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I   Preliminaries: Set theory and categories

1   Naive set theory

1.1. If there is a set V such that $\forall x, x \in V$, let $A = \{x \in V \mid x \notin x\}$.

Now, $A \in A \iff A \in V \wedge A \notin A \iff A \notin A$ and this is a contradiction.

Finally, $\forall x, \exists y, y \notin x$.

1.2. $\mathscr{P}_\sim = \{[a]_\sim \mid a \in S\}$

$\sim$ is symmetric, so given $a \in S$, $a \in [a]_\sim$ so $[a]_\sim \neq \emptyset$

As $\sim$ is reflexive, $\bigcup_{a \in S}[a]_\sim = S$.

Finally, if $[a]_\sim \neq [b]_\sim$, let us prove that $[a]_\sim \cap [b]_\sim = \emptyset$.

Indeed, if $x \sim a$ and $x \sim b$ then by transitivity and symmetry, $a \sim b$ which is a contradiction.

1.3. If $\mathscr{P}$ is a partition on S, let us define $a \sim b$ iff $\exists p \in P, a \in P \wedge b \in P$.

This relation is reflexive because $\forall p \in P, p \neq \emptyset$

It is obviously symmetric.

If $a \sim b$ and $b \sim c$ then there is $p \in P$ such that $a, b \in p$ and $p' \in P$ such that $b, c \in p$. $b \in p \cap p'$ so $p \cap p' \neq \emptyset$ so $p = p'$ and $b \sim c$.

Remark. Let $\mathbf{Eq}(S)$ the set of all relations on S and $\mathfrak{P}$ the set of all partitions of S.

$$\mathbf{Eq}(S) \to \mathfrak{P}$$
$$\sim \mapsto \mathscr{P}_\sim$$

is a bijection which has as inverse the 'operation' of the previous exercise.

1.4. An equivalence relation $\sim$ on $\{1, 2, 3\}$ is a subset of $\{1, 2, 3\}^2$ which necessarily includes the diagonal $\{(1, 1), (2, 2), (3, 3)\}$.

Then we have three choices as far as distinct unordered couples are concerned:

- There is no distinct a and b such that $a \sim b$,

- There are distinct a, b, c such that $a \sim b$ and $b \sim c$, in which case, we necessarily have $a \sim c$,

- There is only two distinct a and b such that $a \sim b$. We have then $\binom{3}{2} = 3$ choices.

We can verify that in each case we indeed have an equivalence relation on $\{1, 2, 3\}$ so there are 5 equivalence relations on $\{1, 2, 3\}$.

1.5. Let R a relation on $\{1, 2, 3\}$ such that $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$

We can verify that R is reflexive, symmetric, but not transitive.

Trying to define $\mathscr{P}_R$ will not lead to a partition, as $[1]_R \neq [3]_R$ (because $(1, 3) \notin R$) but $[1]_R \cap [3]_R \neq \emptyset$ (because $2 \in [1]_R \cap [3]_R$).

1.6. $a - a = 0 \in \mathbb{Z}$ so $\sim$ is reflexive.

If $a - b \in \mathbb{Z}$ then $b - a \in \mathbb{Z}$, so it is symmetric.

If $a - b \in \mathbb{Z}$ and $b - c \in \mathbb{Z}$ then $a - c = a - b + (b - c) \in \mathbb{Z}$, so $\sim$ is an equivalence relation.

$$f : \mathbb{R} \to [0,1[$$
$$x \mapsto \text{the only } y \in [0,1[\text{ such that } y - x \in \mathbb{Z}$$

f is clearly surjective.

We remark that $x \sim y$ iff $f(x) = f(y)$.

We can thus define:

$$f' : \mathbb{R}\big/_{\sim} \to [0,1[$$
$$[x]_{\sim} \mapsto f(x)$$

f' is a bijection.

Because $\sim$ is an equivalence relation, $\approx$ is clearly an equivalence relation.

Let

$$g : \mathbb{R}^2 \to [0,1[^2$$
$$(x,y) \mapsto (f(x), f(y))$$

Because f is a bijection it is easy to prove that g is a bijection.

Remark. We could have factored f using Claim 5.5, page 34.

The situation for $\mathbb{R}^2$ can be represented by the following diagram:

$$
\begin{array}{ccccc}
[0,1[& \longleftarrow & [0,1[^2 & \longrightarrow & [0,1[\\
\big\uparrow f & & \exists! f \& f \big\uparrow & & \big\uparrow f \\
\mathbb{R} & \longleftarrow & \mathbb{R}^2 & \longrightarrow & \mathbb{R}
\end{array}
$$

$(f,g) \mapsto f \wr\!\! g$ is bijective and $f \wr\!\! f$ is bijective iff f is bijective.

2 Functions between sets

2.1. Let us prove on the set of bijections from a set S to a set S' has cardinality $n!$ if $|S| = |S'| = n$.

By induction on $n \in \mathbb{N}$:

If $n = 0$, $\emptyset : S \to S'$ is the only bijection. The statement is true.

Suppose the statement is true for any set S, S' such that $|S| = |S'| = n$. Let T, T' be such that $|T| = |T'| = n + 1$.

Pick $x \in T$. A bijection from T to T' is the datum of an image $x' \in T'$ for x and a bijection from $T \setminus \{x\} \to T' \setminus x'$. There are $n!$ such bijections.

Finally, there are $(n+1)n! = (n+1)!$ bijections from $T \to T'$

Remark. The previous situation is represented by the following diagram:

$$
\begin{array}{ccccc}
\{x'\} & \longrightarrow & T' \cong \{x'\} \bigsqcup T' \setminus \{x'\} & \longleftarrow & T' \setminus \{x'\} \\
\big\uparrow f & & \exists! f \wr\!\! g \big\uparrow & & \big\uparrow g \\
\{x\} & \longrightarrow & T \cong \{x\} \bigsqcup T \setminus \{x\} & \longleftarrow & T \setminus \{x\}
\end{array}
$$

$f \wr\!\! g$ is bijective iff f and g are bijective.

2.2. If f has a right inverse g, $f \circ g = \text{id}_B$.
  $f(g(B)) = (f \circ g)(B) = \text{id}_B(B) = B$, but $g(B) \subseteq A$ so $f(A) = B$. Hence f is surjective.
  If f is surjective, let $\alpha : \mathscr{P}(A) \setminus \{\emptyset\} \to A$ be a choice function.
  $\forall x \in B, \exists z \in A, f(z) = x$. Let $g(x) = \alpha(\{z \in A \mid f(z) = x\})$.
  It is clear that $f \circ g = \text{id}_B$.

2.3. If $f : A \to B$ is a bijection, $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$ but Corollary 2.2 tells us that if a function has a two-sided inverse, then it is a bijection so f^{-1} is a bijection.
  If $f : A \to B$ and $g : B \to C$ are bijections, $g \circ f : A \to C$ is also a bijection.
  Indeed, $(g \circ f) \circ (f^{-1} \circ g^{-1}) = \text{id}_C$ and $(f^{-1} \circ g^{-1}) \circ (g \circ f) = \text{id}_A$.

2.4. Let $\sim$ be the relation 'isomorphism'. $X \sim X$ because $id_X : X \to X$ and id_X is clearly a bijection.
  $X \sim Y$ implies that $Y \sim X$ because the inverse of a bijection is a bijection.
  $\sim$ is transitive because the composition of two bijections is a bijection.

2.5. $f : A \to B$ is an epimorphism iff $\forall g, g' : B \to C, g \circ f = g' \circ f \implies g = g'$.
  Let us prove that f is an epimorphism iff it is surjective.
  If f is surjective. Let $y \in B$, there is $x \in A$ such that $y = f(x)$.
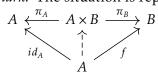 $g(y) = g \circ f(x) = g' \circ f(x) = g'(y)$. Hence $g = g'$.
  If f is not surjective then, there is $y \in B \setminus f(A)$. Let $g, g' : B \to \{0,1\}$ such that $\forall x \in B, g(x) = 0$, $\forall x \in B \setminus \{y\}, g'(x) = 1$ and $g'(y) = 0$.
  $g \circ f = g' \circ f$ but $g \neq g'$ so f is not an epimorphism.

2.6. If $f : A \to B$. Define $\gamma : A \to A \times B$ such that $\gamma(a) = (a, f(a))$. It is obvious that $f \circ \gamma = \text{id}_A$.

Remark. The situation is represented by the following commutative diagram:

$$
\begin{array}{ccccc}
A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B \\
 & \nwarrow & \uparrow & \nearrow & \\
 & {\scriptstyle id_A} & \vdots & {\scriptstyle f} & \\
 & & A & &
\end{array}
$$

2.7. Let $\Gamma_f \subseteq A \times B$. Let $i : \Gamma_f \to A \times B$ the inclusion map.
  $\pi_A \circ i : \Gamma_f \to A$ is a bijection.
  Indeed, $\forall a \in A, \exists! y \in B, (a, y) \in \Gamma_f$
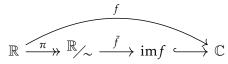 That is: $\forall a \in A, \exists! z \in \Gamma_f, \pi_A(z) = a$ (this z is (a, y))

2.8.

$$f : \mathbb{R} \to \mathbb{C}$$
$$r \mapsto e^{2\pi i r}$$

The canonical decomposition of f gives:

$$
\mathbb{R} \xrightarrow{\pi} \mathbb{R}/_\sim \xrightarrow{\tilde{f}} \text{im} f \hookrightarrow \mathbb{C}
$$
with f spanning over the diagram.

$$\alpha : \mathbb{R}/_\sim \to S_2$$
$$[r]_\sim \mapsto e^{2\pi i r}$$

where S_2 is the circle with radius 1 in $\mathbb{C}$ is well defined and is a bijection.
  An alternative decomposition would have been:

$$\mathbb{R} \xrightarrow[\alpha \circ \pi]{\overset{f}{\frown}} S_2 \xrightarrow{\tilde{f} \circ \alpha^{-1}} \operatorname{im} f \hookrightarrow \mathbb{C}$$

This exercise matches Exercise 1.6.

2.9. Suppose $A' \xrightarrow[\cong]{\alpha} A''$, $B' \xrightarrow[\cong]{\beta} B''$, $A' \cap B' \neq \emptyset$ and $A'' \cap B'' \neq \emptyset$.
Let

$$\gamma : A \cup B \to A'' \cup B''$$
$$x \mapsto \alpha(x) \text{ if } x \in A$$
$$y \mapsto \beta(y) \text{ if } x \in B$$

γ is well defined because $A \cap B = \emptyset$, surjective because α and β are surjective, and injective because α and β are injective and because $A'' \cap B'' = \emptyset$.

Hence, however we take 'copies' of A and B such that their copies are disjoint, we always obtain isomophic sets.

2.10. Let us prove by induction on n that if $|A| = n$, we have $|B^A| = |B|^{|A|}$.
$|B^\emptyset| = \{\emptyset\} = |B|^\emptyset$ so the statement is true for $n = 0$
If $|A| = n + 1$, pick $x \in A$
A function from $A \to B$ is the datum of a function of $A \to B \setminus \{x\}$ and image $x' \in B$ for x. There are $|B|^{|A \setminus \{x\}|} = |B|^{|A|-1}$ such functions.
Hence $|B|^{|A|} = |B||B|^{|A|-1} = |B|^{|A|}$.

Remark. The previous situation is represented by the following diagram:

$$\begin{array}{ccccc}
\{x'\} & \longrightarrow & B \cong \{x'\} \bigsqcup B \setminus \{x'\} & \longleftarrow & B \setminus \{x'\} \\
f \uparrow & & \exists! f ⅋ g \uparrow & & g \uparrow \\
\{x\} & \longrightarrow & A \cong \{x\} \bigsqcup A \setminus \{x\} & \longleftarrow & A \setminus \{x\}
\end{array}$$

$(f, g) \mapsto f ⅋ g$ is a bijection.

2.11. Let

$$f : \mathscr{P}(A) \to 2^A$$
$$X \mapsto \quad x \mapsto 1 \text{ if } x \in X$$
$$x \mapsto 0 \text{ otherwise}$$

$f(X)$ is called the characteristic function of X. It is easy to prove that f is a bijection.

3 Categories

3.1. Let $\circ'$ be the operation of $\mathbf{C}^{op}$. If $f : A \to B$ and $g : B \to C$ in $\mathbf{C}$, $f \circ' g = g \circ f$.
$(f \circ' g) \circ' h = h \circ (g \circ f) = (h \circ g) \circ f = f \circ' (g \circ' h)$
If $f : A \to B$ in $\mathbf{C}$: $f \circ' \operatorname{id}_B = \operatorname{id}_B \circ f = f$ and $\operatorname{id}_A \circ' f = f \circ A = f$
Finally, it is clear that $\operatorname{Hom}_{\mathbf{C}^{op}}(A, B) = \operatorname{Hom}_{\mathbf{C}^{op}}(A', B')$ iff $A = A'$ and $B = B'$.

3.2. If A is finite, as $\operatorname{End}_{\mathbf{Set}}(A) = A^A$, $|\operatorname{End}_{\mathbf{Set}}(A)| = |A|^{|A|}$ (see Exercise 2.10).

3.3. $1_a = (a, a)$ is the identity of a. Indeed, let $f : a \to b$, Necessarily, if f exists, $f = (a, b)$.
$f \circ 1_a = (a, b) \circ (a, a) = (a, b) = f$
Similarly, if $f : b \to a$, $1_a \circ f = f$.

3.4. $<$ is not reflexive, so for example, $(3,3)$ would not be in the category. But, necessarily, it should be as the only possible choice for 1_3 (think about it).

3.5. For every preorder, we can define a category in the style of Example 3.3.

Remark. In fact, writing $\mathbf{Proset}\!\restriction_A$ the set of all categories whose set of objects is A, whose set of morphisms is a subset of A^2 and such that $\forall a,b \in A, \exists^{\leq 1} f : a \to b$, there is a bijection :

$$\phi : \text{Preorders on } A \to \mathbf{Proset}\!\restriction_A$$

$$R \mapsto \text{The category such that} \qquad \text{Hom}(a,b) = \{(a,b)\} \text{ if } aRb$$
$$\text{Hom}(a,b) = \emptyset \text{ otherwise}$$
$$(b,c) \circ (a,b) = (a,c)$$

R is an order iff $\phi(R)$ is skeletal.
R is an equivalence relation iff $\phi(R)$ is a groupoid.

3.6. The set of matrices with n rows and m columns is $\mathbb{R}^{m \times n}$ where $m = \{0,\ldots,m-1\}$ and $n = \{0,\ldots,n-1\}$ and $\times$ is the product of sets.

Hence, $\mathbb{R}^{0 \times n} = \mathbb{R}^{\emptyset \times n} = \mathbb{R}^{\emptyset} = \{\emptyset\}$. There is only one matrix with 0 lines is $\emptyset$, and, similarly, it is the only matrix with 0 columns.

If $A : m \to n$ (A has n lines and m columns) and $B : n \to p$ (B has p lines and n columns), then $B \circ A = B \times A$.

The product of matrices is associative and $I_n = \text{id}_n$.

It feels familiar. If we define the category $\mathbb{N}$ with $\text{Obj}(\mathbb{N}) = \mathbb{N}$ and with $\text{Hom}_{\mathbb{N}}(n,m) = m^n$. In both of the categories, it is true that if $\text{dom}(f) = \text{im}(f)$ then f is a monomorphism iff it is an epimorphism.