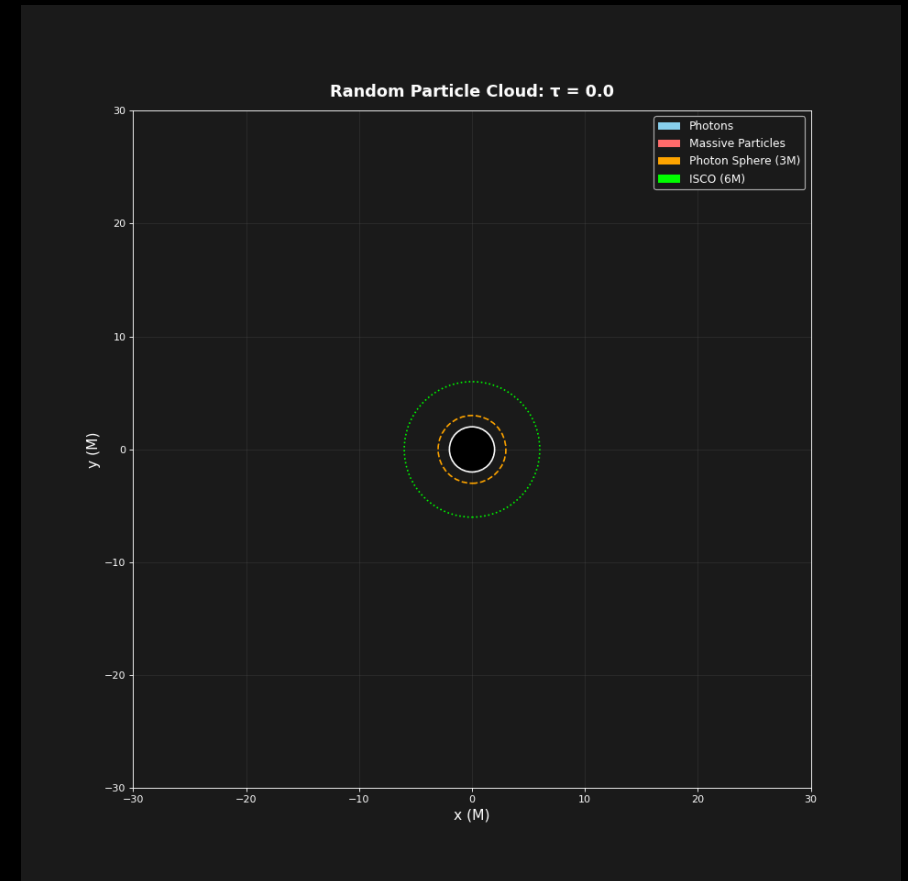# Simulating Schwarzschild and Kerr Black Hole Geodesics

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MSc Physics

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Table of Contents

Introduction

- **Problem – Solve geodesic equations in curved spacetime for timelike and null trajectories**

- **Geodesic Equation:**
  - $$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

- **Christoffel Symbols:**
  - $$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\right)$$

- **Metrics Considered:**
  - Schwarzschild (a = 0): Spherical Symmetry
  - Kerr (a ≠ 0): Axial Symmetry

- **Numerical Method:**
  - DOP853 (8th order RK)

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Schwarzschild – Theory and Implementation

- **Schwarzschild Metric (Boyer-Lindquist, $\theta = \pi/2$):**

  - $ds^2 = -f(r)\,dt^2 + f(r)^{-1}dr^2 + r^2 d\phi^2, \quad \text{where } f(r) = 1 - \frac{r_s}{r} = 1 - \frac{2M}{r}$

- **Killing Vectors & Conserved Quantities:**

  - Time translation: $E = -p_t = f(r)\,u^t$

  - Axial rotation: $L = p_\phi = r^2 u^\phi$

- **Geodesic Equations (equatorial, $\theta = \pi/2$):**

  - $\frac{d^2\phi}{d\tau^2} = -\frac{2}{r} u^r u^\phi$

- **Normalization:**

  - $g_{\mu\nu}u^\mu u^\nu = -1$ **(timelike)** | $g_{\mu\nu}u^\mu u^\nu = 0$ **(null)**

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Schwarzschild – Theory and Implementation

Implementation Details:

- State Vector:
    - $y = \left[\, t,\, r,\, \phi,\, u^t,\, u^r,\, u^\phi \,\right]$

- Initial Conditions (from conserved quantities):
    - Given: $r_0, \varphi_0, E, L\,(orb = L/E)$
    - $u_t = E/f(r_0)$
    - $u^\varphi = L/r_0^2$

    - from normalization constraints:
        - Null: $(u^r)^2 = E^2 - f(r_0)\dfrac{L^2}{r_0^2}$

        - Timelike: $(u^r)^2 = E^2 - \left(1 + \dfrac{L^2}{r_0^2}\right)f(r_0)$

Schwarzschild – Results: Null Geodesics

Results:
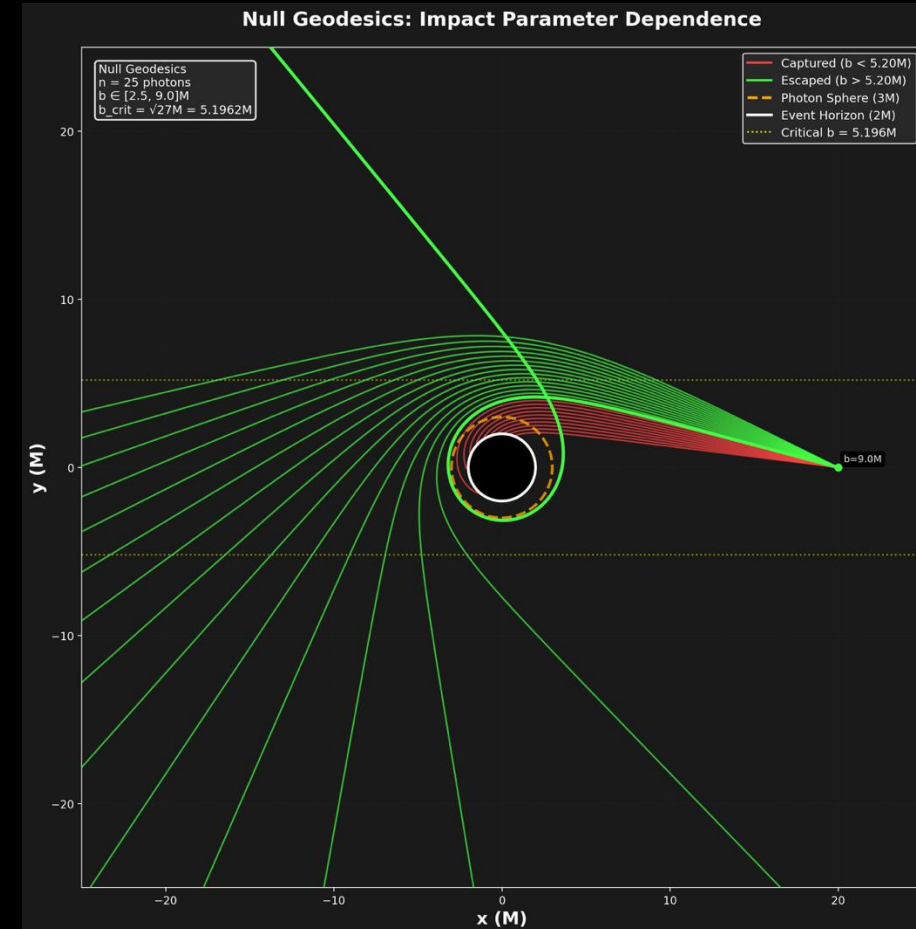
- $b < b_{crit} \rightarrow$ captured, eventually crosses horizon

- $b > b_{crit} \rightarrow$ scattered, returns to $r \rightarrow \infty$

- $b \approx b_{crit} \rightarrow$ unstable circular orbit, sensitive to initial cond.

- Clear bifurication at critical impact parameter

Constraint Violations (over full trajectory):

Energy: $\left|\frac{\Delta E}{E}\right| < 10^{-15}$

Angular Momentum: $\left|\frac{\Delta L}{L}\right| < 10^{-15}$

Null constraint: $\left|g_{\mu\nu}u^{\mu}u^{\nu}\right| < 10^{-14}$

Null Geodesics: Impact Parameter Dependence

Schwarzschild – Results: Null Geodesics

Results:

- $b \, < \, b_{crit} \, \rightarrow \,$ captured, eventually crosses horizon
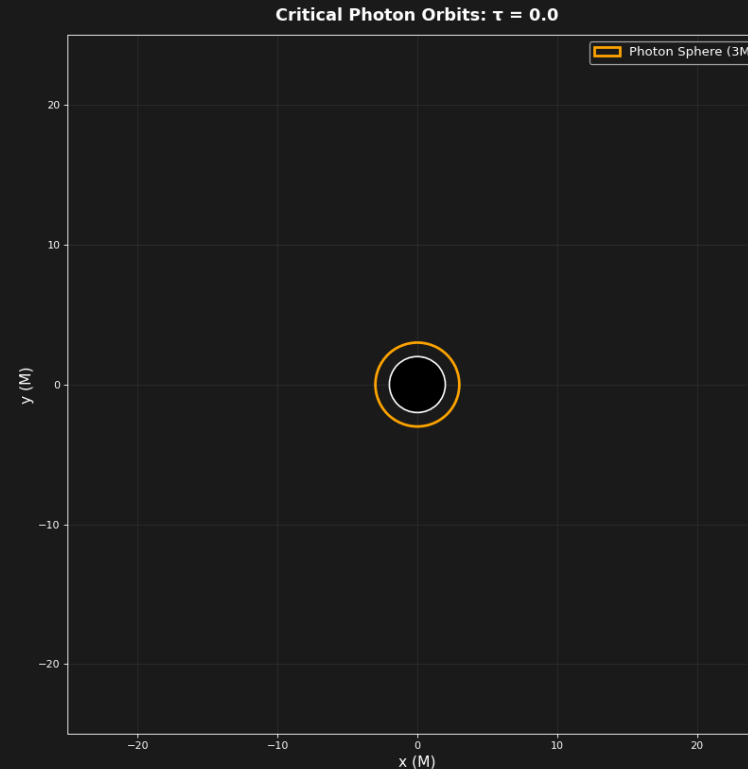
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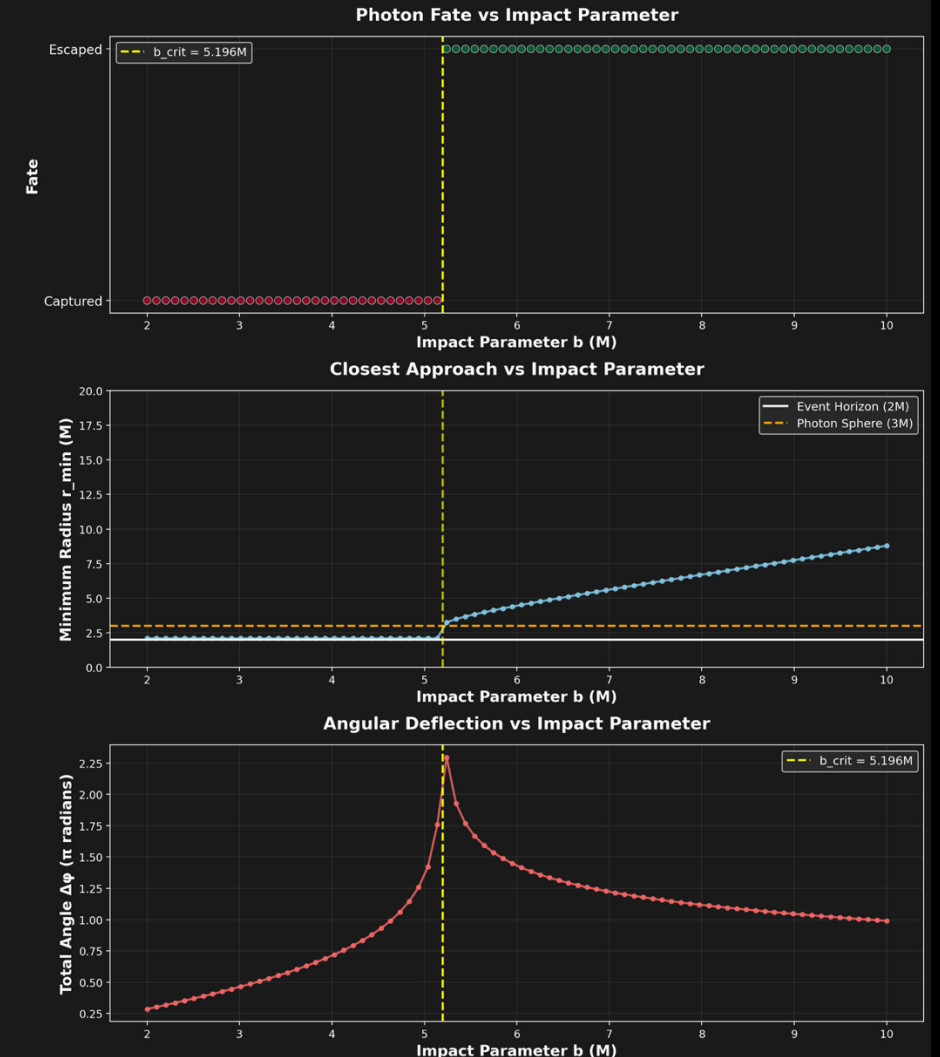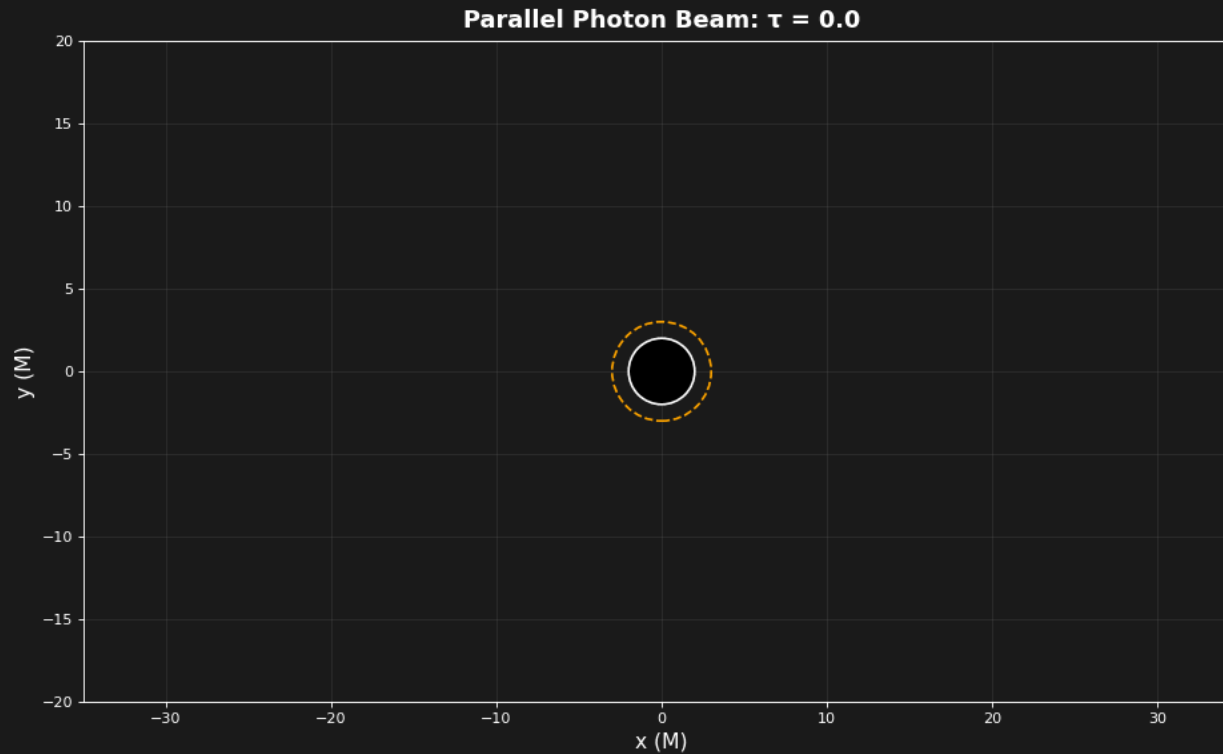
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Angular Momentum: $\left| \frac{\Delta L}{L} \right| < 10^{-15}$

Null constraint: $\left| g_{\mu\nu} u^{\mu} u^{\nu} \right| < 10^{-14}$

Critical Photon Orbits: τ = 0.0

Schwarzschild – Scattering Experiment

Schwarzschild – Results: Timelike Geodesics

Energy Constraint Analysis:

Effective Potential:

$$V_{eff}^{massive} = (1 + L^2/r^2)(1 - 2M/r)$$

Radial turning points satisfy:

$$E^2 = V_{eff}(r_{crit})$$

For circular orbits at r:
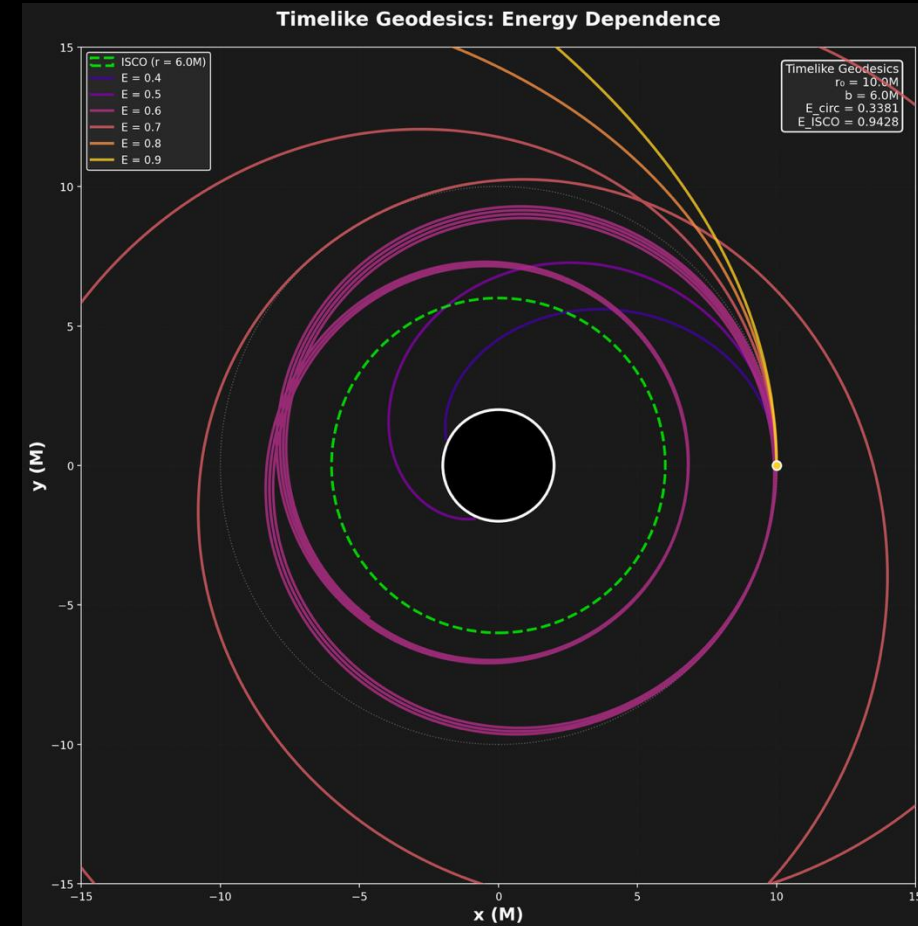
$$E_{circ}(r) = \sqrt{(r - 2M)/(r - 3M)}/\sqrt{r}$$
$$L_{circ}(r) = M\sqrt{r/(r - 3M)}$$

ISCO Analysis (r=6M):

$$E_{ISCO} \approx 0.943$$
$$L_{ISCO} \approx 3.464M$$

$$E < E_{circ}(r) \rightarrow \text{Bound orbits}$$

Timelike Geodesics: Energy Dependence

Schwarzschild – Results: Timelike Geodesics

Energy Constraint Analysis:

Effective Potential:
$$V_{eff}^{massive} = (1 + L^2/r^2)(1 - 2M/r)$$

Radial turning points satisfy:
$$E^2 = V_{eff}(r_{crit})$$

For circular orbits at r:
$$E_{circ}(r) = \sqrt{(r - 2M)/(r - 3M)}/\sqrt{r}$$
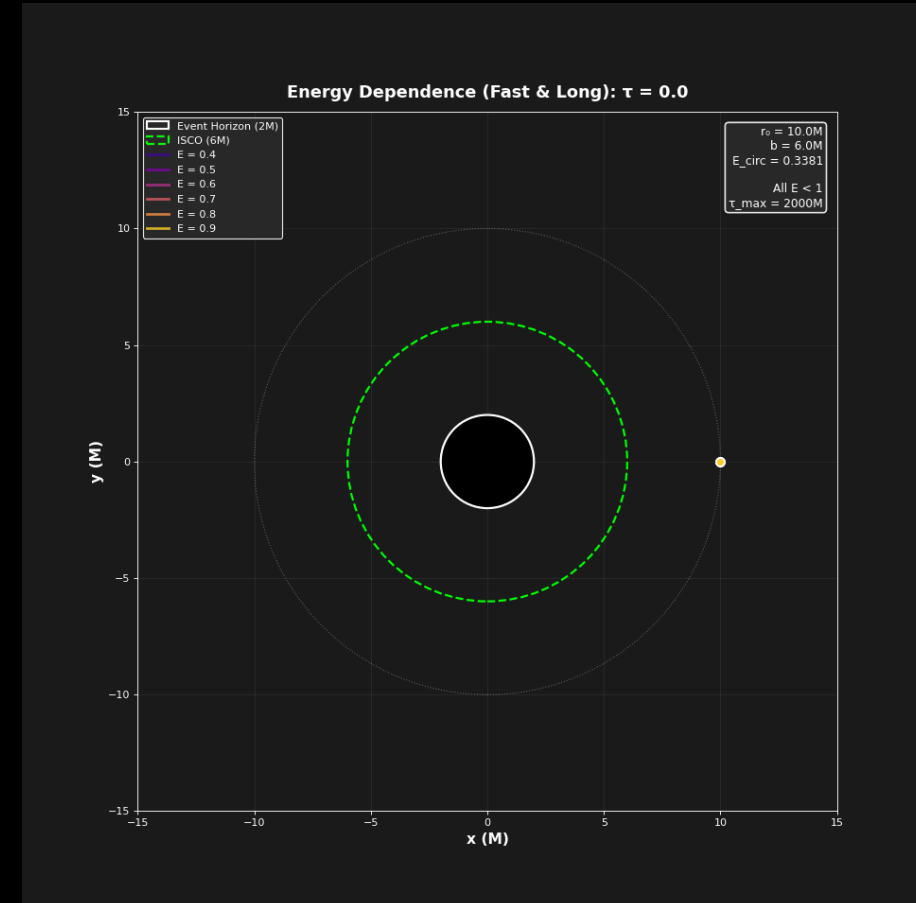$$L_{circ}(r) = M\sqrt{r/(r - 3M)}$$

ISCO (innermost stable circular orbit) Analysis (r=6M):
$$E_{ISCO} \approx 0.943$$
$$L_{ISCO} \approx 3.464M$$

$$E < E_{circ}(r) \rightarrow \text{Bound orbits}$$

Kerr – Theory and Implementation

- **Kerr Metric (Boyer-Lindquist, $\theta = \pi/2$):**
  - $ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mra}{\Sigma}\,dt\,d\phi + \frac{\Sigma}{\Delta}\,dr^2 + \frac{A}{\Sigma}\,d\phi^2$
  - Where:
    - $\Sigma(r,\theta) = r^2 + a^2\cos^2\theta$, $\Delta(r) = r^2 - 2Mr + a^2$, $A(r,\theta) = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$
  - At equator ($\theta = \pi/2$):

- **Metric components (equatorial):**

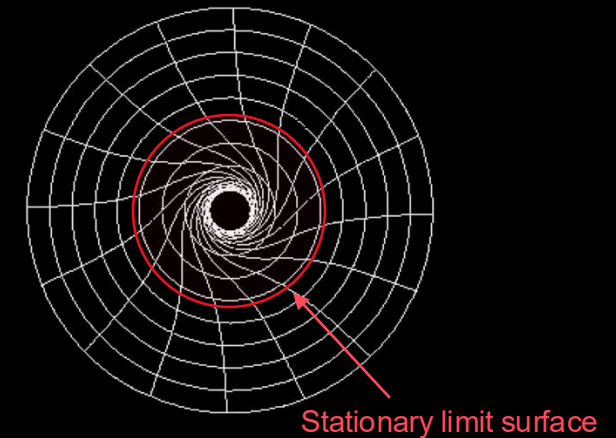**Event Horizons ($\Delta = 0$):**

- Inner horizon: r_-,

Extremal limit: $a = M \Rightarrow r_\pm = M$

Ergosphere ($g_{tt} = 0$):

- At equator

Frame Dragging:

Visualization of Frame Dragging

Stationary limit surface

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Kerr – Theory and Implementation

Non-zero Christoffel Symbols (equatorial):

$$\text{Limit } a \to 0: \text{ Schwarzschild} \Gamma^{\phi}_{tr} = -\frac{Ma}{\Sigma^2 \Delta}, \qquad \Gamma^{\phi}_{r\phi} = \frac{1}{r}, \qquad \Gamma^{r}_{t\phi}$$

$$= \frac{2aM^2 r}{\Sigma^3}, \qquad \Gamma^{r}_{\phi\phi} = -\frac{\Delta r}{\Sigma}, \qquad \Gamma^{r}_{rr} = \frac{Mr^2 - Ma^2 - r\Delta}{\Sigma \Delta}, \qquad \Gamma^{r}_{tt}$$

$$= \frac{M\Delta(r^2 - a^2)}{\Sigma^3}, \qquad \Gamma^{t}_{r\phi} = \frac{Ma(r^2 - 2Mr + a^2)}{\Sigma^2 \Delta}, \qquad \Gamma^{t}_{tr} = \frac{Mr}{\Sigma^2 \Delta},$$
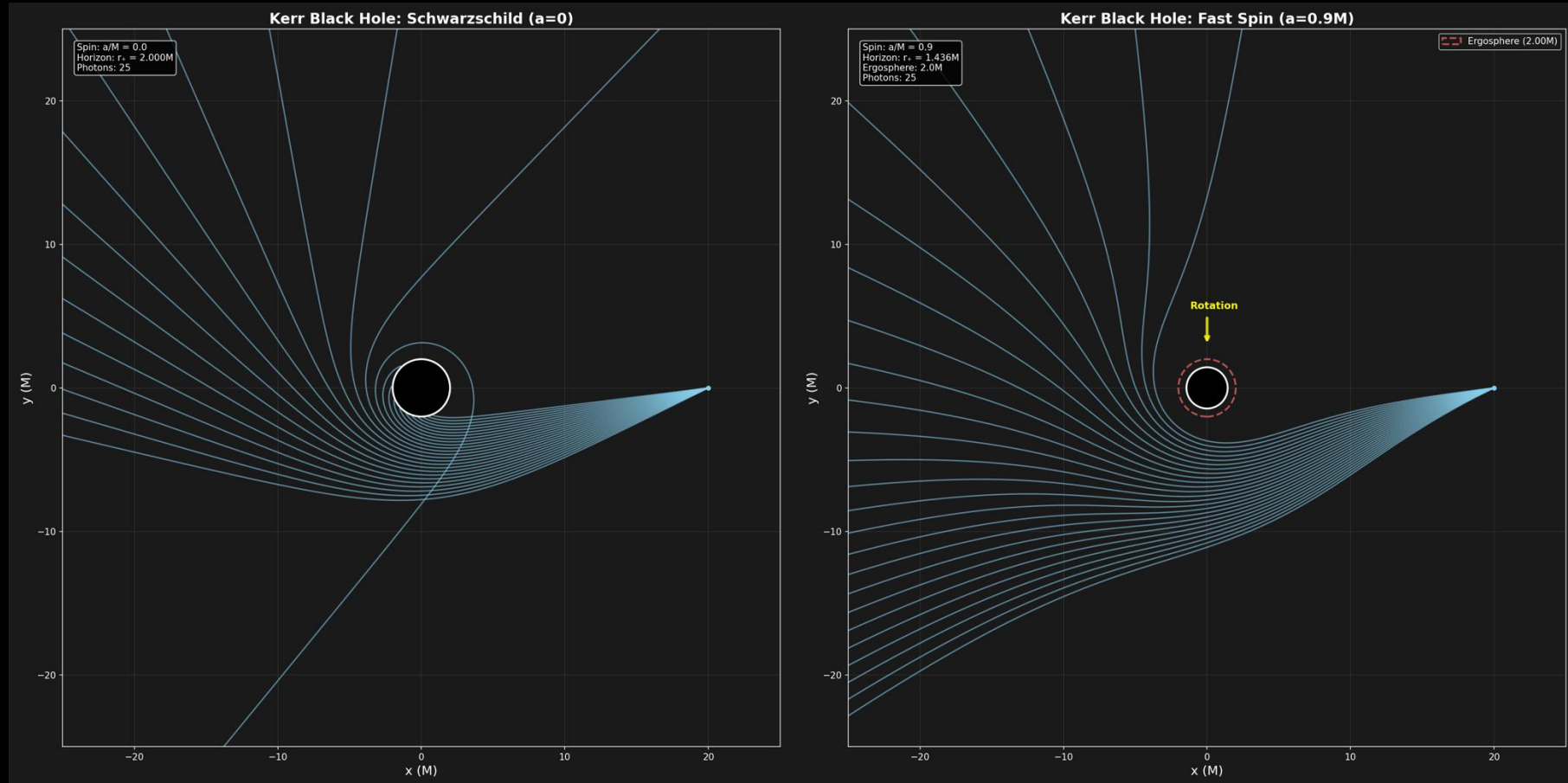
Geodesic Equations[1] (equatorial):

$$\frac{d^2\phi}{d\tau^2} = -2, \Gamma^{\phi}_{r\phi}, u^r u^\phi - 2, \Gamma^{\phi}_{tr}, u^t u^r, \qquad \frac{d^2 r}{d\tau^2}$$

$$= -\Gamma^{r}_{tt}(u^t)^2 - \Gamma^{r}_{rr}(u^r)^2 - \Gamma^{r}_{\phi\phi}(u^\phi)^2 - 2, \Gamma^{r}_{t\phi}, u^t u^\phi, \qquad \frac{d^2 t}{d\tau^2}$$

$$= -2, \Gamma^{t}_{tr}, u^t u^r - 2, \Gamma^{t}_{r\phi}, u^r u^\phi,$$
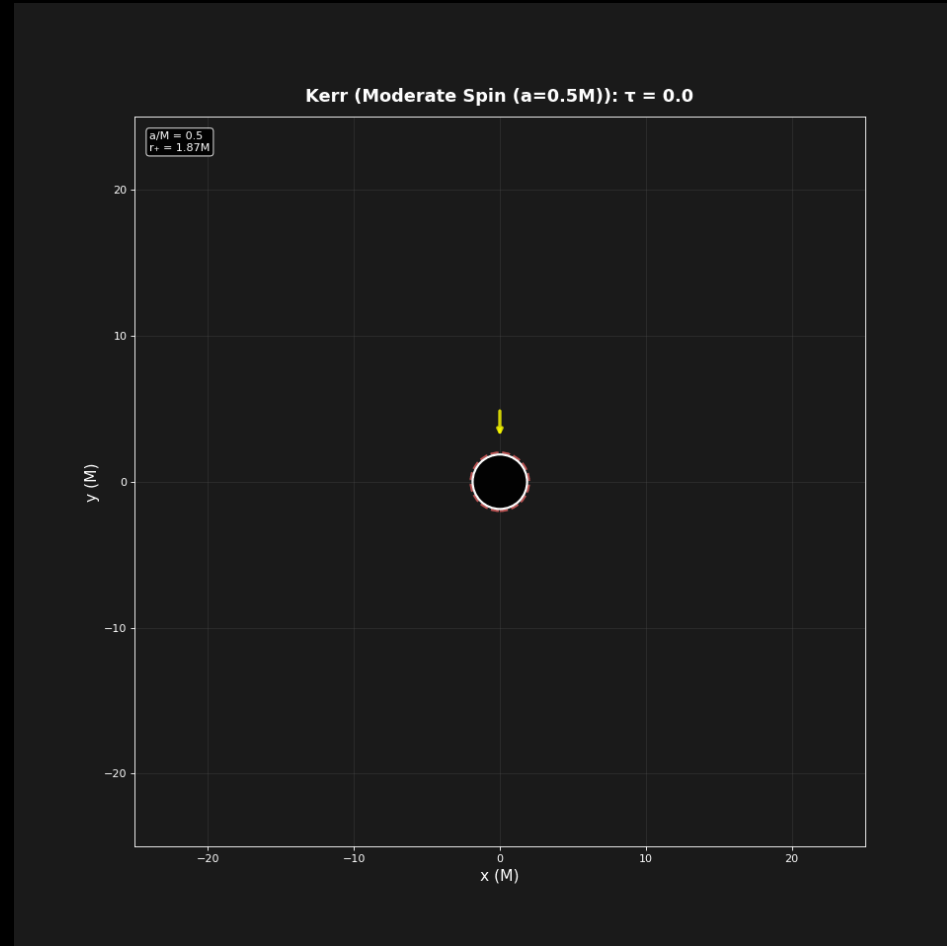
Conservation (from Killing vectors):

$$L = p_\phi = g_{t\phi} u^t + g_{\phi\phi} u^\phi E = -p_t = -(g_{tt} u^t + g_{t\phi} u^\phi),$$

PHY 387M – Relativity Theory 1 – University of Texas at Austin | 1) Reference: Bardeen et al. (1972), Eq. 2.2-2.5

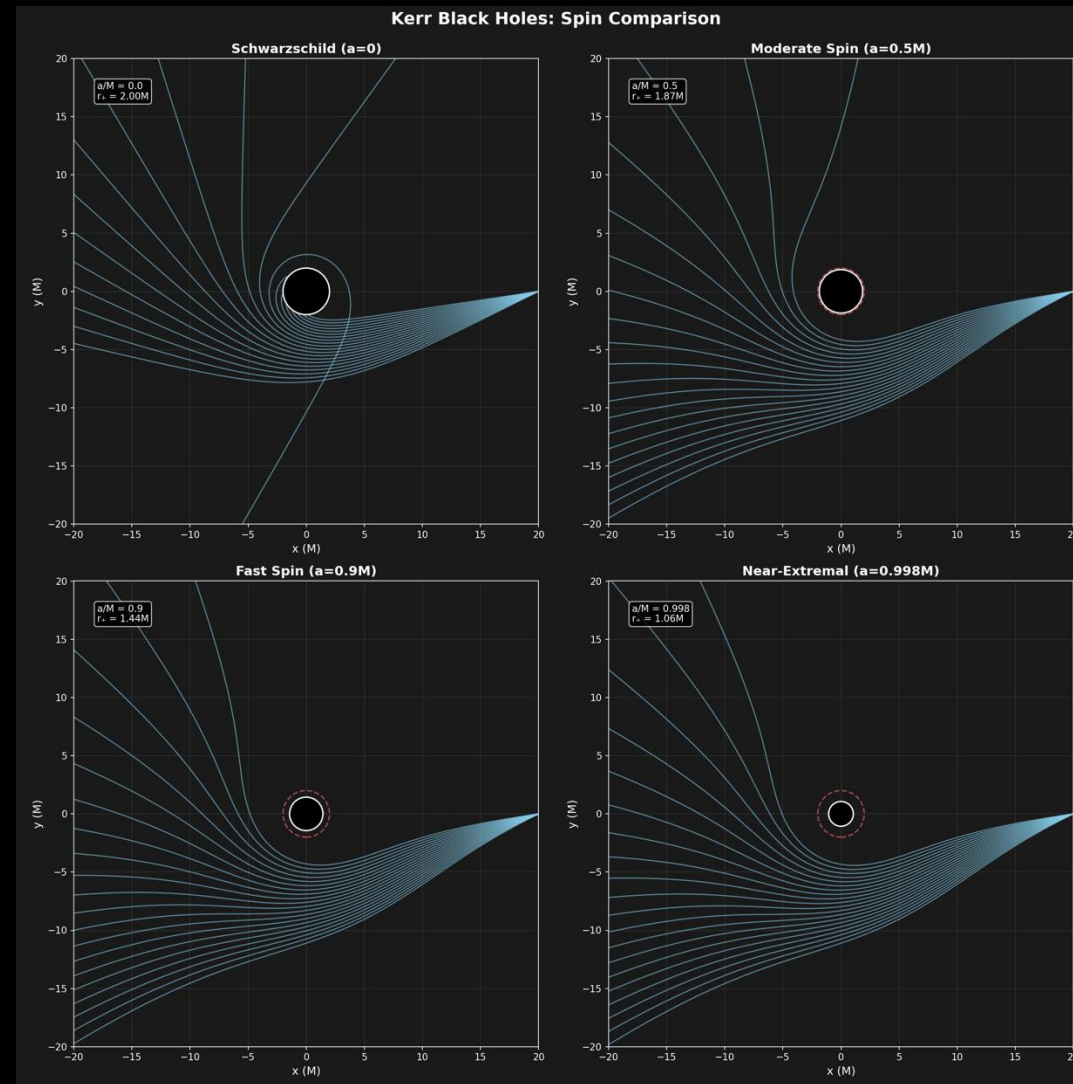ETH zürich

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Kerr – Results: Null Geodesics

Constraint Violations (over full trajectory): Energy: $\left|\frac{\Delta E}{E}\right| < 10^{-14}$, Angular Momentum: $\left|\frac{\Delta L}{L}\right| < 10^{-14}$, Null constraint: $\left|g_{\mu\nu}u^{\mu}u^{\nu}\right| < 10^{-13}$

Kerr – Results: Null Geodesics

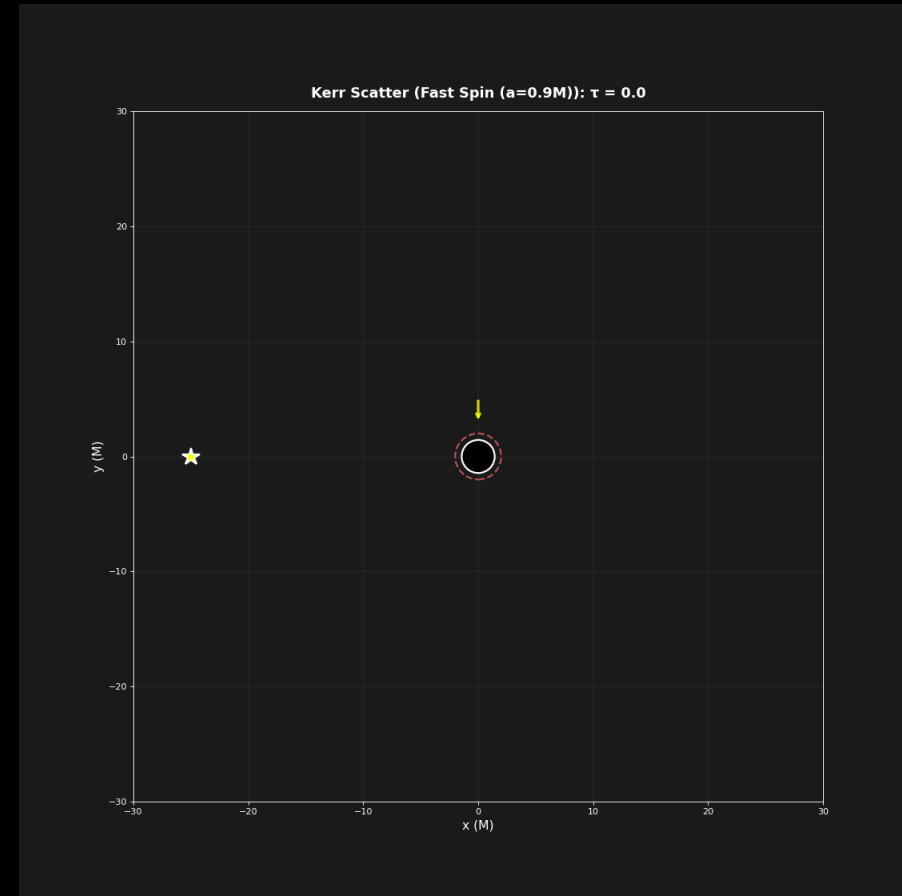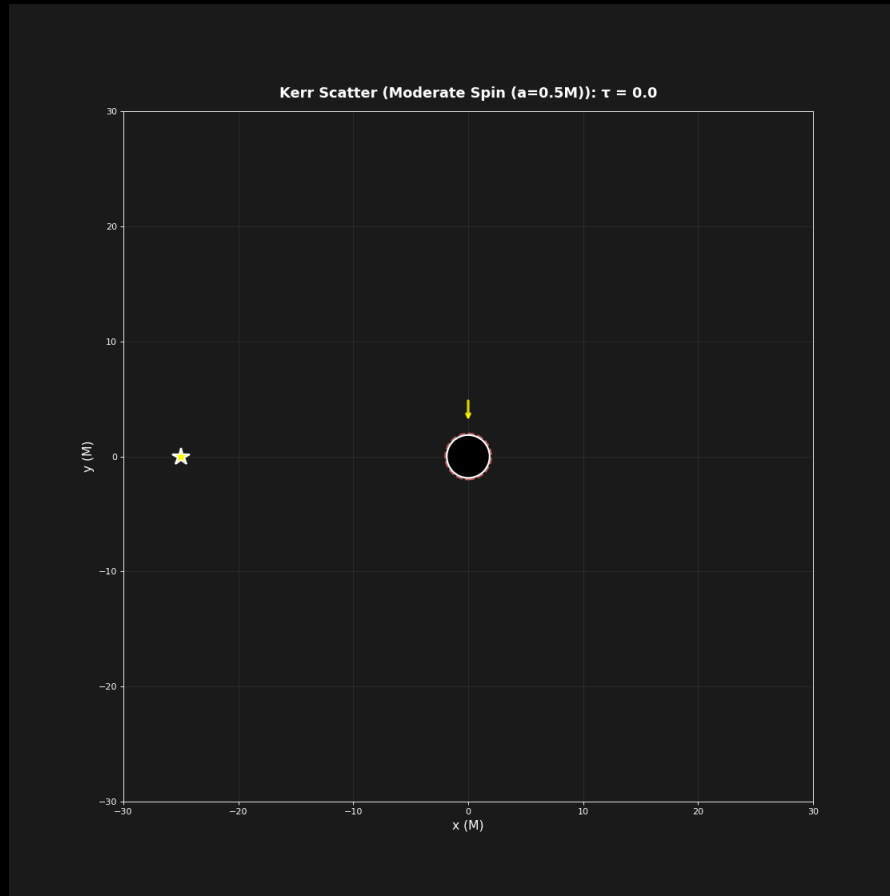**Kerr (Moderate Spin (a=0.5M)): τ = 0.0**

a/M = 0.5
r₊ = 1.87M

Constraint Violations (over full trajectory): Energy: $\left|\frac{\Delta E}{E}\right| < 10^{-14}$, Angular Momentum: $\left|\frac{\Delta L}{L}\right| < 10^{-14}$, Null constraint: $\left|g_{\mu\nu}u^{\mu}u^{\nu}\right| < 10^{-13}$

Kerr – Results: Null Geodesics

Kerr Black Holes: Spin Comparison

Kerr – Results: Null Geodesics

Conclusion

- Developed a fully general **geodesic simulator** for Schwarzschild and Kerr black holes

- Verified relativistic phenomena:
    - Photon Sphere, ISCO, impact parameters
    - Ergosphere behavior, frame dragging

- ~4000 loc

- Source code, simulations Jupyter Notebook user demo and references can be found on GitHub:

    https://github.com/fynnhufler/General-Relativity-Simulations

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