Calculus - Lecture 3

Continuity of a Function and Introduction to Derivatives

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Basic properties of limits

- Let's assume we have two functions $f(x)$ and $g(x)$, and that they both admit a limit

$$\lim_{x\to c} f(x) \qquad \lim_{x\to c} g(x)$$

- Limit of a sum of functions

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

- Limit of function scaled by a constant k

$$\lim_{x\to c} k * f(x) = k * \lim_{x\to c} f(x)$$

- Limit of product of functions

$$\lim_{x\to c} f(x) * g(x) = \lim_{x\to c} f(x) * \lim_{x\to c} g(x)$$

- Limit of quotient of functions (only posible if $\lim_{x\to c} g(x) \neq 0$)

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

- Limit of powers and roots of a function

$$\lim_{x\to c}(f(x))^n = (\lim_{x\to c} f(x))^n$$

Continuity of a function

When is a function continuous?

“The function does not have any interruptions”

“You can draw the graph of the function without ever raising your pen from the paper”

“It probably has something to do with limits… But I guess this time the value of the function at that point is important.”

Continuity of a function

- A function $f(x)$ defined on an open interval containing $x = c$ is **continuous** at $x = c$ if:

$$\lim_{x \to c} f(x) = f(c)$$

But…

1) If the limit does not exist

$$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$$

or

2) The value of the limit is not equal to the value of the function at that point.

$$\neq$$

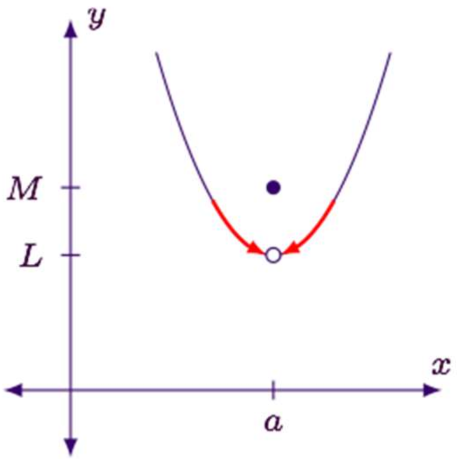
or

3) The function is not defined for $x = c$

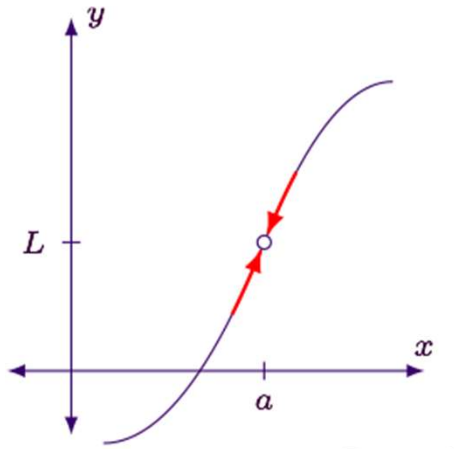
D: $\{x \in \mathbb{R}\} \setminus \{c\}$

The function $f(x)$ has a **discontinuity** (or is **discontinuous**) at $x = c$

Types of discontinuity



$\lim_{x \to c} f(x) \neq f(c)$



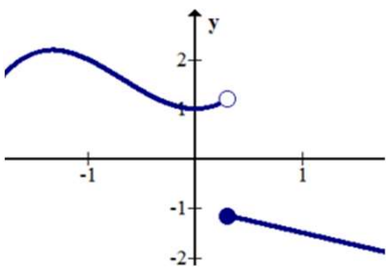
$f(c)$ is not defined

Removable discontinuity:

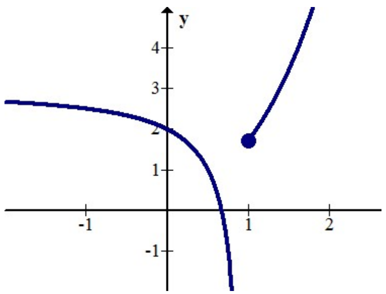
the limit exists but:

- it is not equal to $f(c)$
- or $f(c)$ is not defined.

Why Removable? Because the function can be made continuous by simply defining $f(c)$ to take the value of the limit.



Jump discontinuity



Infinity discontinuity

$$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$$

Non-Removable discontinuity:

the limit does not exist because its left and right limits are different.

Why Non-Removable? There is no way to avoid the discontinuity other than create a new function $f(x)$.

Continuity of a function

Definition

A function $f(x)$ defined on an interval I is **continuous on** I

if $f(x)$ is continuous at each point $x \in I$

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c) \quad x \in I$$

Note: there can be boundary conditions if the interval I is an open interval.
(i.e., if it does not include one of the boundary numbers).

Continuity of a function

From the definition of continuity and the basic properties of limits (remember last lecture) we can derive some properties of continuous functions.

- If $f(x)$ and $g(x)$ are continuous at $x = c$,
  then all these functions are continuous at $x = c$.

- If $f(x)$ and $g(x)$ are continuous on an interval I,
  then all these functions are continuous on I .

- $\underbrace{f(x) \pm g(x)}_{\checkmark}$
- $k * f(x)$

  for k constant
- $f(x) * g(x)$
- $\dfrac{f(x)}{g(x)}$

  $g(c) \neq 0$

Intermediate Value Theorem

Theorem

Let's assume that a *continuous* function $f(x)$ is defined on a *closed* interval $[a,b]$. Then, if w is a value between $f(a)$ and $f(b)$, there must exist at least one $c \in [a,b]$ for which $f(c) = w$.

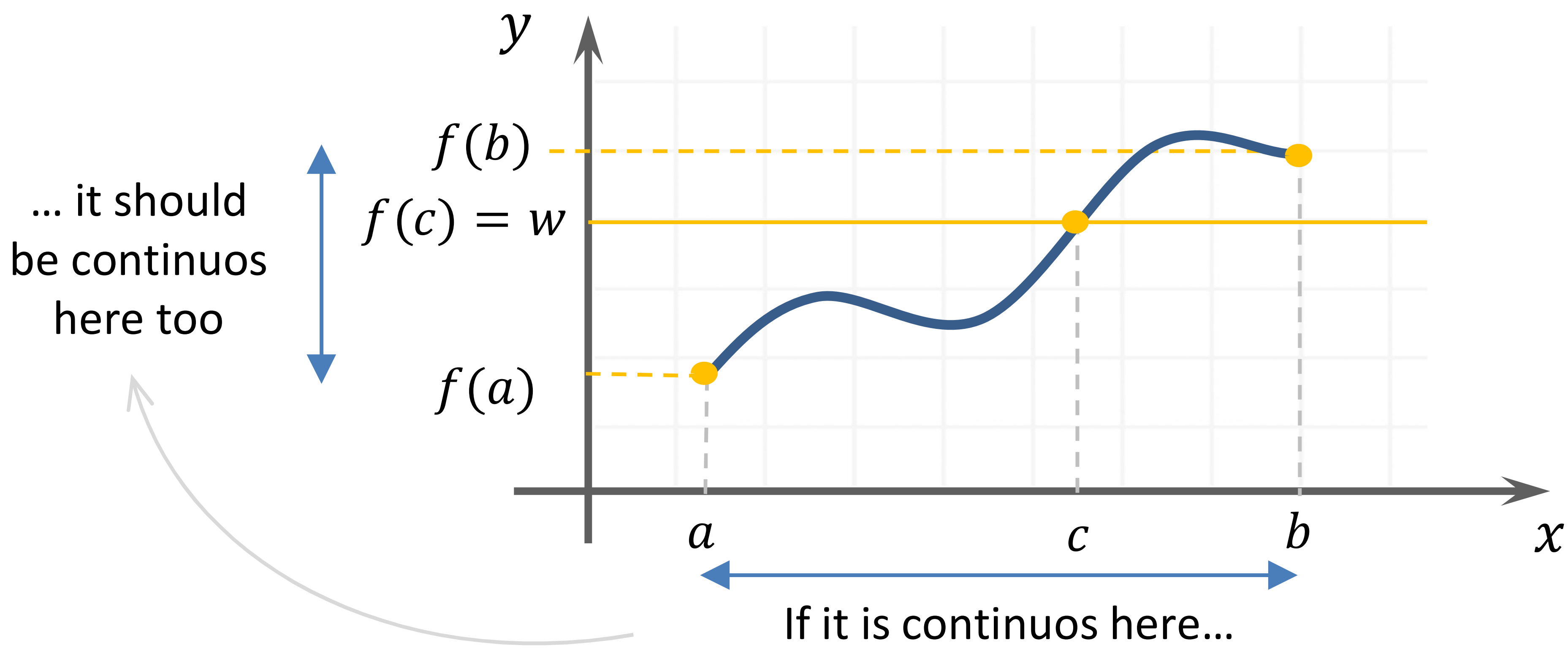
Google Translate

A continuous function "cannot skip values"

Intermediate Value Theorem

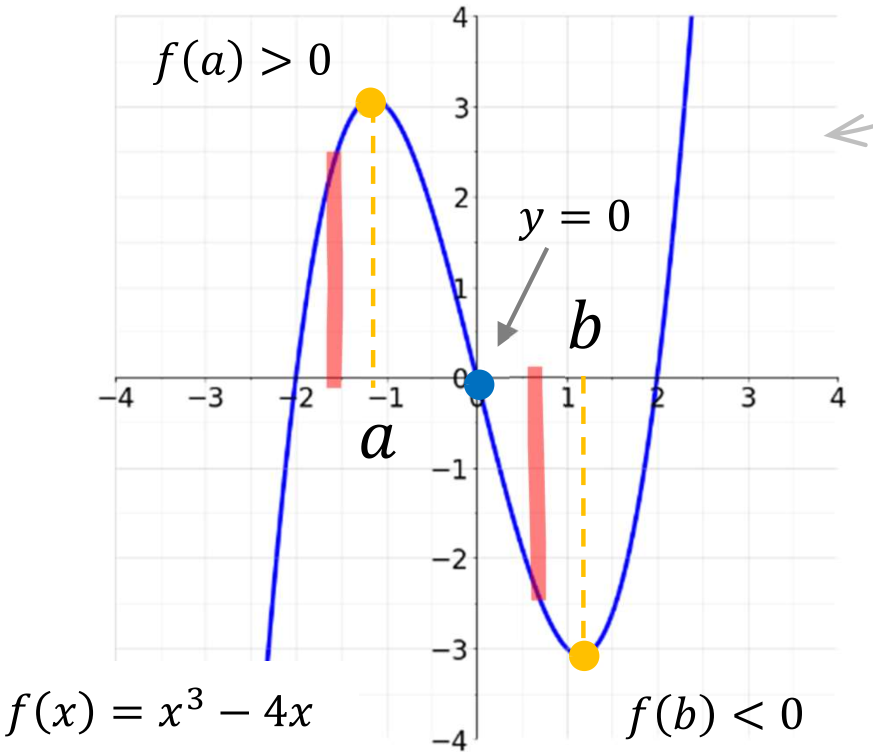
Theorem

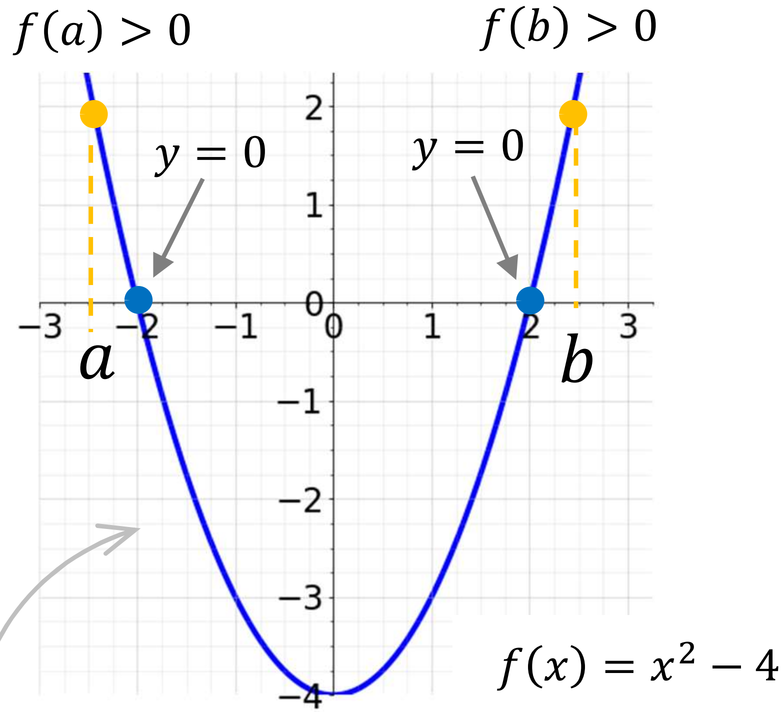
Let's assume that a continuous function $f(x)$ is defined on a *closed* interval $[a, b]$. Then, if w is a value between $f(a)$ and $f(b)$, there must exist at least one $c \in [a, b]$ for which $f(c) = w$.



Existence of zeros of a function

Corollary: for a continuous function $f(x)$ defined on a closed interval $[a, b]$, if $f(a)$ and $f(b)$ have different sign (i.e., $f(a) * f(b) < 0$) then by the Intermediate Value Theorem there must exist an x for which $f(x) = 0$.





What happens if the conditions do not hold?

There may still be roots in the interval (e.g., if the number of roots in the interval is even).

The Bisection Method

- The corollary can be used to numerically compute zeros of a function:
  - If the conditions apply:
    - Continuity over a closed interval $[a, b]$
    - Different sign $f(a) * f(b) < 0$
  - Then there must be at least a zero of the function within the interval.

What happens if we split the interval in two halves?

$$c = \frac{a+b}{2}$$

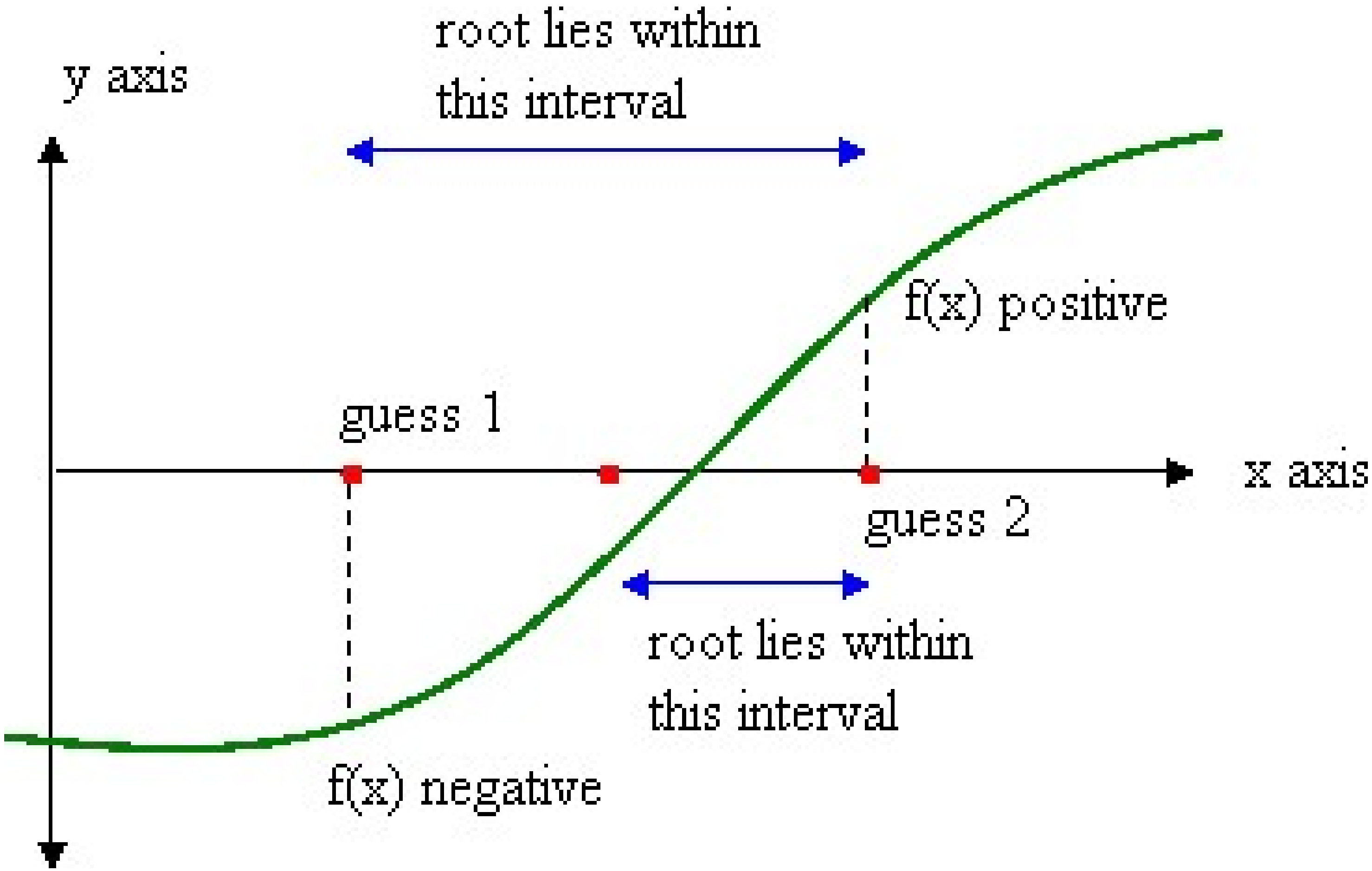
$f(c) = 0$ We found the zero of the function. Great!

$f(c) < 0$ or $f(c) > 0$ Then one of the intervals $[a, c]$ or $[c, b]$ will retain the different sign condition, and thus there exist a zero inside that interval.

$$c' = \frac{a+c}{2} \qquad c'' = \frac{c+b}{2}$$

By iterating the process, the interval tightens around the zero of the function.

The Bisection Method

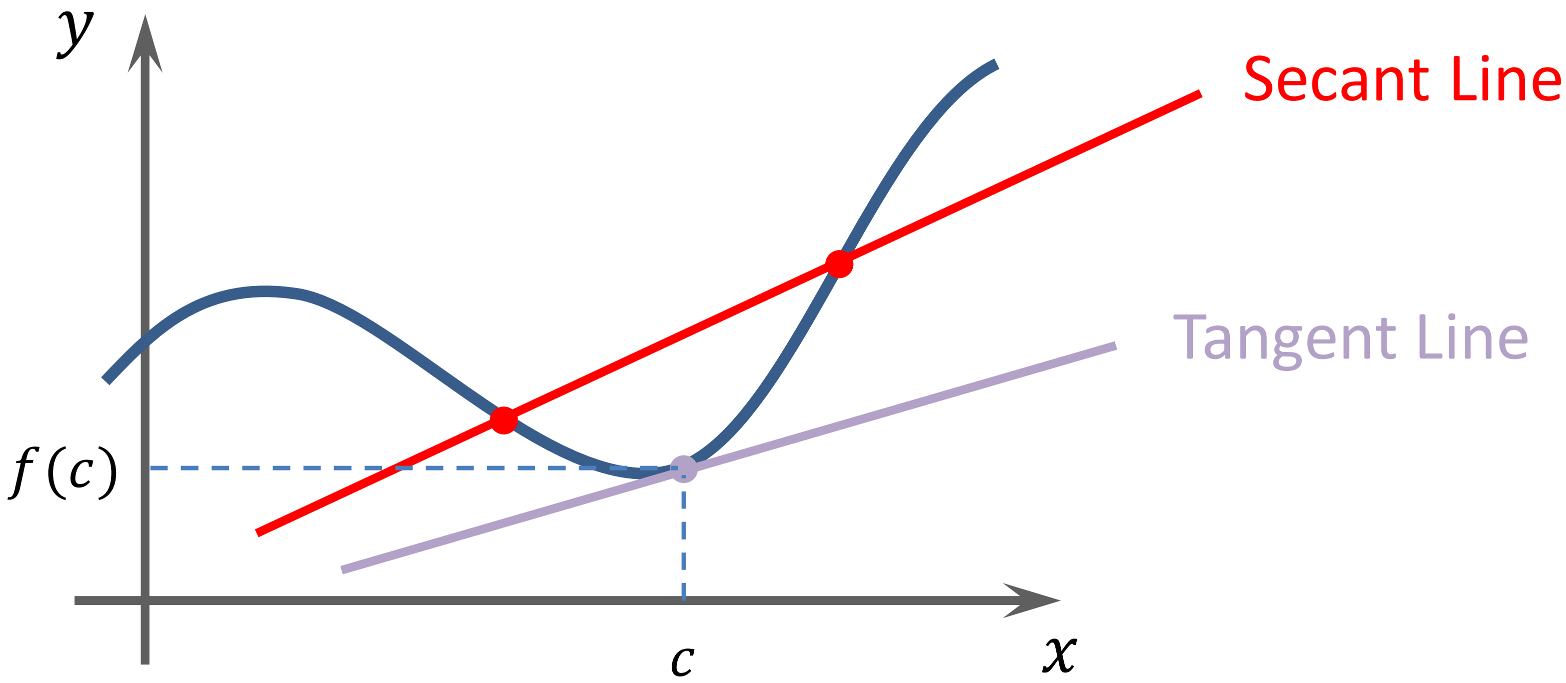


Derivatives

And finally… **Derivatives**

- Tangent lines and introduction to derivatives
  (and its graphical interpretation of derivatives)
- Calculating derivatives with limits
- Properties of derivatives
- Calculating derivatives with formulas
  (more about it next week, but best to start getting used to it!)

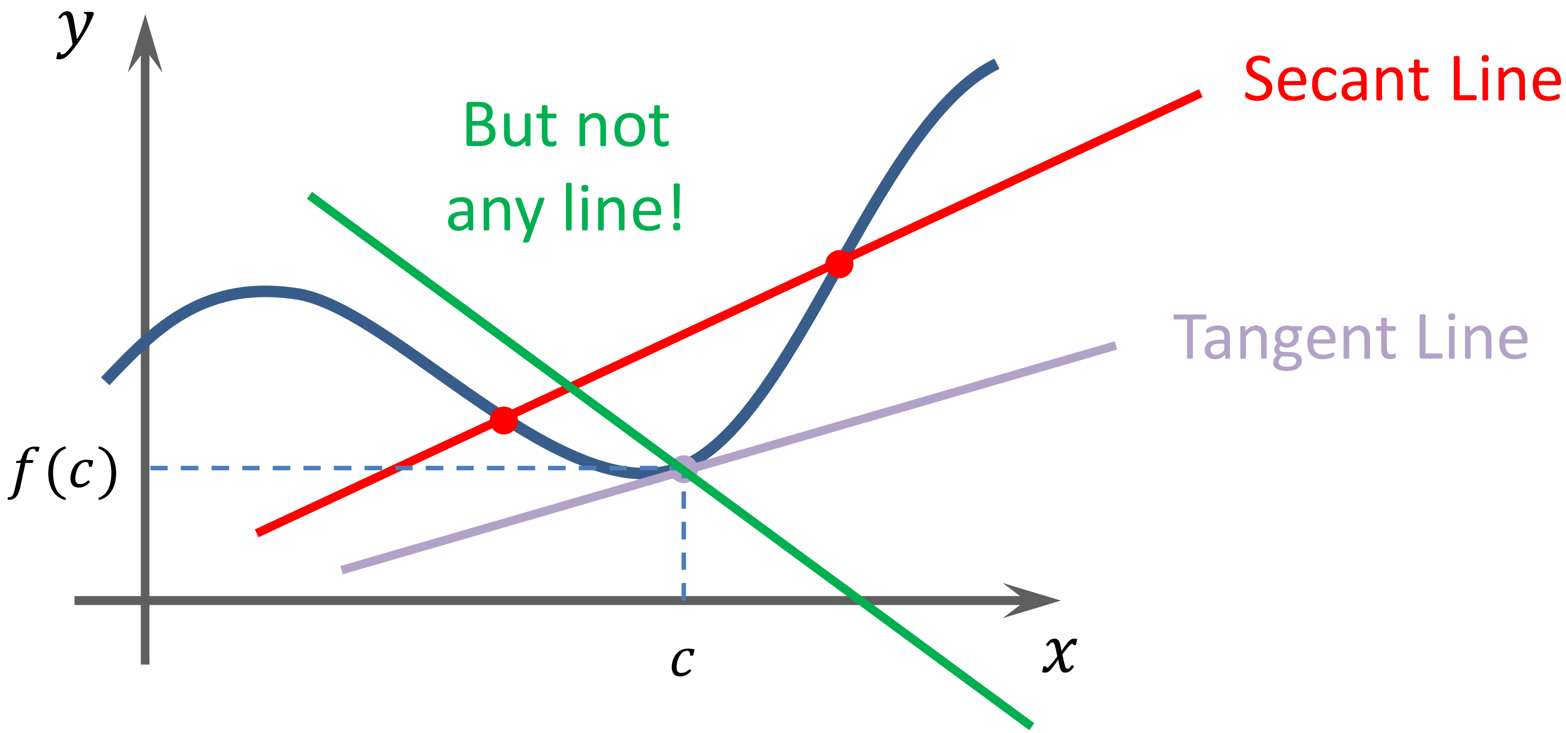
Tangent lines



- A secant line intersects with the curve at multiple points.
- A tangent is a line that "touches" the curve at a single point.

The tangent of a curve $f(x)$ at $x = c$ is the line that passes through the point $(c, f(c))$

Tangent lines

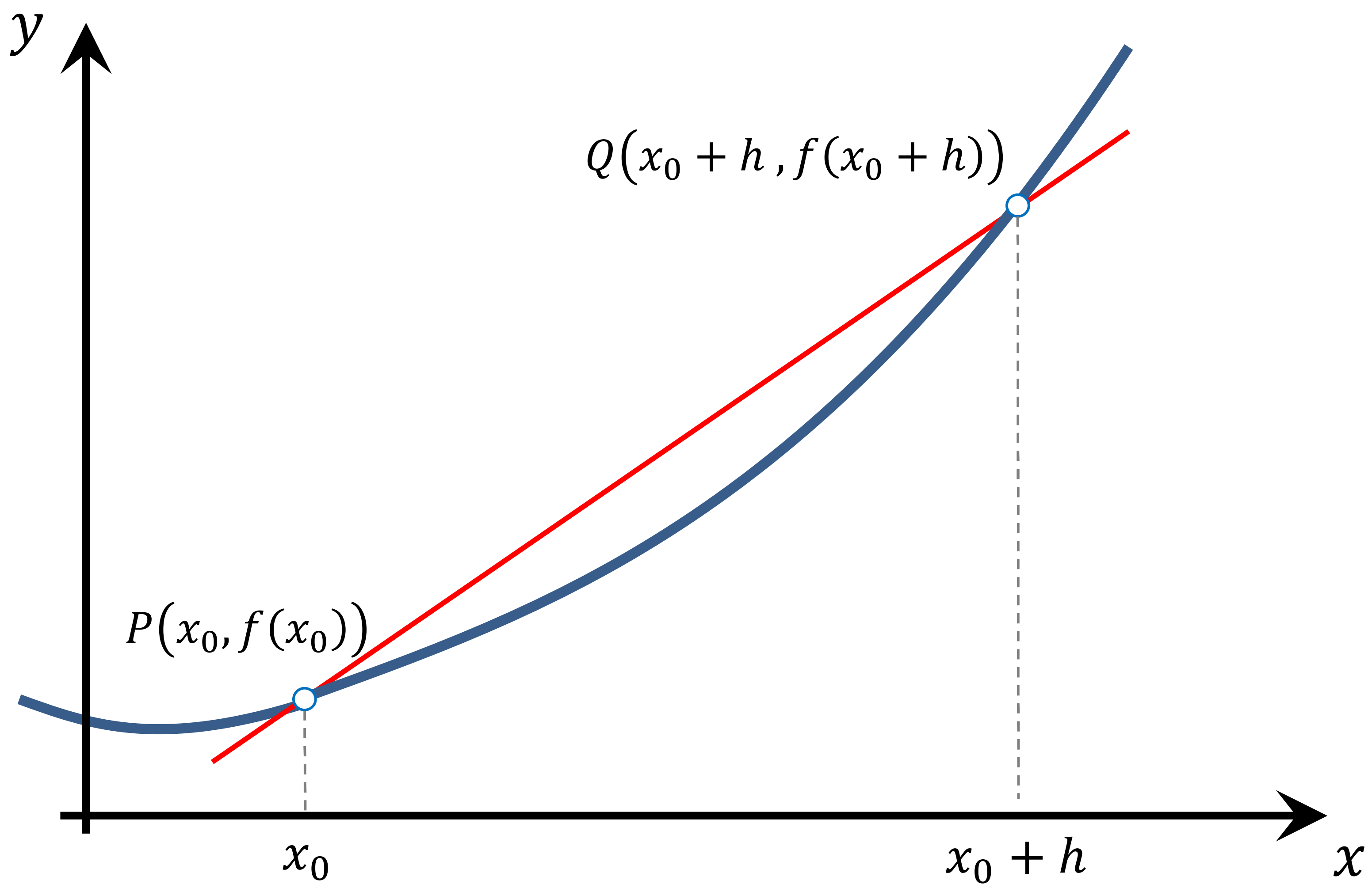


- A secant line intersects with the curve at multiple points.
- A tangent is a line that “touches” the curve at a single point.

The tangent of a curve $f(x)$ at $x = c$ is the line that passes through the point $(c, f(c))$, and by definition, it has the same slope that the curve of the function at that point.

Tangent lines

- Tangents are "limit secants" where the points of intersection with the curve are infinitely close to each other.



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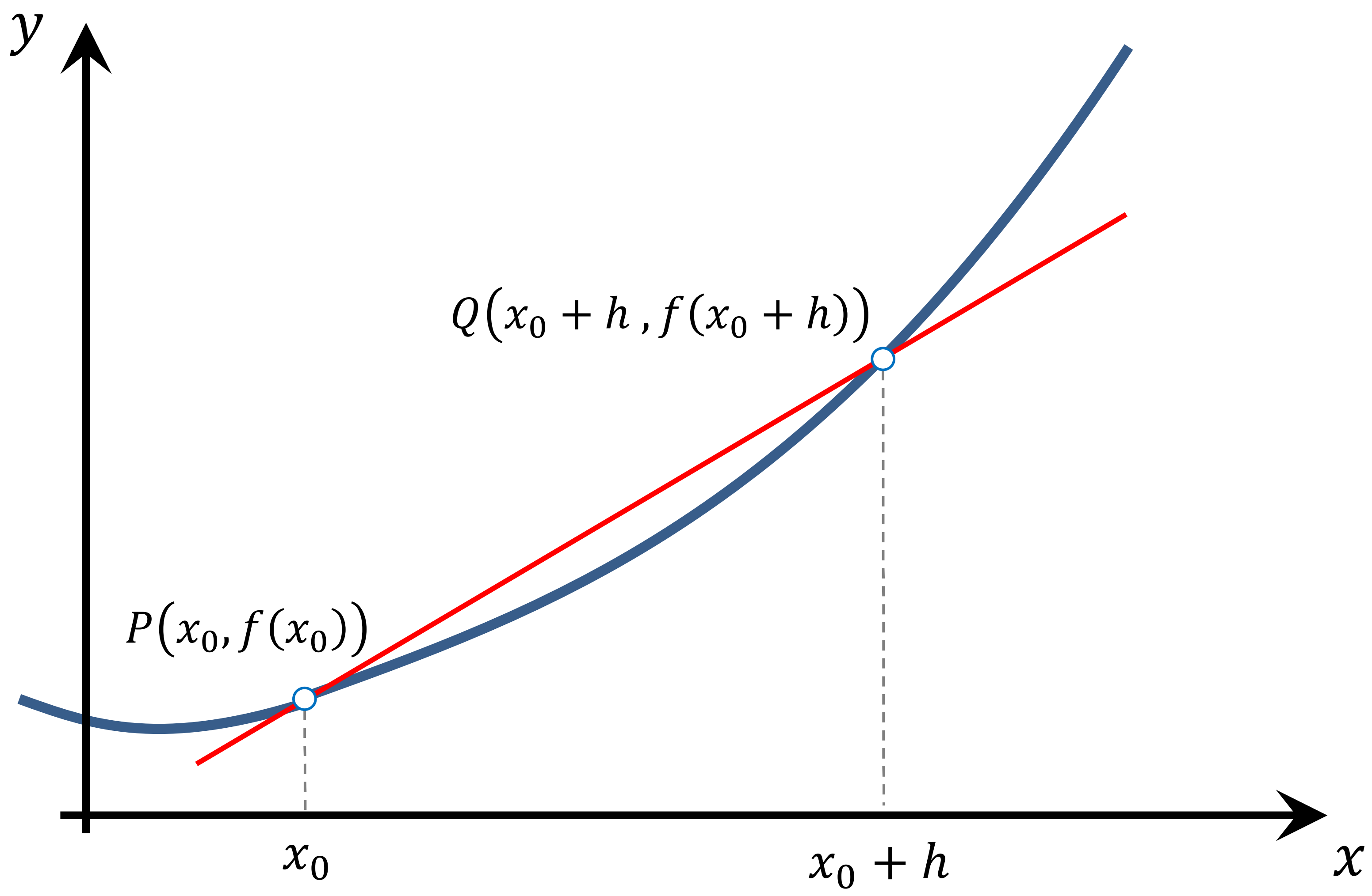

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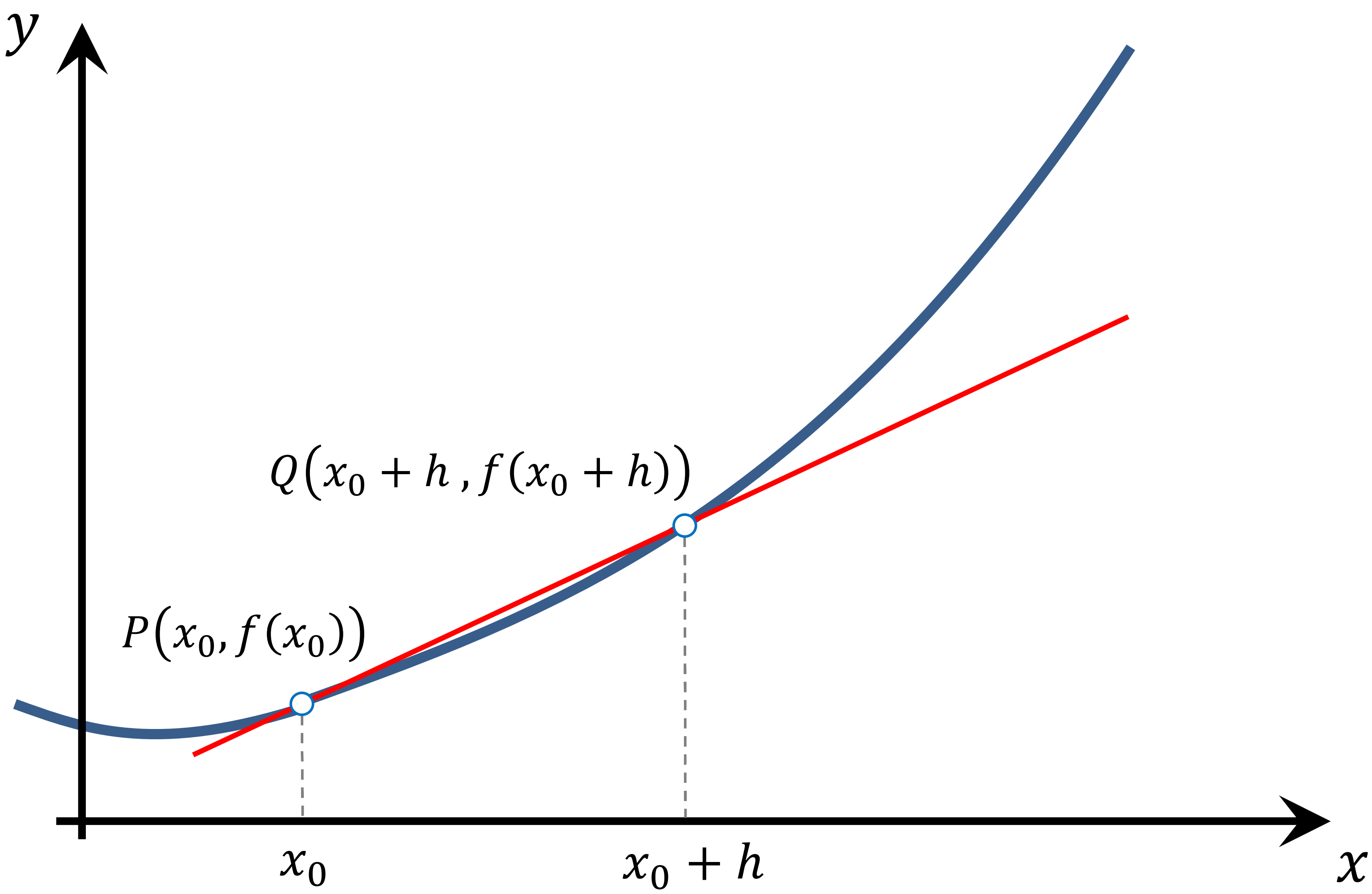

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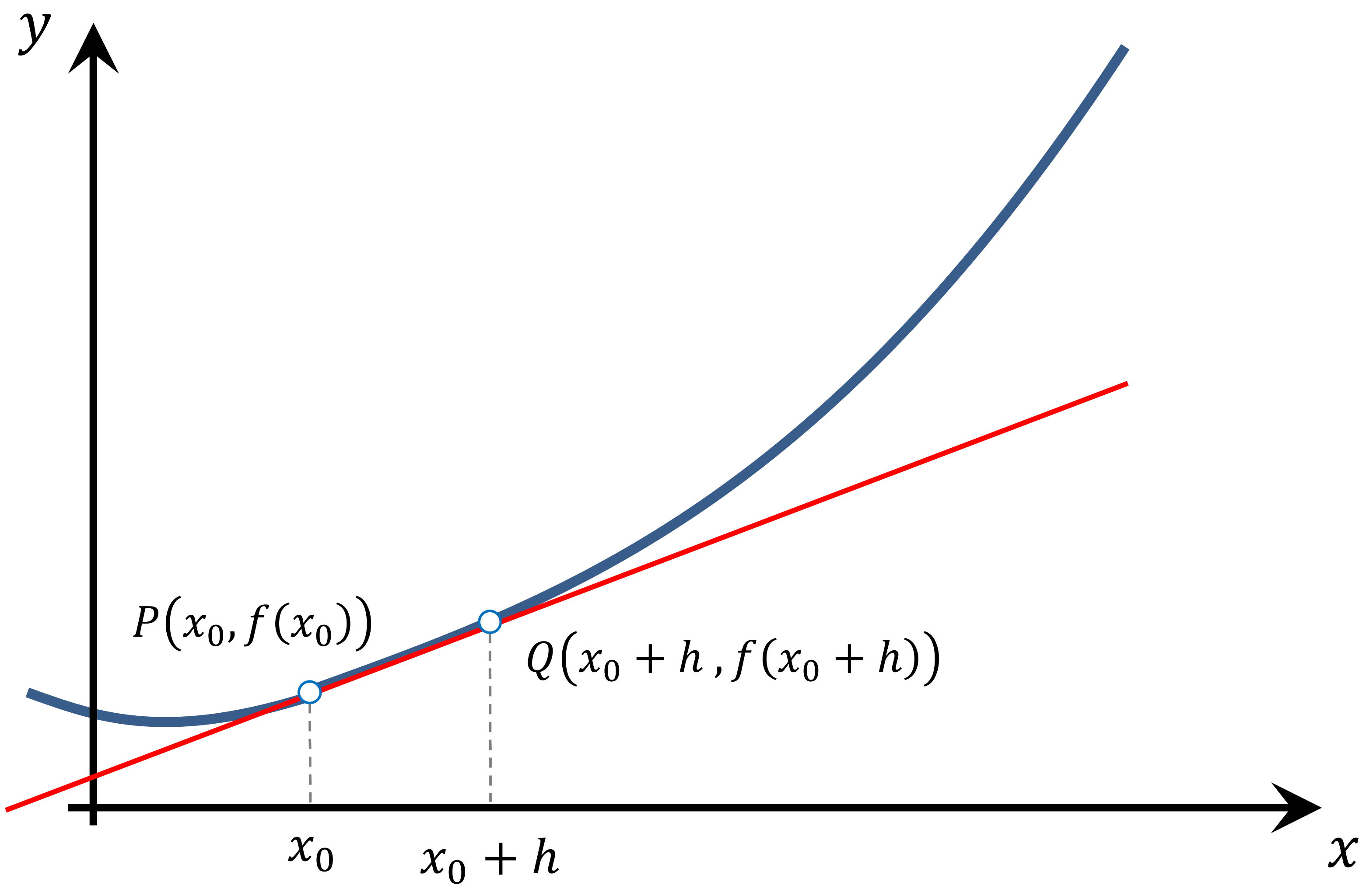
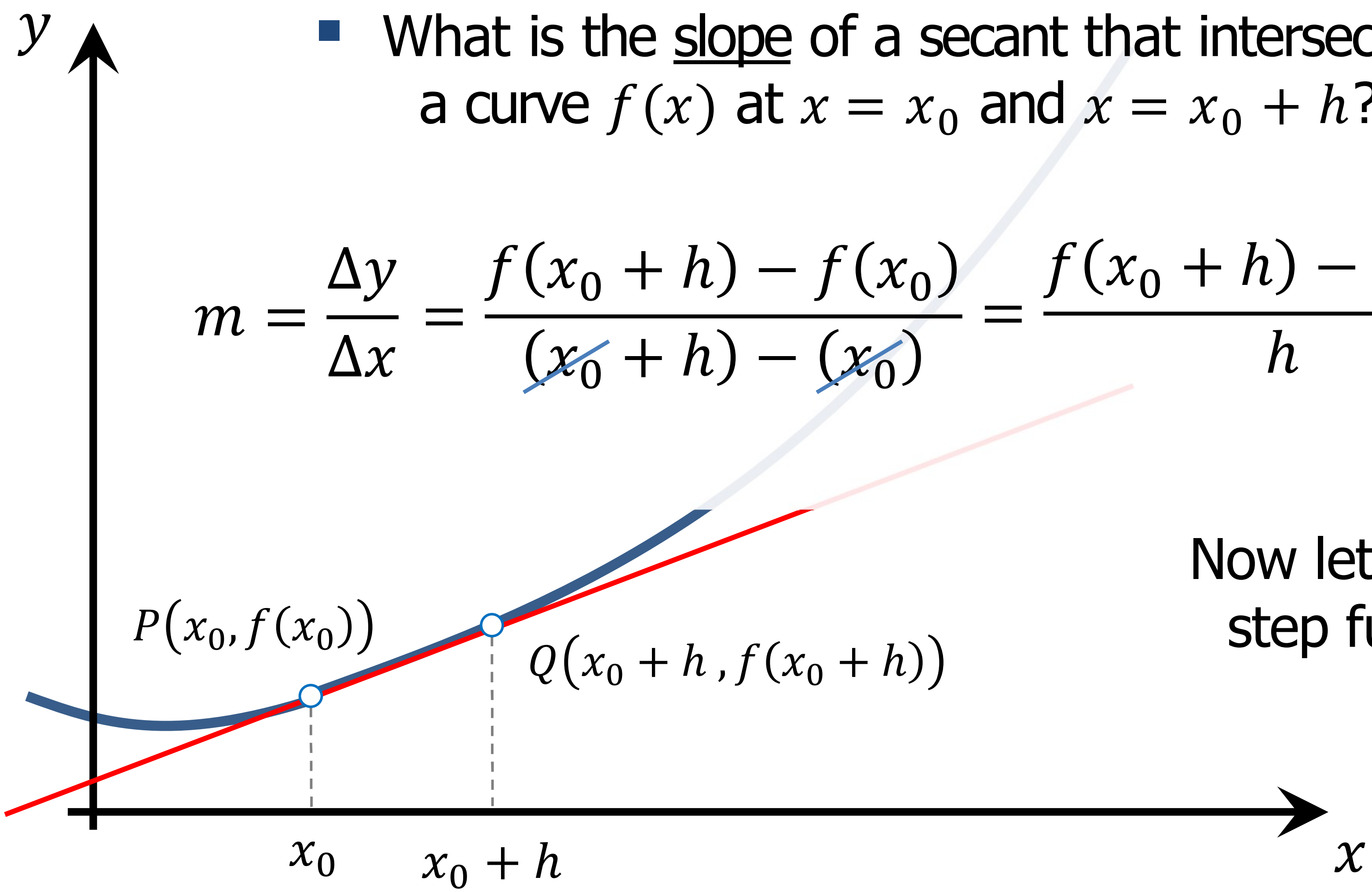


Tangent lines

- Tangents are "limit secants" where the points of intersection with the curve are infinitely close to each other.

- What is the <u>slope</u> of a secant that intersects a curve $f(x)$ at $x = x_0$ and $x = x_0 + h$?

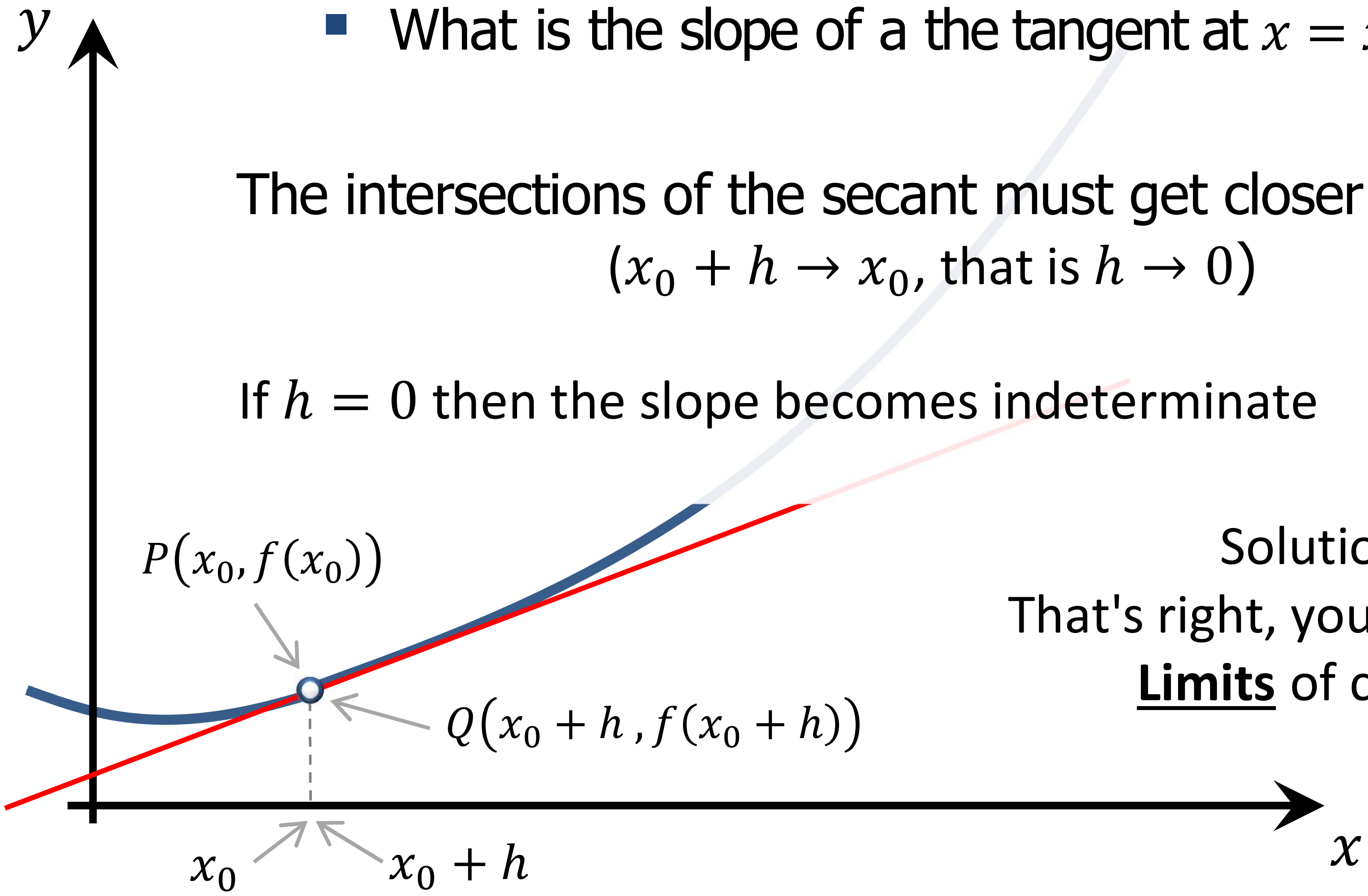
$$m = \frac{\Delta y}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{(\cancel{x_0} + h) - (\cancel{x_0})} = \frac{f(x_0 + h) - f(x_0)}{h}$$



Now let's go one step further…

Tangent lines

- Tangents are "limit secants" where the points of intersection with the curve are infinitely close to each other.



- What is the slope of a the tangent at $x = x_0$?

The intersections of the secant must get closer and closer $(x_0 + h \to x_0$, that is $h \to 0)$

If $h = 0$ then the slope becomes indeterminate $\frac{0}{0}$

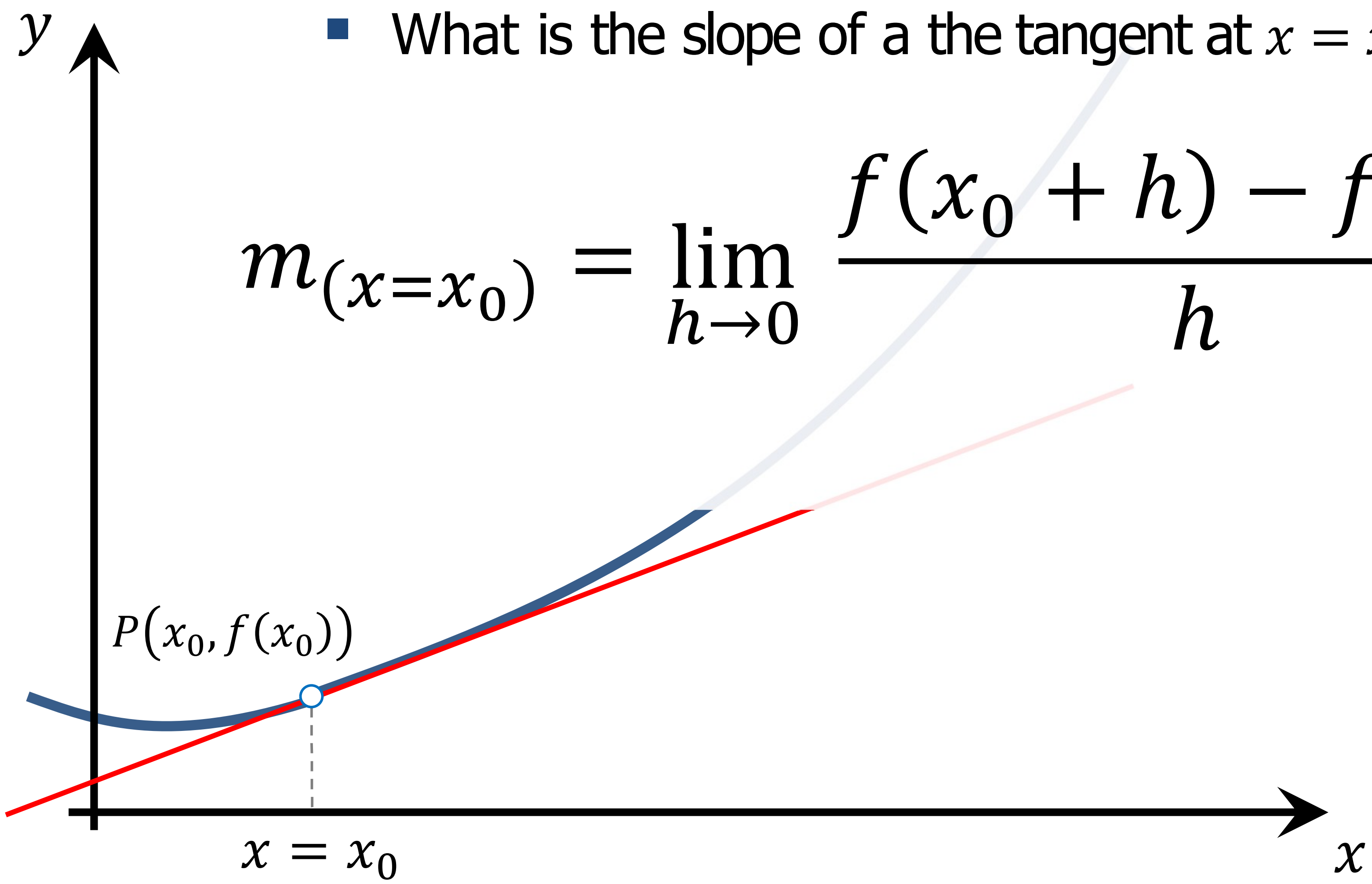
Solution?
That's right, you guessed it,
Limits of course

Tangent lines

- Tangents are "limit secants" where the points of intersection with the curve are infinitely close to each other.

- What is the slope of a the tangent at $x = x_0$?

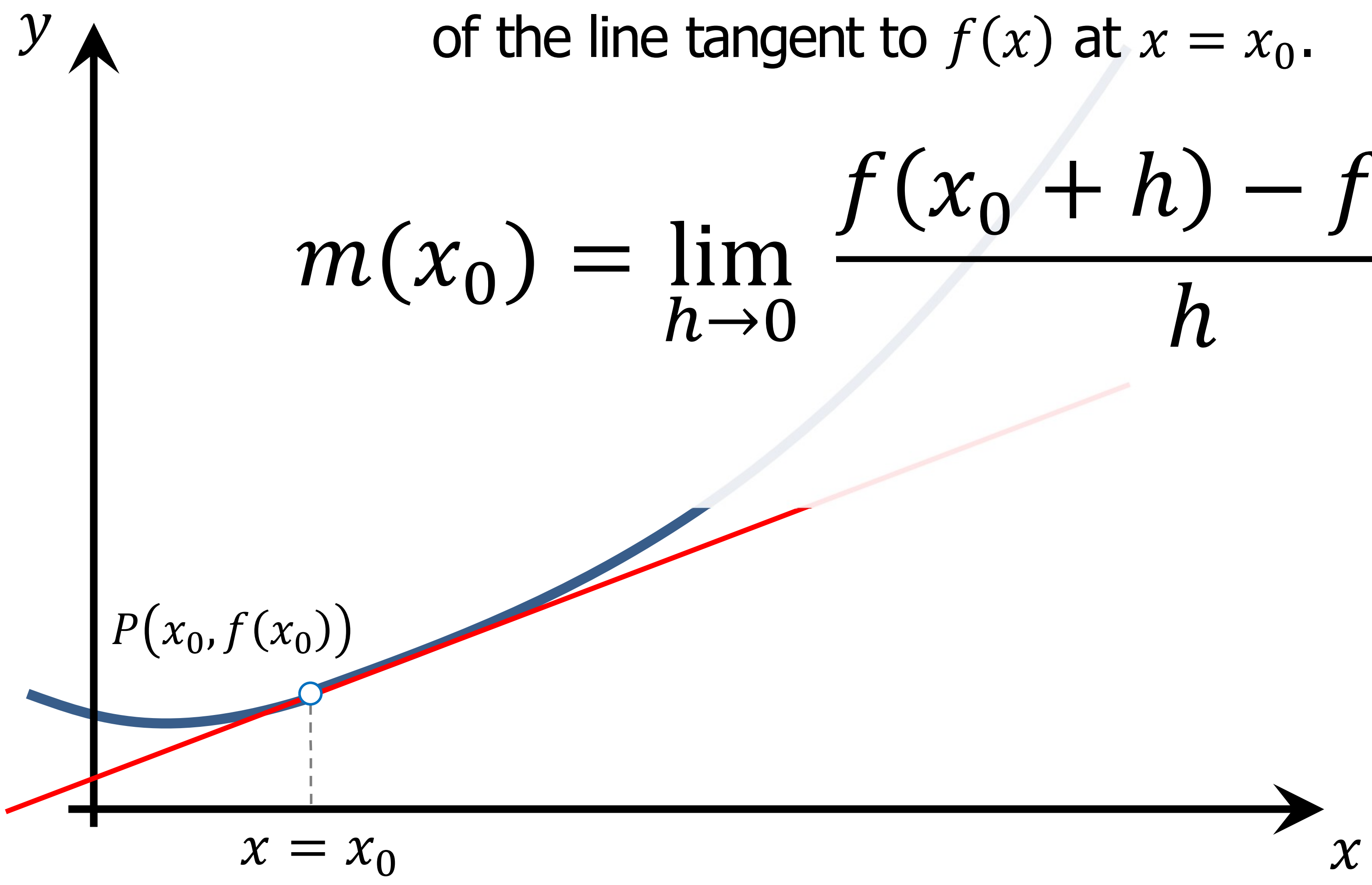
$$m_{(x=x_0)} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$



Tangent lines

- For each x_0 this function gives us the slope of the line tangent to $f(x)$ at $x = x_0$.

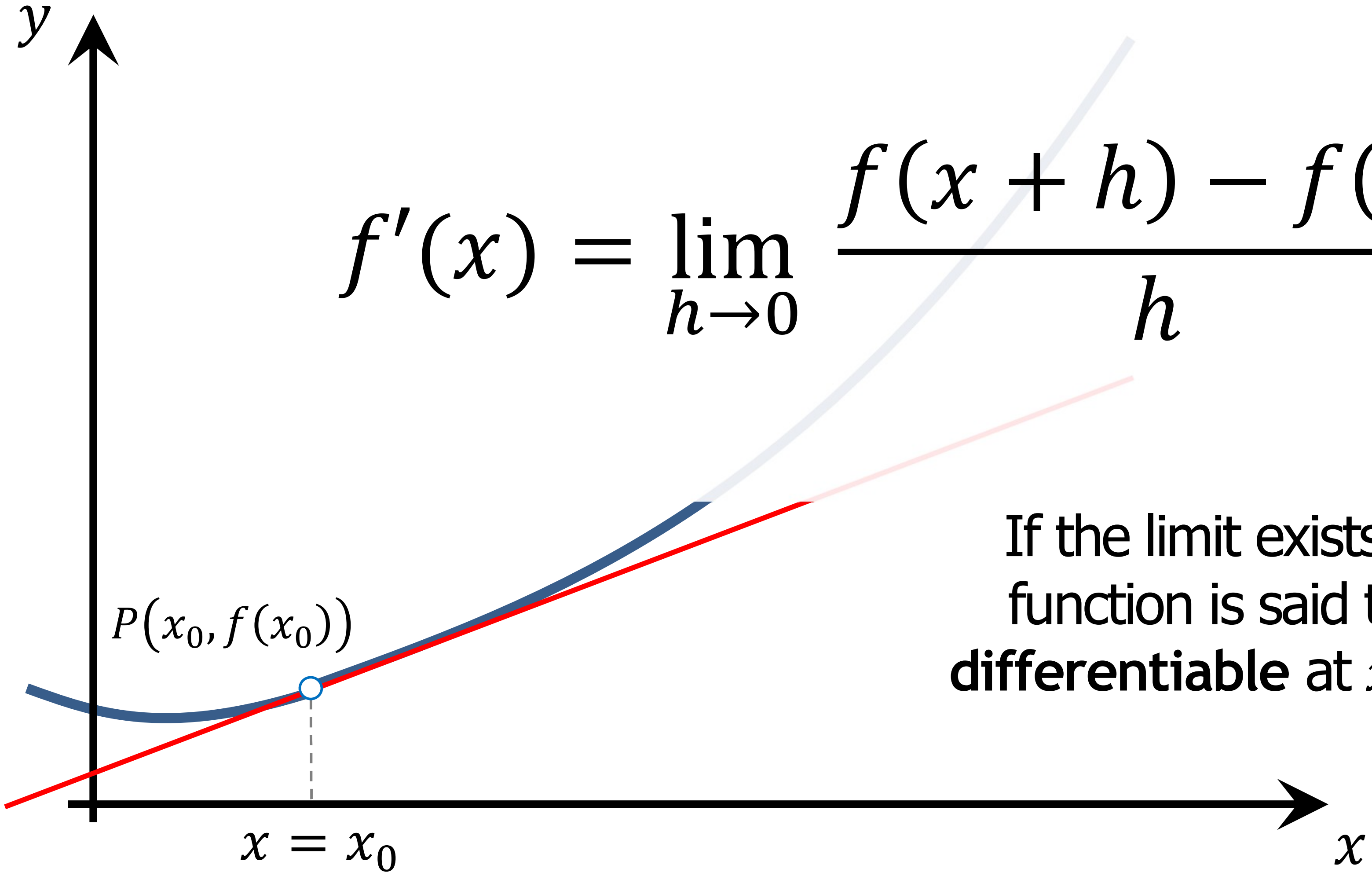
$$m(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$



Tangent lines

- This function is called **derivative** of $f(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$



If the limit exists, the function is said to be **differentiable** at $x = x_0$.

Tangent lines

- Notation and definition:

(Intuitively: the increment h is done on the x axis)

We say that $f'(x)$ is the derivative of $f(x)$ **with respect to** x

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0}$$

Lagrange's notation

Leibniz's notation

Where dx and dy denote the infinitesimals in the ratio of the slope of the secant approaching the tangent line.

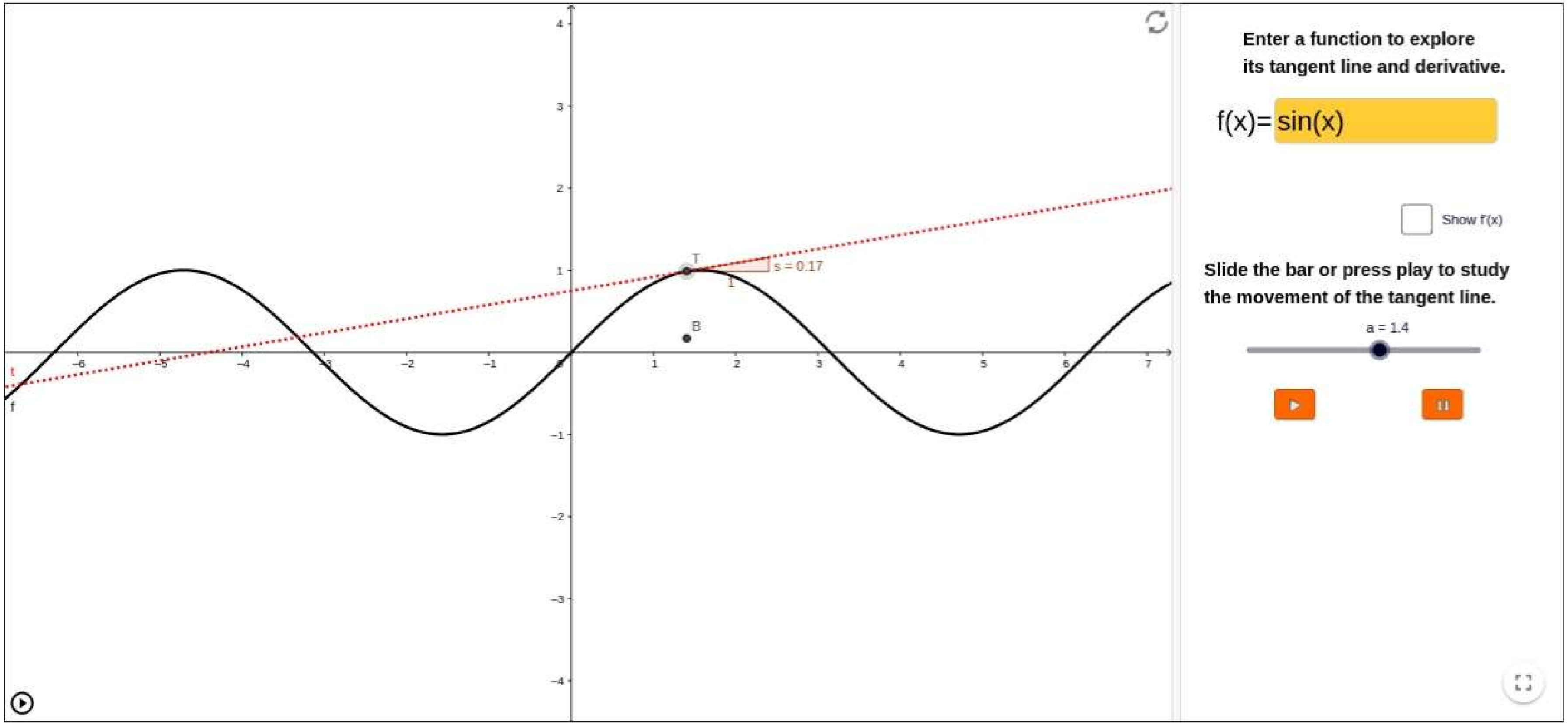
Derivative evaluated at a point $x = x_0$

Other notations:

Newton's notation: $\dot{y}$
(used in physics and engineering)

Euler's notation: $D_x f(x)$
(note: D_x is an operator)

Useful applet



https://www.geogebra.org/m/NNnd6y4H

Why derivatives?

LOTS of reasons! One of the most important is to pass this course.

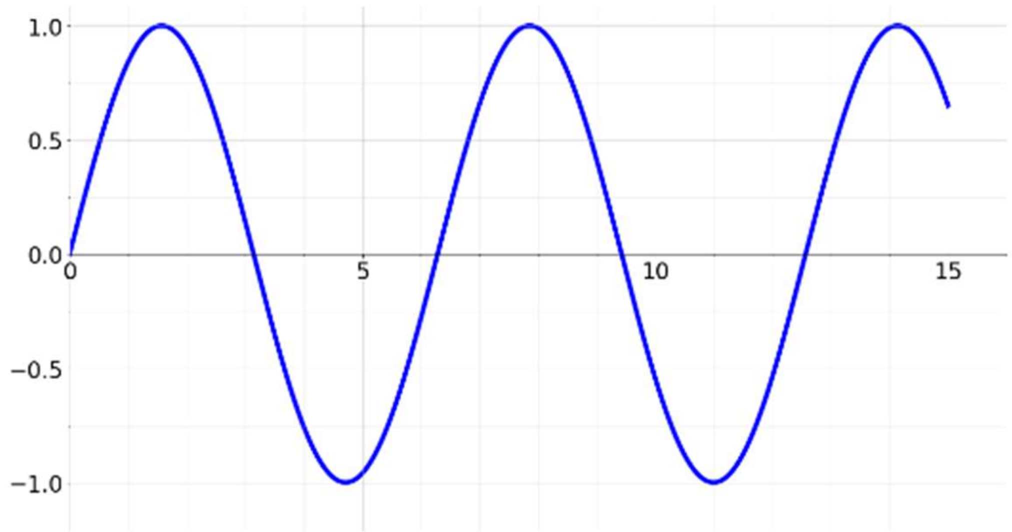
We will see them throughout the rest of the course.

For the moment, look at the following for some ideas

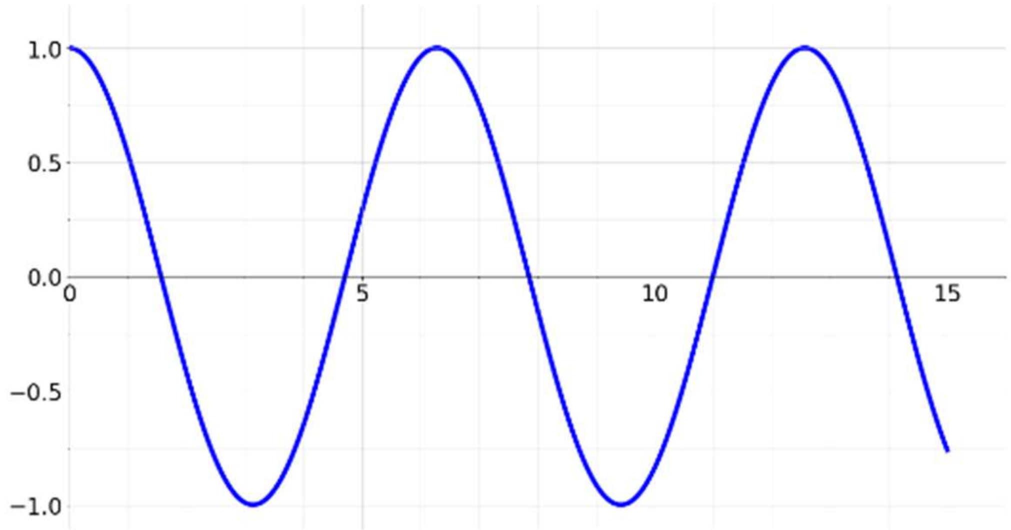
Why derivatives?

- What is the derivative of my function telling me?

$f(x)$

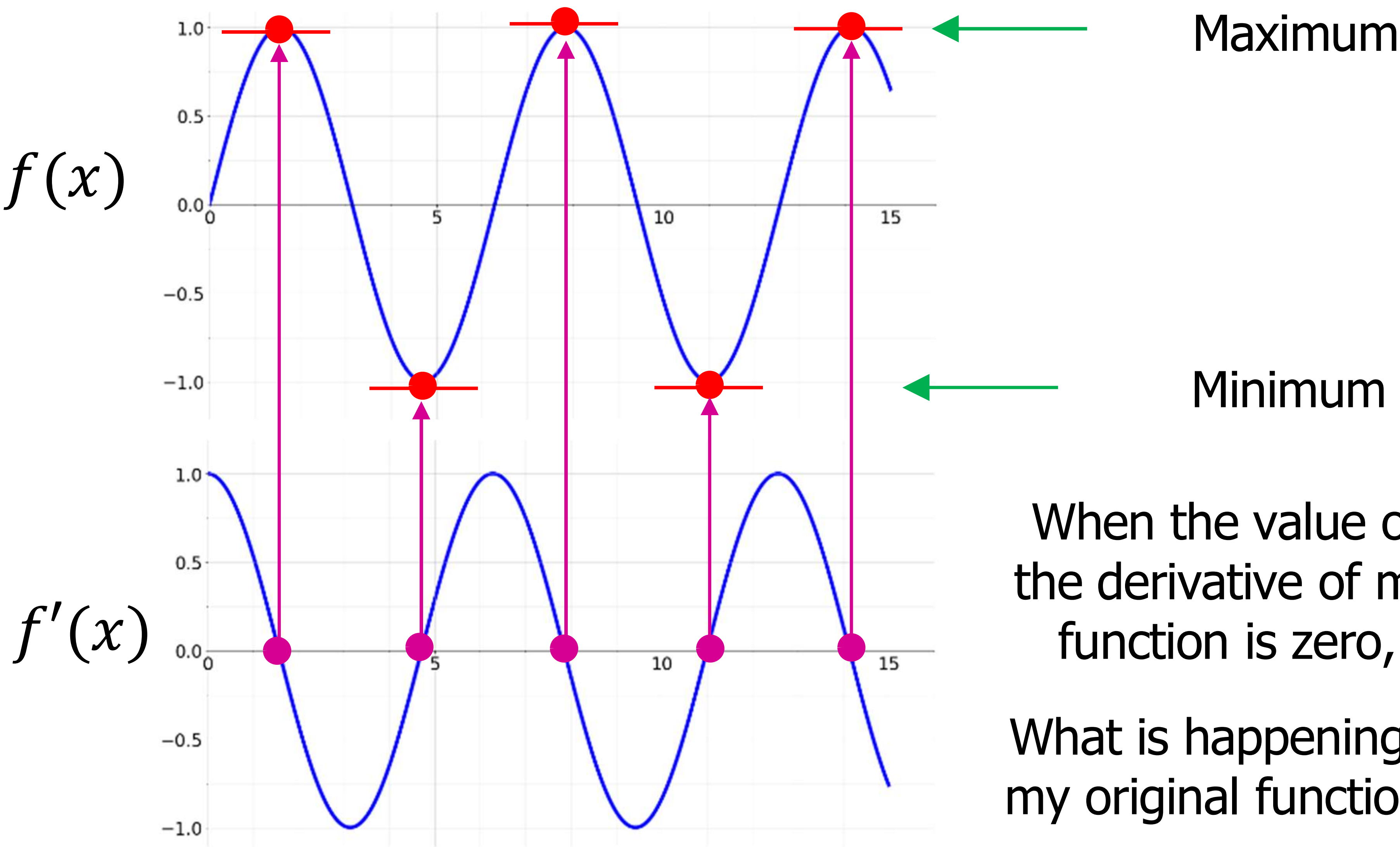


$f'(x)$



Why derivatives?

- What is the derivative of my function telling me?

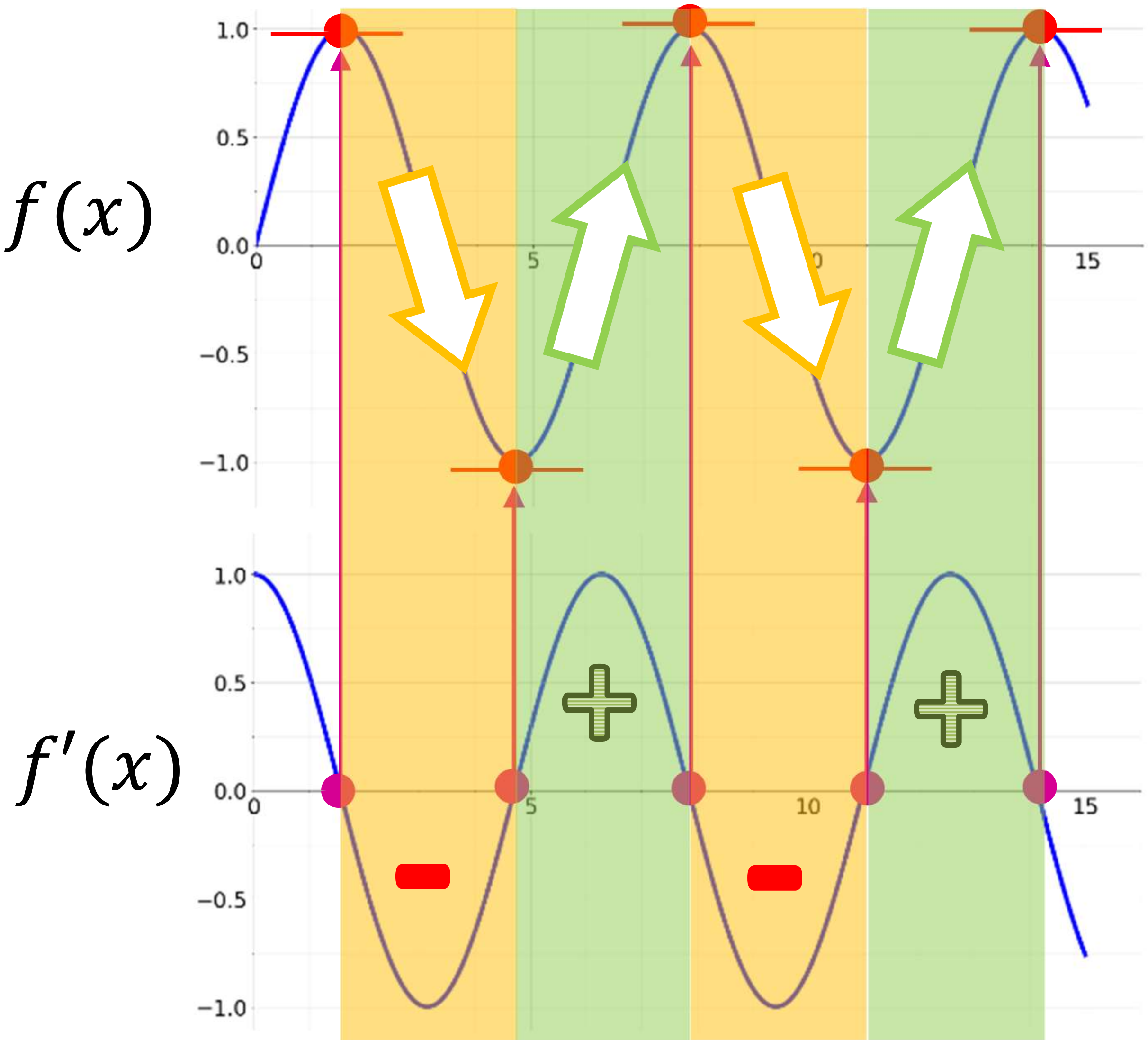


When the value of the derivative of my function is zero,

What is happening in my original function?

Why derivatives?

- What is the derivative of my function telling me?



When the value of the derivative of my function is zero,

What is happening in my original function?

Calculating derivatives with limits

- So, how do we calculate derivatives?
- How do we solve the limit in the definition of a derivative?

Calculating derivatives with limits

- If the limit exist, we can approximate it *numerically* by plugging-in a small value of h .

Example: $f(x) = x^2$, derivative $f'(2)$ at $x = 2$:

$$f'(x) = \frac{d(x^2)}{dx} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \longrightarrow f'(2) = \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h}$$

$$h = \{\,1\,, 0.1\,, 0.01\,, 0.001\,, \dots\} \longrightarrow f'(2) \cong \{\,5\,, 4.1\,, 4.01\,, 4.001\,, \dots\}$$

Exact solution: $f'(2) = 4$

- Used in practice. Numerical analysis finds application in all fields of engineering and sciences (i.e., method of finite-differences)

- Especially useful if the function is defined by data rather than an expression or if the function takes discrete arguments x.

Calculating derivatives with limits

We can do better than this in most cases.

Let's see how we can calculate derivatives in general cases (e.g., when the function is a line, a polynomial, etc. . .)

- Linear functions $f(x) = mx + b$

The tangent to a line is always the line itself, so $f'(x) = m$ regardless of x.

Indeed we can show that:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{m(x+h) + b - (mx+b)}{h} = \lim_{h \to 0} \frac{m\,h}{h} = m$$

- Special case: constant values $f(x) = b \longrightarrow f'(x) = 0$

  Same as before, $m = 0$.

Calculating derivatives with limits

- Quadratic functions

$$f(x) = x^2$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} =$$

$$= \lim_{h \to 0} \frac{(\cancel{x^2} + 2hx + h^2) - \cancel{x^2}}{h} =$$

$$= \lim_{h \to 0} \frac{2\cancel{h}x + h^{\cancel{2}}}{\cancel{h}} =$$

$$= \lim_{h \to 0} 2x + \overset{0}{\cancel{h}} \longrightarrow f'(x) = 2x$$

- Cubic functions

$$f(x) = x^3$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} =$$

$$(x+h) * (x+h)^2 = (x+h) * (x^2 + 2hx + h^2)$$

$$= \lim_{h \to 0} \frac{(\cancel{x^3} + 3x^2h + 3xh^2 + h^3) - \cancel{x^3}}{h} =$$

$$= \lim_{h \to 0} \frac{3x^2\cancel{h} + 3xh^{\cancel{2}} + h^{\overset{2}{\cancel{3}}}}{\cancel{h}} =$$

$$= \lim_{h \to 0} 3x^2 + \overset{0}{\cancel{3xh}} + \overset{0}{\cancel{h^2}} \longrightarrow f'(x) = 3x^2$$

Calculating derivatives with limits

- Will could continue…

$$f(x) = x^4$$

$$(x+h)^4 = (x+h) * (x+h)^3 =$$

$$= (x+h) * (x^3 + 3x^2h + 3xh^2 + h^3)$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h} =$$

$$= \lim_{h \to 0} \frac{(\cancel{x^4} + 4x^3h + 6x^2h^2 + 4x^2h^3 + h^4) - \cancel{x^4}}{h} =$$

$$= \lim_{h \to 0} \frac{4x^3\cancel{h} + 6x^2h^{\cancel{2}} + 4x^2h^{\overset{2}{\cancel{3}}} + h^{\overset{3}{\cancel{4}}}}{\cancel{h}} =$$

$$= \lim_{h \to 0} 4x^3 + \overset{0}{\cancel{6x^2h}} + \overset{0}{\cancel{4x^2h^2}} + \overset{0}{\cancel{h^3}}$$

$$\longrightarrow \quad f'(x) = 4x^3$$

- We can see a pattern emerging

$$f(x) = x^2 \longrightarrow f'(x) = 2x^1$$

$$f(x) = x^3 \longrightarrow f'(x) = 3x^2$$

$$f(x) = x^4 \longrightarrow f'(x) = 4x^3$$

$$f(x) = x^n \longrightarrow f'(x) = n\,x^{n-1}$$

Calculating derivatives with limits

- Power functions $f(x) = x^n$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \xrightarrow{?} n\, x^{n-1}$$

- Example:

$$f(x) = x^6 \longrightarrow f'(x) = \lim_{h\to 0} \frac{\overset{?}{(x+h)^6} - x^6}{h}$$

$$(x+h)^0 = 1$$

$$(x+h)^1 = (x+h)$$

$$(x+h)^2 = (x+h) * (x+h)$$

$$(x+h)^3 = (x+h) * (x+h) * (x+h)$$

$$(x+h)^4 = (x+h) * (x+h) * (x+h) * (x+h)$$

$$(x+h)^5 = (x+h) * (x+h) * (x+h) * (x+b) * (x+h)$$

$$(x+h)^6 = (x+h) * (x+h) * (x+h) * (x+h) * (x+h) * (x+h)$$

$$(x+h)^0 = 1$$

$$(x+h)^1 = x + h$$

$$(x+h)^2 = x^2 + 2xh + h^2$$

$$(x+h)^3 = (x+h) * (x+h)^2$$

$$(x+h)^4 = (x+h) * (x+h)^3$$

$$(x+h)^5 = (x+h) * (x+h)^4$$

$$(x+h)^6 = (x+h) * (x+h)^5$$

$$(x+h)^0 = 1$$

$$(x+h)^1 = x + h$$

$$(x+h)^2 = x^2 + 2xh + h^2 \;=$$

$$(x+h)^3 = (x+h) * (x^2 + 2xh + h^2) \;=$$

$$(x+h)^4 = (x+h) * (x^3 + 3x^2h + 3xh^2 + h^3) \;=$$

$$(x+h)^5 = (x+h) * (x^4 + 4x^3h + 6x^2h^2 + 4x^2h^3 + h^4) \;=$$

$$(x+h)^6 = (x+h) * (x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5)$$

$$(x+h)^0 = 1$$

$$(x+h)^1 = x + h$$

$$(x+h)^2 = x^2 + 2xh + h^2$$

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$$(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4x^2h^3 + h^2$$

$$(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5$$

$$(x+h)^6 = x^6 + 6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + h^6$$

$$(x+h)^0 = 1$$

$$(x+h)^1 = x^1 + h^1$$

$$(x+h)^2 = x^2 + 2xh + h^2$$

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$$(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4x^2h^3 + h^4$$

$$(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5$$

$$(x+h)^6 = x^6 + 6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + h^6$$

Binomial Expansion

$$(x+h)^0 = 1$$

$$(x+h)^1 = 1x^1 + 1h^1$$

$$(x+h)^2 = 1x^2 + 2xh + 1h^2$$

$$(x+h)^3 = 1x^3 + 3x^2h + 3xh^2 + 1h^3$$

$$(x+h)^4 = 1x^4 + 4x^3h + 6x^2h^2 + 4x^2h^3 + 1h^4$$

$$(x+h)^5 = 1x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + 1h^5$$

$$(x+h)^6 = 1x^6 + 6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + 1h^6$$

Pascal's Triangle

$$(x+h)^0 = 1$$

$$(x+h)^1 = x^1 + h^1$$

$$(x+h)^2 = x^2 + 2xh + h^2$$

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$$(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4x^2h^3 + h^4$$

$$(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5$$

$$(x+h)^6 = x^6 + 6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + h^6$$

$$(x+h)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \; h^k$$

Remember: Combinatorial Number : $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

(order doesn't matter)

Example:

$$\binom{4}{2} = \frac{4!}{2!\,(4-2)!} = \frac{4*3*2*1}{(2*1)*(2*1)} = \frac{24}{4} = 6$$

There are 6 ways to choose 2 elements from {1,2,3,4},

namely {1,2} , {1,3} , {1,4} , {2,3} , {2,4} and {3,4}

$$(x+h)^0 = 1$$

$$(x+h)^1 = x^1 + h^1$$

$$(x+h)^2 = x^2 + 2xh + h^2$$

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$$(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4x^2h^3 + h^4$$

$$(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5$$

$$(x+h)^6 =$$

$$x^6 + 6x^5h + {\color{red}15}x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + h$$

$$\underbrace{\cancel{\binom{6}{0}}}_{1} \underbrace{x^{6-0}}_{} \underbrace{\cancel{h^0}}_{1} + \underbrace{\cancel{\binom{6}{1}}}_{6} x^{6-1}\, h^1 + \binom{6}{2} x^{6-2}\, h^2 + \binom{6}{3} x^{6-3}\, h^3 + \binom{6}{4} x^{6-4}\, h^4 + \underbrace{\cancel{\binom{6}{5}}}_{6} x^{6-5}\, h^5 + \underbrace{\cancel{\binom{6}{6}}}_{1} \underbrace{\cancel{x^{6-6}}}_{1} h^6$$

Example: $\binom{6}{2} = \dfrac{6!}{2!\,(6-2)!} = \dfrac{6*5*4*3*2*1}{(2*1)*(4*3*2*1)} = \dfrac{720}{48} = {\color{red}15}$

Calculating derivatives with limits

- Example: $f(x) = x^6$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^6 - x^6}{h} = \lim_{h\to 0} \frac{\left(\sum_{k=0}^{6} \binom{6}{k} x^{6-k}\ h^k\right) - x^6}{h} =$$

$$= \lim_{h\to 0} \frac{(\cancel{x^6} + 6x^5\cancel{h} + 15x^4h^{\cancel{2}} + 20x^3h^{\cancelto{2}{3}} + 15x^2h^{\cancelto{3}{4}} + 6xh^{\cancelto{4}{5}} + h^{\cancelto{5}{6}}) - \cancel{x^6}}{\cancel{h}} =$$

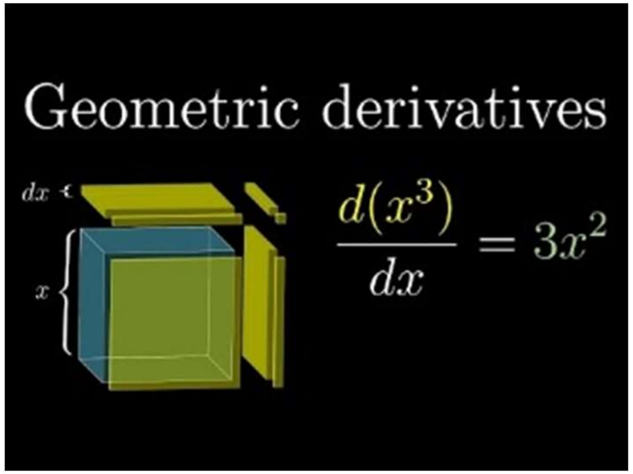
$$= \lim_{h\to 0}(6x^5 + \cancelto{0}{15x^4h} + \cancelto{0}{20x^3h^2} + \cancelto{0}{15x^2h^3} + \cancelto{0}{6xh^4} + \cancelto{0}{h^5}) = 6x^5$$

$$= 6x^5$$

Calculating derivatives with limits

- "The Power Rule"

$$f(x) = x^n \longrightarrow f'(x) = n\,x^{n-1}$$



3Blue1Brown

Essence of calculus, Chapter 3

Derivative formulas through geometry

YouTube

https://youtu.be/S0_qX4VJhMQ

Calculating derivatives with limits

- Power functions $f(x) = x^n$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = n\, x^{n-1}$$

- Square root

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \longrightarrow f'(x) = n\, x^{n-1} = -\, x^{-2} = \frac{-1}{x^2}$$

- One divided by x

$$f(x) = \frac{1}{x} = x^{-1} \longrightarrow f'(x) = n\, x^{n-1} = \frac{1}{2}\, x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Calculating derivatives with formulas

- You can (and actually you should) learn the derivatives of the most common functions
- You should not compute the limit explicitly if the function is a common one.

You should however know how to solve these limits!

E.g., $f(x) = 5x^3$

- **DO** (best)

Realize it's a power function, so the derivative is…

$$f'(x) = n\, x^{n-1} = 15x^2$$

- **DON'T**

Much slower and unnecessary if you remember a few common derivatives

$$f'(x) = \lim_{h \to 0} \frac{5(x+h)^3 - 5x^3}{h}$$

Calculating derivatives with limits

- Exponential function

$$f(x) = e^x$$

$$e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}$$

$$e^x = \left(\lim_{x \to 0} (1+x)^{\frac{1}{x}}\right)^x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$e^h = \left(\lim_{h \to 0} (1+h)^{\frac{1}{h}}\right)^h$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} =$$

$$= \lim_{h \to 0} \frac{e^x\, e^h - e^x}{h} = \lim_{h \to 0} \frac{e^x\left(e^h - 1\right)}{h} = \lim_{h \to 0} \frac{e^x\left((\cancel{1} + h)^{\cancel{h/h}} \cancel{- 1}\right)}{h}$$

$$= \lim_{h \to 0} \frac{e^x\, \cancel{h}}{\cancel{h}} \longrightarrow f'(x) = e^x$$

Calculating derivatives with limits

- Logarithmic functions

$$f(x) = \ln(x)$$

$$\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$$

$$f'(x) = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h \to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h \to 0} \frac{\ln\left(\frac{x}{x} + \frac{h}{x}\right)}{h} =$$

$$\ln(a)/_b = \ln(a)^{1/b}$$

Replace $n = \frac{x}{h}$

So $\frac{h}{x} = \frac{1}{n}$ and $\frac{1}{h} = \frac{n}{x}$

$n \to \infty$ as $h \to 0$

$$= \lim_{h \to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{h} = \lim_{h \to 0} \ln\left(1 + \frac{h}{x}\right)^{\frac{1}{h}} = \lim_{n \to \infty} \ln\left(1 + \frac{1}{n}\right)^{\frac{n}{x}} =$$

$$e = \lim_{x \to \infty} \ln\left(1 + \frac{1}{x}\right)^{x}$$

$$= \lim_{n \to \infty} \ln\left(1 + \frac{1}{n}\right)^{\frac{n}{x}} = \lim_{n \to \infty} \frac{1}{x} \ln\left(1 + \frac{1}{n}\right)^{n} = \frac{1}{x} \cancel{\ln e} = \frac{1}{x} \longrightarrow f'(x) = \frac{1}{x}$$

Calculating derivatives with formulas

- Trigonometric functions are a bit tricky. Following is just a taster.

$$\frac{d\sin x}{dx} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} =$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \sin h\cos x - \sin x}{h} =$$

$$= \lim_{h\to 0}\frac{\sin x\,(\cos h - 1) + \sin h\cos x}{h} =$$

$$= \sin x \lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h} =$$

Calculating derivatives with formulas

- The first part goes as

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = \lim_{h\to 0} -\frac{1-\cos h}{h}\frac{1+\cos h}{1+\cos h} =$$
$$= \lim_{h\to 0} -\frac{1-\cos^2 h}{h(1+\cos h)} =$$
$$= \lim_{h\to 0} -\frac{\sin^2 h}{h(1+\cos h)} =$$
$$= \lim_{h\to 0} -\frac{\sin h}{h}\frac{\sin h}{1+\cos h} =$$
$$= -1 \lim_{h\to 0}\frac{\sin h}{1+\cos h} = 0$$

Calculating derivatives with formulas

- Putting everything together:

$$\frac{d \sin x}{dx} =$$

$$= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h} =$$

$$= \cos x$$

Required Materials

Continuity of a Function:

- (OpenStax Calculus Volume 1) Sections 2.4, 2.5.
- Suggested exercises: Section 2.4 ex. 131-138, 139, 141, 143, 144, 145-149, 150, 154, 156, (highly advised) **158**.
  Section 2.5 ex. 180, 188.

Introduction to Derivatives:

- (OpenStax Calculus Volume 1) Sections 3.1, 3.2, 3.3, 3.5, 3.9.
  Note: limits of products/quotients will be covered in the next lecture, but you are welcome to read the full sections anyway.
- Optional but recommended: Section 3.4.
- Suggested exercises: section 3.1 ex. 13. 18, 19; section 3.2 ex. 55, 59, 62, 63, 66, 68, 70, 75, 82; section 3.3 ex. 106, 109, 110, 111, 119.