## PURE MATHEMATICS

*Mensuration*

Volume of sphere = $\frac{4}{3}\pi r^3$

Surface area of sphere = $4\pi r^2$

Volume of cone or pyramid = $\frac{1}{3} \times \text{base area} \times \text{height}$

Area of curved surface of cone = $\pi r \times \text{slant height}$

Arc length of circle = $r\theta$ ($\theta$ in radians)

Area of sector of circle = $\frac{1}{2}r^2\theta$ ($\theta$ in radians)

*Algebra*

For the quadratic equation $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For an arithmetic series:

$$u_n = a + (n-1)d, \qquad S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\{2a + (n-1)d\}$$

For a geometric series:

$$u_n = ar^{n-1}, \qquad S_n = \frac{a(1-r^n)}{1-r} \quad (r \neq 1), \qquad S_\infty = \frac{a}{1-r} \quad (|r| < 1)$$

Binomial series:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \dots + b^n$$, where $n$ is a positive integer

and $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$, where $n$ is rational and $|x| < 1$

*Trigonometry*

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1, \qquad 1 + \tan^2\theta \equiv \sec^2\theta, \qquad \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$$

$$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

$$\sin 2A \equiv 2\sin A\cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

Principal values:

$$-\tfrac{1}{2}\pi \leqslant \sin^{-1}x \leqslant \tfrac{1}{2}\pi, \qquad 0 \leqslant \cos^{-1}x \leqslant \pi, \qquad -\tfrac{1}{2}\pi < \tan^{-1}x < \tfrac{1}{2}\pi$$

*Differentiation*

| $\mathbf{f(x)}$ | $\mathbf{f'(x)}$ |
|---|---|
| $x^n$ | $nx^{n-1}$ |
| $\ln x$ | $\frac{1}{x}$ |
| $e^x$ | $e^x$ |
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $\tan x$ | $\sec^2 x$ |
| $\sec x$ | $\sec x\tan x$ |
| $\operatorname{cosec} x$ | $-\operatorname{cosec} x\cot x$ |
| $\cot x$ | $-\operatorname{cosec}^2 x$ |
| $\tan^{-1}x$ | $\frac{1}{1+x^2}$ |
| $uv$ | $v\frac{du}{dx} + u\frac{dv}{dx}$ |
| $\frac{u}{v}$ | $\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ |

If $x = \mathrm{f}(t)$ and $y = \mathrm{g}(t)$ then $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

*Integration*

(Arbitrary constants are omitted; $a$ denotes a positive constant.)

| $f(x)$ | $\int f(x)\,dx$ | |
| --- | --- | --- |
| $x^n$ | $\dfrac{x^{n+1}}{n+1}$ | $(n \neq -1)$ |
| $\dfrac{1}{x}$ | $\ln\lvert x\rvert$ | |
| $e^x$ | $e^x$ | |
| $\sin x$ | $-\cos x$ | |
| $\cos x$ | $\sin x$ | |
| $\sec^2 x$ | $\tan x$ | |
| $\dfrac{1}{x^2+a^2}$ | $\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$ | |
| $\dfrac{1}{x^2-a^2}$ | $\dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert$ | $(x > a)$ |
| $\dfrac{1}{a^2-x^2}$ | $\dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert$ | $(\lvert x\rvert < a)$ |

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert$$

*Vectors*

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$$

## FURTHER PURE MATHEMATICS

*Algebra*

Summations:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Maclaurin's series:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^r}{r!} f^{(r)}(0) + \ldots$$

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \qquad \text{(all } x\text{)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \qquad (-1 < x \leqslant 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^r \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 \leqslant x \leqslant 1)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 < x < 1)$$

*Trigonometry*

If $t = \tan\frac{1}{2}x$ then:

$$\sin x = \frac{2t}{1+t^2} \quad \text{and} \quad \cos x = \frac{1-t^2}{1+t^2}$$

*Hyperbolic functions*

$$\cosh^2 x - \sinh^2 x \equiv 1, \qquad \sinh 2x \equiv 2\sinh x \cosh x, \qquad \cosh 2x \equiv \cosh^2 x + \sinh^2 x$$

$$\sinh^{-1} x \equiv \ln\{x + \sqrt{x^2+1}\}$$

$$\cosh^{-1} x \equiv \ln\{x + \sqrt{x^2-1}\} \qquad (x \geqslant 1)$$

$$\tanh^{-1} x \equiv \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad (|x| < 1)$$

*Differentiation*

| $\mathrm{f}(x)$ | $\mathrm{f}'(x)$ |
|---|---|
| $\sin^{-1} x$ | $\dfrac{1}{\sqrt{1-x^2}}$ |
| $\cos^{-1} x$ | $-\dfrac{1}{\sqrt{1-x^2}}$ |
| $\sinh x$ | $\cosh x$ |
| $\cosh x$ | $\sinh x$ |
| $\tanh x$ | $\operatorname{sech}^2 x$ |
| $\sinh^{-1} x$ | $\dfrac{1}{\sqrt{1+x^2}}$ |
| $\cosh^{-1} x$ | $\dfrac{1}{\sqrt{x^2-1}}$ |
| $\tanh^{-1} x$ | $\dfrac{1}{1-x^2}$ |

*Integration*

(Arbitrary constants are omitted; $a$ denotes a positive constant.)

| $\mathrm{f}(x)$ | $\int \mathrm{f}(x)\,\mathrm{d}x$ | |
|---|---|---|
| $\sec x$ | $\ln\lvert\sec x+\tan x\rvert = \ln\lvert\tan(\tfrac{1}{2}x+\tfrac{1}{4}\pi)\rvert$ | $(\lvert x\rvert<\tfrac{1}{2}\pi)$ |
| $\operatorname{cosec} x$ | $-\ln\lvert\operatorname{cosec} x+\cot x\rvert = \ln\lvert\tan(\tfrac{1}{2}x)\rvert$ | $(0<x<\pi)$ |
| $\sinh x$ | $\cosh x$ | |
| $\cosh x$ | $\sinh x$ | |
| $\operatorname{sech}^2 x$ | $\tanh x$ | |
| $\dfrac{1}{\sqrt{a^2-x^2}}$ | $\sin^{-1}\left(\dfrac{x}{a}\right)$ | $(\lvert x\rvert<a)$ |
| $\dfrac{1}{\sqrt{x^2-a^2}}$ | $\cosh^{-1}\left(\dfrac{x}{a}\right)$ | $(x>a)$ |
| $\dfrac{1}{\sqrt{a^2+x^2}}$ | $\sinh^{-1}\left(\dfrac{x}{a}\right)$ | |

## MECHANICS

*Uniformly accelerated motion*

$v = u + at$, $s = \frac{1}{2}(u+v)t$, $s = ut + 12at2$, – $v^2 = u^2 + 2as$

## FURTHER MECHANICS

*Motion of a projectile*

Equation of trajectory is:

$y = x \tan\theta -$ $\frac{gx2}{2V\cos 2}$

$\theta$

*Elastic strings and springs*

$T\lambda x E \lambda x 2 =, =$

$l2l$

*Motion in a circle*

For uniform circular motion, the acceleration is directed towards the centre and has magnitude

$\omega 2r$ or $v$ $^2$

$r$

*Centres of mass of uniform bodies*

Triangular lamina: 2

3 along median from vertex

Solid hemisphere of radius $r$: 3

8$r$ from centre

Hemispherical shell of radius $r$: 1

2$r$ from centre

Circular arc of radius $r$ and angle 2: $r$sin from centre $\alpha$ $\frac{\alpha}{\alpha}$

Circular sector of radius $r$ and angle 2 $\alpha$ $\frac{2r\sin\alpha}{3\alpha}$ from centre

Solid cone or pyramid of height $h$: $\frac{3}{4}h$ from vertex

## PROBABILITY & STATISTICS

*Summary statistics*

For ungrouped data:

$$\bar{x} = \frac{\Sigma x}{n}, \qquad \text{standard deviation} = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2}$$

For grouped data:

$$\bar{x} = \frac{\Sigma xf}{\Sigma f}, \qquad \text{standard deviation} = \sqrt{\frac{\Sigma(x-\bar{x})^2 f}{\Sigma f}} = \sqrt{\frac{\Sigma x^2 f}{\Sigma f} - \bar{x}^2}$$

*Discrete random variables*

$$\mathrm{E}(X) = \Sigma xp, \qquad \mathrm{Var}(X) = \Sigma x^2 p - \{\mathrm{E}(X)\}^2$$

For the binomial distribution $\mathrm{B}(n, p)$:

$$p_r = \binom{n}{r} p^r (1-p)^{n-r}, \qquad \mu = np, \qquad \sigma^2 = np(1-p)$$

For the geometric distribution Geo($p$):

$$p_r = p(1-p)^{r-1}, \qquad \mu = \frac{1}{p}$$

For the Poisson distribution $\mathrm{Po}(\lambda)$

$$p_r = \mathrm{e}^{-\lambda}\frac{\lambda^r}{r!}, \qquad \mu = \lambda, \qquad \sigma^2 = \lambda$$

*Continuous random variables*

$$\mathrm{E}(X) = \int x\,\mathrm{f}(x)\,\mathrm{d}x, \qquad \mathrm{Var}(X) = \int x^2\,\mathrm{f}(x)\,\mathrm{d}x - \{\mathrm{E}(X)\}^2$$

*Sampling and testing*

Unbiased estimators:

$$\bar{x} = \frac{\Sigma x}{n}, \qquad s^2 = \frac{\Sigma(x-\bar{x})^2}{n-1} = \frac{1}{n-1}\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)$$

Central Limit Theorem:

$$\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

Approximate distribution of sample proportion:

$$\mathrm{N}\left(p, \frac{p(1-p)}{n}\right)$$

## FURTHER PROBABILITY & STATISTICS

*Sampling and testing*

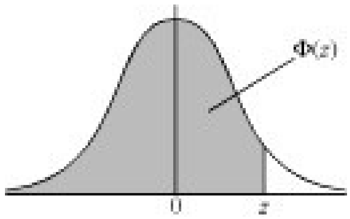
Two-sample estimate of a common variance:

$$s^2 = \frac{\Sigma(x_1 - \bar{x}_1)^2 + \Sigma(x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}$$

*Probability generating functions*

$\mathrm{G}_X(t) = \mathrm{E}(t^X)$, $\quad \mathrm{E}(X) = \mathrm{G}'_X(1)$, $\quad \mathrm{Var}(X) = \mathrm{G}''_X(1) + \mathrm{G}'_X(1) - \{\mathrm{G}'_X(1)\}^2$

**THE NORMAL DISTRIBUTION FUNCTION**




If $Z$ has a normal distribution with mean 0 and variance 1, then, for each value of $z$, the table gives the value of $\Phi(z)$, where

$\Phi(z) = P(Z$

$\leqslant z)$. For negative values of $z$, use $\Phi(-z) = 1 - \Phi(z)$.

| z | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 5 6 ADD 16 20 24 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 | 4 | 8 | 12 | 16 20 24 | 2 | 3 | 3 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 | 4 | 8 | 12 | 15 19 23 | 8 | 2 | 6 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 | 4 | 8 | 12 | 15 19 22 | 2 | 3 | 3 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 | 4 | 7 | 11 | 14 18 22 | 8 | 2 | 6 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 | 4 | 7 | 11 | 14 17 20 | 2 | 3 | 3 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 | 3 | 7 | 10 | 13 16 19 | 7 | 1 | 5 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 | 3 | 7 | 10 | 12 15 18 | 2 | 3 | 3 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 | 3 | 6 | 9 | 11 14 16 | 6 | 0 | 4 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 | 3 | 5 | 8 | 10 13 15 | 2 | 2 | 3 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 | 3 | 5 | 8 | 9 12 14 | 5 | 9 | 2 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 | 2 | 5 | 7 | 8 10 12 | 2 4 | 2 7 | 3 1 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 | 2 | 4 | 6 | 7 9 11 | 2 | 2 | 2 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 | 2 | 4 | 6 | 6 8 10 | | | |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 | 2 | 3 | 5 | 6 7 8 | 3 | 6 | 9 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 | 1 | 3 | 4 | 5 6 7 | 2 1 | 2 4 | 2 7 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 | 1 | 2 | 4 | 4 5 6 | 1 | 2 | 2 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 | 1 | 2 | 3 | 4 4 5 | 9 | 2 | 5 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 | 1 | 2 | 3 | 3 4 4 | 1 | 2 | 2 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 | 1 | 1 | 2 | 2 3 4 | 8 | 0 | 3 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 | 1 | 1 | 2 | 2 2 3 | 1 6 | 1 9 | 2 1 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 | 0 | 1 | 1 | 2 2 2 | 1 | 1 | 1 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 | 0 | 1 | 1 | 1 2 2 | | | |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 | 0 | 1 | 1 | 1 1 2 | 4 | 6 | 8 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 | 0 | 1 | 1 | 1 1 1 | 1 3 | 1 5 | 1 7 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 | 0 | 0 | 1 | 1 1 1 | 1 | 1 | 1 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 | 0 | 0 | 0 | 0 1 1 | 1 | 3 | 4 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 | 0 | 0 | 0 | 0 0 1 | 1 0 | 1 1 | 1 3 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 | 0 | 0 | 0 | 0 0 0 | | | |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 | 0 | 0 | 0 | 0 0 0 | 8 | 1 | 1 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 | 0 | 0 | 0 | | 7 | 0 | 1 |

6 8 9
5 7 8
4 6 6
5 5
3 4 4
3 4 4
2 3 3
2 2 2
1 2 2
2 2
1
1 1 1
1 1 1
1 1 1
0 1 1
0 0 0

**Critical values for the normal distribution**

If $Z$ has a normal distribution with mean 0 and variance 1, then, for each value of $p$, the table gives the value of $z$ such that

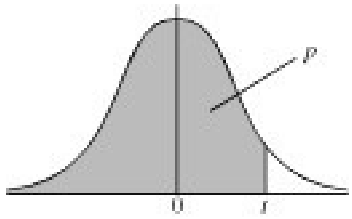
$P(Z$

$\leqslant z) = p$.

| p | 0.75 | 0.90 | 0.95 | 0.975 | 0.99 | 0.995 | 0.9975 | 0.999 | 0.9995 |
|---|---|---|---|---|---|---|---|---|---|
| z | 0.674 | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 | 2.807 | 3.090 | 3.291 |

## CRITICAL VALUES FOR THE *t*-DISTRIBUTION

If $T$ has a $t$-distribution with $\nu$ degrees of freedom, then, for each pair of values of $p$ and $\nu$, the table gives the value of $t$ such that:

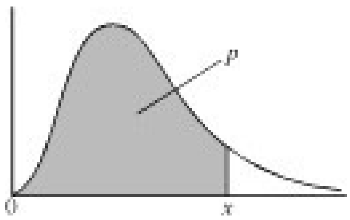



$P(T \leqslant t) = p.$

| $p$ | 0.75 | 0.90 | 0.95 | 0.975 | 0.99 | 0.995 | 0.9975 | 0.999 | 0.9995 |
|---|---|---|---|---|---|---|---|---|---|
| $\nu$ = 1 | 1.000 | 3.078 | 6.314 | 12.71 | 31.82 | 63.66 | 127.3 | 318.3 | 636.6 |
| 2 | 0.816 | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 | 14.09 | 22.33 | 31.60 |
| 3 | 0.765 | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 | 7.453 | 10.21 | 12.92 |
| 4 | 0.741 | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 | 5.598 | 7.173 | 8.610 |
| 5 | 0.727 | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 | 4.773 | 5.894 | 6.869 |
| 6 | 0.718 | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 | 4.317 | 5.208 | 5.959 |
| 7 | 0.711 | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 | 4.029 | 4.785 | 5.408 |
| 8 | 0.706 | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 | 3.833 | 4.501 | 5.041 |
| 9 | 0.703 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 | 3.690 | 4.297 | 4.781 |
| 10 | 0.700 | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 | 3.581 | 4.144 | 4.587 |
| 11 | 0.697 | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 | 3.497 | 4.025 | 4.437 |
| 12 | 0.695 | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 | 3.428 | 3.930 | 4.318 |
| 13 | 0.694 | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 | 3.372 | 3.852 | 4.221 |
| 14 | 0.692 | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 | 3.326 | 3.787 | 4.140 |
| 15 | 0.691 | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 | 3.286 | 3.733 | 4.073 |
| 16 | 0.690 | 1.337 | 1.746 | 2.120 | 2.583 | 2.921 | 3.252 | 3.686 | 4.015 |
| 17 | 0.689 | 1.333 | 1.740 | 2.110 | 2.567 | 2.898 | 3.222 | 3.646 | 3.965 |
| 18 | 0.688 | 1.330 | 1.734 | 2.101 | 2.552 | 2.878 | 3.197 | 3.610 | 3.922 |
| 19 | 0.688 | 1.328 | 1.729 | 2.093 | 2.539 | 2.861 | 3.174 | 3.579 | 3.883 |
| 20 | 0.687 | 1.325 | 1.725 | 2.086 | 2.528 | 2.845 | 3.153 | 3.552 | 3.850 |
| 21 | 0.686 | 1.323 | 1.721 | 2.080 | 2.518 | 2.831 | 3.135 | 3.527 | 3.819 |
| 22 | 0.686 | 1.321 | 1.717 | 2.074 | 2.508 | 2.819 | 3.119 | 3.505 | 3.792 |
| 23 | 0.685 | 1.319 | 1.714 | 2.069 | 2.500 | 2.807 | 3.104 | 3.485 | 3.768 |
| 24 | 0.685 | 1.318 | 1.711 | 2.064 | 2.492 | 2.797 | 3.091 | 3.467 | 3.745 |
| 25 | 0.684 | 1.316 | 1.708 | 2.060 | 2.485 | 2.787 | 3.078 | 3.450 | 3.725 |
| 26 | 0.684 | 1.315 | 1.706 | 2.056 | 2.479 | 2.779 | 3.067 | 3.435 | 3.707 |
| 27 | 0.684 | 1.314 | 1.703 | 2.052 | 2.473 | 2.771 | 3.057 | 3.421 | 3.689 |
| 28 | 0.683 | 1.313 | 1.701 | 2.048 | 2.467 | 2.763 | 3.047 | 3.408 | 3.674 |
| 29 | 0.683 | 1.311 | 1.699 | 2.045 | 2.462 | 2.756 | 3.038 | 3.396 | 3.660 |
| 30 | 0.683 | 1.310 | 1.697 | 2.042 | 2.457 | 2.750 | 3.030 | 3.385 | 3.646 |
| 40 | 0.681 | 1.303 | 1.684 | 2.021 | 2.423 | 2.704 | 2.971 | 3.307 | 3.551 |
| 60 | 0.679 | 1.296 | 1.671 | 2.000 | 2.390 | 2.660 | 2.915 | 3.232 | 3.460 |
| 120 | 0.677 | 1.289 | 1.658 | 1.980 | 2.358 | 2.617 | 2.860 | 3.160 | 3.373 |
| $\infty$ | 0.674 | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 | 2.807 | 3.090 | 3.291 |

## CRITICAL VALUES FOR THE $\chi^2$ -DISTRIBUTION

If $X$ has a $\chi$2-distribution with $\nu$ degrees of freedom then, for each pair of values of $p$ and ,
t e ive
$\leqslant\nu$h table gs the value of $x$ such that P($X$ $x$) = $p$.




ν 3 3 2

| $p$ | 0.01 | 0.025 | 0.05 | 0.9 | 0.95 | 0.975 | 0.99 | 0.995 | 0.999 |
|---|---|---|---|---|---|---|---|---|---|
| = 1 | 0.01571 | 0.09821 | 0.03932 | 2.706 | 3.841 | 5.024 | 6.635 | 7.879 | 10.83 |
| 2 | 0.02010 | 0.05064 | 0.1026 | 4.605 | 5.991 | 7.378 | 9.210 | 10.60 | 13.82 |
| 3 | 0.1148 | 0.2158 | 0.3518 | 6.251 | 7.815 | 9.348 | 11.34 | 12.84 | 16.27 |
| 4 | 0.2971 | 0.4844 | 0.7107 | 7.779 | 9.488 | 11.14 | 13.28 | 14.86 | 18.47 |
| 5 | 0.5543 | 0.8312 | 1.145 | 9.236 | 11.07 | 12.83 | 15.09 | 16.75 | 20.51 |
| 6 | 0.8721 | 1.237 | 1.635 | 10.64 | 12.59 | 14.45 | 16.81 | 18.55 | 22.46 |
| 7 | 1.239 | 1.690 | 2.167 | 12.02 | 14.07 | 16.01 | 18.48 | 20.28 | 24.32 |
| 8 | 1.647 | 2.180 | 2.733 | 13.36 | 15.51 | 17.53 | 20.09 | 21.95 | 26.12 |
| 9 | 2.088 | 2.700 | 3.325 | 14.68 | 16.92 | 19.02 | 21.67 | 23.59 | 27.88 |
| 10 | 2.558 | 3.247 | 3.940 | 15.99 | 18.31 | 20.48 | 23.21 | 25.19 | 29.59 |
| 11 | 3.053 | 3.816 | 4.575 | 17.28 | 19.68 | 21.92 | 24.73 | 26.76 | 31.26 |
| 12 | 3.571 | 4.404 | 5.226 | 18.55 | 21.03 | 23.34 | 26.22 | 28.30 | 32.91 |
| 13 | 4.107 | 5.009 | 5.892 | 19.81 | 22.36 | 24.74 | 27.69 | 29.82 | 34.53 |
| 14 | 4.660 | 5.629 | 6.571 | 21.06 | 23.68 | 26.12 | 29.14 | 31.32 | 36.12 |
| 15 | 5.229 | 6.262 | 7.261 | 22.31 | 25.00 | 27.49 | 30.58 | 32.80 | 37.70 |
| 16 | 5.812 | 6.908 | 7.962 | 23.54 | 26.30 | 28.85 | 32.00 | 34.27 | 39.25 |
| 17 | 6.408 | 7.564 | 8.672 | 24.77 | 27.59 | 30.19 | 33.41 | 35.72 | 40.79 |
| 18 | 7.015 | 8.231 | 9.390 | 25.99 | 28.87 | 31.53 | 34.81 | 37.16 | 42.31 |
| 19 | 7.633 | 8.907 | 10.12 | 27.20 | 30.14 | 32.85 | 36.19 | 38.58 | 43.82 |
| 20 | 8.260 | 9.591 | 10.85 | 28.41 | 31.41 | 34.17 | 37.57 | 40.00 | 45.31 |
| 21 | 8.897 | 10.28 | 11.59 | 29.62 | 32.67 | 35.48 | 38.93 | 41.40 | 46.80 |
| 22 | 9.542 | 10.98 | 12.34 | 30.81 | 33.92 | 36.78 | 40.29 | 42.80 | 48.27 |
| 23 | 10.20 | 11.69 | 13.09 | 32.01 | 35.17 | 38.08 | 41.64 | 44.18 | 49.73 |
| 24 | 10.86 | 12.40 | 13.85 | 33.20 | 36.42 | 39.36 | 42.98 | 45.56 | 51.18 |
| 25 | 11.52 | 13.12 | 14.61 | 34.38 | 37.65 | 40.65 | 44.31 | 46.93 | 52.62 |
| 30 | 14.95 | 16.79 | 18.49 | 40.26 | 43.77 | 46.98 | 50.89 | 53.67 | 59.70 |
| 40 | 22.16 | 24.43 | 26.51 | 51.81 | 55.76 | 59.34 | 63.69 | 66.77 | 73.40 |
| 50 | 29.71 | 32.36 | 34.76 | 63.17 | 67.50 | 71.42 | 76.15 | 79.49 | 86.66 |
| 60 | 37.48 | 40.48 | 43.19 | 74.40 | 79.08 | 83.30 | 88.38 | 91.95 | 99.61 |
| 70 | 45.44 | 48.76 | 51.74 | 85.53 | 90.53 | 95.02 | 100.4 | 104.2 | 112.3 |
| 80 | 53.54 | 57.15 | 60.39 | 96.58 | 101.9 | 106.6 | 112.3 | 116.3 | 124.8 |
| 90 | 61.75 | 65.65 | 69.13 | 107.6 | 113.1 | 118.1 | 124.1 | 128.3 | 137.2 |
| 100 | 70.06 | 74.22 | 77.93 | 118.5 | 124.3 | 129.6 | 135.8 | 140.2 | 149.4 |

## WILCOXON SIGNED-RANK TEST

The sample has size $n$.

$P$ is the sum of the ranks corresponding to the positive differences.

$Q$ is the sum of the ranks corresponding to the negative differences.

$T$ is the smaller of $P$ and $Q$.

For each value of $n$ the table gives the **largest** value of $T$ which will lead to rejection of the null hypothesis at the level of significance indicated.

**Critical values of *T***

| Level of significance | | | | |
|---|---|---|---|---|
| One-tailed | 0.05 | 0.025 | 0.01 | 0.005 |
| Two-tailed | 0.1 | 0.05 | 0.02 | 0.01 |
| $n = 6$ | 2 | 0 | | |
| 7 | 3 | 2 | 0 | |
| 8 | 5 | 3 | 1 | 0 |
| 9 | 8 | 5 | 3 | 1 |
| 10 | 10 | 8 | 5 | 3 |
| 11 | 13 | 10 | 7 | 5 |
| 12 | 17 | 13 | 9 | 7 |
| 13 | 21 | 17 | 12 | 9 |
| 14 | 25 | 21 | 15 | 12 |
| 15 | 30 | 25 | 19 | 15 |
| 16 | 35 | 29 | 23 | 19 |
| 17 | 41 | 34 | 27 | 23 |
| 18 | 47 | 40 | 32 | 27 |
| 19 | 53 | 46 | 37 | 32 |
| 20 | 60 | 52 | 43 | 37 |

For larger values of $n$, each of $P$ and $Q$ can be approximated by the normal distribution with mean $\frac{1}{4}n(n+1)$ and variance $\frac{1}{24}n(n+1)(2n+1)$.

**WILCOXN RANK-SUM TEST**

The two samples have sizes $m$ and $n$, where $m \leqslant n$.

$R_m$ is the sum of the ranks of the items in the sample of size $m$.

$W$ is the smaller of $R_m$ and $m(n + m + 1) - R_m$.

For each pair of values of $m$ and $n$, the table gives the **largest** value of $W$ which will lead to rejection of the null hypothesis at the level of significance indicated.

**Critical values of *W***

| Level of significance | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| One-tailed | 0.05 | 0.025 | 0.01 | 0.05 | 0.025 | 0.01 | 0.05 | 0.025 | 0.01 | 0.05 | 0.025 | 0.01 |
| Two-tailed | 0.1 | 0.05 | 0.02 | 0.1 | 0.05 | 0.02 | 0.1 | 0.05 | 0.02 | 0.1 | 0.05 | 0.02 |
| $n$ | $m = 3$ | | | $m = 4$ | | | $m = 5$ | | | $m = 6$ | | |
| 3 | 6 | – | – | | | | | | | | | |
| 4 | 6 | – | – | 11 | 10 | – | | | | | | |
| 5 | 7 | 6 | – | 12 | 11 | 10 | 19 | 17 | 16 | | | |
| 6 | 8 | 7 | – | 13 | 12 | 11 | 20 | 18 | 17 | 28 | 26 | 24 |
| 7 | 8 | 7 | 6 | 14 | 13 | 11 | 21 | 20 | 18 | 29 | 27 | 25 |
| 8 | 9 | 8 | 6 | 15 | 14 | 12 | 23 | 21 | 19 | 31 | 29 | 27 |
| 9 | 10 | 8 | 7 | 16 | 14 | 13 | 24 | 22 | 20 | 33 | 31 | 28 |
| 10 | 10 | 9 | 7 | 17 | 15 | 13 | 26 | 23 | 21 | 35 | 32 | 29 |

| Level of significance | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| One-tailed | 0.05 | 0.025 | 0.01 | 0.05 | 0.025 | 0.01 | 0.05 | 0.025 | 0.01 | 0.05 | 0.025 | 0.01 |
| Two-tailed | 0.1 | 0.05 | 0.02 | 0.1 | 0.05 | 0.02 | 0.1 | 0.05 | 0.02 | 0.1 | 0.05 | 0.02 |
| $n$ | $m = 7$ | | | $m = 8$ | | | $m = 9$ | | | $m = 10$ | | |
| 7 | 39 | 36 | 34 | | | | | | | | | |
| 8 | 41 | 38 | 35 | 51 | 49 | 45 | | | | | | |
| 9 | 43 | 40 | 37 | 54 | 51 | 47 | 66 | 62 | 59 | | | |
| 10 | 45 | 42 | 39 | 56 | 53 | 49 | 69 | 65 | 61 | 82 | 78 | 74 |

For larger values of $m$ and $n$, the normal distribution with mean $\frac{1}{2}m(m+n+1)$ and variance $\frac{1}{12}mn(m+n+1)$ should be used as an approximation to the distribution of $R_m$.