Phase Transition of Ising Model with Belief Propegation

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Abstract

Ising models have traditionally been solved with Markov chain Monte Carlo methods, the most popular of which is the Gibbs' sampler. However, MCMC suffers from correlated samples and long mixing times. In recent years, a new class of algorithms based on variational inference has come to light. These algorithms promise faster convergence abit at the cost of convergence. Belief propagation is an inference algorithm that uses message passing in order to approximate the probability distribution over a factor graph. In this paper, we apply belief propagation to solve the Ising model and show that during belief propagation, the 2D Ising model exhibits a phase transition similar to simulations using MCMC. We compare metrics of phase transitions to MCMC approaches and known theoretical results and conclude that belief propagation can solve the Ising model to a high degree of accuracy and retain many interesting properties that the Ising model is known for.

1. Introduction

The Ising model is a mathematical model of ferromagnestim in statistical mechanics. Invented first by Wilhelm Lenz in 1920 and the equations of the one-dimensional Ising model were solved in by Lenz's student Ernst Ising in 1924. In 1944, Lars Onsager solved the equations for the two-dimensional case. As of today, no analytic solution has been found in dimensions three or higher. Over the last 100 years, many other problems such as percolation, min cut max flow, error correction, and neurodegenerative diseases were shown to be very closely mathematically related to the Ising model. The Ising model's relatively simple formulation yet rich and general mathematical properties makes a *drosophila* of physics, mathematics, statistics, and computer science.

Since Onsager's solution, it is known that the Ising model exhibits a phase transition just like magnets in the real world. Above the a special temperature, the critical temperature (or more generally the critical point), the system under goes a rapid change, changing from an ordered magnetic state to a disordered non-magnetic state. This rapid change of order is fundementally related to the relationship between energy, entropy and free energy. When the system is at the critical point, it is said to be at criticality and many interesting properties such as scale invariance and long range correlations are present during the phase transition.

However, due to the difficulty of solving the Ising models in higher dimensions, many approxiomation methods have been devolved over the years. Statistical physicists developed mean field theory and the cavity method to find approximate analytic solutions. In more recent times, the rise of computational power allowed computer algorithms to approximate solutions to the Ising model. The most well known class of algorithms are Markov Chain Monte Carlo (MCMC) algorithms such as Metropolis-Hasting algorithm and Gibbs' sampling. Although these algorithms are powerful and still widely used, they suffer long convergence times and autocorrelated samples.

In this paper, we explore an alternative algorithm to approximately solve the Ising model called belief propagation. Belief propagation works by sending messages from each node of a factor graph to iteratively find the probability distrubition. We show that this method can calculate solutions comparable to Metropolis-Hastings and Gibbs sampling and show that solutions given by belief propegation exhibit a phase translation like predicted theoretically.

2. Background

2.1. Ising Model

The Ising model models ferromagnetic materials as a lattice where each lattice site represents an electron that is either spin up $\sigma_i = +1$ or spin down $\sigma_i = -1$. Each lattice point only interacts with it's adjacent neighbors. Thus, the total energy is given by the Hamiltonian function

$$H(\sigma) = -J \sum_{(i,j)\in\mathcal{N}} \sigma_i \sigma_j - B \sum_j \sigma_j \tag{1}$$

where $(i, j) \in \mathcal{N}$ is understood to be the sum over all pairs of (i, j) that are adjacent neighbors, J is the physical con-

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stant knows as the coupling strength and B is the strength of the external magnetic field. We assume $J = 1$ for simplicity and no external magnetic field $B = 0$. Thus other Hamiltonian reduces to

$$H(\sigma) = -\sum_{(i,j)\in\mathcal{N}} \sigma_i \sigma_j \tag{2}$$

We notice that the energy is lower when σ_i and σ_j are the same sign and greater when σ_i and σ_j are different signs. The energy is minimized when the spins are alligned in the same direction, corresponding to a magnetized state. However, the entropy is maximized when the spins are misalligned corresponding to a demagnetized state.

The PMF of the system is given by the Boltzmann distribution

$$P(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z} \tag{3}$$

where the inverse temperature $\beta = 1/(kT)$ and the normalization constant $Z = \sum e^{-\beta H(\sigma)}$. Given an initial square lattice of σ and inverse temperature β the goal is to find the equalibrium distrubition over σ.

2.2. Phase Transition

3. Methods

3.1. Metropolis-Hasting

3.2. Gibbs Sampler

3.3. Belief Propegation

4. Experiments

5. Discussion

6. Conclusion