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The derivation of scaling relationship between acoustic and electromagnetic scattering by spheres

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The rigorous theory of the conversion between the scattering of uniform plane electromagnetic wave by a perfectly conducting sphere and the scattering of uniform plane acoustic wave by a rigid sphere is studied in this paper. The conversion formula between these two different scattering based on two calibration curves is derived, which describes the quantitative relationship between acoustic and electromagnetic wave scattering at arbitrary frequencies by spheres of arbitrary sizes. In addition, the scaling relationship of the sizes of those two spheres and the corresponding frequencies are discussed in detail, and an indirect method of measurement of electromagnetic scattering by the spheres is proposed. [http://dx.doi.org/10.1063/1.4837395]

As is well understood, Radar cross section (RCS) is an important concept in the microwave electromagnetic field, which is a measurement of how detectable an object is with Radar. Obviously, the RCS of a stealth aircraft is one of the most important indexes in its stealth performance. The smaller the RCS, the better the stealth performance is. In general, the RCS of an object should be measured in a microwave anechoic chamber. However, it is difficult to put a very large object like an aircraft into a common microwave anechoic chamber, for the reason that not only should the chamber be large enough to hold the object, but also the chamber should be reserved enough measuring distance to meet the far field condition of the Maxwell's Equations.

In resolving this problem, an indirect method of electromagnetic scattering is proposed. In fact, although the acoustic and electromagnetic wave is respectively mechanical longitudinal wave and electromagnetic transverse wave, they both satisfy the same wave equations. Therefore, they have similar solutions in form, which can be transformed from one to the other. This makes it possible that the measuring device could be simplified and the measuring space, also, would be greatly reduced. Wu Nan *et al.* proposes the concept of conversion between acoustic and electromagnetic scattering at a certain scaling frequency and the compares of the nature between acoustic and electromagnetic waves.[1] Song Dong-an, *et al.* refers to electromagnetic scattering and gives the concept of RCS of acoustic scattering and a detail comparison of nature between the two waves.[2] However, these authors do not give a detail process of derivation and simulation. Wang Wei, *et al.* explicitly defines the method of RCS based on acoustic simulation, and gives a certain illumination of the theory by formulas, nevertheless, the theory ignores the mutual coupling phenomenon between each part, which greatly limits its application scope.[3]

In this paper, the conversion formula between the acoustic scattering by a rigid sphere and the electromagnetic scattering by a conducting sphere is derived in detail. Further more, the scaling relationship of the sizes of the two different spheres, and the corresponding frequencies is discussed

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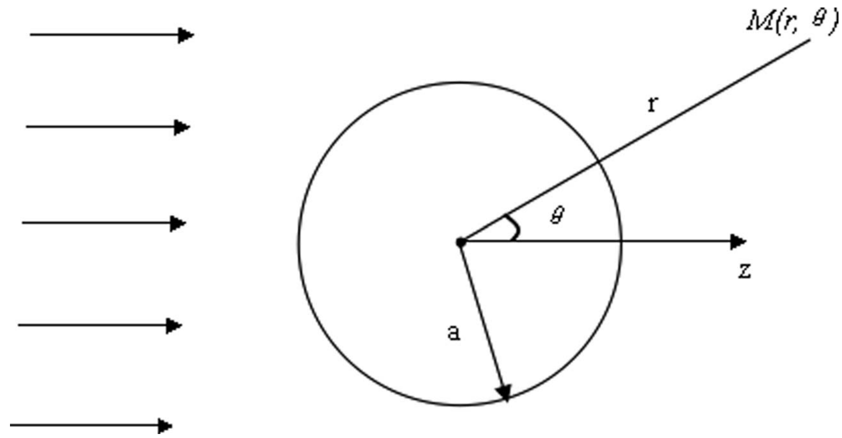

FIG. 1. Acoustic scattering by a rigid sphere.

and applied in practice. It indicates a conclusion that the scattering of an electromagnetic wave at a higher frequency by a larger conducting sphere could be expressed by the scattering of an acoustic wave at a lower frequency by a smaller rigid sphere. Furthermore, the transformation between an acoustic and electromagnetic scattering is simulated by software and an ideal result attained.

The RCS is defined as:[4]

$$\sigma = \lim_{r\to\infty} 4\pi r^2 \frac{\overrightarrow{E_s}}{\overrightarrow{E_i}} = \lim_{r\to\infty} 4\pi r^2 \frac{\overrightarrow{H_s}}{\overrightarrow{H_i}} \tag{1}$$

where, σ is the RCS of an object, $\vec{E}_i$ and $\vec{H}_i$ is the incident electromagnetic wave near the object, $\vec{E}_s$ and $\vec{H}_s$ is the electromagnetic wave of scattering near the radar, r is the distance between the object and radar.

According to the definition of RCS, the work to study the scattering of the $\vec{E}$ field is sufficient to derive a value for the RCS. The discussion of the $\vec{E}$ field pertinent. Besides, here we suppose the target of the scattering is a sphere.

Let the center of the rigid sphere with a a_p radius be the origin, the direction of propagation of the incident plane wave is z and the time factor is$e^{j\omega_p t}$, just as Fig. 1.

The incident acoustic wave of pressure can be expressed in the following series of Legendre polynomials:[5]

$$p_i(z) = p_0 e^{ik_p z} = p_0 \sum_{n=0}^{\infty} i^n (2n+1) j_n(k_p r_p) P_n(\cos\theta) \tag{2}$$

where, $j_n(x)$ is the spherical Bessel function, $P_n(x)$ is the Legendre's polynomial, k_p is the phase number of acoustic wave and $x = \cos\theta$.

The scattering wave p_s satisfies the boundary conditions that the velocity of the vertical component of the boundary surface is zero.

By applying mode-matching technique, the acoustic scattering is obtained as

$$p_s(\theta, \phi, r_p) = -p_0 \sum_{n=0}^{\infty} i^n (2n+1) \frac{j_n'(k_p a_p)}{h_n^{(1)\prime}(k_p a_p)} h_n^{(1)}(k_p r_p) P_n(\cos\theta) \tag{3}$$

Under the far field conditions ($k_p r_p \to \infty$), we derive

$$h_n^{(1)}(k_e^1 r_e) \approx \frac{i^{-(n+1)} e^{jk_p r_p}}{k_p r_p} \tag{4}$$

Similarly, let the center of the conducting sphere with a a_e radius be the origin, the direction of propagation of the incident plane wave is z. We assume that the incident wave is polarized with its

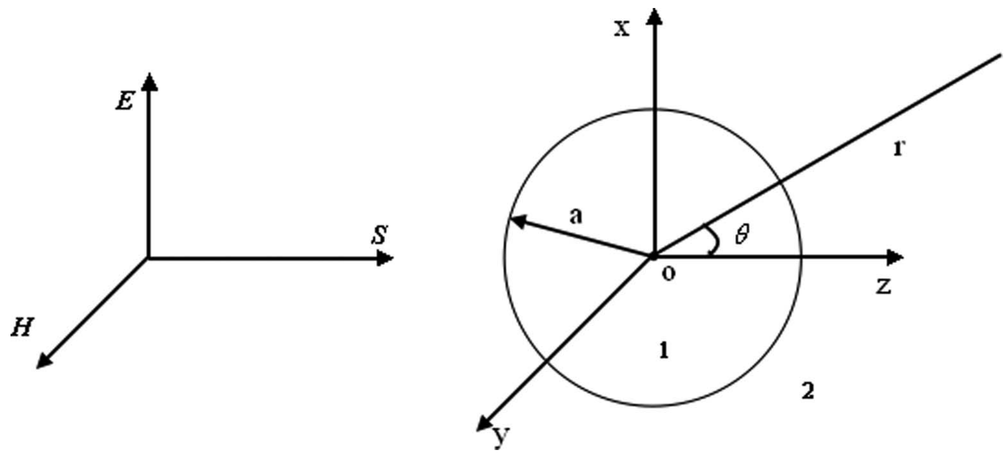

FIG. 2. Electromagnetic scattering by a conducting sphere.

field $\vec{E}$ vibrating along x axis, just as Fig. 2. Let medium 1 have the values ε_1, $\mu_1 = 4\pi \times 10^{-7}$, $\sigma_1 = 0$ and medium 2 the values ε_2, $\mu_2 = \mu_1$,σ_2. Incident wave can be expressed as:

$$E_i = E_0 e^{ik_1 z} = E_0 \sum_{n=0}^{\infty} i^n (2n+1) j_n(k_1 r) P_n(\cos\theta) \tag{5}$$

By the Mie theory, we arrive at the analytic solution of electromagnetic scattering:[6]

$$E_{sr}(\theta, \phi, r_e) = \frac{E_0 \cos\phi}{(k_e^1 r_e)^2} \sum_{n=0}^{\infty} i^{n+1}(2n+1) a_n \eta_n(k_e^1 r_e) P_n^1(\cos\theta) \tag{6}$$

$$E_{s\theta}(\theta, \phi, r_e) = \frac{E_0 \cos\phi}{k_e^1 r_e} \sum_{n=0}^{\infty} i^n \frac{2n+1}{n(n+1)} [i a_n \eta_n'(k_e^1 r_e) \tau_n(\theta) - b_n \eta_n(k_e^1 r_e) \chi_n(\theta)] \tag{7}$$

$$E_{s\phi}(\theta, \phi, r_e) = \frac{E_0 \sin\phi}{k_e^1 r_e} \sum_{n=0}^{\infty} i^n \frac{2n+1}{n(n+1)} [-i a_n \eta_n'(k_e^1 r_e) \chi_n(\theta) + b_n \eta_n(k_e^1 r_e) \tau_n(\theta)] \tag{8}$$

where,

$$a_n = \frac{n_1 \phi_n(k_e^1 a_e) \phi_n'(k_e^2 a_e) - n_2 \phi_n(k_e^2 a) \phi_n'(k_e^2 a_e)}{n_1 \eta_n(k_e^1 a_e) \phi_n'(k_e^2 a_e) - n_2 \eta_n'(k_e^1 a_e) \phi_n(k_e^2 a_e)} \tag{9}$$

$$b_n = \frac{n_2 \phi_n(k_e^1 a_e) \phi_n'(k_e^2 a_e) - n_1 \phi_n(k_e^2 a_e) \phi_n'(k_e^1 a_e)}{n_2 \eta_n(k_e^1 a_e) \phi_n'(k_e^2 a_e) - n_1 \eta_n'(k_e^1 a_e) \phi_n(k_e^2 a_e)} \tag{10}$$

$$\tau_n(\theta) = \frac{d P_n^1(\cos\theta)}{d\theta} \tag{11}$$

$$\chi_n(\theta) = \frac{P_n^1(\cos\theta)}{\sin\theta} \tag{12}$$

Where, $n_1 = \sqrt{\varepsilon_1}$, $n_2 = \sqrt{\varepsilon_2 + i\frac{\sigma_2}{\omega\varepsilon_0}}$, $\phi_n(x) = x j_n(x)$ and $\eta_n(x) = x h_n^{(1)}(x)$.
For a perfectly conducting sphere, we can get $\sigma_2 = \infty$ and $|n_2| = \infty$.
Under the far field condition ($k_e^1 r_e \to \infty$),

We have,

$$\eta_n(k_e^1 r_e) \approx i^{-(n+1)} e^{jk_e^1 r_e} \tag{13}$$

$$\eta_n'(k_e^1 r_e) \approx i^{-n} e^{jk_e^1 r_e} \tag{14}$$

It is obvious that $E_{sr} \to 0$.

For the reason that $E_{s\theta}$ and $E_{s\phi}$ has the similar form, only $E_{s\theta}$ needs to be discussed. For simplicity, $E_{s\theta}$ is substituted as:

$$E_{s\theta} = \frac{iE_0 \cos\phi}{k_e r_e} e^{ik_e r_e} \sum_{n=0}^{\infty} \frac{2n+1}{n(n+1)} [\frac{(k_e a_e j_n(k_e a_e))'}{(k_e a_e h_n^{(1)}(k_e a_e))'} \frac{dP_n^1(\cos\theta)}{d\theta} + \frac{j_n(k_e a_e)}{h_n^{(1)}(k_e a_e)} \frac{P_n^1(\cos\theta)}{\sin\theta}] \tag{15}$$

Let

$$E_{s\theta}^1 = \frac{iE_0 \cos\phi}{k_e r_e} e^{ik_e r_e} \sum_{n=0}^{\infty} \frac{2n+1}{n(n+1)} \frac{(k_e a_e j_n(k_e a_e))'}{(k_e a_e h_n^{(1)}(k_e a_e))'} \frac{dP_n^1(\cos\theta)}{d\theta} \tag{16}$$

$$E_{s\theta}^2 = \frac{iE_0 \cos\phi}{k_e r_e} e^{ik_e r_e} \sum_{n=0}^{\infty} \frac{2n+1}{n(n+1)} \frac{j_n(k_e a_e)}{h_n^{(1)}(k_e a_e)} \frac{P_n^1(\cos\theta)}{\sin\theta} \tag{17}$$

If the acoustic scattering and the electromagnetic scattering satisfies

$$k_e a_e = k_p a_p = ka \tag{18}$$

$$k_e r_e = k_p r_p = kr \tag{19}$$

By the nature of the Legendre's polynomial, we can easily get

$$P_n^1(\cos\theta) = -\frac{dP_n(\cos\theta)}{d\theta} \tag{20}$$

$$\frac{dP_n^1(\cos\theta)}{d\theta} = -\frac{d^2 P_n(\cos\theta)}{d\theta^2} \tag{21}$$

Here we define two calibration functions:

$$\Delta = \tilde{p}_s + err_1 \tag{22}$$

$$\Gamma = \tilde{p}_s + err_2 \tag{23}$$

Where

$$\tilde{p}_s = \frac{p_s}{p_0} \tag{24}$$

$$err_1 = -\frac{i}{kr} e^{jkr} \sum_{n=0}^{\infty} (2n+1) [\frac{1}{n(n+1)} \frac{(kaj_n(ka))'}{(kah_n^{(1)}(ka))'} - \frac{j_n\,'(ka)}{h_n^{(1)}\,'(ka)}] P_n(\cos\theta) \tag{25}$$

$$err_2 = -\frac{i}{kr} e^{ikr} \sum_{n=0}^{\infty} (2n+1) [\frac{1}{n(n+1)} \frac{j_n(ka)}{h_n^{(1)}(ka)} - \frac{j_n\,'(ka)}{h_n^{(1)}\,'(ka)}] P_n(\cos\theta) \tag{26}$$

So normalized $E_{s\theta}$ can be expressed as

$$\tilde{E}_{s\theta} = \cos\phi \frac{d^2\Delta}{d\theta^2} + \frac{\cos\phi}{\sin\theta} \frac{d\Gamma}{d\theta} \tag{27}$$

TABLE I. Acoustic velocity and corresponding frequencies ratio of electromagnetic and acoustic wave.

medium	velocity (m/s)	fe/fp (1.0 × 105)
air (15 °C)	340	8.823
water (normal temperature)	1450	2.069
sea (25 °C)	1531	1.960
ice	3160	0.949

TABLE II. Frequency bands of the optimum transmission for an acoustic wave and the corresponding bands of the electromagnetic wave.

medium	Acoustic frequency band	Corresponding electromagnetic frequency
air	20 KHz ~ 3 MHz	17.646 GHz ~ 2.647 THz
sea	40 KHz ~ 80 KHz	7.84 GHz ~ 15.68 GHz

Where $\tilde{E}_{s\theta}^{1} = \frac{E_{s\theta}^{1}}{E_0}$, $\tilde{E}_{s\theta}^{2} = \frac{E_{s\theta}^{2}}{E_0}$.
If $a_p = a_e$, then

$$k_p = k_e \tag{28}$$

$$r_p = r_e \tag{29}$$

From $k = \frac{2\pi f}{v}$, we can get

$$\frac{f_p}{v_p} = \frac{f_e}{v_e} \tag{30}$$

where f_p, v_p is the frequency and phase velocity of an acoustic wave, f_e and $v_e = 3 \times 10^8$ m/s is the frequency and phase velocity of an electromagnetic wave.

We can obtain the velocities of acoustic wave propagation in common mediums, and get the corresponding ratio of the frequency of the electromagnetic wave to the frequency of the acoustic wave, which is shown in Table I.

We conclude from Table I that the frequency of an electromagnetic wave is nearly five orders of magnitude higher than that of the corresponding acoustic wave in common transmission medias.

In fact, the frequency band of the optimum transmission for an acoustic wave in a certain media is very limited. The Table II shows us the frequency bands of the optimum transmission for an acoustic wave in the air and sea and the corresponding frequency band of the electromagnetic wave in the vacuum.

In conclusion, we can get the following points from the measuring of the scattering by the same size spheres:

(1) The materials of the objects of electromagnetic scattering should be good conductors; while those of acoustic scattering should be rigid, which suggests that the measuring of electromagnetic scattering cost will be greatly reduced;
(2) The frequency of the electromagnetic wave is almost five orders of magnitude higher than the corresponding frequency of the acoustic wave in common media;
(3) The measuring distance of scattering for both is equivalent.

If $a_p \neq a_e$, then

$$\frac{f_p a_p}{v_p} = \frac{f_e a_e}{c_0} \tag{31}$$

$$\frac{r_p}{r_e} = \frac{a_p}{a_e} \tag{32}$$

TABLE III. The scaling relationship of electromagnetic wave and acoustic wave in different frequencies.

medium	Acoustic frequencies	Electromagnetic frequencies	Ratios of the scale
air	20 KHz ∼ 3 MHz	1 GHz	17.65 ∼ 2647.06
		35 GHz	0.50 ∼ 75.63
		94 GHz	0.19–28.16
sea	40 KHz ∼ 80 KHz	1 GHz	7.84 ∼ 15.68
		5G Hz	1.57 ∼ 3.14
		9.4 GHz	0.83 ∼ 1.67

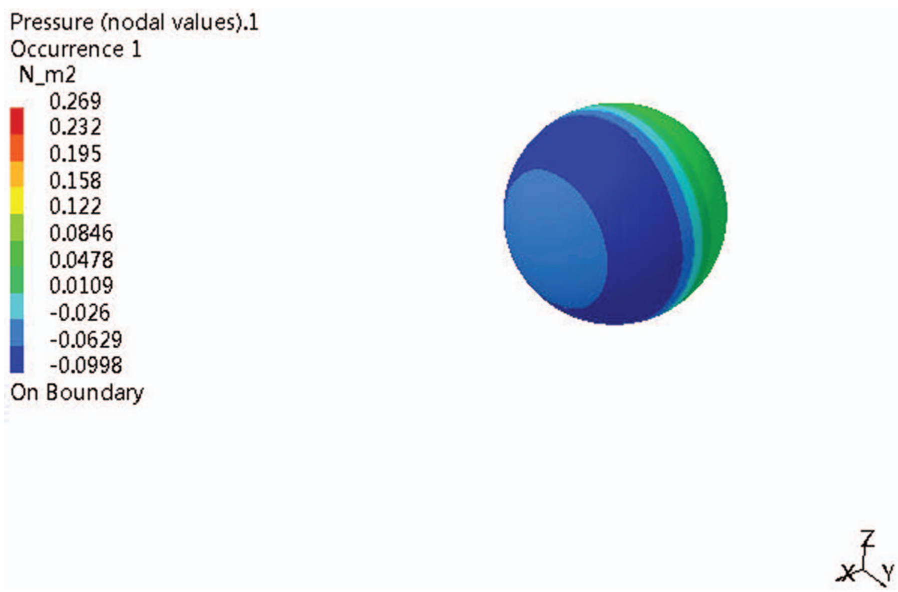

FIG. 3. The simulation of acoustic scattering.

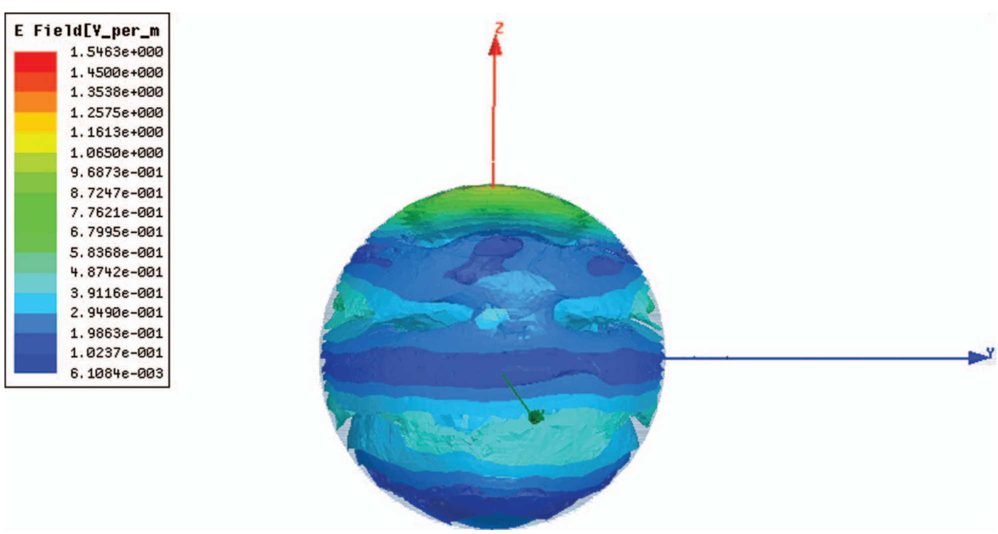

FIG. 4. The simulation of electromagnetic scattering.

The Table III shows us the scaling relationship of electromagnetic wave and acoustic wave in different frequencies.

TABLE IV. The conditions of simulations of acoustic and electromagnetic scattering.

	Acoustic Field			Electromagnetic Field		
Frequency	270.56 KHz	27.056 KHz	27.056 KHz	23.873 GHz	23.873 GHz	238.73 GHz
Radius/mm	0.6	10	14	6	10	1.4
ka	3	5	7	3	5	7

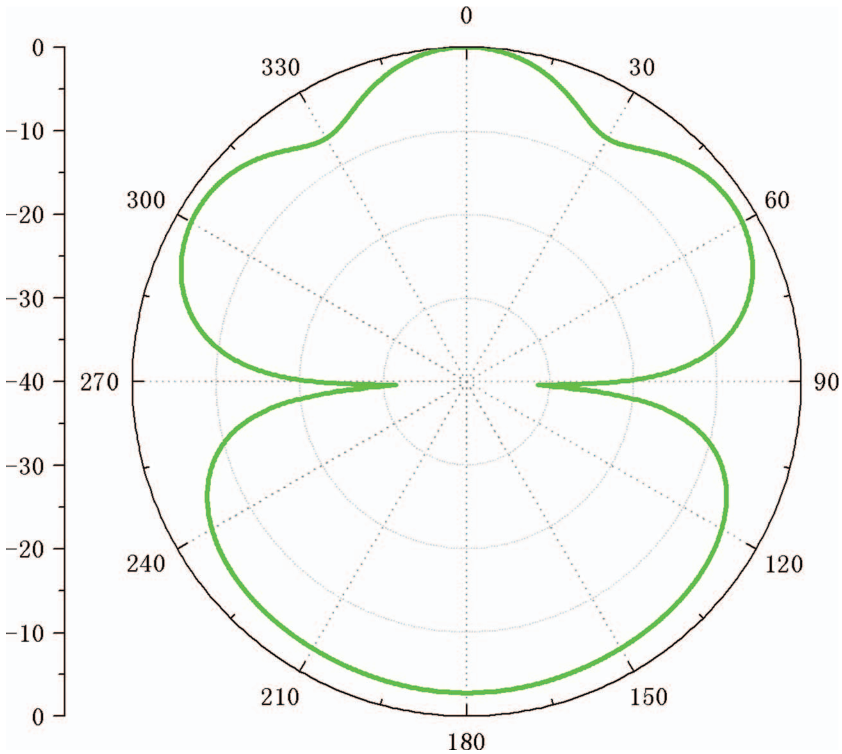

FIG. 5. The normalized field of acoustic scattering (a = 0.6 mm) at 270.56 KHz.

From above, we can get the conclusions:

(1) An electromagnetic scattering by a conducting sphere with arbitrary sizes could be expressed by an acoustic scattering by a rigid sphere with a certain scaling sizes, which will make it possible that the measuring of electromagnetic scattering by a very large sphere can be measured in a common microwave anechoic chamber;
(2) The frequency of acoustic scattering is much lower than that of the electromagnetic scattering;
(3) The scaling relationship of both the size and the distance of the measuring of both acoustic and electromagnetic scattering is equivalent.

From the discussion above, we conclude the advantages of the method of measurement of electromagnetic scattering with acoustic wave:

(1) It is much easier and cheaper to get rigid materials than conducting materials. The rigid materials include cement, glass, steel, etc. while conducting materials are mainly precious metals, such as gold, silver, etc.;
(2) The objects can be enlarged or shrunk by much more than even 100 times to suitable sizes to measure. It means if we want to get electromagnetic scattering by a macro conducting object with a dimension of 50 m such as airplane, then it can be substituted by acoustic scattering by a much smaller rigid model with a dimension of only 0.5 m; However, if we want to get electromagnetic scattering by a very tiny object with a dimension of 500 μm such as ice

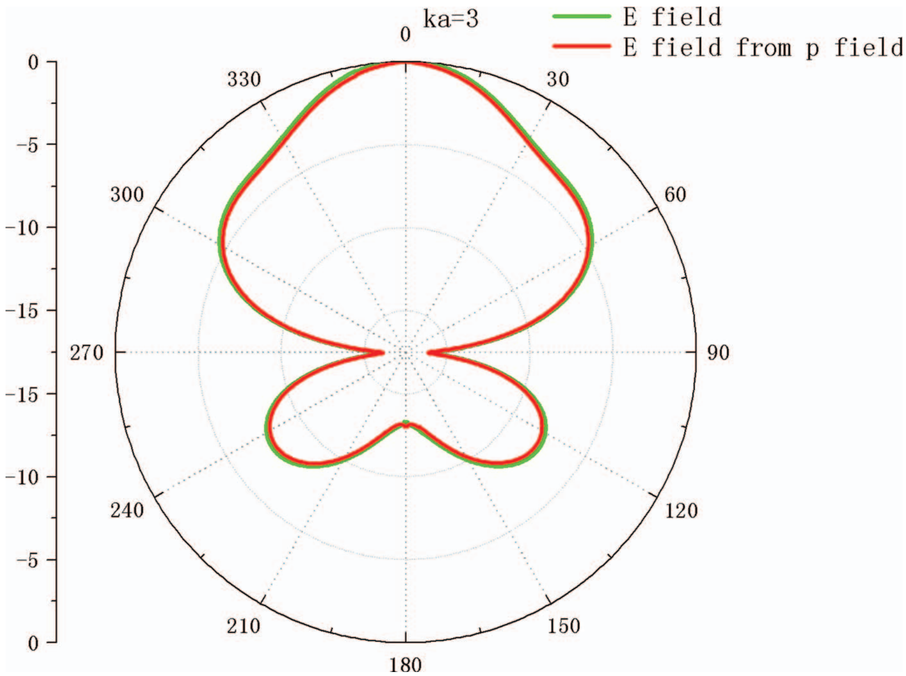

FIG. 6. The normalized field of electromagnetic scattering (a = 6 mm) at 23.873 GHz.

particles, then it can also be replaced by acoustic scattering by a much bigger rigid model with a dimension of 5 cm, which will be more convenient for measurement;

(3) This method makes it possible to put objects into the laboratory. As is well understood, the measurements of scattering should meet the far-field condition so that the scattering wave can be considered as plane wave and at the same time, the measurement environment should be in an anechoic chamber, so it is almost impossible to build a large enough chamber to hold an airplane and meet the far-field condition. However, the scaling rigid model can be easily put into an anechoic chamber and accurately measured by the acoustic wave;

(4) The required frequencies for measurement are significantly reduced. When the frequencies of the electromagnetic wave are required a very high level like THz, it is extremely difficult to make the transmitter and antenna and meanwhile, the return loss is very significant. By this method of measurement, the frequencies of the acoustic wave can be reduced to KHz and return loss is very low.

In order to demonstrate the conversion between acoustic and electromagnetic scattering, acoustic and electromagnetic scattering by spheres with the same sizes is simulated with ACTRAN 12 (produced by Free Field Technologies) and High Frequency Structure Simulator 14 (HFSS14, produced by Ansoft). Suppose a rigid sphere is illuminated by an incident acoustic plane wave in the air with a density of 1.225 kg/m^3, as is shown in Fig. 3. Meanwhile, consider a perfectly conducting sphere illuminated by an incident electromagnetic plane wave in the vacuum space, as is shown in Fig. 4. To demonstrate electromagnetic scatterings by conducting sphere can be substituted by acoustic scatterings by rigid sphere model with scaling dimensions at scaling frequencies, three typical simulations have been done and the data are listed in Table IV. The results of the simulations are shown in Figs. 5–10.

From these figures above, it shows that the curves of both acoustic and electromagnetic scatterings by spheres with the scaling dimensions at scaling frequencies are very approximately identical, which implies that electromagnetic scatterings by a conducting sphere can be obtained by acoustic scatterings by a rigid sphere with a scaling size.

In this paper, the conversion formula between the scattering of an acoustic plane wave by a rigid sphere and the scattering of an electromagnetic plane wave by a conducting sphere under the

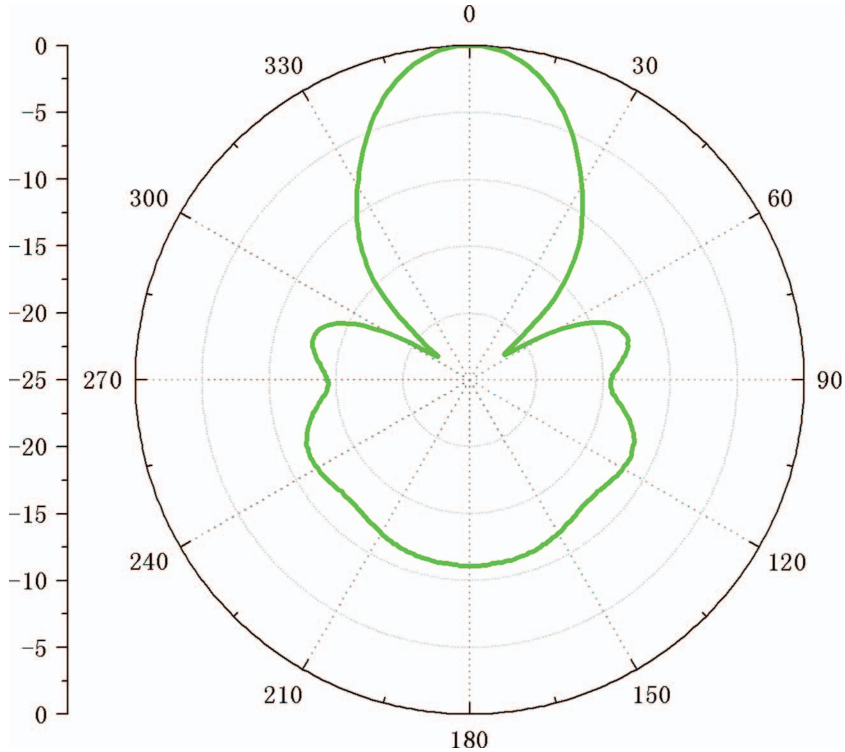

FIG. 7. The normalized field of acoustic scattering (a = 10 mm) at 27.056 KHz.

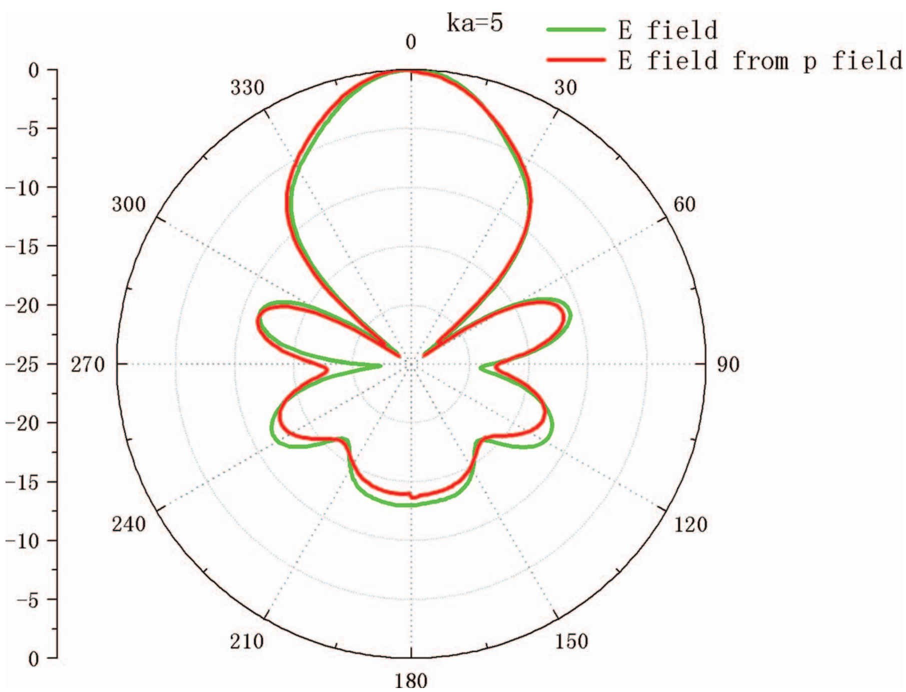

FIG. 8. The normalized field of electromagnetic scattering (a = 10 mm) at 23.873 GHz.

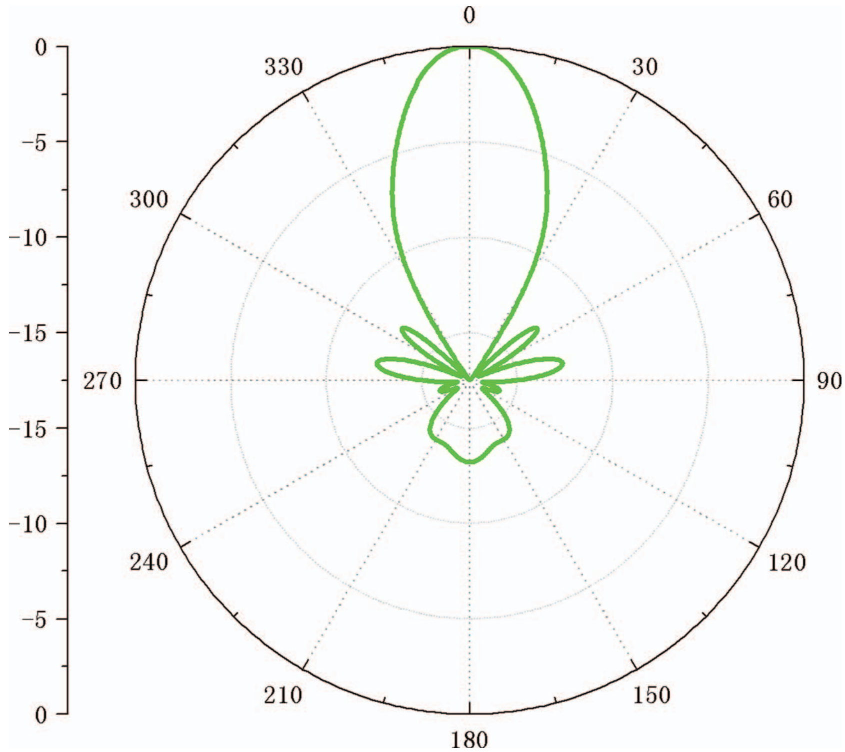

FIG. 9. The normalized field of acoustic scattering (a = 14 mm) at 27.056 KHz.

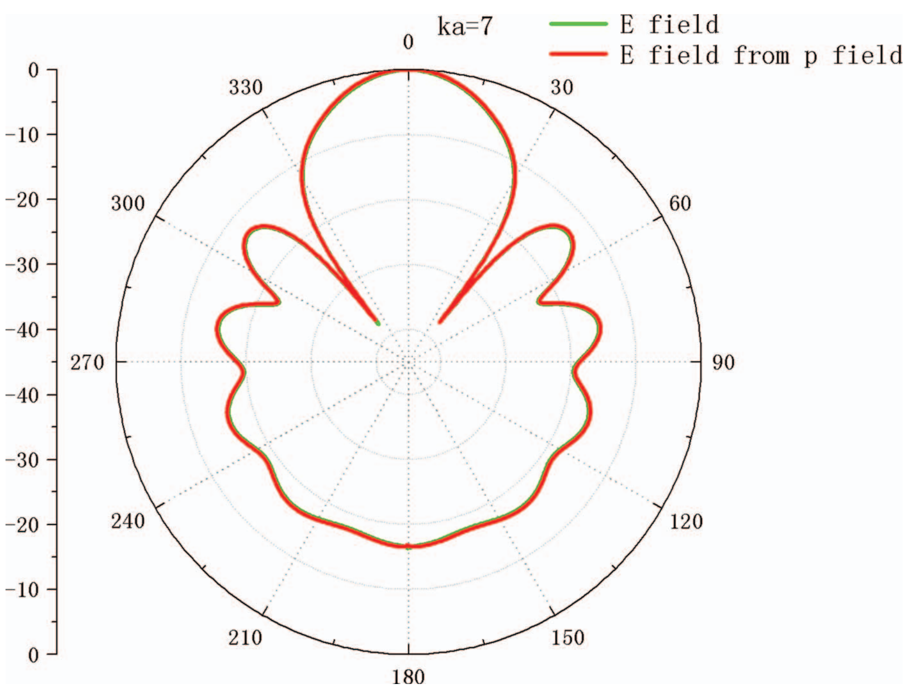

FIG. 10. The normalized field of electromagnetic scattering (a = 1.4 mm) at 238.73 GHz.

far field condition was derived. In addition, the scaling transformation of the sizes between two different spheres and the frequencies between different waves was discussed. With a co-simulation, the electromagnetic scattering derived from the result of simulation of the acoustic scattering is compared with the result of direct simulation of electric scattering, and a very similar curve is obtained, which greatly confirms the conversion formula.

ACKNOWLEDGMENTS

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