Online Companion to "Dynamic portfolio selection for nonlinear law-dependent preferences"

Zongxia Liang* Jianming Xia† Fengyi Yuan‡

November 12, 2023

In this Online Companion, we apply Condition (a) in Theorem 3.2 of the main context to recover some results of Basak and Chabakauri (2010) and Hu et al. (2021) on portfolio selections for the mean-variance (MV) preference with stochastic factor models and for the rank dependent utility (RDU) with the Black-Scholes model, respectively.

OC1 Mean variance preference

Let us now investigate the dynamic MV problems. For simplicity, the interest rate is 0 and there is only one stock and only one stochastic factor; the extension to the case of multiple stocks and stochastic factors is straightforward. The stock price process S and the stochastic factor Y satisfy the following system of SDEs:

$$\begin{aligned}
&\mathrm{d}S_t/S_t = \theta(t,S_t,Y_t)\mathrm{d}t + \sigma(t,S_t,Y_t)\mathrm{d}W_t^{\mathcal{S}},\\
&\mathrm{d}Y_t = m(t,Y_t)\mathrm{d}t + \nu(t,Y_t)\mathrm{d}W_t^{Y},\\
&W^Y = \rho W^{\mathcal{S}} + \sqrt{1-\rho^2}W^{\mathcal{O}},
\end{aligned}$$

where $\rho \in [-1,1]$, Brownian motion $W^{\mathcal{S}}$ is independent of Brownian motion $W^{\mathcal{O}}$, and all of θ, σ, m and ν are Borel functions such that the above system of SDEs has a unique solution.

The MV preference functional g is given by

$$g(\mathbb{P}_X) = \mathbb{E}[X] - \frac{\gamma}{2}\mathbb{E}[X^2] + \frac{\gamma}{2}(\mathbb{E}[X])^2,$$

where $\gamma > 0$ is the coefficient of risk aversion. Moreover, $\mathcal{P}_0 = \{\mu \in \mathcal{P}(\mathbb{R}) : \int_{\mathbb{R}} x^2\mu(dx) < \infty\}$.

It is straightforward to obtain

$$\partial_x \nabla g(\mathbb{P}_X, x) = 1 - \gamma x + \gamma\mathbb{E}[X].$$

Taking $\xi^t = \partial_x \nabla g(\mathbb{P}^t_{\bar{X}_T}, \bar{X}_T)$, (3.8) turns into a very simple form:

$$Z^{\bar{X}_T,\mathcal{S}}(t) = \frac{\kappa(t)}{\gamma}.$$

*Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China; Email: liangzongxia@mail.tsinghua.edu.cn

†RCSDS, NCMIS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Email: xia@amss.ac.cn

‡Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China; Email: yfy19@mails.tsinghua.edu.cn

That is, there exist a process ϕ such that

$$\bar{X}_T = \mathbb{E}[\bar{X}_T] + \int_0^T \frac{\kappa(s)}{\gamma} \mathrm{d}W_s^{\mathcal{S}} + \int_0^T \phi_s \mathrm{d}W_s^{\mathcal{O}}.$$

Now the problem is to replicate $\bar{X}_T$, i.e., to find appropriate π such that the corresponding terminal wealth is $\bar{X}_T$. Note that for $H_{\cdot} = \exp(-\frac{1}{2}\int_0^{\cdot} |\kappa_s|^2 \mathrm{d}s - \int_0^{\cdot} \kappa_s \mathrm{d}W_s^{\mathcal{S}})$ (only $W^{\mathcal{S}}$ is involved), $H_{\cdot}\bar{X}_{\cdot}$ is always a local martingale, we aim to find a π such that it is a martingale. Therefore,

$$\begin{aligned} X_t &= \frac{1}{H_t}\left(\mathbb{E}_t\left[H_T \int_0^T \frac{\kappa(s)}{\gamma}\mathrm{d}W_s^{\mathcal{S}}\right] + \mathbb{E}_t\left[H_T \int_0^T \phi_s \mathrm{d}W_s^{\mathcal{O}}\right]\right) \\ &= \int_0^t \frac{\kappa(s)}{\gamma}\mathrm{d}W_s^{\mathcal{S}} + \int_0^t \phi_s \mathrm{d}W_s^{\mathcal{O}} - \frac{1}{H_t}\left\{\mathbb{E}_t\left[H_T \int_t^T \frac{\kappa(s)}{\gamma}\mathrm{d}W_s^{\mathcal{S}}\right] + \mathbb{E}_t\left[H_T \int_t^T \phi_s \mathrm{d}W_s^{\mathcal{O}}\right]\right\}. \end{aligned}$$

Because

$$H_s - H_t = -\int_t^s H_r \kappa(r) \mathrm{d}W_r^{\mathcal{S}},$$

we have

$$\begin{aligned} &\frac{1}{H_t}\mathbb{E}_t\left[H_T \int_t^T \frac{\kappa(s)}{\gamma}\mathrm{d}W_s^{\mathcal{S}}\right] = -\frac{1}{H_t}\mathbb{E}_t\left[\int_t^T \frac{|\kappa(s)|^2 H_s}{\gamma}\mathrm{d}s\right] = -\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{|\kappa(s)|^2}{\gamma}\mathrm{d}s\right], \\ &\frac{1}{H_t}\mathbb{E}_t\left[H_T \int_t^T \phi_s \mathrm{d}W_s^{\mathcal{O}}\right] = 0, \end{aligned}$$

where $\mathbb{Q}$ is the risk-neutral measure induced by H. To derive the second identity, we have used the fact that $W^{\mathcal{S}}$ and $W^{\mathcal{O}}$ are independent. Then

$$\bar{X}_t = \int_0^t \frac{\kappa(s)}{\gamma}\mathrm{d}W_s^{\mathcal{S}} + \int_0^t \phi_s \mathrm{d}W_s^{\mathcal{O}} - \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{|\kappa(s)|^2}{\gamma}\mathrm{d}s\right]. \tag{OC. 1}$$

Let us assume now that there exists a function $f(t, s, y)$ which is smooth enough[1] such that

$$f(t, S_t, Y_t) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{|\kappa(s)|^2}{\gamma}\mathrm{d}s\right].$$

Differentiating on both sides of (OC. 1) and comparing it to (2.4) yield

$$\bar{\pi}_t = \frac{\kappa(t)}{\gamma\sigma(t)} - \left(S_t \partial_s f_t + \frac{\rho\nu_t}{\sigma_t}\partial_y f_t\right), \tag{OC. 2}$$

where $\partial_j f_t = \partial_j f(t, S_t, Y_t)$, $j = s, y$, and $\nu_t = \nu(t, Y_t)$. In addition, as a byproduct, we have

$$\phi_t = \sqrt{1-\rho^2}\partial_y f_t \nu_t.$$

To verify that the strategy given by (OC. 2) is indeed an equilibrium, we only need to check Conditions (a)-(c) in Theorem 3.2. In fact, because the structure of the MV preference is rather simple, all conditions are straightforward except for (3.1), (3.4) and (3.5). Indeed, because $g(\mu) = \int_{\mathbb{R}}\left(x - \frac{\gamma}{2}x^2\right)\mu(\mathrm{d}x) + \frac{\gamma}{2}\left(\int_{\mathbb{R}} x\mu(\mathrm{d}x)\right)^2$ and $\nabla g(\mu, x) = x - \frac{\gamma}{2}x^2 + \gamma\left(\int_{\mathbb{R}} y\mu(\mathrm{d}y)\right)x$, we have

$$g(\mu_1) - g(\mu_0) = \int_{\mathbb{R}}\left(x - \frac{\gamma}{2}x^2\right)(\mu_1 - \mu_0)(\mathrm{d}x) + \frac{\gamma}{2}\left(\int_{\mathbb{R}} x(\mu_1 - \mu_0)(\mathrm{d}x)\right)\left(\int_{\mathbb{R}} y(\mu_1 + \mu_0)(\mathrm{d}y)\right)$$

[1] This is also assumed in Basak and Chabakauri (2010), and it can be verified directly in specific models. Moreover, f can be obtained by solving a PDE resulting from the Feynman-Kac formula.

$$
\begin{aligned}
&= \int_{\mathbb{R}} \left[x - \frac{\gamma}{2} x^2 + \gamma \left(\int_{\mathbb{R}} y \mu_0(\mathrm{d}y) \right) \right] (\mu_1 - \mu_0)(\mathrm{d}x) + \frac{\gamma}{2} \left(\int_{\mathbb{R}} x (\mu_1 - \mu_0)(\mathrm{d}x) \right)^2 \\
&= \int_{\mathbb{R}} \nabla g(\mu_0, x) \mu_0(\mathrm{d}x) + M_0(\mu_0, \mu_1),
\end{aligned}
$$

with

$$
M_0(\mu_0, \mu_1) := \frac{\gamma}{2} \left(\int_{\mathbb{R}} x (\mu_1 - \mu_0)(\mathrm{d}x) \right)^2 .
$$

Therefore, (3.1) holds with $M_1 \equiv 1$ and M_0 as above. Furthermore,

$$
\begin{aligned}
M_0(\mathbb{P}^t_{\bar{X}_T}, \mathbb{P}^t_{\bar{X}_T^{t,\varepsilon,\varphi}}) &= \frac{\gamma}{2} (\mathbb{E}_t[\bar{X}^{t,\varepsilon,\varphi} - \bar{X}_T])^2 \\
&= \|\varphi\|^2 \frac{\gamma}{2} \left(\mathbb{E} \left[\int_t^{t+\varepsilon} (\theta(s)\mathrm{d}s + \sigma(s)\mathrm{d}W_s^{\mathcal{S}}) \right] \right)^2 \\
&= \|\varphi\|^2 \frac{\gamma}{2} \left(\mathbb{E} \left[\int_t^{t+\varepsilon} \theta(s)\mathrm{d}s \right] \right)^2 \\
&= O(\varepsilon^2).
\end{aligned}
$$

Therefore (3.4) and (3.5) are all verified. As such, Theorem 3.2 is applicable to prove that the strategy given by (OC. 2) is an equilibrium. Thus, we obtain the same results as in Basak and Chabakauri (2010).

OC2 Rank dependent utility

Time-consistent portfolio selection with RDU has been studied in Hu et al. (2021), assuming that the market is complete and both θ and σ are deterministic. We shall recover their ODE (5) using Condition (a) of our Theorem 3.2. Here, the distortion functions are time-dependent. Hence, we need a slight extension of the results in the main context. In fact, such an extension is straightforward. Furthermore, we emphasize that, because the preference is now generally non-concave, Theorem 3.2 can not be applied to verify that the solution of the ODE gives an equilibrium. Nevertheless, it can be used to find the candidate equilibrium strategy. The rigorous verification has been provided in Hu et al. (2021), which is rather delicate and technical.

The RDU at time t is given by

$$
g(t, \mathbb{P}_X) = \int_0^\infty w(t, \mathbb{P}(U(X) > y))\mathrm{d}y + \int_{-\infty}^0 [w(t, \mathbb{P}(U(X) > y)) - 1]\,\mathrm{d}y,
$$

where U is the utility function and w is the distortion function.

To proceed, we need the following result about the derivatives:

$$
\nabla g(t, \mathbb{P}_X, x) = \int_{-\infty}^{U(x)} w_p'(t, \mathbb{P}(U(X) > y))\mathrm{d}y \tag{OC. 3}
$$

and

$$
\partial_x \nabla g(t, \mathbb{P}_X, x) = w_p'(t, 1 - F_X(x)) U'(x), \tag{OC. 4}
$$

where $F_X(x) = \mathbb{P}(X \leq x)$, $x \in \mathbb{R}$. Indeed, suppose that we have imposed enough assumptions on w and U such that we can interchange the differentials and integrals freely. For μ_0 and μ_1, denote

$\mu_s = s\mu_1 + (1-s)\mu_0,\ s \in [0,1]$. For simplicity, we use the notation $p^{\mu}(y) := \mu(x : U(x) > y)$. Clearly,

$$g(t,\mu_s) = \int_0^\infty w(t, sp^{\mu_1}(y) + (1-s)p^{\mu_0}(y))\mathrm{d}y + \int_{-\infty}^0 [w(t, sp^{\mu_1}(y) + (1-s)p^{\mu_0}(y)) - 1]\mathrm{d}y.$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} g(t,\mu_s) &= \int_{-\infty}^{\infty} w_p'(t, p^{\mu_s}(y))(p^{\mu_1}(y) - p^{\mu_0}(y))\mathrm{d}y \\ &= \int_{-\infty}^{\infty} w_p'(t, p^{\mu_s}(y)) \int_{-\infty}^{\infty} \mathbb{1}_{\{U(x)>y\}}(\mu_1 - \mu_0)(\mathrm{d}x)\mathrm{d}y \\ &= \int_{\mathbb{R}} \left(\int_{-\infty}^{U(x)} w_p'(t, p^{\mu_s}(y))\mathrm{d}y \right) (\mu_1 - \mu_0)(\mathrm{d}x).\end{aligned}$$

By Definition 3.1, we obtain (OC. 3). Taking derivative with respect to the variable x obtains (OC. 4).

Now we are going to derive ODE (5) in Hu et al. (2021). Following Hu et al. (2021), we make an ansatz $\bar{X}_T = (U')^{-1}(\nu \mathcal{Z}_T^W(-\lambda\kappa))$, where $\nu > 0$ is a parameter related to the initial endowment, and λ is a deterministic function to be determined. For simplicity, we use the following notations:

$$\Lambda(t) = \int_t^T |\lambda(s)\kappa(s)|^2 \mathrm{d}s,\ \mathcal{E}_{s,t} = \int_s^t \lambda(s)\kappa(s)\mathrm{d}W_s,\ G(x) = -\frac{1}{2}\Lambda(0) + \log\nu - \log U'(x).$$

We first calculate $\partial_x \nabla g(t, \mathbb{P}^t_{\bar{X}_T}, x)$. As $\mathcal{E}_{t,T} \sim \mathcal{N}(0, \Lambda(t))$ and is independent of $\mathcal{F}_t$, we have

$$1 - F^{\mathbb{P}^t_{\bar{X}_T}}(x) = \mathbb{P}_t(\bar{X}_T > x) = \mathbb{P}_t(G(x) - \mathcal{E}_{0,t} < \mathcal{E}_{t,T}) = N\left(\frac{\mathcal{E}_{0,t} - G(x)}{\sqrt{\Lambda(t)}}\right),$$

where we require κ to be deterministic, N is the cumulative distribution function of the standard normal distribution $\mathcal{N}(0,1)$. To use Theorem 3.2, we take $\xi^t = \partial_x \nabla g(t, \mathbb{P}^t_{\bar{X}_T}, \bar{X}_T)$.

By (OC. 3) and (OC. 4), we have, for $s \geq t$,

$$\begin{aligned}\mathbb{E}_s[\xi^t] &= \mathbb{E}_s\left[w_p'\left(t, N\left(\frac{\mathcal{E}_{0,t} - G(\bar{X}_T)}{\sqrt{\Lambda(t)}}\right)\right)\nu\mathcal{E}_T(\lambda\kappa)\right] \\ &= \mathbb{E}_s\left[w_p'\left(t, N\left(\frac{-\mathcal{E}_{t,T}}{\sqrt{\Lambda(t)}}\right)\right)\nu e^{-\mathcal{E}_{0,T} + \frac{1}{2}\Lambda(0)}\right] \\ &= \nu e^{-\mathcal{E}_{0,s} + \frac{1}{2}\Lambda(0)}\mathbb{E}\left[w_p'\left(t, N\left(\frac{-\mathcal{E}_{t,s} - \sqrt{\Lambda(s)}\xi}{\sqrt{\Lambda(t)}}\right)\right)e^{\sqrt{\Lambda(s)}\xi}\right],\end{aligned} \tag{OC. 5}$$

where $\xi \sim \mathcal{N}(0,1)$. Therefore, by Itô's formula, we have

$$\begin{aligned}Z^{\xi^t}(t) &= -\nu e^{-\frac{1}{2}\Lambda(0) - \mathcal{E}_{0,t}}\lambda(t)\kappa(t)\mathbb{E}\left[w_p''(t, N(\xi))N'(\xi)\frac{e^{\sqrt{\Lambda(t)}\xi}}{\sqrt{\Lambda(t)}} + w_p'(t, N(\xi))e^{\sqrt{\Lambda(t)}\xi}\right] \\ &= -\nu e^{-\frac{1}{2}\Lambda(0) - \mathcal{E}_{0,t}}\lambda(t)\kappa(t)\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}z}\left(w_p'(t, N(z))e^{\sqrt{\Lambda(t)}z}\right)\frac{N'(z)}{\sqrt{\Lambda(t)}}\mathrm{d}z.\end{aligned}$$

Using integration by part and the fact that $N''(z) = -zN'(z)$ yields

$$Z^{\xi^t}(t) = -\nu e^{-\frac{1}{2}\Lambda(0)-\mathcal{E}_{0,t}}\lambda(t)\kappa(t)\frac{\mathbb{E}\left[w_p'(t, N(\xi))\xi e^{\sqrt{\Lambda(t)}\xi}\right]}{\sqrt{\Lambda(t)}}. \tag{OC. 6}$$

Denote $h(t,x) = \mathbb{E}[w_p'(t, N(\xi))e^{x\xi}]$. Combing (OC. 5) and (OC. 6), the equilibrium condition (3.8) becomes

$$\lambda(t)h_x'(t, \sqrt{\Lambda(t)}) = \sqrt{\Lambda(t)}h(t, \sqrt{\Lambda(t)}).$$

This is in fact an ODE about Λ. Therefore, the ODE (5) in Hu et al. (2021) is recovered.

Because the problem with RDU is non-concave, the second-order condition is needed (see inequality (8) in Hu et al. (2021)). We believe that this can be derived by calculating the second-order derivative $\partial_x^2 \nabla g(t, \mu, x)$. To completely accommodate the RDU, it requires a verification theorem for non-concave preferences, which is beyond the scope of this paper and left for the future research.

References

S. Basak and G. Chabakauri. Dynamic Mean-Variance Asset Allocation. *The Review of Financial Studies*, 23(8):2970–3016, 2010.

Y. Hu, H. Jin, and X. Y. Zhou. Consistent investment of sophisticated rank-dependent utility agents in continuous time. *Mathematical Finance*, 31(3):1056–1095, 2021.