MS-E1653 Finite element method **Hannukainen / Kuutela**

Problem set 4

Return written solutions either to the box outside M330a or online by Monday (11.2.) by 23:59 in order to obtain 0–2 additional points through grading.

1. Let $x_i, i = 1, \ldots, N_n$ be the nodes of your $1D$ - finite element mesh. On element $K = (x_{n_1}^K, x_{n_2}^K)$ define node

$$x_{K+N_n} = \frac{x_{n_1}^K + x_{n_2}^K}{2}, \quad K = 1, \ldots, N_e$$

where N_e is the number of elements. Denote the space of second order piecewise polynomials by

$$V_h = \{u \in H_0^1(0,1) \mid u_{|K} \in P^2(K) \text{ for } K = 1, \ldots, N_e \}.$$

One possible basis for the space V_h is

$$V_h = \operatorname{span}\{\varphi_1, \ldots, \varphi_{N_n}, \varphi_{N_n+1}, \ldots, \varphi_{N_n+N_e}\}$$

where functions φ_i are defined such that

$$\varphi_i(x_j) = \delta_{ij}, \quad i, j = 1, \ldots, N_n + N_e.$$

(a) Let $\hat{K} = (0,1)$ and $K = (x_{n_1}^K, x_{n_2}^K)$. Define a linear mapping $r_K(t)$ such that $r_K(\hat{K}) = K$.

(b) Let $K = (x_{n_1}^K, x_{n_2}^K)$ and r_K be as defined in (a). Give a formula for reference basis functions $\hat{\varphi}_i : \hat{K} \to \mathbb{R}$ such that

$$\begin{aligned}\hat{\varphi}_1(\hat{t}) &= \varphi_{n_1}(r(\hat{t}))\\ \hat{\varphi}_2(\hat{t}) &= \varphi_{n_2}(r(\hat{t}))\\ \hat{\varphi}_3(\hat{t}) &= \varphi_{K+N_n}(r(\hat{t})).\end{aligned}$$

for $\hat{t} \in (0,1)$.

(c) Using the change-of-variables from element $K = (x_{n_1}^K, x_{n_1}^K)$ to $\hat{K}$, the mapping from (a), and basisfunctions from (b) give a formula for computing the integral

$$A_{ij} = \int_K \varphi_i'(t)\varphi_j'(t)\, dt \quad \text{where} \quad i, j \in \{n_1, n_2, K + N_n\}.$$

using some quadrature rule over reference element $(0,1)$.

2. Consider the problem: find $u \in H_0^1(0,1)$ such that

$$\int_0^1 u'v' \, dt = \int_0^1 \sin \pi t v \, dt \quad \forall v \in H_0^1(0,1).$$

The exact solution to this problem is $u = \pi^{-2} \sin \pi t$.

(a) Modify the 1D-finite element solver written as a lecture example to use the space

$$V_h = \text{span}\{\varphi_1, \ldots, \varphi_{N_n}, \varphi_{N_n+1}, \ldots, \varphi_{N_n+N_e}\}$$

in which $\varphi_1, \ldots, \varphi_{N_n+N_e}$ are as defined in Problem 1. One way to manage the indices is to define a mapping from the "local" basis function numbering to the global numbering. In the example solver, this would be

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l2g = [1:(N_n-1) ; 2:(N_n) ; N_n + [1:N_e] ];
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(b) Solve the problem using different numbers of elements. Plot the error in H^1-norm as a function of the mesh size

$$h = \max_K |x_{n_1}^K - x_{n_2}^K|.$$

using logarithmic plot.

3. Let $\Omega = (0, L)$ and consider the Poincare-Friedrichs inequality : There exists $C(L) > 0$ independent on u, but dependent on L such that

$$\left(\int_0^L u^2 \, dx\right)^{1/2} \le C(L) \left(\int_0^L (u')^2 \, dx\right)^{1/2} \quad \forall u \in H_0^1(\Omega).$$

(a) Show that the smallest possible constant $C(L)$ in the P-F inequality can be characterized as

$$C(L)^{-2} = \min_{u \in H_0^1(\Omega)} G(u),$$

in which

$$G(u) = \frac{\int_0^L (u')^2 \, dx}{\int_0^L u^2 \, dx}.$$

(b) Show that C^{-2} is the smallest eigenvalue λ_i of the problem: find $(\lambda_i, u_i) \in (\mathbb{R}, H_0^1(\Omega) \setminus \{0\})$ such that

$$\int_0^L u_i'v' \, dx = \lambda_i \int_0^L u_i v \, dx \quad \forall v \in H_0^1(\Omega).$$

Hint : the minimum is located at the critical point u_i that can be characterized as $\frac{d}{dt}G(u_i + tv)_{|t} = 0 \quad \forall v \in H_0^1(\Omega)$. Also, note the each eigenvalue satisfies

$$\lambda_i = \frac{\int_0^L (u_i')^2 \, dx}{\int_0^L u_i^2 \, dx}$$

in which u_i is some eigenfunction corresponding to λ_i.

(c) Use finite element method to approximate the constant $C(L)$. Plot the constant as a function of L. Study, the dependency between $C(L)$ and L using different different plots.

Hint: The eigenvalue problem that you need to solve is $A\boldsymbol{x} = \lambda M\boldsymbol{x}$, in which $A_{ij} = (\varphi_j', \varphi_i')$ and $M_{ij} = (\varphi_j, \varphi_i)$. The smallest eiganvalue of such generalised eigenvalue problem can be solved with command $eigs(A, M, 1, 'SM')$.