Homework list 2

Complex Analysis

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1 Problems

Problem 1.1 (Exercise 1)**.** Mark each of the following statements as true or false.

1. Every power series $\Sigma a_n z^n$ converges for some $z \in \mathbb{C}$.
2. The radius of convergence of $\Sigma a_n z^n$ is at most equal to $\max\{|a_n| : n \in \mathbb{N}\}$.
3. $\exp : \mathbb{C} \to \mathbb{C}^\times$ is a surjective function.
4. $\log_\alpha z \neq \log_\beta z$ for all real numbers $\alpha \neq \beta$ and all $z \in \mathbb{C}$
5. $\log(zw) = \log z + \log w$ for all $z, w \in U_{-1}$.

Solution 1.1.1.

1. True (let $z = 0$)
2. False (consider the exponential function then $\max\{|a_n| : n \in \mathbb{N}\} = 1$, but the radius of convergence is infinite)
3. True
4. False
5. True

Problem 1.2 (Exercise 2)**.**

1. Determine the radius R of convergence for $\sum_{n=0}^{\infty} (-2)^n z^n$. Show that for $z \in D_R(0)$, we have $\sum_{n=0}^{\infty} (-2)^n z^n = \frac{1}{1+2z}$.
2. Determine the radius R of convergence for

$$1 + \sum_{n \geq 2, \text{ even}} \frac{i^n 2^{n-1}}{n!} z^n$$

and show that it equals $(\cos z)^2$ as a power series. Show that $(\cos z)^2 + (\sin z)^2 = 1$ as power series.

Solution 1.2.1. We have that

$$\begin{aligned} R &= \frac{1}{\limsup |(-2)^n|^{\frac{1}{n}}} \\ &= \frac{1}{2}. \end{aligned}$$

Let $z \in D_{\frac{1}{2}}(0)$ and define $x = -2z$ we then have that $x \in D_1(0)$. Thus we can expand as the following:

$$\begin{aligned}\frac{1}{1+2z} &= \frac{1}{1-x} \\ &= \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} (-2z)^n \\ &= \sum_{n=0}^{\infty} (-2) z^n.\end{aligned}$$

Lemma 1.1.

$$(m!)^{\frac{1}{m}} \geq$$

Solution 1.2.2. We have that

$$a_n = \frac{i^n 2^{n-1}}{n!}$$

(technically we want to investigate a_{2n} but we'll mention it again later).
First we have that

$$|a_n| = \frac{2^{n-1}}{n!}.$$

From the lecture we saw that we can reason as following:

$$(n!)^{\frac{1}{n}} \geq \left(\left(\frac{n}{2}\right)^n\right)^{\frac{1}{n}} = \frac{n}{2}.$$

Thus we have that

$$\begin{aligned}|a_n|^{\frac{1}{n}} &= \frac{2^{\frac{n-1}{n}}}{(n!)^{\frac{1}{n}}} \\ &\leq \frac{2^{\frac{n-1}{n}}}{n/2} \\ &= \frac{2^{\frac{n-1}{n}+1}}{n} \\ &= \frac{2^{\frac{2n-1}{n}}}{n} \\ &= \frac{2^{2-\frac{1}{n}}}{n} \\ &= \frac{4 \cdot 2^{\frac{1}{n}}}{n}.\end{aligned}$$

The last line converges to 0 and since $0 \leq |a_n|^{\frac{1}{n}}$ we have that $|a_n|^{\frac{1}{n}}$ converges to 0 which means the subsequence $|a_{2n}|^{\frac{1}{2n}}$ also converges to zero, which also means that the upper limit of a_{2n} is zero. Thus we get the radius of convergence to be $\frac{1}{0}$ which we defined to be all of $\mathbb{C}$.

We have that

$$\cos z = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} z^{2m}.$$

Let $n = 2m$ then we have that

$$\cos z = \sum_{n \geq 0, n \text{ even}}^{\infty} \frac{(-1)^{\frac{n}{2}}}{n!} z^n$$

Using the Cauchy Multiplication theorem we get that

$$\begin{aligned}
(\cos z)^2 &= (\cos z)(\cos z) \\
&= \sum_{i>0,\text{ i } even}^{\infty} \sum_{k \geq 0}^{\infty} \frac{(-1)^{\frac{i}{2}}}{i!} z^i \cdot \frac{(-1)^{\frac{i-k}{2}}}{(i-k)!} z^{i-k} \\
&= \sum_{i>0,\text{ i } even}^{\infty} \sum_{k \geq 0}^{\infty} \frac{(-1)^{\frac{2i-k}{2}}}{i!\,(i-k)!} z^{2i-k}
\end{aligned}$$