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THE CHARACTERIZATION OF MONADIC LOGIC

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The first section of this paper is concerned with the intrinsic properties of elementary monadic logic (EM), and characterizations in the spirit of Lindström [2] are given. His proofs do not apply to monadic logic since relations are used, and intrinsic properties of EM turn out to differ in certain ways from those of the elementary logic of relations (i.e., the predicate calculus), which we shall call EL. In the second section we investigate connections between higher-order monadic and polyadic logics.

§1. EM is the subsystem of EL which results by the restriction to one-place predicate letters. We omit constants (for simplicity) but take EM to contain identity. Let a *type* be any finite sequence (possibly empty) of one-place predicate letters. A model M of type $\mathscr{T}$ has a nonempty universe $|M|$ and assigns to each predicate letter P of $\mathscr{T}$ a subset P_M of $|M|$.

Let us take a monadic logic L to be any collection of classes of models, called L-classes, satisfying the following:

1. All models in a given L-class are of the same type (called the type of the class).
2. Isomorphic models lie in the same L-classes.
3. If $\mathscr{K}_1$ and $\mathscr{K}_2$ are L-classes of the same type, then $\overline{\mathscr{K}_1}$ and $\mathscr{K}_1 \cap \mathscr{K}_2$ are L-classes.

An L-class, of course, is to be thought of as the class of models satisfying a given closed formula. Clearly EM satisfies this definition of logic, as do logics constructed by adding to EM generalized quantifiers (see Lindström [3]). These conditions are somewhat simpler than those needed for polyadic logics.

A logic L is *compact* $=_{\mathrm{DF}}$. For any L-classes $\mathscr{K}_i$ $(i \in \omega)$, all of the same type, $\bigcap_{i<\omega}\mathscr{K}_i \neq \varnothing$ provided that, for all n, $\bigcap_{i<n}\mathscr{K}_i \neq \varnothing$.

In general it is necessary to distinguish two versions of the Löwenheim-Skolem property:

LS_1 $=_{\mathrm{DF}}$ Every nonempty L-class has a countable (i.e., finite or denumerably infinite) member.

LS_ω $=_{\mathrm{DF}}$ Given any L-classes $\mathscr{K}_i$ $(i \in \omega)$, all of the same type, if $\bigcap_{i<\omega}\mathscr{K}_i \neq \varnothing$ then $\bigcap_{i<\omega}\mathscr{K}_i$ has a countable member.

LS_ω says that if a countable set of formulas of the same type has a model, then it has a countable model. Trivially, LS_ω implies LS_1.

As is well known, EM is compact and satisfies LS_ω as well as the further condition $\mathscr{C}$: Suppose ψ has p predicate letters and v variables. If $M \models \psi$, then there is a $J \subseteq M$, of cardinality $\leq 2^p \cdot v$, such that for any J', if $J \subseteq J' \subseteq M$, then $J' \models \psi$. (See Jensen [4].) This condition clearly gives a decision procedure for validity.

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Let us say $L_1 \subseteq L_2$ if all L_1-classes are L_2-classes; $L_1 \equiv L_2$ if $L_1 \subseteq L_2$ and $L_2 \subseteq L_1$.

LEMMA. *Suppose* EM $\subseteq L$ *and* $\mathscr{K}$ *is an* L-*class but not an* EM-*class. Let* $\mathscr{M}_1, \mathscr{M}_2, \cdots$ *be a complete list of the* EM *classes of the type of* $\mathscr{K}$. *Let* $\mathscr{M}^\varepsilon$ *be* $\mathscr{M}$ *or* $\overline{\mathscr{M}}$ *according to whether* ε *is* 0 *or* 1. *Then there is a sequence* $\varepsilon_1, \varepsilon_2, \cdots$ *such that, for all* n, $\mathscr{K} \cap \mathscr{M}_1^{\varepsilon_1} \cap \cdots \cap \mathscr{M}_n^{\varepsilon_n}$ (*call it* $\mathscr{K}_n$) *and* $\overline{\mathscr{K}} \cap \mathscr{M}_1^{\varepsilon_1} \cap \cdots \cap \mathscr{M}_n^{\varepsilon_n}$ (*call it* $\mathscr{K}_n'$) *are, neither of them*, EM-*classes*.

PROOF. By induction on n. For $n = 0$, $\mathscr{K}_n = \mathscr{K}$ which is not EM. $\mathscr{K}_n' = \overline{\mathscr{K}}$, and if $\overline{\mathscr{K}}$ were EM, $\overline{\overline{\mathscr{K}}} = \mathscr{K}$ would be EM also.

Assume the conclusion for n. $\mathscr{K}_n = (\mathscr{K}_n \cap \mathscr{M}_{n+1}) \cup (\mathscr{K}_n \cap \overline{\mathscr{M}}_{n+1})$. If both $\mathscr{K}_n \cap \mathscr{M}_{n+1}$ and $\mathscr{K}_n \cap \overline{\mathscr{M}}_{n+1}$ were EM, their union $\mathscr{K}_n$ would be EM. Take one of them which is not EM as $\mathscr{K}_{n+1}$. This gives a value for ε_{n+1} and it is only necessary to check that

$$\mathscr{K}_{n+1}' = \overline{\mathscr{K}} \cap \mathscr{M}_1^{\varepsilon_1} \cap \cdots \cap \mathscr{M}_n^{\varepsilon_n} \cap \mathscr{M}_{n+1}^{\varepsilon_{n+1}}$$

is not EM. If it were, then $[\mathscr{M}_1^{\varepsilon_1} \cap \cdots \cap \mathscr{M}_n^{\varepsilon_n} \cap \mathscr{M}_{n+1}^{\varepsilon_{n+1}}] - [\overline{\mathscr{K}} \cap \mathscr{M}_1^{\varepsilon_1} \cap \cdots \cap \mathscr{M}_n^{\varepsilon_n} \cap \mathscr{M}_{n+1}^{\varepsilon_{n+1}}]$ would be EM. But this difference is just $\mathscr{K}_{n+1}$.

THEOREM 1. *Let* L *be a compact monadic logic satisfying* LS_ω. *Then if* EM $\subseteq L$, EM $\equiv L$.

PROOF. If not EM $\equiv L$, then there is an L-class $\mathscr{K}$ which is not an EM-class. By the lemma, the L-classes $\mathscr{K}_n$ and $\mathscr{K}_n'$ are all nonempty since they are not EM classes. Thus, since L is compact, $\mathscr{K} \cap \bigcap_{i<\omega} \mathscr{M}_i^{\varepsilon_i}$ is nonempty and so is $\overline{\mathscr{K}} \cap \bigcap_{i<\omega} \mathscr{M}_i^{\varepsilon_i}$.

Since L satisfies LS_ω both classes contain countable members. Let M be a countable member of $\mathscr{K} \cap \bigcap_{i<\omega} \mathscr{M}_i^{\varepsilon_i}$ and let N be a countable member of $\overline{\mathscr{K}} \cap \bigcap_{i<\omega} \mathscr{M}_i^{\varepsilon_i}$. M and N satisfy the same EM formulas, and we shall next show that they are isomorphic since they are countable.

Suppose the type of $\mathscr{K}$ is $\langle P_1, P_2, \cdots, P_p \rangle$, where $p \geq 0$. If $p > 0$ let a *monad* be any formula of the form $P_1^{\delta_1}(v_0) \wedge \cdots \wedge P_p^{\delta_p}(v_0)$ (where P^δ is P or $\neg P$, according as $\delta = 0$ or $\delta = 1$). If $p = 0$, let $v_0 = v_0$ be the only *monad*. In any case there are 2^p monads, and the universe of a model is partitioned into 2^p disjoint sets by the 2^p formulas.

Let $|\mu| = k$, $|\mu| \geq k$ and $|\mu| < k$ be the EM formulas stating that the monad μ has cardinality k, cardinality $\geq k$, and cardinality $< k$. Since M and N satisfy the same EM formulas, corresponding sets in the partitions of $|M|$ and $|N|$ defined by the monads either have the same finite cardinality or are both infinite. Since M and N are countably infinite, infinite subsets have cardinality $\aleph_0$. Thus there is a 1:1 onto function f from $|M|$ to $|N|$ which preserves the partitions. Such a function preserves the relations of the models and so it is an isomorphism. Thus M and N are isomorphic, but $M \in \mathscr{K}$ and $N \in \overline{\mathscr{K}}$, which yields the desired contradiction.

The above proof is a much simplified version, due to D. A. Martin, of my original proof. Martin also showed that the assumption of compactness can be weakened to "compact for recursive sets of formulas" under reasonable assumptions about the syntax of L. Assume (for instance) that the formulas of L are a set of numbers containing the formulas of EM; an EM formula is to define the same class of models in L as in EM.

THEOREM 2 (MARTIN). *If L satisfies* LS_ω *and is compact on recursive sets, then* $\mathrm{EM} \equiv L$.

PROOF. For simplicity, say that a formula of L is EM if it defines the same class of models as some EM formula. If L properly contains EM, then there is a formula ψ which is not EM. Assume that ψ uses the letters $P_1, P_2, \cdots, P_p$, $p \geq 0$. Let $\mu_1, \mu_2, \cdots, \mu_{2^p}$ be the monads.

We shall partition the set $\{1, 2, \cdots, 2^p\}$ into two disjoint sets F and I. For $i \in F$ ($i \in I$) the monad μ_i will be true of finitely (infinitely) many points in the ultimate models.

The partition is defined in a sequence of stages. At stage 1, if $\psi \wedge (|\mu_1| = k)$ is not EM for some k, let k_1 be the first such k, and put 1 in F. Otherwise put 1 in I.

To proceed from stage n to stage $n + 1$, suppose F_n is the subset of F already defined, and consider the formulas $\psi \wedge \bigwedge_{i \in F_n}(|\mu_i| = k_i) \wedge |\mu_{n+1}| = k$. If, for some k, the corresponding formula is not EM, let k_{n+1} be the first such k and put $n + 1$ in F. Otherwise put $n + 1$ in I. This procedure clearly partitions $\{1, 2, \cdots, 2^p\}$.

$\psi \wedge \bigwedge_{i \in F}(|\mu_i| = k_i)$ is not EM by construction. Neither is $\neg\psi \wedge \bigwedge_{i \in F}(|\mu_i| = k_i)$, for if it were, so would be $\bigwedge_{i \in F}(|\mu_i| = k_i) \wedge \neg(\neg\psi \wedge \bigwedge_{i \in F}(|\mu_i| = k_i))$ which is equivalent to $\psi \wedge \bigwedge_{i \in F}(|\mu_i| = k_i)$.

Let

$$X = \Big\{\psi, \bigwedge_{i \in F} (|\mu_i| = k_i)\Big\} \cup \Big\{\bigwedge_{j \in I} (|\mu_j| \geq n) : n \in \omega\Big\},$$

$$X' = \Big\{\neg\psi, \bigwedge_{i \in F} (|\mu_i| = k_i)\Big\} \cup \Big\{\bigwedge_{j \in I} (|\mu_j| \geq n) : n \in \omega\Big\}.$$

X and X' are both recursive. If both have models, then both have countable models by LS_ω. As in the preceding theorem these countable models must be isomorphic, which is a contradiction, since one of them satisfies ψ and the other satisfies $\neg\psi$.

If X has no model, then by compactness for recursive sets, for some n the formula

$$\psi \wedge \bigwedge_{i \in F} (|\mu_i| = k_i) \wedge \bigwedge_{j \in I} (|\mu_j| \geq n) \tag{1}$$

has no model, and is EM.

By the definition of F and I, for each $j \in I$ there is a subset F_{j-1} of F such that $\psi \wedge \bigwedge_{i \in F_{j-1}}(|\mu_i| = k_i) \wedge |\mu_j| = m$ is EM for every m. Hence for every m, $\psi \wedge \bigwedge_{i \in F}(|\mu_i| = k_i) \wedge |\mu_j| = m$ is EM by the intersection of EM classes. Hence $\bigvee_{j \in I} \bigvee_{m < n}[\psi \wedge \bigwedge_{i \in F}(|\mu_i| = k_i) \wedge |\mu_j| = m]$ is EM. This formula is equivalent to

$$\psi \wedge \bigwedge_{i \in F} (|\mu_i| = k_i) \wedge \bigvee_{j \in I} (|\mu_j| < n). \tag{2}$$

The disjunction of (1) and (2) thus is EM. But this disjunction is equivalent to $\psi \wedge \bigwedge_{i \in F}(|\mu_i| = k_i)$ which is not EM. This is a contradiction, so X has a model. By a similar argument X' has a model. This concludes the proof.

The intrinsic properties of EM differ from those of EL in that, roughly speaking, condition LS_1 plus either axiomatizability or compactness suffice to characterize EL. Also, Lindström has informed me that for extensions of EL defined by the addition of finitely many generalized quantifiers, axiomatizability implies compactness on r.e. sets of formulas.

By adding generalized quantifiers to EM it is easy to give counterexamples to various weakened versions of the hypotheses used to characterize EM.

EXAMPLE 1. There is a monadic logic L which is decidable, satisfies LS_ω, but properly contains EM.

PROOF. Add to EM the quantifier (i.e., sentential letter) E which is true in models of even finite cardinality.

L satisfies LS_ω: If $\{\psi_1, \psi_2, \cdots\}$ has an uncountable model M, then in M the letter E is false. Let ψ^F be the EM formula resulting from ψ by replacing E by some contradictory EM formula. M satisfies $\{\psi_1^F, \psi_2^F, \cdots\}$, so there is a countably infinite model N of $\{\psi_1^F, \psi_2^F, \cdots\}$ by LS_ω for EM. N is also a model for $\{\psi_1, \psi_2, \cdots\}$ since E is false in N.

L is decidable: If $M \models \psi$, we claim that there is a model of ψ whose cardinality is less than some finite bound calculable from ψ. If E is true in M, let ψ' be ψ^T, the result of replacing E in ψ by a valid EM formula. Otherwise let ψ' be ψ^F. $M \models \psi'$, and since EM satisfies the condition $\mathscr{C}$ mentioned above, there is a model $J \subseteq M$ satisfying ψ' of cardinality $\leq 2^p \cdot v$. If E has a different value in J than in M, extend J to J' by adding a point of M; otherwise let J' be J. In either case, $J' \models \psi'$ and the cardinality of J' is no greater than $2^p \cdot v + 1$.

L is clearly not compact: Consider the recursive set $\{E, \exists \geq nx(x = x) : n \in \omega\}$. Even in a compact logic the condition LS_ω cannot be weakened to LS_1.

EXAMPLE 2. There is a monadic logic L extending EM which is compact, decidable, and every formula with a model has a finite model.

PROOF. Define L by adding to EM the quantifier S, which is true in models of even finite or uncountable cardinality. As in the previous example, every formula ψ with a model has a finite model whose cardinality is bounded by a number calculable from ψ. This proves the last two assertions.

To show that L is compact, consider a given set $\{\psi_0, \psi_1, \cdots\}$ such that each finite subset has a model. Let $\theta_i = \bigwedge_{j<i}\psi_j$. Let $f(i)$ be the smallest cardinal of any model of θ_i. f has finite values and is monotone increasing. Suppose f is bounded by some number k. Since there are only finitely many models of cardinality $\leq k$, some one model satisfies infinitely many θ_i and hence satisfies all θ_i.

On the other hand, if f is unbounded, then either there are infinitely many i such that S comes out true in some smallest model of θ_i, or else S comes out false infinitely often. In the former case, each of the formulas θ_i^T has models of arbitrarily large finite cardinality, so by compactness for EM, $\{\theta_i^T : i \in \omega\}$ has an uncountable model M. M satisfies $\{\theta_i : i \in \omega\}$ since the quantifier S is true in M. The case where S comes out false infinitely often is similar, since one gets a countably infinite model of $\{\theta_i^F : i \in \omega\}$ which also satisfies $\{\theta_i : i \in \omega\}$.

The condition $\mathscr{C}$ may be viewed as a strong continuity property of EM. More generally, consider the condition $\mathscr{D}$: There is a function f mapping the formulas into ω such that if M satisfies ψ, then there is a $J \subseteq M$, of cardinality $\leq f(\psi)$, such that, for all J', if $J \subseteq J' \subseteq M$ then J' satisfies ψ.[1]

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[1] A similar continuity notion can be applied to quantifiers of polyadic logics. See my abstract *The uniqueness of elementary logic*, *Notices of the American Mathematical Society*, vol. 20 (1973).

It is not hard to see, by using I-sequences (see Lindström [2]), that condition $\mathscr{D}$ characterizes EM. This result also follows readily from Martin's construction in Theorem 2.

THEOREM 3. *If* EM $\subseteq L$ *and* L *satisfies* $\mathscr{D}$, *then* EM $\equiv L$.

PROOF. If not EM $\equiv L$, consider some ψ which is not EM. Let $m = \max(f(\psi), f(\neg\psi))$. Without using compactness one has a model M of

$$\psi \wedge \bigwedge_{i \in F} (|\mu_i| = k_i) \wedge \bigwedge_{j \in I} (|\mu_j| \geq m),$$

and a model N of $\neg\psi \wedge \bigwedge_{i \in F}(|\mu_i| = k_i) \wedge \bigwedge_{j \in I}(|\mu_j| \geq m)$.

Let $J \subseteq M$ and $K \subseteq N$ be the submodels given by condition $\mathscr{D}$, which have cardinality $\leq m$. M has k_i points satisfying the monad μ_i $(i \in F)$ and at least m points satisfying the monad μ_j $(j \in I)$. Extend J to J' by adding points until J' satisfies $\bigwedge_{i \in F}|\mu_i| = k_i \wedge \bigwedge_{j \in I}|\mu_j| = m$. Likewise extend K to K'. Then J' and K' are isomorphic, but J' satisfies ψ and K' satisfies $\neg\psi$.

In this theorem one is not assuming that f is recursive. Indeed the formulas of L may be any set such that there is a function mapping the formulas onto the L-classes.

§2. Let EM^2 be second-order monadic logic, which differs from EM by allowing quantification over predicate letters. It is well known that $\mathrm{EM}^2 \equiv \mathrm{EM}$, and that the equivalence is effective (see Ackermann [5]). Consider next EM^3 where one has letters for predicates of predicates, and allows quantification over them, as well as over predicate letters and variables. A typical formula might be $\exists\mathscr{P}\forall P[\mathscr{P}(P) \leftrightarrow \exists x P(x)]$, which is clearly valid. A formula of EM^3 which has only first-order predicate letters free admits models of the same sort as a formula of EM, so one may ask whether $\mathrm{EM} \equiv \mathrm{EM}^3$. We shall show that this is false in the course of a construction which demonstrates that EM^3 is, in a reasonable sense, the same as the second-order logic of relations, which we call EL^2.

THEOREM 4. EL^2 *can be interpreted in* EM^3. *In particular* $\mathrm{Val}(\mathrm{EL}^2) \approx_r \mathrm{Val}(\mathrm{EM}^3)$ *where* $\approx_r$ *means "is recursively isomorphic to"*.

PROOF. We sketch the proof since the details are cumbersome, first defining several formulas of EM^3.

1. $\mathscr{P}$ is a linear ordering:

$$\mathrm{Lo}(\mathscr{P}) =_{\mathrm{DF}} \forall P \forall Q[(\mathscr{P}(P) \wedge \mathscr{P}(Q)) \to (P \subseteq Q \vee Q \subseteq P)].$$

We shall consider well-orderings similar to the von Neumann ordinals. If $\mathscr{P}$ satisfies Wo, which we define below, then $\mathscr{P}$ will look like $\{\varnothing, \{\bar{0}\}, \cdots, \{\bar{0}, \bar{1}, \cdots, \bar{\gamma}, \cdots\}_{\gamma<\beta}, \cdots\}_{\beta<\alpha}$ where the $\bar{\gamma}$ are elements of the universe of the model.

2. (a) $\mathscr{P}$ is well-founded:

$$\mathrm{Wf}(P) =_{\mathrm{DF}} \forall \mathscr{Q}[(\mathscr{Q} \subseteq \mathscr{P} \wedge \mathscr{Q} \neq \varnothing) \to \exists P(\mathscr{Q}(P) \wedge \forall Q(Q \subsetneq P \to \neg\mathscr{Q}(Q)))].$$

(b) P is a successor of $\mathscr{P}$:

$$\mathrm{Succ}(\mathscr{P}, P) =_{\mathrm{DF}} \exists Q[\mathscr{P}(Q) \wedge Q \subsetneq P \wedge \exists v_0(P = Q \cup \{v_0\})].$$

(c) P is a limit member of $\mathscr{P}$:

$$\mathrm{Lim}(\mathscr{P}, P) =_{\mathrm{DF}} \forall v_0[P(v_0) \to \exists Q(\mathscr{P}(Q) \wedge Q \subsetneq P \wedge Q(v_0))].$$

(d) $\mathrm{Wo}(\mathscr{P}) =_{\mathrm{DF}} \mathrm{Lo}(\mathscr{P}) \wedge \mathrm{Wf}(\mathscr{P}) \wedge \forall P[\mathscr{P}(P) \to (\mathrm{Succ}(\mathscr{P}, P) \vee \mathrm{Lim}(\mathscr{P}, P))].$

3. $\mathscr{P}$ is the natural numbers:

$$\mathrm{Nat}(\mathscr{P}) =_{\mathrm{DF}} \mathrm{Wo}(\mathscr{P}) \wedge \forall P[\mathscr{P}(P) \to (\mathrm{Succ}(\mathscr{P}, P) \vee P = \varnothing)] \wedge \forall P[\mathscr{P}(P) \to \exists v_0(\neg P(v_0) \wedge \mathscr{P}(P \cup \{v_0\}))].$$

To say that a model is infinite is to say $\exists\mathscr{P}\ \mathrm{Nat}(\mathscr{P})$. Thus clearly $\mathrm{EM}^3 \not\equiv \mathrm{EM}^2$ and moreover EM^3 is not compact, since one can say the universe is finite.

We shall be interested in models whose universe is split into two disjoint pieces, one piece being the field of $\mathscr{P}$ where $\mathrm{Nat}(\mathscr{P})$, and the other piece being the (non-empty) field of $\mathscr{Q}$ where $\mathrm{Wo}(\mathscr{Q})$.

4. (a) $\mathrm{Fld}(v_0, \mathscr{P}) =_{\mathrm{DF}} \exists P(\mathscr{P}(P) \wedge P(v_0))$,

(b) $\mathrm{Cond}(\mathscr{P}, \mathscr{Q}) =_{\mathrm{DF}} \mathrm{Nat}(\mathscr{P}) \wedge \mathrm{Wo}(\mathscr{Q}) \wedge \exists v_0\, \mathrm{Fld}(v_0, \mathscr{Q}) \wedge \forall v_0(\mathrm{Fld}(v_0, \mathscr{P}) \leftrightarrow \neg\mathrm{Fld}(v_0, \mathscr{Q}))$.

A $\mathscr{Q}$ such that $\mathrm{Wo}(\mathscr{Q})$ defines a well-ordering of its field:

$$v_0 <_{\mathscr{Q}} v_1 =_{\mathrm{DF}} \exists P(\mathscr{Q}(P) \wedge P(v_0) \wedge \neg P(v_1)).$$

In particular, if $\mathrm{Nat}(\mathscr{P})$ then one has a well-ordering of the field of $\mathscr{P}$, and one may identify n with $\bar{n}$ where $\bar{0}, \bar{1}, \cdots$ is the sequence determined by $<_{\mathscr{P}}$.

Given $\mathscr{Q}$, such that $\mathrm{Wo}(\mathscr{Q})$, and $x_1, \cdots, x_k$ such that $x_1 <_{\mathscr{Q}} x_2 <_{\mathscr{Q}} \cdots <_{\mathscr{Q}} x_k$, we may regard the n-tuple $\langle x_{i_1}, \cdots, x_{i_n}\rangle$ $(n \geq k)$ as the set $\{x_1, \cdots, x_k, \bar{c}\}$ where c is a number coding instructions for ordering $\{x_1, \cdots, x_k\}$ relative to $<_{\mathscr{Q}}$. Thus given $x_1 <_{\mathscr{Q}} x_2 <_{\mathscr{Q}} x_3$, and $c = 2^3 \cdot 3^1 \cdot 5^1 \cdot 7^2$ we take $\{x_1, x_{23}, x, \bar{c}\}$ to be $\langle x_3, x_1, x_1, x_2\rangle$.

We thus define the predicate of membership, $\mathrm{Mem}(\mathscr{P}, \mathscr{Q}, v_1, \cdots, v_n, \mathscr{G})$ which is to mean $\langle v_1, \cdots, v_n\rangle \in \mathscr{G}$ for v_i in the field of $\mathscr{Q}$. There are finitely many ways that the v_i may be identified and ordered relative to $<_{\mathscr{Q}}$. For each way there is a code c. Let θ_c state that $v_1, \cdots, v_n$ are identified and ordered in the manner coded by c. Then the disjunction of $(\theta_c \wedge \{v_1, \cdots, v_n, \bar{c}\} \in \mathscr{G})$ defines the predicate

$$\mathrm{Mem}(\mathscr{P}, \mathscr{Q}, v_1, \cdots, v_n, \mathscr{G}).$$

(We are ignoring those members of $\mathscr{G}$ which do not encode n-tuples.) Clearly every $\mathscr{G}$ encodes a unique n-place relation, and every n-place relation is encoded by some $\mathscr{G}$.

Given any formula $\psi(G_1, \cdots, G_p)$ of EL^2 construct the formula

$$\psi'(\mathscr{P}, \mathscr{Q}, \mathscr{G}_1, \cdots, \mathscr{G}_p)$$

of EM^3 by first replacing atomic formulas $G_j(v_{i_1}, \cdots, v_{i_n})$ by

$$\mathrm{Mem}(\mathscr{P}, \mathscr{Q}, v_{i_1}, \cdots, v_{i_n}, \mathscr{G}_j).$$

Then change first-order quantifications to quantifications over the field of $\mathscr{Q}$, and second-order quantifications to third-order quantifications.

Given a model M for EL^2 of type $\langle G_1, \cdots, G_p\rangle$ construct a model N for EM^3 of type $\langle \mathscr{P}, \mathscr{Q}, \mathscr{G}_1, \cdots, \mathscr{G}_p\rangle$: Let $|N| = \omega \cup |M|$ (where we assume $\omega \cap |M| = \varnothing$). Let $\mathscr{P}$ and $\mathscr{Q}$ be such that $N \models \mathrm{Cond}(\mathscr{P}, \mathscr{Q})$, and let $(\mathscr{G}_j)_N$ encode the relation $(G_j)_M$. Then $M \models \psi \Leftrightarrow N \models \psi'$. Further, if ψ is closed, then ψ is valid $\Leftrightarrow$ $\forall\mathscr{P}\forall\mathscr{Q}[\mathrm{Cond}(\mathscr{P}, \mathscr{Q}) \to \psi']$ is valid.

Thus one has a 1:1 recursive function mapping $\mathrm{Val}(\mathrm{EL}^2)$ into $\mathrm{Val}(\mathrm{EM}^3)$. It is known that $\mathrm{Val}(\mathrm{EL}^k) \approx_r \mathrm{Val}(\mathrm{EL}^2)$ for $k > 2$ (since EL^k can be interpreted in a certain second-order set theory). Obviously $\mathrm{Val}(\mathrm{EM}^3) \leq_1 \mathrm{Val}(\mathrm{EL}^3)$, so one has $\mathrm{Val}(\mathrm{EM}^3) \approx_r \mathrm{Val}(\mathrm{EL}^2)$. This completes the proof.

The result that $\mathrm{Val}(\mathrm{EL}^k) \approx_r \mathrm{Val}(\mathrm{EL}^2)$ may be extended into the transfinite. Recall, however, that $\mathrm{Val}(L_1) \approx_r \mathrm{Val}(L_2)$ does not imply $L_1 \equiv L_2$: For example, the class of measurable cardinals is an EL^3-class but not an EL^2-class. Some light is shed on the status of $\mathrm{Val}(\mathrm{EL}^2)$ by showing that it is a natural set of numbers in the classificatory scheme given by Lévy [6].

THEOREM 5. *$\mathrm{Val}(\mathrm{EL}^2)$ is Π_2-complete.*

PROOF. To show $\mathrm{Val}(\mathrm{EL}^2)$ is Π_2 it suffices to show that the relation "M satisfies n" is Π_2. This follows by considering negations of formulas, if one can show "M satisfies n" is Σ_2. However, this is clear since the relation can be defined by the formula

$$\exists T[\mathrm{Trans}(T) \wedge \forall y \forall x(x \in T \wedge y \subseteq x \rightarrow y \in T) \wedge M \in T \wedge \beta(T) \wedge \tau(M, n, T)]$$

where $\mathrm{Trans}(T)$ means T is transitive, β is a formula bounded by T saying that T is closed under certain simple operations such as pairing, union, and crossproducts, and τ is the truth definition relativized to T. The closure conditions $x \in T \wedge y \subseteq x \rightarrow y \in T$ and $\beta(T)$ are strong enough to ensure that if n is true in M relative to T, then n is true in M.

To show that any Π_2 predicate can be reduced to $\mathrm{Val}(\mathrm{EL}^2)$, first note that there is a formula θ of EL^2 such that M satisfies θ iff M is isomorphic to an $R(\alpha)$ satisfying

(1) α is a cardinal $> \aleph_0$ and

(2) $y \in R(\alpha) \rightarrow \mathrm{Card}\, y < \alpha$.

It suffices to let θ be the conjunction of the axiom of extensionality, the axiom of infinity, $\forall x \exists \gamma(x \in R(\gamma))$, $\forall x \exists \gamma(\gamma = \mathrm{Card}\, x)$, and the second-order formulas stating that ϵ is well-founded and that every subclass of a set is a set.

Then we claim that $\forall x \exists y \psi(\bar{n}, x, y)$ is true iff the formula $(\theta \rightarrow \forall x \exists y \psi(\bar{n}, x, y))$ is valid, i.e., iff $\forall x \exists y \psi(\bar{n}, x, y)$ holds in all $R(\alpha)$ satisfying (1) and (2). (ψ is bounded and $\bar{n}$ is a constant.)

$\Leftarrow$ If not $\forall x \exists y \psi(\bar{n}, x, y)$, then, by the reflection principle of ZF, there is an $R(\alpha)$ such that $R(\alpha) \models \neg \forall x \exists y \psi(\bar{n}, x, y)$. Since we may assume $R(\alpha)$ satisfies any finite list of formulas of ZF, it follows that we may assume $R(\alpha)$ has properties (1) and (2).

$\Rightarrow$ Suppose $\forall x \exists y \psi(\bar{n}, x, y)$ but there is an $R(\alpha)$ satisfying (1) and (2) such that $R(\alpha) \models \exists x \forall y \neg \psi(\bar{n}, x, y)$. Then $R(\alpha) \models \forall y \neg \psi(\bar{n}, x_0, y)$ for some $x_0 \in R(\alpha)$. Since $R(\alpha)$ is a limit ordinal, $x_0 \in R(\gamma) \in R(\alpha)$ for a certain γ. Since $\exists y \psi(\bar{n}, x_0, y)$, by a Löwenheim-Skolem construction there is a transitive set T, $R(\gamma) \subseteq T$, $\mathrm{Card}\, T \leq \mathrm{Card}\, R(\gamma) + \aleph_0$, and $T \models [\exists y \psi(\bar{n}, x_0, y) \wedge \forall x \exists \delta(x \in R(\delta))]$.

We claim that $T \subseteq R(\alpha)$: If $x \in T$ then for a certain $\delta \in T$, $T \models [x \in R(\delta)]$. In fact $x \in R(\delta)$ is true because a power set relative to T is contained in the absolute power set. But $\delta \in T \rightarrow \delta \subseteq T \rightarrow \mathrm{Card}\, \delta \leq \mathrm{Card}\, T$, and $\mathrm{Card}\, T \leq \mathrm{Card}\, R(\gamma) + \aleph_0 < \alpha$ since $R(\alpha)$ satisfies (1) and (2). So $\mathrm{Card}\, \delta < \alpha$ and $\delta < \alpha$ since α is a cardinal. Hence $R(\delta) \subseteq R(\alpha)$ and $x \in R(\alpha)$.

Thus we have $T \subseteq R(\alpha)$, and since $T \models [\exists y \psi(\bar{n}, x_0, y)]$, $R(\alpha)$ must satisfy $\exists y \psi(\bar{n}, x_0, y)$. This is a contradiction.

If the satisfaction relation of a logic L is Π_2, then the set $\mathrm{Val}(L)$ will be Π_2, and so $\mathrm{Val}(L)$ will be 1:1 reducible to $\mathrm{Val}(\mathrm{EL}^2)$. In particular, the satisfaction relations of EL^k $(k \geq 1)$ are obviously Π_2.

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