Homework 2

Metric and Topological Spaces

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1 Problems

Problem 1.1 (Exercise 5.1)**.** Given points x, y, z in a metric space (X, d) prove that

$$|d(x, z) - d(y, z)| \leq d(x, y).$$

Solution 1.1.1. Let x, y, z be points in (X, d). Since it is a metric it follows the triangle inequality so:

$$d(x, z) \leq d(x, y) + d(y, z)$$

which implies that

$$d(x, z) - d(y, z) \leq d(x, y).$$

Using the triangle inequality again but now with x as middle point we can reason that:

$$\begin{aligned} & d(y, z) \leq d(y, x) + d(x, z) \\ \implies & d(y, z) - d(x, z) \leq d(y, x) \\ \implies & -(d(x, z) - d(y, z)) \leq d(x, y). \end{aligned}$$

We can now conclude that

$$|d(x, z) - d(y, z)| \leq d(x, y).$$

Problem 1.2 (Exercise 5.8)**.** Suppose that (X, d) is a metric space, $A \subseteq X$. Show that A is bounded iff there is some constant Δ such that $d(a, a') \leq \Delta$ for all $a, a' \in A$.

Solution 1.2.1. Let (X, d) be a metric space and let $A \subseteq X$. We want to prove:

$$\exists M : \exists c \in A : \forall x \in A : d(c, x) \leq M$$
$$\iff$$
$$\exists \Delta : \forall a{,}a' \in A, d(a, a') \leq \Delta.$$

Proof of $\impliedby$:
Let $M = \Delta$ and let c be any point of A.
Proof of $\implies$:
Let $\Delta = 2M$. Let $a, a' \in A$. Because of the triangle inequality we will yield:

$$\begin{aligned} & d(a, a') \\ \leq & d(a, c) + d(c, a') \\ \leq & M + M \\ = & 2M. \end{aligned}$$

Lemma 1.1 (For exercise 5.9)**.** If $A \subseteq B$ then $\sup A \leq \sup B$ (given that they exist).

Proof. Assume $\sup B < \sup A$. Per defintion the $\sup B$ must be an upper bound of all values in B and thus an upper bound of all values in A. Since $\sup B < \sup A$, there must exist a $c \in A$ such that $\sup B < c$, but this is a contradiction since $\sup B$

was supposed to be an upper bound for all values in A. $\square$

Problem 1.3 (Exercise 5.9)**.** Suppose that $A \subseteq B$ where B is a bounded subset of a metric space. Prove that A is bounded and $\operatorname{diam}(A) \leq \operatorname{diam}(B)$.

Proof. If B is bounded then by Exercise 5.8 there exists a Δ such that $d(a, a') \leq \Delta$ for all $a, a' \in B$. But this must also be true for A since for any $a, a' \in A$ it is also true that $a, a' \in B$. Since this requirement was equivalent with being bounded, it means A is bounded.
Let $S_d := \{d(x, y) : x, y \in S\}$.
We have that if $A \subseteq B$ then $A_d \subseteq B_d$, because if $D \in A$ then there exists $x, y \in A \subseteq B$ such that $D = d(x, y)$. And since $x, y \in B$ then $D \in B_d$.
From the lemma 1.1 we have that since $A_d \subseteq B_d$ it must be true that $\sup A_d \leq \sup B_d$, but this is the same as saying that $\operatorname{diam}(A) \leq \operatorname{diam}(B)$. $\square$

Problem 1.4 (Exercise 5.13)**.** Prove that a subsset of a metric space is open iff it is a union of open balls.

Proof. Let A be a subset of the metric space (X, d). We want to prove that

$$A \text{ is open}$$
$$\iff$$
$$\exists \mathcal{C} \subseteq \{ \text{ open subsets } \} : A = \bigcup_{U \in \mathcal{C}} U.$$

Proof of $\Longleftarrow$:
Let $x \in A$, then there exists an open U such that $x \in U \subseteq A$. And that means there exists an open ball B such that $x \in B \subseteq U \subseteq A$.
Proof of $\Longrightarrow$: Since A is open we can for each $x \in A$ find an open ball U_x. Consider the union $S = \bigcup_{x \in A} U_x$, since each $U_x \subseteq A$ we have that $S \subseteq A$. But since for all $x \in A$, $x \in S$ since $x \in U_x \subseteq A$. Thus we have that $A = \bigcup_{x \in A} U_x$ which means that A can be described as a union of open balls. $\square$

Problem 1.5 (Exercise 5.17)**.** Let (X, d) be a metric space and consider $X \times X$ as a metric space with metric d_1 of Exercise 5.16. Show that $d : X \times X \to \mathbb{R}$ is continuous.

Proof. Let $(x_0, y_0) \in X \times X$. We want to find an δ for each ϵ such that $d_1((x, y), (x_0, y_0)) < \delta$ implies $|d(x, y) - d(x_0, y_0)| < \epsilon$.
Let us rewrite $d_1((x, y), (x_0, y_0)) < \delta$ as $d(x, x_0) + d(y, y_0) < \delta$.
Let just $\delta < \epsilon$ so for example $\delta = \frac{\epsilon}{2}$. Then that means if $d(x, x_0) + d(y, y_0) < \delta$ then because of exercise 5.2 $|d(x, y) - d(x_0, y_0)| < d(x, x_0) + d(y, y_0) < \delta < \epsilon$.
This means that d is continuous on (x_0, y_0). And since we chose (x_0, y_0) arbitrary, this is true for all points in $X \times X$. $\square$

Problem 1.6 (Exercise 5.18)**.** Suppose that in a metric space X we have $B_r(x) = B_s(y)$ for som $x, y \in X$ and some positive real numbers r, s. Is $x = y$? Is $r = s$?

Solution 1.6.1. Counterexample for $x = y$ and $r = s$.
Let $(\mathbb{R}, d)$ be a metric space where d is the discrete metric. Then $B_2(2) = B_3(3) = \mathbb{R}$.