Two-Period Arbitration Model

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Environment

- Two countries dividing a pie of size 1.
- Each country has a type indicating the strength: H for high and L for low.
- Dividing by war: the pie will shrink to $\theta < 1$.

Assumptions

- Type are exogenous, the probability of one country is a high type is $q \in (0, 1)$.
- Countries' payoff perfectly correspond to how much they get from the pie.
- Symmetry:
  - (H, H) or (L, L) gets $(\frac{\theta}{2}, \frac{\theta}{2})$ from the war.
  - (H, L) gets $(p\theta, (1-p)\theta)$ from the war, where $p\theta > \frac{1}{2}$ with $p > \frac{1}{2}$.

Arbitration

- Two countries can participate in the arbitration model instead of war.
- There are 2 periods and 2 short lived arbitrators.
- Myopic arbitrators' objective is to minimise the war probability in their periods.
- Arbitrators make binding decisions in their periods.

The First Arbitration

A Direct Mechanism

- Two countries observe their own type at the beginning.
- Then, each country reports their type $r_i \in \{h, l\}$ privately to the first arbitrator (referred as A1). $(i \in \{1, 2\})$.
- A1 can randomise:
  - proposes a split $(x, 1-x)$ with probability $p(r)$, $r = (r_1, r_2)$.
  - postpones the two countries to the second period with probability $1 - p(r)$.
- Two countries have to accept the split if proposed.

The First Arbitration

Symmetry Assumption

Since the arbitrators make binding decision, by symmetry we have:

- If two countries' types are:
  - (H, H) or (L, L), then the peaceful split is $(\frac{1}{2}, \frac{1}{2})$.
  - (H, L), then the peaceful split is $(b, 1-b)$, with the high type gets $b \in [\frac{1}{2}, 1]$.
- Let $p_L = p(l, l)$, $p_M = p(h, l) = p(l, h)$, and $p_H = p(h, h)$.

The First Arbitration
A1's Objective

- To minimise the ex-ante war probability.

$$\min_{b,p_L,p_M,p_H} V \equiv \min_{b,p_L,p_M,p_H} \{q^2(1-p_H) + 2q(1-q)(1-p_M) + (1-q)^2(1-p_L)\}$$

- Revelation principle then implies two groups of constraints:
  - The individual rationality constraints.
  - The incentive compatibility constraints.

The First Arbitration

Individual Rationality Constraints

- Let $z_{k,j}$ be type $k \in \{H, L\}$'s expected payoff from the second arbitration when reporting $j \in \{H, L\}$ in the first arbitration.
- Outside option: war.
- Time discount factor: $\delta \in (0, 1)$
- For the low type:

$$(1-q)\frac{p_L}{2} + qp_M(1-b) + ((1-q)(1-p_L) + q(1-p_M))\delta z_{L,L} \geqslant (1-q)\frac{\theta}{2} + q(1-p)\theta \tag{LIR1}$$

- For the high type:

$$(1-q)p_M b + q\frac{p_H}{2} + ((1-q)(1-p_M) + q(1-p_H))\delta z_{H,H} \geqslant (1-q)p\theta + q\frac{\theta}{2} \tag{HIR1}$$

- For the low type:

$$
\begin{aligned}
&(1-q)\frac{p_L}{2} + qp_M(1-b) + ((1-q)(1-p_L) + q(1-p_M))\delta z_{L,L} \\
\geqslant &(1-q)p_M b + q\frac{p_H}{2} + ((1-q)(1-p_M) + q(1-p_H))\delta z_{L,H}
\end{aligned} \tag{LIC1}
$$

- For the high type:

$$
\begin{aligned}
&(1-q)p_M b + q\frac{p_H}{2} + ((1-q)(1-p_M) + q(1-p_H))\delta z_{H,H} \\
\geqslant &(1-q)\frac{p_L}{2} + qp_M(1-b) + ((1-q)(1-p_L) + q(1-p_M))\delta z_{H,L}
\end{aligned} \tag{HIR1}
$$

The Second Arbitration

A Direct Mechanism and Symmetry Assumption

If A1 postpones, then two countries go to the second period.

- Outside option: war.
- In second period, the total size of pie shrinks to δ.
- Same as in period 1, each country reports their type $s_i \in \{h, l\}$ privately to the second arbitrator (referred as A2). $(i \in \{1, 2\})$
- A2 can randomise:
  - proposes a split $(y, \delta - y)$ with probability $q(s)$, $s = (s_1, s_2)$.
  - proposes war with probability $1 - q(s)$.

The Second Arbitration

Symmetry Assumption

- Similar to period 1, we impose symmetry assumption. If the two countries' types are:
  - (H, H) or (L, L), then the peaceful split is $(\frac{\delta}{2}, \frac{\delta}{2})$.
  - (H, L), then the peaceful split is $(a, 1-a)$, with the high type gets $a \in [\frac{\delta}{2}, \delta]$.
- Let $q_L = q(l, l)$, $q_M = q(h, l) = p(l, h)$, and $q_H = p(h, h)$.
- Let $c \equiv \frac{a}{\delta}$, then $c \in [\frac{1}{2}, 1]$, and we can get rid of δ when dealing with the second period.

The Second Arbitration

Belief Updating - A2

- A2 can neither observe two countries' types nor their reports to A1.
- But she can update her belief on two countries' type by the fact that they were postponed in period 1.
- Let w be A2's belief in a player is a high type, then:

$$w = \frac{q(1-q)(1-p_M) + q^2(1-p_H)}{(1-q)^2(1-p_L) + 2q(1-q)(1-p_M) + q^2(1-p_H)}$$

- This is the probability that one country is of a high type conditional on being proposed to wait by A1.

The Second Arbitration

Belief Updating - Two countries

- Each country can also update their belief on the opponent.
- Here, assume they reported truthfully to A1.
- If we use μ_k to denote the type k's belief on the opponent is a high type $(k \in \{H, L\})$, then:

$$\mu_L = \frac{q(1-p_M)}{q(1-p_M)+(1-q)(1-p_L)}$$
$$\mu_H = \frac{q(1-p_H)}{q(1-p_H)+(1-q)(1-p_M)}$$

- Given the correlation between w, μ_L and μ_H, we can simplify 3 variables to 2 variables.

The Second Arbitration

Belief Updating

- Define:
  - $m \equiv Prob(s = (h, h))$
  - $n \equiv Prob(s = (h, l)) = Prob(s = (l, h))$
- then observe:

$$\begin{aligned} w &= m + n \\ \mu_L &= \frac{n}{1 - m} \\ \mu_H &= \frac{m}{m + n} \end{aligned}$$

The Second Arbitration

Belief Updating

- In addition, we can write down m and n:

$$m = \frac{q^2(1-p_H)}{(1-q)^2(1-p_L) + 2q(1-q)(1-p_M) + q^2(1-p_H)}$$

$$n = \frac{q(1-q)(1-p_M)}{(1-q)^2(1-p_L) + 2q(1-q)(1-p_M) + q^2(1-p_H)}$$

The Second Arbitration

A2's Objective

- To minimise the ex-ante war probability in period 2.

$$\min_{c,q_L,q_M,q_H} W \equiv \min_{c,q_L,q_M,q_H} \{m(1-q_H) + 2n(1-q_M) + (1-m-n)(1-q_L)\}$$

- Revelation principle again implies two groups of constraints:
  - The individual rationality constraints.
  - The incentive compatibility constraints.

The Second Arbitration

Individual Rationality Constraints

- Notice the beliefs are updated.
- For the low type:

$$\frac{1-m-n}{1-m}(\frac{q_L}{2}+(1-q_L)\frac{\theta}{2})+\frac{n}{1-m}(q_M(1-c)+(1-q_M)(1-p)\theta)$$
$$\geqslant\frac{1-m-n}{1-m}\frac{\theta}{2}+\frac{n}{1-m}(1-p)\theta \tag{LIR2}$$

- For the high type:

$$\frac{n}{m+n}(q_Mc+(1-q_M)p\theta)+\frac{m}{m+n}(\frac{q_H}{2}+(1-q_H)\frac{\theta}{2})$$
$$\geqslant\frac{n}{m+n}p\theta+\frac{m}{m+n}\frac{\theta}{2} \tag{HIR2}$$

- Note that the time discount factor δs are cancelled on both sides.

The Second Arbitration

Incentive Compatibility Constraints

- For the low type:

$$
\begin{aligned}
&\frac{1-m-n}{1-m}\left(\frac{q_L}{2}+(1-q_L)\frac{\theta}{2}\right)+\frac{n}{1-m}\left(q_M(1-c)+(1-q_M)(1-p)\theta\right) \\
\geqslant &\frac{1-m-n}{1-m}\left(q_M c+(1-q_M)\frac{\theta}{2}\right)+\frac{n}{1-m}\left(\frac{q_H}{2}+(1-q_H)(1-p)\theta\right) \quad \text{(LIC2)}
\end{aligned}
$$

- For the high type:

$$
\begin{aligned}
&\frac{n}{m+n}\left(q_M c+(1-q_M)p\theta\right)+\frac{m}{m+n}\left(\frac{q_H}{2}+(1-q_H)\frac{\theta}{2}\right) \\
\geqslant &\frac{n}{m+n}\left(\frac{q_L}{2}+(1-q_L)p\theta\right)+\frac{m}{m+n}\left(q_M(1-c)+(1-q_M)\frac{\theta}{2}\right) \quad \text{(HIC2)}
\end{aligned}
$$

- Note that the time discount factor δs are cancelled on both sides.

Solve Backwards

- Take the first mechanism as given, and assume players reported truthfully in the first period.
- Solve for the second optimal arbitration programme, expressed with variables from the first arbitration programme.
- Accordingly, calculate out $z_{k,j}$.
- Then plug in and solve the first arbitration problem.
- With first optimal arbitration programme, again solve for the second arbitration programme.

The Reduced Second Arbitration Problem

- Note that if (LIC2) is satisfied, (LIR2) will always be satisfied:
  - By assumption $c \geqslant \frac{1}{2}$, thus $c \geqslant \frac{\theta}{2}$.
  - Similarly, $p\theta > \frac{1}{2}$, thus $(1-p)\theta < 1 - p\theta < \frac{1}{2}$.
- We speculate (HIC2) will slack by intuitive argument. We will come back to check it.
- With only (LIC2) and (HIR2) bind, $q_L^* = 1$:
  - q_L only appears in the lhs of (LIC2).
  - If $q_L < 1$, then we can increase q_L to further decrease the objective function without violating constraints.

The Reduced Second Arbitration Problem

- Then we can write down the reduced second arbitration problem:

$$\min_{c,q_M,q_H} W \equiv \min_{c,q_M,q_H} \{m(1-q_H)+2n(1-q_M)\}$$

subject to

$$n(c-p\theta)q_M + m(\frac{1}{2}-\frac{\theta}{2})q_H = 0 \tag{HIR2}$$

$$(1-m-n)(\frac{1}{2}-q_Mc-(1-q_M)\frac{\theta}{2}) + n(q_M(1-c)-\frac{q_H}{2}+(q_H-q_M)(1-p)\theta) = 0 \tag{LIC2}$$

$$\frac{1}{2} \leqslant c \leqslant p\theta$$

$$0 \leqslant q_H \leqslant 1$$

$$0 \leqslant q_M \leqslant 1$$

- Try solve by Kuhn-Tucker conditions.

The Reduced Second Arbitration Problem

The Lagrangian:

$$
\begin{aligned}
&\mathscr{L}(c, q_M, q_H, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8) \\
=&m(1-q_H) + 2n(1-q_M) - \lambda_1(n(c-p\theta)q_M + m(\frac{1}{2}-\frac{\theta}{2})q_H) \\
-&\lambda_2((1-m-n)(\frac{1}{2} - q_M c - (1-q_M)\frac{\theta}{2}) + n(q_M(1-c) - \frac{q_H}{2} + (q_H - q_M)(1-p)\theta)) \\
-&\lambda_3(\frac{1}{2} - c) - \lambda_4(c - p\theta) - \lambda_5(-q_H) - \lambda_6(q_H - 1) - \lambda_7(-q_M) - \lambda_8(q_M - 1)
\end{aligned}
$$

The Reduced Second Arbitration Problem

- Take FOCs:

$$\lambda_1 n q_M + \lambda_4 = \lambda_2(1-m)q_M + \lambda_3 \tag{1.1}$$

$$\lambda_1 n(p\theta - c) - 2n + \lambda_7 = \lambda_2((1-m-n)(\frac{\theta}{2} - c) + n(1-c-(1-p)\theta)) + \lambda_8 \tag{1.2}$$

$$\lambda_5 - \lambda_1 m(\frac{1}{2} - \frac{\theta}{2}) = \lambda_2 n((1-p)\theta - \frac{1}{2}) + m + \lambda_6 \tag{1.3}$$

- Additionally:

$$\lambda_3(\frac{1}{2} - c) = 0 \tag{1.4}$$

$$\lambda_4(c - p\theta) = 0 \tag{1.5}$$

$$\lambda_5 q_H = 0 \tag{1.6}$$

$$\lambda_6(q_H - 1) = 0 \tag{1.7}$$

$$\lambda_7 q_M = 0 \tag{1.8}$$

$$\lambda_8(q_M - 1) = 0 \tag{1.8}$$