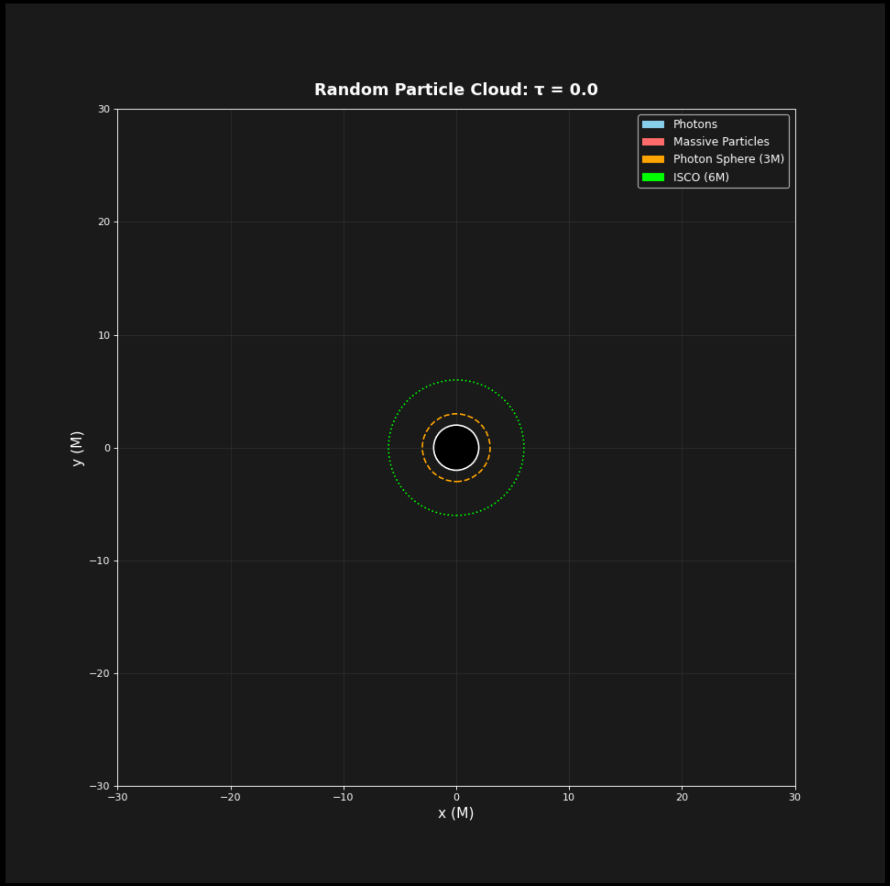
Simulating Schwarzschild and Kerr Black Hole Geodesics



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MSc Physics

Dec 3rd 2025, Austin

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Introduction

- **Problem – Solve geodesic equations in curved spacetime for timelike and null trajectories**
- **Geodesic Equation:**
  - $\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$
- **Christoffel Symbols:**
  - $\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\right)$
- **Metrics Considered:**
  - Schwarzschild (a = 0): Spherical Symmetry
  - Kerr (a $\neq$ 0): Axial Symmetry
- **Numerical Method:**
  - DOP853 (8th order RK)

Schwarzschild – Theory and Implementation

- **Schwarzschild Metric (Boyer-Lindquist, $\boldsymbol{\theta = \pi/2}$)**:
  - $ds^2 = -f(r)\,dt^2 + f(r)^{-1}dr^2 + r^2 d\phi^2, \quad \text{where } f(r) = 1 - \frac{r_s}{r} = 1 - \frac{2M}{r}$
- **Killing Vectors & Conserved Quantities:**
  - Time translation: $E = -p_t = f(r)\,u^t$
  - Axial rotation: $L = p_\phi = r^2 u^\Phi$
- **Geodesic Equations from Christoffel Symbols (equatorial, $\boldsymbol{\theta} = \boldsymbol{\pi}/2$):**
  - $\frac{\mathrm{d}^2\mathrm{t}}{\mathrm{d}\tau^2} = (2\mathrm{M}/\mathrm{r}^2\mathrm{f})\mathrm{u}^\mathrm{t}\mathrm{u}^\mathrm{r}$
  - $\frac{d^2r}{d\tau^2} = -(Mf/r^2)u^t u^t + (M/r^2 f)u^r u^r + (r-2M)u^\Phi u^\Phi$
  - $\frac{d^2\phi}{d\tau^2} = -\frac{2}{r}u^r u^\phi$
- **Normalization:**
  - $g_{\mu\nu}u^\mu u^\nu = -1$ **(timelike) |** $g_{\mu\nu}u^\mu u^\nu = 0$ **(null)**

Schwarzschild – Theory and Implementation

Implementation Details:

- State Vector:
  - $\boldsymbol{y} = [\, \boldsymbol{t},\ \boldsymbol{r},\ \boldsymbol{\phi},\ \boldsymbol{u^t},\ \boldsymbol{u^r},\ \boldsymbol{u^\phi} \,]$
- Initial Conditions (from conserved quantities):
  - Given: $r_0, \varphi_0, E, L(orb = L/E)$
  - $u_t = E/f(r_0)$
  - $\mathrm{u}^\varphi = \mathrm{L}/\mathrm{r}_0^2$

  - from normalization constraints:
    - Null: $(u^r)^2 = E^2 - f(r_0)\frac{L^2}{r_0^2}$
    - Timelike: $(u^r)^2 = E^2 - \left(1 + \frac{L^2}{r_0^2}\right) f(r_0)$

Schwarzschild – Results: Null Geodesics

Results:

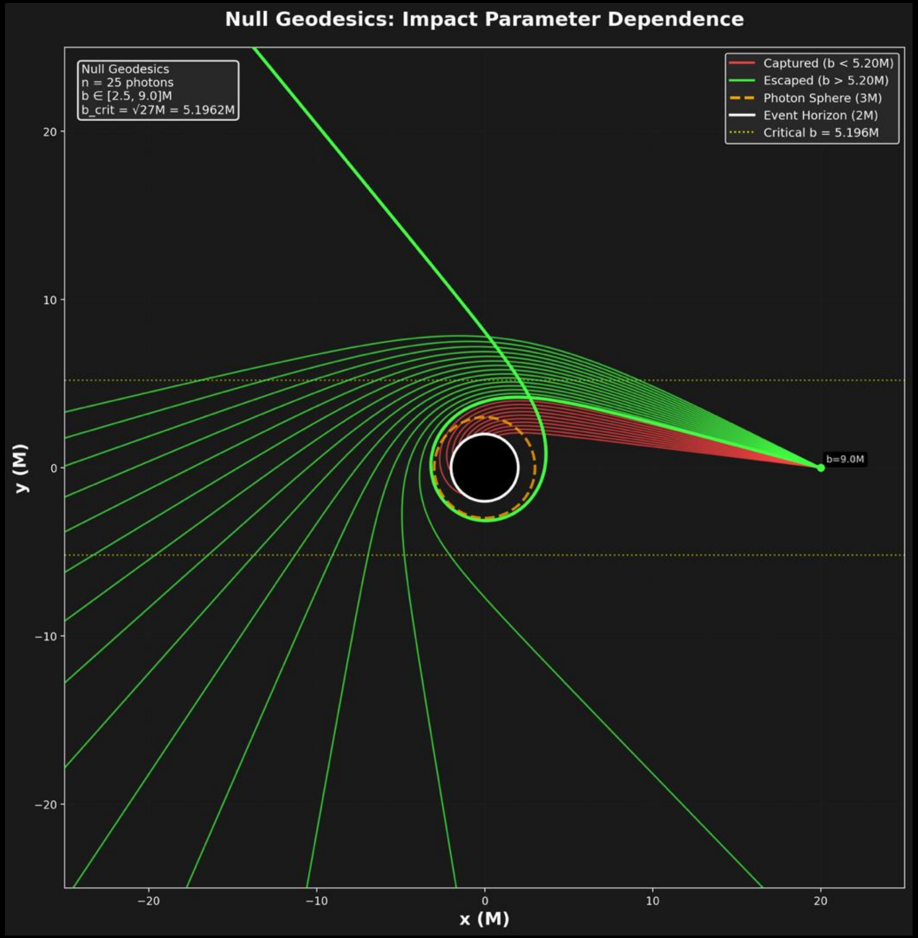
- $b < b_{crit} \rightarrow$ captured, eventually crosses horizon
- $b > b_{crit} \rightarrow$ scattered, returns to $r \rightarrow \infty$
- $\mathrm{b} \approx \mathrm{b_{crit}} \rightarrow$ unstable circular orbit, sensitive to initial cond.
- Clear bifurication at critical impact parameter

Constraint Violations (over full trajectory):

Energy: $\left|\frac{\Delta E}{E}\right| < 10^{-15}$

Angular Momentum: $\left|\frac{\Delta L}{L}\right| < 10^{-15}$

Null constraint: $\left|g_{\mu\nu}u^{\mu}u^{\nu}\right| < 10^{-14}$



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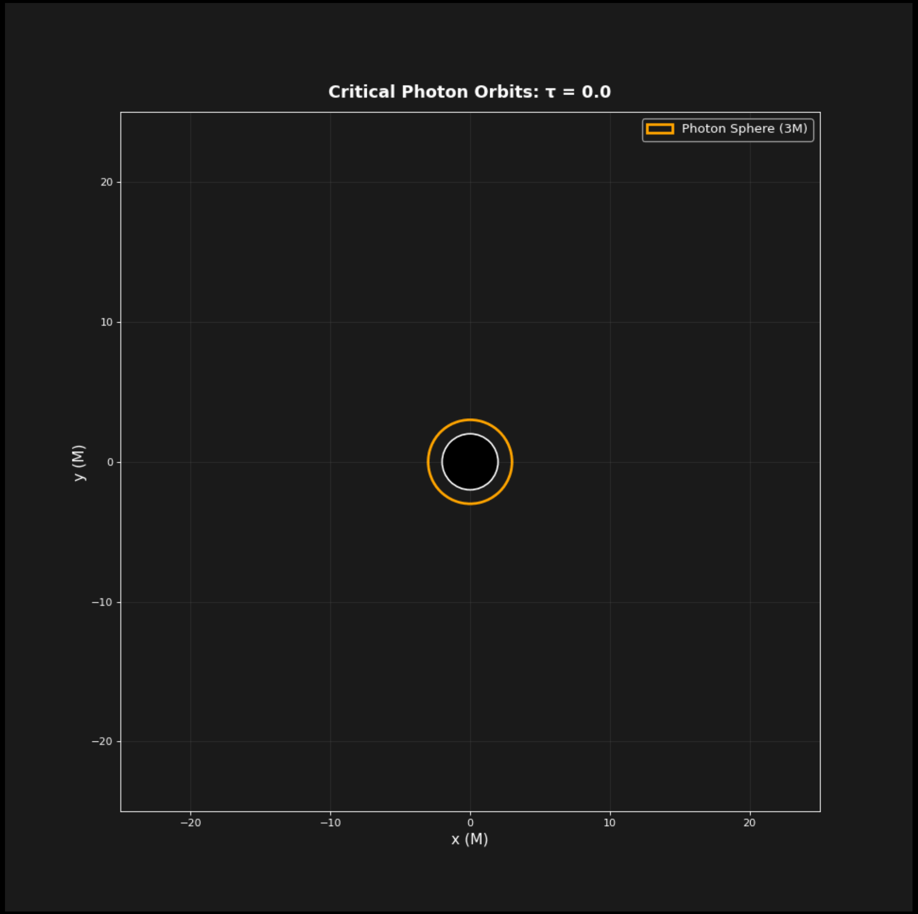
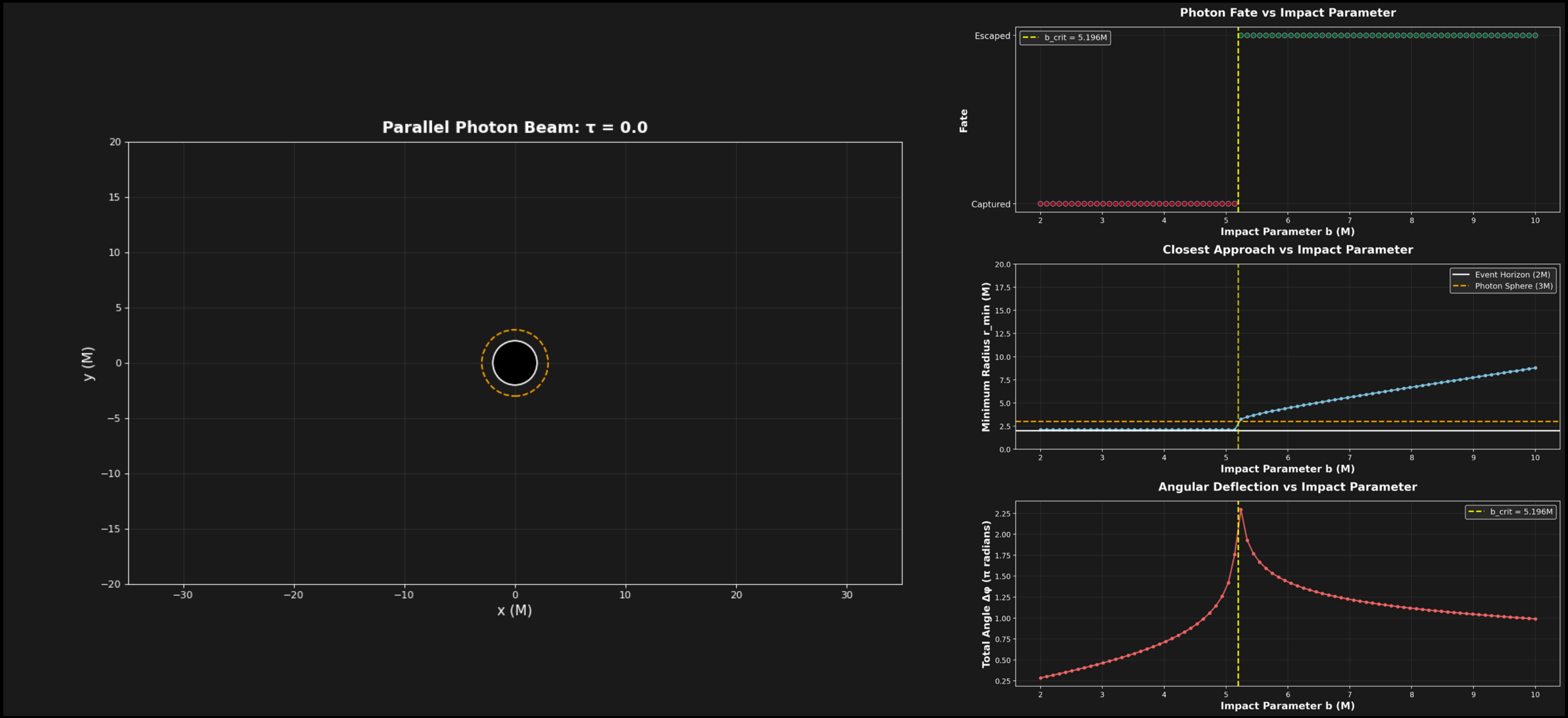


Schwarzschild – Scattering Experiment



Schwarzschild – Results: Timelike Geodesics

Energy Constraint Analysis:

Effective Potential:

$$V_{\text{eff}}^{\text{massive}} = (\mathbf{1} + L^2/r^2)(\mathbf{1} - 2M/r)$$

Radial turning points satisfy:

$$E^2 = V_{\text{eff}}(r_{\text{crit}})$$

For circular orbits at r:

$$E_{\text{circ}}(r) = \sqrt{(r-2M)/(r-3M)}\,/\sqrt{r}$$

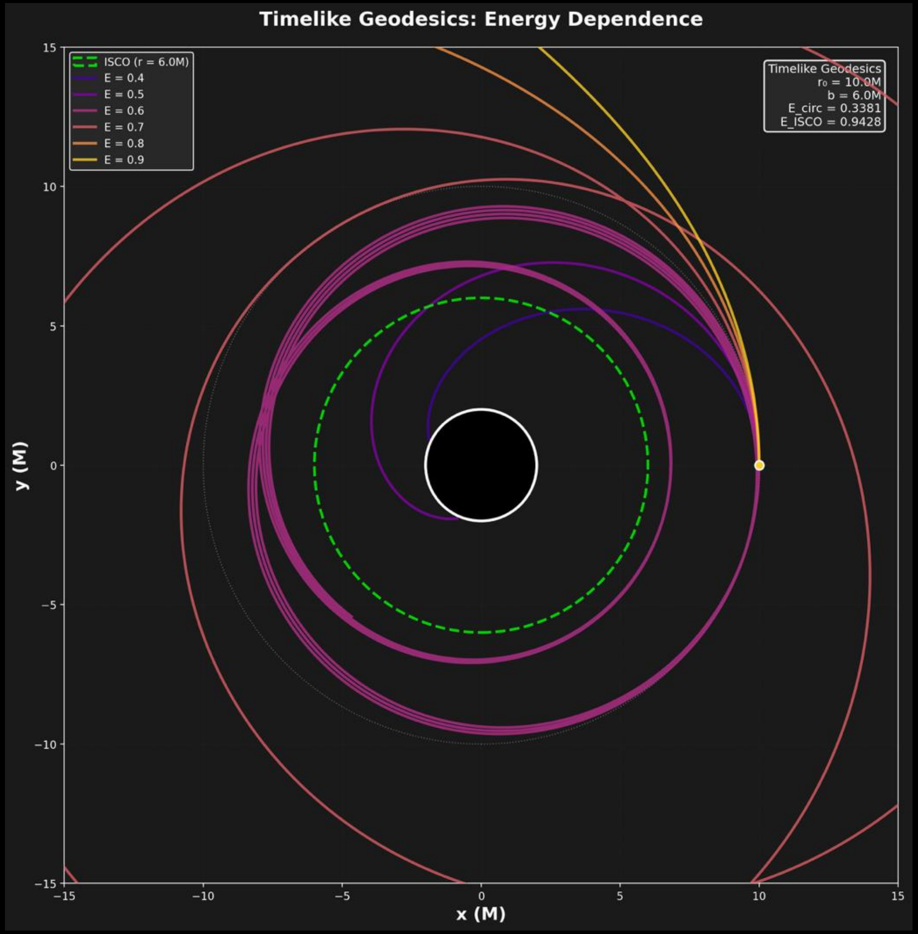
$$L_{\text{circ}}(r) = M\sqrt{r/(r-3M)}$$

ISCO (innermost stable circular orbit) Analysis (r=6M):

$$E_{\text{ISCO}} \approx 0.943$$

$$L_{\text{ISCO}} \approx 3.464M$$

$$E < E_{\text{circ}}(r) \rightarrow \text{Bound orbits}$$



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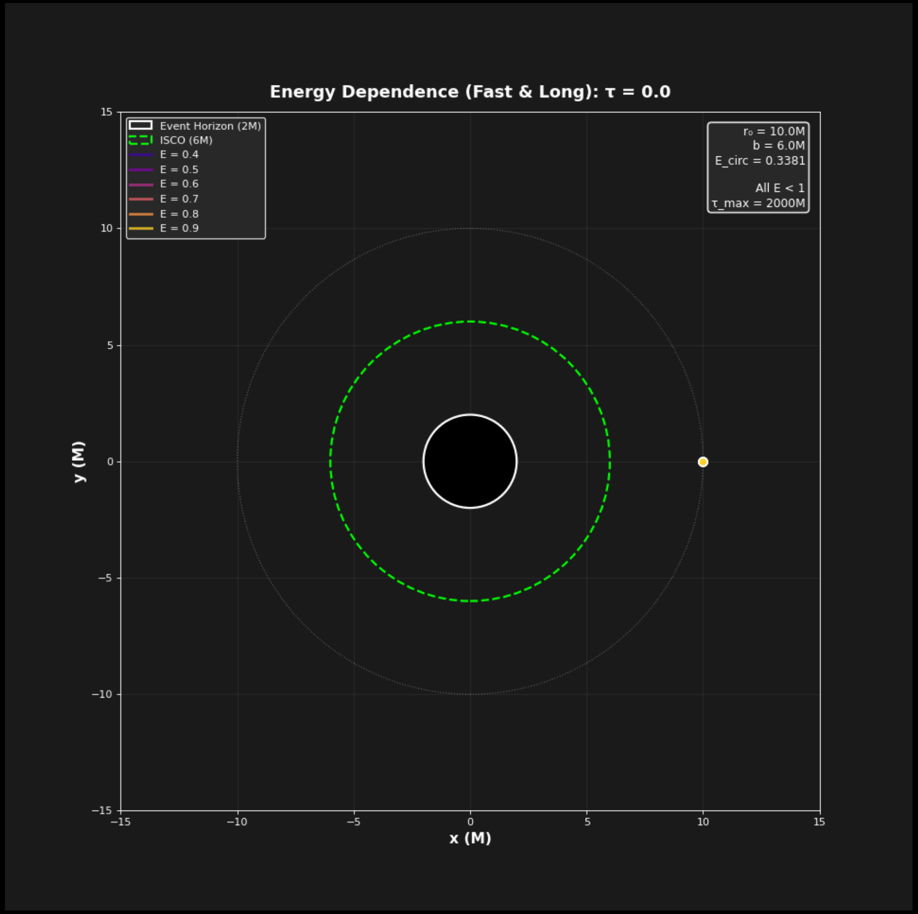


Kerr – Theory and Implementation

- **Kerr Metric (Boyer-Lindquist, $\boldsymbol{\theta = \pi/2}$):**
  - $ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mra}{\Sigma}\, dt\, d\phi + \frac{\Sigma}{\Delta}\, dr^2 + \frac{A}{\Sigma}\, d\phi^2$
  - Where:
    - $\Sigma(r,\theta) = r^2 + a^2 \cos^2\theta$,
    - $\Delta(r) = r^2 - 2Mr + a^2$,
    - $A(r,\theta) = (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta$
  - At equator ($\theta = \pi/2$):
    - $\Sigma = r^2$
    - $A = (r^2 + a^2)^2 - a^2\Delta = r^4 + a^2 r^2 + 2Ma^2 r$

- **Metric components (equatorial):**
  - $g_{\phi\phi} = r^2 + a^2 + \frac{2Ma^2}{r}$
  - $g_{rr} = \frac{r^2}{\Delta}$
  - $g_{t\phi} = -\frac{2Ma}{r}$
  - $g_{tt} = -\left(1 - \frac{2M}{r}\right)$

Event Horizons ($\Delta = 0$):

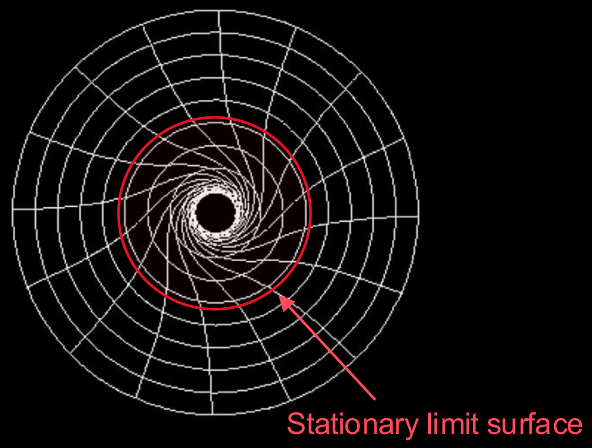
- $\boldsymbol{r_\pm = M \pm \sqrt{M^2 - a^2}}$
- Outer horizon: r_+
- Inner horizon: r_-,

Extremal limit: $a = M \Rightarrow r_\pm = M$

Ergosphere ($\mathrm{g_{tt}} = 0$):

- $\mathrm{r_{ergo}} = 2M$
- $\mathrm{r}_+ < r < \mathrm{r_{ergo}}$

Visualization of Frame Dragging



Frame Dragging (angular velocity of ZAMO[1]):

- $\omega_H = \frac{a}{r_+^2 + a^2}$, at horizon
- $\omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2Mra}{A}$

Kerr – Theory and Implementation

Non-zero Christoffel Symbols (equatorial):

$$\textbf{Limit } \boldsymbol{a \to 0}\textbf{: Schwarzschild}$$

$$\Gamma^{\phi}_{tr} = -\frac{Ma}{\Sigma^2\Delta},$$
$$\Gamma^{\phi}_{r\phi} = \frac{1}{r},$$
$$\Gamma^{r}_{t\phi} = \frac{2aM^2r}{\Sigma^3},$$
$$\Gamma^{r}_{\phi\phi} = -\frac{\Delta r}{\Sigma},$$
$$\Gamma^{r}_{rr} = \frac{Mr^2 - Ma^2 - r\Delta}{\Sigma\Delta},$$
$$\Gamma^{r}_{tt} = \frac{M\Delta(r^2 - a^2)}{\Sigma^3},$$
$$\Gamma^{t}_{r\phi} = \frac{Ma(r^2 - 2Mr + a^2)}{\Sigma^2\Delta},$$
$$\Gamma^{t}_{tr} = \frac{Mr}{\Sigma^2\Delta},$$

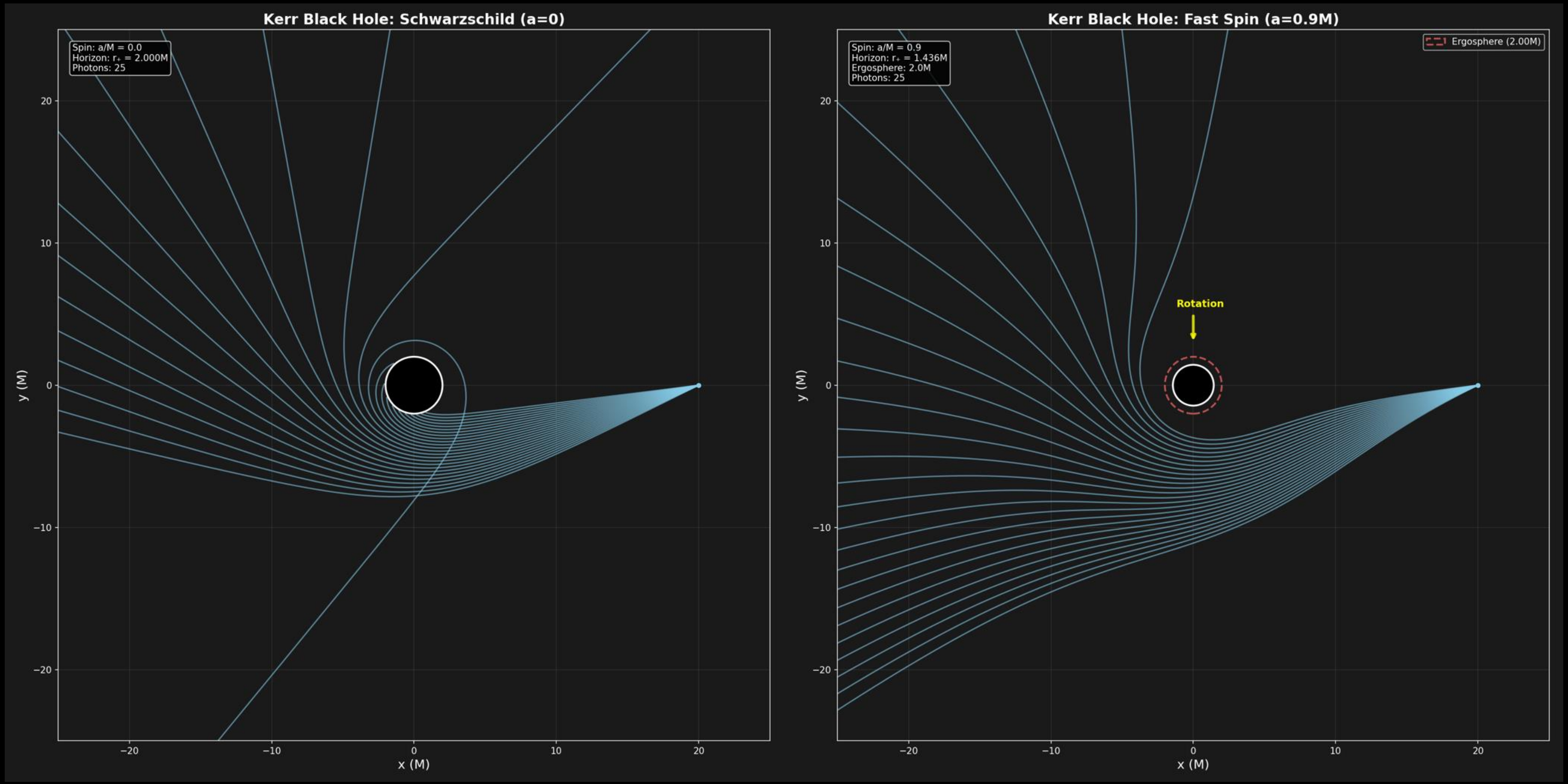
Geodesic Equations[1] (equatorial):

$$\frac{d^2\phi}{d\tau^2} = -2, \Gamma^{\phi}_{r\phi}, u^r u^\phi - 2, \Gamma^{\phi}_{tr}, u^t u^r,$$
$$\frac{d^2r}{d\tau^2} = -\Gamma^{r}_{tt}(u^t)^2 - \Gamma^{r}_{rr}(u^r)^2 - \Gamma^{r}_{\phi\phi}\left(u^\phi\right)^2 - 2, \Gamma^{r}_{t\phi}, u^t u^\phi,$$
$$\frac{d^2t}{d\tau^2} = -2, \Gamma^{t}_{tr}, u^t u^r - 2, \Gamma^{t}_{r\phi}, u^r u^\phi,$$

Conservation (from Killing vectors):

$$L = p_\phi = g_{t\phi}u^t + g_{\phi\phi}u^\phi$$
$$E = -p_t = -\left(g_{tt}u^t + g_{t\phi}u^\phi\right),$$

Kerr – Results: Null Geodesics



Constraint Violations (over full trajectory): Energy: $\left|\frac{\Delta E}{E}\right| < 10^{-14}$, Angular Momentum: $\left|\frac{\Delta L}{L}\right| < 10^{-14}$, Null constraint: $\left|g_{\mu\nu}u^{\mu}u^{\nu}\right| < 10^{-13}$

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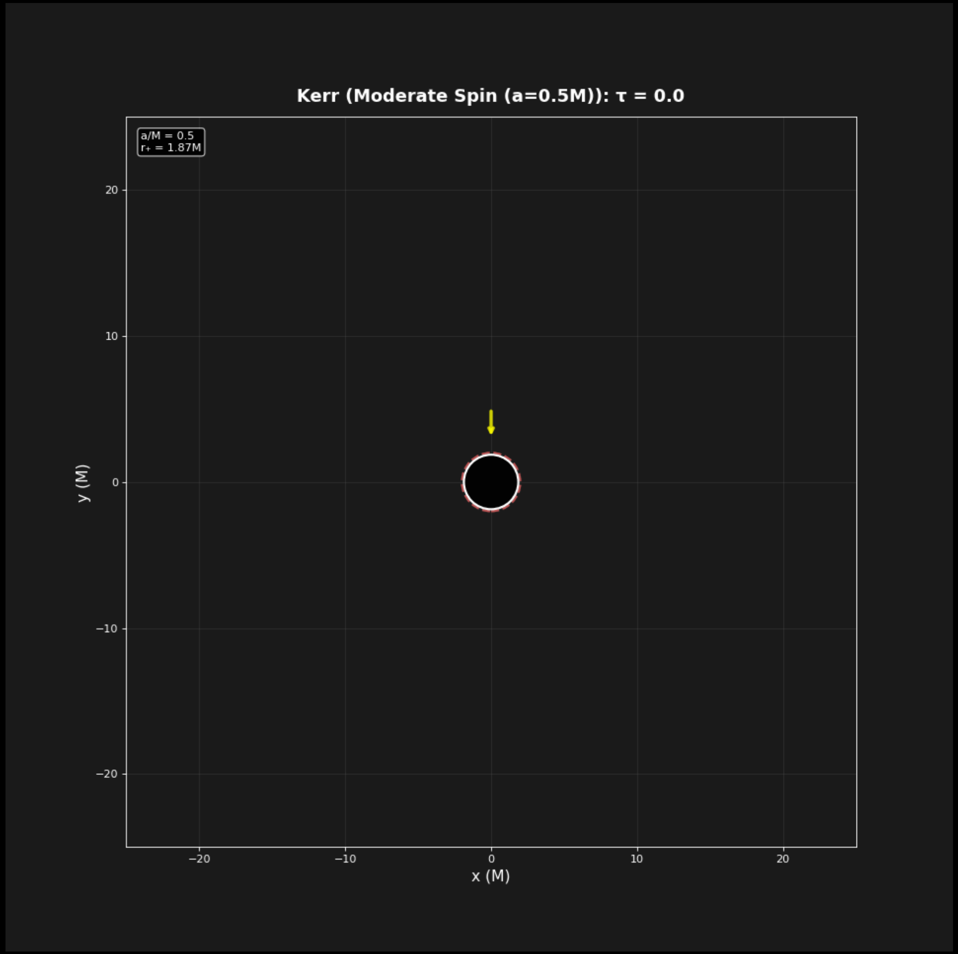

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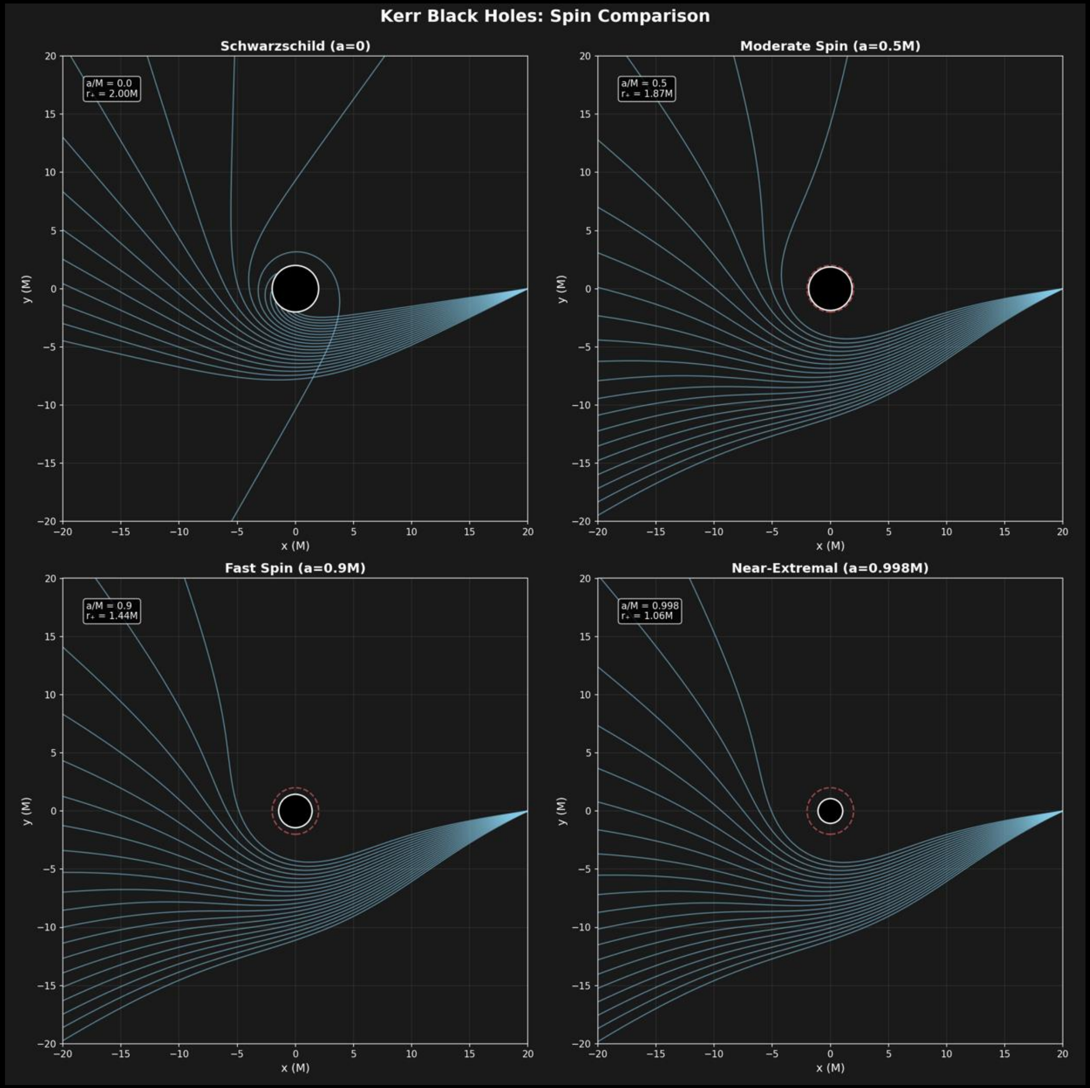

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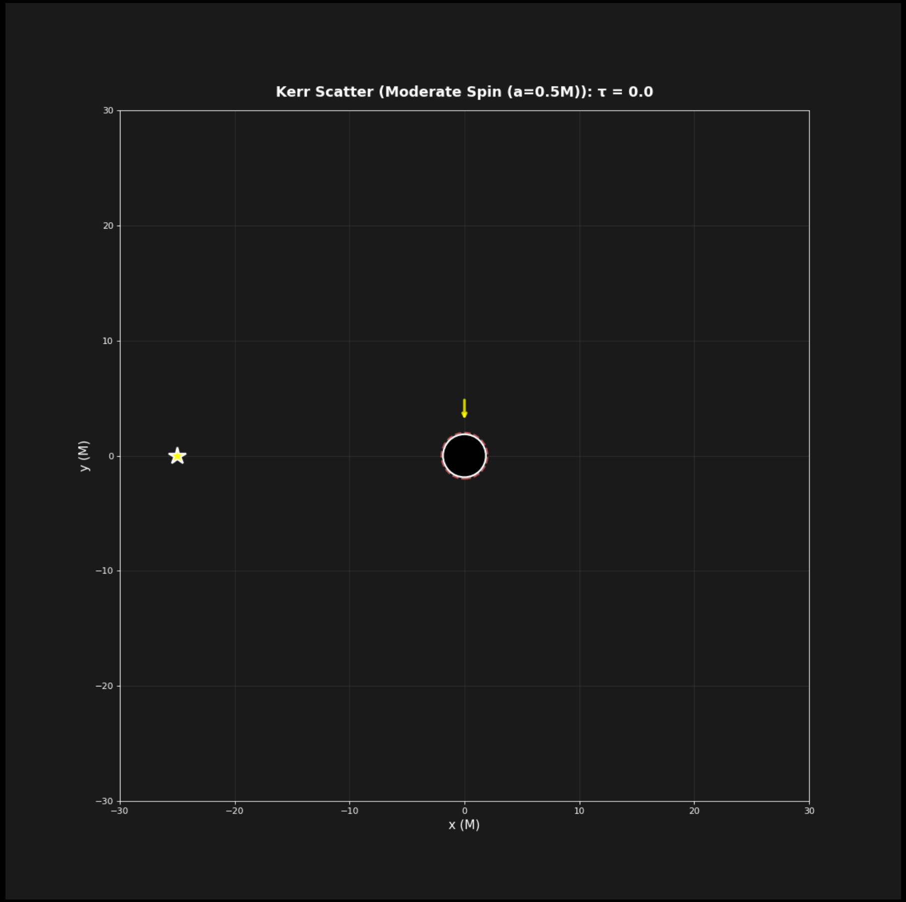
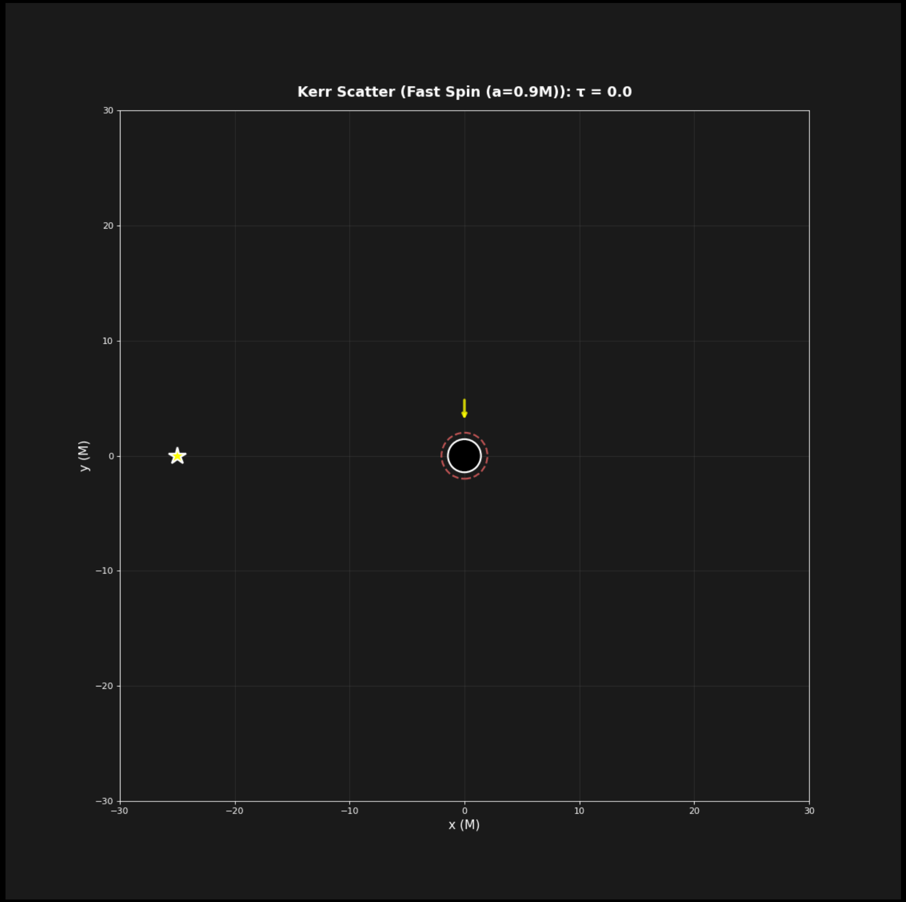


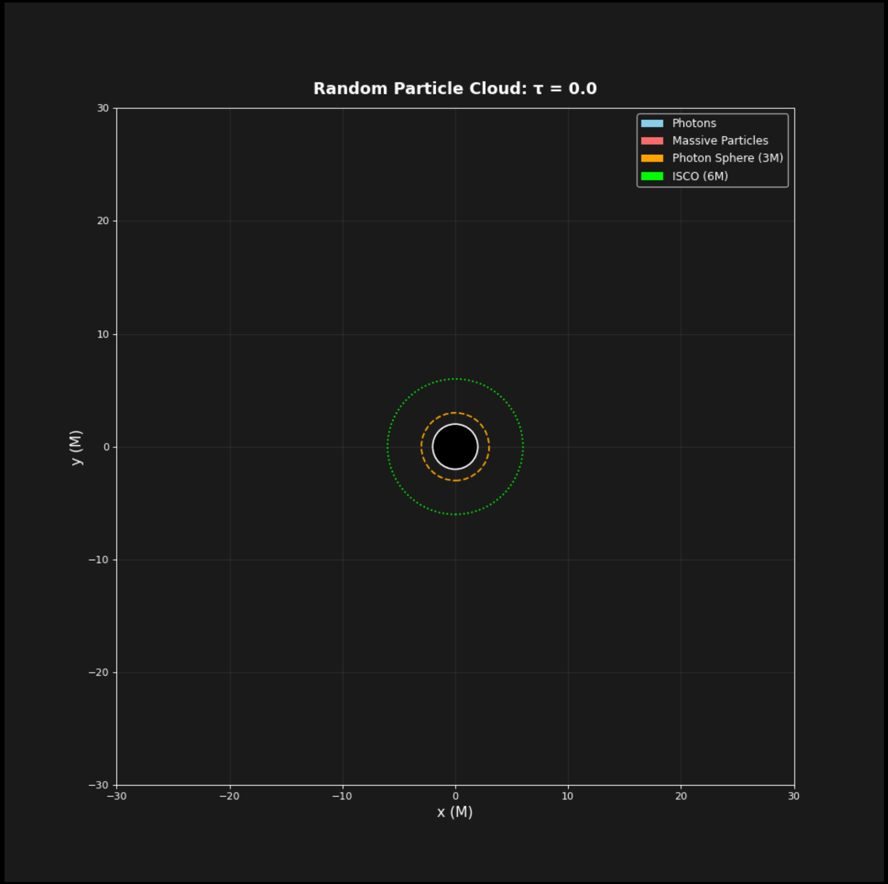


Conclusion

- Developed a fully general **geodesic simulator** for Schwarzschild and Kerr black holes
- Verified relativistic phenomena:
  - Photon Sphere, ISCO, impact parameters
  - Ergosphere behavior, frame dragging
- ~4000 loc

- Source code, simulations Jupyter Notebook user demo and references can be found on GitHub:

https://github.com/fynnhufler/General-Relativity-Simulations



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