数值最优化方法 **(Due: 13/4/20)**

Chapter 6 Problems

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Problem 10

Solution:

(1)

$$\mathcal{L}(x,\lambda)=(x_1-1)^2+(x_2-2)^2-\lambda[(x_1-1)^2-5x_2]$$

KKT 条件为

$$\frac{\partial\mathcal{L}}{\partial x_1}=2(1-\lambda)(x_1-1)=0$$

$$\frac{\partial\mathcal{L}}{\partial x_2}=2(x_2-2)+5\lambda=0$$

$$(x_1-1)^2-5x_2=0$$

解得

$$x^*=(1,0)^T,\lambda=\frac{4}{5}$$

$$\mathcal{F}_1^*=\{d|d\neq 0,(0,-5)^Td=0\}$$

可以设 $d=(m,0)^T,m\neq 0$

$$W^*=\begin{pmatrix}\frac{2}{5} & 0\\ 0 & 2\end{pmatrix}$$

所以

$$d^TW^*d=\frac{2}{5}m^2>0$$

这个 KKT 点为最优解

(2)

$$\mathcal{L}(x,\lambda)=(x_1+x_2)^2+2x_1+x_2^2-\lambda_1(4-x_1-3x_2)-\lambda_2(3-2x_1-x_2)-\lambda_3x_1-\lambda_4x_2$$

KKT 条件为

$$\frac{\partial\mathcal{L}}{\partial x_1}=2(x_1+x_2)+2+\lambda_1+2\lambda_2-\lambda_3=0$$

$$\frac{\partial\mathcal{L}}{\partial x_2}=w(x_1+x_2)+2x_2=3\lambda_1+\lambda_2-\lambda_4=0$$

$$\lambda_1(4-x_1-3x_2)=\lambda_2(3-2x_1-x_2)=\lambda_3x_1=\lambda_3x_2=0$$

$$\lambda_i\geqslant 0,i=1,2,3,4$$

$$4-x_1-3x_2\geqslant 0$$

$$3-2x_1-x_2\geqslant 0$$

$$x_1\geqslant 0,x_2\geqslant 0$$

解得

$$x^*=(0,0)^T,\lambda=(0,0,2,0)^T$$

$$\mathcal{F}_1^*=\{d|d\neq 0,(1,0)^Td=0\}$$

可以设 $d=(0,m)^T, m\neq 0$

$$W^* = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$$

所以

$$d^T W^* d = 2m^2 > 0$$

这个 KKT 点为最优解

Problem 11

Solution:

(1)

$$\mathcal{L}(x,\lambda) = x_1^2 + 4x_2^2 + 16x_3^2 - \lambda(x_1 - 1)$$

KKT 条件为

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 8x_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_3} = 32x_3 = 0$$
$$x_1 - 1 = 0$$

解得

$$x^* = (1,0,0)^T, \lambda = 2$$
$$\mathcal{F}_1^* = \{d | d \neq 0, (1,0,0)^T d = 0\}$$

不妨设 $d=(0,m,n)^T, m,n\neq 0$

$$W^* = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 32 \end{pmatrix}$$

所以

$$d^T W^* d = 8m^2 + 32n^2 > 0$$

均为最优解

(2)

$$\mathcal{L}(x,\lambda) = x_1^2 + 4x_2^2 + 16x_3^2 - \lambda(x_1 x_2 - 1)$$

KKT 条件为

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 - \lambda x_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 8x_2 - \lambda x_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_3} = 32x_3 = 0$$
$$x_1 x_2 - 1 = 0$$

解得

$$x_1^* = (\sqrt{2}, \frac{1}{\sqrt{2}}, 0)^T, x_2^* = (-\sqrt{2}, -\frac{1}{\sqrt{2}}, 0)^T, \lambda = 4$$

$$\mathcal{F}_1^* = \{d | d \neq 0, (\frac{1}{\sqrt{2}}, \sqrt{2}, 0)^T d = 0\}$$

不妨设 $d=(2,-1,m)^T, m\neq 0$

$$W^*=\begin{pmatrix}2 & -4 & 0\\ -4 & 8 & 0\\ 0 & 0 & 32\end{pmatrix}$$

所以

$$d^TW^*d=32(1+m^2)>0$$

均为最优解

(3)

$$\mathcal{L}(x,\lambda)=x_1^2+4x_2^2+16x_3^2-\lambda(x_1x_2x_3-1)$$

KKT 条件为

$$\frac{\partial\mathcal{L}}{\partial x_1}=2x_1-\lambda x_2x_3=0$$
$$\frac{\partial\mathcal{L}}{\partial x_2}=8x_2-\lambda x_1x_3=0$$
$$\frac{\partial\mathcal{L}}{\partial x_3}=32x_3-\lambda x_1x_2=0$$
$$x_1x_2x_3-1=0$$

解得

$$x_1^*=(2,1,\frac{1}{2})^T, x_2^*=(2,-1,-\frac{1}{2})^T, x_3^*=(-2,1,-\frac{1}{2})^T, x_4^*=(-2,-1,\frac{1}{2})^T, \lambda=8$$
$$\mathcal{F}_1^*=\{d|d\neq 0,(x_2x_3,x_1x_3,x_1x_2)^Td=0\}$$

不妨设 $d=(m,n,q)^T, d\neq 0$

$$W^*=\begin{pmatrix}2 & -8x_3 & -8x_2\\ -8x_3 & 8 & -8x_1\\ -8x_2 & -8x_1 & 32\end{pmatrix}$$

所以，化简后

$$d^TW^*d=2(2m\pm 4n)^2\geqslant 0$$

当 $2m\pm 4n=0$ 时取 0
故均不为最优解

$\boxed{\textbf{Problem 12}}$

Solution:

(1)

$$\mathcal{L}(x,\lambda)=c_yx_1^{\frac{1}{4}}x_2^{\frac{1}{4}}-c_1(x_1-k_1)-c_2(x_2-k_2)-\lambda_1x_1-\lambda_2x_2$$

KKT 条件为

$$\frac{\partial\mathcal{L}}{\partial x_1}=\frac{1}{4}c_yx_1^{-\frac{3}{4}}x_2^{\frac{1}{4}}-c_1-\lambda_1=0$$
$$\frac{\partial\mathcal{L}}{\partial x_2}=\frac{1}{4}c_yx_1^{\frac{1}{4}}x_2^{-\frac{3}{4}}-c_2-\lambda_2=0$$
$$\lambda_1x_1=0$$
$$\lambda_2x_2=0$$
$$x_1,x_2>0$$
$$\lambda_i\geqslant 0, i=1,2$$

解得

$$x^*=(\frac{1}{16}c_y^2(c_1c_2)^{-\frac{1}{2}}c_1^{-1},\frac{1}{16}c_y^2(c_1c_2)^{-\frac{1}{2}}c_2^{-1})^T,\lambda=(0,0)^T$$

(2)

$$x_1 = x_2 = \frac{1}{16}$$

$$y = \frac{1}{4}$$

(3)

$$x_1 = x_2 = \frac{1}{16}$$

$$y = \frac{1}{4}$$

与 (2) 没有区别，因为 y 的最优值、该问题的 KKT 点、约束条件和目标函数均与 k_1, k_2 的取值无关