Problem 10

Solution:

(1)

$$\mathcal{L}(x, \lambda) = (x_1 - 1)^2 + (x_2 - 2)^2 - \lambda[(x_1 - 1)^2 - 5x_2]$$

KKT 条件为

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2(1 - \lambda)(x_1 - 1) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 2(x_2 - 2) + 5\lambda = 0$$

$$(x_1 - 1)^2 - 5x_2 = 0$$

解得

$$x^* = (1, 0)^T, \lambda = \frac{4}{5}$$

$$\mathcal{F}_1^* = \{d | d \neq 0, (0, -5)^T d = 0\}$$

可以设 $d = (m, 0)^T, m \neq 0$

$$W^* = \begin{pmatrix} \frac{2}{5} & 0 \\ 0 & 2 \end{pmatrix}$$

所以

$$d^T W^* d = \frac{2}{5} m^2 > 0$$

这个 KKT 点为最优解

(2)

$$\mathcal{L}(x, \lambda) = (x_1 + x_2)^2 + 2x_1 + x_2^2 - \lambda_1(4 - x_1 - 3x_2) - \lambda_2(3 - 2x_1 - x_2) - \lambda_3 x_1 - \lambda_4 x_2$$

KKT 条件为

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2(x_1 + x_2) + 2 + \lambda_1 + 2\lambda_2 - \lambda_3 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = w(x_1 + x_2) + 2x_2 = 3\lambda_1 + \lambda_2 - \lambda_4 = 0$$

$$\lambda_1(4 - x_1 - 3x_2) = \lambda_2(3 - 2x_1 - x_2) = \lambda_3 x_1 = \lambda_3 x_2 = 0$$

$$\lambda_i \geqslant 0, i = 1, 2, 3, 4$$

$$4 - x_1 - 3x_2 \geqslant 0$$

$$3 - 2x_1 - x_2 \geqslant 0$$

$$x_1 \geqslant 0, x_2 \geqslant 0$$

解得

$$x^* = (0, 0)^T, \lambda = (0, 0, 2, 0)^T$$

$$\mathcal{F}_1^* = \{d | d \neq 0, (1, 0)^T d = 0\}$$

可以设 $d = (0, m)^T, m \neq 0$

$$W^* = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$$

所以

$$d^T W^* d = 2m^2 > 0$$

这个 KKT 点为最优解

Problem 11

Solution:

(1)

$$\mathcal{L}(x, \lambda) = x_1^2 + 4x_2^2 + 16x_3^2 - \lambda(x_1 - 1)$$

KKT 条件为

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 8x_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_3} = 32x_3 = 0$$

$$x_1 - 1 = 0$$

解得

$$x^* = (1, 0, 0)^T, \lambda = 2$$

$$\mathcal{F}_1^* = \{d | d \neq 0, (1, 0, 0)^T d = 0\}$$

不妨设 $d = (0, m, n)^T, m, n \neq 0$

$$W^* = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 32 \end{pmatrix}$$

所以

$$d^T W^* d = 8m^2 + 32n^2 > 0$$

均为最优解

(2)

$$\mathcal{L}(x, \lambda) = x_1^2 + 4x_2^2 + 16x_3^2 - \lambda(x_1 x_2 - 1)$$

KKT 条件为

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 - \lambda x_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 8x_2 - \lambda x_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_3} = 32x_3 = 0$$

$$x_1 x_2 - 1 = 0$$

解得

$$x_1^* = (\sqrt{2}, \frac{1}{\sqrt{2}}, 0)^T, x_2^* = (-\sqrt{2}, -\frac{1}{\sqrt{2}}, 0)^T, \lambda = 4$$

$$\mathcal{F}_1^* = \{d | d \neq 0, (\frac{1}{\sqrt{2}}, \sqrt{2}, 0)^T d = 0\}$$

不妨设 $d = (2, -1, m)^T, m \neq 0$

$$W^* = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 8 & 0 \\ 0 & 0 & 32 \end{pmatrix}$$

所以

$$d^T W^* d = 32(1 + m^2) > 0$$

均为最优解

(3)

$$\mathcal{L}(x, \lambda) = x_1^2 + 4x_2^2 + 16x_3^2 - \lambda(x_1 x_2 x_3 - 1)$$

KKT 条件为

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 - \lambda x_2 x_3 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 8x_2 - \lambda x_1 x_3 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_3} = 32x_3 - \lambda x_1 x_2 = 0$$

$$x_1 x_2 x_3 - 1 = 0$$

解得

$$x_1^* = (2, 1, \frac{1}{2})^T, x_2^* = (2, -1, -\frac{1}{2})^T, x_3^* = (-2, 1, -\frac{1}{2})^T, x_4^* = (-2, -1, \frac{1}{2})^T, \lambda = 8$$

$$\mathcal{F}_1^* = \{d | d \neq 0, (x_2 x_3, x_1 x_3, x_1 x_2)^T d = 0\}$$

不妨设 $d = (m, n, q)^T, d \neq 0$

$$W^* = \begin{pmatrix} 2 & -8x_3 & -8x_2 \\ -8x_3 & 8 & -8x_1 \\ -8x_2 & -8x_1 & 32 \end{pmatrix}$$

所以，化简后

$$d^T W^* d = 2(2m \pm 4n)^2 \geqslant 0$$

当 $2m \pm 4n = 0$ 时取 0
  故均不为最优解

Problem 12

Solution:

(1)

$$\mathcal{L}(x, \lambda) = c_y x_1^{\frac{1}{4}} x_2^{\frac{1}{4}} - c_1(x_1 - k_1) - c_2(x_2 - k_2) - \lambda_1 x_1 - \lambda_2 x_2$$

KKT 条件为

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{4} c_y x_1^{-\frac{3}{4}} x_2^{\frac{1}{4}} - c_1 - \lambda_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{4} c_y x_1^{\frac{1}{4}} x_2^{-\frac{3}{4}} - c_2 - \lambda_2 = 0$$

$$\lambda_1 x_1 = 0$$

$$\lambda_2 x_2 = 0$$

$$x_1, x_2 > 0$$

$$\lambda_i \geqslant 0, i = 1, 2$$

解得

$$x^* = (\frac{1}{16} c_y^2 (c_1 c_2)^{-\frac{1}{2}} c_1^{-1}, \frac{1}{16} c_y^2 (c_1 c_2)^{-\frac{1}{2}} c_2^{-1})^T, \lambda = (0, 0)^T$$

(2)

$$x_1 = x_2 = \frac{1}{16}$$

$$y = \frac{1}{4}$$

(3)

$$x_1 = x_2 = \frac{1}{16}$$

$$y = \frac{1}{4}$$

与 (2) 没有区别，因为 y 的最优值、该问题的 KKT 点均与 k_1, k_2 的取值无关