Homework # 2, Q6 (a-d) Solution

6. Consider the Poisson loglinear model with

$$\log \mu_{ijk} = \lambda + \lambda_i^X + \lambda_j^Y + \lambda_k^Z + \lambda_{ik}^{XZ} + \lambda_{jk}^{YZ}.$$

(a) Describe in words the type of independence assumed in this model.

This model assumes that categorical variables X and Y are conditionally independent given Z. That is, $\pi_{ij|k} = \pi_{i+|k}\pi_{+j|k}$ for all i, j, k.

(b) Write the loglikelihood for this model.

$$\begin{aligned}
l(\{\mu_{ijk}\}|\{n_{ijk}\}) &= \log\left[\prod_i\prod_j\prod_k \frac{exp^{-\mu_{ijk}}\mu_{ijk}^{n_{ijk}}}{n_{ijk}!}\right] \\
&= \sum_i\sum_j\sum_k [-\mu_{ijk} + n_{ijk}\log\mu_{ijk} - \log(n_{ijk}!)] \\
&\propto \sum_i\sum_j\sum_k \left[-e^{\lambda+\lambda_i^X+\lambda_j^Y+\lambda_k^Z+\lambda_{ik}^{XZ}+\lambda_{jk}^{YZ}} + n_{ijk}(\lambda + \lambda_i^X + \lambda_j^Y + \lambda_k^Z + \lambda_{ik}^{XZ} + \lambda_{jk}^{YZ})\right] \\
&= n\lambda + \sum_i n_{i++}\lambda_i^X + \sum_j n_{+j+}\lambda_j^Y + \sum_i n_{++k}\lambda_k^Z + \sum_i\sum_k n_{i+k}\lambda_{ik}^{XZ} \\
&\quad + \sum_j\sum_k n_{+jk}\lambda_{jk}^{YZ} - \sum_i\sum_j\sum_k e^{\lambda+\lambda_i^X+\lambda_j^Y+\lambda_k^Z+\lambda_{ik}^{XZ}+\lambda_{jk}^{YZ}}
\end{aligned}$$

(c) Find the set of likelihood equations for this model.

$$\begin{aligned}
\frac{\partial l}{\partial \lambda} &= n - \sum_i\sum_j\sum_k e^{\lambda+\lambda_i^X+\lambda_j^Y+\lambda_k^Z+\lambda_{ik}^{XZ}+\lambda_{jk}^{YZ}} = n - \sum_i\sum_j\sum_k \mu_{ijk} = 0 \quad\Rightarrow\quad \hat{\mu}_{+++} = n \\
\frac{\partial l}{\partial \lambda_i^X} &= n_{i++} - \sum_j\sum_k e^{\lambda+\lambda_i^X+\lambda_j^Y+\lambda_k^Z+\lambda_{ik}^{XZ}+\lambda_{jk}^{YZ}} = n_{i++} - \sum_j\sum_k \mu_{ijk} = 0 \quad\Rightarrow\quad \hat{\mu}_{i++} = n_{i++} \\
\frac{\partial l}{\partial \lambda_j^Y} &= n_{+j+} - \sum_i\sum_k e^{\lambda+\lambda_i^X+\lambda_j^Y+\lambda_k^Z+\lambda_{ik}^{XZ}+\lambda_{jk}^{YZ}} = n_{+j+} - \sum_i\sum_k \mu_{ijk} = 0 \quad\Rightarrow\quad \hat{\mu}_{+j+} = n_{+j+} \\
\frac{\partial l}{\partial \lambda_k^Z} &= n_{++k} - \sum_i\sum_j e^{\lambda+\lambda_i^X+\lambda_j^Y+\lambda_k^Z+\lambda_{ik}^{XZ}+\lambda_{jk}^{YZ}} = n_{++k} - \sum_i\sum_j \mu_{ijk} = 0 \quad\Rightarrow\quad \hat{\mu}_{++k} = n_{++k} \\
\frac{\partial l}{\partial \lambda_{ik}^{XZ}} &= n_{i+k} - \sum_j e^{\lambda+\lambda_i^X+\lambda_j^Y+\lambda_k^Z+\lambda_{ik}^{XZ}+\lambda_{jk}^{YZ}} = n_{i+k} - \sum_j \mu_{ijk} = 0 \quad\Rightarrow\quad \hat{\mu}_{i+k} = n_{i+k} \\
\frac{\partial l}{\partial \lambda_{jk}^{YZ}} &= n_{+jk} - \sum_i e^{\lambda+\lambda_i^X+\lambda_j^Y+\lambda_k^Z+\lambda_{ik}^{XZ}+\lambda_{jk}^{YZ}} = n_{+jk} - \sum_i \mu_{ijk} = 0 \quad\Rightarrow\quad \hat{\mu}_{+jk} = n_{+jk}
\end{aligned}$$

(d) Show that the likelihood equations derived from derivatives with respect to the interaction terms determine the MLEs. That is, show that they imply the other equations.

$$\frac{\partial l}{\partial \lambda_{jk}^{YZ}} = 0 \Rightarrow \frac{\partial l}{\partial \lambda_{k}^{Z}} = 0 : \qquad \hat{\mu}_{+jk} = n_{+jk} \Rightarrow \sum_{j} \hat{\mu}_{+jk} = \sum_{j} n_{+jk} \Rightarrow \hat{\mu}_{++k} = n_{++k}$$
$$\frac{\partial l}{\partial \lambda_{jk}^{YZ}} = 0 \Rightarrow \frac{\partial l}{\partial \lambda_{j}^{Y}} = 0 : \qquad \hat{\mu}_{+jk} = n_{+jk} \Rightarrow \sum_{k} \hat{\mu}_{+jk} = \sum_{k} n_{+jk} \Rightarrow \hat{\mu}_{+j+} = n_{+j+}$$
$$\frac{\partial l}{\partial \lambda_{ik}^{XZ}} = 0 \Rightarrow \frac{\partial l}{\partial \lambda_{i}^{X}} = 0 : \qquad \hat{\mu}_{i+k} = n_{i+k} \Rightarrow \sum_{k} \hat{\mu}_{i+k} = \sum_{k} n_{i+k} \Rightarrow \hat{\mu}_{++k} = n_{++k}$$
$$\frac{\partial l}{\partial \lambda_{ik}^{XZ}} = 0 \Rightarrow \frac{\partial l}{\partial \lambda} = 0 : \qquad \hat{\mu}_{i+k} = n_{i+k} \Rightarrow \sum_{i} \sum_{k} \hat{\mu}_{i+k} = \sum_{i} \sum_{k} n_{i+k} \Rightarrow \hat{\mu}_{+++} = n_{+++}$$

The likelihood equations based on the lower order derivatives can be derived from $\frac{\partial l}{\partial \lambda_{jk}^{YZ}} = 0$ and $\frac{\partial l}{\partial \lambda_{ik}^{XZ}} = 0$, making them redundant. Therefore, only the likelihood equations derived from the derivatives with respect to the interaction terms are necessary to solve for the MLE.