EHRENFEUCHT GAMES FOR GENERALIZED QUANTIFIERS

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§ 0. Introduction

In [4] Ehrenfeucht gave a game theoretical characterization of elementary equivalence with respect to a first order language. A similar characterizations for the language $L(Q_\alpha)$ are due independently to Lipner [7], Brown [2], and Vinner [9], for the language with Henkin quantifier to Krynicki [5], for the language with Malitz quantifier to Badger [1] (as we read in [8]). The idea of all these games is rather uniform. Our paper grew out this observation. This paper was born during the 1975 Conference in Helsinki, where the authors could not get accustomed to polar night. We wish to thank our Finnish Friends for the inwitation to this conference. The authors wish to thank prof. A. Lachlan for fruitful discussions.

Structure $\langle A, A_1, \dots, A_k \rangle$ s.t. $A_i \subseteq A^{n_i}$ we call structure of type $\langle n_1, \dots, n_k \rangle$. Let L denote a first order language with equality allowing constant symbols but without functions and quantifier symbols. By a quantifier Q of type $\langle n_1, \dots, n_k \rangle$ we mean some class of structures of this type. Formulae of language $L(Q)$ are defined by induction:

1. $L \subseteq L(Q)$

2. If $\varphi_0, \dots, \varphi_{k+2}$ $L(Q)$ then $\varphi_{k+1} \vee \varphi_{k+2}$, $\neg \varphi_{k+1}$, $(Q\bar{x}_1, \dots, \bar{x}_k)\,(\varphi_1, \dots, \varphi_k) \in L(Q)$ provided no free variable of φ_i occurs in $\bar{x}_j$ for $i \neq j$.

We use Lindström [6] definition of satisfaction:

$$\underline{\underline{A}} \models (Q\overline{x}_1,\ldots,\overline{x}_k)(\varphi_1,\ldots,\varphi_k) \quad \Longleftrightarrow$$

$$\langle A, \{\overline{a} \in A^{n_1} : \underline{\underline{A}} \models \varphi_1[\overline{a}]\}, \ldots, \{\overline{a} \in A^{n_k} : \underline{\underline{A}} \models \varphi_k[\overline{a}]\}\rangle \quad \in Q$$

By Q_c we denote the Chang quantifier, by Q_I the Hartig quantifier, by Q_α^n the Malitz one. For definitions see [8]. By Q_H we denote the Henkin quantifier defined in [3] as follows:

$$(Qxyuv)\ \varphi(x,y,u,v) \Longleftrightarrow \begin{pmatrix} \forall x \ \exists u \\ \forall y \ \exists v \end{pmatrix} \varphi(x,y,u,v)$$

Suppose the formulae and semantics of the language $L(Q_1,\ldots,Q_n)$ have been defined and Q_{n+1} is a quantifier of some finite type we obtain $L(Q_1,\ldots,Q_{n+1})$ from $L(Q_1,\ldots,Q_n)$ in the same way as $L(Q)$ from L.

We use standard model theoretical notation.

§ 1. Quantifiers of the type $\langle q\rangle$

Definition 1.1. The quantifier Q of type $\langle q\rangle$ is called monotone if for every $\langle A,R\rangle \in Q$ and every $R \subseteq R' \subseteq A^q$ $\langle A,R'\rangle \in Q$

For example $\exists, \forall, Q_\alpha, Q_H, Q_\alpha^n$ are monotone.

Definition 1.2. The class $K \subseteq Q$ is a support for Q if for arbitrary $\langle A,R\rangle \in Q$ exists R' s.t. $R' \subseteq R$ and $\langle A,R'\rangle \in K$.

Examples: The quantifier "there existe exactly $\aleph_\alpha$" is a support for Q_α. The Machover quantifier $Q_M = \{\langle A,R\rangle : \overline{\overline{R}} = \overline{\overline{A-R}}\}$ is a support for Q_c. The class K s.t. $\langle A,R\rangle \in K$ iff $R \subsetneq A^4$ and there are $f,g \in A^A$ s.t. $\langle a,b,c,d\rangle \in R$ $f(a) = c$ and $g(b) = d$ is a support for Q_H.

Let $\underline{\underline{A}}$, $\underline{\underline{B}}$ be structures with the same signature. By partial isomorphism we mean a partial function from the universe of $\underline{\underline{A}}$ into the universe of $\underline{\underline{B}}$ s.t. the domain of that function contain constans of $\underline{\underline{A}}$ and function preserve relations.

Assume now that Q is fixed monotonic quantifier of type $\langle q\rangle$ and K its support. The Ehrenfeucht game appropriate to such a quantifier is given by the following:

<u>Definition</u> 1.3. Consider the game $G_n(\underline{\underline{A}},\underline{\underline{B}})$ played by two players and having n moves. In each move player 1 chooses a structure (e.g. $\underline{\underline{A}}$) and its subset $(R \subseteq A^q)$ s.t. $\langle A,R\rangle \in K$, then player 2 chooses a subset $R' \subseteq B^q$ (if $R \subseteq B^q$ then $R' \subseteq A^q$). Later player 1 chooses a q-tuple $\langle b_1,\dots,b_q\rangle \in R'$ and then player 2 chooses a q-tuple $\langle a_1,\dots,a_q\rangle \in R$. So, we may, that after the n moves we have a partial mapping $\{\langle a_i,b_i\rangle : i=1,\dots,qn\}$ from the universe of $\underline{\underline{A}}$ into the universe of $\underline{\underline{B}}$. If this mapping is extendable to a partial isomorphism player 2 has won. Otherwise player 1 won.

<u>Definition</u> 1.4. $\underline{\underline{A}} \sim_n \underline{\underline{B}}$ iff player 2 has a winning strategy in the game $G_n(\underline{\underline{A}},\underline{\underline{B}})$.

By the rank of the formula $\varphi \in L(Q)$ $(\mathrm{rk}\,(\varphi))$ we mean the number of quantifiers occuring in φ. By $\underline{\underline{A}} \equiv_n \underline{\underline{B}}$ we denote that $Th_n(\underline{\underline{A}}) = Th_n(\underline{\underline{B}})$, where $Th_n\,\underline{\underline{A}} = \{\varphi \in L(Q) :\ \varphi\text{-sentence},\ \mathrm{rk}(\varphi) \leq n,\ \underline{\underline{A}} \models \varphi\}$.

<u>Theorem</u> 1.1. If $\underline{\underline{A}} \sim_n \underline{\underline{B}}$ then $\underline{\underline{A}} \equiv_n \underline{\underline{B}}$.

<u>Proof</u>. By induction with respect to n. The case $n=0$ is obvious. Assume that $\underline{\underline{A}} \sim_{n+1} \underline{\underline{B}}$ and there is a sentence φ of rank $n+1$ s.t. $\underline{\underline{A}} \models \varphi$, $\underline{\underline{B}} \models \neg\varphi$. We may assume that φ is of the form $(Q\overline{x})\,\phi(\overline{x})$. Let $R_1 = \{\overline{a} \in A^q\colon \underline{\underline{A}} \models \phi[\overline{a}]\}$. Hence $\langle A,R_1\rangle \in Q$ and there is $R \subseteq R_1$ s.t. $\langle A,R\rangle \in K$. In the first move of the game $G_{n+1}(\underline{\underline{A}},\underline{\underline{B}})$ let player 1 choose this R. So player 2 using winning strategy choose R'. Claim that there is $\overline{b} \in R'$ s.t. $\underline{\underline{B}} \models \neg\,\phi[\overline{b}]$. Indeed otherwise $R' \subseteq \{\overline{b}\colon \underline{\underline{B}} \models \phi[\overline{b}]\}$ and by monotonicity of Q $\underline{\underline{B}} \models (Q\overline{x})\,\phi(\overline{x})$. Let player 1 choose such $\overline{b}$ and player 2 applaying his winning strategy returb $\overline{a}$. Hence $\langle \underline{\underline{A}},\overline{a}\rangle \sim_n \langle \underline{\underline{B}},\overline{b}\rangle$ and by induction hypothesis $\langle \underline{\underline{A}},\overline{a}\rangle \equiv_n \langle \underline{\underline{B}},\overline{b}\rangle$. This contradicts the fact that $\underline{\underline{A}} \models \phi[\overline{a}]$ and $\underline{\underline{B}} \models \neg\,\phi[\overline{b}]$.

<u>Corollary</u>. If for every n $\underline{\underline{A}} \sim_n \underline{\underline{B}}$ then $\underline{\underline{A}} \equiv_{L(Q)} \underline{\underline{B}}$.

It is well known that converse implication doesn't hold without additional assumption.

Now let τ denote a fixed finite signature. We define two finite sequences $\{l_p^{n,\tau}\}_{p=1,..,n}$, $\{m_p^{n,\tau}\}_{p=1,..,n}$ for each n.

$l_0^{n,\tau} = 0$

$m_{p+1}^{n,\tau}$ = "the number of formulas of rank $l_p^{n,\tau}$ in which occur at most the first $l_p^{n,\tau} + (n-p)\,q$ variables".

$l_{p+1}^{n,\tau} = 2^{m_{p+1}^{p,\tau}} \cdot l_p^{p,\tau} \cdot m_{p+1}^{n,\tau} + 1$

Claim: If τ' is an extension of τ and contains at most q new constants then for $p = 0,\ldots,n-1$

(1) $$l_p^{n,\tau} \geqslant l_p^{n-1,\tau'}$$

<u>Theorem</u> 1.2. Let $\underline{\underline{A}},\underline{\underline{B}}$ be structures of type τ. If $\underline{\underline{A}} \equiv_{l_p^{n,\tau}} \underline{\underline{B}}$ then $\underline{\underline{A}} \sim_n \underline{\underline{B}}$.

<u>Proof</u>. By induction with respect to n. For $n = 0$ it is obvious. Let $\underline{\underline{A}} \equiv_{l_{n+1}^{n+1,\tau}} \underline{\underline{B}}$ and $\varphi_0,\ldots,\varphi_{s-1}$ be formulas of rank $l_n^{n+1,\tau}$ in which occur at most the first $l_n^{n+1,\tau} + q$ variables and exactly q free variables. Naturally $s \leqslant m_{n+1}^{n+1,\tau}$.

Let for $\varepsilon \in 2^s$ $\Theta_\varepsilon = \bigwedge_{i<s} \varphi_i^{\varepsilon(i)}$ where $\varphi_i^0 = \varphi_i$, $\varphi_i^1 = \neg\varphi_i$

Note that:

(2) $$\varepsilon_1 \neq \varepsilon_2 \Rightarrow \text{ for all } \bar{a} \in A^q \quad \underline{\underline{A}} \models \neg\, \Theta_{\varepsilon_1} \wedge \Theta_{\varepsilon_2}[\bar{a}]$$

and $\underline{\underline{A}} \models \bigvee_{\varepsilon\in 2^s} \Theta_\varepsilon[\bar{a}]$

(3) $$\mathrm{rk}(\Theta_\varepsilon) \leqslant s \cdot l_n^{n+1,\tau} \leqslant m_{n+1}^{n+1,\tau} \cdot l_n^{n+1,\tau}$$

Assume that player 1 chooses $A_1 \subseteq A^q$ s.t. $\langle A,A_1\rangle \in K$. Let $J = \{\varepsilon \in 2^s : (\exists \bar{a})_{A_1}\ \underline{\underline{A}} \models \Theta_\varepsilon[\bar{a}]\}$. Hence $A_1 \subseteq \{\bar{a} \in A^q : \underline{\underline{A}} \models \bigvee_{\varepsilon\in J} \Theta_\varepsilon[\bar{a}]\}$

But Q is monotonic so $\underline{\underline{A}} \models (Q\bar{x}) \bigvee_{\varepsilon\in J} \Theta_\varepsilon$. On the other hand

$$\mathrm{rk}((Q\bar{x}) \bigvee_{\varepsilon\in J}\Theta_\varepsilon) \leqslant 2^s \cdot \mathrm{rk}(\Theta_\varepsilon) + 1 \leqslant 2^{m_{n+1}^{n+1,\tau}} \cdot m_{n+1}^{n+1,\tau} \cdot l_n^{n+1,\tau} + 1 = l_{n+1}^{n+1,\tau}$$

and $\underline{\underline{B}} \models (Q\bar{x}) \bigvee_{\varepsilon\in J}\Theta_\varepsilon$. By choice of the class K there exists $B_1 \subseteq B^q$ s.t. $B_1 \subseteq \{\bar{b} \in B^q : \underline{\underline{B}} \models \bigvee_{\varepsilon\in J}\Theta_\varepsilon[\bar{b}]\}$ and $\langle B,B_1\rangle \in K$. This B_1 is adequate for player 2 reply. Now let $\overline{b} \in B_1$ be chosen by player 1, by (2) and definition B_1 there is exactly one $\varepsilon \in J$ such that $\underline{\underline{B}} \models \Theta_\varepsilon[\overline{b}]$ so player 2 should choose $\overline{a} \in A_1$ s.t.. $\underline{\underline{A}} \models \Theta_\varepsilon[\overline{a}]$. Then $\langle \underline{\underline{A}}, a\rangle \equiv_{1_n^{n+1},\tau} \langle \underline{\underline{B}}, \overline{b}\rangle$ which by (1) implice $\langle \underline{\underline{A}}, \overline{a}\rangle \equiv_{1_n^{n},\tau'} \langle \underline{\underline{B}}, \overline{b}\rangle$ where τ' is an expansion of τ and contains q new constans. By inductive assumption we obtain that $\langle \underline{\underline{A}}, \overline{a}\rangle \sim_n \langle \underline{\underline{B}}, \overline{b}\rangle$ which completes the proof.

Note that the relation $\sim_n$ for a monotonic quantifier does not depend on the choice of support.

Definition 1.5. Let $Q_1,\dots,Q_k$ be monotonic quantifiers, $\underline{\underline{A}},\underline{\underline{B}}$ structures of the same type. The game $G^n_{Q_1,..,Q_k}(\underline{\underline{A}},\underline{\underline{B}})$ is the following game for two gamblers. In every move player 1 chooses $i \leq k$ and both players play according to Definition 1.3 with Q_i in place of Q .

Theorem 1.3. Let $Q_1,\dots,Q_k$ be monotonic quantifiers, $\underline{\underline{A}},\underline{\underline{B}}$ structures of the same finite signature. Then $\underline{\underline{A}} \equiv_{L(Q_1,..,Q_k)} \underline{\underline{B}}$ iff for each n player 2 has winning strategy in the game $G^n_{Q_1,..,Q_k}(\underline{\underline{A}},\underline{\underline{B}})$.

The proof is a slight elaboration of the preceding one.

Until now we have assumed that Q is a monotonic quantifier. Now we prove that defined games doesn't work for arbitrary quantifier. More precisely we prove the following theorem,

Theorem 1.4. Let Q be a quantifier of type $\langle q\rangle$, $Q' = \{\langle A,R\rangle : \text{exists } R' \subseteq R \text{ and } \langle A,R'\rangle \in Q\}$. Then either $\underline{\underline{A}} \equiv_{L(Q)} \underline{\underline{B}}$ iff $\underline{\underline{A}} \equiv_{L(Q')} \underline{\underline{B}}$ or the games $G^n(\underline{\underline{A}},\underline{\underline{B}})$ are not adequate for the quantifier Q .

Proof. It is enough note that Q is support for Q' .

Question. Is it possible to find a "natural" game theoretical characterization of elementary equivalence for an arbitrary quantifier of type $\langle q\rangle$ (e.g. for quantifier "there exists exactly $\aleph_\alpha$")?

§ 2. Quantifier of the type $\langle q_1, q_2 \rangle$

We restrict our discussion to the case of a quantifier of type $\langle 1,1 \rangle$, but all results hold for general case as well.

Definition 2.1. Quantifier Q of type $\langle 1,1 \rangle$ is monotonic if for arbitrary $\langle A, A_1, A_2 \rangle \in Q$ and arbitrary $A_1' \subseteq A_1$, $A \supseteq A_2' \supseteq A_2$ $\langle A, A_1', A_2' \rangle \in Q$

Note that by this definition Q_I is not monotonic but the following quantifier is monotonic: $(Qx)(\varphi,\psi) \Longleftrightarrow \overline{\overline{\{x\colon \varphi(x)\}}} \leqslant \overline{\overline{\{x\colon \psi(x)\}}}$. The language with this quantifier is a little stonger then the language with Q_I.

Definition 2.2. Now we define the games $G^n(\underline{\underline{A}}, \underline{\underline{B}})$ as in Definition 1.3 with the following changes: the first step player 1 chooses one of the two structures (e.g. $\underline{\underline{A}}$) and A_1, A_2 s.t. $\langle A, A_1, A_2 \rangle \in Q$, player 2 replye choosing B_1, B_2 s.t. $\langle B, B_1, B_2 \rangle \in Q$ and player 1 chooses $i \in \{1,2,-1,-2\}$ and $b \in B_i$ where $B_{-j} = B - B_j$. Finally player 2 chooses $a \in A_i$.

Theorem 2.1. Let Q be a monotonic quantifier of type $\langle 1,1 \rangle$. Then $\underline{\underline{A}} \sim_n \underline{\underline{B}}$ implies $\underline{\underline{A}} \equiv_n \underline{\underline{B}}$.

Proof. As in Theorem 1.1. Let $rk(\varphi) = n+1$, $\underline{\underline{A}} \models \varphi$, $\underline{\underline{B}} \models \neg\varphi$, $\varphi = (Qx)(\varphi_1, \varphi_2)$ and player 1 choose $A_i = \{a \in A\colon \underline{\underline{A}} \models \varphi_i[a]\}$, player 2 using winning strAregy reply B_1, B_2. Let $B_i' = \{b \in B\colon \underline{\underline{B}} \models \varphi_i[b]\}$ hence $B_1' \neq B_1$ or $B_2' \neq B_2$. If for example $B_1' - B_1 \neq \emptyset$ then player 1 choose -1 and $x \in B_1' - B_1$. One can prove that this procedure followed the contradiction.

A converse theorem analogous to Thm. 1.2 holds as well.

Let τ be a fixed finite signature. Similarly as above we define sequences $\{l_p^{n,\tau}\}_{p=0,..,n}$, $\{m_p^{n,\tau}\}_{p=1,..,n}$

$$l_0^{n,\tau} = 0$$

$$m_{p+1}^{n,\tau} = \text{"the number of formulas of the rank } l_p^{n,\tau} \text{ in which occur at most the first } l_p^{n,\tau} + n - p \text{ variables".}$$

$$l_{p+1}^{m,} = (3 \cdot l_p^{n,\tau} \cdot m_{p+1}^{n,\tau} + 2) \cdot 2^{m_{p+1}^{n,\tau}} + 1$$

<u>Theorem</u> 2.2. Let $\underline{\underline{A}},\underline{\underline{B}}$ be a structures of the signature τ. Then for each monotonic quantifier Q of type $\langle 1,1\rangle$ we have:

If $\underline{\underline{A}} \equiv_{l_n^{n,\tau}} \underline{\underline{B}}$ then $\underline{\underline{A}} \sim_n \underline{\underline{B}}$.

<u>Proof</u>. By induction. For $n=0$ it is obvious.
Assume $\underline{\underline{A}} \equiv_{l_{n+1}^{n+1,\tau}} \underline{\underline{B}}$. Let $\varphi_0,\varphi_1,\dots,\varphi_{s-1}$ be all formulas of the rank $l_n^{n+1,\tau}$ with one free variable and which occur at most first $l_n^{n+1,\tau}+1$ variables.
As above, for $\varepsilon \in 2^s$ let $\Theta_\varepsilon = \bigwedge_{i<s} \varphi_i^{\varepsilon(i)}$. Note, that for $l_p^{n,\tau}$ $m_p^{n,\tau}$ and Θ_ε defined in such a manner the conditions (1),(2),(3) from section 1 hold.
We will give a winning stategy for player 2 in the game $G^{m+1}(\underline{\underline{A}},\underline{\underline{B}})$.
In the first move let player 1 choose $A_1, A_2 \subseteq A$ s.t.

$$(4) \qquad \langle A, A_1, A_2\rangle \in Q$$

For $i=1,2$ let

$$J_0^i = \{\varepsilon \in 2^s\colon \text{ for all } a\in A \text{ if } \underline{\underline{A}} \models \Theta_\varepsilon[a] \text{ then } a \in A_i\}$$

$$J_1^i = \{\varepsilon \in 2^s\colon \text{ there are } a \in A_i \text{ and } b \notin A_i \text{ s.t. } \underline{\underline{A}} \models \Theta_\varepsilon[a] \wedge \Theta_\varepsilon[b]\}$$

Let r (t) be a number of elements of J_1^1 (J_1^2).
By (4) and monotonicity of Q we have:

$$\underline{\underline{A}} \models (\exists y_1\dots y_r)(\exists z_1\dots z_t) \bigwedge_{\substack{i\leqslant r\\ \varepsilon^i\in J_1^1}} \Theta_{\varepsilon^i}(y_i) \wedge \bigwedge_{\substack{i\leqslant t\\ \varepsilon^i\in J_1^2}} \Theta_{\varepsilon^i}(z_i) \wedge$$

$$\wedge (Qx)\Big[\bigvee_{\varepsilon\in J_0^1} \Theta_\varepsilon(x) \vee \bigvee_{i\leqslant r} x=y_i \,,\, \bigvee_{\varepsilon\in J_0^2\cup J_1^2} \Theta_\varepsilon(x) \wedge \bigwedge_{i\leqslant t} x\neq z_i\Big]$$

It is easy to see that $rk(\Theta) \leqslant l_{n+1}^{n+1,\tau}$. Hence $\underline{\underline{B}} \models \Theta$ and there are $b_1,\dots,b_r$, $c_1,\dots,c_t \in B$ s.t.

$$\underline{\underline{B}} \models (Qx)\Big[\bigvee_{\varepsilon\in J_0^1} \Theta_\varepsilon(x) \vee \bigvee_{i\leqslant r} x=b_i \,,\, \bigvee_{\varepsilon\in J_0^2\cup J_1^2} \Theta_\varepsilon(x) \wedge \bigwedge_{j\leqslant t} x\neq c_j\Big]$$

and

$$\underline{\underline{B}} \models \bigwedge_{i\leq r} \Theta_{\varepsilon^i}[b_i] \wedge \bigwedge_{j\leq t} \Theta_{\varepsilon^j}[c_j]$$

Let player 2 choose the following subsets

$$B_1 = \{b : \underline{\underline{B}} \models \bigvee_{\varepsilon\in J_0^1} \Theta_\varepsilon[b] \vee \bigvee_{i\leq r} b = b_i\}$$

$$B_2 = \{b : \underline{\underline{B}} \models \bigvee_{\varepsilon\in J_0^2\cup J_1^2} \Theta_\varepsilon[b] \wedge \bigwedge_{j\leq t} b \neq c_j\}$$

Let player 1 choose $i \in \{1,2,-1,-2\}$ and $b \in B_i$. There exists exactly one ε s.t. $\underline{\underline{B}} \models \Theta_\varepsilon[b]$. From definition of B_1 follows that we may find $a \in A_i$ s.t. $\underline{\underline{A}} \models \Theta_\varepsilon[a]$. Hence $\langle A,a\rangle \equiv_{1_n^{n+1},\tau} \langle \underline{\underline{B}},b\rangle$ and $\langle \underline{\underline{A}},a\rangle \equiv_{1_n^n,\tau'} \langle \underline{\underline{B}},b\rangle$ where $\tau' = \tau \cup \{c\}$.

So, from the inductive assumption $\langle \underline{\underline{A}},a\rangle \sim_n \langle \underline{\underline{B}},b\rangle$.

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