ and the vector of expected excess returns $(\boldsymbol{\mu} - \mathbf{1}r_f)$; and second, a scalar term that is a function of the desired expected return, μ_p. This decomposition highlights that, when focusing on investments solely in the N risky assets, the relative proportions of the assets remain invariant regardless of the chosen expected return, μ_p. In simpler terms, if an investor aims for a higher expected return, μ_p, the scalar term $(\mu_p - r_f)/H$ indicates that they must allocate proportionally more capital to all risky assets. This adjustment ensures that the relative weights of the assets remain constant, thereby maintaining the portfolio's desired risk–return profile despite changes in the expected return. The solution for w_f, representing the weight allocated to the risk-free asset, can be expressed as $(1 - \mathbf{w}'\mathbf{1})$.

2.5.2 The Efficient Frontier

Substituting the vector defined in Eq. (2.50) into the expression for the portfolio return variance results in the following:[11]

$$\sigma_p^2 = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} = \frac{1}{H}(\mu_p - r_f)^2.$$

Taking the square root of this expression and rearranging, we obtain the equation characterizing the *portfolio frontier* in the (σ_p, μ_p) space:

$$\mu_p = r_f \pm \sqrt{H}\sigma_p. \tag{2.51}$$

A graphical representation of this "$N + 1$" *portfolio frontier* is provided in Fig. 2.3. Clearly, only allocations lying on the upper segment of the *portfolio frontier* are efficient. Therefore, the new EF is represented by the green segment in Fig. 2.3. In finance, we refer to this straight line, with slope $\sqrt{H}$, also as the capital market line (CML). The slope of the CML indicates the trade-off between risk and return along the EF. Moreover, it corresponds to the *Sharpe ratio* measured as the ratio between the expected excess return and standard deviation of any EP, i.e., $(\mu_p - r_f)/\sigma_p$. In other words, the slope of the EF measures the benefit in terms of a larger expected excess return an investor would obtain from a given increase in risk exposure.

[11] Alternatively, one can pre-multiply both sides of Eq. (2.50) by $\mathbf{w}'\boldsymbol{\Sigma}$, which yields:

$$\begin{aligned}\left(\mathbf{w}'\boldsymbol{\Sigma}\right)\mathbf{w} = \sigma_p^2 &= \left(\mathbf{w}'\boldsymbol{\Sigma}\right)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)\frac{\mu_p - r_f}{H} = \mathbf{w}'(\boldsymbol{\mu} - \mathbf{1}r_f)\frac{\mu_p - r_f}{H}\\ &= \underbrace{(\boldsymbol{\mu} - \mathbf{1}r_f)'\boldsymbol{\Sigma}^{-1}\frac{\mu_p - r_f}{H}}_{\mathbf{w}'}(\boldsymbol{\mu} - \mathbf{1}r_f)\frac{\mu_p - r_f}{H}.\end{aligned}$$

Rearranging terms and simplifying, we then obtain:

$$\sigma_p^2 = \underbrace{(\boldsymbol{\mu} - \mathbf{1}r_f)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}_{H}\frac{\mu_p - r_f}{H}\frac{\mu_p - r_f}{H} = H\frac{(\mu_p - r_f)^2}{H^2} = \frac{1}{H}(\mu_p - r_f)^2. \ (Q.E.D.)$$

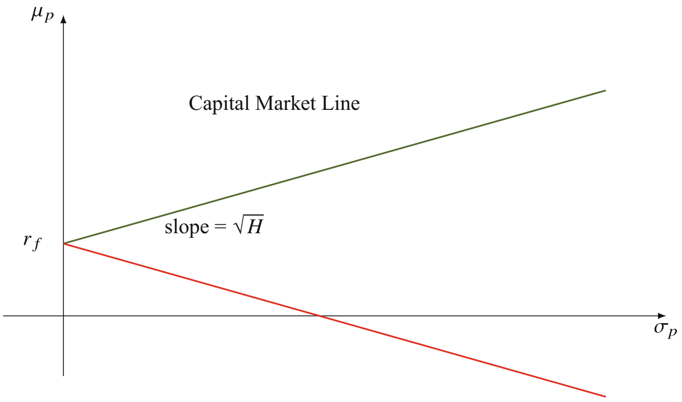


Fig. 2.3 The portfolio frontier (N risky assets + 1 riskless asset)

2.5.3 The Tangency Portfolio

An essential characteristic of the CML, which represents the *portfolio frontier* in an economy with N risky assets and one riskless asset, denoted as EF_{N+1}, is its tangency to the efficient frontier of portfolios comprising only risky assets, denoted as EF_N. The point of tangency identifies a portfolio, referred to as T, which is entirely composed of risky assets and lies on the efficient segment of the frontier of risky portfolios (see Fig. 2.4). A key feature of the *tangency portfolio* is that its *Sharpe ratio*, given by $(\mu_T - r_f)/\sigma_T$, corresponds to the slope of the CML and, therefore, is equal to $\sqrt{H}$. Since the *tangency portfolio* invests solely in risky assets, its allocation can be determined by imposing the condition $\mathbf{w}'\mathbf{1} = \mathbf{1}'\mathbf{w} = 1$ in Eq. (2.50). To demonstrate this, we multiply both sides of Eq. (2.50) by $\mathbf{1}'$. This operation yields:

$$1 = (\mathbf{1}')\mathbf{w} = (\mathbf{1}')\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)\frac{\mu_p - r_f}{H} \quad \Rightarrow \quad (\mu_p - r_f) = \frac{H}{(\mathbf{1}')\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}.$$

Substituting this result back into the original expression (2.50), we obtain:

$$\mathbf{w} \equiv \mathbf{w_T} = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)\frac{1}{H}\frac{H}{(\mathbf{1}')\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}.$$

Consequently, the solution for the *tangency portfolio*, $\mathbf{w_T}$, assumes the following straightforward form:[12]

$$\mathbf{w_T} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}. \tag{2.52}$$

[12] To ensure completeness, Boxes (2.5) and (2.6) derive the vector of weights of the *tangency portfolio* using two alternative approaches.

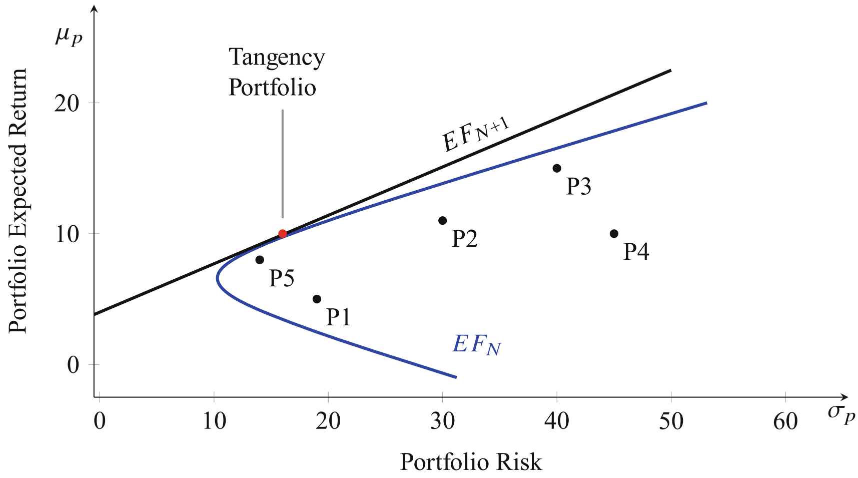


Fig. 2.4 The tangency portfolio

It is notable that the formula for the *tangency portfolio* (2.52) bears a resemblance to the formula for the *global minimum variance portfolio* (2.43). In both formulas, we observe $\boldsymbol{\Sigma}^{-1}$ in the numerator and $\mathbf{1}'\boldsymbol{\Sigma}^{-1}$ in the denominator. It is worth emphasizing that the positioning of the *tangency portfolio* and the sign of the *Sharpe ratio* hinge upon the relationship between the risk-free rate, r_f, and the expected return on the *global minimum variance portfolio*, $\mu_{p,gmv}$. Typically, when $\mu_{p,gmv} > r_f$, the *tangency portfolio* boasts a positive *Sharpe ratio*. Conversely, during periods of economic downturns or financial crises, if $\mu_{p,gmv} < r_f$, the *tangency portfolio* carries a negative *Sharpe ratio*. In such scenarios, EPs involve shorting the *tangency portfolio* and allocating the proceeds into riskless assets such as Treasury bills.

Substituting (2.52) into the definition of the portfolio variance, $\sigma_p^2 = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$, yields:

$$\sigma_T^2 = \mathbf{w}_\mathbf{T}'\boldsymbol{\Sigma}\mathbf{w}_\mathbf{T} = \left(\frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}\right)' \boldsymbol{\Sigma} \left(\frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}\right)$$

$$= \frac{(\boldsymbol{\mu} - \mathbf{1}r_f)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}{[(\boldsymbol{\mu} - \mathbf{1}r_f)'\boldsymbol{\Sigma}^{-1}\mathbf{1}][\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)]}$$

where $\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1} = 1$. Using $H = (\boldsymbol{\mu} - \mathbf{1}r_f)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)$ and imposing $(\boldsymbol{\mu} - \mathbf{1}r_f)'\boldsymbol{\Sigma}^{-1}\mathbf{1} = \mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f) = \Upsilon$, we obtain

$$\sigma_T^2 = \frac{H}{\Upsilon^2} \;\Rightarrow\; \left(\sigma_T^2\right)^{1/2} = \frac{H^{1/2}}{(\Upsilon^2)^{1/2}} \;\Rightarrow\; \sigma_T = \frac{\sqrt{H}}{\Upsilon}.$$

Hence, the standard deviation and expected return of the *tangency portfolio* are:

$$\sigma_T = \frac{\sqrt{H}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}, \quad \mu_T = r_f \pm \sqrt{H} \cdot \sigma_T = r_f \pm \frac{H}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}. \tag{2.53}$$

N.B. H and $\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)$ are scalars.

Box (2.5)

Alternative Derivation of the Tangency Portfolio I

Consider the construction of portfolios consisting of N risky assets, characterized by a return vector $\tilde{\boldsymbol{r}}$, alongside a risk-free asset such as Treasury bills, offering a constant return denoted by r_f. Let $\mathbf{w}$ represent the vector of weights assigned to the risky assets, and let w_f denote the weight allocated to the risk-free asset. We assume that $\mathbf{w}'\mathbf{1} + w_f = 1$, ensuring that the investor's entire wealth is fully invested across these assets. The resulting (random) portfolio return is given by:

$$\tilde{r}_p = \mathbf{w}'\tilde{\boldsymbol{r}} + w_f r_f = \mathbf{w}'\tilde{\boldsymbol{r}} + (1 - \mathbf{w}'\mathbf{1})r_f = r_f + \mathbf{w}'(\tilde{\boldsymbol{r}} - r_f\mathbf{1}).$$

From the aforementioned equation, we can define the expected excess return of the portfolio as:

$$(\mu_p - r_f) = \mathbf{w}'(\boldsymbol{\mu} - \mathbf{1}rf)$$

where $\mu_p = \mathbb{E}(\tilde{r}_p)$ and $\boldsymbol{\mu} = \mathbb{E}(\tilde{\boldsymbol{r}})$. The portfolio variance is $\sigma_p^2 = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$. For simplicity in notation, let us define the following terms:

- $\breve{\mu}_p = \mathbf{w}'\breve{\boldsymbol{\mu}} = (\mu_p - r_f) \;\rightarrow$ expected excess return of the portfolio
- $\breve{\boldsymbol{\mu}} = (\boldsymbol{\mu} - \mathbf{1}rf) \;\rightarrow (N \times 1)$ vector of expected excess returns
- $\breve{\mu}_{p,target} \;\rightarrow$ target expected excess return

With these definitions in place, the minimization problem can be formulated as follows:

$$\min_{\{\mathbf{w}\}} \quad \sigma_p^2 = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \quad s.t. \quad \mathbf{w}'\breve{\boldsymbol{\mu}} = \breve{\mu}_{p,target}$$

The Lagrangian function for this optimization problem is defined as:

$$\Lambda(\mathbf{w}, \lambda) = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \lambda(\mathbf{w}'\breve{\boldsymbol{\mu}} - \breve{\mu}_{p,target}),$$

(continued)

Box (2.5) (continued)

where $\mathbf{w}$ is the vector of portfolio weights, λ is the Lagrange multiplier, $\boldsymbol{\Sigma}$ represents the covariance matrix of asset returns, $\breve{\boldsymbol{\mu}}$ is the vector of expected excess returns, and $\breve{\mu}_{p,target}$ is the target expected excess return.
The FOCs are given by:

$$\frac{\partial \Lambda}{\partial \mathbf{w}} = 0 = 2\boldsymbol{\Sigma}\mathbf{w} + \lambda\breve{\boldsymbol{\mu}} \quad \Rightarrow \quad (*) \;\; \mathbf{w} = -\frac{1}{2}\lambda\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}},$$

$$\frac{\partial \Lambda}{\partial \lambda} = 0 = \mathbf{w}'\breve{\boldsymbol{\mu}} - \breve{\mu}_{p,target} \quad \Rightarrow \quad (**) \;\; \mathbf{w}'\breve{\boldsymbol{\mu}} = \breve{\mu}_{p,target}.$$

Pre-multiplying both sides of (*) by ($\breve{\boldsymbol{\mu}}'$), we have:

$$(\breve{\boldsymbol{\mu}}')\mathbf{w} = -\frac{1}{2}\lambda(\breve{\boldsymbol{\mu}}')\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}.$$

Since $\breve{\boldsymbol{\mu}}'\mathbf{w} = \mathbf{w}'\breve{\boldsymbol{\mu}} = \breve{\mu}_{p,target}$, a solution for λ is:

$$\lambda = -\frac{2\breve{\mu}_{p,target}}{\breve{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}},$$

which can be substituted back into equation (*) to yield:

$$\mathbf{w} = -\frac{1}{2}\left(-\frac{2\breve{\mu}_{p,target}}{\breve{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}}\right)\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}} \quad \Rightarrow \quad \mathbf{w} = \breve{\mu}_{p,target}\frac{\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}}{\breve{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}} \quad (***)$$

Given that the *tangency portfolio* is fully invested in risky assets, it follows that $\mathbf{w}'\mathbf{1} = 1$ and $w_f = 0$. Therefore, the following relationship holds:

$$\mathbf{1}'\mathbf{w_T} = \breve{\mu}_{p,target}\frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}}{\breve{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}} = 1 \quad \Rightarrow \quad \breve{\mu}_{p,target} = \frac{\breve{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}}.$$

We can now substitute this definition for $\breve{\mu}_{p,target}$ into equation (***) to obtain:

$$\mathbf{w_T} = \left(\frac{\breve{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}}\right)\cdot\frac{\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}}{\breve{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}} = \frac{\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}}.$$

(continued)

Box (2.5) (continued)

Using the definition $\breve{\boldsymbol{\mu}} = (\boldsymbol{\mu} - \mathbf{1}rf)$, we obtain an explicit solution for $\mathbf{w_T}$ as follows:

$$\mathbf{w_T} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}rf)}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}rf)}.\blacksquare \tag{2.54}$$

This solution corresponds to the one we derived in Sect. 2.5.3 (see Eq. (2.52)).

Box (2.6)

Alternative Derivation of the Tangency Portfolio II

The *tangency portfolio* represents the portfolio of risky assets with the highest *Sharpe ratio*. Let $\mathbf{w_T}$ denote the vector of weights of the *tangency portfolio*. If the objective is to construct a portfolio with the maximum *Sharpe ratio* (Θ), then $\mathbf{w_T}$ solves the following maximization problem:

$$\max_{\{\mathbf{w_T}\}} \quad \Theta = \frac{\mathbf{w_T}'(\boldsymbol{\mu} - \mathbf{1}r_f)}{(\mathbf{w_T}'\boldsymbol{\Sigma}\mathbf{w_T})^{1/2}} = \frac{(\mu_p - r_f)}{\sigma_p} \tag{2.55}$$

where the constraint $\mathbf{w}'\boldsymbol{\mu} + (1 - \mathbf{w}'\mathbf{1})r_f = \mu_p$ or $\mathbf{w}'(\boldsymbol{\mu} - \mathbf{1}r_f) = \mu_p - r_f$ is utilized. The FOC is given by:

$$\frac{d\Theta}{d\mathbf{w_T}} = (\boldsymbol{\mu} - \mathbf{1}r_f)(\mathbf{w_T}'\boldsymbol{\Sigma}\mathbf{w_T})^{-1/2} - \frac{1}{2}\mathbf{w_T}'(\boldsymbol{\mu} - \mathbf{1}r_f)\left(\mathbf{w_T}'\boldsymbol{\Sigma}\mathbf{w_T}\right)^{-\frac{1}{2}-1}(2\boldsymbol{\Sigma}\mathbf{w_T}) = 0$$

where $\frac{d(\mathbf{w_T}'\boldsymbol{\Sigma}\mathbf{w_T})}{d\mathbf{w_T}} = 2\boldsymbol{\Sigma}\mathbf{w_T}$. Pre-multiplying both sides of the aforementioned FOC by $(\mathbf{w_T}'\boldsymbol{\Sigma}\mathbf{w_T})^{1/2}$ results in:

$$(\boldsymbol{\mu} - \mathbf{1}r_f) = \left[\frac{\mathbf{w_T}'(\boldsymbol{\mu} - \mathbf{1}r_f)}{(\mathbf{w_T}'\boldsymbol{\Sigma}\mathbf{w_T})}\right]\boldsymbol{\Sigma}\mathbf{w_T} \quad \Rightarrow \quad (\mathbf{1}')\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f) = \frac{\mu_p - rf}{\sigma_p^2}(\mathbf{1}')\mathbf{w_T}.$$

Given that $(\mathbf{1}')\mathbf{w_T} = 1$ (i.e., 100% invested in the risky portfolio), the following condition holds:

$$\frac{\mu_p - rf}{\sigma_p^2} = \mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f),$$

which can be substituted to obtain:

(continued)

Box (2.6) (continued)

$$(\boldsymbol{\mu} - \mathbf{1}r_f) = \left[\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)\right]\boldsymbol{\Sigma}\mathbf{w_T}.$$

Multiplying both sides of this latter expression by $\boldsymbol{\Sigma}^{-1}$ and solving for $\mathbf{w_T}$ yield:

$$\mathbf{w_T} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}.\blacksquare \tag{2.56}$$

2.5.4 The "One-Fund" Theorem

When a riskless asset is available, the CML, which represents the EF of portfolios comprising both riskless and risky assets, consists of portfolios that combine the riskless asset, providing a fixed return r_f, with the *tangency portfolio* T, which offers a random return $\tilde{r}_T$. These portfolios, denoted as p, allocate a portion α (where $\alpha \in \mathbb{R}$) of the investor's wealth to the riskless asset and the remaining fraction $1 - \alpha$ to the *tangency portfolio*. The expected return μ_p and standard deviation σ_p of such EPs are given by:

$$\mu_p = \alpha r_f + (1 - \alpha)\mu_T, \tag{2.57}$$

$$\sigma_p = (1 - \alpha)\sigma_T, \tag{2.58}$$

where μ_T represents a weighted average of the risk-free rate and the expected return of the *tangency portfolio*, and σ_T equals the standard deviation of the *tangency portfolio*'s return multiplied by its weight. Solving Equation (2.58) for $(1-\alpha)$ yields:

$$(1 - \alpha) = \frac{\sigma_p}{\sigma_T} \quad \Rightarrow \quad \alpha = 1 - \frac{\sigma_p}{\sigma_T},$$

which, when substituted into Eq. (2.57), yields a simplified expression for the CML in terms of the *Sharpe ratio* of the *tangency portfolio*:

$$\mu_p = r_f + \frac{\mu_T - r_f}{\sigma_T}\sigma_p. \tag{2.59}$$

This equation encapsulates the expected return μ_p of the portfolio p as a function of the risk-free rate r_f and the *Sharpe ratio* of the *tangency portfolio*.

Through borrowing and lending mechanisms, any EP, represented as any point along the upper tangent line in Fig. 2.5, can be decomposed into a combination of

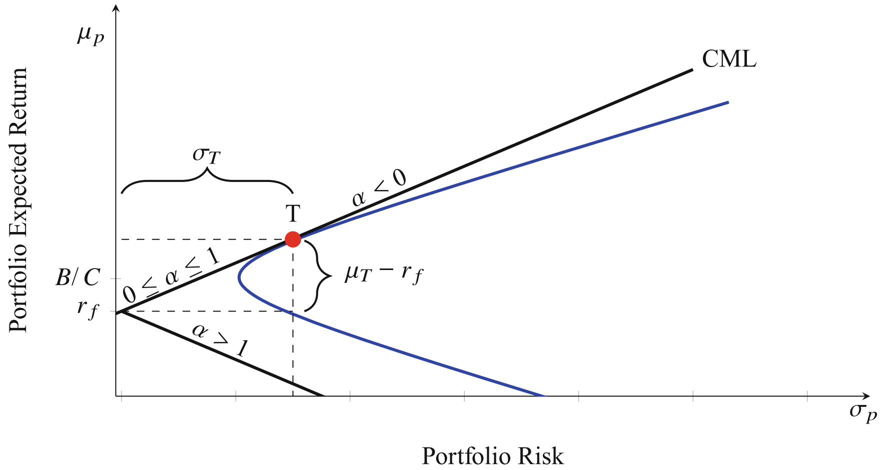


Fig. 2.5 The CML and the frontier of risky assets

the riskless asset and the portfolio T. Put differently, a singular fund T comprising risky assets exists such that any EP can be formed as a blend of this fund T and the riskless asset. This principle, known as the *one-fund theorem*, revises the previous notion of the *two-fund theorem*. Notably, investing in the riskless asset, such as Treasury bills (T-bills), can be regarded as investing in a fund exclusively composed of riskless assets, thereby invoking the concept of the *two-fund theorem* once more. In practical terms, given the availability of a publicly traded fund of risky assets and a riskless asset, investors can attain EPs by allocating a portion of their wealth to the fund of risky assets and the remainder to T-bills.

The graphical representation depicted in Fig. 2.5 presents an alternative perspective on illustrating how the CML encapsulates the EF when augmented with a risk-free asset. As previously established, the combination of risk and return along the CML can be delineated by holding specific portfolios comprised of both the riskless asset and the *tangency portfolio*. Nevertheless, it remains conceivable to combine the riskless asset with any other portfolio of risky assets. Let us designate a different portfolio, denoted as k, situated on the *portfolio frontier*.

The risk–return relationship for portfolios formed by combining the riskless asset with portfolio k is depicted in Fig. 2.6 as CML$_k$. This graph illustrates that the *Sharpe ratio* of portfolio k is lower than that of the *tangency portfolio*, i.e.,

$$\frac{\mu_k - r_f}{\sigma_k} < \frac{\mu_T - r_f}{\sigma_T}.$$

It follows that any portfolio resulting from the combination of the riskless asset with portfolio k cannot achieve efficiency, as it is always dominated by a portfolio on the CML. To substantiate this claim, let us revisit our earlier derivation. Consider an arbitrary $\alpha \in \mathbb{R}$ and define a portfolio p where a fraction α is allocated to the risk-

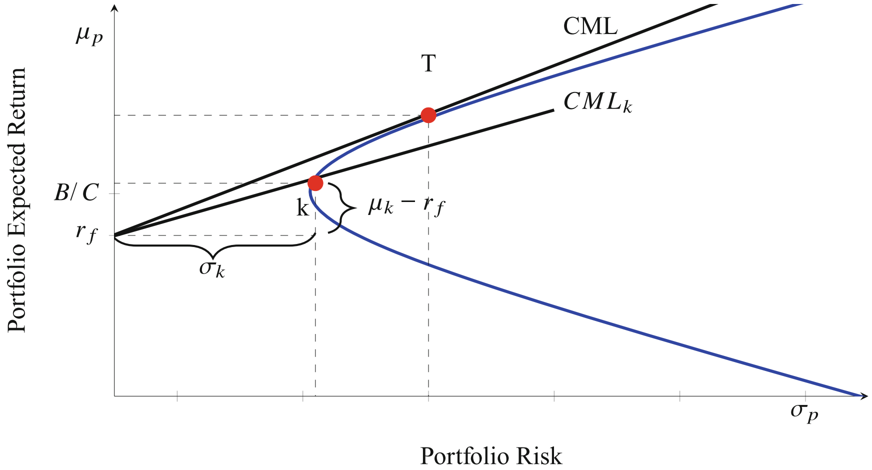


Fig. 2.6 The CML and other portfolios of assets

free asset, while the remaining fraction $1-\alpha$ is allocated to portfolio k, rather than the *tangency portfolio*. Under these conditions, the mean and standard deviation of this portfolio are given by:

$$\mu_p = \alpha r_f + (1-\alpha)\mu_k,$$
$$\sigma_p = (1-\alpha)\sigma_k.$$

Solving for $(1-\alpha)$ and substituting give the expression for the new line depicted in Fig. 2.6:

$$\mu_p = r_f + \frac{\mu_k - r_f}{\sigma_k}\sigma_p,$$

where its slope corresponds to the *Sharpe ratio* of portfolio k, hence lying below the CML. The points on this line denote portfolios that lack efficiency. By repeating this procedure for any portfolio k consisting of risky assets but distinct from the *tangency portfolio*, one can deduce that the CML aligns with the EF in an economy featuring N risky assets and a single riskless security.

A fundamental property of EF_{N+1} is its inherent convexity, ensuring that any convex combination of EPs associated with the CML maintains efficiency. This characteristic is particularly significant when considering investors, specifically those positioned along the CML, who maintain EPs. Upon aggregation into the *market portfolio*, this composite portfolio retains its efficiency. It is crucial to note that, under specific assumptions including M-V optimization and a consensus on the probabilistic distribution of assets, along with the existence of a riskless asset, investors uniformly opt for investment in the *market portfolio*.

2.6 Bringing Theory to Practice: Implementing Mean-Variance Allocation

This section provides an empirical application of the Markowitz (1952)'s M-V portfolio allocation. We aim to apply this theoretical framework to real-world financial data, computing the EF and optimal portfolios for two distinct economic scenarios. The first scenario involves an economy solely comprising N risky assets. We then extend the analysis to include a risk-free asset, offering a broader perspective on portfolio optimization strategies. Before proceeding, it is crucial to clarify our objective. We aim to present a simple empirical application that directly connects theory with practice. This highlights the practical implementation of theoretical concepts, rather than offering economic interpretations of observed asset allocations. Essentially, we treat this exercise as a basic empirical example, similar to a classroom illustration. In keeping with the practical focus of this book, all the code used for our M-V portfolio analyses, developed in a variety of programming languages (such as MATLAB, Python, Julia, and R), will be made available on the following GitHub repository: https://github.com/DoCoGu/EoFE_DCG_Springer.

2.6.1 Economy with $N = 5$ Risky Assets

For our empirical study, we concentrate on analyzing five prominent stocks that are constituents of the Dow Jones Industrial Average (DJIA): Procter&Gamble, 3M, IBM, Merck, and American Express. It is worth highlighting that these five selected stocks hold the distinction of being the oldest listed stocks within the DJIA. To ensure robustness and historical relevance, we extract stock price data for these companies from February 2000 to December 2022. This dataset underpins our empirical analysis of M-V portfolio allocation's real-world effectiveness.

Prior to constructing the efficient frontier and determining optimal weights via M-V, we present an overview of the descriptive statistics for the five selected securities. Mean and standard deviation values, in addition to the *Sharpe ratio*, Skewness, and Kurtosis, are meticulously presented in Table 2.1. Analyzing the entries in Table 2.1, several significant empirical facts emerge. For instance, significant disparities are evident in the long-term average performance of the five stocks. For example, IBM shows an average monthly return of 0.91%, equating to an annual return of nearly 11%. This performance exceeds American Express by more than 4% and surpasses Procter&Gamble and 3M by almost 2%. However, IBM also displays a relatively high level of risk, with a monthly volatility of 9.15% (or 32% annually). As a result, IBM's average performance per unit of risk is comparatively lower. Specifically, its *Sharpe Ratio* stands at 0.0856, which is below that of both Procter&Gamble (0.1248) and 3M (0.1035).

Furthermore, the data presented in Table 2.1 underscore significant discrepancies among stocks concerning the degree of kurtosis and skewness, offering additional insights into the diverse risk and return profiles inherent within the analyzed securi-

Table 2.1 The five oldest stocks in the DJIA: descriptive statistics

Company (Symbol)	Procter&Gamble (PG)	3M (MMM)	IBM (IBM)	Merck (MRK)	American Express (AXP)
Date added	26.05.1932	09.08.1976	29.06.1979	29.06.1979	30.08.1982
Mean	0.0076	0.0074	0.0091	0.0069	0.0056
StDev	0.0512	0.0595	0.0915	0.0690	0.0726
Sharpe ratio	0.1248	0.1035	0.0856	0.0823	0.0601
Skweness	− 1.4188	− 0.0292	2.5058	− 0.2304	0.4314
Kurtosis	11.3913	3.5590	28.4326	4.2254	6.2922

Notes: Monthly returns computed from price adjusted for dividends and splits retrieved from Yahoo.finance. Sharpe ratio $:= \frac{r_i - r_f}{\sigma_i}$ where r_i is the average return of asset i, r_f is the risk-free rate proxied by the one-month T-bill and σ_i is the standard deviation of the stock return i. Companies market capitalization as of May 10th, 2024: PG = \$393.79B; MMM = \$54.74B; IBM = \$153.54B; MRK = \$329.42B; AXP = \$174.29B (B = Billions). Sample period: January 2000–December 2022 (276 observations).
Data and codes—written in , , julia, and —for replication are available at the following link: https://github.com/DoCoGu/EoFE_DCG_Springer.

ties. Specifically, our analysis reveals that Procter&Gamble, 3M, and Merck exhibit negative skewness, implying that, on average, investors may anticipate frequent small gains coupled with occasional large losses from these stocks. Conversely, IBM and American Express demonstrate positive skewness. Notably, Procter&Gamble and IBM exhibit significant leptokurtosis, suggesting a higher likelihood of extreme events. This characteristic warrants consideration in risk assessment and portfolio management.

The descriptive statistics presented in Table 2.1 are calculated based on monthly stock returns observed over the period from January 2000 to December 2022. It is essential to acknowledge that the specific temporal window selected for analysis can exert a substantial impact on these statistical measures. As a result, the mean, standard deviation, skewness, and kurtosis values of stock returns may exhibit considerable variability across different time periods.

The risk–return relationships of the five stocks are visually depicted by orange dots in Fig. 2.7. As anticipated, each individual stock is outperformed, within an M-V framework, by all portfolios lying on the EF. Despite all stocks being situated within the feasible set, each one can be surpassed by a portfolio offering a higher expected return for an equivalent level of risk (i.e., an EP). To identify the EF, we resort to Eq. (2.40), which furnishes the expected return of the portfolio corresponding to a specific volatility level. In practice, we plot this equation on a grid of expected portfolio volatilities. The resulting EF is depicted as a black line in Fig. 2.7. Notably, the expected return on the portfolio comprising all five stocks consistently surpasses the returns of individual stocks across various volatility levels.

A key aspect of M-V portfolio allocation is the *global minimum variance portfolio*, which minimizes volatility. This portfolio, marked by the red dot in Fig. 2.7, offers a 0.7% expected return and 3.94% volatility in our analysis. While

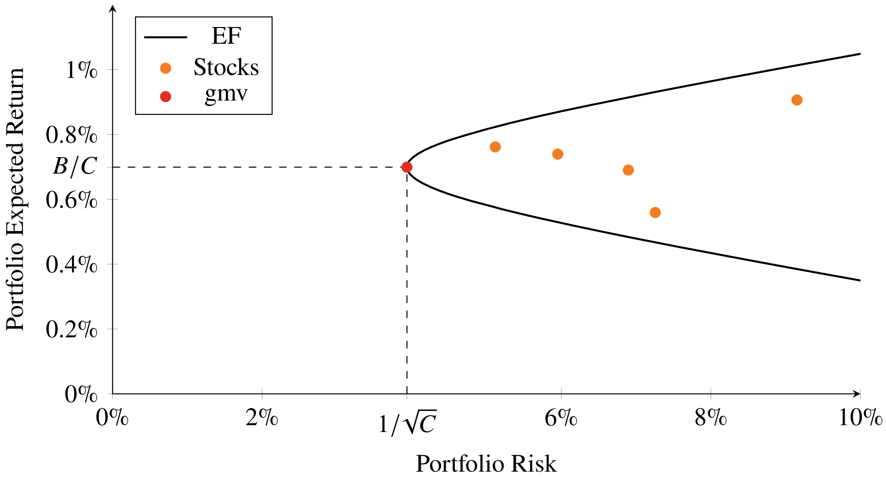


Fig. 2.7 Efficient frontier. *Notes:* This figure depicts the efficient frontier (black line) and the *global minimum variance portfolio* (red dot) computed using the returns and covariances of the five oldest stocks listed on the DJIA. Sample period: January 2000–December 2022 (276 observations). Data and codes—written in , , julia, and R—for replication are available at the following link: https://github.com/DoCoGu/EoFE_DCG_Springer

Table 2.2 Optimal portfolio weights for different target expected returns (N risky assets)

Asset/ μ_p	$\mu_{gmv} \equiv \frac{B}{C} = 0.7\%$	0.1%	0.64%	1.19%	1.73%	2.28%	2.82%	3.37%	3.91%	4.46%	5%
PG	0.449	− 0.162	0.393	0.948	1.503	2.058	2.613	3.168	3.723	4.278	4.833
MMM	0.212	0.184	0.210	0.236	0.262	0.288	0.314	0.340	0.366	0.392	0.418
IBM	0.203	1.962	0.363	− 1.235	− 2.834	− 4.433	− 6.032	− 7.631	− 9.229	− 10.828	− 12.427
MRK	0.168	0.546	0.203	− 0.141	− 0.484	− 0.827	− 1.170	− 1.514	− 1.857	− 2.200	− 2.543
AXP	− 0.032	− 1.530	− 0.169	1.192	2.554	3.915	5.276	6.637	7.998	9.359	10.720

Notes: This table presents the optimal portfolio allocation for various (monthly) expected portfolio returns, computed using Eq. (2.37). Sample period: January 2000–December 2022 (276 observations).
Data and codes—written in , , julia, and R—for replication are available at the following link: https://github.com/DoCoGu/EoFE_DCG_Springer.

its expected return is similar to individual stocks, its lower volatility highlights the benefits of diversification.

To determine optimal portfolio weights for a range of target expected returns, we capitalize on the potential diversification benefits inherent in our selection of five stocks from distinct industries. Table 2.2 presents the calculated weights allocated to each of the five assets for both the *global minimum variance portfolio* and portfolios targeting expected returns ranging from 0.1% to 5%. These optimal weights are derived through the application of the closed-form solution presented in Sect. 2.4:

$$\mathbf{w} = \mathbf{g} + \mathbf{h}\mu_p,$$

where $\mathbf{g} = \frac{1}{\Delta}[A(\boldsymbol{\Sigma}^{-1}\mathbf{1}) - B(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})]$ and $\mathbf{h} = \frac{1}{\Delta}[C(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) - B(\boldsymbol{\Sigma}^{-1}\mathbf{1})]$.

Substituting the *global minimum variance portfolio*'s expected return, B/C, into the optimal solution yields the weights that minimize portfolio variance. These weights, which are presented in the first column of Table 2.2, are calculated to ensure that they sum up to one by construction. Importantly, these weights can assume both positive and negative values, thereby enabling the implementation of short selling strategies. This feature becomes particularly evident when considering target return levels that exceed 1.19%, as evidenced by the negative weights assigned to IBM and Merck in these cases.

The distribution of weights across different target levels of expected returns reveals a heterogeneous allocation strategy, driven by the varied risk–return profiles of the five stocks from distinct sectors. The M-V optimizer suggests allocating wealth across all stocks, albeit in varying proportions. Nevertheless, regardless of the targeted expected return level, we observe instances of extreme portfolio allocations. Specifically, the M-V optimizer advocates for significant overweighting of certain stocks while heavily short-selling others. This phenomenon, commonly referred to as the "corner solution issue" within the finance realm, is a well-known drawback of M-V allocation. As an illustration, if we adhere to a monthly target return of 0.1%, the optimization recommends short-selling American Express by 153.0% and establishing a long position on IBM by 196.2%. The remaining wealth is allocated as follows: 54.6%, 18.4%, and $-$ 16.2% in Merck, 3M, and Procter&Gamble, respectively. It is noteworthy that as we progress toward higher target return levels, extreme allocations become even more pronounced, thereby exacerbating the "corner solution issue".[13]

2.6.2 Economy with $N = 5$ Risky Assets + 1 Riskless Asset

Building on the theoretical framework in Sect. 2.5, we construct the efficient frontier by allocating wealth to N risky assets and a risk-free security. This involves combining the risk-free asset with our five DJIA stocks to achieve specific target returns. The resultant EF, commonly referred to as the CML, is graphically depicted in Fig. 2.8 (see blue line). By definition, the CML intersects the EF derived from the M-V optimal portfolio problem considering solely the five aforementioned risky assets (as demonstrated in Fig. 2.7). The juncture at which the CML tangentially meets the EF delineates the *tangency portfolio* (see orange dot), a configuration where wealth is exclusively allocated to risky assets, devoid of any investment in the riskless security. This *tangency portfolio* represents the optimal risky portfolio, as it offers the highest Sharpe ratio among all possible portfolios composed solely of risky assets.

[13] In the context of Markowitz mean-variance optimization, "corner solution issues" arise when the optimal solution involves extreme allocations, such as heavily weighting a few assets or short-selling others. These occur when constraints or assumptions limit the feasible region.

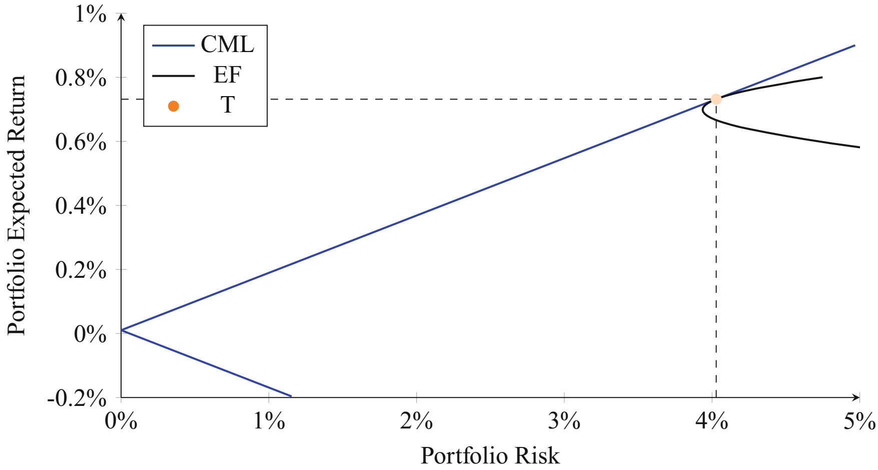


Fig. 2.8 The capital market line, the EF and the *tangency portfolio*. *Notes:* This figure illustrates the CML (blue line), the EF (black line), and the *tangency portfolio* (orange dot). These elements are calculated using the returns and covariance of the five oldest stocks still listed in the DJIA, in addition to a risk-free asset represented by the 1-month Treasury bill rate, which is sourced from the Fama-French Data Library. Sample period: January 2000–December 2022 (276 observations). Data and codes—written in , , julia, and R—for replication are available at the following link: https://github.com/DoCoGu/EoFE_DCG_Springer

From a practical standpoint, the availability of a risk-free bond, along with the option to borrow against it, opens avenues for constructing portfolios that exhibit varying levels of exposure to the equity market. These portfolios are encapsulated within the domain of the CML, representing an array of optimal investment options. The CML is distinguished by an intercept equivalent to the risk-free rate, denoting the absence of systematic risk, and its alignment is tangential with the efficient frontier, showcasing the highest expected returns for a given level of risk. It is crucial to underscore that portfolios yielding returns below those of the *tangency portfolio* can be attained by decreasing exposure to equity markets in favor of investment in the risk-free bond. Conversely, returns surpassing those of the *tangency portfolio* can be achieved by leveraging the risk-free rate, thereby augmenting exposure to risky assets. This latter strategy involves borrowing at the risk-free rate to invest in the *tangency portfolio*, effectively amplifying both the potential returns and risks associated with the investment.[14]

The subsequent step involves determining the optimal allocation weights for the six assets: the five oldest stocks listed on the DJIA and the risk-free bond. We begin with the calculation of the weights for the *tangency portfolio*. By definition, the *tangency portfolio* is exclusively composed of risky assets, implying that the weight

[14] This concept is clearly illustrated in Fig. 2.5, where $\alpha < 0$ denotes the short-selling of the risk-free asset.

assigned to the risk-free bond is zero. In practice, these weights are computed using the following formula:

$$\mathbf{w_T} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}.$$

The computed weights for the *tangency portfolio*, $\mathbf{w_T}$, are presented in the column $\mathbf{w_T}$ of Table 2.3. Subsequent columns of the table detail the weights on the assets for a range of expected portfolio returns, calculated as follows:

$$\mathbf{w} = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)\frac{\mu_p - r_f}{H}.$$

In general, it is observed that as the expected portfolio return increases, the allocation to the risk-free bond decreases. This shift signifies a greater emphasis on exposure to the equity market to achieve the desired return. Conversely, portfolios targeting lower expected returns allocate a larger proportion of wealth to the risk-free bond, reflecting a more conservative investment strategy focused on capital preservation. Let us consider the optimal weights for a specific expected portfolio return, say 1.73%. According to entries in Table 2.3:

- For Procter&Gamble (PG), the optimal weight is 115.1%. This suggests that for every dollar invested in the portfolio, approximately \$1.151 is allocated to PG stock.
- For 3M (MMM), the optimal weight is 51.1%, indicating a slightly lower allocation compared to Procter&Gamble.
- For IBM, the optimal weight is 25.7%, reflecting a smaller allocation to this stock.
- For Merck (MRK), the optimal weight is 35.3%, representing a moderate allocation.
- For American Express (AXP), the optimal weight is 11.7%, indicating a relatively smaller allocation compared to the other stocks.

Notably, the weight on the risk-free bond (Rf) is -139%. A negative weight on the risk-free bond implies borrowing at the risk-free rate to finance investments in the equity market, indicating a leveraged position in the portfolio. As the expected portfolio return increases, the allocation to the risk-free bond decreases, while the allocations to the stocks generally increase. For instance, at a higher expected return of 5%, the allocation to the risk-free bond becomes -592%, indicating a more significant leveraged position, while the allocations to the stocks correspondingly increase. In fact, the M-V optimizer in this case suggests to allocate a 333% (148%) in Procter&Gamble (3M). Conversely, at lower expected returns, such as 0.1%, the weight on the risk-free bond increases to 88%, indicating a risk-averse strategy with a higher allocation to the risk-free asset. These examples illustrate how the

Table 2.3 Optimal portfolio weights for different target expected returns (N risky assets + 1 Rf)

Asset/μ_p	w_T	0.1%	0.64%	1.19%	1.73%	2.28%	2.82%	3.37%	3.91%	4.46%	5%
PG	0.48	0.06	0.424	0.788	1.151	1.515	1.879	2.243	2.606	2.97	3.33
MMM	0.21	0.03	0.188	0.349	0.511	0.672	0.833	0.995	1.156	1.317	1.48
IBM	0.11	0.01	0.095	0.176	0.257	0.338	0.419	0.5	0.581	0.662	0.74
MRK	0.15	0.02	0.13	0.241	0.353	0.464	0.576	0.687	0.799	0.91	1.02
AXP	0.05	0.01	0.043	0.08	0.117	0.155	0.192	0.229	0.266	0.303	0.34
Rf	0.00	0.88	0.121	− 0.63	− 1.39	− 2.14	− 2.9	− 3.65	− 4.41	− 5.16	− 5.92

Notes: This table presents the optimal weights for the five oldest stocks listed on the Dow Jones Industrial Average (DJIA) and the risk-free bond across a spectrum of target expected returns. The initial column delineates the weights for the *tangency portfolio*, denoted as $\mathbf{w_T}$. The vector of optimal weights for each distinct target expected return is computed using Eq. (2.50), while the determination of the *tangency portfolio* is derived from Eq. (2.52). Sample period: January 2000–December 2022 (276 observations).
Data and codes—written in , , julia, and —for replication are available at the following link: https://github.com/DoCoGu/EoFE_DCG_Springer.

optimal weights vary across different expected portfolio returns, reflecting the trade-off between risk and return in portfolio construction.

Let us remark that the M-V allocations implemented so far rely exclusively on historical observations of returns. However, historical returns are based on past market conditions, which may not accurately reflect future performance. Over-reliance on historical data can lead to biased estimations of expected returns, volatilities, and correlations, resulting in suboptimal portfolio allocations that may underperform in future market conditions. Moreover, historical returns may be subject to non-stationarity, meaning that the statistical properties of returns change over time, further distorting risk and return estimations. To mitigate these issues, practitioners often employ techniques such as robust optimization or Bayesian methods, which incorporate additional information or assumptions about the future behavior of asset returns. An alternative approach proposed by Black and Litterman (1992) involves incorporating investors' views into the determination of expected returns, offering a more intuitive and powerful framework.[15] Additionally, ongoing monitoring and adjustment of portfolio allocations based on updated information can help address the limitations of relying solely on historical observations.

References

Black, F., & Litterman, R. (1992). Global portfolio optimization. *Financial Analysts Journal, 48*(5), 28–43.
Markowitz, H. (1952). Portfolio selection. *Journal of Finance, 7*, 77–91.

[15] A formal and detailed derivation of the Black–Litterman Model, along with a related empirical application, will be provided in Sect. 8.