A Concise Introduction to College Algebra with Preliminaries

2025 Edition

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Preface

The aim of this book is to provide a concise introduction to college algebra for students preparing for precalculus or other STEM courses. It covers the essential topics of a college algebra course, beginning with preliminaries and continuing through expressions, equations, and functions. Trigonometric functions are omitted so that the material can be completed in a single semester.

Each section begins with clear learning goals and a *think-about-it* question (except in Part I: Preliminaries), followed by definitions, properties, examples, and exercises. Most examples are partially completed, requiring students to fill in the missing steps. This active-learning approach fosters engagement and strengthens problem-solving skills. Students are also encouraged to attempt most exercises independently, with answers provided for self-checking.

The content of each section is closely aligned with the stated learning goals, helping both students and instructors focus on key concepts and track progress. Instructors may skip examples or exercises that fall outside their course objectives.

Despite careful preparation, minor errors may remain. The author welcomes and appreciates comments, corrections, and suggestions, which are invaluable for refining future editions and enhancing the book's effectiveness.

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Contents

Part I

Preliminaries

Topic 1 Arithmetics

Learning Goals

- I can add, subtract, multiply, divide and simplify fractions.
- I can evaluate and simplify arithmetic expressions without the use of a calculator.

1.1 Fractions

A **fraction** is a number of the form $\frac{n}{d}$, where n and d are real numbers and $d \neq 0$. The number n is called the **numerator** and d is called the **denominator**. The fraction $\frac{n}{d}$ represents the division of n by d, or the ratio of n to d. A rational number is a fraction whose numerator and denominator are both integers. A rational number is **reduced** or **simplified** if the numerator and denominator have no common factors other than 1 or -1. A rational number $\frac{1}{d}$ can be considered a unit of the size $\dfrac{1}{d}$, and $\frac{n}{d}$ means n units of the size $\frac{1}{d}$.

Properties of Fractions

Equivalence

$$\frac{n}{d} = \frac{n \cdot k}{d \cdot k} = \frac{n \div k}{d \div k}$$

for any real number $k \neq 0$.

Multiplication

$$\frac{n}{d} \cdot \frac{a}{b} = \frac{an}{bd},$$

in particular, $m \cdot \dfrac{n}{d} = \dfrac{mn}{d}$ and $-\dfrac{n}{d} = \dfrac{-n}{d} = \dfrac{n}{-d}$.

Division

$$\frac{n}{d} \div \frac{a}{b} = \frac{nb}{da}.$$

Addition/Subtraction (with the same denominator)

$$\frac{m}{d} \pm \frac{n}{d} = \frac{m \pm n}{d}.$$

Addition/Subtraction (with different denominators)

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ap \pm cq}{m},$$

where m is a common multiple of b and d, $p = \dfrac{m}{b}$ and $q = \dfrac{m}{d}$. For simplicity, m is usually taken as the **lease common multiple**, also known as the **least common denominator**, that is, the smallest common multiple.

Example 1.1. Simplify.

1) $\dfrac{5}{6} \cdot \dfrac{3}{10}$ 2) $\dfrac{5}{6} \div \dfrac{3}{10}$ 3) $\dfrac{7}{15} - \dfrac{13}{15}$ 4) $\dfrac{1}{8} + \dfrac{5}{12}$

Solution.

1) Since $6 = 2 \cdot 3$ and $10 = 2 \cdot 5$,
$$\frac{5}{6} \cdot \frac{3}{10}. = \frac{5 \cdot 3}{6 \cdot 10} = \frac{5 \cdot 3}{2 \cdot \_\_\_\_ \cdot (2 \cdot 5)} = \frac{1}{2 \cdot 2} = \frac{1}{\_\_\_\_}.$$

2) Since $6 = 2 \cdot 3$ and $10 = 2 \cdot 5$,
$$\frac{5}{6} \div \frac{3}{10} = \frac{5}{6} \cdot \frac{10}{\_\_\_\_} = \frac{5 \cdot (2 \cdot \_\_\_\_)}{(\_\_\_\_ \cdot 2) \cdot 3} = \frac{25}{9}.$$

3) Since the denominators are the same,
$$\frac{7}{15} - \frac{13}{15} = \frac{7 - \_\_\_\_}{15} = \frac{-6}{15} = -\frac{2 \cdot \_\_\_\_}{\_\_\_\_ \cdot 5} = -\frac{2}{5}.$$

4) Since $8 = 2 \cdot 2 \cdot 2$ and $12 = 2 \cdot 2 \cdot 3$, the least common multiple of 8 and 12 is $2 \cdot 2 \cdot 2 \cdot 3 = 24$. Then
$$\frac{1}{8} + \frac{5}{12} = \frac{1 \cdot \_\_\_\_}{24} + \frac{5 \cdot \_\_\_\_}{24} = \frac{3 + \_\_\_\_}{24} = \frac{13}{24}.$$

1.2 Order of Operations

The order of operations is a set of rules that defines the sequence in which operations are performed in an arithmetic expression. The standard order of operations is:

1) **Parentheses:** Perform operations inside parentheses first or remove parentheses by applying the distributive property first.
2) **Exponents:** Evaluate exponents (powers and roots) next.
3) **Multiplication and Division:** Perform multiplication and division from left to right.
4) **Addition and Subtraction:** Finally, perform addition and subtraction from left to right.

Example 1.2. Simplify.

1) $(2+3) \cdot 4 - 5$ 2) $2 + ((3 \div 4) - 5)$ 3) $1 + (2-3) \div (-4) \div 5$

Solution.

1)
$$\begin{aligned}(2+3) \cdot 4 - 5 &= \_\_\_\_ \cdot 4 - 5 \\ &= 5 \cdot 4 - 5 = \_\_\_\_ - 5 = \_\_\_\_.\end{aligned}$$

2)

$$\begin{aligned}
2+(3\div 4-5) &= 2+\left(\frac{3}{4}-5\right)\\
&= 2+\frac{3}{4}+(\underline{\hspace{1.5em}})\\
&= \underline{\hspace{1.5em}}+\frac{3}{4}\\
&= -3+\frac{3}{4}\\
&= \frac{\underline{\hspace{1.5em}}}{4}+\frac{3}{4}\\
&= -\frac{9}{4}.
\end{aligned}$$

3)

$$\begin{aligned}
1+(2-3)\div(-4)\div 5 &= 1+(\underline{\hspace{1.5em}})\div(-4)\div 5\\
&= 1+\frac{\underline{\hspace{1.5em}}}{4}\div 5\\
&= 1+\frac{\underline{\hspace{1.5em}}}{4}\cdot\frac{1}{\underline{\hspace{1.5em}}}\\
&= 1+\frac{\underline{\hspace{1.5em}}}{20}\\
&= \frac{\underline{\hspace{1.5em}}}{20}+\frac{1}{20}\\
&= \frac{21}{20}.
\end{aligned}$$

Exercises

Exercise 1.1. Simplify.

1) $\frac{7}{10} \cdot \frac{15}{21}$ 2) $\frac{10}{3} \div \frac{5}{12}$ 3) $\frac{5}{6} + \frac{7}{8}$ 4) $\frac{13}{12} - \frac{5}{8}$

Answer: 1) $\frac{1}{2}$ 2) 8 3) $\frac{41}{24}$ 4) $\frac{11}{24}$

Exercise 1.2. Simplify.

1) $(2+3) \cdot 4 - 5$ 2) $(4 - (5 \div 3)) - 2$ 3) $1 - (2-3) \div ((-4) \div 5)$

Answer: 1) 15 2) $\frac{1}{3}$ 3) $-\frac{1}{4}$

Topic 2 Integer Exponents of Numbers

Learning Goals

- I can demonstrate understanding of integral exponents of whole numbers.
- I can use the properties of integral exponents to simplify and evaluate expressions with integral exponents.

2.1 Exponentiation

For an integer n and a real number b, the multiplication of b with itself n times is call **exponentiation** of b to the n-th power and written as b^n, that is,

$$b^n = \underbrace{b \cdot b \cdot \cdots \cdot b}_{n \text{ factors of } b}.$$

In the notation, b is called the **base** and n is called the **exponent**.

From the definition, we have the following rules for exponentiation:

Multiplication $b^m \cdot b^n = b^{m+n}$

Division $\dfrac{b^m}{b^n} = b^{m-n}$ (for $b \neq 0$ if $m < n$)

Zero exponent $b^0 = 1$ (for $b \neq 0$)

Negative exponent $b^{-n} = \dfrac{1}{b^n}$ (for $b \neq 0$ if $n > 0$)

Power to a power $(b^m)^n = b^{mn}$

Product raised to a power $(bc)^n = b^n c^n$

Quotient raised to a power $\left(\dfrac{b}{c}\right)^n = \dfrac{b^n}{c^n}$ (for $c \neq 0$)

Example 2.1. Evaluate the expression without using a calculator.

1) $(-2)^3$ 2) $(-2)^{-3}$ 3) $\left(\dfrac{-5}{3}\right)^{-2}$ 4) 2^{2^2}

Solution.

1)

$$(-2)^3 = (-2) \cdot (-2) \cdot (\underline{\hspace{2em}}) = -8.$$

2)

$$(-2)^{-3} = \frac{1}{(-2)^{\underline{\hspace{1.5em}}}} = \frac{1}{\underline{\hspace{2em}}} = -\frac{1}{8}.$$

3)

$$\left(\frac{-5}{3}\right)^{-2} = \left(\frac{3}{-5}\right)^{\underline{\quad}} = \frac{3^{\underline{\quad}}}{(-5)^2} = \frac{9}{25}.$$

Example 2.2. Evaluate the expression without using a calculator.

1) $\left(3^{-2} \cdot (-3)^3\right)^2$ 2) $\dfrac{\left(3^2\right)^3 \cdot 3^4}{3^{12}}$ 3) $\left(-\dfrac{4}{(-2)^3}\right)^{-3}$

Solution.

1)

$$\left(3^{-2} \cdot (-3)^3\right)^2 = \left(3^{-2} \cdot (-3^{\underline{\quad}})\right)^2 = \left(-(3^{-2} \cdot 3^3)\right)^2 = (-3^{\underline{\quad}})^2 = 3^{\underline{\quad}} = \frac{1}{9}.$$

2)

$$\frac{\left(3^2\right)^3 \cdot 3^4}{3^{12}} = \frac{3^{\underline{\quad}} \cdot 3^4}{3^{12}} = \frac{3^{\underline{\quad}}}{3^{12}} = \frac{1}{3^{\underline{\quad}}} = \frac{1}{9}.$$

3)

$$\left(-\frac{4}{(-2)^3}\right)^{-3} = \left(-\frac{4}{\underline{\quad}}\right)^{-3} = \left(\frac{1}{\underline{\quad}}\right)^{-3} = (\underline{\quad})^3 = 8.$$

Exercises

Exercise 2.1. Evaluate the expression without using a calculator.

1) $(-2)^4 \cdot (-2)^3$

2) $\left(-\dfrac{6}{3^2}\right)^{-2}$

3) $\left(\dfrac{-3^{-2} \cdot 3^3}{(-8)^0(-9)}\right)^{-5}$

Topic 3 Basics of Polynomials

Learning Goals

- I can identify the degree of a polynomial and describe coefficients.
- I can differentiate between a "term" and a "factor".
- I can evaluate polynomials with given variable values.
- I can simplify expressions invovling polynomials using the order of operations, associative, commutative, and distributive properties, and combining like terms.

3.1 Terminology

A (single variable) **polynomial** is an algebraic expression that is a sum of one or more terms, where each **term** is a product of a constant (called the **coefficient**) and a variable raised to a non-negative integer exponent. The **degree of a polynomial** is the highest exponent of the variable in the polynomial. The **leading coefficient** is the coefficient of the term with the highest degree. The **constant term** is the term with no variable (i.e., the term with degree 0).

Example 3.1. Identify the degree, leading coefficient, the constant term, and the term of degree 3 of the polynomial $3x^4 - 5x^3 + 2x^2 + 7$.

Solution. The degree of the polynomial is 4, the leading coefficient is 3, the constant term is 7, and the term of degree 3 is $-5x^3$.

3.2 Evaluating Polynomials

To evaluate a polynomial at a given value of the variable, substitute the value into the polynomial and simplify.

Example 3.2. Evaluate the polynomial $2x^3 - 4x^2 + 3x - 5$ at $x = 2$.

Solution. Substitute x by 2 in the polynomial and simplify:

$$\begin{aligned} 2(2)^3 - 4(2)^2 + 3(2) - 5 &= 2(\underline{\qquad}) - 4(\underline{\qquad}) + 6 - 5 \\ &= 16 - \underline{\qquad} + 6 - 5 \\ &= 1. \end{aligned}$$

3.3 Arithmetic Operations of Polynomials

Arithmetic operations of polynomials, more generally, algebraic expressions, follow the same rules as operations of real numbers. The order of operations, associative, commutative, and distributive properties apply.

Addition and subtraction of polynomials are performed by combining **like terms** that are terms with the same variable raised to the same exponent. For example, $3x^2$ and $-5x^2$ are like terms, while $3x^2$ and $-5x^3$ are not.

A **factor** of a polynomial is a polynomial that divides the polynomial evenly. In other words, if $P(x)$ is a polynomial and $Q(x)$ is a factor of $P(x)$, then there exists another polynomial $R(x)$ such that $P(x) = Q(x) \cdot R(x)$. A polynomial is **irreducible** if it has no factors other than itself and 1 (or -1).

Example 3.3. Simplify the expression.

$$(2x^3 + 3x^2) - (5x^3 + 4x^2 - 7)$$

Solution. To simplify the expression, we first distribute the negative sign across the second polynomial:

$$\begin{aligned}(2x^3 + 3x^2) - (5x^3 + 4x^2 - 7) &= 2x^3 + 3x^2 + (\underline{\quad\quad})x^3 + (\underline{\quad\quad})x^2 + 7 \\ &= (2 + \underline{\quad\quad})x^3 + (3 + (\underline{\quad\quad})x^2 + 7 \\ &= -3x^3 - x^2 + 7.\end{aligned}$$

Example 3.4. Multiply and simplify the expression.

$$(2x^2 - 3)(4x + 5)$$

Solution. To multiply the two polynomials, we use the distributive property:

$$\begin{aligned}(2x^2 - 3)(4x + 5) &= \underline{\quad\quad\quad}(4x + 5) + (\underline{\quad\quad})(4x + 5) \\ &= (8x^3 + \underline{\quad\quad}x^2) + ((\underline{\quad\quad})x + (-15)) \\ &= 8x^3 + 10x^2 - 12x - 15.\end{aligned}$$

Example 3.5. Determine a degree 2 factor and the degree 2 term of the polynomial

$$(x^2 + 1)(1 - x).$$

Solution. Since the polynomial is a product of two polynomials of degree 2 and 1, respectively, a degree 2 factor is $x^2 + 1$.

To find the degree 2 term, we first multiply the two polynomials and simplify:

$$\begin{aligned}(x^2 + 1)(1 - x) &= x^2(\underline{\quad\quad\quad\quad}) + 1(1 - x) \\ &= x^2 + (\underline{\quad\quad\quad}) + 1 + (-x) \\ &= -x^3 + \underline{\quad\quad} - x + 1.\end{aligned}$$

So the degree 2 term is x^2.

Exercises

Exercise 3.1. Identify the degree, leading coefficient, constant term, and term of degree 4 of the polynomial $4x^5 - 2x^3 + 7x^2 + 9$.

Answer: The degree is 5, the leading coefficient is 4, the constant term is 9, and the term of degree 4 is 0.

Exercise 3.2. Evaluate the polynomial $-x^3 + 2x^2 + 4x - 6$ at $x = -1$.

Answer: -7

Exercise 3.3. Simplify the expression.

1) $(3x^3 + 4x - 5) - (2x^2 - 3x + 6)$ 2) $(x^3 - 2x^2 + 3)(x - 5)$

Answer: 1) $3x^3 - 2x^2 + 7x - 11$ 2) $x^4 - 7x^3 + 10x^2 + 3x - 15$

Exercise 3.4. Determine a degree 3 factor and the degree 3 term of the polynomial

$$(x^3 + 2x^2 - 1)(x - 2).$$

Answer: A degree 3 factor is $x^3 + 2x^2 - 1$. The degree 3 term is 0.

Topic 4 Square Roots

Learning Goals

- ☐ I can demonstrate understanding of square roots of real numbers.
- ☐ I can simplify and evaluate expressions involving square roots.

4.1 Square Roots and Properties

For a nonnegative real number a, a **square root** of a is a number b such that

$$b^2 = a.$$

Because squaring a number always produces a nonnegative number, the square root of a negative number is not a real number.

For every positive number, there are **two** square roots: a positive and a negative one. For example, both 3 and -3 are square roots of 9.

By convention, the nonnegative square root is written with the **radical symbol** $\sqrt{}$.

Example 4.1. Find all square roots of the given number if they exist. Otherwise, state that the square root is not a real number.

1) 16 2) 0 3) 0.25 4) -4

Solution.

1) The square roots of 16 are 4 and ____ because $4^2 = 16$ and $(-4)^2 = 16$.
2) The only square root of 0 is ____ because $0^2 = 0$.
3) The square roots of 0.25 are ____ and ____ because $(0.5)^2 = 0.25$ and $(-0.5)^2 = 0.25$.
4) The square root of -4 is not a real number because there is no real number b such that $b^2 = -4$.

The nonnegative square root $\sqrt{a}$ of a nonnegative number a is called the **principal square root**.

Properties of Square Roots

Inverse of the Square $\sqrt{a^2} = |a|$.

Multiplication $\sqrt{ab} = \sqrt{a}\sqrt{b}$ for $a, b \neq 0$.

Division $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ for $b > 0$.

A square root is **simplified** if the radicand (the number or expression inside the radical symbol) has **no perfect square factors** other than 1.

To **simplify a radical**, factor out perfect squares from the radicand and use the inverse of the square and multiplication property of square roots to rewrite the expression.

Example 4.2. Determine whether the square root is simplified. If not, find its simplified form.

1) $\sqrt{(-2)^2}$ 2) $\sqrt{20}$ 3) $\sqrt{13}$

Solution.

1) $\sqrt{(-2)^2}$ is not simplified because $(-2)^2$ is a perfect square. By the property of square roots, we have $\sqrt{(-2)^2} = |-2| =$ ____.

2) $\sqrt{20}$ is not simplified because $20 = 4 \cdot$ ____ and $4 =$ ____^{2}. The simplified form is

$$\sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{(\_\_\_\_)^2 \cdot 5} = 2\sqrt{5}.$$

1) $\sqrt{13}$ is simplified because 13 has no perfect square factors other than 1.

Example 4.3. Simplify or evaluate.

1) $\sqrt{8} \cdot \sqrt{2}$ 2) $\dfrac{\sqrt{45}}{\sqrt{5}}$

Solution.

1)

$$\sqrt{8} \cdot \sqrt{2} = \sqrt{\_\_\_\_} = \sqrt{(\_\_\_\_)^2} = 4.$$

2)

$$\frac{\sqrt{45}}{\sqrt{5}} = \sqrt{\frac{45}{5}} = \sqrt{\_\_\_\_} = 3.$$

Exercises

Exercise 4.1. Find the square root of the given number if it exists. Otherwise, state that the square root is not a real number.

1) 25 2) 0.16 3) -9 4) $\sqrt{4-(-2)^2}$

Answer: 1) 5 2) 0.4 3) Not a real number 4) 0

Exercise 4.2. Evaluate or simplify the following without using a calculator.

1) $\sqrt{200}$ 2) $\sqrt{(-5)^2}$ 3) $\sqrt{12}\cdot\sqrt{3}$ 4) $\dfrac{\sqrt{98}}{\sqrt{2}}$

Answer: 1) $10\sqrt{2}$ 2) 5 3) 6 4) 7

Topic 5 Geometry Essentials

Learning Goals

- I can use the Pythagorean Theorem.
- I know and can apply geometry formulas (perimeter/circumference/area/volume) related to squares, triangles, circles, and boxes.

5.1 Pythagorean Theorem and Its Inverse

Theorem 5.1.1 (Pythagorean Theorem). *In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.*

Theorem 5.1.2: Pythagorean Theorem

Theorem 5.1.2 (Inverse Pythagorean Theorem). *If the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of the other two sides, then the triangle is a right triangle.*

Theorem 5.1.3: Inverse Pythagorean Theorem

Distance Formula

The distance between two points (x_1, y_1) and (x_2, y_2) in the plane is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 5.1. Find the length of the hypotenuse of a right triangle with legs of lengths 5 and 12.

Solution. By the Pythagorean Theorem, the length of the hypotenuse c is given by

$$c^2 = 5^2 + 12^2.$$

Calculating, we find

$$c^2 = \underline{\qquad} + 144 = 169.$$

Taking the square root, we have

$$c = \sqrt{169} = \underline{\qquad}.$$

5.2 Geometry Formulas

Formulas from Geometry

Area of a rectangle

$$\text{Area of a rectangle} = \text{length} \cdot \text{width}$$

Area of a triangle

$$\text{Area of a triangle} = \frac{1}{2} \cdot \text{base} \cdot \text{height}$$

Area of a circle

$$\text{Area of a circle} = \pi \cdot \text{radius}^2$$

Circumference of a circle

$$\text{Circumference of a circle} = 2\pi \cdot \text{radius}$$

Volume of a rectangular box

$$\text{Volume of a rectangular box} = \text{length} \cdot \text{width} \cdot \text{height}$$

Volume of a cylinder

$$\text{Volume of a cylinder} = \pi \cdot \text{radius}^2 \cdot \text{height}$$

Volume of a sphere

$$\text{Volume of a sphere} = \frac{4}{3}\pi \cdot \text{radius}^3$$

Surface area of a sphere

$$\text{Surface area of a sphere} = 4\pi \cdot \text{radius}^2$$

Volume of a cone

$$\text{Volume of a cone} = \frac{1}{3}\pi \cdot \text{radius}^3 \cdot \text{height}$$

Example 5.2. A 15cm^2 rectangle has a length of 5 cm. What is its width?

Solution. The area of a rectangle is given by the formula

$$\text{Area of a rectangle} = \text{length} \cdot \text{width}.$$

Substituting the known values, we have

$$15 = 5 \cdot \text{width}.$$

Dividing both sides by 5, we find

$$\text{width} = 3 \text{ cm}.$$

Example 5.3. A triangle has a base of 8 cm and area 24. What is its height?

Solution. The area of a triangle is given by the formula

$$\text{Area of a triangle} = \frac{1}{2} \cdot \text{base} \cdot \text{height}.$$

Substituting the known values, we have

$$24 = \frac{1}{2} \cdot 8 \cdot \text{height}.$$

Simplifying, we find

$$24 = 4 \cdot \text{height}.$$

Dividing both sides by 4, we find

$$\text{height} = 6 \text{ cm}.$$

Exercises

Exercise 5.1. A right triangle has a leg of length 6 and a hypotenuse of length 10. What is the length of the other leg?

Answer: 8

Exercise 5.2. A right triangle has a leg of the length 5 and the area 25.

1) What is the length of the other leg?

2) What is the length of the hypotenuse?

Answer: 1) 10 cm 2) 11.18 cm (approximately)

Exercise 5.3. A rectangle has a length of 10 cm and an area of $50\mathrm{cm}^2$. What is its width?

Answer: 5 cm

Exercise 5.4. A circle has a circumference of 4π. What is its area?

Answer: 4π

Part II

Expressions

Topic 6 Expressions with Integer Exponents

Learning Goals

- [ ] I can use the properties of integer exponents to simplify and evaluate expressions with integer exponents.

6.1 Think about It

Think Twice

A pizza shop sales larger size pizza at the price $\$24$/each and the smaller size pizza at the $\$12$/each. The larger size pizza has a dimeter 1.5 times the smaller size pizza. With $\$24$, would you like to order one larger pizza or two smaller pizzas? Why?

6.2 Exponentiation with Integer Exponents

Exponentiation is a mathematical operation that generalizes the idea of repeated multiplication, just as multiplication generalizes the idea of repeated addition. The resulting value is called a **power**.

For an positive integer n and an expression x, the n-th **power** of x, is defined as

$$x^n = \underbrace{x \cdot x \cdot \cdots \cdot x}_{n \text{ factors of } x}.$$

In the expression x^n, n is called **the (integer) exponent** and x is called **the base**. The expression x^n is often read as "x to the n-th power", "x raised to the n-th power", "x to the power of n", or simply "x to the n".

The zero-th power of x, if $x \neq 0$, is defined as

$$x^0 = 1.$$

For a negative integer $-n$, if $x \neq 0$, the $-n$-th power of x is defined as

$$x^{-n} = \underbrace{\frac{1}{x} \cdot \frac{1}{x} \cdot \cdots \cdot \frac{1}{x}}_{n \text{ factors of } \frac{1}{x}} = \frac{1}{\underbrace{x \cdot x \cdot \cdots \cdot x}_{n \text{ factors of } x}} = \frac{1}{x^n}.$$

Example 6.1. For positive integers m and n, using the definition of exponentiation to write the product $x^m \cdot x^n$ as a single exponentiation expression.

Solution. By the definition of exponentiation with positive integer,

$$x^m = \underbrace{x \cdot x \cdot \cdots \cdot x}_{m \text{ factors of } x} \quad \text{and} \quad x^n = \underbrace{x \cdot x \cdot \cdots \cdot x}_{n \text{ factors of } x}.$$

Therefore, the product $x^m \cdot x^n$ is the multiplication of

$$x^m \cdot x^n = \underbrace{(x \cdot x \cdot \cdots \cdot x)}_{m \text{ factors of } x} \cdot \underbrace{(x \cdot x \cdot \cdots \cdot x)}_{m \text{ factors of } x} = \underbrace{x \cdot x \cdot \cdots \cdot x}_{\_\_\_\_ \text{ factors of } x} = x^{\_\_\_\_}.$$

The Properties of Exponentiation with Integer Exponents

Product rule

$$x^m \cdot x^n = x^{m+n}$$

Zero exponent rule (for $x \neq 0$)

$$x^0 = 1$$

Quotient rule (for $x \neq 0$)

$$\frac{x^m}{x^n} = x^{m-n} = \begin{cases} x^{m-n} & \text{if } m \geq n \\ \frac{1}{x^{n-m}} & \text{if } m < n \end{cases}$$

Negative exponent rule (for $x \neq 0$)

$$x^{-n} = \frac{1}{x^n} = \left(\frac{1}{x}\right)^n \text{ and } \left(\frac{1}{x}\right)^{-n} = \frac{1}{x^{-n}} = x^n$$

Power-to-power rule

$$(x^m)^n = x^{mn}$$

Product-to-power rule

$$(xy)^n = x^n y^n$$

Quotient-to-power rule (for $y \neq 0$)

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

Example 6.2. Simplify. **Write with positive exponents.**

1) $(2x^3y^2) \cdot (5x^2y^4)$ 2) $\dfrac{6a^4b^2}{ab^3}$ 3) $x^{-4}y^2$

Solution.

1) Apply rules of multiplication and the product rule:

$$(2x^3y^2) \cdot (5x^2y^4) = 2 \cdot 5 \cdot x^3 \cdot x^2 \cdot y^2 \cdot y^4 = 10x^{\underline{\quad}}y^{\underline{\quad}}$$

2) Apply the multiplication rule of factions and the quotient rule:

$$\frac{6a^4b^2}{ab^3} = \frac{6}{1} \cdot \frac{a^4}{a} \cdot \frac{b^2}{b^3} = 6 \cdot a^{\underline{\quad}} \cdot b^{\underline{\quad}} = 6 \cdot (a^3) \cdot \frac{1}{\underline{\quad}} = \frac{6a^3}{b}$$

3) Apply the negative exponent rule and multiplication rule of fractions:

$$x^{-4}y^2 = \frac{1}{\underline{\quad}} \cdot y^2 = \frac{y^2}{x^4}$$

Example 6.3. Simplify. **Write with positive exponents.**

1) $(2xy^{-2})^3$ 2) $\left(\dfrac{a^2}{b^3}\right)^{-2}$ 3) $(x^{-2}y^3)^{-4}$

Solution.

1) Apply the product-to-power rule, the power-to-power rule, and the negative exponent rule:

$$(2xy^{-2})^3 = 2^3 \cdot (x^1)^3 \cdot (y^{-2})^3 = \underline{\quad} x^3y^{\underline{\quad}} = \frac{8x^3}{y^6}$$

2) Apply the negative exponent rule, the quotient-to-power rule, and the power-to-power rule:

$$\left(\frac{a^2}{b^3}\right)^{-2} = \left(\frac{\quad}{\underline{\quad}}\right)^2 = \frac{b^{3\cdot 2}}{a^{2\cdot 2}} = \frac{b^6}{a^4}$$

3) Apply the product-to-power rule, the power-to-power rule, and the negative exponent rule:

$$(x^{-2}y^3)^{-4} = (x^{-2})^{-4} \cdot (y^3)^{-4} = x^{\underline{\quad}} \cdot y^{\underline{\quad}} = \frac{x^8}{y^{12}}$$

Order of Basic Mathematical Operations

In mathematics, the order of operations reflects conventions about which procedure should be performed first. There are four levels (from the highest to the lowest):

Parenthesis; **Exponentiation**; **Multiplication and Division**; **Addition and Subtraction**.

Within the same level, the convention is to perform from the left to the right.

Example 6.4. Simplify. **Write with positive exponents.**

$$\left(\frac{2y^{-2}z^{-5}}{4x^{-3}y^6}\right)^{-4}.$$

Solution. As the base expression is complicated and the outer exponent is negative, it's better first simplify the base expression, apply negative exponent rule to change the negative exponent to positive exponent, and then simplify the result using the quotation to power rule, the power to power rule, and the power to power rule.

$$\left(\frac{2y^{-2}z^{-5}}{4x^{-3}y^6}\right)^{-4} = \left(\frac{x^3}{\underline{\qquad\qquad}}\right)^{-4} = \left(\frac{2z^5y^8}{x^3}\right)^{\underline{\quad}} = \frac{2^4(z^5)^4(y^8)^4}{(x^3)^4} = \frac{16y^{32}z^{20}}{x^{12}}$$

Simplify (at least partially) the problem first

To avoid mistakes when working with negative exponents, it's better to apply the negative exponent rule to change negative exponents to positive exponents and simplify the base first.

Exercises

Exercise 6.1. Simplify. **Write with positive exponents.**

1) $(2x^2y^3) \cdot (3xy^2)$ 2) $z^0 \cdot w^{-2}$ 3) $\dfrac{8a^5b^2}{a^2b}$ 4) $(3a^2b^{-1})^2$

Answer: 1) $6x^3y^5$ 2) $\dfrac{1}{w^2}$ 3) $8a^3b$ 4) $\dfrac{9a^4}{b^2}$

Exercise 6.2. Simplify. **Write with positive exponents.**

1) $\dfrac{x^4y^{-3}}{xy^{-1}}$

2) $(-2)^{-3} \cdot \left(\dfrac{1}{x}\right)^2$

3) $\dfrac{(4a)^{-2}b^4}{a^3b^{-1}}$

4) $\left(\dfrac{2a^3}{b^{-2}}\right)^2$

Answer: 1) $\dfrac{x^3}{y^2}$ 2) $-\dfrac{1}{8x^2}$ 3) $\dfrac{b^5}{16a^5}$ 4) $4a^6b^4$

Exercise 6.3. Simplify. **Write with positive exponents.**

1) $(x^{-2}y^3)^{-3}$ 2) $\left(\dfrac{-2x^{-1}y^2}{z}\right)^{-2}$ 3) $\left(\dfrac{3a^{-3}b^2}{6c^{-2}a^4}\right)^{-3}$

Answer: 1) $\frac{x^6}{y^9}$ 2) $\frac{x^2z^2}{4y^4}$ 3) $\frac{8a^{21}}{b^6c^6}$

Exercise 6.4. Simplify. **Write with positive exponents.**

1) $\dfrac{\left(2a^2b^{-3}\right)^{-2}}{a^{-1}b^2}$

2) $\dfrac{\left(x^{-2}y^3\right)^{-2}\cdot\left(xy^{-1}\right)^3}{\left(x^{-3}y^2\right)^{-2}}$

Answer: 1) $\dfrac{b^4}{4a^3}$ 2) $\dfrac{x}{y^5}$

Exercise 6.5. A store has large size and small size watermelons. A large one cost \$4 and a small one \$1. Putting on the same table, a smaller watermelons has only half the height of the larger one. Given \$4, will you buy a large watermelon or 4 smaller ones? Why?

Topic 7 Factoring Polynomials

Learning Goals

- I can identify a greatest common factor (GCF) and use it to factor polynomials.
- I can apply the grouping method to factor a polynomial with four terms.
- I can factor a binomial using formulas (difference of squares, difference or sum of cubes).
- I can apply the trial and error method to factor a trinomial.
- I can factor a polynomial completely using various methods.
- I can use the substitution idea to factor a polynomial whose terms contain powers of another polynomial.

7.1 Think about It

Can You Outsmart a Calculator?

Can you find a faster way to evaluate these expressions—perhaps by using factoring, special products, or mental math?

1) Evaluate the polynomial $2x^3 - 98x$ at $x = -7$.
2) Evaluate the polynomial $x^2 - 9x - 22$ at $x = 11$.
3) Evaluate the polynomial $x^3 - 2x^2 - 9x + 18$ at $x = -3$.
4) Compute the difference $16^2 - 14^2$.

7.2 Factoring out the GCF

Factorization or **factoring** is the process of breaking down a mathematical expression —especially a polynomial—into a product of simpler expressions (called **factors**) that is equivalent to the original expression.

We typically begin by factoring out the **greatest common factor (GCF)** which is the largest polynomial that divides all terms. Here, "largest" refers to having the *greatest numerical coefficient* and the *highest possible degree* of any common variables.

Example 7.1. Factor $4x^3y - 8x^2y^2 + 12x^4y^3$.

Solution.

1) Find the GCF of all terms.
   The GCF of $4x^3y$, $-8x^2y^2$ and $12x^4y^3$ is ____ $x^{\_\_}y$.
2) Write each term as the product of the GCF and the remaining factor.
   $4x^3y = (4x^2y) \cdot$ ____, $\quad -8x^2y^2 = (4x^2y) \cdot$ ____, $\quad 12x^4y^3 = (4x^2y) \cdot$ ____

3) Factor out the GCF from each term.

$$4x^3y - 8x^2y^2 + 12x^4y^3 = 4x^2y \cdot (\underline{\hspace{3cm}})$$

7.3 Factoring by Grouping

The **grouping method** is a technique to factor a polynomial with four or more terms. It involves grouping the terms into pairs or groups, factoring out the GCF from each group, and then factoring out the common binomial factor.

Example 7.2. Factor $2x^2 - 6xy + xz - 3yz$.

Solution.

1) Group the first two terms and the last two terms.

$$\begin{aligned} &2x^2 - 6xy + xz - 3yz \\ &= (2x^2 - 6xy) + (xz - 3yz) \end{aligned}$$

2) Factor out the GCF from each group.

$$= 2x(x - 3y) + z\ \underline{\hspace{1.5cm}}$$

3) Factor out the binomial GCF.

$$= (x - 3y)(2x + z)$$

Think Ahead

After factoring one group, anticipate that the other group will share the same (binomial) factor. Comparing the common factor with the second group will help you determine the corresponding cofactor.

Example 7.3. Factor $ax + 4b - 2a - 2bx$.

Solution.

1) Group the first term with the third term and group the second term with the last term.

$$\begin{aligned} &ax + 4b - 2a - 2bx \\ &= (ax - 2a) + \underline{\hspace{1.5cm}} \end{aligned}$$

2) Factor out the GCF from each group.

$$= a(x - 2) + \underline{\hspace{0.8cm}}\ (x - 2)$$

3) Factor out the binomial GCF.

$$= (x - 2)(a - 2b)$$

7.4 Factoring Difference of Powers

Factorization is closely related to polynomial equations. Factoring a polynomial helps solving the polynomial equation. Backward, a solution of a polynomial equation gives a linear factor of the polynomial. For example, the polynomial equation $x^2 - 1 = 0$ has a solution $x = 1$. On the other hand, we know that the polynomial $x^2 - 1$ factors as $(x - 1)(x + 1)$. In general, r is a solution of the polynomial equation $P(x) = 0$ if and only if $(x - r)$ is a factor of the polynomial $P(x)$ and vice versa. This is known as the **Factor Theorem** and leads to the following formulas for factoring the difference of powers.

Difference of the n-th Powers

$$a^n - b^n = (a - b)\left(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}\right)$$

Difference of Squares or Cubes

Difference of squares:

$$a^2 - b^2 = (a - b)(a + b)$$

Difference of cubes:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Sum of cubes (from difference of cubes with b replaced by $-b$):

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Remark

There are also formulas for the **sum of even powers** but involves complex numbers.

Example 7.4. Factor $25x^2 - 16$.

Solution.

1) Recognize the binomial as a difference of squares.

$$(\_\_\_\_)^2 - 4^2$$

2) Apply the difference of squares formula.

$$(5x - 4)(\_\_\_\_\_\_)$$

Example 7.5. Factor $8x^3 + 27y^3$ completely.

Solution.

$$\begin{aligned} 8x^3 + 27y^3 &= (2x)^3 + (\_\_\_\_)^{\_\_\_} \\ &= (\_\_\_\_ + \_\_\_\_)\left((\_\_\_\_)^2 - (2x)(3y) + (\_\_\_\_)^2\right) \\ &= (\_\_\_\_ + \_\_\_\_)(\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_) \end{aligned}$$

7.5 Factoring a Trinomial

If a trinomial $ax^2 + bx + c$ with $a \neq 0$ (also known as a **quadratic polynomial**) can be factored, then we can expect a product of two binomials:

$$ax^2 + bx + c = (mx + p)(nx + q)$$

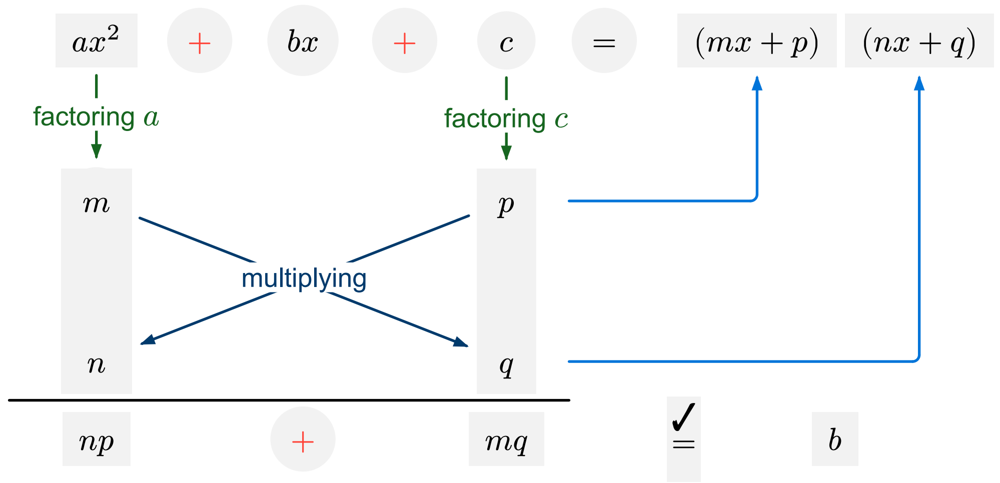
Simplifying the product and comparing coefficients leads to the following equations that m, n, p and q must satisfy:

$$a = mn \qquad b = mq + np \qquad c = pq$$

Solving the equations for m, n, p and q gives us the factors of the trinomial. This approach is known as the **method of undetermined coefficients**. However, since the equations are nonlinear, they cannot be solved directly using standard algebraic methods. When the coefficients a, b and c are integers, the structure of the equations provides a guideline for finding the factors of the trinomial. We can systematically test combinations of factors of a and c by **trial-and-error method** to find suitable values for m, n, p, and q. A effective way is to use a diagram.

A diagram algorithm for factoring a trinomial

In the following diagram, arrows indicate the steps.



Example 7.6. Factor $x^2 + 6x + 8$.

Solution (Method of undetermined coefficients).

1) Factoring a.

$$a = 1 = 1 \cdot 1$$

2) Factoring c.

$$c = 8 = 1 \cdot 8 = 2 \cdot 4$$

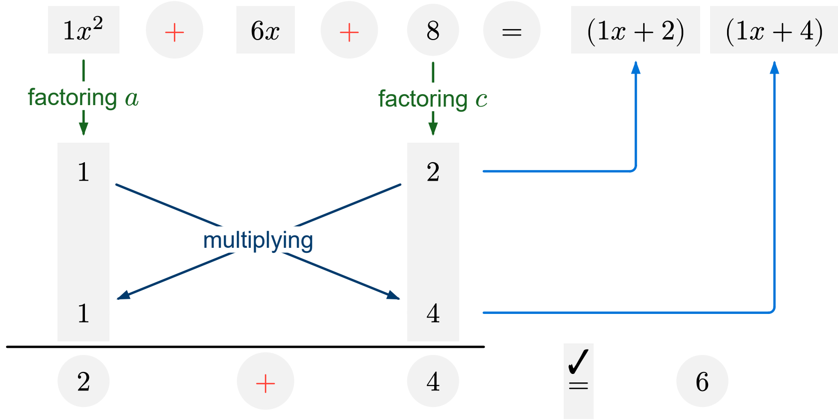
3) Choose a proper combination of pairs of factors and check if the sum of cross product equals b.

$$1 \cdot 4 + 1 \cdot 2 = 6 = b$$

4) Factor the trinomial.

$$x^2 + 6x + 8 = (x + 2)(x + 4)$$

The above procedure can be visualized using the diagram algorithm:



Remark (The Decomposition (or Grouping) Method)

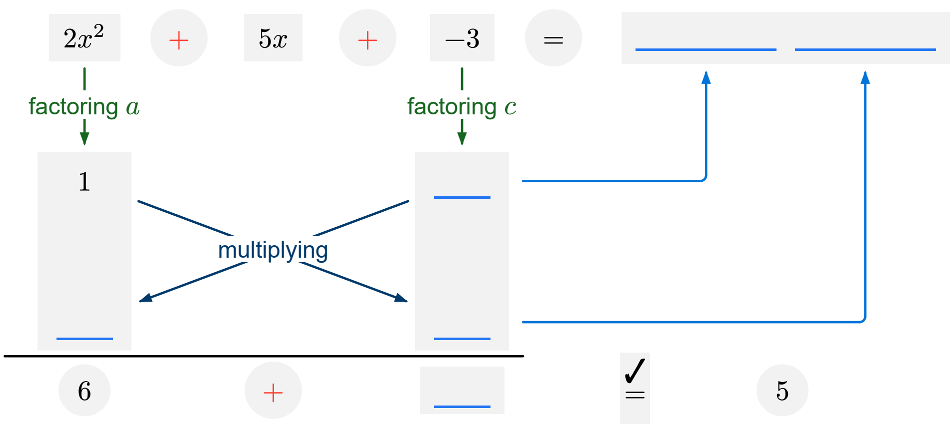
Note that $(mq)(np) = (mn)(pq) = ac$ and $mq + np = b$. Therefore, one can also factor the trinomial by finding a pair of factors f and g of the product ac such that bx as $fx + gx$, and then factoring by grouping.

Solution (Grouping method). Since $ac = 8 = 2 \cdot 4$ and $b = 6 = 2 + 4$, decomposing the middle term $6x$ leads to the factorization:

$$\begin{aligned} x^2 + 6x + 8 &= x^2 + \underline{\quad\quad} + \underline{\quad\quad} + 8 \\ &= (x^2 + \underline{\quad\quad}) + (\underline{\quad\quad} + 8) \\ &= x(x + \underline{\quad\quad}) + \underline{\quad\quad}\ (\underline{\quad\quad\quad\quad}) \\ &= (x + 2)(x + 4). \end{aligned}$$

Example 7.7. Factor $2x^2 + 5x - 3$.

Solution.

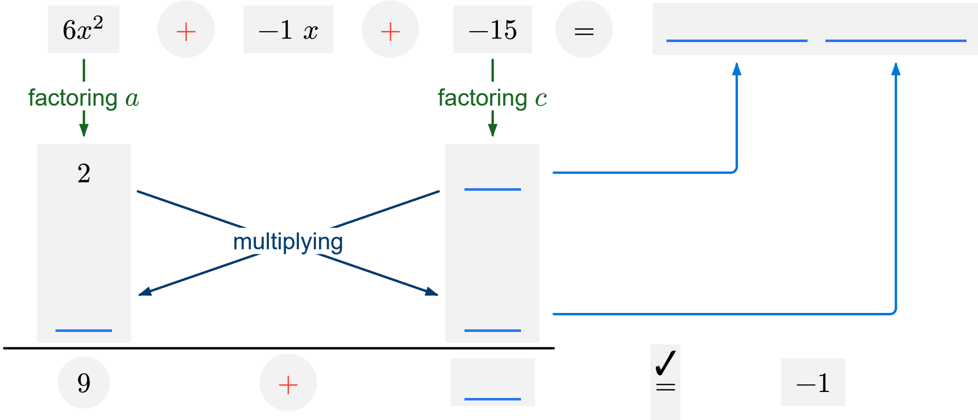


Using the diagram, we can factor the trinomial as follows:

$$2x^2 + 5x - 3 = (2x - 1)(x + 3).$$

Example 7.8. Factor $6x^2 - x - 15$.

Solution.



Using the diagram, we can factor the trinomial as follows:

$$6x^2 - x - 15 = (2x + 3)(3x - 5).$$

Factoring Perfect-Square Trinomials

A **perfect-square trinomial** is a trinomial of the form $A^2 \pm 2AB + B^2$, where A and B are expressions.A perfect-square trinomial can be factored as

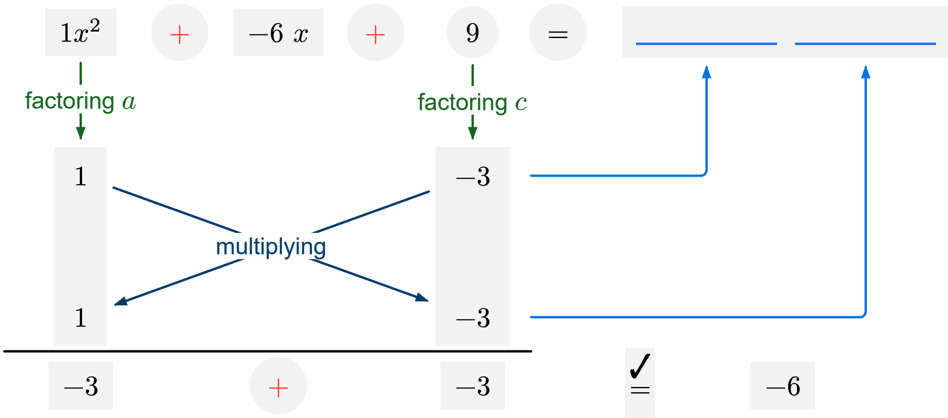
$$A^2 \pm 2AB + B^2 = (A \pm B)^2$$

Example 7.9. Factor $x^2 - 6x + 9$.

Solution. Note that $a = 1 = 1^2$, $c = 9 = 3^2$ and $b = -6 = -2 \cdot 1 \cdot 3$. The trinomial is a perfect square trinomial, which can be factored as follows:

$$x^2 - 6x + 9 = (x - \underline{\quad\quad})^2.$$

This can also be obtained using the diagram:



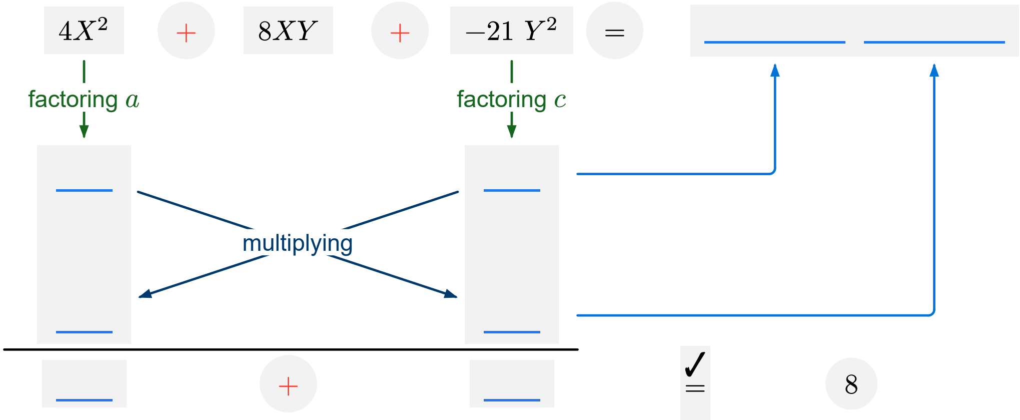
Remark (Trinomial with Two Variables)

The reason we focus only on the coefficients is that we expect to factor x^2 as $x \cdot x$.

If a trinomial is given in the form $aX^2 + bXY + cY^2$, then we expect its factorization to be $(mX + pY)(nX + qY)$, where X and Y are expressions. Once again, we only need to consider the coefficients a, b, and c to determine the factors of the trinomial.

Example 7.10. Factor $4X^2 + 8XY - 21Y^2$.

Solution.



So,

$$4X^2 + 8XY - 21Y^2 = (2X - 3Y)(2X + 7Y).$$

7.6 Factoring Completely

In general, factoring a polynomial completely may require using more than one strategy we have learned.

The following systematic approach helps ensure that the polynomial is factored completely:

1) Factor out the greatest common factor (GCF).
2) Look for special products (e.g., difference of squares, perfect square trinomials).
3) Factor trinomials.
4) Repeat the process until no further factoring is possible.

Example 7.11. Factor $2x^3y - 32xy^5$ completely.

Solution.

$$\begin{aligned} 2x^3y - 32xy^5 &= 2xy(\underline{\hspace{3cm}}) \\ &= 2xy\left(x^2 - (\underline{\hspace{1cm}})^2\right) \\ &= 2xy(x - 4y^2)(\underline{\hspace{2cm}}) \end{aligned}$$

Example 7.12. Factor completely.

$$4x^4 - 4x^2y^2 + 12x^3z - 12xy^2z$$

Solution.

$$\begin{aligned}
&4x^4 - 4x^2y^2 + 12x^3z - 12xy^2z \\
&= \underline{\hspace{2em}}\,(x^3 - xy^2 + 3x^2z - 3y^2z) \\
&= \underline{\hspace{2em}}\,((x^3 - xy^2) + \underline{\hspace{6em}}) \\
&= \underline{\hspace{2em}}\,(x(\underline{\hspace{5em}}) + \underline{\hspace{2em}}\,(x^2 - y^2)) \\
&= \underline{\hspace{2em}}\,(x + 3z)(\underline{\hspace{5em}}) \\
&= 4x(x + 3z)(x - y)(x + y)
\end{aligned}$$

7.7 Quadratic-like Polynomials

When all terms have even degrees, or share a common factor in their degrees (e.g., $x^4 - x^2 - 6$ or $x^9 - y^6$), we can replace a common power with a new variable to simplify the expression. This technique, known as the **substitution method**, is especially useful for factoring and solving equations.

Example 7.13. Factor the trinomial completely.

$$4x^4 - x^2 - 3$$

Solution. One approach is to use a substitute.

1) Let $Y = \underline{\hspace{2em}}$. Then $4x^4 - x^2 - 3 = 4Y^2 - Y - 3$.
2) Factor the trinomial in Y: $4Y^2 - Y - 3 = (\underline{\hspace{4em}})(Y - 1)$.
3) Replace Y by x^2 and factor further.

$$\begin{aligned}
4x^4 - x^2 - 3 &= 4Y^2 - Y - 3 \\
&= (\underline{\hspace{4em}})(Y - 1) \\
&= (4\underline{\hspace{2em}} + 3)(x^2 - 1) \\
&= (\underline{\hspace{4em}})(\underline{\hspace{4em}})(\underline{\hspace{4em}})
\end{aligned}$$

Example 7.14. Factor the polynomial $3(a - 5)^2 - a + 3$.

Solution. One approach is to simplify the polynomial first and then factor it. However, this often results in larger coefficients and makes the factoring process more complicated. A better approach is to use substitution: set $Y = a - 5$. Then

$$\begin{aligned}
3(a - 5)^2 - a + 3 &= 3Y^2 - (Y + 5) + 3 \\
&= 3Y^2 - Y - 2 \\
&= (\underline{\hspace{4em}})(Y - 1) \\
&= (\underline{\hspace{2em}}\,(a - 5) + \underline{\hspace{2em}})((a - 5) - 1) \\
&= (\underline{\hspace{3em}})(a - 6).
\end{aligned}$$

Exercises

Exercise 7.1. Factor out the GCF.

1) $18x^2y^2 - 12xy^3 - 6x^3y^4$ 2) $5x(x-7) + 3y(x-7)$ 3) $-2a^2(x+y) + 3a(x+y)$

Answer: 1) $6xy^2(3x - 2y - x^2y^2)$ 2) $(x-7)(5x+3y)$ 3) $a(x+y)(-2a+3)$

Exercise 7.2. Factor by grouping.

1) $12xy - 10y + 18x - 15$ 2) $12ac - 18bc - 10ad + 15bd$ 3) $5ax - 4bx - 5ay + 4by$

Answer: 1) $(6x-5)(2y+3)$ 2) $(2a-3b)(6c-5d)$ 3) $(x-y)(5a-4b)$

Exercise 7.3. Factor into polynomials with integer coefficients.

1) $25x^2 - 4$ 2) $8x^3 - y^3$

Answer: 1) $(5x-2)(5x+2)$ 2) $(2x-y)(4x^2+2xy+y^2)$

Exercise 7.4. Factor into polynomials with integer coefficients.

1) $9m^2 - 4n^2$ 2) $27A^3 + 125B^3$

Answer: 1) $(3m-2n)(3m+2n)$ 2) $(3A+5B)(9A^2-15AB+25B^2)$

Exercise 7.5. Factor the trinomial.

1) $x^2 + 4x + 3$ 2) $x^2 + 6x - 7$ 3) $x^2 - 3x - 10$

Answer: 1) $(x+1)(x+3)$ 2) $(x-1)(x+7)$ 3) $(x+2)(x-5)$

Exercise 7.6. Factor the trinomial.

1) $5x^2 + 7x + 2$ 2) $2x^2 + 5x - 12$ 3) $3x^2 - 10x - 8$

Answer: 1) $(x+1)(5x+2)$ 2) $(x+4)(2x-3)$ 3) $(x-4)(3x+2)$

Exercise 7.7. Factor completely into polynomials with integer coefficients.

1) $25x^3y - 4xy^3$ 2) $24a^5 - 3a^2$

Answer: 1) $xy(5x-2y)(5x+2y)$ 2) $3a^2(2a-1)(4a^2+2a+1)$

Exercise 7.8. Factor completely into polynomials with integer coefficients

1) $3x^3 + 6x^2 - 12x - 24$ 2) $x^4 + 3x^3 - 4x^2 - 12x$

Answer: 1) $3(x-2)(x+2)^2$ 2) $x(x+3)(x-2)(x+2)$

Exercise 7.9. Factor completely into polynomials with integer coefficients.

1) $(x-1)^3 - 4(x-1)$ 2) $4x^4 + 19x^2 - 5$ 3) $2x^3y - 9x^2y^2 - 5xy^3$

Answer: 1) $(x-1)(x-3)(x+1)$ 2) $(2x+1)(2x-1)(x^2+5)$ 3) $xy(2x+5)(x-y)$

Exercise 7.10. Each trinomial below has a factor in the table. Match the letter on the left of a factor with the number on the left of a trinomial to decipher the following quotation.

> $\underline{\;13\;}\quad \underline{\;10\quad 2\quad 9\quad 15\;},\quad \underline{\;9\quad 5\quad 14\;}\quad \underline{\;13\;}\quad \underline{\;4\quad 3\quad 15\quad 7\quad 2\quad 1\;};$
>
> $\underline{\;13\;}\quad \underline{\;11\quad 2\quad 2\;},\quad \underline{\;9\quad 5\quad 14\;}\quad \underline{\;13\;}\quad \underline{\;15\quad 2\quad 8\quad 2\quad 8\quad 6\quad 2\quad 15\;};$
>
> $\underline{\;13\;}\quad \underline{\;14\quad 3\;},\quad \underline{\;9\quad 5\quad 14\;}\quad \underline{\;13\;}\quad \underline{\;12\quad 5\quad 14\quad 2\quad 15\quad 11\quad 1\quad 9\quad 5\quad 14\;}.$

— Chinese proverb

1) $x^2 - 2x - 24$
2) $6x^2 + x - 2$
3) $x^2 - 16x + 39$
4) $6x^2 + 13x - 5$
5) $x^2 - 5x - 14$
6) $2x^2 - 5x - 3$
7) $x^2 - x - 110$
8) $x^2 + 7x + 10$
9) $-3x^2 + 11x - 6$
10) $x^2 - 10x + 16$
11) $-2x^2 + 5x + 12$
12) $42x^2 - x - 1$
13) $-2x^2 - 3x + 27$
14) $x^2 + 14x + 49$
15) $x^2 - 81$
16) $x^2 + 8x + 12$
17) $x^2 + x - 2$
18) $x^2 + 5x + 6$
19) $2x^2 - 3x - 5$
20) $5x^2 + 7x - 6$
21) $4x^2 - 3x - 22$
22) $3x^2 - 4x - 15$
23) $3x^2 + 10x + 8$
24) $8x^2 - 37x - 15$
25) $x^2 - 12x - 28$
26) $5x^2 + 4x - 12$

A: $3x-2$	**B:** $2x+1$	**C:** $x+6$	**D:** $x+7$	**E:** $2x-1$	**F:** $3x-1$
G: $x+10$	**H:** $x-8$	**I:** $2x+9$	**J:** $x-1$	**K:** $x+3$	**L:** $2x-5$
M: $x+5$	**N:** $x-7$	**O:** $x-13$	**P:** $5x-3$	**Q:** $4x-11$	**R:** $x-9$
S: $2x+3$	**T:** $x+4$	**U:** $7x+1$	**V:** $3x+5$	**W:** $3x+4$	**X:** $8x+3$
Y: $x-14$	**Z:** $5x-6$				

Answer: I hear, and I forget; I see, and I remember; I do, and I understand.

Topic 8 Rational Expressions

Learning Goals

- I can determine the domain of a rational expression by identifying values that make the denominator zero.
- I can use the equivalence property of rational expressions to simplify rational expressions.
- I can add, subtract, multiply, and divide rational expressions and express the result in reduced form.
- I can simplify complex rational expressions.

8.1 Think about It

Surprising But It's True

Tim and Jim refill their cars at the same gas station twice last month. Each time Tim got $30 worth of gas and Jim got 10 gallons. Suppose they refill their cars on the same days. The price was $3 per gallon the first time. The price on the second time changed. Determine who would have the lower average price (dollar per gallon).

8.2 Basics of Rational Expressions

A **rational expression** is a fraction where both the **numerator** and **denominator** are polynomials. The **domain** of a rational expression is the set of all real numbers except those that make the denominator zero.

Example 8.1. Find the domain of the rational expression $\dfrac{x^2+3x+2}{x-1}$.

Solution. The denominator is $x-1$. Setting it to zero gives $x-1=0$, or $x=1$. Thus, the domain is all real numbers except $x=1$.

Remark

In set-builder notation, the domain is $\{x \in \mathbb{R} \mid x \neq 1\}$, where $\mathbb{R}$ is the set of real numbers.

In the interval notation, the domain is $(-\infty, 1) \cup (1, \infty)$, where $(-\infty, 1)$ represents the set real numbers less than 1 and $(1, \infty)$ represents the sets real numbers greater than 1, the symbol $\cup$, read as "union", means combining two sets. We will have more detailed discussion on interval notation in the topic on linear inequalities.

A rational expression is **simplified** if the numerator and the denominator have no common factors other than 1 or -1.

Fundamental Fraction Properties

Let A, B, and C are real numbers such that $B \neq 0$.

Multiplication Property of Equality (with fractions)

$$\frac{A}{B} = C \iff A = B \cdot C.$$

Property of Equivalent Fractions If $C \neq 0$, then

$$\frac{A \cdot C}{B \cdot C} = \frac{A}{B} \qquad \text{and} \qquad \frac{\frac{A}{C}}{\frac{B}{C}} = \frac{A}{B}.$$

Remark

The equivalence property allows us to simplify rational expressions by multiplying or dividing out common nonzero factors. However, a simplified rational expression may have a larger domain than the original. Therefore, when writing an equality between the original and simplified expressions, we are asserting that the two are equal only on their **common domain**, that is, the set of values where both expressions are defined.

Example 8.2. Simplify $\dfrac{x^2+4x+3}{x^2+3x+2}$

Solution. Factor both the numerator and denominator, and then remove common factors:

$$\frac{x^2+4x+3}{x^2+3x+2} = \frac{(x+1)(\_\_\_\_\_\_)}{(x+1)(\_\_\_\_\_\_)} = \frac{\cancel{(x+1)}(\_\_\_\_\_\_)}{\cancel{(x+1)}(\_\_\_\_\_\_)} = \_\_\_\_\_\_$$

8.3 Multiplication and Division of Rational Expressions

Multiplication and Division of Rational Expressions

Suppose P, S, Q, T are expressions such that $Q \neq 0$ and $T \neq 0$. We have the following rules of multiplication and division of rational expressions:

Multiplication

$$\frac{P}{Q} \cdot \frac{S}{T} = \frac{P \cdot S}{Q \cdot T}$$

Division

$$\frac{\frac{P}{Q}}{\frac{S}{T}} = \frac{P}{Q} \div \frac{S}{T} = \frac{P}{Q} \cdot \frac{T}{S} \quad \text{if} \quad S \neq 0.$$

Example 8.3. Multiply and simplify $\dfrac{3x^2}{x^2+x-6} \cdot \dfrac{x^2-4}{6x}$.

Solution. Multiply, factor numerators and denominators, and then remove common factors:

$$\frac{3x^2}{x^2+x-6} \cdot \frac{x^2-4}{6x} = \frac{3x^2(x^2-4)}{6x(x^2+x-6)} = \frac{3x^2(\underline{\qquad})(x+2)}{6x(x-2)(\underline{\qquad})} = \underline{\qquad\qquad}.$$

Remark

The multiplication can be done at any step, for example, factor first and then multiply, or factor first, remove, and then multiply. Usually, it is better to factor first, then multiply/divide, and finally remove common factors. This approach can help avoid mistakes.

Example 8.4. Divide and simplify $\dfrac{2x+6}{x^2-6x-7} \div \dfrac{6x-2}{2x^2-x-3}$.

Solution. Factor both rational expressions, then multiply by the reciprocal of the second expression, and finally remove common factors:

$$\begin{aligned}\frac{2x+6}{x^2-6x-7} \div \frac{6x-2}{2x^2-x-3} &= \frac{2(x+3)}{(x-7)(x+1)} \div \frac{2(3x-1)}{(2x-3)(x+1)} \\ &= \frac{2(x+3)}{(x-7)(x+1)} \cdot \frac{\underline{\qquad\qquad}}{\underline{\qquad\qquad}} \\ &= \frac{(x+3)(\underline{\qquad})}{(x-7)(\underline{\qquad})}\end{aligned}$$

8.4 Adding/Subtracting Rational Expressions

8.4.1 With Common Denominators

Addition and Subtraction of Rational Expressions

Suppose $R \neq 0$:

$$\frac{P}{R} + \frac{Q}{R} = \frac{P+Q}{R} \quad \text{and} \quad \frac{P}{R} - \frac{Q}{R} = \frac{P-Q}{R}.$$

Example 8.5. Add and simplify $\dfrac{x^2+4}{x^2+3x+2} + \dfrac{5x}{x^2+3x+2}$.

Solution. Since the denominators are the same, we combine the numerators, keep the common denominator, and simplify:

$$\begin{aligned}\frac{x^2+4}{x^2+3x+2} + \frac{5x}{x^2+3x+2} &= \frac{\underline{\qquad\qquad}}{x^2+3x+2} \\ &= \frac{(x+1)(x+4)}{(x+1)(x+2)} = \frac{x+4}{x+2}\end{aligned}$$

Example 8.6. Subtract $\dfrac{2x^2}{2x^2-x-3} - \dfrac{3x+5}{2x^2-x-3}$

Solution. Since the denominators are the same, we subtract the numerators, keep the common denominator, and simplify:

$$\begin{aligned}\frac{2x^2}{2x^2-x-3} - \frac{3x+5}{2x^2-x-3} &= \frac{\underline{\qquad\qquad}}{2x^2-x-3} \\ &= \frac{(\underline{\qquad})(x+1)}{(\underline{\qquad})(x+1)} \\ &= \underline{\qquad\quad}\end{aligned}$$

8.4.2 With Different Denominators

When the denominators of two rational expressions are different, we can use the equivalence property of fractions to rewrite the expressions with a common denominator in order to add or subtract them. To simplify the process, we look for the **least common denominator** (LCD) that is the **least common multiple (LCM)** of the denominators.

The **least common multiple of two polynomials** is a polynomial that is divisible by both polynomials and that divides any common multiple of the two.

The relation between LCM and GCF

Given two polynomial P an Q, the least common multiple $\mathrm{LCM}(P,Q)$ and greatest common factor $\mathrm{GCF}(P,Q)$ are related by the formula:

$$\mathrm{LCM}(P(x),Q(x)) = \frac{P(x)\cdot Q(x)}{\mathrm{GCF}(P(x),Q(x))}.$$

Example 8.7. Find LCD of $\dfrac{3}{x^2-x-6}$ and $\dfrac{6}{x^2-4}$

Solution (Using the formula).

1) Factor denominators:

$$x^2-x-6 = (x+2)(x-3)$$
$$x^2-4 = (x-2)(x+2)$$

2) Since there is only one nontrivial common factor $(x+2)$, the GCF is $x+2$.

3) Find the LCD using the formula:

$$\mathrm{LCD} = \frac{(x+2)(x-3)\cdot(x-2)(x+2)}{(x+2)} = \underline{\qquad\quad}\ (x-3)(x-2)$$

LCM of Two Polynomials as the Product of Factors with Higher Exponents

Given two polynomials, P and Q, the LCM is also the product of all factors of P and Q that have the higher exponents. For example, the LCM of $(x-2)(x+2)$ and $(x-2)^2(x+3)$ is $(x-2)^2(x+2)(x+3)$.

The method described above suggested an approach to find the LCM of two polynomials using a table which consists of three rows:

Row 1 List of factors of one polynomial

Row 2 List of factors of the other polynomial with already appeared factors aligned in columns.

Row 3 Combined list of factors that have the higher exponents.

The LCM is the product of all factors in the combined list.

Solution (Using a table).

1) Factor denominators:

$$x^2 - x - 6 = (x+2)(x-3)$$
$$x^2 - 4 = (x-2)(x+2)$$

2) List the factors of each denominator:

Factors of $x^2 - x - 6$	$x+2$	$x-3$	
Factors of $x^2 - 4$	$x+2$		$x-2$
Combined list	$x+2$	$x-3$	$x-2$

3) The LCD is the product of all factors in the combined list:

$$\text{LCD} = (x+2)(x-3)(x-2)$$

Example 8.8. Subtract and simplify $\dfrac{x-3}{x^2-2x-8} - \dfrac{1}{x^2-4}$

Solution.

1) Factor denominators:

$$x^2 - 2x - 8 = (x+2)(x-4)$$
$$x^2 - 4 = (x-2)(x+2)$$

2) The LCD is $(x+2)(x-2)(x-4)$:

3) Rewrite the two rational expressions with the common denominator, replacing the denominators with the LCD and multiplying the numerators by the new factors in the denominators:

$$\frac{x-3}{x^2-2x-8} - \frac{1}{x^2-4} = \frac{(x-3)(\underline{\qquad\qquad})}{(x+2)(x-4)(x-2)} - \frac{\underline{\qquad\qquad}}{(x+2)(x-2)(x-4)}$$

4) Subtract and simplify:

$$= \frac{(x-3)(\underline{\qquad\qquad}) - (\underline{\qquad\qquad})}{(x+2)(x-4)(x-2)}$$

$$= \frac{x^2 - 6x + 10}{(x+2)(x-2)(x-4)}$$

Example 8.9. Add and simplify $\dfrac{3y+4}{y^2-5y-6}+\dfrac{y-3}{6-y}$

Solution.

1) Factor denominators:

$$y^2-5y-6=(y+1)(y-6)$$
$$6-y=\underline{\qquad}(y-6)$$

2) The LCD can be taken as $(y-6)(y+1)$.

3) Before, we rewrite the rational expressions into equivalent expressions with the same denominator, we want to rewrite the second rational expression so that leading coefficient of the denominator is positive. We can do this by multiplying the numerator and denominator by -1:

$$\frac{3y+4}{y^2-5y-6}+\frac{y-3}{6-y}=\frac{3y+4}{y^2-5y-6}\ \underline{\qquad}\ \frac{y-3}{y-6}$$

4) Rewrite the two rational expressions with the common denominator, replacing the denominators with the LCD and multiplying the numerators by the new factors in the denominators:

$$=\frac{3y+4}{(y+1)(y-6)}\ \underline{\qquad}\ \frac{(y-3)(\underline{\qquad\qquad})}{(y+1)(y-6)}$$

5) Add and simplify:

$$=\frac{(3y+4)\ \underline{\qquad}\ (y-3)(\underline{\qquad\qquad})}{(y+1)(y-6)}$$
$$=\frac{-y^2+\underline{\qquad}+\underline{\qquad}}{(y+1)(y-6)}$$
$$=-\frac{(\underline{\qquad})(y-7)}{(y+1)(y-6)}$$
$$=-\frac{y-7}{y-6}$$

Problem-solving Strategy: Equivalent Reduction

The idea of the **equivalent reduction strategy** is to intentionally transform an expression or equation into a different but mathematically equivalent form that is simpler and easier to manipulate or solve. The reduction in the above examples is to rewrite the rational expression into equivalent rational expressions with a common denominator.

8.5 Simplifying Complex Rational Expressions

A **complex rational expression** is a fraction whose denominator or numerator contains a rational expression. The fractional bar separating the rational expressions on the top and the bottom is called the **major fractional bar**. Since dividing by a number is equivalent to multiplying by its reciprocal, one strategy to rewrite a complex rational expression as a multiplication of the numerator with the reciprocal of the denominator after simplifying the numerator and denominator.

Example 8.10. Simplify

$$\frac{\dfrac{2x-1}{x^2-1}+\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}-\dfrac{1}{x^2-1}}$$

Solution.

1) Factor all denominators and then simplify the numerator and the denominator with respect to the major fractional bar.

$$\frac{\dfrac{2x-1}{x^2-1}+\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}-\dfrac{1}{x^2-1}} = \frac{\dfrac{2x-1}{(x-1)(x+1)}+\dfrac{(x-1)(x-1)}{(x-1)(x+1)}}{\dfrac{(x+1)(x+1)}{(x-1)(x+1)}-\dfrac{1}{(x-1)(x+1)}}$$

$$= \frac{\dfrac{\underline{\qquad\qquad}}{(x-1)(x+1)}}{\dfrac{\underline{\qquad\qquad}}{(x-1)(x+1)}}$$

2) Rewrite the fraction of rational expressions as a product of rational expressions and then simplify.

$$= \frac{\underline{\qquad\qquad}}{(x-1)(x+1)} \cdot \frac{(x-1)(x+1)}{\underline{\qquad\qquad}}$$

$$= \frac{\underline{\qquad\qquad}}{\underline{\qquad\qquad}}$$

$$= \frac{x^2}{x^2+2x}$$

$$= \underline{\qquad\qquad}$$

Simplifying Complex Rational Expression by Clearing All Denominators

Another way to simplify a complex rational expression is to multiply the LCD of all denominators to both the denominator and numerator and then simplify. This approach works better if the denominators share most factors.

Example 8.11. Simplify the following complex rational expression.

$$\frac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}}$$

Solution.

1) Since the denominators are either $x+1$ or $x-1$, the LCD is the product of the two distinct factors, that is, $(x-1)(x+1)$.

2) Multiply each rational expression by the LCD simplify.

$$\begin{aligned}\frac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}} &= \frac{\dfrac{x+1}{x-1}\cdot(x+1)(x-1)+\dfrac{x-1}{x+1}\cdot(x+1)(x-1)}{\dfrac{x+1}{x-1}\cdot(x+1)(x-1)-\dfrac{x-1}{x+1}\cdot(x+1)(x-1)}\\ &= \frac{(x+1)^2+(x-1)^2}{(x+1)^2-(x-1)^2}\\ &= \frac{\underline{\qquad\qquad}}{\underline{\qquad\qquad}}\\ &= \frac{\underline{\qquad\qquad}}{2x}\end{aligned}$$

Problem-solving strategies: Change the Viewpoint

Changing the viewpoint of a problem can often lead to a much simpler solution. In the case of complex rational expressions, rewriting the expression as a multiplication by the reciprocal of the denominator reduce the question to more familiar and easier to simplify question.

Exercises

Exercise 8.1. Find the domain of the following rational expressions.

1) $\dfrac{x^2+3x+2}{x+1}$ 2) $2x^2+\dfrac{-5x+3}{x-2}$ 3) $\dfrac{3x-1}{x+2}-\dfrac{5x-7}{x-3}$

Answer: 1) $x \neq -1$ 2) $x \neq 2$ 3) $x \neq -2, x \neq 3$

Exercise 8.2. Simplify. Express the result in reduced form.

1) $\dfrac{3x^2 - x - 4}{x + 1}$ 2) $\dfrac{2x^2 - x - 3}{2x^2 + 3x + 1}$ 3) $\dfrac{x^2 - 9}{3x^2 - 8x - 3}$

Answer: 1) $3x - 4$ 2) $\dfrac{2x - 3}{2x + 1}$ 3) $\dfrac{x + 3}{3x + 1}$

Exercise 8.3. Multiply and simplify. Express the result in reduced form.

1) $\dfrac{x+5}{x+4} \cdot \dfrac{x^2+3x-4}{x^2-25}$ 2) $\dfrac{3x^2-2x}{x+2} \cdot \dfrac{3x^2-4x-4}{9x^2-4}$ 3) $\dfrac{6y-2}{3-y} \cdot \dfrac{y^2-6y+9}{3y^2-y}$

Answer: 1) $\dfrac{x-1}{x-5}$ 2) $\dfrac{x(x-2)}{x+2}$ 3) $-\dfrac{2(y-3)}{y}$

Exercise 8.4. Divide and simplify. Express the result in reduced form.

1) $\dfrac{9x^2-49}{6} \div \dfrac{3x^2-x-14}{2x+4}$ 2) $\dfrac{x^2+3x-10}{2x-2} \div \dfrac{x^2-5x+6}{x^2-4x+3}$ 3) $\dfrac{y-x}{xy} \div \dfrac{x^2-y^2}{y^2}$

Answer: 1) $\dfrac{3x+7}{3}$ 2) $\dfrac{x+5}{2}$ 3) $-\dfrac{y}{x(x+y)}$

Exercise 8.5. Simplify. Express the result in reduced form.

$$\frac{-x^2+11x-18}{x^2-4x+4} \div \frac{x^2-5x-36}{x^2-7x+12} \cdot \frac{2x^2+7x-4}{x^2+2x-15}$$

Answer: $-\dfrac{(x-4)(2x-1)}{(x-2)(x+5)}$

Exercise 8.6. Add/subtract and simplify. Express the result in reduced form.

1) $\dfrac{x^2+2x-2}{x^2+2x-15}+\dfrac{5x+12}{x^2+2x-15}$

2) $\dfrac{3x^2-10}{x^2-25}-\dfrac{14x-5}{x^2-25}$

Answer: 1) $\dfrac{x+2}{x-3}$ 2) $\dfrac{3x+1}{x+5}$

Exercise 8.7. Find the LCD of the following rational expressions.

1) $\dfrac{2x}{2x^2-5x-3}$ and $\dfrac{x-1}{x^2-x-6}$

2) $\dfrac{9}{7x^2-28x}$ and $\dfrac{2}{x^2-8x+16}$

Answer: 1) $(2x+1)(x-3)(x+2)$ 2) $7x(x-4)^2$

Exercise 8.8. Add and simplify. Express the result in reduced form.

1) $\dfrac{2}{x-2}+\dfrac{x-10}{x^2-4}$ 2) $\dfrac{x}{x-5}+\dfrac{3x+3}{x^2-4x-5}$ 3) $\dfrac{4x+1}{2x^2+x-1}+\dfrac{2x+7}{3x^2+x-2}$

Answer: 1) $\dfrac{3}{x+2}$ 2) $\dfrac{x+3}{x-5}$ 3) $\dfrac{16x-9}{(2x-1)(3x-2)}$

Exercise 8.9. Subtract and simplify. Express the result in reduced form.

1) $\dfrac{2x-3}{x^2-4}-\dfrac{x+2}{x^2+2x-8}$ 2) $\dfrac{3x-11}{x^2-7x+12}-\dfrac{2}{x-3}$ 3) $\dfrac{y+8}{y^2-5y-6}-\dfrac{6}{y^2-4x-5}$

Answer: 1) $\dfrac{x^2+x-16}{(x-2)(x+2)(x+4)}$ 2) $\dfrac{1}{x-4}$ 3) $\dfrac{y-4}{(y-6)(y-5)}$

Exercise 8.10. Simplify. Express the result in reduced form.

1) $\dfrac{x+11}{7x^2-2x-5}+\dfrac{x-2}{x-1}-\dfrac{x}{7x+5}$

2) $\dfrac{x-1}{x^2-3x}+\dfrac{4}{x^2-2x-3}-\dfrac{1}{x(x+1)}$

Answer: 1) $\dfrac{6x-1}{7x+5}$ 2) $\dfrac{x+2}{x(x-3)}$

Exercise 8.11. Simplify. Express the result in reduced form.

1) $\dfrac{\frac{x^2-y^2}{y^2}}{\frac{1}{x}-\frac{1}{y}}$

2) $\dfrac{\frac{x+1}{x-1}+\frac{x-1}{x+1}}{\frac{x+1}{x-1}-\frac{x-1}{x+1}}$

3) $\dfrac{\frac{2}{x^2+2x+1}-\frac{1}{x+1}}{1-\frac{4}{x^2+2x+1}}$

Answer: 1) $-\dfrac{x(x+y)}{y}$ 2) $\dfrac{x^2+1}{2x}$ 3) $-\dfrac{1}{x+3}$

Exercise 8.12. Alice and Bob refill their cars at the same gas station twice last month. Each time, Alice purchased \$41.88 worth of gas, and Bob purchased 12 gallons. They refilled on the same days. The price was \$3.49 per gallon the first time. The price on the second time changed. Determine who would have the better average price.

Hint: Express the average price per gallon for each person in terms of p, simplify the difference of the two average prices, then factor the numerator.

Topic 9 Radicals and Rational Exponents

Learning Goals

- I can evaluate and simplify a radical without using a calculator.
- I can perform arithmetic operations (multiply, divide, combine like radicals) on radical expressions.
- I can rationalize the denominator or numerator of a given (radical) expression.

9.1 Think about It

Enough Coffee or Not

You drink a cup of 8 oz coffee that contains about 90 mg of caffeine. Your body eliminates half of the caffeine every 5 hours. Research suggests that as little as 30-50 mg of caffeine can noticeably improve alertness and cognitive performance in most people. After 7 hours, how much caffeine is left in your system? Would you still feel the effects?

9.2 Radical Expressions

A number x is called an **n-th root** of a number a if $x^n = a$. When $n = 2$, it is called a **square root**; when $n = 3$, it is called a **cube root**.

The square root of a positive number has two values: one positive and one negative. **The square root of a negative number is not real.** The cube root of a number is defined for all real numbers and has only one real value.

The nonnegative square root of a nonnegative number a is called the **principal square root** and is denoted by $\sqrt{a}$. The cube root of a number a is denoted by $\sqrt[3]{a}$.

In the notation $\sqrt[n]{a}$, the **radical sign** (or **root symbol**) is denoted as $\sqrt{}$, a is the **radicand**, and n is the **index**.

The Existence and Fundamental Property of n-th Roots

Let n be a positive integer and a be a real number.

- If n is odd, then a has a unique real n-th root, denoted as $\sqrt[n]{a}$.
- If n is even and $a \geq 0$, then a has a unique nonnegative real n-th root, called the **principal n-th root** of a and denoted as $\sqrt[n]{a}$.
- If n is even and $a < 0$, then a has no real n-th root.

Fundamental property

$$\left(\sqrt[n]{a}\right)^n = a \qquad \text{and} \qquad \sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ is even,} \\ a & \text{if } n \text{ is odd.} \end{cases}$$

A radical is **simplified** if the radicand has no perfect power factors (other than 1) matching the index.

Example 9.1. Simplify the radical expression.

1) $\sqrt[3]{x^3}$ 2) $\sqrt[4]{x^{12}}$ 3) $\sqrt[3]{-x^3}$ 4) $\left(\sqrt[3]{8x}\right)^6$

Solution.

1) $\sqrt[3]{x^3} = \underline{\qquad}$

2) $\sqrt[4]{x^{12}} = \sqrt[4]{(\underline{\qquad})^4} = x^3$

3) $\sqrt[3]{-x^3} = \sqrt[3]{(-x)^3} = \underline{\qquad}$

4) $\left(\sqrt[3]{8x}\right)^6 = \left(\left(\sqrt[3]{8x}\right)^3\right)^{\underline{\quad}} = (\underline{\qquad})^2 = 64x^2$

Example 9.2. Simplify the radical expression using the fundamental property.

1) $\sqrt{4(y-1)^2}$ 2) $\sqrt[3]{-8x^3y^6}$ 3) $\sqrt[4]{2x^2y^6}$

Solution.

1) $(4(y-1)^2) = \sqrt{(2(\underline{\qquad}))^2} = 2\underline{\qquad}$

2) $\sqrt[3]{-8x^3y^6} = \sqrt[3]{(\underline{\quad}x^{\underline{\quad}}y^{\underline{\quad}})^3} = \underline{\qquad}$

3) $\sqrt[4]{32x^2y^6} = \sqrt[4]{2^{\underline{\quad}}x^2y^{\underline{\quad}}} = \sqrt[4]{2x^2y^{\underline{\quad}}(2y^{\underline{\quad}})^4} = 2\underline{\quad}\sqrt[4]{2x^2y^{\underline{\quad}}}$

9.3 Rational Exponents

For a positive integer n, a real number a such that $\sqrt[n]{a}$ is real, and any integer m, the expression $a^{\frac{1}{n}}$, where $\frac{1}{n}$ is called a **rational exponent**, is defined as

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

More generally, for a rational number $\frac{m}{n}$, the expression $a^{\frac{m}{n}}$ is defined as

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m \quad \text{or} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

Important Fact: Rational exponents obeys the laws of integer exponents.

Warning

If $\sqrt[n]{a}$ is not real, then $\sqrt[n]{a^m}$ may not equal $a^{\frac{m}{n}}$. For example, $\sqrt{(-1)^2} \neq (-1)^{\frac{2}{2}}$.

Example 9.3. Rewrite the rational exponent in radical notation and evaluate.

1) $4^{\frac{3}{2}}$ 2) $-81^{\frac{3}{4}}$ 3) $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$

Solution.

1) $4^{\frac{3}{2}} = (\sqrt{\underline{\quad}})^{\underline{\quad}} = (\underline{\quad})^{\underline{\quad}} = 8$

2) $-81^{\frac{3}{4}} = -(\sqrt[\underline{\quad}]{\underline{\quad}})^3 = -(\underline{\quad})^3 = -27$

3) $\left(\frac{27}{8}\right)^{-\frac{2}{3}} = \left(\sqrt[\underline{\quad}]{\underline{\quad}}\right)^{-2} = \left(\underline{\quad}\right)^2 = \underline{\quad}$

Example 9.4. Simplify. Write in radical notation.

1) $\sqrt{x}\sqrt[3]{x^2}$ 2) $\sqrt[3]{\sqrt{x^3}}$ 3) $\left(\frac{x^{\frac{1}{2}}}{x^{-\frac{5}{6}}}\right)^{\frac{1}{4}}$ 4) $\sqrt{\frac{x^{-\frac{1}{2}}y^2}{x^{\frac{3}{2}}}}$

Solution.

1) $\sqrt{x}\sqrt[3]{x^2} = x^{\underline{\quad}}x^{\underline{\quad}} = x^{\underline{\quad}} = x\sqrt[6]{x}$

2) $\sqrt[3]{\sqrt{x^3}} = \left((x^3)^{\underline{\quad}}\right)^{\underline{\quad}} = x^{\underline{\quad}} = \underline{\quad\quad}$

3) $\left(\frac{x^{\frac{1}{2}}}{x^{-\frac{5}{6}}}\right)^{\frac{1}{4}} = (x^{\underline{\quad}})^{\frac{1}{4}} = x^{\underline{\quad}} = \underline{\quad\quad}$

4) $\sqrt{\frac{x^{-\frac{1}{2}}y^2}{x^{\frac{3}{2}}}} = \sqrt{\frac{\underline{\quad}}{\underline{\quad}}} = \left|\frac{y}{x}\right|$

Choosing the Appropriate Form to Use

In general, expressing radicals using rational exponents makes simplification easier.

9.4 Product and Quotient Rules for Radicals

Product and Quotient Rules for Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are both real numbers, then by the definition of n-th roots or the laws of rational exponents (which are the same as laws of integer exponents), we have:

$$\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}, \qquad \text{and} \qquad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \quad \text{if} \quad b \neq 0.$$

Warning

If both $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are not real, then the product or quotient rule may not hold. For example, $\sqrt{-1}\sqrt{-1} \neq \sqrt{(-1)(-1)}$.

Example 9.5. Multiply/Divide and Simplify. Write powers in positive exponents.

1) $\sqrt[4]{8xy^4}\,\sqrt[4]{2x^7y}$ 2) $\dfrac{\sqrt[5]{96x^9y^3}}{\sqrt[5]{3x^{-1}y}}$

Solution.

1) $\sqrt[4]{8xy^4 \cdot 2x^7y} = \sqrt[4]{16x^{\underline{\quad}}y^{\underline{\quad}}} = \sqrt[4]{\underline{\quad\quad}\left(\underline{\quad\quad}\right)^4} = \underline{\quad\quad\quad\quad}$

1) $\dfrac{\sqrt[5]{96x^9y^3}}{\sqrt[5]{3x^{-1}y}} = \sqrt[5]{\dfrac{96x^9y^3}{3x^{-1}y}} = \sqrt[5]{32x^{\underline{\quad}}y^2} = \sqrt[5]{\left(\underline{\quad\quad\quad}\right)^5y^2} = \underline{\quad\quad\quad}\sqrt[5]{y^2}$

9.5 Combining Like Radicals

Like radicals are radical expressions that have the same index and the same radicand. Like radicals can be combined by adding or subtracting their coefficients, similar to combining like terms in polynomials. To simplify a radical expression, first simplify each radical to see if they are like radicals.

Example 9.6. Simplify the radical expression.

1) $\sqrt{2}+\sqrt{8}-\sqrt{12}$ 2) $\sqrt{8x^3}-\sqrt{(-2)^2x^4}+\sqrt{2x^5}$

Solution. First simplify each radical term and then combine like radicals.

1)
$$\begin{aligned}&\sqrt{2}+\sqrt{8}-\sqrt{32}\\ &=\sqrt{2}+\underline{\quad\quad}\sqrt{2}-\underline{\quad\quad}\sqrt{2}\\ &=\underline{\quad\quad\quad\quad}\sqrt{2}\end{aligned}$$

2)
$$\begin{aligned}&\sqrt{8x^3}-\sqrt{(-2)^2x^4}+\sqrt{2x^5}\\ &=\underline{\quad\quad\quad}\sqrt{2x}-\underline{\quad\quad\quad}+x^2\sqrt{\underline{\quad\quad\quad}}\\ &=\left(\underline{\quad\quad\quad\quad}\right)\sqrt{2x}-2x^2\end{aligned}$$

9.6 Arithmetic Operations on Radical Expressions

Arithmetic operations on radical expressions are similar to those on polynomials, but we simplify radicals before combine like radicals.

Example 9.7. Simplify. Assume all variables are positive.

$$\left(\sqrt{2x}+2\sqrt{x}\right)\left(\sqrt{2x}-3\sqrt{x}\right)$$

Solution.

$$\begin{aligned}&\left(\sqrt{2x}+2\sqrt{x}\right)\left(\sqrt{2x}-3\sqrt{x}\right)\\ &=\sqrt{2x}\sqrt{2x}-3\sqrt{x}\sqrt{2x}+2\sqrt{x}\sqrt{2x}-6\sqrt{x}\sqrt{x}\\ &=\underline{\quad\quad\quad}-\underline{\quad\quad\quad}+\underline{\quad\quad\quad}-\underline{\quad\quad\quad}\\ &=\left(\underline{\quad\quad\quad}-\sqrt{\underline{\quad\quad\quad}}\right)x\end{aligned}$$

9.7 Rationalizing Denominators

Rationalizing the denominator means rewriting a fraction so that there are no radicals in the denominator. The new expression is mathematically equivalent to the original one for all real numbers where both expressions are defined.

To rationalize the denominator, we multiply the numerator and denominator by a suitable expression that will eliminate the radical in the denominator. The choice of this expression depends on whether the denominator is a single radical or a binomial involving radicals.

Example 9.8. Rationalize the denominator and simplify.

1) $\dfrac{1}{\sqrt[3]{2}}$ 2) $\dfrac{1}{2\sqrt{x^3}}$ 3) $\dfrac{\sqrt{2x^2}}{\sqrt{3x^3}}$ 4) $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

Solution.

1) Multiply both the numerator and denominator by $\sqrt[3]{2^2}$ and simplify:

$$\frac{1}{\sqrt[3]{2}} = \frac{1}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}} = \frac{\sqrt[3]{4}}{\underline{\qquad}}$$

2) Multiply the expression by $\dfrac{\sqrt{x}}{\sqrt{x}}$ and simplify:

$$\frac{1}{2\sqrt{x^3}} = \frac{1}{2\sqrt{x^3}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{x}}{2\sqrt{x^3}\sqrt{x}} = \frac{\sqrt{x}}{\underline{\qquad}}$$

3) Multiply the expression by $\dfrac{\sqrt{3x}}{\sqrt{3x}}$ and simplify:

$$\frac{\sqrt{2x^2}}{\sqrt{3x^3}} = \frac{\sqrt{2x^2}}{\sqrt{3x^3}} \cdot \frac{\sqrt{3x}}{\sqrt{3x}} = \frac{\sqrt{\underline{\qquad}}}{\underline{\qquad}} = \frac{\underline{\qquad}\sqrt{6x}}{\underline{\qquad}} = \underline{\qquad}$$

4) Multiply both the numerator and denominator by the conjugate $\sqrt{x}+\sqrt{y}$ of the denominator $\sqrt{x}-\sqrt{y}$ and simplify:

$$\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \frac{\left(\sqrt{x}+\sqrt{y}\right)^2}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} = \frac{\underline{\qquad\qquad}}{\underline{\qquad\qquad}}$$

The "Why" Behind the Conjugate

When rationalizing a denominator or numerator like $\sqrt{X}+Y$ or $\sqrt{X}-Y$, just multiplying $\sqrt{X}$ won't work. Why? Because Y will also get multiplied by $\sqrt{X}$, and the radical stays.

Our goal is to turn $\sqrt{X}$ into $\left(\sqrt{X}\right)^2 = X$ so the square root disappears. To do that, we use a trick called **multiplying by the conjugate**.

The **conjugate** of $A + B$ is $A - B$, and vice versa. These pairs are special because of the difference of squares formula:

$$(A - B)(A + B) = A^2 - B^2.$$

So multiplying the conjugate $\sqrt{X} - Y$ or $\sqrt{X} + Y$ eliminates the radical:

$$\left(\sqrt{X} - Y\right)\left(\sqrt{X} + Y\right) = X - Y^2.$$

That's why the conjugate trick works!

Exercises

Exercise 9.1. Evaluate the square root. If it is not a real number, state so.

1) $-\sqrt{\dfrac{4}{25}}$ 2) $\sqrt{49}-\sqrt{9}$ 3) $-\sqrt{-1}$

Answer: 1) $-\frac{2}{5}$ 2) 4 3) $-\sqrt{-1}$ is not a real number.

Exercise 9.2. Simplify the radical expression.

1) $\sqrt{(-7x^2)^2}$ 2) $\sqrt{(x+2)^2}$ 3) $\sqrt[3]{-27x^3}$ 4) $\sqrt[4]{16x^8}$

Answer: 1) $7x^2$ 2) $|x+2|$ 3) $-3x$ 4) $4x^2$

Exercise 9.3. Simplify the radical expression. Assume all variables are positive.

1) $\sqrt{50x^3}$ 2) $\sqrt[3]{-8x^2y^3}$ 3) $\sqrt[5]{32x^{12}y^2z^8}$

Answer: 1) $5x\sqrt{2x}$ 2) $-2y\sqrt[3]{x^2}$ 3) $2x^2z\sqrt[5]{x^2y^2z^3}$

Exercise 9.4. Rewrite the rational exponent in radical notation and evaluate.

1) $16^{\frac{3}{4}}$ 2) $(-64)^{\frac{2}{3}}$ 3) $\left(\frac{81}{16}\right)^{-\frac{3}{4}}$

Answer: 1) 8 2) 16 3) $\frac{16}{27}$

Exercise 9.5. Rewrite the expression using rational exponents and simplify. Write your answer in radical notation.

1) $\sqrt{y^3}\,\sqrt[5]{y^3}$ 2) $\sqrt[4]{\sqrt[3]{x^8}}$ 3) $\left(\frac{x^{\frac{1}{3}}}{x^{-\frac{3}{5}}}\right)^{\frac{1}{2}}$ 4) $\sqrt[5]{\frac{x^2y^{\frac{10}{3}}}{x^{\frac{1}{3}}}}$

Answer: 1) $y^2\,\sqrt[10]{y}$ 2) $\sqrt[3]{x^2}$ 3) $\sqrt[15]{x^7}$ 4) $\sqrt[3]{xy^2}$

Exercise 9.6. Simplify the radical expression. Assume all variables are positive.

1) $\sqrt{20xy}\sqrt{4xy^2}$ 2) $\sqrt{8x^2y^3}\sqrt{2x^5y}$ 3) $\sqrt[3]{16x^5}\,\sqrt[3]{2x^2}$ 4) $\sqrt[5]{8x^4y^3z^3}\,\sqrt[5]{8xy^4z^8}$

Answer: 1) $4xy\sqrt{5y}$ 2) $4x^3y^2\sqrt{x}$ 3) $2x^2\,\sqrt[3]{4x}$ 4) $2xyz^2\,\sqrt[5]{2y^2z}$

Exercise 9.7. Divide and simplify. Assume all variables are positive.

1) $\sqrt{\dfrac{9x^3}{y^8}}$ 2) $\sqrt[3]{\dfrac{32x^4}{x}}$ 3) $\dfrac{\sqrt{40x^5}}{\sqrt{2x}}$ 4) $\dfrac{\sqrt[3]{24a^6b^4}}{\sqrt[3]{3b}}$

Answer: 1) $\dfrac{3x\sqrt{x}}{y^4}$ 2) $2x\sqrt[3]{4}$ 3) $2x^2\sqrt{5}$ 4) $2a^2b$

Exercise 9.8. Add or subtract and simplify. Assume all variables are positive.

1) $5\sqrt{6}+3\sqrt{6}$ 2) $4\sqrt{20}-2\sqrt{5}$ 3) $3\sqrt{32x^2}-5x\sqrt{8}$

Answer: 1) $8\sqrt{6}$ 2) $6\sqrt{5}$ 3) $2x\sqrt{2}$

Exercise 9.9. Add or subtract and simplify. Assume all variables are positive.

1) $7\sqrt{4x^2}+2\sqrt{25x}-\sqrt{16x}$ 2) $5\sqrt[3]{x^2y}+\sqrt[3]{27x^5y^4}$ 3) $3\sqrt{9y^3}-3y\sqrt{16y}+\sqrt{25y^3}$

Answer: 1) $14x+6\sqrt{x}$ 2) $(5+3xy)\sqrt[3]{x^2y}$ 3) $2y\sqrt{y}$

Exercise 9.10. Multiply and simplify. Assume all variables are positive.

1) $\sqrt{2}\left(3\sqrt{3}-2\sqrt{2}\right)$ 2) $\left(\sqrt{5}+\sqrt{7}\right)\left(3\sqrt{5}-2\sqrt{7}\right)$ 3) $\left(\sqrt{6}-2\sqrt{5}\right)\left(\sqrt{6}+2\sqrt{5}\right)$

Answer: 1) $3\sqrt{6}-4$ 2) $1+\sqrt{35}$ 3) -14

Exercise 9.11. Multiply and simplify. Assume all variables are positive.

1) $\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)$ 2) $\left(3-2\sqrt{x-5}\right)^2$ 3) $\left(2\sqrt[3]{x}-1\right)\left(\sqrt[3]{x}+6\right)$

Answer: 1) x 2) $4x-11-12\sqrt{x-5}$ 3) $2\sqrt[3]{x^2}+11\sqrt[3]{x}-6$

Exercise 9.12. Rationalize the denominator and simplify. Assume all variables are positive.

1) $\sqrt[3]{\dfrac{2}{25}}$ 2) $\sqrt{\dfrac{x}{5y}}$ 3) $\dfrac{\sqrt[3]{x}}{\sqrt[3]{3y^2}}$ 4) $\dfrac{3x}{\sqrt[4]{x^3y}}$

Answer: 1) $\dfrac{\sqrt[3]{10}}{5}$ 2) $\dfrac{\sqrt{5xy}}{5y}$ 3) $\dfrac{\sqrt[3]{9xy}}{3y}$ 4) $\dfrac{3\sqrt[4]{xy^3}}{y}$

Exercise 9.13. Rationalize the denominator and simplify. Assume all variables are positive.

1) $\dfrac{6\sqrt{3}}{\sqrt{3}-1}$ 2) $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$ 3) $\dfrac{2\sqrt{x}}{\sqrt{x}-\sqrt{y}}$

Answer: 1) $9+3\sqrt{3}$ 2) $4-\sqrt{15}$ 3) $\dfrac{2x+2\sqrt{xy}}{x-y}$

Exercise 9.14. Simplify and rationalize the denominator. Assume all variables are positive.

1) $\dfrac{\sqrt{x}}{\sqrt{x}-1}+\dfrac{1}{\sqrt{x}+1}$

2) $\dfrac{\sqrt{x}+1}{\sqrt{x}}-\dfrac{1}{\sqrt{x}-1}$

Answer: 1) $\dfrac{x-1+2\sqrt{x}}{x-1}$ 2) $\dfrac{x^2+\sqrt{x}}{x^2-x}$

Topic 10 Complex Numbers

Learning Goals

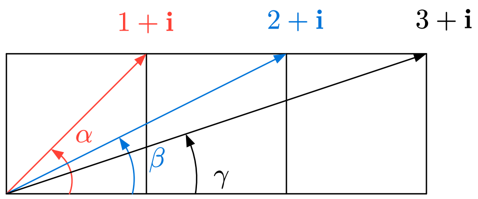
- I can perform arithmetic operations with complex numbers.
- I can simplify expressions involving powers of the imaginary unit $\mathbf{i}$.
- I can evaluate polynomials and other functions at complex numbers.

Complex Made Easy

In 18 century, mathematician, Leonhard Euler, introduced a formula $e^{\mathbf{i}x} = (\cos(x) + \mathbf{i}\sin(x))$ that builds a connection between exponentiation and trigonometry, which has significantly influenced modern mathematics, physics, and engineering.

Fact: the rules of exponents hold for complex numbers, that is, .$e^{a+\mathbf{i}b}e^{x+\mathbf{i}y} = e^{(a+x)+\mathbf{i}(b+y)} = e^{a+x}e^{\mathbf{i}(b+y)}$.

Can you find the sum of the angles α, β, and γ in the diagram below?



The **imaginary unit** $\mathbf{i}$ is defined as $\mathbf{i} = \sqrt{-1}$, so $\mathbf{i}^2 = -1$. In general, for a positive real number b, the expression $\sqrt{-b}$ is defined as $\mathbf{i}\sqrt{b}$. So,

$$\sqrt{-b}\sqrt{-b} = \left(\sqrt{b}\mathbf{i}\right)\left(\sqrt{b}\mathbf{i}\right) = \left(\sqrt{b}\right)^2 \cdot \mathbf{i}^2 = -b.$$

A **complex number** is a number of the form $a + b\mathbf{i}$, where $\mathbf{i}$ is the imaginary unit, and a and b are real numbers called the **real part** and the **imaginary part**, respectively. The complex number $b\mathbf{i}$ with $b \neq 0$ is called a **purely imaginary number**.

Geometry of Complex Numbers

A complex number $z = a + b\mathbf{i}$ represents the point (a, b) in the Cartesian plane, with the **real axis** and **imaginary axis** corresponding to a and b.

The **magnitude** $|z| = \sqrt{a^2 + b^2}$ is the distance from the origin to (a, b), and the **argument** $\arg(z)$ is the angle from the positive real axis to the ray connecting the origin to (a, b). In polar form, $z = |z| \left(\cos\left(\arg(z)\right) + \mathbf{i}\sin(\arg(z))\right)$.

By Euler's formula,

$$z = |z| \left(\cos\left(\arg(z)\right) + \mathbf{i}\sin(\arg(z))\right) = |z|\ e^{\mathbf{i}\arg(z)}.$$

Arithmetics of Complex Numbers

Adding, subtracting, multiplying, and dividing complex numbers are similar to those operations with radicals.

If $z_1 = a + b\mathbf{i}$ and $z_2 = c + d\mathbf{i}$ are complex numbers, then

- **Addition:** $z_1 + z_2 = (a + c) + (b + d)\mathbf{i}$
- **Subtraction:** $z_1 - z_2 = (a - c) + (b - d)\mathbf{i}$
- **Multiplication:** $z_1 z_2 = (ac - bd) + (ad + bc)\mathbf{i}$
- **Division:** $\dfrac{z_1}{z_2} = \dfrac{(a + b\mathbf{i})(c - d\mathbf{i})}{(c + d\mathbf{i})(c - d\mathbf{i})} = \dfrac{(ac + bd) + (bc - ad)\mathbf{i}}{c^2 + d^2}$

The **conjugate** $\overline{z}$ of a complex numbers $z = a + b\mathbf{i}$ is defined as $\overline{z} = a - b\mathbf{i}$. The product of a complex number $z = a + b\mathbf{i}$ and its conjugate $\overline{z}$ is a real number,

$$z\overline{z} = (a + b\mathbf{i})(a - b\mathbf{i}) = a^2 + b^2 = |z|^2,$$

where $|z|$ is the magnitude of z.

Example 10.1. Simplify and write the result in the form $a + b\mathbf{i}$.

1) $\sqrt{-3}\sqrt{-4}$ 2) $(4\mathbf{i} - 3)(-2 + \mathbf{i})$ 3) $\dfrac{-2 + 5\mathbf{i}}{\mathbf{i}}$ 4) $\dfrac{3i - 1}{1 - 2\mathbf{i}}$

Solution.

1)

$$\sqrt{-3}\sqrt{-4} = \mathbf{i}\,\_\_\_\_ \cdot \mathbf{i}\,\_\_\_\_ = \_\_\_\_ \cdot 2\sqrt{3} = \_\_\_\_\_\_$$

2)

$$\begin{aligned}(4\mathbf{i} - 3)(-2 + \mathbf{i}) &= -8\mathbf{i} + \_\_\_\_\,\mathbf{i}^2 + 6 + \_\_\_\_\,\mathbf{i} \\ &= (\_\_\_\_ + \_\_\_\_)\mathbf{i} + \_\_\_\_ + 6 \\ &= \_\_\_\_\_\_\_\_\end{aligned}$$

3)

$$\frac{-2 + 5\mathbf{i}}{\mathbf{i}} = \frac{(-2 + 5\mathbf{i})\_\_\_\_\_\_}{\mathbf{i}^2} = \frac{\_\_\_\_\_\_}{-1} = \_\_\_\_\_\_\_\_$$

4)

$$\begin{aligned}\frac{3\mathbf{i} - 2}{1 - 2\mathbf{i}} &= \frac{(3\mathbf{i} - 2)(\_\_\_\_\_\_)}{(1 - 2\mathbf{i})(1 + 2\mathbf{i})} \\ &= \frac{\_\_\_\_\_\_\_\_}{\_\_\_\_} \\ &= \_\_\_\_\_\_\_\_\end{aligned}$$

Powers of the Imaginary Unit

The powers of the imaginary unit $\mathbf{i}$ cycle every four terms:

$$\mathbf{i}^1 = \mathbf{i} \qquad \mathbf{i}^2 = -1 \qquad \mathbf{i}^3 = -\mathbf{i} \qquad \mathbf{i}^4 = 1.$$

So for any integer n, $\mathbf{i}^n = \mathbf{i}^r$, where r is the remainder of n divided by 4.

Example 10.2. Evaluate $\mathbf{i}^{2025}$.

Solution.

$$\mathbf{i}^{2025} = (\mathbf{i}^{\underline{\quad}})^{506} \cdot \mathbf{i}^{\underline{\quad}} = \underline{\qquad}^{506} \cdot \mathbf{i} = \mathbf{i}$$

Example 10.3. Evaluate $z^2 + \dfrac{z-1}{z+1}$ for $z = 1 + \mathbf{i}$. Write in the form $a + b\mathbf{i}$.

Solution. Replace z with $1 + \mathbf{i}$ in the expression and simplify.

$$\begin{aligned}
(1+\mathbf{i})^2 + \frac{(1+\mathbf{i})-1}{2+\mathbf{i}} &= (1+\mathbf{i})^2 + \frac{\mathbf{i}}{2+\mathbf{i}} \\
&= \underline{\qquad\qquad} + \frac{\mathbf{i}(\underline{\qquad\qquad})}{(2+\mathbf{i})(\underline{\qquad\qquad})} \\
&= 1 + 2\mathbf{i} - 1 + \frac{2\mathbf{i} - \mathbf{i}^2}{4+1} \\
&= \underline{\qquad\qquad} + \frac{\underline{\qquad\qquad}}{5} \\
&= \underline{\qquad} + \underline{\qquad}\,\mathbf{i}
\end{aligned}$$

Exercises

Exercise 10.1. Add, subtract, or multiply complex numbers. Write your answer in the form $a + b\mathbf{i}$.

1) $\sqrt{-2}\sqrt{-3}$ 2) $\sqrt{2}\sqrt{-8}$ 3) $(5 - 2\mathbf{i}) + (3 + 3\mathbf{i})$ 4) $(2 + 6\mathbf{i}) - (12 - 4\mathbf{i})$

Answer: 1) $-\sqrt{6}$ 2) $4\mathbf{i}$ 3) $8 + \mathbf{i}$ 4) $-10 + 10\mathbf{i}$

Exercise 10.2. Add, subtract, or multiply complex numbers. Write your answer in the form $a + b\mathbf{i}$.

1) $(3 + \mathbf{i})(4 + 5\mathbf{i})$ 2) $(7 - 2\mathbf{i})(-3 + 6\mathbf{i})$ 3) $\left(3 - x\sqrt{-1}\right)\left(3 + x\sqrt{-1}\right)$ 4) $(2 + 3\mathbf{i})^2$

Answer: 1) $7 + 19\mathbf{i}$ 2) $-9 + 48\mathbf{i}$ 3) $9 + x^2$ 4) $-5 + 12\mathbf{i}$

Exercise 10.3. Divide the complex number. Write your answer in the form $a + b\mathbf{i}$.

1) $\dfrac{2\mathbf{i}}{1+\mathbf{i}}$ 2) $\dfrac{5-2\mathbf{i}}{3+2\mathbf{i}}$ 3) $\dfrac{2+3\mathbf{i}}{3-\mathbf{i}}$ 4) $\dfrac{4+7\mathbf{i}}{-3\mathbf{i}}$

Answer: 1) $1+\mathbf{i}$ 2) $\frac{11}{13}-\frac{16}{13}\mathbf{i}$ 3) $\frac{3}{10}+\frac{11}{10}\mathbf{i}$ 4) $-\frac{7}{3}+\frac{4}{3}\mathbf{i}$

Exercise 10.4. Simplify the expression.

1) $(-\mathbf{i})^8$ 2) $\mathbf{i}^{67}$ 3) $\mathbf{i}^{2030}$ 4) $\dfrac{1}{\mathbf{i}^{2027}}$

Answer: 1) 1 2) $-\mathbf{i}$ 3) -1 4) $\mathbf{i}$

Exercise 10.5. Evaluate the polynomial $2x^2 - 3x + 5$ at $x = 1 - \mathbf{i}$. Write your answer in the form $a + b\mathbf{i}$.

Answer: $2 - \mathrm{i}$

Exercise 10.6. Evaluate the expression $\mathbf{i}x^2 - x + \dfrac{2}{x-1}$ at $x = \mathbf{i} - 1$. Write your answer in the form $a + b\mathbf{i}$.

Answer: $\frac{11}{5} - \frac{7}{5}\mathrm{i}$

Part III

Equations and Inequalities

Topic 11 Quadratic Equations

Learning Goals

- I can simplify and solve quadratic equations by factoring.
- I can solve a quadratic equation by completing the square.
- I can solve a quadratic equation by using the quadratic formula.
- I can solve equations that can be written in quadratic form by substitution.
- I can build quadratic equations for word problems, solve the equations, and interpret the solutions in context.

11.1 Think about It

Handshaking Problem

In a meeting room, a group of people all shook hands with one another. In total, 15 handshakes occurred. Do you know how many people were in the group?

11.2 Basics of Equations

An **equation** is a mathematical statement asserting that two expressions are equal. It involves one or more **unknowns**—variables whose values are to be determined. It may also include **known quantities**, often called **constants**, **coefficients**, or **parameters**.

An **identity** is an equation that holds true for all permissible values of the variables it contains. For example, $x^2 - y^2 = (x + y)(x - y)$ is true for all x and y.

Solving an equation means finding all values of the unknowns that make the equation true. These values are called **solutions**. Two equations are **equivalent** if they have exactly the same solution set.

The **domain of an equation** is the set of values for which the expressions involved are defined.

An **extraneous solution** is a value that is obtained during the process of solving an equation but is not a valid solution to the original equation.

Common Techniques for Solving Equations

The following operations produce an equivalent equation only when restricted to the domain of the original equation and when the operation is logically reversible:

- **Applying the same arithmetic operation** (e.g., addition, subtraction, or multiplication by a nonzero constant) to both sides of the equation.

  Caution: Multiplying or dividing both sides by an expression involving variables may alter the domain. It is essential to ensure that the expression is nonzero over the intended domain.

- **Applying algebraic identities**, such as factoring, expanding, or simplifying expressions on either side.

  These transformations preserve equivalence when they do not introduce or remove domain restrictions.

- **Applying the same function to both sides**, such as taking the square root, raising to the n-th power, or using logarithmic or exponential functions.

  Caution: If the function is not one-to-one, it may introduce extraneous solutions, even if the values lie within the domain of the original equation. For example, $\sqrt{x^2} = |x|$ which implies that squaring is not reversible without considering the sign of x.

Example 11.1. Solve the equation.

1) $2x + 3 = 7$ 2) $x^2 - x - 2 = 0$ 3) $x^2 = 3$

Solution.

1) Linear equations can usually be solved by applying arithmetic operations to both sides to isolate the variable.

$$\begin{aligned} 2x + 3 &= 7 \\ \underline{\qquad} &= 4 \\ x &= \underline{\qquad} \end{aligned}$$

2) Many polynomial equations can be solved by factoring and applying the zero-product property. This is an example of using identity.

$$\begin{aligned} x^2 - x - 2 &= 0 \\ (x + \underline{\qquad})(x - \underline{\qquad}) &= 0 \\ x + 1 &= 0 \quad \text{or } x - 2 = 0 \\ x &= \underline{\qquad} \quad \text{or} \quad x = \underline{\qquad} \end{aligned}$$

3) In many cases, we may apply a function to both sides to reduce the equation to an easier form. In this example, we may take the square root of both sides.

$$\begin{aligned} x^2 &= 3 \\ \sqrt{x^2} &= \sqrt{3} \\ \underline{\qquad} &= \sqrt{3} \\ x = \sqrt{3} \quad &\text{or} \quad x = \underline{\qquad} \end{aligned}$$

11.3 Solving by Factoring

A **polynomial equation** is an equation that can be written in the form
$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0,$$
where n is a positive integer and $a_n \neq 0$.

A polynomial equation is called a **quadratic equation** if $n = 2$. We often write a quadratic equation in its **standard form** (or **general form**) as
$$ax^2 + bx + c = 0,$$
where a, b, and c are numbers, and $a \neq 0$.

Zero Product Property

If A and B are expressions, then
$$A \cdot B = 0 \quad \text{if and only if} \quad A = 0 \quad \text{or} \quad B = 0.$$

For a quadratic equation, if $ax^2 + bx + c = (mx - p)(nx - q)$, then any solution of the quadratic equation $ax^2 + bx + c = 0$ must satisfy either $mx - p = 0$ or $nx - q = 0$. That is, the solutions are $x = \dfrac{p}{m}$ or $x = \dfrac{q}{n}$.

A simpler way to present the solutions is as a set: $\left\{\dfrac{p}{m}, \dfrac{q}{n}\right\}$.

Example 11.2. Solve the equation
$$2x^2 + 5x = 3.$$

Solution.

1) Rewrite the equation into "**Expression** = 0" form and factor.
$$2x^2 + 5x = 3$$
$$2x^2 + 5x - 3 = 0$$
$$(\underline{\qquad\quad})(x + 3) = 0$$

2) Apply the zero product property.
$$\underline{\qquad\quad} = 0 \quad \text{or} \quad x + 3 = 0.$$

3) Solve each equation.
$$2x = \underline{\qquad} \qquad x = -3$$
$$x = \underline{\qquad} \quad \text{or} \quad x = -3$$

4) The solution set is $\{-3, \underline{\qquad}\}$.

Example 11.3. Solve the equation
$$(x - 2)(x + 3) = -4.$$

Solution.

1) Rewrite the equation into "**Expression** = 0" form and factor.

$$
\begin{aligned}
(x-2)(x+3) &= -4 \\
x^2 + \underline{\qquad\qquad} &= -4 \\
\underline{\qquad\qquad\qquad} &= 0 \\
(x-1)(x+2) &= 0
\end{aligned}
$$

2) Apply the zero product property.

$$x - 1 = 0 \quad \text{or} \quad x + 2 = 0.$$

3) Solve each equation.

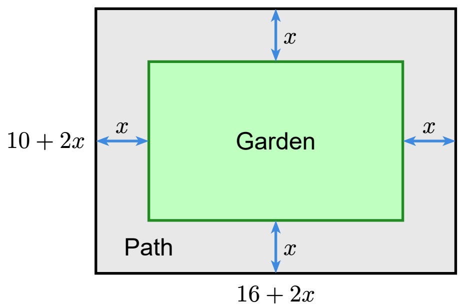
$$x = 1 \quad \text{or} \quad x = -2$$

4) The solution set is $\{-2, 1\}$.

Example 11.4. A rectangular garden is surrounded by a path of uniform width. If the dimensions of the garden are 10 meters by 16 meters and the total area is 216 square meters, determine the width of the path.

Solution.

1) Analyze the question and represent information using mathematical expressions: suppose that the width of the frame is x meters. The information can be modeled by the picture below, from which we see that the total width of the garden plus the path is $2x + 10$ meters, and the total length is $2x + 16$ meters.



2) Build an equation: using the formula for the area of a rectangle, we can write the equation

$$(2x + 10)(2x + 16) = \underline{\qquad}.$$

3) Solve the equation.

$$
\begin{aligned}
(2x+10)(2x+16) &= 216 \\
\underline{\qquad\qquad} + 160 &= 216 \\
4x^2 + 52x + \underline{\qquad} &= 0 \\
x^2 + \underline{\qquad\qquad} &= 0 \\
(x+14)(x-1) &= 0 \\
x = -14 \quad \text{or} \quad x &= 1
\end{aligned}
$$

4) Interpret the answer: since the width cannot be negative, the path is 1 meter wide.

Problem-Solving Step 1: Understand the Problem

When solving a word problem, first identify what is known and what is unknown, and restate the problem using algebraic expressions. Once the problem has been reformulated algebraically, you can solve it using your mathematical knowledge.

11.4 Solving by Completing the Square

Square Root Property

The square root property states that if $X^2 = d$, then $X = \sqrt{d}$ or $X = -\sqrt{d}$, or simply $X = \pm\sqrt{d}$.

Example 11.5. Solve the equation using square root property.

1) $(x-1)^2 = 2$ 2) $(x+1)^2 = -1$

Solution. From the square root property, we have

1)

$$\begin{aligned} x - 1 &= \sqrt{2} && \text{or} & x - 1 &= -\sqrt{2} \\ x &= \underline{\hspace{3em}} && \text{or} & x &= \underline{\hspace{3em}} \end{aligned}$$

2)

$$\begin{aligned} x + 1 &= \sqrt{-1} && \text{or} & x + 1 &= -\sqrt{-1} \\ x &= \underline{\hspace{2em}} + \mathbf{i} && \text{or} & x &= -1 - \underline{\hspace{2em}} \end{aligned}$$

The square root property provides another approach to solve a quadratic equation, **by completing the square**. This method is based on the following observations:

$$ax^2 + bx + c = a(x-h)^2 + k,$$

where $h = -\dfrac{b}{2a}$ and

$$k = ah^2 + bh + c = -ah^2 + c = \frac{4ac - b^2}{4a^2}.$$

The procedure to rewrite a trinomial as the sum of a perfect square and a constant is called **completing the square**.

Remark (The value of k)

The equality $k = ah^2 + bh + c$ is obtained by evaluating both sides at $x = h$. The equality $k = -ah^2 + c$ is obtained by evaluating both sides at $x = 0$. The equality $k = \dfrac{4ac - b^2}{4a^2}$ is obtained by substituting $h = -\dfrac{b}{2a}$ into the first equality or the second inequality.

Example 11.6. Solve the equation $x^2 + 2x - 1 = 0$ by completing the square.

Solution.

1) Find h and k:

$$h = -\frac{b}{2a} = -\frac{2}{2 \cdot 1} = -1$$

$$k = ah^2 + bh + c = (\_\_\_\_)^2 + 2(\_\_\_\_) - 1 = \_\_\_\_$$

2) Completing the square for the equation and rewriting it in the form $(x-h)^2 = -\frac{k}{a}$:

$$\begin{aligned} x^2 + 2x - 1 &= 0 \\ (x+1)^2 + \_\_\_\_ &= 0 \\ (x+1)^2 &= 2 \end{aligned}$$

3) Solve the resulting equation using the square root property.

$$x + 1 = \sqrt{2} \qquad \text{or} \qquad x + 1 = \_\_\_\_$$

$$x = \_\_\_\_ + \sqrt{2} \qquad\qquad x = \_\_\_\_ - \sqrt{2}$$

The solutions can also be written as $x = -1 \pm \sqrt{2}$.

Example 11.7. Solve the equation $-2x^2 + 8x - 9 = 0$ by completing the square.

Solution.

1) Find h and k:

$$h = -\frac{b}{2a} = -\frac{\_\_\_\_}{2 \cdot (\_\_\_\_)} = 2$$

$$k = ah^2 + bh + c = -2(2)^2 + 8(2) - 9 = \_\_\_\_ + 16 - 9 = -1$$

2) Completing the square for the equation and rewriting it in the form $(x-h)^2 = -\frac{k}{a}$:

$$\begin{aligned} -2x^2 + 8x - 9 &= 0 \\ -2(x-2)^2 + (-1) &= 0 \\ -2(x-2)^2 &= \_\_\_\_ \\ (x-2)^2 &= \_\_\_\_ \end{aligned}$$

3) Solve the resulting equation using the square root property.

$$x - 2 = \frac{\_\_\_\_}{\sqrt{2}} \qquad \text{or} \qquad x - 2 = -\frac{\_\_\_\_}{\sqrt{2}}$$

$$x = 2 + \frac{\_\_\_\_}{2}\mathbf{i} \qquad\qquad x = 2 - \frac{\_\_\_\_}{2}\mathbf{i}$$

11.5 The Quadratic Formula

Completing the square for $ax^2 + bx + c = 0$ with $a \neq 0$ and solving for x using the square root property leads to the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity $\Delta = b^2 - 4ac$ is called the **discriminant** of the quadratic equation. The symbol Δ reads as "delta" and is the fourth letter of the Greek alphabet.

Discriminant and Numbers of Solutions

- If $b^2 - 4ac > 0$, the equation has two real solutions.
- If $b^2 - 4ac = 0$, the equation has one real solution.
- If $b^2 - 4ac < 0$, the equation has two imaginary solutions (no real solutions).

Example 11.8. Determine the type and the number of solutions of the equation $(x-1)(x+2) = -3$.

Solution.

1) Rewrite the equation in the form $ax^2 + bx + c = 0$.

$$\begin{aligned} (x-1)(x+2) &= -3 \\ x^2 + \underline{\qquad\qquad} &= -3 \\ x^2 + \underline{\qquad\qquad} &= 0 \end{aligned}$$

2) Determined the values of a, b, and c and find the discriminant $b^2 - 4ac$.

$$a = 1, \quad b = \underline{\qquad} \quad \text{and} \quad c = 3$$

$$b^2 - 4ac = (\underline{\qquad})^2 - 4 \cdot 1 \cdot 3 = \underline{\qquad}$$

Since the discriminant is ______, the equation has two imaginary solutions.

Example 11.9. Solve the equation $2x^2 - 4x + 7 = 0$.

Solution. Since the equation is in the form $ax^2 + bx + c = 0$, we can apply the quadratic formula.

1) Determine the values of a, b, and c and find the discriminant $b^2 - 4ac$.

$$a = 2, \quad b = -4 \quad \text{and} \quad c = 7$$

$$\begin{aligned} b^2 - 4ac &= (-4)^2 - 4 \cdot 2 \cdot 7 \\ &= 16 - 56 = -40 \end{aligned}$$

2) Apply the quadratic formula and simplify.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(\underline{\qquad}) \pm \sqrt{\underline{\qquad}}}{2 \cdot 2} \\ &= \frac{4 \pm 2 \cdot \underline{\qquad\quad} \mathbf{i}}{4} \\ &= 1 \pm \underline{\qquad} \mathbf{i} \end{aligned}$$

Example 11.10. Find the base and the height of a **triangle** whose base is three inches more than twice its height and whose area is 5 square inches. Round your answer to the nearest tenth of an inch.

Solution.

1) Suppose the height is x inches. The base can be expressed as ______ inches.

2) By the area formula for a triangle, we have an equation.

$$\frac{1}{2}x(2x+3) = \underline{\qquad}$$

3) Rewrite the equation in $ax^2+bx+c=0$ form.

$$x(2x+3) = \underline{\qquad}$$
$$\underline{\qquad} + 3x - 10 = 0$$

4) The discriminant is

$$b^2-4ac = 3^2-4\cdot 2\cdot(-10) = 9+80 = \underline{\qquad}$$

4) By the quadratic formula, we have

$$x = \frac{-\underline{\qquad} \pm \sqrt{\underline{\qquad}}}{2\cdot 2} = \frac{\underline{\qquad\qquad}}{4}$$

Since x cannot be negative, $x = \dfrac{-3+\sqrt{89}}{4} \approx \underline{\qquad}$ and $2x+3 \approx \underline{\qquad}$. The height and base of the triangle are approximately 1.6 inches and ____ inches, respectively.

11.6 Vieta's Formulas (Optional)

Vieta's formulas relate the coefficients of a quadratic expression to sums and products of its roots.

For a quadratic equation $ax^2+bx+c=0$, if x_1 and x_2 are the roots, then:

$$x_1+x_2 = -\frac{b}{a} \qquad \text{and} \qquad x_1x_2 = \frac{c}{a}.$$

Vieta's formulas can be used to simplify problems and solve problems efficiently.

Example 11.11. Denote the two solutions of the quadratic equation $x^2-5x+6=0$ as r and s. Find the value of r^2+s^2.

Solution. By Vieta's formulas, we have $r+s=5$ and $rs=6$.

Using the perfect square formula, we can find the value of r^2+s^2:

$$r^2+s^2 = (r+s)^2-2rs = 5^2-26 = 25-12 = 13$$

Example 11.12. Denote the two solutions of the quadratic equation $x^2-5x+3=0$ as r and s. Find the value of $r-s$ assuming that $r>s$.

Solution. By Vieta's formulas, we have $r+s=5$ and $rs=3$.

Using the perfect square formula, we can find the value of $(r-s)^2$:

$$(r-s)^2 = (r+s)^2 - 4rs = 5^2 - 4 \cdot 3 = 25 - 12 = 13$$

Since $r > s$, we have

$$r - s = \sqrt{(r-s)^2} = \sqrt{13}$$

Exercises

Exercise 11.1. Solve the equation by factoring.

1) $x^2 - 3x + 2 = 0$ 2) $2x^2 - 3x = 5$ 3) $(x-1)(x+3) = 5$ 4) $\frac{1}{3}(2-x)(x+5) = 4$

Answer: 1) $x = 1$ or $x = 2$ 2) $x = \dfrac{5}{2}$ or $x = -1$ 3) $x = 2$ or $x = -4$ 4) $x = -1$ or $x = -2$

Exercise 11.2. Solve the quadratic equation by the square root property.

1) $2x^2 - 6 = 0$ 2) $(x-3)^2 = 10$ 3) $4(x+1)^2 + 25 = 0$

Answer: 1) $x = \pm\sqrt{3}$ 2) $x = 3 \pm \sqrt{10}$ 3) $x = -1 \pm \dfrac{5}{2}\mathrm{i}$

Exercise 11.3. Solve the quadratic equation by completing the square.

1) $x^2 + 8x + 13 = 0$ 2) $x^2 + x - 1 = 0$ 3) $3x^2 + 6x - 1 = 0$

Answer: 1) $x = -4 \pm \sqrt{3}$ 2) $x = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ 3) $x = -1 \pm \frac{2\sqrt{3}}{3}$

Exercise 11.4. Determine the number and the type of solutions of the given equation.

1) $x^2 + 8x + 3 = 0$ 2) $3x^2 - 2x + 4 = 0$ 3) $2x^2 - 4x + 2 = 0$

Answer: 1) two real 2) two complex 3) one real

Exercise 11.5. Solve using the quadratic formula.

1) $x^2 + 3x - 7 = 0$ 2) $2x^2 = -4x + 5$ 3) $2x^2 = x - 3$

Answer: 1) $x = \frac{-3 \pm \sqrt{37}}{2}$ 2) $x = \frac{-2 \pm \sqrt{14}}{2}$ 3) $x = \frac{1 \pm \sqrt{23}\mathbf{i}}{4}$

Exercise 11.6. Solve using the quadratic formula. Simplify your answer if possible.

1) $(x-1)(x+2) = 3$ 2) $2x^2 - x = (x+2)(x-2)$ 3) $\frac{1}{2}x^2 + x = \frac{1}{3}$

Answer: 1) $x = \frac{-1 \pm \sqrt{21}}{2}$ 2) $x = \frac{1 \pm \sqrt{15}\mathbf{i}}{2}$ 3) $x = \frac{-3 \pm \sqrt{15}}{3}$

Exercise 11.7 (Optional). Find all **real** solutions of the equation by factoring.

1) $4(x-2)^2-9=0$ 2) $2x^3-18x=0$ 3) $3x^4-2x^2=1$ 4) $x^3-3x^2-4x+12=0$

Answer: 1) $x=\frac{7}{2}$ or $x=\frac{1}{2}$ 2) $x=0$ or $x=\pm 3$ 3) $x=\pm 1$ 4) $x=3$ or $x=\pm 2$

Exercise 11.8. A painting measuring 7 inches by 9 inches is surrounded by a frame of uniform width. If the combined area of the painting and the frame is 120 square inches, determine the width of the frame.

Answer: 1.5 in

Exercise 11.9. The product of two **consecutive negative odd** numbers is 35. Find the numbers.

Answer: $-5, -7$

Exercise 11.10. In a right triangle, the longer leg is 2 inches more than three times the length of the shorter leg. The hypotenuse is 1 inch longer than the longer leg. Find the length of the shortest side.

Answer: $3 + \sqrt{14}$ in

Exercise 11.11. A ball is thrown upwards from a rooftop. It will reach a maximum vertical height and then fall back to the ground. The height h of the ball from the ground after time t seconds is $h = -16t^2 + 48t + 160$ feet. How long will it take the ball to hit the ground?

Answer: 5 s

Exercise 11.12. A **triangle** whose area is 7.5 square meters has a base that is one meter less than three times the height. Find the length of its base and height. Round to the nearest hundredth of a meter.

Answer: base: 6.23 m, height: 2.41 m

Exercise 11.13. A **rectangular** garden whose length is 1 feet longer than the double of its width has an area of 50 square feet. Find the dimensions of the garden, rounded to the nearest hundredth of a foot.

Answer: 4.76 ft by 10.52 ft

Exercise 11.14 (Optional). Denote the two solutions of the quadratic equation $x^2 - 2x + 3 = 0$ as r and s. Find the value of $\frac{1}{r} + \frac{1}{s}$.

Answer: $-\frac{2}{3}$

Topic 12 Rational Equations

Learning Goals

- I can solve rational equations.
- I can identify and check for extraneous solutions.
- I can solve literal equations for a specified variable.
- I can solve word problems involving rational equations.

12.1 Think about It

A Problem of the Father of Algebra

The father of algebra, Muḥammad ibn Musa al-Khwarizmi, in his book "Algebra", answered the following problem:

> … I have divided ten into two parts; and have divided the first by the second, and the second by the first, and the sum of the quotient is two and one-sixth;…
>
> — Muḥammad ibn Musa al-Khwarizmi

Do you know what are those two parts?

12.2 Solving Rational Equations

A **rational equation** is an equation that involves one or more rational expressions.

To solve a rational equation, a common strategy is to eliminate denominators by multiplying both sides by the **least common denominator** (LCD). This converts the equation into a polynomial equation. However, this step may introduce **extraneous solutions** that will make the LCD zero. Therefore, it is crucial to check each solution to ensure it is a valid solution of the original equation.

Example 12.1. Solve the rational equation.

$$\frac{5}{x^2-9} = \frac{3}{x-3} - \frac{2}{x+3}$$

Solution. (by Clearing Denominators).

1) Find the LCD: since $x^2 - 9 = (x+3)(x-3)$, the LCD is ____________.

2) Clear denominators: multiply each rational expression on both sides by the LCD $(x+3)(x-3)$ and simplify.

$$(x+3)(x-3)\cdot\frac{5}{x^2-9}=(x+3)(x-3)\cdot\frac{3}{x-3}-(x+3)(x-3)\cdot\frac{2}{x+3}$$
$$5=3(\underline{\qquad\qquad})-2(\underline{\qquad\qquad})$$

3) Solve the resulting equation.

$$\begin{aligned}5&=3(x+3)-2(x-3)\\5&=\underline{\qquad\qquad}\\x&=\underline{\qquad}\end{aligned}$$

4) Check for any extraneous solution: plugging the solution into the LCD, if the solution makes the LCD zero, then it is an extraneous solution.

$$(-10+3)(-10-3)\neq 0$$

So $x=-10$ is a valid solution of the original equation.

Solving by Simplifying into a Reduced Equation

Another method for solving a rational equation is to combine terms using addition or subtraction, then simplify it into the form $\dfrac{A}{B}=0$, where the fraction is **fully reduced**. Then the rational equation is equivalent to the equation $A=0$.

Solution. (by Reduction).

1) Simplify the right-hand side, then subtract it from both sides and simply the left-hand side:

$$\begin{aligned}\frac{5}{x^2-9}&=\frac{3}{x-3}-\frac{2}{x+3}\\\frac{5}{x^2-9}&=\frac{3(x+3)-2(x-3)}{(x-3)(x+3)}\\\frac{5}{x^2-9}&=\frac{x+15}{x^2-9}\\\frac{-x-10}{x^2-9}&=0\end{aligned}$$

2) The equation is equivalent to $-x-10=0$.

3) Solve the resulting equation.

$$\begin{aligned}-x-10&=0\\x&=-10\end{aligned}$$

So $x=-10$ is the solution of the original equation.

Problem-Solving Strategy: Using Additional Conditions

When solving a rational equation, we may assume the LCD is nonzero and clear the denominators to simplify the equation. However, this can introduce extra conditions, so it is essential to check the solutions and eliminate any **extraneous solutions**.

More broadly, if a problem is difficult to solve under its original conditions, try solving it under additional assumptions. Afterward, eliminate those assumptions and any consequences they impose to ensure that the solution remains valid.

12.3 Literal Equations

A **literal equation** is an equation involving two or more variables. When solving a literal equation for one variable, other variables can be viewed as constants.

Example 12.2. Solve for x from the equation.

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{z}$$

Solution.

1) The LCD is $\underline{\qquad\quad}$.

2) Clear denominators.

$$xyz\cdot\frac{1}{x}+xyz\cdot\frac{1}{y}=xyz\cdot\frac{1}{z}$$
$$\underline{\qquad}+xz=\underline{\qquad}$$

3) Solve the resulting equation.

$$\begin{aligned} yz+xz&=xy\\ yz&=\underline{\qquad\qquad}\\ yz&=x(\underline{\qquad\quad})\\ \frac{yz}{y-z}&=x \quad \text{if } y\neq z \end{aligned}$$

4) The solution is $x=\dfrac{yz}{y-z}$ if $y\neq z$ and $xyz\neq 0$. If $y=z$, $x=0$, $y=0$, or $z=0$, the equation has no solution.

Exercises

Exercise 12.1. Solve.

1) $\dfrac{1}{x+1}+\dfrac{1}{x-1}=\dfrac{4}{x^2-1}$

2) $\dfrac{30}{x^2-25}=\dfrac{3}{x+5}+\dfrac{2}{x-5}$

Answer: 1) $x=2$ 2) $x=7$

Exercise 12.2. Solve.

1) $\dfrac{6}{x^2+2x-8}=\dfrac{1}{x-2}+\dfrac{x}{x+4}$

2) $\dfrac{3x}{x-5}=\dfrac{2x}{x+1}-\dfrac{42}{x^2-4x-5}$

Answer: 1) $x=-1$. 2) $x=-6$ or $x=-7$.

Exercise 12.3. Solve a variable from a formula.

1) Solve for f from $\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$.

2) Solve for x from $A=\frac{f+cx}{x}$.

Answer: 1) $f=\frac{pq}{p+q}$ 2) $x=\frac{f}{A-c}$

Exercise 12.4. Solve for x from the equation.

1) $2(x+1)^{-1}+x^{-1}=2$

2) $\frac{a^2x+2a}{x^{-1}}=-1$ with $a\neq 0$

Answer: 1) $x=-\frac{1}{2}$ or $x=1$ 2) $x=-\frac{1}{a}$

Exercise 12.5. David can row 3 miles per hour in still water. It takes him 90 minutes to row 2 miles upstream and then back. How fast is the current?

Answer: 1 mile per hour

Exercise 12.6. A swimming pool fills in 15 hours when both the faucet and the drain are open. If the drain takes 4 hours longer than the faucet to empty the pool, how long does the faucet take to fill the pool alone?

Answer: 6 hours

Topic 13 Radical Equations

Learning Goals

- I can solve radical equations involving only one radical.
- I can identify and check for extraneous solutions.
- I can solve radical equations involving rational expressions.
- I can solve quadratic-like equations involving radicals.
- I can solve word problems involving radical equations.

13.1 Think about It

Design a Pendulum Clock

A pendulum clock is a clock that uses a pendulum, a swinging weight, as its timekeeping element. Galileo Galilei discovered in the early 17th century the relation between the length L of a pendulum and the period T of the pendulum. For a pendulum clock, the relation is approximately determined by the following rule of thumb formula:

$$T \approx 2\sqrt{L}$$

given that L and T are measured in meters and seconds, respectively.

If the period of a pendulum clock is 2 seconds, how long should the pendulum be?

13.2 Solving Radical Equations by Taking a Power

To solve a radical equation of the form $\sqrt[n]{X} = a$, raise both sides to the nth power to eliminate the radical:

$$\sqrt[n]{X} = a \quad \Longrightarrow \quad X = a^n$$

Then solve the resulting equation and check for **extraneous solutions**, as taking powers may introduce invalid solutions. For example, $\sqrt{x-1} = -1$ has no solution, but squaring both sides gives $x - 1 = 1$, leading to $x = 2$. However, $x = 2$ is not in the domain. Substituting $x = 2$ back gives $\sqrt{2-1} = \sqrt{1} = 1 \neq -1$ which is a contradiction. Therefore, $x = 2$ is extraneous.

To avoid complications, **always isolate one radical before applying a power**. If radicals remain afterward, repeat the process until the variable is isolated and all radicals are eliminated.

Example 13.1. Solve the equation $x - \sqrt{x+1} = 1$.

Solution.

1) Arrange terms so that one radical is isolated on one side of the equation.

$$\underline{\hspace{2cm}} = \sqrt{x+1}$$

2) Square both sides to eliminate the square root.

$$(\underline{\hspace{2cm}})^2 = x+1$$

3) Solve the resulting equation.

$$\begin{aligned} x^2 + \underline{\hspace{2cm}} &= x+1 \\ x^2 + \underline{\hspace{2cm}} &= 0 \\ x(\underline{\hspace{1.5cm}}) &= 0 \\ x = 0 \quad \text{or} \quad \underline{\hspace{1.5cm}} &= 0 \\ x = 0 \qquad x &= 3 \end{aligned}$$

4) Check for extraneous solution(s).

When $x = 0$, the left-hand side is

$$0 - \sqrt{0+1} = 0 - \sqrt{1} = 0 - 1 = -1$$

which does not equal the right-hand side. So $x = 0$ is not a solution.

When $x = 3$, the left-hand side is

$$3 - \sqrt{3+1} = 3 - \sqrt{4} = 3 - 2 = 1.$$

which equals the right-hand side. So $x = 3$ is a solution.

Example 13.2. Solve the equation $\sqrt{x-1} - \sqrt{x-6} = 1$.

Solution.

1) Isolate one radical.

$$\sqrt{\underline{\hspace{2cm}}} = \sqrt{x-6} + 1$$

2) Square both sides to remove the radical sign and then isolate the remaining radical.

$$\begin{aligned} x - 1 &= (\underline{\hspace{2cm}}) + 2\sqrt{x-6} + 1 \\ x - 1 &= x - 5 + 2\sqrt{x-6} \\ \underline{\hspace{1cm}} &= 2\sqrt{x-6} \\ \underline{\hspace{1cm}} &= \sqrt{x-6} \end{aligned}$$

3) Square both sides to remove the radical sign and then solve.

$$\begin{aligned} \underline{\hspace{2cm}} &= 4 \\ x &= \underline{\hspace{1cm}} \end{aligned}$$

4) Check the solution. Substituting $x = 10$ into the original equation gives an equality

$$\sqrt{10-1} - \sqrt{10-6} = \sqrt{9} - \sqrt{4} = 3 - 2 = 1.$$

Therefore, $x = 10$ is the solution of the equation.

Example 13.3. Solve the equation $-2\sqrt[3]{x-4} = 6$.

Solution.

1) Isolate the radical.

$$\sqrt[3]{x-4} = \underline{\qquad}$$

1) Cube both sides to eliminate the cube root and then solve the resulting equation.

$$\begin{aligned} x - 4 &= (\underline{\qquad})^3 \\ x - 4 &= \underline{\qquad} \\ x &= -23 \end{aligned}$$

The solution is $x = -23$.

Problem-Solving Strategy: Reduce to Simpler Forms

The goal of solving a single-variable equation is to isolate the variable.

When radical expressions are involved, this often requires eliminating the radical sign—typically by raising both sides to a power. However, it's best to **isolate the radical first**, as taking powers prematurely may produce more complicated expressions or additional radicals.

Use powers carefully, and always check your solutions in the original equation to rule out extraneous results.

13.3 Equations Involving Rational Exponents

Equations involving rational exponents can be solved similarly. However, we must be careful with rational exponents, especially when the denominator is even. The reason is that $(x^2)^{\frac{1}{2}} = |x|$ not x, unless x is non-negative.

When m is even and n is odd, $\left(X^{\frac{m}{n}}\right)^{\frac{n}{m}} = |X|$. Otherwise, $\left(X^{\frac{m}{n}}\right)^{\frac{n}{m}} = X$.

Example 13.4. Solve the equation $(x-1)^{\frac{2}{3}} = 4$.

Solution (Taking reciprocal power). Take the reciprocal power of both sides and solve:

$$\begin{gathered} (x-1)^{\frac{2}{3}} = 4 \\ \left((x-1)^{\frac{2}{3}}\right)^{\frac{3}{2}} = 4^{\frac{3}{2}} \\ |\underline{\qquad\quad}| = \underline{\qquad} \\ x - 1 = \underline{\qquad} \quad \text{or} \quad x - 1 = \underline{\qquad} \\ x = 9 \quad \text{or} \quad x = -7 \end{gathered}$$

Check each solution:

$$(9-1)^{\frac{2}{3}} = 8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4$$

$$(-7-1)^{\frac{2}{3}} = (-8)^{\frac{2}{3}} = \left((-8)^{\frac{1}{3}}\right)^2 = (-2)^2 = 4$$

So the equation has two solutions $x = 9$ and $x = -7$.

Break into Multiple Steps

When solving equations with rational exponents, it is often helpful to break the process into steps. For example, to solve $X^{\frac{m}{n}} = Y$, first raise both sides to the n-th power to eliminate the denominator of the exponent, giving $X^m = Y^n$. Next, take the m-th root of both sides to isolate X. This step-by-step approach helps prevent missing possible solutions.

Solution (Taking n-th power and then m-th root). Take third power of both sides and then take square root of both sides:

$$\begin{aligned}
(x-1)^{\frac{2}{3}} &= 4 \\
(x-1)^2 &= 4^3 \\
\sqrt{(x-1)^2} &= \sqrt{2\underline{\hspace{2em}}} \\
|x-1| &= \underline{\hspace{2em}} \\
x-1 = \underline{\hspace{2em}} \quad &\text{or} \quad x-1 = \underline{\hspace{2em}} \\
x = 9 \quad &\text{or} \quad x = -7
\end{aligned}$$

Check each solution:

$$(9-1)^{\frac{2}{3}} = 8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4$$

$$(-7-1)^{\frac{2}{3}} = (-8)^{\frac{2}{3}} = \left((-8)^{\frac{1}{3}}\right)^2 = (-2)^2 = 4$$

So the equation has two solutions $x = 9$ and $x = -7$.

Example 13.5. Solve the equation $(x+2)^{\frac{1}{2}} - (x-3)^{\frac{1}{2}} = 1$.

Solution. Since there is more than one term involving rational exponents, to solve the equation, we isolate one term, take a power, and proceed accordingly.

$$\begin{aligned}
(x+2)^{\frac{1}{2}} - (x-3)^{\frac{1}{2}} &= 1 \\
(x+2)^{\frac{1}{2}} &= (x-3)^{\frac{1}{2}} + 1 \\
x+2 &= \left((x-3)^{\frac{1}{2}} + 1\right)^2 \\
x+2 &= (x-3) + 2(x-3)^{\frac{1}{2}} + 1 \\
2(x-3)^{\frac{1}{2}} &= 4 \\
(x-3)^{\frac{1}{2}} &= 2 \\
x-3 &= 4 \\
x &= 7
\end{aligned}$$

Check:

$$(7+2)^{\frac{1}{2}} - (7-3)^{\frac{1}{2}} = \sqrt{9} - \sqrt{4} = 3 - 2 = 1.$$

So the equation has one solution $x = 7$.

13.4 Radical Equations Solvable by Substitution

Some radical or rational exponent equations can be transformed into polynomial or rational equations by substitution. To solve these, we substitute a repeated algebraic expression with a new variable, solve the resulting equation, and then substitute back to find the solution to the original equation.

Example 13.6. Solve the equation $\sqrt[3]{(x+1)^2} + \sqrt[3]{x+1} = 6$.

Solution.

1) Substitute $\sqrt[3]{x+1}$ by t and solve the resulting equation.

$$\begin{aligned} t^2 + t &= 6 \\ t^2 + t - 6 &= 0 \\ (t+3)(t-2) &= 0 \\ t + 3 = 0 \quad &\text{or} \quad t - 2 = 0 \\ t = -3 \quad &\qquad\quad t = 2 \end{aligned}$$

2) Substitute back to get the solutions of the original equation.

$$\begin{aligned} \sqrt[3]{x+1} &= -3 \qquad & \sqrt[3]{x+1} &= 2 \\ x + 1 &= -27 & x + 1 &= 8 \\ x &= -28 & x &= 7 \end{aligned}$$

13.5 Applications of Radical Equations

Example 13.7. The surface area S of a cone with radius r and height h is given by the formula

$$S = \pi r \sqrt{h^2 + r^2}.$$

If the surface area of a cone is 15π square centimeters and the radius is 3 centimeters, what is the height of the cone?

Solution. Substitute $S = 15\pi$ and $r = 3$ into the formula and solve for h.

$$\begin{aligned} 15\pi &= \pi \cdot 3 \cdot \sqrt{h^2 + 3^2} \\ 5 &= \sqrt{h^2 + 9} \\ 25 &= h^2 + 9 \\ h^2 &= 161 \\ h &= 4 \end{aligned}$$

Thus, the height of the cone is 4 centimeters.

13.6 Learn from Mistakes

This example shows some common mistakes when solving radical equations.

Example 13.8. Can you identify the mistakes in the given solution and fix the solution?

Solve the radical equation.

$$\sqrt{x-2}+2=x$$

Incorrect Solution:

$$\begin{aligned} \sqrt{x-2}+2&=x \\ \left(\sqrt{x-2}\right)^2+2^2&=x^2 \\ x-2+4&=x^2 \\ x+2&=x^2 \\ x^2-x-2&=0 \\ (x-2)(x+1)&=0 \\ x=2 \quad \text{or} \quad x&=-1 \end{aligned}$$

Answer: the equation has two solutions $x=2$ and $x=-1$.

Solution. A few errors appear in the incorrect solution:

1) The square of the left-hand side was computed incorrectly. The correct expansion of $(a+b)^2$ is $a^2+2ab+b^2$, not a^2+b^2.
2) The possible solutions were not checked in the original equation.

Correct solution:

$$\begin{aligned} \sqrt{x-2}+2&=x \\ \sqrt{x-2}&=x-2 \\ x-2&=(x-2)^2 \\ x-2&=x^2+\underline{\qquad}+4 \\ x^2+\underline{\qquad}+\underline{\quad}&=0 \\ (x-2)(x-3)&=0 \\ x=2 \quad \text{or} \quad x&=3 \end{aligned}$$

Now check both possible solutions in the **original** equation:

- For $x=2$: $\sqrt{2-2}+2=0+2=2$
- For $x=3$: $\sqrt{3-2}+2=\sqrt{1}+2=3$

Conclusion: Both solutions check correctly. The equation has two solutions: $x=2$ and $x=3$.

Checking solutions confirms that no extraneous solutions occurred.

Exercises

Exercise 13.1. Solve each radical equation.

1) $\sqrt{3x+1}=4$

2) $\sqrt{2x-1}-5=0$

Answer: 1) $x=5$ 2) $x=13$

Exercise 13.2. Solve each radical equation.

1) $\sqrt{5x+1}=x+1$

2) $x=3-\sqrt{7-3x}$

3) $\sqrt{x-1}-2x=-3$

Answer: 1) $x=0$ or $x=3$ 2) $x=1$ or $x=2$ 3) $x=2$

Exercise 13.3. Solve each radical equation.

1) $\sqrt{2x-1} = \sqrt{3x+1} - 1$ 2) $\sqrt{x+2} + \sqrt{x-1} = 3$ 3) $\sqrt{x+5} + \sqrt{x-3} = 2$

Answer: 1) $x = 1$ or $x = 5$ 2) $x = 2$ 3) No solution

Exercise 13.4. Solve each radical equation.

1) $\sqrt[3]{-2x-7} + 3 = 0$ 2) $3\sqrt[3]{3x-1} = 6$

Answer: 1) $x = -10$ 2) $x = 3$

Exercise 13.5. Solve each radical equation.

1) $(x+3)^{\frac{1}{2}} = x+1$

2) $2(x-1)^{\frac{1}{2}} - (x-1)^{-\frac{1}{2}} = 1$

Answer: 1) $x=1$ 2) $x=2$

Exercise 13.6. Solve each radical equation.

1) $(x-1)^{\frac{3}{2}} = 8$

2) $(x+1)^{\frac{2}{3}} - 8 = 1$

Answer: 1) $x=5$ 2) $x=-28$ or $x=26$

Exercise 13.7. Solve each radical equation.

1) $\sqrt[3]{(x+2)^2} - 2\sqrt{x+2} = 8$

2) $\sqrt{2x+1} - \dfrac{12}{\sqrt{2x+1}} = 1$

Answer: 1) $x = -6$ or $x = 62$ 2) $x = \dfrac{15}{2}$

Exercise 13.8. The distance d, in kilo-meters, that a person can see to the horizon is modeled by the equation:

$$d = \sqrt{13h},$$

where h is the height (in meters) of the observer above sea level.

A lighthouse is to be constructed on top of a cliff that is 12 meters above sea level. How tall must the lighthouse be so that its light can be seen from a distance of 15 kilometers?

Answer: 5.31 meters

Topic 14 Absolute Value Equations

Learning Goals

- I can evaluate absolute values and use properties to simplify expressions.
- I can solve simple absolute value equations involving one absolute value expression.
- I can solve absolute value equations involving two absolute value expressions.

14.1 Think about It

The Direction of a Number

Given a real number x, can you write an expression that represents the **direction** of x on the number line?

14.2 Properties of Absolute Values

The **absolute value** of a real number a, denoted by $|a|$, is the distance from 0 to a on the number line. In particular, $|a|$ is nonnegative, that is, $|a| \geq 0$.

Properties of Absolute Values

$$|-a| = |a|, \qquad |ab| = |a|\ |b| \qquad \text{and} \qquad \left|\frac{a}{b}\right| = \frac{|a|}{|b|}.$$

Example 14.1. Evaluate the absolute values.

1) $-|-1-(-4)|$ 2) $|-2\cdot(-5)|$ 3) $-\dfrac{6}{|-3|}$ 4) $\dfrac{|2-5|+5}{|3-1|}$

Solution.

1) $-|-1-(-4)| = -\ |\underline{\qquad}| = \underline{\qquad}$

2) $|-2\cdot(-5)| = |\underline{\qquad}| = \underline{\qquad}$

3) $-\dfrac{6}{|-3|} = -\dfrac{6}{\underline{\qquad}} = \underline{\qquad}$

4) $\dfrac{|2-5|+5}{|3-1|} = \dfrac{|\underline{\qquad}|+5}{|\underline{\qquad}|} = \dfrac{\underline{\qquad}+5}{\underline{\qquad}} = \underline{\qquad}$

14.3 Solving Absolute Value Equations

An **absolute value equation** is an equation that contains an absolute value expression.

Equivalent Equations of Absolute Value Equations

Consider the absolute value equation in the form $|X| = c$, where X is an expression and c is a real number.

- If $c > 0$, then
$$|X| = c \iff X = \pm c \iff X = c \text{ or } X = -c.$$
Warning: if c is replaced with an expression Y, then only the solutions such that $Y \geq 0$ is valid.
- If $c = 0$, then the equation $|X| = 0$ is equivalent to $X = 0$.
- If $c < 0$, then the equation $|X| = c$ has no solution.
- If Y is another expression, then
$$|X| = |Y| \iff X = Y \text{ or } X = -Y.$$
- By the definition of square root,
$$|X| = c \iff \sqrt{X^2} = c.$$
In particular, if $c > 0$, then $|X| = c$ is equivalent to $X^2 = c^2$.

Since $|X| = \sqrt{X^2}$, solving absolute value equations is similar to solving radical equations. The first step it to isolate an about value expression on one side of the equation. The apply the properties above or the definition to solve it.

Example 14.2. Solve the equation
$$|2x - 3| = 7.$$

Solution. The equation is equivalent to
$$\begin{aligned} 2x - 3 &= \underline{\quad\quad} &\quad \text{or} \quad 2x - 3 &= \underline{\quad\quad} \\ 2x &= \underline{\quad\quad} & 2x &= \underline{\quad\quad} \\ x &= -2 & x &= 5 \end{aligned}$$
The solutions are $x = -2$ or $x = 5$. In set-builder notation, the solution set is $\{-2, 5\}$.

Example 14.3. Solve the equation
$$|2x - 1| - 3 = 8.$$

Solution.

1) Rewrite the equation into $|X| = c$ form.
$$|2x - 1| = \underline{\quad\quad}$$

1) Solve the equation.

$$\begin{aligned} 2x - 1 &= \underline{\qquad} \quad \text{or} \quad & 2x - 1 &= \underline{\qquad} \\ 2x &= \underline{\qquad} & 2x &= \underline{\qquad} \\ x &= -5 & x &= 6 \end{aligned}$$

The solutions are $x = -5$ or $x = 6$. In set-builder notation, the solution set is $\{-5, 6\}$.

Example 14.4. Solve the equation

$$3|2x - 5| = 9.$$

Solution.

1) Rewrite the equation into $|X| = c$ form.

$$|2x - 5| = \underline{\qquad}$$

2) Solve the equation.

$$\begin{aligned} 2x - 5 &= \underline{\qquad} \quad \text{or} \quad & 2x - 5 &= \underline{\qquad} \\ 2x &= \underline{\qquad} & 2x &= \underline{\qquad} \\ x &= 1 & x &= 4 \end{aligned}$$

The solutions are $x = 1$ or $x = 4$. In set-builder notation, the solution set is $\{1, 4\}$.

Example 14.5. Solve the equation

$$2|1 - 2x| - 3 = 7.$$

Solution.

1) Rewrite the equation into $|X| = c$ form.

$$\begin{aligned} 2|1 - 2x| - 3 &= 7 \\ 2|2x - 1| &= \underline{\qquad} \\ |2x - 1| &= \underline{\qquad} \end{aligned}$$

2) Solve the equation.

$$\begin{aligned} 2x - 1 &= \underline{\qquad} \quad \text{or} \quad & 2x - 1 &= \underline{\qquad} \\ 2x &= \underline{\qquad} & 2x &= \underline{\qquad} \\ x &= -2 & x &= 3 \end{aligned}$$

The solutions are $x = -2$ or $x = 3$. In set-builder notation, the solution set is $\{-2, 3\}$.

Example 14.6. Solve the equation
$$|3x-2| = |x+2|.$$

Solution. (Using $|X| = |Y| \iff X = \pm Y$). The equation is equivalent to
$$3x-2 = x+2 \quad \text{or} \quad 3x-2 = -(x+2).$$
Solve the equations:
$$\begin{aligned} 3x-2 &= x+2 &\quad\text{or}\quad 3x-2 &= -(x+2) \\ 2x &= 4 & 4x &= 0 \\ x &= 2 & x &= 0 \end{aligned}$$
The solutions are $x = 2$ and $x = 0$. In set-builder notation, the solution set is $\{0, 2\}$.

Solution. (Using $|X| = \sqrt{X^2}$). Since $|X| = \sqrt{X^2}$, we can rewrite the equation as
$$\sqrt{(3x-2)^2} = \sqrt{(x+2)^2}$$
Square both sides and solve:
$$\begin{aligned} (3x-2)^2 &= (x+2)^2 \\ (3x-2)^2 - (x+2)^2 &= \underline{\hspace{2em}} \\ ((3x-2)-(x+2))((3x-2) + \underline{\hspace{3em}}) &= 0 \\ (\underline{\hspace{4em}})(\underline{\hspace{4em}}) &= 0 \end{aligned}$$
$$\begin{aligned} 2x-4 &= 0 &\quad\text{or}\quad 4x &= 0 \\ 2x &= 4 & x &= 0 \\ x &= 2 \end{aligned}$$
The solutions are $x = 2$ and $x = 0$. In set-builder notation, the solution set is $\{0, 2\}$.

Example 14.7. Solve the equation.
$$2\,|1-x| = |2x+10|$$

Solution. Since $2 = |2|$,
$$2|1-x| = |2|\,|1-x| = |2(1-x)| = |2-2x|.$$
Moreover, because $|-X| = |X|$, $|2-2x| = |-(2x-2)| = |2x-2|$. The equation is then equivalent to
$$|2x-2| = |2x+10|.$$
Solve the equation.
$$\begin{aligned} 2x-2 &= 2x+10 &\quad\text{or}\quad 2x-2 &= -(2x+10) \\ -2 &= 10 & 4x &= -8 \\ &\text{impossible} & x &= -2 \end{aligned}$$
The equation has one solution $x = -2$. In set-builder notation, the solution set is $\{-2\}$.

Exercises

Exercise 14.1. Find the solution set for the equation.

1) $|2x-1|=5$

2) $\left|\dfrac{3x-9}{2}\right|=3$

Answer: 1) $\{3,-2\}$ 2) $\{5,1\}$

Exercise 14.2. Find the solution set for the equation.

1) $|3x-6|+4=13$

2) $3\,|2x-5|=9$

Answer: 1) $\{5,-1\}$ 2) $\{4,1\}$

Exercise 14.3. Find the solution set for the equation.

1) $|5x-10|+6=6$

2) $-3\,|3x-11|=5$

Answer: 1) $\{2\}$ 2) no solution

Exercise 14.4. Find the solution set for the equation.

1) $3\,|5x-2|-4=8$ 2) $-2\,|3x+1|+5=-3$

Answer: 1) $\left\{\frac{6}{5},-\frac{2}{5}\right\}$ 2) $\left\{-\frac{5}{3},1\right\}$

Exercise 14.5. Find the solution set for the equation.

1) $|5x-12|=|3x-4|$ 2) $|x-1|=-5\,|(2-x)-1|$

Answer: 1) $\{4,2\}$ 2) $\{1\}$

Exercise 14.6 (Optional). Find the solution set for the equation.

1) $|2x-1|=5-x$ 2) $-2x=|x+3|$

Answer: 1) $\{-4,2\}$ 2) $\{-1\}$

Topic 15 Linear Inequalities

Learning Goals

- I can convert between set builder notation, interval notation, inequality notation, and graphical notation for subsets of real numbers.
- I can use properties of inequalities to solve linear inequalities and present the solution set in different notations.
- I can solve compound linear inequalities and present the solution set in different notations.

15.1 Think about It

Minimum Score Needed on the Final

A course has three types of assessments: homework, monthly test, and the final exam. The grading policy of the course states that homework counts 20%, monthly test counts 45%, and the final exam counts for 35%. On the last day of class, a student wants to know the minimum grade needed on the final to achieve a grade C or better, equivalently, an overall grade of 74 or above. The student earned 100 on homework and 80 on the monthly test.

1) What is the minimum grade the student must earn on the final to achieve a C or better?

2) If, in addition, the final exam must be at least 55 to earn a C or better, what would be the minimum grade needed?

15.2 Properties of Inequalities

An **inequality** is a mathematical statement that compares two expressions using inequality signs. The most common inequality signs are:

Inequality Sign	Meaning
$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to

Geometrically, an inequality shows the relative position of two points on a number line. For example, $x < 2$ means that x is to the left of 2, and $x > 2$ means that x is to the right of 2.

- Adding a number moves everything to left (if the number is negative) or right (if the number is positive).
- Multiplying a positive number scales everything.
- Multiplying -1 flips statement horizontally.

These geometric interpretations help us understand the following basic properties of inequalities (exampled with the less than inequality sign).

Properties of Inequalities

For any real numbers a, b, and c:

- **The additive property**
  If $a < b$, then $a + c < b + c$, for any real number c.
  If $a < b$, then $a - c < b - c$, for any real number c.
- **The positive multiplication property**
  If $a < b$ and c is positive, then $ac < bc$.
  If $a < b$ and c is positive, then $\frac{a}{c} < \frac{b}{c}$.
- **The negative multiplication property**
  If $a < b$ and c is negative, then $ac > bc$.
  If $a < b$ and c is negative, then $\frac{a}{c} > \frac{b}{c}$.

Properties for Other Inequality Signs

These properties for the inequality sign $<$ also apply to $\leq$, $>$, and $\geq$.

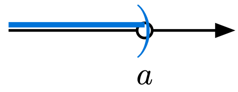
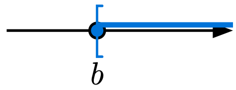
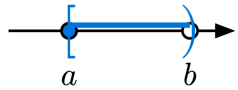
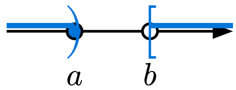
Subtraction is Addition

It is always better to view $a - c$ as $a + (-c)$. Because addition has the commutative property but subtraction does not. For example $1 - 2 \neq 2 - 1$ but $1 + (-2) = (-2) + 1$.

15.3 Solving Linear Inequalities

A **linear inequality** is an inequality that contains only linear expressions (degree 1 polynomials) in one or both sides of the inequality sign. The solution to a linear inequality is a set of real numbers that satisfy the inequality. The solution set can be expressed in different notations, such as interval notation, set-builder notation, or graphical representation.

The following table summarizes the common notations for expressing the solution set of a linear inequality.

Inequality	Interval Notation	Set-builder Notation	Graphical Representation
$x < a$	$(-\infty, a)$	$\{x \mid x < a\}$	
$x \geq b$	$[b, \infty)$	$\{x \mid x \geq b\}$	
$a \leq x < b$	$[a, b)$	$\{x \mid a \leq x < b\}$	
$x < a$ or $x \geq b$	$(-\infty, a) \cup [b, \infty)$	$\{x \mid x < a \text{ or } x \geq b\}$	

To solve a linear inequality, we isolate the variable on one side of the inequality sign using the arithmetic properties of inequalities. Make sure to switch the inequality sign when multiplying or dividing both sides by a negative number.

Example 15.1. Solve the linear inequality.

$$2x + 4 > 0.$$

Solution. To isolate x, we first subtract 4 from both sides and then divide by 2.

$$\begin{aligned} 2x + 4 &> 0 \\ \underline{\hspace{1.5cm}} \ \underline{\hspace{1cm}} &-4 \\ x \ \underline{\hspace{1cm}} &-2 \end{aligned}$$

Example 15.2. Solve the linear inequality and present the solution in set-builder and interval notations, and graphical presentation.

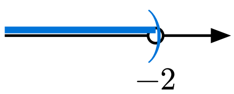
$$-3x - 4 < 2$$

Solution. To isolate x, we first add 4 to both sides and then divide by -3. Remember to switch the inequality sign when dividing by a negative number.

$$\begin{aligned} -3x - 4 &> 2 \\ \underline{\hspace{1.5cm}} \ \underline{\hspace{1cm}} &\ 6 \\ x \ \underline{\hspace{1cm}} &-2 \end{aligned}$$

In set-builder notation, the solution set is $\{x \mid \underline{\hspace{2.5cm}}\}$.

In graphical representation, the solution set is all numbers to the right of -2 on the number line.



In interval notation, the solution set is $(\underline{\hspace{1cm}}, \underline{\hspace{1cm}})$.

15.4 Compound Inequalities

A **compound inequality** is formed by two inequalities with the word **and** or the word **or**. For example, the following are three commonly seen types of compound inequalities:

- $x - 1 > 2 \quad \text{and} \quad 2x + 1 < 3$
- $3x - 5 < 4 \quad \text{or} \quad 3x - 2 > 10$
- $-3 \leq \dfrac{2x - 4}{3} < 2$

The last compound inequality is a simplified expression for the compound inequality $-3 \leq \dfrac{2x-4}{3}$ and $\dfrac{2x-4}{3} < 2$.

To solve a compound inequality, we solve each part of the inequality separately and then combine the results. If the compound inequality uses the word **and**, the solution set is the **intersection** of the two parts. If it uses the word **or**, the solution set is the **union** of the two parts.

Example 15.3. Solve the compound linear inequality

$$x + 2 < 3 \quad \text{and} \quad -2x - 3 < 1.$$

Solution. Solve each part of the compound inequality separately and combine the results.

$$\begin{aligned} x + 2 < 3 \quad &\text{and} \quad -2x - 3 < 1 \\ & \qquad\quad -2x < 4 \\ x < 1 \quad &\text{and} \qquad\quad x \,\underline{\qquad}\, -2 \end{aligned}$$

Equivalently, $-2 < x < 1$.

Example 15.4. Solve the compound linear inequality and present the solution in set-builder and interval notations, and graphical presentation.

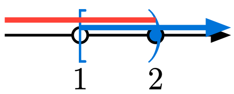
$$-x + 4 > 2 \quad \text{or} \quad 2x - 5 \geq -3.$$

Solution. Solve each part of the compound inequality separately and combine the results.

$$\begin{aligned} -x + 4 > 2 \quad &\text{or} \quad 2x - 5 \geq -3 \\ -x > -2 \quad & \qquad\quad 2x \geq 2 \\ x \,\underline{\qquad}\, 2 \quad &\text{or} \qquad\quad x \geq 1 \end{aligned}$$

In set-builder notation, $\{x \mid x \geq 1 \ \text{ or } \ x < 2\}$.

In graphical representation, the solution set is all numbers to the right of 1 **or** to the left of 2 on the number line, that is, the join of the two parts $x \geq 1$ and $x < -2$.



In interval notation, the solution set is $(-\infty, \infty)$.

Example 15.5. Solve the compound linear inequality and write the solution in interval notation.

$$-4 \leq \frac{2x-4}{3} < 2.$$

Solution. For this type of compound inequality, we can isolate the variable in the middle expression and solve the two inequalities together.

$$\begin{array}{rcccl} -4 \leq & & \dfrac{2x-4}{3} & & < 2 \\ -12 & \underline{\hspace{2em}} & 2x-4 & \underline{\hspace{2em}} & 6 \\ -8 & \underline{\hspace{2em}} & 2x & \underline{\hspace{2em}} & 10 \\ -4 & \underline{\hspace{2em}} & x & \underline{\hspace{2em}} & 5 \end{array}$$

Since x is in between, the solution set in interval notation is $[-4, 5)$.

Example 15.6. Solve the compound linear inequality and write the solution in interval notation.

$$-1 \leq \frac{-3x+4}{2} < 3.$$

Solution. Similar to the previous example, we can isolate the variable in the middle expression and solve the two inequalities together.

$$\begin{array}{rcccl} -1 \leq & & \dfrac{-3x+4}{2} & & < 3 \\ -2 & \underline{\hspace{2em}} & -3x+4 & \underline{\hspace{2em}} & 6 \\ -6 & \underline{\hspace{2em}} & -3x & \underline{\hspace{2em}} & 2 \\ 2 & \underline{\hspace{2em}} & x & \underline{\hspace{2em}} & -\dfrac{2}{3} \end{array}$$

Since x is in between, the solution set in interval notation is $\left(-\frac{2}{3}, 2\right]$.

Example 15.7. Suppose that $-1 \leq x < 2$. Find the range of $5-3x$. Write your answer in interval notation.

Solution. To get $5-3x$ from x, we need to first multiply x by -3 and then add 5.

$$\begin{array}{rcccl} -1 \leq & & x & & < 2 \\ 3 & \underline{\hspace{2em}} & -3x & \underline{\hspace{2em}} & -6 \\ 8 & \underline{\hspace{2em}} & 5-3x & \underline{\hspace{2em}} & -1 \end{array}$$

The range of $5-3x$ is $(-1, 8]$.

Problem-solving Strategy: Bridging

To solve a problem, knowing the goal helps you narrow the gap step by step by comparing the target with your current progress.

An inequality (or equation) is solved when the unknown variable is isolated—that is what you aim for. To isolate the unknown, use comparisons to determine which mathematical operations should be applied. When an operation is applied to one side, the same operation must also be applied to the other side. For inequalities, you must also decide whether the inequality sign should be preserved or reversed depending on the operation.

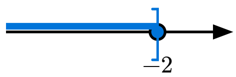
Exercises

Exercise 15.1. Solve the linear inequality. **Write your answer in set-builder notation, interval notation and graphical representation.**

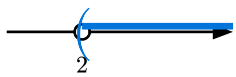
1) $3x + 7 \leq 1$

2) $2x - 3 > 1$

Answer: 1) $\{x \mid x \leq -2\}$; $(-\infty, -2]$;



2) $\{x \mid x > 2\}$; $(2, \infty)$;

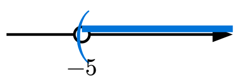


Exercise 15.2. Solve the linear inequality. **Write your answer in set-builder notation, interval notation and graphical representation.**

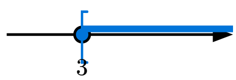
1) $4x + 7 > 2x - 3$

2) $3 - 2x \leq x - 6$

Answer: 1) $\{x \mid x > -5\}$; $(-5, \infty)$;



2) $\{x \mid x \geq 3\}$; $[3, \infty)$;



Exercise 15.3. Solve the compound linear inequality. **Write your answer in interval notation.**

1) $3x + 2 > -1 \quad \text{and} \quad 2x - 7 \leq 1$

2) $4x - 7 < 5 \quad \text{and} \quad -5x + 2 \geq 3$

Answer: 1) $(-1, 4]$ 2) $(-\infty, 1]$

Exercise 15.4. Solve the compound linear inequality. **Write your answer in interval notation.**

1) $-4 \leq 3x + 5 < 11$

2) $7 \geq 2x - 3 \geq -7$

Answer: 1) $[-3, 2)$ 2) $[-2, 5]$

Exercise 15.5. Solve the compound linear inequality. **Write your answer in interval notation.**

1) $3x - 5 > -2 \quad \text{or} \quad 10 - 2x \leq 4$

2) $2x + 7 < 5 \quad \text{or} \quad 3x - 8 \geq x - 2$

Answer: 1) $(1, \infty)$ 2) $(-\infty, -1) \cup [3, \infty)$

Exercise 15.6. Solve the compound linear inequality. **Write your answer in interval notation.**

1) $-2 \leq \dfrac{2x - 5}{3} < 3$

2) $-1 < \dfrac{3x + 7}{2} \leq 4$

Answer: 1) $\left[-\frac{1}{2}, 7\right)$ 2) $\left(-3, \frac{1}{3}\right]$

Exercise 15.7. Solve the linear inequality. **Write your answer in interval notation.**

$$\frac{1}{3}x + 1 < \frac{1}{2}(2x - 3) - 1$$

Answer: $\left(\frac{21}{4}, \infty\right)$

Exercise 15.8. Solve the compound linear inequality. **Write your answer in interval notation.**

$$0 \leq \frac{2}{5} - \frac{x+1}{3} < 1$$

Answer: $\left(-\frac{14}{5}, \frac{1}{5}\right]$

Exercise 15.9. Suppose $0 < x \leq 1$. Find the range of $-2x + 1$. **Write your answer in interval notation.**

Answer: $[-1, 1)$

Exercise 15.10. Suppose that $x + 2y = 1$ and $1 \leq x < 3$. Find the range of y. **Write your answer in interval notation.**

Answer: $(-1, 0]$

Exercise 15.11 (Optional). A toy store has a promotion “Buy one get the second one half price” on a certain popular toy. The sale price of the toy is \$20 each. The store earns more profit when a customer buys **two** toys under the promotion than when buying just **one** at full price. Based on this information, what can you conclude about the store’s purchase price (cost per toy)?

Answer: Less than \$10

Exercise 15.12 (Optional). A college student has completed 40 credits with a cumulative GPA of 1.8. To graduate, the student must meet both of the following requirements:

- A cumulative GPA of at least 2.0
- A minimum of 60 attempted credits

What is the **minimum GPA** the student must earn in all future courses to meet these graduation requirements?

Cumulative GPA = $\dfrac{\text{Total Quality Points}}{\text{Total Attempted Credits}}$

Total Quality Points = sum of $\text{Credits Attempted} \times \text{GPA}$

Answer: 2.4 if the student takes 20 more credits

Topic 16 Systems of Linear Equations

Learning Goals

- I can solve systems of linear equations in two variables.
- I can identify inconsistent systems of linear equations in two variables.
- I can identify dependent system of linear equations in two variables and express the solution in terms of one variable.
- I can solve a word problem involving a system of equations in two variables.

16.1 Think about It

A Pharmacist's Solution

A pharmacist needs to prepare 250 mL of a 25% alcohol solution. Unfortunately, only two stock solutions are available: one with 40% alcohol and another with 10% alcohol.

The pharmacist plan to mix these two solutions to prepare the desired solution.

How many milliliters of each solution should be mixed to obtain the desired concentration?

16.2 Introduction

A **system of linear equations** in two variables consists of two linear equations involving the same variables. A **solution** to the system is an ordered pair that satisfies both equations. Geometrically, a solution corresponds to the point of intersection of the two lines defined by the equations.

Example 16.1. Check if the ordered pair $(1, -3)$ is a solution to the system of linear equations.

$$2x - 3y = 11$$
$$5x + 2y = -1$$

Solution. Plug $x = 1$ and $y = -3$ into the left-hand side of first equation:

$$2 \cdot \underline{\quad\quad} - 3 \cdot (\underline{\quad\quad}) = 2 + 9 = 11$$

Plug $x = 1$ and $y = -3$ into the left-hand side of second equation:

$$5 \cdot \underline{\quad\quad} + 2 \cdot (\underline{\quad\quad}) = 5 - 6 = -1$$

Since both equations are satisfied, the ordered pair $(1, -3)$ is a solution.

Two common algebraic methods for solving systems of linear equations are the substitution method and the elimination method.

- **Substitution Method:** This method treats one equation as defining a variable in terms of the other. By solving one equation for one variable and substituting it into the other, the system is reduced to a single-variable equation. This method works best when a variable is already isolated or can be easily isolated.
- **Elimination Method:** This method eliminates one variable by combining the equations. This is done by adding or subtracting the equations—sometimes after multiplying one or both—to reduce the system to a single equation. It is especially efficient when both equations are in standard form.

16.3 Substitution Method

Example 16.2. Solve the system of linear equations using the substitution method.

$$x + y = 3 \qquad (1)$$
$$2x + y = 4 \qquad (2)$$

Solution.

1) Solve one variable from one equation. For example, solve y from equation (1).

$$y = \underline{\qquad\qquad}$$

2) Plug $y = 3 - x$ into equation (2) and solve for x.

$$\begin{aligned} 2x + (\underline{\qquad\qquad}) &= 4 \\ x + 3 &= 4 \\ x &= 1 \end{aligned}$$

3) Plug the solution $x = 1$ into the equation from step 1 to solve for y.

$$y = 3 - x = 3 - 1 = 2$$

The solution of the system is $(1, 2)$.

16.4 Elimination Method

Example 16.3. Solve the system of linear equations using the elimination method.

$$5x + 2y = 7 \qquad (1)$$
$$3x - y = 13 \qquad (2)$$

Solution.

1) Eliminate one variable and solve for the other. For example, choose to eliminate y. In order to eliminate y, we **add the opposite**. Multiply both sides of the second equation by 2 to get the opposite $-2y$.

$$\begin{aligned} 2(3x) - 2y &= 2 \cdot 13 \\ \underline{\qquad\qquad} &= 26 \qquad (3) \end{aligned}$$

Adding equations (1) and (3) will eliminate y.

$$\begin{array}{rr} & 5x + 2y = 7 \\ + & 6x - 2y = 26 \\ \hline & \underline{\qquad\qquad} = 33 \\ & x = 3 \end{array}$$

2) Find the missing variable by plugging $x = 3$ into equation (1) and solve for y.

$$\begin{aligned} 5 \cdot 3 + 2y &= 7 \\ \underline{\qquad\qquad} &= 7 \\ 2y &= \underline{\qquad} \\ y &= -4 \end{aligned}$$

3) Check the proposed solution. Plug $(3, -4)$ into the first equation:

$$5 \cdot 3 + 2 \cdot (-4) = 15 - 8 = 7.$$

Plug $(3, -4)$ into the second equation:

$$3 \cdot 3 - (-4) = 9 + 4 = 13.$$

Possible Number of Solutions of a System of Linear Equations

A linear system can have **one solution**, **no solution**, or **infinitely many solutions**.

- Geometrically, if the equations define lines with **different slopes**, the lines intersect at a single point, and the system has **one solution**. Algebraically, this means that substitution or elimination leads to a unique solution for both variables.
- Geometrically, if the equations define lines with the **same slope but different y-intercepts**—that is, the lines are **parallel** and never intersect—or algebraically, if substitution or elimination leads to a contradiction such as $0 = c$ (where $c \neq 0$), the system has **no solution**. In this case, the system is called **inconsistent**.
- Geometrically, if the equations define lines with the **same slope and same y-intercept**—that is, they represent the **same line**—or algebraically, if substitution or elimination results in an identity such as $0 = 0$, the system has **infinitely many solutions**.

  In this case, the system is called **dependent**, and the solution can be expressed as

$$(x, f(x)) = (x, mx + b).$$

16.5 Inconsistent or Dependent Systems

Example 16.4. Determine whether the system of linear equations is inconsistent, dependent, or has a unique solution. If it has a unique solution, find it. If it is dependent, express the solution in terms of one variable.

$$3x - 2y = 1 \qquad (1)$$
$$6x - 4y = 3 \qquad (2)$$

Solution.

1) We will eliminate the variable x. Since the coefficients of x are 2 and 4, to eliminate x, we first multiply equation (1) by -2.

$$-2(3x - 2y) = -2 \cdot 1$$
$$\underline{\qquad\qquad} = -2 \qquad (3)$$

2) Add equation (2) to equation (2).

$$\begin{array}{rr} & -6x + 4y = -2 \\ + & 6x - 4y = 3 \\ \hline & \underline{\qquad\quad} = 1 \end{array}$$

Since the result is a contradiction, the system is inconsistent and has no solution.

Example 16.5. Determine whether the system of linear equations is inconsistent, dependent, or has a unique solution. If it has a unique solution, find it. If it is dependent, express the solution in terms of one variable.

$$2x + 3y = 6 \qquad (1)$$
$$4x + 6y = 12 \qquad (2)$$

Solution.

1) We will eliminate the variable x. Since the coefficients of x are 2 and 4, to eliminate x, we first multiply equation (1) by -2.

$$-2(2x + 3y) = -2 \cdot 6$$
$$\underline{\qquad\qquad} = -12 \qquad (3)$$

2) Add equation (3) to equation (2).

$$\begin{array}{rr} & -4x - 6y = -12 \\ + & 4x + 6y = 12 \\ \hline & 0 = \underline{\qquad} \end{array}$$

Since the result is an identity, the system is dependent and has infinitely many solutions. The solution can be expressed as $\left(x, -\frac{2}{3}x + 2\right)$.

16.6 Applications of Linear Systems

Example 16.6. Sarah and John are planning a party. Sarah buys 3 pizzas and 5 drinks for \$$76$. John buys 2 pizzas and 4 drinks for \$$52$. What is the price of one pizza and one drink, respectively?

Solution.

1) Let p be the price of one pizza and d be the price of one drink. Set up the system of equations based on the purchases:

$$3p + 5d = 76 \qquad (1)$$
$$2p + 4d = 52 \qquad (2)$$

2) Solve the system using the elimination method. Multiply equation (1) by -2 and equation (2) by 3 to align the coefficients of p:

$$-2(3p + 5d) = -2 \cdot 76$$
$$\underline{\qquad\qquad} = \underline{\qquad} \qquad (3)$$
$$3(2p + 4d) = 3 \cdot 52$$
$$\underline{\qquad\qquad} = \underline{\qquad} \qquad (4)$$

3) Add equation (3) from equation (4) to eliminate p:

$$\begin{array}{rr} & -6p - \underline{\qquad} = -152 \\ + & \underline{\qquad} + 12d = 156 \\ \hline & \underline{\qquad\quad} = 4 \\ & d = \underline{\qquad} \end{array}$$

4) Substitute $d = 2$ into equation (1) to find p:

$$3p + \underline{\qquad} = 76$$
$$3p = \underline{\qquad}$$
$$p = \underline{\qquad}$$

The price of one pizza is \$$22$ each and the price of one drink is \$$2$ each.

Exercises

Exercise 16.1. Solve.

1) $$\begin{aligned} 2x - y &= 8 \\ -3x - 5y &= 1 \end{aligned}$$

2) $$\begin{aligned} x + 4y &= 10 \\ 3x - 2y &= -12 \end{aligned}$$

Answer: 1) $(3, -2)$ 2) $(-2, 3)$

Exercise 16.2. Solve.

1) $$\begin{aligned} -x - 5y &= 29 \\ 7x + 3y &= -43 \end{aligned}$$

2) $$\begin{aligned} 2x + 15y &= 9 \\ x - 18y &= -21 \end{aligned}$$

Answer: 1) $(-4, -5)$ 2) $(-3, 1)$

Exercise 16.3. Solve.

1) $2x + 7y = 5$
$3x + 2y = 16$

2) $4x + 3y = -10$
$-2x + 5y = 18$

Answer: 1) $(6, -1)$ 2) $(-4, 2)$

Exercise 16.4. Determine whether the system of linear equations is inconsistent, dependent, or has a unique solution. If it has a unique solution, find it. If it is dependent, express the solution in terms of one variable.

1) $3x + 2y = 6$
$6x + 4y = 16$

2) $2x - 3y = -6$
$-4x + 6y = 12$

Answer: 1) Inconsistent, no solution. 2) Dependent, infinitely many solutions: $(x, \frac{2}{3}x - 2)$.

Exercise 16.5. Last week, Mike bought 5 apples and 4 oranges for \$$7$. This week, with the same prices, he bought 3 apples and 6 oranges for \$$6$. What is the price of one apple and one orange, respectively?

Answer: \$$1$ per apple and \$$0.5$ per orange.

Exercise 16.6. The sum of the digits of a certain two-digit number is 7. Reversing its digits increases the number by 27. What is the number?

Answer: The number is 25.

Part IV

Functions

Topic 17 Introduction to Functions

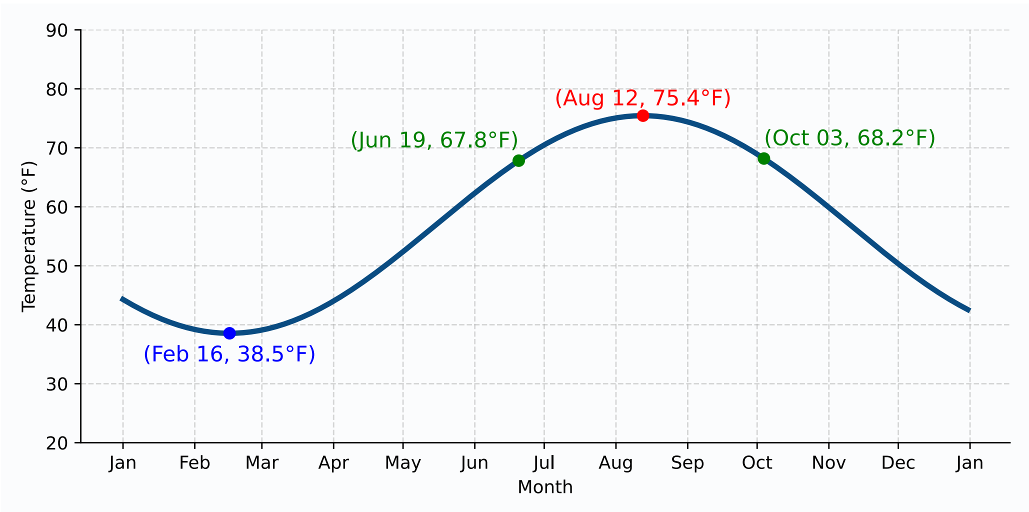
Learning Goals

- I can determine when relations are functions given a graph, table, or real-world situation.
- I can define a function and use function notations correctly.
- I can evaluate a function for a given input and find inputs for a given output.
- I can evaluate and simplify expressions involving function notation, e.g. difference quotient.
- I can find the domain and range of a function defined by a graph.
- I can estimate and interpret intercepts, locate extremum, and determine intervals of increase/decrease and where a function is positive or negative using its graph.

17.1 Think about It

Is It a Good Time to Swim in the Sea?

The graph below shows the average monthly water temperature of the Atlantic Ocean in New York City throughout the year.



1) How does the temperature change over the year?
2) What is the coldest water?
3) When is the hottest water?
4) When is the best time to swim in the ocean (≥ 68°F)?

17.2 Definition of Function

A **relation** is a set of ordered pairs. The set of all first elements in the pairs is called the **domain**, and the set of all second elements is called the **range**.

A **function** is a special type of relation in which each element of the domain corresponds to **exactly one** element in the range.

Example 17.1. Consider the relation $L = \{(1,3),(2,5),(3,7),(4,9)\}$.

1) Write the domain and range of L.

2) Is L a function? Why or why not?

Solution.

1) The domain of L is $\{1, 2, \underline{\qquad\quad}\}$ and the range is $\{3, 5, \underline{\qquad\quad}\}$.

2) Yes, L is a function because each element in the domain corresponds to exactly one element in the range. For example, 1 corresponds to 3, 2 corresponds to 5, and so on.

Example 17.2. Consider the equation $y^2 = x$. Determine whether this equation defines y as a function of x.

Solution. The equation $y^2 = x$ does not define y as a function of x because for some values of x, there are two corresponding values of y. For example, if $x = 1$, then y can be either $\underline{\qquad}$ or $\underline{\qquad}$. Therefore, this equation does not satisfy the definition of a function.

Geometrically, an ordered pair (x, y) can be represented as a point in the rectangular coordinate system. A relation can be represented as a set of points, called the **graph**, in the coordinate system. **The graph of a function** is the graph of its ordered pairs.

Vertical Line Test

A relation defines y as a function of x if any vertical line crosses the graph of the relation at most once.

How to Find the Domain and Range of a Graph

To find the **domain**, trace where vertical lines intersect the graph from left to right.

- The domain begins at the leftmost point of intersection and ends at the rightmost point. If there is break in the graph, the domain is the union of the domains of the disjoint pieces of the graph.
- If there is no leftmost point, the domain starts at $-\infty$.

- If there is no rightmost point, the domain ends at ∞.

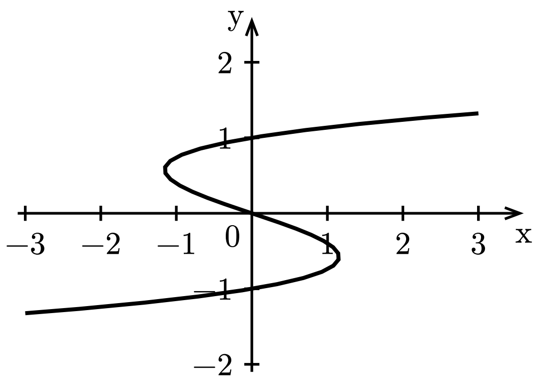
To find the **range**, trace where horizontal lines intersect the graph from bottom to top.

- The range begins at the lowest point of intersection and ends at the highest point. If there is break in the graph, the range is the union of the ranges of the disjoint pieces of the graph.
- If there is no lowest point, the range starts at $-\infty$.
- If there is no highest point, the range ends at ∞.

Note: Use parentheses with $-\infty$ or ∞ in interval notation, since infinity is not a fixed number.

Example 17.3. Consider the relation represented by the graph below.

1) Find the domain and range of the relation.
2) Determine whether this relation defines y as a function of x



Solution. Since there are no leftmost or rightmost points, the domain is ________.

Since there are no lowest or highest points, the range is ________.

The relation does not define y as a function of x because there are vertical lines that intersect the graph at three points, such as the vertical line at $x =$ ____.

17.3 Function notations

In functions, the variable x typically represents an input value from the domain and is called the **independent variable**. The corresponding output is denoted by y and is called the **dependent variable**. In this context, we say that **y is a function of x**.

To distinguish between different functions, we often name them using letters such as f, g, or F. The notation $f(x)$, read as "f of x" or "f at x", represents the output value of the function f when the input is x. The notation $f(x)$ is called **function notation**.

- The **domain** of a function is the set of all valid input values.
- The **range** is the set of all possible output values.

Functions can be represented in various ways, such as by equations, tables, or graphs. For example, the equation $f(x) = 2x + 3$ defines a function where each input x is mapped to an output calculated by multiplying x by 2 and then adding 3.

The process of finding the output of a function for a specific input value is called **evaluating the function** at that input. This process is called **finding the input for a given output**. To find the input value x such that $f(x) = y$, we solve the equation $f(x) = y$.

Example 17.4. Find the indicated function value.

1) $f(-2)$,
$f(x) = 2x + 1$

2) $g(2)$,
$g(x) = 3x^2 - 10$

3) $h(a - t)$,
$h(x) = 3x + 5$.

Solution.

1)
$$\begin{aligned} & f(-2) \\ &= 2 \cdot (\underline{\qquad}) + 1 \\ &= \underline{\qquad} + 1 \\ &= -3. \end{aligned}$$

2)
$$\begin{aligned} & g(2) \\ &= 3 \cdot (\underline{\qquad})^2 - 10 \\ &= \underline{\qquad} - 10 \\ &= 2. \end{aligned}$$

3)
$$\begin{aligned} & h(a-t) \\ &= 3 \cdot (\underline{\qquad\quad}) + 5 \\ &= \underline{\qquad\quad} + 5. \end{aligned}$$

Example 17.5. Consider the function $f(x) = x^2 - 3x + 1$. Evaluate the **difference quotient**

$$\frac{f(x+h) - f(x)}{h}$$

and simplify your answer.

Solution. Evaluate $f(x + h)$ and simplify the difference quotient leads to:

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 - 3(x+h) + 1 - (x^2 - 3x + 1)}{h} \\ &= \frac{(x^2 + \underline{\qquad\qquad}) - 3x - \underline{\qquad} + 1 - (x^2 - 3x + 1)}{h} \\ &= \frac{\underline{\qquad\qquad} - 3h}{h} \\ &= \frac{h(2x + h - \underline{\qquad})}{h} \\ &= 2x + \underline{\qquad} - 3 \end{aligned}$$

Difference Quotient and Average Rate of Change

Given a function f, the **difference quotient** $\dfrac{f(x+h) - f(x)}{h}$, or the **average rate of change** $\dfrac{f(a) - f(b)}{a - b}$ over an interval $[a, b]$ is a fundamental concept in calculus and is used to derive the derivative of a function.

Example 17.6. Find the input value x such that $f(x) = 5$ for the function $f(x) = 2x^2 - 3x$.

Solution. To find the input value x such that $f(x) = 5$, we solve the equation

$$\begin{aligned} 2x^2 - 3x &= 5 \\ 2x^2 - 3x + \underline{\qquad} &= 0 \\ (2x + \underline{\qquad})(x + \underline{\qquad}) &= 0 \\ 2x + \underline{\qquad} = 0 \quad &\text{or} \quad x + \underline{\qquad} = 0 \\ x = \underline{\qquad} \quad &\text{or} \quad x = \underline{\qquad} \end{aligned}$$

Thus, there are two values $x = \frac{5}{2}$ and $x = -1$ such that $f(x) = 5$.

17.4 Graph Reading

When a function is represented by its graph, we can identify important features such as the coordinates of specific points, intercepts with the axes, intervals where the function is increasing or decreasing, and any maximum or minimum values.

How to Read a Graph

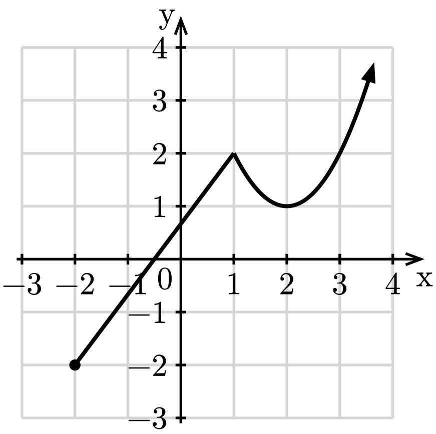
- To find the **coordinates of a point** on the graph, draw vertical and horizontal lines through the point.
  - The x-value is where the vertical line intersects the x-axis.
  - The y-value is where the horizontal line intersects the y-axis.
- To **find the y-value** for a given x-value, draw a vertical line at the given x and find its intersection with the graph.
  - The y-value is the y-coordinate of the point where the graph and the line intersect.
- To **find the x-value** for a given y-value, draw a horizontal line at the given y and find where it intersects the graph.
  - The x-value(s) are the x-coordinate(s) of those intersection points.
- The **y-intercept** is the point where the graph crosses the y-axis, which is where $x = 0$.
- The **x-intercept(s)** are the point(s) where the graph crosses the x-axis, which is where $y = 0$.
- A function is **increasing** where the graph rises as x moves from left to right, and **decreasing** where it falls.
- **Maximum** and **minimum** values are the y-coordinates of the highest and lowest points of the graph, respectively.

- To determine where the graph is **above** or **below** a specific horizontal line, identify the intervals where the graph lies above or below that line.

  In particular, a function is **positive** where the graph is above the x-axis and **negative** where it is below.

Example 17.7. Use the graph in the picture to answer the following questions.

1) Determine whether the graph is a function and explain your answer.
2) Find the domain (in interval notation) of the graph.
3) Find the range (in interval notation) of the graph.
4) Find the interval where the graph is above 2.
5) Find the interval where the graph is decreasing.
6) Find all maximum and minimum values of the function if they exist.



7) Find the value of y such that the point $(3, y)$ is on the graph.
8) Find the value of x such that $(x, 0)$ is on the graph.

Solution.

1) The graph ______ a function. Because every vertical line crosses the graph at most once.
2) The graph has the left-most point at $(-2, \_\_\_\_)$ and no right-most point. So the domain is $[\_\_\_\_, +\infty)$.
3) The graph has a lowest point $(\_\_\_\_, -2)$ but no highest point. So the range is $[\_\_\_\_, +\infty)$.
4) The graph intersects with $y = 2$ at $(1, 2)$ and $(\_\_\_\_, 2)$. So it is above 2 only over the interval $(3, \_\_\_\_)$.
5) The graph is only going down when x moves from 1 to 2. So it is decreasing over the interval ________.
6) The lowest point of the graph is $(-2, -2)$. So, the minimum value is the y-coordinate ____. Since there is no highest point, there ________ maximum value.
7) When draw a vertical line through $x = 3$, it intersects with the graph at $(3, \_\_\_\_)$. So the y-value of the point $(3, y)$ on the graph is 2.
8) Since $y = 0$, the point is an x-intercept, that is the point where the graph crosses x-axis. Here, it is $(\_\_\_\_, 0)$. So, the x-value of the point $(x, 0)$ on the graph is -0.5.

Exercises

Exercise 17.1. Find the indicated function values for the functions

$$f(x) = -x^2 + x - 1 \quad \text{and} \qquad g(x) = 2x - 1.$$

Simplify your answer.

1) $f(2)$ 2) $f(-a)$ 3) $g(-1)$ 4) $g(f(1))$

Answer: 1) $-3.$ 2) $-a^2 - a - 1.$ 3) $-3.$ 4) $-3.$

Exercise 17.2. Suppose $g(x) = -3x + 1$.

1) Compute $\dfrac{g(4) - g(1)}{4 - 1}$ 2) Compute $\dfrac{g(a + h) - g(a)}{h}$

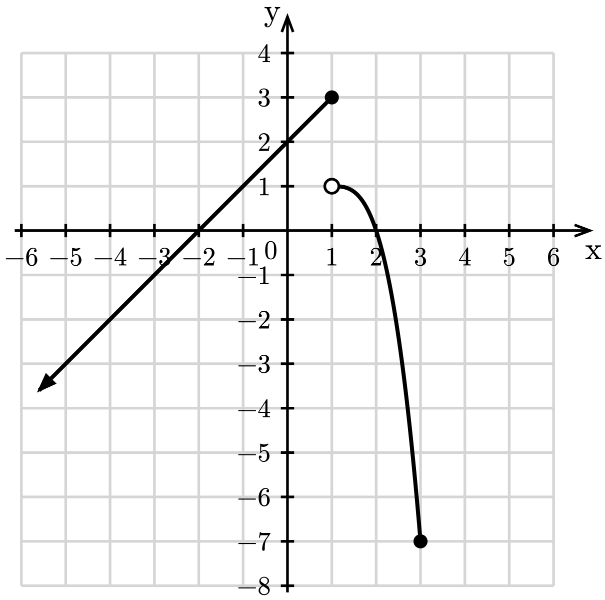
Answer: 1) $-3.$ 2) $-3.$

Exercise 17.3. Suppose the domain of the linear function $l(x) = 1 - 2x$ is $(0, 1)$. Find the range of the function.

Answer: $(-1, 1)$

Exercise 17.4. Use the graph in the picture to answer the following questions.

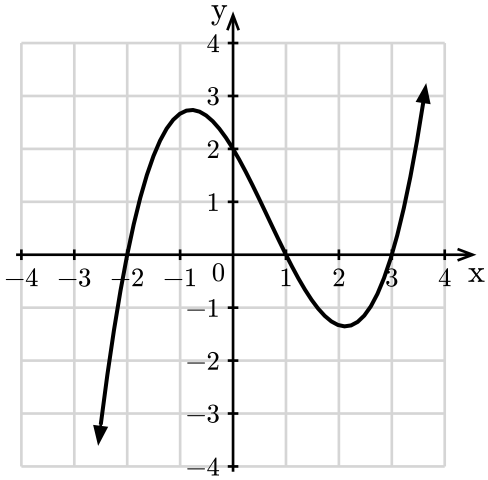
1) Determine whether the graph is a function and explain your answer.
2) Find the domain of the graph (write the domain in interval notation).
3) Find the range of the graph (write the range in interval notation).
4) Find the interval where the graph is above the x-axis.
5) Find all points where the graph reaches a maximum or a minimum.
6) Find all values of x such that the point $(x, 1)$ is on the graph.



Answer: 1) Yes 2) $(-\infty, 3]$ 3) $[-\infty, 3]$ 4) $(-2, 2)$ 5) $(1, 3)$ 6) -1

Exercise 17.5. Use the graph of the function f in the picture to answer the following questions.

1) Find the coordinates of the y-intercept.
2) Find the value $\dfrac{f(3)-f(0)}{3}$.
3) Find the values x such that $f(x)=0$.
4) Find the solution to the inequality $f(x)>0$. Write in interval notation.



Answer: 1) $(0,2)$ 2) $-\frac{2}{3}$ 3) $-2,1,3$ 4) $(-2,1)\cup(3,+\infty)$

Topic 18 Linear Functions

Learning Goals

- I can find the slope and identify the type of slope (negative, positive, zero, undefined) of a line passing through two points.
- I can find the slope and intercepts of a line from its equation or graph.
- I can write the equation of a linear function or a line (including vertical line) in the slope-intercept, the point-slope form, or the general form.
- I can find equations of parallel or perpendicular lines.
- I can construct a linear function model a real-world scenario and interpret values and results in context.

18.1 Think about It

Cost, Revenue and Profit

A company has fixed costs of $\$10,000$ for equipment and variable costs of $\$ 15$ for each unit of output. The sale price for each unit is $\$25$. What is total cost, total revenue and total profit at varying levels of output?

18.2 The Slope of a Line

A **line** is a straight path that extends infinitely in both directions. It is determined by two distinct points or by a point and a slope. The slope of a line is a measure of its steepness, the rise over the run. In terms of coordinates, the **slope** m **of a line** is determined by any two distinct points (x_1, y_1) and (x_2, y_2) on the line and defined by the formula

$$m = \frac{\overset{\text{one point}}{y_2} - y_1}{x_2 - \underset{\text{another point}}{x_1}}$$

Example 18.1. Find the slope of the line passing through the points $(2, 3)$ and $(4, 7)$.

Solution. The slope is given by

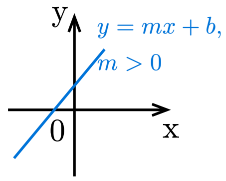
$$m = \frac{7 - \underline{\hspace{1.5cm}}}{4 - \underline{\hspace{1.5cm}}} = \frac{4}{2} = 2.$$

Remark

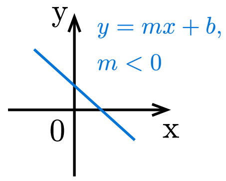
The slope of a line does not depend on the choice of points used in the calculation.

The slope can be positive, negative, zero, or undefined:

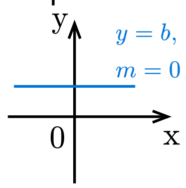
- A **positive slope** means the line is increasing, that is, rises as it moves from left to right.



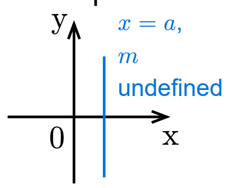
- A **negative slope** means the line is decreasing, that is, falls as it moves from left to right.



- A **zero slope** means the line is **horizontal**.



- An **undefined slope** means the line is **vertical**.



Example 18.2. Determine whether the line passing through the given points is increasing, decreasing, horizontal, or vertical:

1) $(\frac{7}{4}, -2), (-1, \frac{1}{4})$ 2) $(3, 5), (3, -1)$ 3) $(3, \frac{3}{2}), (2, -\frac{5}{4})$ 4) $(-3.14, 1), (2.73, 1)$

Solution.

1) The slope is

$$m = \frac{\frac{1}{4} - (-2)}{-1 - \frac{7}{4}} = \frac{\frac{9}{4}}{\underline{\qquad}} = -\frac{9}{11}$$

Because the slope is ________, the line is decreasing.

2) The slope is

$$m = \frac{-1 - 5}{3 - 3} = \frac{-6}{\underline{\qquad}}$$

Because the slope is ________, the line is vertical.

3) The slope is

$$m = \frac{-\frac{5}{4} - \frac{3}{2}}{2 - 3} = \frac{-\frac{5}{4} - \frac{6}{4}}{-1} = \frac{\underline{\qquad}}{-1} = \frac{11}{4}$$

Because the slope is positive, the line is ________.

4) The slope is

$$m = \frac{1 - 1}{2.73 - (-3.14)} = \frac{\underline{\qquad}}{5.87} = 0$$

Because the slope is ________, the line is horizontal.

18.3 Equations of a Line

Given a slope m and a point (x_0, y_0) on the line, a point (x, y) is on the line if and only if

$$m = \frac{y - y_0}{x - x_0} \quad \text{or equivalently,} \quad y - y_0 = m(x - x_0).$$

Point-Slope Form Equation

The equation

$$y - y_0 = m(x - x_0) \qquad \text{or} \qquad y = m(x - x_0) + y_0$$

is called the **point-slope form equation** of the line.

Example 18.3. Find the point-slope form equation of the line with slope 3 that passes through the point $(2, 5)$.

Solution. The point-slope form equation is given by

$$y - \underline{\quad\quad} = \underline{\quad\quad}(x - \underline{\quad\quad}).$$

Slope-Intercept Form Equation

Given the y-intercept $(0, b)$ and the slope m, the point-slope equation simplifies to **the slope-intercept form equation**:

$$y = mx + b.$$

Example 18.4. Find the slope-intercept form equation of the line with slope -2 that passes through the point $(0, 3)$.

Solution. The slope-intercept form equation is given by

$$y = \underline{\quad\quad}\, x + \underline{\quad\quad}.$$

Example 18.5. Find the slope-intercept form equation of the line passing through the points $(1, -2)$ and $(-3, 5)$.

Solution.

1) Find the slope:

$$m = \frac{5 - \underline{\quad\quad}}{\underline{\quad\quad} - 1} = \frac{7}{-4} = -\frac{7}{4}$$

2) Use point-slope form with $(1, -2)$:

$$y - \underline{\quad\quad} = -\frac{7}{4}(x - \underline{\quad\quad}).$$

3) Simplify to slope-intercept form:

$$y = \underline{\quad\quad}\, x + \frac{7}{4} - 2 = -\frac{7}{4}x - \frac{1}{4}$$

General/Standard Form Equation

The equation of a line can also be expressed in the **general form** or **standard form**:

$$Ax + By + C = 0, \quad \text{or} \quad Ax + By = C$$

where A, B, and C are real numbers, and A, B cannot be both zero.

- If $B \neq 0$, the slope of the line is $-\dfrac{A}{B}$.
- If $B = 0$, the slope is undefined, the equation can be simplified into the form $x = a$, and defines a **vertical line**, where a is the x-coordinate of all points on the line.
- If $A = 0$, the slope is zero, the equation can be simplified into the form $y = b$, and defines a **horizontal line**, where b is the y-coordinates of all points on the line.

Example 18.6. Find the slope and y-intercept of the line given by the equation

$$3x + 4y = 12.$$

Solution.

1) Rewrite the equation in slope-intercept form:

$$4y = \underline{\qquad} + 12$$

$$y = -\frac{\underline{\quad}}{4}x + 3$$

2) The slope is $-\dfrac{3}{4}$ and the y-intercept is $(0, 3)$.

Example 18.7. Find an equation of the line that passes through the given points.

1) $(2, 3)$ and $(2, 7)$

2) $(1, -2)$ and $(3, -2)$

Solution.

1) The slope is

$$\frac{7-3}{2-\underline{\quad}} = \frac{\underline{\quad}}{0}$$

Since the slope is undefined, the line is a vertical line, and it can be defined by the equation $x = \underline{\quad}$.

2) The slope is

$$\frac{-2-\underline{\quad}}{3-1} = \frac{0}{\underline{\quad}} = 0$$

Since the slope is 0, the line is a horizontal line, and it can be defined by the equation $y = \underline{\quad}$.

18.4 Linear Function

A **linear function** f is a function whose graph is a line. An equation for f can be written in the slope-intercept form:

$$f(x) = mx + b, \quad \text{where } m \text{ is the slope and } b = f(0) \text{ is the } y\text{-intercept}$$

or in the point-slope form:

$$f(x) = m(x - x_0) + f(x_0), \quad \text{where } m \text{ is the slope and } (x_0, f(x_0)) \text{ is a point on the line.}$$

When is a Function Linear

A function f is linear if

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_3) - f(x_1)}{x_3 - x_1}$$

for any three real numbers x_1, x_2, and x_3.

Example 18.8. Find the slope-intercept form equation for the linear function f such that $f(2) = 5$ and $f(-1) = 2$.

Solution.

1) Find the slope m: $m = \dfrac{5-2}{2-(-1)} = \dfrac{\underline{\qquad}}{\underline{\qquad}} = 1.$

2) Use point-slope form with $(2, 5)$:

$$f(x) = 1(x - \underline{\qquad}) + 5.$$

3) Simplify: $f(x) = \underline{\qquad\qquad}$

18.5 Graphing Linear functions

To graph a linear function, we need find two points on its graph. We can do this by evaluating the function at two different x values or by using "the rise over the run" to find another point from a known point.

Example 18.9. Sketch the graph of the linear function $f(x) = -\dfrac{1}{2}x + 1$.

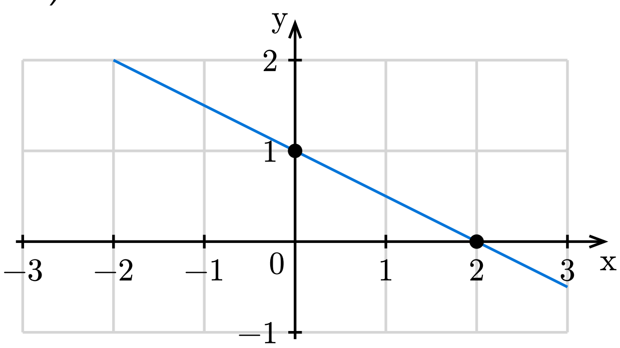
Solution. (Evaluating f at different values).

1) Choose, for example, $x = 0$ and $x = 2$.

2) Evaluate:

$$f(0) = -\frac{1}{2} \cdot 0 + 1 = \underline{\qquad},$$

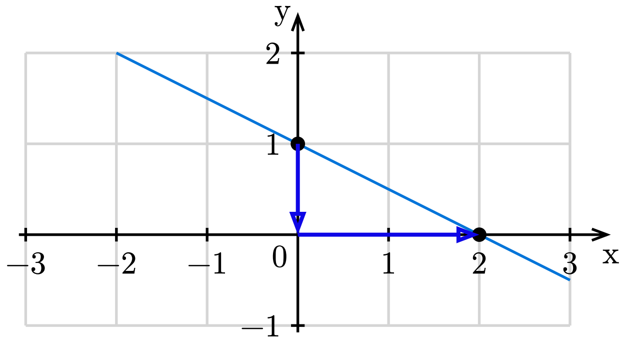
$$f(2) = -\frac{1}{2} \cdot \underline{\qquad} + 1 = \underline{\qquad}.$$



3) Plot the points $(0, 1)$ and $(2, 0)$ and draw a straight line through them.

Solution. (Using rise and run).

1) Plot y-intercept: $(0,1)$.
2) Use slope $-\frac{1}{2} = \frac{\text{rise}}{\text{run}}$ to find another point: take, for example,
$$\text{rise} = -1 \quad \text{and} \quad \text{run} = 2.$$
From $(0,1)$, going down 1 unit, and then right 2 units gives an point $(2,0)$.



3) Plot the points $(0,1)$ and $(2,0)$ and draw a straight line through them.

18.6 Perpendicular and Parallel Lines

Two lines are **parallel** if they never intersect. Two lines are **perpendicular** if they intersect at a right angle.

Algebraic Criteria of Parallel and Perpendicular Lines

Consider two different lines L defined by $Ax + By = C$.

- A parallel line $L_{\parallel}$ has an equation
$$Ax + By = D \quad \text{with} \quad D \neq C.$$
- A perpendicular line $L_{\perp}$ has an equation
$$Bx - Ay = D.$$

In particular, if $AB \neq 0$, then L and $L_{\parallel}$ has the same slope $m_{L_{\parallel}} = m_L$, and the product of the slopes is
$$m_L m_{L_{\perp}} = -1 \quad \text{or equivalently,} \quad m_{L_{\perp}} = -\frac{1}{m_L}.$$

Example 18.10. Find an equation of the line L that is parallel to $4x + 2y = 1$ and passes through $(-3,1)$.

Solution. (Find slopes first).

1) Solve original for y to get the slope m_L:
$$4x + 2y = 1$$
$$2y = \underline{\qquad} x + 1$$
$$y = -2x + \underline{\qquad}.$$
So the slope is $m_L = -2$.
2) The parallel line has same slope:
$$m_{L_{\parallel}} = m_L = -2.$$
3) Get the point-slope form equation:
$$y = -2(x+3) + 1.$$

Solution. (Use the general form).

1) The parallel line has an equation in the form
$$4x + 2y = D$$

2) Plug the point $(-3, 1)$ into the equation:
$$\begin{aligned} 4(\underline{\quad\quad}) + 2(\underline{\quad\quad}) &= D \\ -12 + 2 &= D \\ D &= -10 \end{aligned}$$

3) An equation of the parallel line is
$$4x + 2y = -10 \quad \text{or} \quad 2x + y = -5$$

Example 18.11. Find an equation of the line L that is perpendicular to $4x - 2y = 1$ and passes through $(-2, 3)$.

Solution. (Find the slope first).

1) Solve original for y to get the slope m_L:
$$\begin{aligned} 4x - 2y &= 1 \\ -2y &= \underline{\quad\quad} x + 1 \\ y &= 2x + \underline{\quad\quad} \end{aligned}$$
So $m_L = 2$.

2) The slope of the perpendicular line is
$$m_{L_\perp} = -\frac{1}{m_L} = \underline{\quad\quad}$$

3) Get the point-slope form equation:
$$y = -\frac{1}{2}(x + 2) + 3$$

Solution. (Use the general form).

1) The perpendicular line has an equation in the form
$$2x + 4y = D$$

2) Plug the point $(-2, 3)$ into the equation:
$$\begin{aligned} 2(\underline{\quad\quad}) + 4(\underline{\quad\quad}) &= D \\ -4 + 12 &= D \\ D &= 8 \end{aligned}$$

3) An equation of the perpendicular line is
$$2x + 4y = 8 \quad \text{or} \quad x + 2y = 4$$

Exercises

Exercise 18.1. Find the slope of the line passing through:

1) $(3, 5)$ and $(-1, 1)$

2) $(-2, 4)$ and $(5, -2)$

Answer: 1) 1 2) $-\frac{6}{7}$

Exercise 18.2. Find the point-slope form of the line with slope 5 that passes through $(-2, 1)$.

Answer: $y = 5(x + 2) + 1$

Exercise 18.3. Find the point-slope form of the line through $(3, -2)$ and $(1, 4)$.

Answer: $y = -3(x - 1) + 4$

Exercise 18.4. Find the slope-intercept form of the line through $(6, 3)$ and $(2, 5)$.

Answer: $y = -\frac{1}{2}x + 6$

Exercise 18.5. Let f and g be linear functions such that $f(1) = 4$, $f(3) = 10$, $g(0) = 2$, and $g(2) = 8$. Do $f(x)$ and $h(x)$ define the same function? Explain.

Answer: Yes.

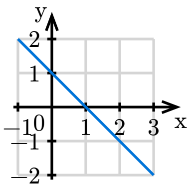
Exercise 18.6. If f is a linear function, and $(5, -1)$ and $(2, 5)$ lie on the graph of f, find $f(-3)$.

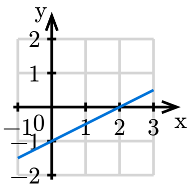
Answer: $f(-3) = 15$

Exercise 18.7. Graph the functions:

1) $f(x) = -x + 1$

2) $f(x) = \dfrac{1}{2}x - 1$

Answer: 1) 

2) 

Exercise 18.8. A rental company charges \$15 base fee and \$x per day. The cost for 10 days is \$20.

1) Write the cost y as a function of the days x.

2) What's the cost for the summer (June, July, August)?

Answer: 1) $y = 15 + 2x$ 2) $\$199$

Exercise 18.9. Find equations for two lines through $(-1, 2)$:

1) Vertical line

2) Horizontal line

Answer: 1) $x = -1$ 2) $y = 2$

Exercise 18.10. Line L: $2x - 5y = -3$. Find $m_{\parallel}$ and $m_{\perp}$.

Answer: $m_{\parallel} = \frac{2}{5}$, $m_{\perp} = -\frac{5}{2}$

Exercise 18.11. Line L_1: $3y + 5x = 7$. Line L_2 passes through $(-1, -3)$ and $(4, -8)$. Are they parallel, perpendicular, or neither?

Answer: They are neither.

Exercise 18.12. Find point-slope and slope-intercept forms of the line parallel to $3x - y = 4$ and through $(2, -3)$.

Answer: Point-slope form: $y = 3(x - 2) - 3$ Slope-intercept form: $y = 3x - 9$

Exercise 18.13. Find slope-intercept form of line perpendicular to $4y - 2x + 3 = 0$ and through $(2, -5)$.

Answer: $y = -2x - 1$

Exercise 18.14. Consider the lines

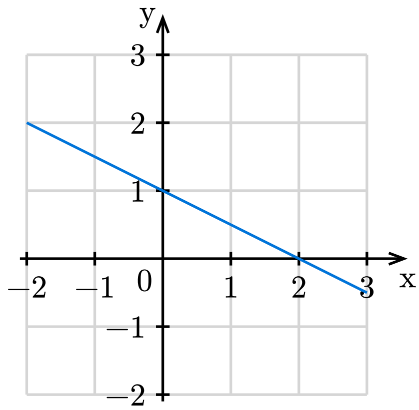
$$L_1 : Ax + By = 3, \qquad L_2 : Ax + By = 2, \qquad L_3 : Bx - Ay = 1.$$

Determine whether L_1, L_2, L_3 are parallel, perpendicular, or neither

Answer: $L_1 \parallel L_2$, $L_1 \perp L_3$, $L_2 \perp L_3$

Exercise 18.15. Use the graph of line L:

1) Find the slope-intercept form equation for L
2) Find the slope-intercept form equation of the perpendicular line $L_\perp$ through $(1, 1)$
3) Find the slope-intercept form equation of the parallel line $L_\parallel$ through $(-2, -1)$



Answer: 1) $y = -\frac{1}{2}x + 1$ 2) $y = 2x - 1$ 3) $y = -\frac{1}{2}x - 2$

Exercise 18.16. Do the points $(-3, 1)$, $(-2, 6)$, $(3, 5)$, and $(2, 0)$ form a square? Explain.

Answer: Yes

Exercise 18.17. A tutoring center has fixed costs of \$1000 and charges \$85 per lesson, with \$35 variable cost. After many lessons, the tutoring center will start making a profit?

Answer: $n = 20$

Exercise 18.18. A school had population 1011 in 2014 and 1281 in 2024. Assume linear growth.

1) Find population $P(t)$ for t years after 2014.
2) Predict population in 2034.

Answer: 1) $P(t) = \dfrac{27}{10}t + 1011$ 2) 1308

Topic 19 Quadratic Functions

Learning Goals

- I can find the vertex, axis of symmetry, intercepts, extremum of a quadratic function using its equation or graph.
- I can convert an equation of a quadratic function from standard form to vertex form.
- I can graph a quadratic function using its axis of symmetry, the vertex and intercepts.
- I can write an equation for a quadratic function based its graph.
- I can model real-world scenarios using quadratic functions and interpret values and results in context.

19.1 Think about It

Maximize the Revenue

When price increases, demand decreases and vice versa. A retail store found that the price p as a function of the demand x for a certain product is $p(x) = 100 - \frac{1}{2}x$. The revenue R of selling x units is $R = x \cdot p(x) = x\left(100 - \frac{1}{2}x\right)$. To maximize the revenue, what should be the price?

19.2 Characteristics of Quadratic Functions

A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

where a, b and c are real numbers and $a \neq 0$. This form is called the **standard form** (or **general form**) of a quadratic function.

From the definition, the **domain** of a quadratic function is the set of all real numbers $\mathbb{R}$.

The graph of a quadratic function is a **parabola**.

The Vertex Form of a Quadratic Function

By completing the square, a quadratic function $f(x) = ax^2 + bx + c$ can always be written in the **vertex form**:

$$f(x) = a(x - h)^2 + k,$$

where

$$h = -\frac{b}{2a}, \quad \text{and} \quad k = f(h) = f\left(-\frac{b}{2a}\right) = \frac{4ac - b^2}{4a}.$$

The point (h, k) is called the **vertex** of the quadratic function.

The Factored Form and Zeros of a Quadratic Function

By expressing the vertex form as difference of squares, a quadratic function f can be written in the **factored form**:

$$f(x) = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right).$$

The numbers $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$ are called the **zeros** or **roots** of f.

Intercepts, Symmetry, Extremum, and the Range of a Quadratic Function

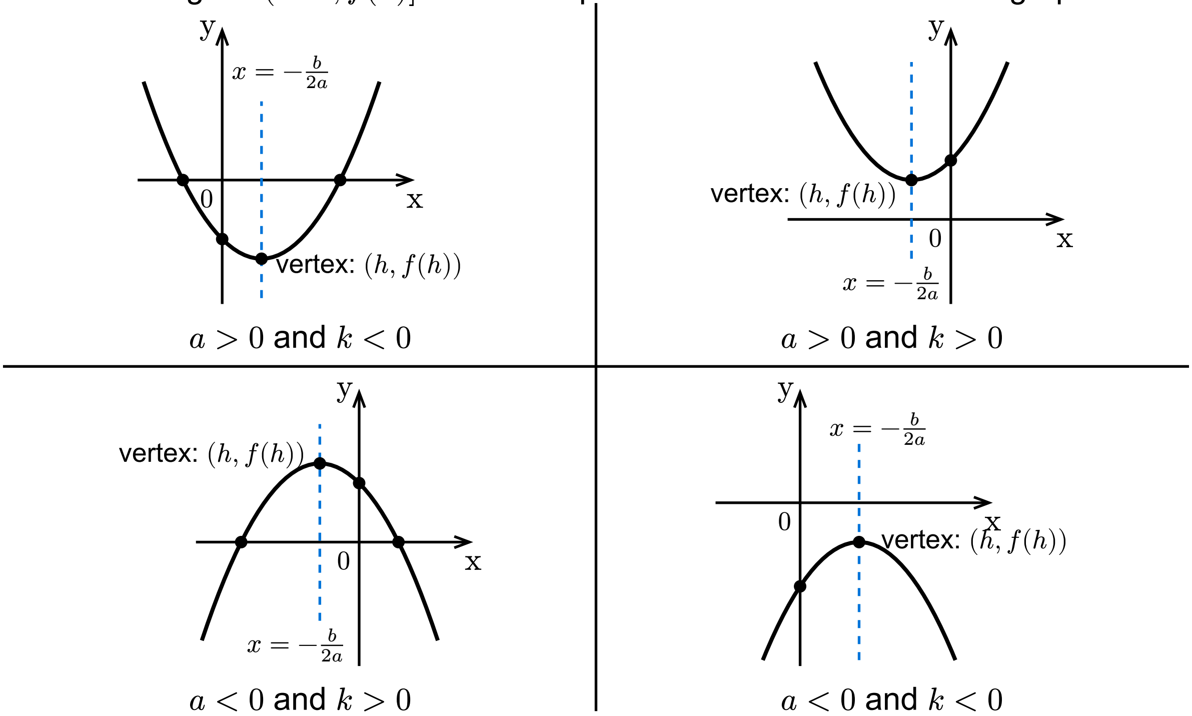
1) A quadratic function f always has a $\boldsymbol{y}$**-intercept** at $(0, f(0)) = (0, c)$. It has an $\boldsymbol{x}$**-intercept** if and only if $b^2 - 4ac = -4ak \geq 0$.

2) Since $f(t + h) = f(-t + h)$ for any t, the function is symmetric about the vertical line
$$x = h = -\frac{b}{2a},$$
called the **axis of symmetry**. Geometrically, the vertex is the intersection of this axis and the parabola.

3) Because $(x - h)^2$ grows as $|x - h|$ increases, $f(x)$ becomes larger or smaller depending on the sign of a:

   - If $\boldsymbol{a > 0}$, f opens **upward**. It decreases on $(-\infty, h]$ and increases on $[h, \infty)$. The vertex $(h, f(h))$ is the lowest point, so $f(h)$ is the **(absolute) minimum**. The range is $[f(h), \infty)$. See examples in the first columns of graphs.

   - If $\boldsymbol{a < 0}$, f opens **downward**. It increases on $(-\infty, h]$ and decreases on $[h, \infty)$. The vertex $(h, f(h))$ is the highest point, so $f(h)$ is the **(absolute) maximum**. The range is $(-\infty, f(h)]$. See examples in the second columns of graphs.



$a > 0$ and $k < 0$ | $a > 0$ and $k > 0$

$a < 0$ and $k > 0$ | $a < 0$ and $k < 0$

Example 19.1. Find the axis of symmetry and the vertex of the quadratic function $f(x) = x^2 - 4x + 5$.

Solution. Since the function is in standard form and the coefficients are $a = \underline{\quad\quad}$, $b = \underline{\quad\quad}$ and $c = \underline{\quad\quad}$, the axis of symmetry is

$$x = -\frac{b}{2a} = -\frac{\underline{\quad\quad}}{2 \cdot 1} = 2$$

and the vertex is

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = (2, f(\underline{\quad\quad})) = \big(2, (\underline{\quad\quad})^2 - 4(\underline{\quad\quad}) + 5\big) = (2, \underline{\quad\quad})$$

Example 19.2. Find the axis of symmetry and the vertex of the quadratic function $f(x) = 2(x-1)^2 - 1$.

Solution. Comparing the vertex form equation of a quadratic function with the given function, we find that

$$h = \underline{\quad\quad} \qquad \text{and} \qquad k = \underline{\quad\quad}$$

Therefore, the axis of symmetry is

$$x = h = \underline{\quad\quad}$$

and the vertex is

$$(h, k) = (\underline{\quad\quad}, \underline{\quad\quad})$$

Example 19.3. Sketch the graph of the quadratic function $f(x) = x^2 - 2x + 2$.

Solution. Plot the line of symmetry, vertex, intercepts, and then sketch the graph.

1) The line of symmetry is

$$x = -\frac{b}{2a} = -\frac{\underline{\quad\quad}}{2 \cdot 1} = 1$$

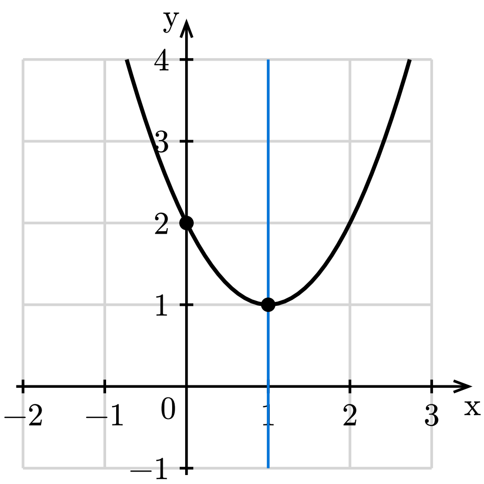
2) The vertex is

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = (1, f(\underline{\quad\quad})) = (1, 2)$$

4) Since $k = f(1) > 0$ and $a = 1 > 0$, the parabola is above the x-axis and there is no x-intercept.

5) The y-intercept is
$(0, f(0)) = (0, c) = (0, \underline{\quad\quad})$.

6) Sketch the left branch using the vertex and the y-intercept, then reflect it about the axis of symmetry to get the right branch.



Example 19.4. Determine whether the function $f(x) = 2x^2 - 4x - 6$ has a maximum or minimum and find its value.

Solution. Since $a = 2 > 0$ and the function is a quadratic function, the graph opens upward and achieves a minimum value

$$f\left(-\frac{b}{2a}\right) = f\left(-\frac{\_\_\_\_}{2 \cdot 2}\right) = f(1) = 2(1)^2 - 4(1) - 6 = \_\_\_\_$$

Example 19.5. Consider the function $f(x) = -x^2 + 3x + 6$. Find values of x such that $f(x) = 2$.

Solution. Substituting $f(x)$ by $2x^2 - 4x - 6$ in the equation $f(x) = 2$ gives us the equation

$$-x^2 + 3x + 6 = \_\_\_\_$$

Solving this equation gives us

$$\begin{aligned} -x^2 + 3x + 6 &= 2 \\ -x^2 + 3x + \_\_\_\_ &= 0 \\ x^2 - 3x - 4 &= 0 \\ (x + \_\_\_\_)(x - 4) &= 0 \\ x + 1 = 0 \quad &\text{or} \quad x - 4 = 0 \\ x = \_\_\_\_ \quad &\text{or} \quad x = 4 \end{aligned}$$

So the values of x such that $f(x) = 2$ are ____ and ____.

Example 19.6. A quadratic function f whose vertex is $(1, 2)$ has a y-intercept $(0, -3)$. Find an equation that defines the function.

Solution. To find the equation of the quadratic function, we can use the vertex form of a quadratic function, which is $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex. Here,

$$h = \_\_\_\_ \quad \text{and} \quad k = \_\_\_\_,$$

so we have an equation for f:

$$f(x) = a(x - \_\_\_\_)^2 + 2$$

Since $f(0) = -3$, evaluating f at $x = 0$ using the above equation gives us

$$\begin{aligned} a(0 - \_\_\_\_)^2 + 2 &= -3 \\ a(\_\_\_\_) + 2 &= -3 \\ a &= \_\_\_\_ \end{aligned}$$

So an equation of f is

$$f(x) = -5(x - 1)^2 + 2$$

Example 19.7. A ball is thrown upward with an initial velocity of v_0 feet per second from a platform 40 feet above the ground. Its height (in feet) after t seconds is modeled by the equation $h(t) = -16t^2 + v_0 t + 40$. The ball hits the ground after 5 seconds.

Find the maximum height the ball reaches and the time when it occurs.

Solution. Since the ball hits the ground after 5 seconds, we have $h(5) = 0$. Substituting $t = 5$ into the equation gives us

$$\begin{aligned} -16(5)^2 + v_0(5) + 40 &= 0 \\ 5v_0 &= \underline{\qquad} \\ v_0 &= 72 \text{ ft/sec} \end{aligned}$$

The maximum height occurs at

$$t = -\frac{v_0}{32} = -\frac{\underline{\qquad}}{32} = \frac{9}{4},$$

which is

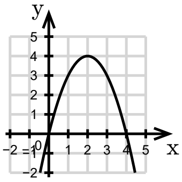
$$h(\underline{\qquad}) = -16(\underline{\qquad})^2 + 72\ \underline{\qquad} + 40 = 121 \text{ ft}$$

Exercises

Exercise 19.1. Consider the quadratic function $f(x) = -(x-2)^2 + 4$.

1) Sketch the graph of f.

2) Using the graph to find the interval where $f(x) \geq 0$.

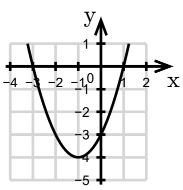
Answer: 1) 

2) [0, 4]

Exercise 19.2. Consider the function $f(x) = x^2 + 2x - 3$.

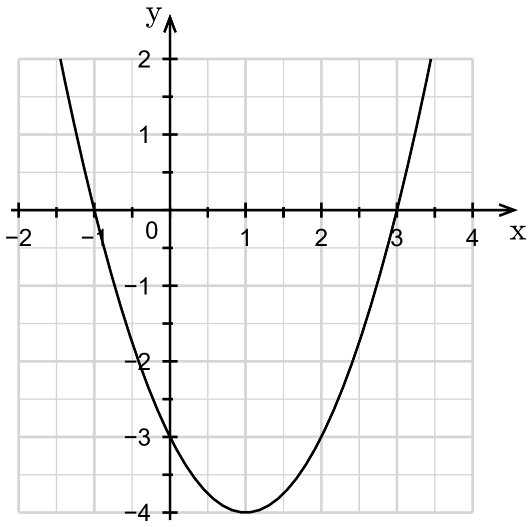
1) Sketch the graph of f using the characteristics of f.

2) Find the interval where $f(x) > 0$.

Answer: 1) 

2) $(-\infty, -3) \cup (1, \infty)$

Exercise 19.3. Consider the parabola in the graph.



Find the following information of the function f:

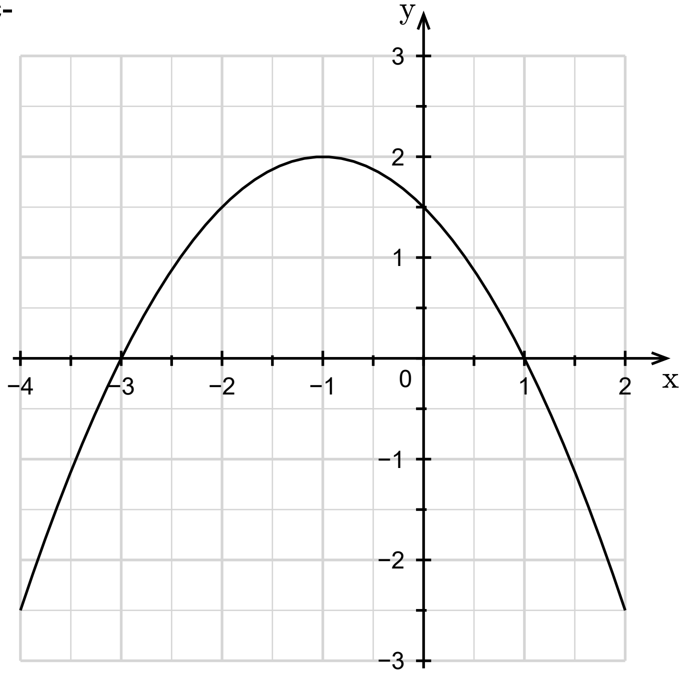
1) x-intercepts
2) y-intercept
3) the vertex
4) values where $f(x) = -3$
5) an equation of the function

Answer: 1) $(-1, 0)$ and $(3, 0)$ 2) $(0, -3)$ 3) $(1, -4)$ 4) $(2, -3)$ and $(0, -3)$ 5) $f(x) = (x - 1)^2 - 4$

Exercise 19.4. Consider the graph of f shown.

Find the following information of the function f:

1) x-intercepts
2) y-intercept
3) vertex
4) domain
5) range
6) values where $f(x) = \frac{3}{2}$
7) interval where f is positive
8) interval where f is decreasing
9) an equation of the function



Answer: 1) $(-3, 0)$ and $(1, 0)$ 2) $(0, 1.5)$ 3) $(-1, 2)$ 4) $(-\infty, \infty)$ 5) $(-\infty, 2]$ 6) -2 and 0 7) $(-3, 1)$ 8) $(-\infty, -1]$ 9) $f(x) = -\frac{1}{2}(x+1)^2 + 2$

Exercise 19.5.

Let $g(x) = x^2 - 3x - 4$. Find:

1) all intercepts
2) the vertex
3) equation of the axis of symmetry
4) domain and range
5) additional point
6) max/min value
7) where the max/min is reached
8) interval where $g(x) < 0$
9) interval where $g(x) > 0$
10) interval where g is increasing

Answer:

1) $(-1, 0)$ and $(4, 0)$; $(0, -4)$
2) $\left(\frac{3}{2}, -\frac{25}{4}\right)$
3) $x = \frac{3}{2}$
4) domain: $(-\infty, \infty)$ range: $\left[-\frac{25}{4}, \infty\right)$
5) $(2, -6)$
6) No Max. Min: $-\frac{25}{4}$
7) at $x = \frac{3}{2}$
8) $(-1, 4)$
9) $(-\infty, -1) \cup (4, \infty)$
10) $\left[\frac{3}{2}, \infty\right)$

Exercise 19.6. A store owner charges x dollars per cell phone case and sells $d(x) = 40 - x$ each week. Revenue is $R(x) = x(40 - x)$. Find the price that maximizes revenue and the amount of maximum revenue.

Answer: For \$$20$ per case, the revenue is maximized with the value \$$400$.

Exercise 19.7. A ball is thrown upward from a rooftop. Its height (in foot) is $h(t) = -16t^2 + 48t + 160$.

1) When is max height reached? What is it?

2) When does it hit the ground?

3) Height after 2 seconds?

4) When is it 96 ft high?

Answer: 1) Max: 196 ft at $t = 1.5$. 2) After 5 seconds. 3) 192 ft. 4) After 4 seconds.

Exercise 19.8. A ball is thrown with initial velocity v_0. Height: $h(t) = -16t^2 + v_0 t$. It hits ground after 4 seconds.

Find max height and when it's reached.

Answer: Max height is 64 ft at $t = 2$ seconds.

Exercise 19.9. A toy has weekly demand $300 - x$ if price is $x. Fixed cost is $40,000.

1) Find an equation for the revenue $R(x)$.

2) Find price range where $R(x) \geq 0$.

Answer: 1) $R(x) = 30x - x^2$ 2) $[0, 30]$

Topic 20 Rational Functions

Learning Goals

- I can find the domain of a rational function.
- I can solve word problems involving rational functions and interpret values and results in context.

20.1 Thank about It

The Law of Lever

"Give me a fulcrum and a place to stand, I will move the world."

In volume I of his book **On the Equilibrium of Planes**, Archimedes proved that magnitudes are in equilibrium at distances reciprocally proportional to their weights. See the video Law of the Lever for an animated explanation.

Suppose there is an infinite long lever with a load of 100 newtons placed 1 meter away from the fulcrum, the pivoting point.

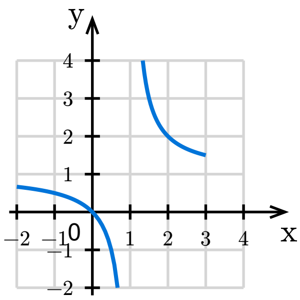
- Can you find the force needed to balance the load in terms of the distance away from the fulcrum?
- How much force will be needed if it is placed 5 meters away from the fulcrum?

20.2 The Domain of a Rational Function

A **rational function** f is defined by an equation $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

Since a fraction is undefined if the denominator is zero, the **domain of a rational function** $f(x) = \dfrac{p(x)}{q(x)}$ consists of all real numbers except the values such that $q(x) = 0$, in set-builder notation, the domain of f is $\{x \mid x \text{ is real and } q(x) \neq 0\}$.

Example: The graph of $f(x) = \dfrac{x}{x-1}$

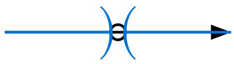


Example 20.1. Find the domain of the function $f(x) = \dfrac{1}{x-1}$.

Solution. Set the denominator to be zero solve for x shows that

$$\begin{aligned} x - 1 &= 0 \\ x &= \underline{\qquad} \end{aligned}$$

Then the domain is $\{x \texttt{ mid } x \neq 1\}$. On graph,



In interval notation, the domain is

$$(-\infty, 1) \cup (1, \infty).$$

Example 20.2. Suppose that the cost in dollars to produce x toy cars is given by $C(x) = 5,000 + 5x$.

1) What is the average cost $A(X)$ (in dollars) of produce x toy cars.

2) How many toy cars would the factory have to produce in order for the average production cost to be \$$10$ per item?

Solution. The average cost $A(x)$ is the total cost $C(x)$ divided by the number x of toy cars produced, that is,

$$A(x) = \frac{C(x)}{x} = \frac{\underline{\qquad\qquad}}{x}$$

The number x of toy cars should be produced satisfies the equation

$$\frac{5000 + 5x}{x} = \underline{\qquad}$$

Solving the equation leads to

$$\begin{aligned} \frac{5000 + 5x}{x} &= 10 \\ 5000 + 5x &= \underline{\qquad} \\ 5x &= \underline{\qquad} \\ x &= 1000 \end{aligned}$$

The average cost would be \$$10$ per toy car if 1000 cars would be produced.

Exercises

Exercise 20.1.

Find the domain of each function. Write in interval notation.

1) $f(x) = \dfrac{x^2}{x-2}$

2) $f(x) = \dfrac{x}{x^2-1}$

Answer: 1) $(-\infty, 2) \cup (2, \infty)$ 2) $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Exercise 20.2. A factory would like to start a new line to produce a popular wood puzzle. A new line would have a fixed post of 10000 per month together with $\$10$ per item cost.

1) What is the average cost $A(X)$ (in dollars) of produce x toy cars.

2) How many toy cars would the factory have to produce so that the average production cost would be at most $\$15$ per item?

Answer: 1) $A(x) = \dfrac{10000 + 10x}{x}$ 2) 2000

Topic 21 Radical Functions

Learning Goals

- I can find the domain of a radical function.
- I can solve word problems involving radical functions and interpret values and results in context.

21.1 Thank about It

Speed of a Tsunami

A tsunami is generally referred to as a series of waves on the ocean caused by earthquakes or other events that cause sudden displacements of large volumes of water. In ideal situations, the velocity v of a wave at a depth d meters is approximately

$$v = \sqrt{9.8d}.$$

The wave slows down as it nears the coast but becomes higher.

Suppose a tsunami was caused by an earthquake somewhere $10,000$ meters away from the coast of California. The depth of the water where the tsunami was generated is $5,000$ meters.

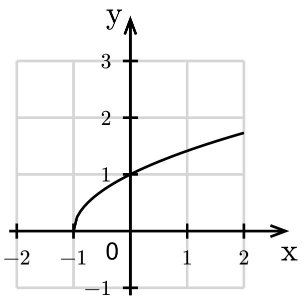
- What's the initial speed of the tsunami?
- What's the speed of the tsunami where the water depth is $2,000$ meters?
- Suppose the speed wouldn't decrease. How long would it take the tsunami to reach the coast?

21.2 The Domain of a Radical Function

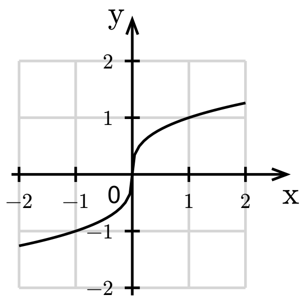
A **radical function** f is defined by an equation $f(x) = \sqrt[n]{r(x)}$, where $r(x)$ is an algebraic expression. The expression $r(x)$ is called the **radicand** of the function. The **index** of the radical is n. The graphs below show two radical functions with different indices.

- When n is odd, $r(x)$ can be any real number, so the domain of f is the same as the **domain of** r.
- When n is *even*, $r(x)$ must be *nonnegative*, so the domain of f is the subset
$$\{x \mid r(x) \geq 0, x \text{ is real}\}$$
of the domain of r.

$f(x) = \sqrt{x+1}$



$f(x) = \sqrt[3]{x}$



Example 21.1. Find the domain of the function $f(x) = \sqrt{x+1}$.

Solution. Since the index is 2 which is even, the function has real outputs only if the radicand

$$x + 1 \geq 0$$

Solving the inequality yields

$$x \geq \underline{\quad\quad}$$

In interval notation, the domain is:

$$[\underline{\quad\quad}, \infty).$$

Example 21.2. For a pendulum clock, the period T of the pendulum is approximately modeled by the following function of the length L of the pendulum:

$$T = 2\sqrt{L}$$

where L and T are measured in meters and seconds respectively.

1) If the length of the pendulum is 4 meters, what is the period?

2) If the period of a pendulum clock is 1 second, how long should the pendulum be?

Solution.

1) When the length is 4 meters, that is, $L = 4$, the period is

$$T = 2 \cdot \sqrt{\underline{\quad\quad}} = 4 \quad \text{seconds}$$

So, the period is 4 seconds if the length of 4 meters.

2) When the period is 1 second, that is, $T = 1$, the length L satisfies the equation $2\sqrt{L} = 1$. Solving it yields

$$\begin{aligned} 2\sqrt{L} &= 1 \\ \sqrt{L} &= \underline{\quad\quad} \\ L &= \left(\frac{1}{2}\right)^2 \\ L &= \underline{\quad\quad} \end{aligned}$$

So, the pendulum should be $\frac{1}{4}$ meters long for a 1-second period.

Exercises

Exercise 21.1. Find the domain of each function. Write in interval notation.

1) $f(x) = \sqrt{1-x}$ 2) $f(x) = \sqrt{x^2+1}$ 3) $f(x) = -\sqrt{\dfrac{1}{x-5}}$

Answer: 1) $(-\infty, 1]$ 2) $(-\infty, \infty)$ 3) $(5, \infty)$

Topic 22 Exponential Functions

Learning Goals

- [ ] I can determine if a function is exponential or not.
- [ ] I can find the domain and range of an exponential function.
- [ ] I can determine whether an exponential function is growth or decay.
- [ ] I can evaluate exponential functions in the form $y = ab^x$ and relate the meaning to real-world contexts.
- [ ] I can set up exponential functions for compounding investments (with finite periods or continuously) and determine outcomes.

22.1 Think about It

Half-life

Half-life is the time required for a quantity to reduce to half of its initial value.

A certain pesticide is used against insects. The half-life of the pesticide is about 12 days. After a month, how much would be left if the initial amount of the pesticide is 10 g? Can you write a function for the remaining quantity of the pesticide after t days?[1]

22.2 Definition and Graphs of Exponential Functions

Let b be a positive number other than 1 (i.e., $b > 0$ and $b \neq 1$). The exponential function f of x with the base b is defined as

$$f(x) = b^x \quad \text{or} \quad y = b^x.$$

Example 22.1. Find the value of each function at $x = \frac{1}{2}$. If the value does not exist, explain why.

1) $f(x) = 2^x$ 2) $g(x) = -2^x$ 3) $h(x) = 2^{-x}$ 4) $k(x) = (-2)^x$.

Solution. To evaluate a function, plugging the given value into the expression and simplify.

1)

$$f\left(\frac{1}{2}\right) = 2^{\frac{1}{2}} = \sqrt{\underline{\qquad}}$$

2)

$$g\left(\frac{1}{2}\right) = -2^{\frac{1}{2}} = -\sqrt{\underline{\qquad}}$$

[1] More examples can be found at http://passyworldofmathematics.com/exponents-in-the-real-world/.

3)

$$h\left(\frac{1}{2}\right) = 2^{-\frac{1}{2}} = \frac{1}{2^{\underline{\quad}}} = \frac{1}{\sqrt{\underline{\quad\quad}}} = \frac{\sqrt{\underline{\quad\quad}}}{2}$$

4)

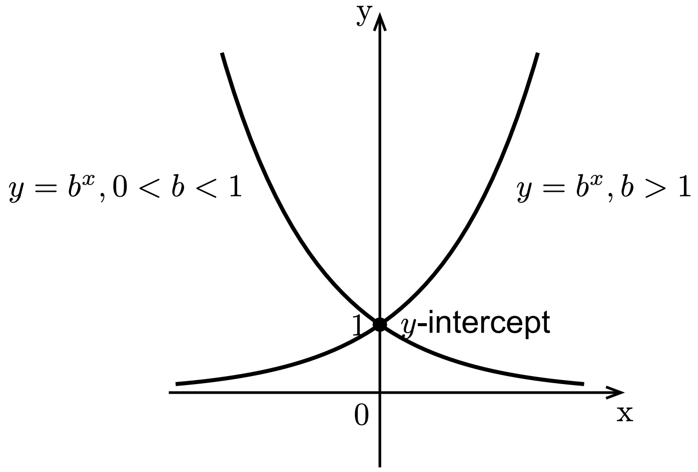
$$k\left(\frac{1}{2}\right) = (-2)^{\frac{1}{2}} = \sqrt{\underline{\quad\quad}}$$

Since the radicand is negative, $k\left(\frac{1}{2}\right)$ is not a real number, that is, undefined.

22.3 Properties of Exponential Functions

Exponential functions can be graphed by plotting points. The following figure shows graphs of two exponential functions. Other exponential functions have similar graphs.

Graphs of Exponential Functions



Properties of Exponential Functions

Consider any exponential function $f(x) = b^x$.

- The domain of f is $(-\infty, \infty)$.
- The range of f is $(0, \infty)$.
- f is a one-to-one function, that is, for any output, there is an unique input, or geometrically, any vertical line or any horizontal line crosses the graph at most once. That also equivalent to that the equation $b^x = c$ has at most one solution for any real number c.
- When x goes to infinity, the value $f(x)$ goes to 0. The horizontal line $y = 0$ is called a **horizontal asymptote** of f.
- The function is increasing if $b > 1$ and decreasing if $0 < b < 1$.

Exponential Functions with Initial Values

In general, an exponential function may be written in the form $f(x) = ab^x$, where $a = f(0)$ is called the **initial value** of f.

22.4 The Natural Number e

The **natural number** e is the number to which the quantity $\left(1 + \frac{1}{n}\right)^n$ approaches as n takes on increasingly large values. Approximately, the natural number is $e \approx 2.718281827$.

22.5 Compound Interests

In finance, compounding means earning interest on both the initial amount of money invested or saved, called **the principal**, and on any interest that has already been added to that principal. A **compounding period** is the length of time between when interest is calculated and added to the principal in an investment or loan.

The **annual interest rate** is the percentage of the principal that is paid as interest in one year.

Given the principal P, annual interest rate r, and the number of compounding periods per year n, the balance $A(t)$ is a function of time t in years and determined by the following formula:

$$A = P\left(1 + \frac{r}{n}\right)^{nt},$$

called the **compounded investment formula**.

In the formula, $\frac{r}{n}$ is the average **period interest rate** and nt is the **total number of compounding periods**.

When the compounding period is as small as possible, or equivalently, the number n of compounding period per year is as large as possible, the above compounded investment formula becomes:

$$A(t) = Pe^{rt},$$

called the **continuously compounded investment formula**.

Example 22.2. A sum of $\$10,000$ is invested at an annual rate of 8%, Find the balance, to the nearest hundredth dollar, in the account after 5 years if the interest is compounded

1) monthly 2) quarterly 3) semiannually 4) continuously

Solution. To calculate the balance, we use the compounded investment formula.

In this question, $P = \underline{\qquad}$, $r = \underline{\qquad}$, $t = \underline{\qquad}$, and n depends on the compounding period.

1) "Monthly" means $n =$ ____. Then

$$A(5) = 10000\left(1 + \frac{0.08}{12}\right)^{\_\_\_\cdot 12} \approx 14898.46.$$

2) "Quarterly" means $n = 4$. Then

$$A(5) = 10000\left(1 + \frac{0.08}{\_\_\_}\right)^{5\cdot \_\_\_} \approx 14859.47.$$

3) "Semiannually" means $n = 2$. Then

$$A(5) = 10000(1 + \_\_\_)^{5\cdot 2} \approx 14802.44.$$

4) For continuously compounded interest, we have

$$A(5) = 10000e^{0.08\cdot \_\_\_} \approx 14918.25.$$

Remark

In the compounded investment module, the $\frac{r}{n}$ is an approximation of the period interest rate. Indeed, if the period rate p satisfies the equation $(1+p)^n = 1+r$, or equivalently

$$p = \sqrt[n]{1+r} - 1.$$

Using the formula $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + ... + x^n$, one may approximately replace $1+r$ by $\left(1+\frac{r}{n}\right)$ and obtain the approximation $p \approx \frac{r}{n}$.

Example 22.3. The population of a country was about 0.78 billion in the year 2015, with an annual growth rate of about 0.4%. The predicted population is $P(t) = 0.78(1.004)^t$ billions after t years since 2015. To the nearest thousandth of a billion, what will the predicted population of the country be in 2030?

Solution. The population is approximately

$$P(15) = 0.78(1.004)^{\_\_\_} \approx 0.828 \text{ billions}.$$

Exercises

Exercise 22.1. The value of a car is depreciating according to the formula: $V = 25000(3.2)^{-0.05x}$, where x is the age of the car in years. Find the value of the car, to the nearest dollar, when it is five years old.

Answer: 18692

Exercise 22.2. A sum of $\$20,000$ is invested at an annual rate of 5.5%. Find the balance, to the nearest dollar, in the account after 5 years subject to

1) monthly compounding,
2) continuously compounding.

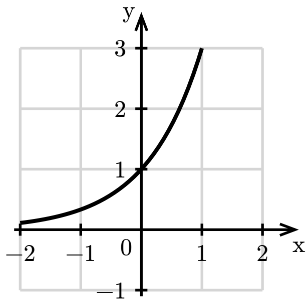
Answer: 1) 26314 2) 26331

Exercise 22.3. Sketch the graph of the function and find its range.

1) $f(x) = 3^x$

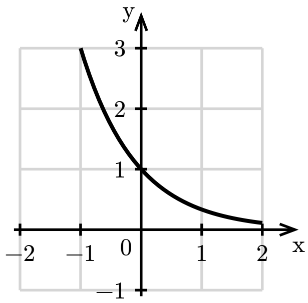
2) $f(x) = \left(\frac{1}{3}\right)^x$

Answer: 1)



Range: $(0, \infty)$

2)



Range: $(0, \infty)$

Exercise 22.4. Use the given function to compare the values of $f(-1.05)$, $f(0)$, and $f(2.4)$ and determine which value is the largest and which value is the smallest. Explain your answer.

1) $f(x) = \left(\frac{5}{2}\right)^x$

2) $f(x) = \left(\frac{2}{3}\right)^x$

Answer: 1) $f(-1.05) \approx 0.4$, $f(0) = 1$, and $f(2.4) \approx 6.25$. The largest value is $f(2.4)$ and the smallest value is $f(-1.05)$.

2) $f(-1.05) =\approx 1.5$, $f(0) = 1$, and $f(2.4) \approx 0.36$. The largest value is $f(-1.05)$ and the smallest value is $f(2.4)$.

Topic 23 Logarithmic Functions

Learning Goals

- [ ] I can convert between exponential and logarithmic forms.
- [ ] I can determine the value of a logarithmic function.
- [ ] I can find the domain of a logarithmic function.
- [ ] I can use basic properties of logarithms to evaluate logarithms.
- [ ] I can use properties of logarithms to expand a logarithm and write a logarithmic expression as a single logarithm.
- [ ] I can evaluate a logarithm using a calculator and the base change property.

23.1 Think about It

Estimate the Number of Digits

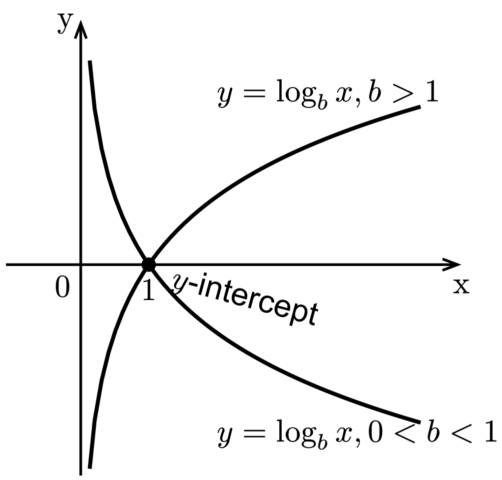
Can you estimate the number of digits in the integer part of the number $2^{15} \times \sqrt{2020} \div 2025$?

23.2 Definition and Graphs of Logarithmic Function

For $x > 0$, $b > 0$ and $b \neq 1$, there is a unique number y satisfying the equation $b^y = x$. We denote the unique number y by $\log_b x$, read as logarithm to the base b of x. In other words, the relation between exponentiation and logarithm is

$$y = \log_b x \quad \text{if and only if} \quad b^y = x.$$

The function $f(x) = \log_b x$ is called the logarithmic function f of x with the base b.



Graphs of logarithmic functions

23.3 Common Logarithms and Natural Logarithms

A logarithmic function $f(x)$ with base 10 is called the common logarithmic function and denoted by $f(x) = \log x$.

A logarithmic function $f(x)$ with base the natural number e is called the natural logarithmic function and denoted by $f(x) = \ln x$.

23.4 Basic Properties of Logarithms

When $b > 0$ and $b \neq 1$, and $x > 0$, we have

1) $b^{\log_b x} = x$.   2) $\log_b(b^x) = x$.   3) $\log_b b = 1$ and $\log_b 1 = 0$.

Example 23.1. Convert between exponential and logarithmic forms.

1) $\log x = \dfrac{1}{2}$ 2) $3^{2x-1} = 5$

Solution. When converting between exponential and logarithmic forms, in practice, we move the base from one side to the other side, then add or drop the $\log$ sign.

1) Move the base 10 to the right side and drop the $\log$ from the left:

$$x = \underline{\qquad}^{\frac{1}{2}}$$

1) Move the 3 to the right and add log to the right:

$$2x - 1 = \log_{\underline{\quad}} 5$$

Example 23.2. Evaluate the logarithms.

1) $\log_4 2$ 2) $10^{\log\left(\frac{1}{2}\right)}$ 3) $\log_5(e^0)$

Solution. The key is to rewrite the log and the power so that they have the same base.

1) $\log_4 2 = \log_4\left(4^{\underline{\quad}}\right) = \dfrac{1}{2}$.

2) $10^{\log\frac{1}{2}} = 10^{\log_{\underline{\quad}}\frac{1}{2}} = \dfrac{1}{2}$.

3) $\log_5(e^0) = \log_5 \underline{\qquad} = 0$.

Example 23.3. Find the domain of the function $f(x) = \ln(2 - 3x)$.

Solution. The function has a real output only if

$$2 - 3x \;\underline{\qquad}\; 0.$$

Solving the inequality, we get

$$-3x > \underline{\qquad}$$
$$x < \underline{\qquad}.$$

So the domain of the function is $(\underline{\qquad}, \frac{2}{3})$.

23.5 Properties of Logarithms

For $M > 0$, $N > 0$, $b > 0$ and $b \neq 1$, we have

1) **The product rule:** $\log_b(MN) = \log_b M + \log_b N$
2) **The quotient rule:** $\log_b\left(\dfrac{M}{N}\right) = \log_b M - \log_b N$.
3) **The power rule:** $\log_b(M^p) = p\log_b M$, where p is any real number.
4) **The change-of-base property:** $\log_b M = \dfrac{\log_a M}{\log_a b}$, where $a > 0$ and $a \neq 1$. In particular,

$$\log_b M = \frac{\log M}{\log b} \quad \text{and} \quad \log_b M = \frac{\ln M}{\ln b}.$$

Example 23.4. Expand and simplify the logarithm $\log_2\left(\dfrac{8\sqrt{y}}{x^3}\right)$.

Solution.

$$\begin{aligned}\log_2\left(\frac{8\sqrt{y}}{x^3}\right) &= \log_2(8\sqrt{y}) \ \underline{\quad\quad}\ \log_2(x^3)\\ &= \log_2 \underline{\quad\quad} + \log_2\left(y^{\frac{1}{2}}\right) - \underline{\quad\quad}\log_2 x\\ &= 3 + \underline{\quad\quad}\log_2 y - 3\log_2 x.\end{aligned}$$

Example 23.5. Write the expression $2\ln(x-1) - \ln(x^2+1)$ as a single logarithm.

Solution.

$$\begin{aligned}2\ln(x-1) - \ln(x^2+1) &= \ln((x-1)^{\underline{\quad}}) - \ln(x^2+1)\\ &= \ln\left(\frac{(x-1)^2}{\underline{\quad\quad}}\right).\end{aligned}$$

Example 23.6. Evaluate the logarithm $\log_3 4$ and round it to the nearest tenth.

Solution. On most scientific calculators, there are only the common logarithmic function $\boxed{\texttt{LOG}}$ and the natural logarithmic function $\boxed{\texttt{LN}}$. To evaluate a logarithm based on a general number, we use the change-of-base property. In this case, the value of $\log_3 4$ is

$$\log_3 4 = \frac{\log \underline{\quad\quad}}{\log 3} \approx 1.3.$$

Example 23.7. Simplify the logarithmic expression

$$\log_2\left(x^{\log 3}\right)\log_3 2.$$

Solution. When simplifying products of logarithms, we use the change-of-base property to rewrite the logarithms so that they have the same base. In this case, we have

$$\begin{aligned}\log_2\left(x^{\log 3}\right)\log_3 2 &= (\underline{\qquad}\log_2 x)\log_3 2\\ &= (\log 3)\left(\frac{\underline{\qquad}}{\log 2}\right)\left(\frac{\log 2}{\underline{\qquad}}\right)\\ &= \log x.\end{aligned}$$

Exercises

Exercise 23.1. Write each equation into equivalent exponential form.

1) $\log_3 7 = y$ 2) $3 = \log_b 64$ 3) $\log x = y$ 4) $\ln(x-1) = c$

Answer: 1) $7 = 3^y$ 2) $b^3 = 64$ 3) $10^y = x$ 4) $x - 1 = e^c$

Exercise 23.2. Write each equation into equivalent logarithmic form.

1) $7^x = 10$ 2) $b^5 = 2$ 3) $e^{2y-1} = x$ 4) $10^x = c^2 + 1$

Answer: 1) $x = \log_7 10$ 2) $5 = \log_b 2$ 3) $2y - 1 = \ln x$ 4) $x = \log(c^2 + 1)$

Exercise 23.3. Evaluate.

1) $\log_2 16$ 2) $\log_9 3$ 3) $\log 10$ 4) $\ln 1$

Answer: 1) 4 2) $\frac{1}{2}$ 3) 1 4) 0

Exercise 23.4. Evaluate.

1) $e^{\ln 2}$ 2) $\log 10^{\frac{1}{3}}$ 3) $\ln(\sqrt{e})$ 4) $\log_2\left(\frac{1}{2}\right)$

Answer: 1) 2 2) $\frac{1}{3}$ 3) $\frac{1}{2}$ 4) -1

Exercise 23.5. Find the domain of the function $f(x) = \log(x-5)$. Write in interval notation.

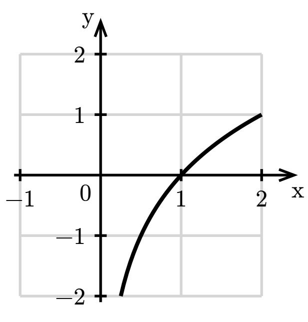
Answer: $(5, \infty)$

Exercise 23.6. Sketch the graph of each function and find its range.

1) $f(x) = \log_2 x$

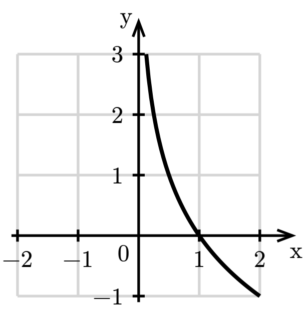
2) $f(x) = \log_{\frac{1}{2}} x$

Answer: 1)



Range: $(-\infty, \infty)$

2)



Range: $(-\infty, \infty)$

Exercise 23.7. Expand the logarithm and simplify. Assume all variables are positive.

1) $\log(100x)$ 2) $\ln\left(\dfrac{10}{e^2}\right)$ 3) $\log_b\left(\sqrt[3]{x}\right)$ 4) $\log_7\left(\dfrac{x^2\sqrt{y}}{z}\right)$

Answer: 1) $2+\log x$ 2) $\ln 10 - 2$ 3) $\dfrac{1}{3}\log_b x$ 4) $2\log_7 x + \dfrac{1}{2}\log_7 y - \log_7 z$

Exercise 23.8. Expand the logarithm and simplify. Assume all variables are positive.

1) $\log_b \sqrt{\dfrac{x^2 y}{5}}$ 2) $\ln\left(\sqrt[3]{(x^2+1)y^{-2}}\right)$ 3) $\log\left(x\sqrt{10x} - \sqrt{10x}\right)$

Answer: 1) $\log_b x + \dfrac{1}{2}\log_b y - \dfrac{1}{2}\log_b 5$ 2) $\dfrac{1}{3}\ln(x^2+1) - \dfrac{2}{3}\ln y$ 3) $\dfrac{1}{2} + \dfrac{1}{2}\log x + \log(x-1)$

Exercise 23.9. Write as a single logarithm.

1) $\dfrac{1}{3}\log x + \log y$ 2) $\dfrac{1}{2}\ln(x^2+1) - 2\ln x$ 3) $\dfrac{1}{3}\log_2 x - 3\log_2(x+1) + 1$

Answer: 1) $\log(y\sqrt[3]{x})$ 2) $\ln\left(\dfrac{\sqrt{x^2+1}}{x^2}\right)$ 3) $\log_2\left(\dfrac{2\sqrt[3]{x}}{(x+1)^3}\right)$

Exercise 23.10. Write as a single logarithm.

1) $2\log(2x+1) - \dfrac{1}{2}\log x$ 2) $3\ln x - 5\ln y + \dfrac{1}{2}\ln z$ 3) $3\log_3 x - 2\log_3(1-x) + \dfrac{1}{3}\log_3(x^2+1)$.

Answer: 1) $\log\left(\dfrac{(2x+1)^2}{\sqrt{x}}\right)$ 2) $\ln\left(\dfrac{x^3\sqrt{z}}{y^5}\right)$ 3) $\log_3\left(\dfrac{x^3\sqrt[3]{x^2+1}}{(1-x)^2}\right)$

Exercise 23.11. Evaluate the logarithm and round it to the nearest hundredth.

1) $\log_2 10$ 2) $\log_3 5$ 3) $\dfrac{1}{\log_5 2}$ 4) $\log_4 5 - \log_2 9$

Answer: 1) 3.32 2) 1.46 3) 0.43 4) -2.01

Exercise 23.12. Simplify the logarithmic expression

$$\frac{\log_3(x^2)\log_y\sqrt{3}}{\log x}.$$

Answer: $\dfrac{1}{\log y}$

Topic 24 Exponential Equations and Applications

Learning Goals

- I can solve a simple exponential equation (in linear form or with no constant terms).
- I can solve a quadratic-like equation involving exponential expressions.
- I can solve applied problems involving exponential equations or functions and interpret parameters and solutions in context.

24.1 Think about It

Newton's Law of Cooling

Suppose an object with an initial temperature T_0 is placed in an environment with surrounding temperature T_s. By Newton's Law of Cooling, after t minutes, the temperature of the object $T(t)$ is given by the exponential function

$$T(t) = T_s + (T_0 - T_s)e^{-rt},$$

where r is a positive constant characteristic of the system.

A cup of coffee is brewed with a temperature $195°F$ and placed in a room with the temperature $60°F$. The cooling constant for a cup of coffee is $r = 0.09\ \mathrm{min}^{-1}$.

1) After 30 minutes, what is the temperature of the coffee?

2) How long does it take for the coffee to cool down to the room temperature?

24.2 Solving Exponential Equations

Solving an exponential is similar to solving an absolute value equation. One strategy is the following:

1) isolate a exponential (where you may need to solve using substitution method),

2) use the relation between exponential functions and logarithmic functions to rewrite the equation into an equivalent equation without exponential expressions,

3) solve the resulting equation.

Example 24.1. Solve the equation $10^{2x-1} - 5 = 0$.

Solution.

1) Rewrite the equation in the form $b^u = c$:

$$10^{2x-1} = \underline{\qquad}$$

3) Solve the resulting equation:

2) Take logarithm of both sides and simplify:

$$2x - 1 = \underline{\qquad}$$

$$\begin{aligned} 2x - 1 &= \log 5 \\ 2x &= \underline{\qquad\quad} \\ x &= \frac{\underline{\qquad\quad}}{2} \end{aligned}$$

So, the solution of the equation is $x = \dfrac{\log 5 + 1}{2}$.

24.3 Applications

Example 24.2. A check of $\$5000$ was deposited in a savings account with an annual interest rate 6% which is compounded monthly. How many years will it take for the money to increase by 20%?

Solution. The balance $A(t)$ after t years is determined by the equation

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

In this question,

$$P = 5000, \quad r = 0.06, \quad n = \underline{\qquad}. \quad \text{and} \quad A(t) = 5000 \cdot (1 + \underline{\qquad}) = 6000.$$

Hence, the number of years t satisfies the following equation:

$$5000\left(1 + \frac{\underline{\qquad}}{12}\right)^{\underline{\qquad}\, t} = 6000$$

This is an exponential equation and can be solved by taking logarithms.

$$\begin{aligned} 5000\left(1 + \frac{0.06}{12}\right)^{12t} &= 6000 \\ \left(1 + \frac{0.06}{12}\right)^{12t} &= \underline{\qquad} \\ \underline{\qquad\quad} \cdot \log\left(1 + \frac{0.06}{12}\right) &= \log(1.2) \\ t &= \frac{\log(1.2)}{12\log(1 + \frac{0.06}{12})} \\ t &\approx \underline{\qquad} \end{aligned}$$

So it takes about 3 years for the savings to increase by 20%.

Solving using the One-to-one Property of Exponential Functions

When solving exponential and logarithmic equations, you may also use the one-to-one property if both sides are powers with the same base or logarithms with the same base.

Example 24.3. Solve the equation $2^{x^2-3} = 4^x$.

Solution. The equation can be rewritten as

$$2^{x^2-3} = 2^{\_\_\_ x}.$$

By the one-to-one property of exponential function,

$$x^2 - 3 = 2x.$$

Solving the equation implies

$$\begin{aligned} x^2 - \_\_\_\_\_\_ - 3 &= 0 \\ (x - \_\_\_\_)(x + \_\_\_\_) &= 0 \\ x - 3 = 0 \quad &\text{or} \quad x + 1 = 0 \\ x = 3 \quad &\text{or} \quad x = -1 \end{aligned}$$

The equation has two solutions $x = 3$ and $x = -1$.

Exercises

Exercise 24.1. Solve the exponential equation.

1) $2^{x-1} = 4$

2) $7e^{2x} - 5 = 58$

Answer: 1) $x = 3$ 2) $x = 1.0986$

Exercise 24.2. Solve the exponential equation.

1) $3^{x^2-2x} = e^{-\ln 3}$

2) $2^{x+1} = 3^{1-x}$

Answer: 1) $x = 1$ 2) $x = \dfrac{\ln 3 - \ln 2}{\ln 2 + \ln 3}$

Exercise 24.3. Find intersections of the given pairs of curves.

$$f(x) = e^{2x} \quad \text{and} \quad g(x) = e^x + 12$$

Answer: $x = 2\ln 2$

Exercise 24.4. Using the formula $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$ to determine how many years, to the nearest hundredth, it will take to double an investment $\$20,000$ at the interest rate 5% compounded monthly.

Answer: 13.89

Exercise 24.5. Newton's Law of Cooling states that the temperature $T(t)$ of an object at any time t satisfies the equation $T(t) = T_s + (T_0 - T_s)e^{-rt}$, where T_s is the temperature of the surrounding environment, T_0 is the initial temperature of the object, and r is a positive constant characteristic of the system, which is in units of time^{-1}. In a room with a temperature of $22°$, a cup of tea of $97°$ was freshly brewed. Suppose that $r = \frac{\ln 5}{20}\ \mathrm{minute}^{-1}$. Approximately, in how many minutes will the temperature of the tea be $37°F$?

Answer: 20

Topic 25 Logarithmic Equations and Applications

Learning Goals

- ☐ I can solve a logarithmic equation and identify extraneous solutions.
- ☐ I can solve applied problems involving logarithmic equations or functions and interpret parameters and solutions in context.

25.1 Think about It

Energy Released from an Earthquake

The energy E released by an earthquake is related to its magnitude M measure on the Richter scale by the equation

$$\log E = 4.8 + 1.5M.$$

Can you imagine how many times more energy released if the magnitude increased by 1 Richter scale?

25.2 Solving Logarithmic Equations

Solving a logarithmic equation is also similar to solving a radical equation. One strategy is the following:

1) isolate a logarithmic expression (involving condensing logarithms),
2) use the relation between exponential functions and logarithmic functions to rewrite the equation into an equivalent equation (equivalent in the sense over the domain of the original equation) without logarithmic expressions,
3) solve the resulting equation.
4) check for extraneous solutions (because the condensing logarithmic functions usually changes the domain.)

Example 25.1. Solve the equation $\log_2 x + \log_2(x-2) = 3$.

Solution.

1) Condense the logarithms and rewrite the equation in the form $\log_b u = c$ using properties of logarithms:

$$\log_2(x(x-2)) = 3$$

2) Rewrite the equation in the exponential form (moving the base):

$$x(x-2) = \underline{\hspace{2em}}$$

3) Solve the resulting equation:

$$
\begin{aligned}
x(x-2) &= 8 \\
x^2 - 2x + \underline{\quad\quad} &= 0 \\
(x+2)(x+\underline{\quad\quad}) &= 0 \\
x+2 = 0 \quad &\text{or} \quad x-4=0 \\
x = \underline{\quad\quad} \quad &\text{or} \quad x = \underline{\quad\quad}
\end{aligned}
$$

4) Check proposed solutions:

Both x and $x-2$ must be positive. So $x = \underline{\quad\quad}$ is not a solution of the original equation. When $x = \underline{\quad\quad}$, we have

$$\log_2 4 + \log_2 2 = 2 + 1 = 3$$

So $x = 4$ is a solution.

25.3 Applications

Example 25.2. A type of virus started spreading. After 10 days, 580 infected cases were confirmed. The natural logarithm of the number of confirmed cases approximately is a linear function of the number of days after the virus started spreading. The slope of this linear function is 0.28. Suppose the pattern continues.

1) Suppose the number of infected cases after t days is $P(t)$. Find an equation of $P(t)$. Round to the nearest hundredth.

2) Estimate the number of infected cases after 20 days. Round to the nearest integer.

Solution. Since 580 infected cases were confirmed after 10 days, we have

$$P(10) = 580$$

Since the natural logarithm of the number of confirmed cases is approximately a linear function with the slope 0.28, we have

$$\ln(P(t)) = \underline{\quad\quad}\, t + b,$$

where b must satisfy the equation

$$\ln(P(10)) = 0.28 \cdot \underline{\quad\quad} + b.$$

Solving for b implies

$$b \approx \underline{\quad\quad\quad}.$$

Therefore,

$$P(t) = e^{\underline{\quad\quad\quad}}.$$

The infected cases after 20 days is

$$
\begin{aligned}
P(20) &\approx e^{0.28\cdot \underline{\quad} +3.56} \\
&\approx 9509.
\end{aligned}
$$

Exercises

Exercise 25.1. Solve the logarithmic equation.

1) $\log_5 x + \log_5(4x-1) = 1$ 2) $\ln\sqrt{x+1} = 1$

Answer: 1) $x = \frac{5}{4}$ 2) $x = e^2 - 1$

Exercise 25.2. Solve the logarithmic equation.

1) $\log_2(x+2) - \log_2(x-5) = 3$ 2) $\log_3(x-5) = 2 - \log_3(x+3)$

Answer: 1) $x = 6$ 2) $x = 6$

Exercise 25.3. For the given function, find values of x satisfying the given equation.

1) $f(x) = \log_4 x - 2\log_4(x+1)$, $f(x) = -1$

2) $g(x) = \log(1-2x) + \log(-x)$, $g(x) = 1$

Answer: 1) $x = 1$ 2) $x = -2$

Exercise 25.4. Find intersections of the given pairs of curves.

$$f(x) = \log_7\left(\tfrac{1}{2}(x+2)\right) \quad \text{and} \quad g(x) = 1 - \log_7(x-3)$$

Answer: $x = 5$

Exercise 25.5. A culture of bacteria began with 1000 bacteria and grows exponentially, that is, the population is $P(t) = P(0)e^{rt}$. An hour later there were 1320 bacteria.

1) After 3 hours, how many bacteria will there be?
2) Approximately, how many hours after starting will there be 3610 bacteria? Round to the nearest tenth.

Answer: 1) 2300 2) 4.6