Lecture 9 notes

Statistical Methods

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1 MVUE

Minimal Variance Unbiased Estimator (MVUE):

Theorem 1.1 (Lehman-Schally). Let $T(X_1, \dots, X_n)$ be an unbiased estimator of $g(\theta)$.

$$E(T(X_1, \dots, X_n)) = g(\theta)$$

with finite variance. T is MVUE for θ iff $E(T(X_1, \dots, X_n) = S(X_1, \dots, X_n)) = 0$ for every S s.t. $E(S(X_1, \dots, X_n)) = 0$, $\mathrm{Var}(S) < \infty$, we can use "special estimators" ex BLUE (Best linear unbiased estimator) in lin reg MSE is usually BLUE. CDF:

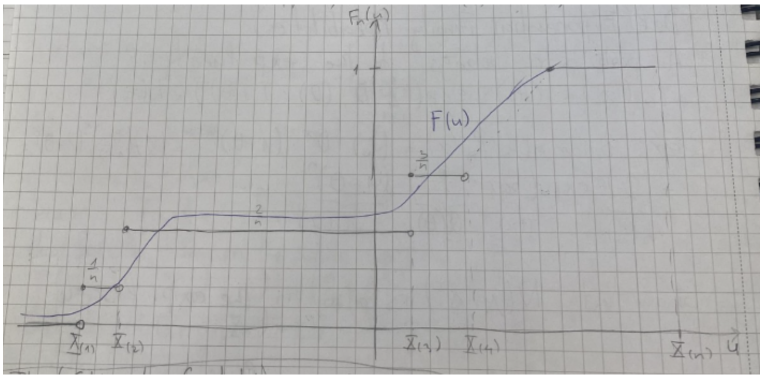
$$F(x) = P(X \leq x)$$

Definition 1.1 (Empirical cumulative distribution function).

$$F_n(u, (X_1, \dots, X_n)) = n^{-1} \sum_{i=0}^{n} I_{(-u,u)}(X_i) = \begin{cases} 0, & u \leq X_{(1)} \\ \frac{k}{n} X_{(k)} \leq u \leq X_{k+1} & \\ 1 & u > X_{(n)}. \end{cases}$$

2 Order statistics

$$X_1, \dots, X_n \Rightarrow X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$$



Theorem 2.1 (Glivenko-Conteli).

$$F_n \to F$$

as $n \to \infty$ "in some sense".

$$X_i \sim \mathrm{N}(\mu, \sigma^2)$$

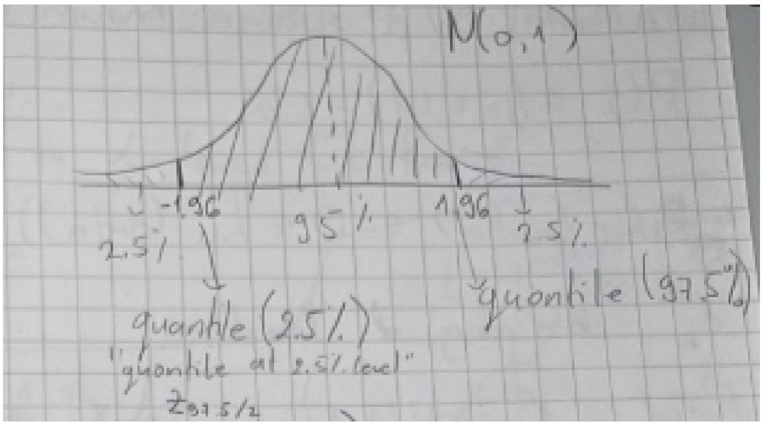
$$\overline{X} \sim N(\mu\frac{\sigma^2}{n}$$

$$\frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Show:

$$\overline{X} \pm \epsilon$$

$$\begin{aligned}
P(-1.96 \leq \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \leq 1.96) &= 95\% / - \frac{\sigma}{\sqrt{n}} \\
P(-1.96\frac{\sigma}{\sqrt{n}} \leq \overline{X} - \mu \leq 1.96\frac{\sigma}{\sqrt{n}}) &= 95\% / - \overline{X} \\
P(-1.96\frac{\sigma}{\sqrt{n}} - \overline{X} \leq -\mu \leq 1.96\frac{\sigma}{\sqrt{n}} - \overline{X}) &= 95\% / - (-1) \\
P(\overline{X} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \overline{X} + 1.96\frac{\sigma}{\sqrt{n}}) &= 95\%.
\end{aligned}$$

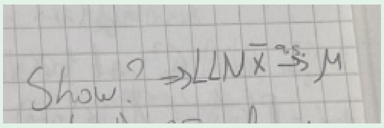


> **Definition 2.1** (Quantile). z_α is defined as to satisfy the following equation:
>
> $$P(z \geq z_\alpha) = \alpha.$$

> **Definition 2.2** (Confidence Interval). A RANDOM interval (L, U)
> $[L, U]$ is a confidence interval for parameter θ with confidence level α if
>
> $$P(L \leq \theta \leq U) = 1 - \alpha$$
>
> Confidence interval (L, U) covers θ with $1 - \alpha$ probability.

$$\overline{X} \pm z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$$

is a α-level confidence interval for μ.

Problem 2.1. 

Problem 2.2. Show $(-\infty, \overline{X} + Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}})$ and $(\overline{X} - Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}, \infty)$ are also $(1-\alpha)$-confidence intervals for μ.

Problem 2.3. Show sizes of confidence intervals.

"Statistic:" $H_\theta = h(\text{ data }, \theta)$, with a known distribution.

$P(L \leq H_\theta \leq U) = 1 - \alpha$.

Fact: If $X_i \sim \mathrm{N}(0,1)$ and $Y = X_1^2 + \cdots + X_n^2$ then $Y \sim \chi_k^2$ where k is the degrees of freedom.

Let $X_i \sim \mathrm{N}(\mu, \sigma^2)$. Since $\frac{X_i - \mu}{\sigma} \sim \mathrm{N}(0,1)$ that means

$$\sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2} \sim \chi_n^2.$$

"More difficult:"

$$\sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{\sigma^2} \sim \chi_{n-1}^2.$$

Definition 2.3 (Sample Variance).

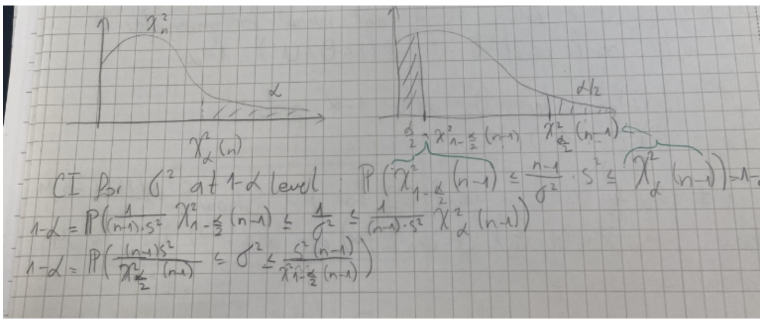
$$s^2 = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1}$$

Thus:

Theorem 2.2.

$$\frac{n-1}{\sigma^2} s^2 \sim \chi_{n-1}^2.$$

Quantile of χ_n^2 of $1-\alpha$ level $\chi_\alpha^2(n)$ is a point s.t. $P(Y \geq \chi_\alpha^2(n)) = \alpha$, $Y \sim X_n^2$



Theorem 2.3. Let $X_1, \ldots, X_n$ be on i.i.d sample from $\mathrm{N}(\mu, \sigma^2)$, where μ and δ are

unknow, then,

$$\left[\frac{(n-1)\cdot s^2}{\chi_{\frac{\alpha}{2}}(n-1)}, \frac{(n-1)s^2}{\chi_{1-\frac{\alpha}{2}}(n-1)}\right]$$

is an α-level confidence interval for σ^2.

Let $X_i \sim \mathrm{N}(\mu, \sigma^2)$ then

$$\frac{\overline{X}-\mu}{s/\sqrt{n}} \sim t(n-1)$$

where $s = \sqrt{s^2}$.

Problem 2.4. Show $\overline{X} \pm t_{\alpha/2}(n-1) \cdot \frac{s}{\sqrt{n}}$ is a $1-\alpha$ level quantile for μ.

3 Statistical Hypothesis Testing

$$\begin{aligned} H_0 :& \mu = \mu_0 \\ H_1 :& \mu \neq \mu_0 \end{aligned}$$

Construct a test statistic

$$T = T(X_1, \ldots, X_n | H_0) \sim \tilde{f}(\mu_0).$$

$$X_1, \ldots, X_n$$

is a random sample

$$T(X_1, \ldots, X_n) \in \text{ critical set } \theta_1$$

If $T \in \theta_1$ then we reject the null hypothesis.
If $T \notin \theta_1$ then we fail to reject the null hypothesis.

$$P(T \in \theta_1 | H_0 \text{ is true }) = \alpha.$$