Homework 4

Metric and Topological Spaces

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1 Exercises

Problem 1.1 (7.2)**.** Give an example of two topologies $\mathcal{T}_1, \mathcal{T}_2$ on the same set such that neither contains the other.

Solution 1.1.1. Sierpinski spaces will provide an example. Let $X = \{0, 1\}$. Let $\mathcal{T}_1 = \{\emptyset, \{0\}, \{0, 1\}\}$ and let $\mathcal{T}_2 = \{\emptyset, \{1\}, \{0, 1\}\}$.
Both $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, but neither contains each other since $\mathcal{T}_1 \cap \mathcal{T}_2 = \{\emptyset, \{0, 1\}\}$ is not equal to either $\mathcal{T}_1$ or $\mathcal{T}_2$.

Problem 1.2 (7.3)**.** Show that the intersection of two topologies on the same set X is also a topology on X, but that their union may or may not be a topology. Does the first result extent to the intersection of an arbitrary family of topologies on X?

Solution 1.2.1. Let $\{\mathcal{T}_i\}_{i\in I}$ be any collection of topologies on X and define

$$\mathcal{T} = \left(\bigcap_{i\in I} \mathcal{T}_i\right).$$

Consider any $j \in I$. Obviously $\mathcal{T} \subseteq \mathcal{T}_j$. Since $\mathcal{T}_j$ is colection of subsets of X that means $\mathcal{T}$ is also a collection of subsets of X.
Since for all $i \in I$ $\mathcal{T}_i$ is a topology that means $X, \emptyset \in \mathcal{T}_i$ for all $i \in I$ which means that $X, \emptyset \in \mathcal{T}$. Thus the first axiom of a topological space is fullfilled.
Let $U, V \in \mathcal{T}$. Since $\mathcal{T} \subseteq \mathcal{T}_i$ for all $i \in I$ we have that $U, V \in \mathcal{T}_i$ for all $i \in I$. Since $\mathcal{T}_i$ are topologies (on X) for all $i \in I$, we have from the second axiom that $U \cap V \in \mathcal{T}_i$ for all $i \in I$ which means that

$$U \cap V \in \left(\bigcap_{i\in I} \mathcal{T}_i\right) = \mathcal{T}.$$

Thus $\mathcal{T}$ fullfills the second axiom of being a topological space.
Let $\{U_j\}_{j\in J}$ be a collection of sets $U_j \in \mathcal{T}$. This must mean that for all $j \in J$, $U_j \in \mathcal{T}_i$ for all $i \in I$. From the third axiom of topological spaces this implies that $(\bigcup_{j\in J} U_j) \in \mathcal{T}_i$ for all $i \in I$ which means that

$$\left(\bigcup_{j\in J} U_i\right) \in \left(\bigcap_{i\in I} \mathcal{T}_i\right) = \mathcal{T}.$$

thus we have shown that $\mathcal{T}$ fullfills the third axiom of being a topological space on X. We have thus shown all axioms and $\mathcal{T}$ is a topological space on X. Which mean the intersection of an arbitrary collection of topologies on X is a topology on X. To prove the case that the intersection of 2 topolgies is a topology we let $I = \{1, 2\}$.

Problem 1.3 (8.1)**.** Prove proposition 8.6.

Solution 1.3.1. a: We want to prove that $id_X : X \to X$ is continuous. Let $U \subseteq X$ be open. Then $id_X^{-1}(U) = U$ which is open by assumption.
b: Let $c \in Y$ and define $f : X \to Y$ as $f(x) = c$ for all $x \in X$. Let $U \subseteq Y$ be open. If U contains c then $f^{-1}(U) = X$ which is open. If U does not contain c, then there are

no values of X that get mapped to U (since all go to c), which means that $f^{-1}(U) = \emptyset$ which is also open. Thus we have shown that given that U is open $f^{-1}(U)$ will also be open.
c: Let $U \subseteq Y$ be open. Since $f^{-1}(U)$ is a subset of X and all subsets of X are open under the discrete topology, then $f^{-1}(U)$ will be open.
d: Let $U \subseteq Y$ be open. This means $U = \emptyset$ or $U = Y$. Assume $U = \emptyset$, then $f^{-1}(U) = \emptyset$ which is open in X. Assume $U = Y$ then $f^{-1}(U) = X$ which is open in X.

Problem 1.4 (10.5)**.** Suppose that $(A, \mathcal{T}_A)$ is a subspace of space $(X, \mathcal{T})$ and that $V \subseteq X$ is closed in X. Prove that $V \cap A$ is closed in $(A, \mathcal{T}_A)$.

Solution 1.4.1. If V is closed in X then $X \backslash V$ is open in X. Since A was a subspace then that means that $A \cap (X \backslash V)$ is open in A. According to exercise 2.2 this is equal to $A \backslash (V \cap A)$ which is thus open. The complement of this in A is $V \cap A$, which is thus closed.

Problem 1.5 (10.6)**.** Suppose that $(A, \mathcal{T}_A)$ is a subspace of a topological space $(X, \mathcal{T})$ and that $W \subseteq A$.
a: If W is open in A, and A is open in X, prove W is open in X.
b: If W is closed in A and A is closed in X the prove W is closed in X.

Solution 1.5.1. a: If W is open in A then there exists an open set U of X such that $U \cap A = W$. If A is open in X then the second topology axiom says $U \cap A = W$ is open in X.
b:
If W is closed in A there exists an open U in A such that $A \backslash U = W$. There exists an open V in X such that $U = V \cap A$. Thus $W = A \backslash (V \cap A)$, and we can rewrite this with exercise 2.2 as $W = A \cap (X \backslash V)$ so if A is closed in X then this is a intersection of closed sets in X which Proposition 9.4 says is also a closed set.