Contents

I Preliminaries: Set theory and categories

1 Naive set theory

1.1. If there is a set V such that $\forall x, x \in V$, let $A = \{x \in V \mid x \notin x\}$.
Now, $A \in A \iff A \in V \wedge A \notin A \iff A \notin A$ and this is a contradiction.
Finally, $\forall x, \exists y, y \notin x$.

1.2. $\mathscr{P}_\sim = \{[a]_\sim \mid a \in S\}$
$\sim$ is symmetric, so given $a \in S$, $a \in [a]_\sim$ so $[a]_\sim \neq \emptyset$
As $\sim$ is reflexive, $\bigcup_{a\in S}[a]_\sim = S$.
Finally, if $[a]_\sim \neq [b]_\sim$, let us prove that $[a]_\sim \cap [b]_\sim = \emptyset$.
Indeed, if $x \sim a$ and $x \sim b$ then by transitivity and symmetry, $a \sim b$ which is a contradiction.

1.3. If $\mathscr{P}$ is a partition on S, let us define $a \sim b$ iff $\exists P \in \mathscr{P}, a \in P \wedge b \in P$.
This relation is reflexive because $\forall p \in \mathscr{P}, p \neq \emptyset$
It is obviously symmetric.
If $a \sim b$ and $b \sim c$ then there is $P \in \mathscr{P}$ such that $a, b \in P$ and $P' \in \mathscr{P}$ such that $b, c \in P'$. $b \in P \cap P'$ so $P \cap P' \neq \emptyset$ so $P = P'$ and $a \sim c$.

Remark. Let $\mathbf{Eq}(S)$ the set of all relations on S and $\mathfrak{P}$ the set of all partitions of S.

$$\begin{aligned} \mathbf{Eq}(S) &\to \mathfrak{P} \\ \sim &\mapsto \mathscr{P}_\sim \end{aligned}$$

is a bijection which has as inverse the 'operation' of the previous exercise.

1.4. An equivalence relation $\sim$ on $\{1,2,3\}$ is a subset of $\{1,2,3\}^2$ which necessarily includes the diagonal $\{(1,1),(2,2),(3,3)\}$.
Then we have three choices as far as distinct unordered couples are concerned:

- There is no distinct a and b such that $a \sim b$,
- There are distinct a, b, c such that $a \sim b$ and $b \sim c$, in which case, we necessarily have $a \sim c$,
- There is only two distinct a and b such that $a \sim b$. We have then $\binom{3}{2} = 3$ choices.

We can verify that in each case we indeed have an equivalence relation on $\{1,2,3\}$ so there are 5 equivalence relations on $\{1,2,3\}$.

1.5. Let R a relation on $\{1,2,3\}$ such that $R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}$
We can verify that R is reflexive, symmetric, but not transitive.
Trying to define $\mathscr{P}_R$ will not lead to a partition, as $[1]_R \neq [3]_R$ (because $(1,3)$ and $(3,1)$ are not in R) but $[1]_R \cap [3]_R \neq \emptyset$ (because $2 \in [1]_R \cap [3]_R$).

1.6. $a - a = 0 \in \mathbb{Z}$ so $\sim$ is reflexive.
If $a - b \in \mathbb{Z}$ then $b - a \in \mathbb{Z}$, so it is symmetric.
If $a - b \in \mathbb{Z}$ and $b - c \in \mathbb{Z}$ then $a - c = a - b + (b - c) \in \mathbb{Z}$, so $\sim$ is an equivalence relation.

$$\begin{aligned} f : \mathbb{R} &\to [0,1[\\ x &\mapsto x - \lfloor x \rfloor \text{ (where } \lfloor x \rfloor \text{ is the floor function)} \end{aligned}$$

f is clearly surjective.
We remark that $x \sim y$ iff $f(x) = f(y)$.
We can thus define:

$$\begin{aligned} f' : \mathbb{R}/\sim &\to [0,1[\\ [x]_\sim &\mapsto f(x) \end{aligned}$$

f' is a bijection.

Because $\sim$ is an equivalence relation, $\approx$ is clearly an equivalence relation.
Let

$$\begin{aligned} g : \mathbb{R}^2 &\to [0,1[^2 \\ (x,y) &\mapsto (f(x), f(y)) \end{aligned}$$

Because f is a bijection it is easy to prove that g is a bijection.

Remark. We could have factored f using Claim 5.5, page 34.

The situation for $\mathbb{R}^2$ can be represented by the following diagram:

$$\begin{tikzcd} {[0,1[} & {[0,1[^2} \arrow[l] \arrow[r] & {[0,1[} \\ \mathbb{R} \arrow[u, "f"] & \mathbb{R}^2 \arrow[l] \arrow[r] \arrow[u, dashed, "\exists! f \& f"] & \mathbb{R} \arrow[u, "f"'] \end{tikzcd}$$

$(f, g) \mapsto f \parr g$ is bijective and $f \parr f$ is bijective iff f is bijective.

2 Functions between sets

2.1. Let us prove on the set of bijections from a set S to a set S' has cardinality $n!$ if $|S| = |S'| = n$.
By induction on $n \in \mathbb{N}$:
If $n = 0$, $\emptyset : S \to S'$ is the only bijection. The statement is true.
Suppose the statement is true for any set S, S' such that $|S| = |S'| = n$. Let T, T' be such that $|T| = |T'| = n + 1$.
Pick $x \in T$. A bijection from T to T' is the datum of an image $x' \in T'$ for x and a bijection from $T \setminus \{x\} \to T' \setminus x'$. There are $n!$ such bijections.
Finally, there are $(n+1)n! = (n+1)!$ bijections from $T \to T'$

Remark. The previous situation is represented by the following diagram:

$$\begin{tikzcd} \{x'\} \arrow[r] & T' \cong \{x'\} \coprod T' \setminus \{x'\} & T' \setminus \{x'\} \arrow[l] \\ \{x\} \arrow[r] \arrow[u, "f"] & T \cong \{x\} \coprod T \setminus \{x\} \arrow[u, dashed, "\exists! f \parr g"] & T \setminus \{x\} \arrow[l] \arrow[u, "g"'] \end{tikzcd}$$

$f \parr g$ is bijective iff f and g are bijective.

2.2. If f has a right inverse g, $f \circ g = \mathrm{id}_B$.
$f(g(B)) = (f \circ g)(B) = \mathrm{id}_B(B) = B$, but $g(B) \subseteq A$ so $f(A) = B$. Hence f is surjective.
If f is surjective, let $\alpha : \mathscr{P}(A) \setminus \{\emptyset\} \to A$ be a choice function.
$\forall x \in B, \exists z \in A, f(z) = x$. Let $g(x) = \alpha(\{z \in A \mid f(z) = x\})$.
It is clear that $f \circ g = \mathrm{id}_B$.

2.3. If $f : A \to B$ is a bijection, $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$ but Corollary 2.2 tells us that if a function has a two-sided inverse, then it is a bijection so f^{-1} is a bijection.
If $f : A \to B$ and $g : B \to C$ are bijections, $g \circ f : A \to C$ is also a bijection.
Indeed, $(g \circ f) \circ (f^{-1} \circ g^{-1}) = \mathrm{id}_C$ and $(f^{-1} \circ g^{-1}) \circ (g \circ f) = \mathrm{id}_A$.

2.4. Let $\sim$ be the relation 'isomorphism'. $X \sim X$ because $id_X : X \to X$ and id_X is clearly a bijection.
$X \sim Y$ implies that $Y \sim X$ because the inverse of a bijection is a bijection.
$\sim$ is transitive because the composition of two bijections is a bijection.

2.5. $f : A \to B$ is an epimorphism iff $\forall g, g' : B \to C, g \circ f = g' \circ f \implies g = g'$.
Let us prove that f is an epimorphism iff it is surjective.
If f is surjective. Let $y \in B$, there is $x \in A$ such that $y = f(x)$.
$g(y) = g \circ f(x) = g' \circ f(x) = g'(y)$. Hence $g = g'$.
If f is not surjective then, there is $y \in B \setminus f(A)$. Let $g, g' : B \to \{0, 1\}$ such that $\forall x \in B, g(x) = 0$, $\forall x \in B \setminus \{y\}, g'(x) = 1$ and $g'(y) = 0$.
$g \circ f = g' \circ f$ but $g \neq g'$ so f is not an epimorphism.

2.6. If $f : A \to B$. Define $\gamma : A \to A \times B$ such that $\gamma(a) = (a, f(a))$. It is obvious that $f \circ \gamma = \mathrm{id}_A$.

Remark. The situation is represented by the following commutative diagram:

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B \\
& A \arrow[ul, "id_A"] \arrow[u, dashed] \arrow[ur, "f"'] &
\end{tikzcd}$$

2.7. Let $\Gamma_f \subseteq A \times B$. Let $i : \Gamma_f \to A \times B$ the inclusion map.
$\pi_A \circ i : \Gamma_f \to A$ is a bijection.
Indeed, $\forall a \in A, \exists! y \in B, (a, y) \in \Gamma_f$
That is: $\forall a \in A, \exists! z \in \Gamma_f, \pi_A(z) = a$ (this z is (a, y))

2.8.

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{C} \\ r &\mapsto e^{2\pi i r} \end{aligned}$$

The canonical decomposition of f gives:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, two heads, "\pi"] \arrow[rrr, bend left, "f"] & \mathbb{R}/\sim \arrow[r, "\tilde{f}"] & \mathrm{im} f \arrow[r, hook] & \mathbb{C}
\end{tikzcd}$$

$$\begin{aligned} \alpha : \mathbb{R}/\sim &\to S_2 \\ [r]_\sim &\mapsto e^{2\pi i r} \end{aligned}$$

where S_2 is the circle with radius 1 in $\mathbb{C}$ is well defined and is a bijection.
An alternative decomposition would have been:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, two heads, "\alpha\circ\pi"] \arrow[rrr, bend left, "f"] & S_2 \arrow[r, "\tilde{f}\circ\alpha^{-1}"] & \operatorname{im} f \arrow[r, hook] & \mathbb{C}
\end{tikzcd}$$

This exercise matches Exercise 1.6.

2.9. Suppose $A' \xrightarrow[\cong]{\alpha} A''$, $B' \xrightarrow[\cong]{\beta} B''$, $A' \cap B' \neq \emptyset$ and $A'' \cap B'' \neq \emptyset$.

Let

$$\begin{aligned}
\gamma : A \cup B &\to A'' \cup B'' \\
x &\mapsto \alpha(x) \text{ if } x \in A \\
y &\mapsto \beta(y) \text{ if } x \in B
\end{aligned}$$

γ is well defined because $A \cap B = \emptyset$, surjective because α and β are surjective, and injective because α and β are injective and because $A'' \cap B'' = \emptyset$.

Hence, however we take 'copies' of A and B such that their copies are disjoint, we always obtain isomophic sets.

2.10. Let us prove by induction on n that if $|A| = n$, we have $|B^A| = |B|^{|A|}$.

$|B^\emptyset| = \{\emptyset\} = |B|^\emptyset$ so the statement is true for $n = 0$

If $|A| = n+1$, pick $x \in A$

A function from $A \to B$ is the datum of a function of $A \to B \setminus \{x\}$ and image $x' \in B$ for x. There are $|B|^{|A\setminus\{x\}|} = |B|^{|A|-1}$ such functions.

Hence $|B|^{|A|} = |B||B|^{|A|-1} = |B|^{|A|}$.

Remark. The previous situation is represented by the following diagram:

$$\begin{tikzcd}
\{x'\} \arrow[r] & B \cong \{x'\} \coprod B \setminus \{x'\} & B \setminus \{x'\} \arrow[l] \\
\{x\} \arrow[r] \arrow[u, "f"] & A \cong \{x\} \coprod A \setminus \{x\} \arrow[u, dashed, "\exists! f \parr g"] & A \setminus \{x\} \arrow[l] \arrow[u, "g"']
\end{tikzcd}$$

$(f,g) \mapsto f \parr g$ is a bijection.

2.11. Let

$$\begin{aligned}
f : \mathscr{P}(A) &\to 2^A \\
X \mapsto \quad & x \mapsto 1 \text{ if } x \in X \\
& x \mapsto 0 \text{ otherwise}
\end{aligned}$$

$f(X)$ is called the characteristic function of X. It is easy to prove that f is a bijection.

3 Categories

3.1. Let $\circ'$ be the operation of $\mathbf{C}^{op}$. If $f : A \to B$ and $g : B \to C$ in $\mathbf{C}$, $f \circ' g = g \circ f$.

$(f \circ' g) \circ' h = h \circ (g \circ f) = (h \circ g) \circ f = f \circ' (g \circ' h)$

If $f : A \to B$ in $\mathbf{C}$: $f \circ' \mathrm{id}_B = \mathrm{id}_B \circ f = f$ and $\mathrm{id}_A \circ' f = f \circ A = f$

Finally, it is clear that $\mathrm{Hom}_{\mathbf{C}^{op}}(A,B) = \mathrm{Hom}_{\mathbf{C}^{op}}(A',B')$ iff $A = A'$ and $B = B'$.

3.2. If A is finite, as $\mathrm{End}_{\mathbf{Set}}(A) = A^A$, $|\mathrm{End}_{\mathbf{Set}}(A)| = |A|^{|A|}$ (see Exercise 2.10).

3.3. $1_a = (a,a)$ is the identity of a. Indeed, let $f : a \to b$, Necessarily, if f exists, $f = (a,b)$.

$f \circ 1_a = (a,b) \circ (a,a) = (a,b) = f$

Similarly, if $f : b \to a$, $1_a \circ f = f$.

3.4. $<$ is not reflexive, so for example, $(3,3)$ would not be in the category. But, necessarily, it should be as the only possible choice for 1_3 (think about it).

3.5. For every preorder, we can define a category in the style of Example 3.3.

Remark. In fact, writing $\mathbf{Proset}\restriction_A$ the set of all categories whose set of objects is A, whose set of morphisms is a subset of A^2 and such that $\forall a, b \in A, \exists^{\leq 1} f : a \to b$, there is a bijection :

$$\begin{aligned} \phi : \text{Preorders on } A &\to \mathbf{Proset}\restriction_A \\ R &\mapsto \text{The category such that} \qquad \begin{aligned} \mathrm{Hom}(a,b) &= \{(a,b)\} \text{ if } aRb \\ \mathrm{Hom}(a,b) &= \emptyset \text{ otherwise} \\ (b,c) \circ (a,b) &= (a,c) \end{aligned} \end{aligned}$$

R is an order iff $\phi(R)$ is skeletal.
R is an equivalence relation iff $\phi(R)$ is a groupoid.

3.6. The set of matrices with n rows and m columns is $\mathbb{R}^{m \times n}$ where $m = \{0, \ldots, m-1\}$ and $n = \{0, \ldots, n-1\}$ and $\times$ is the product of sets.

Hence, $\mathbb{R}^{0 \times n} = \mathbb{R}^{\emptyset \times n} = \mathbb{R}^{\emptyset} = \{\emptyset\}$. There is only one matrix with 0 lines is $\emptyset$, and, similarly, it is the only matrix with 0 columns.

If $A : m \to n$ (A has n lines and m columns) and $B : n \to p$ (B has p lines and n columns), then $B \circ A = B \times A$.

The product of matrices is associative and $\mathrm{I_n} = \mathrm{id_n}$.

It feels familiar. If we define the category $\mathbb{N}$ with $\mathrm{Obj}(\mathbb{N}) = \mathbb{N}$ and with $\mathrm{Hom}_{\mathbb{N}}(n,m) = m^n$. In both of the categories, it is true that if $\mathrm{dom}(f) = \mathrm{im}(f)$ then f is a monomorphism iff it is an epimorphism.