Definition 3.10. We say that $\varphi \in \mathrm{FOE}_1^\infty(A)$ is in *basic form* if $\varphi = \bigvee \nabla^+_{\mathrm{FOE}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ where each disjunct is of the form

$$\nabla^+_{\mathrm{FOE}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) := \nabla^+_{\mathrm{FOE}}(\overline{\mathbf{T}}, \Pi \cup \Sigma) \wedge \nabla^+_\infty(\Sigma)$$
$$\nabla^+_\infty(\Sigma) := \bigwedge_{S \in \Sigma} \exists^\infty y.\tau^+_S(y) \wedge \forall^\infty y. \bigvee_{S \in \Sigma} \tau^+_S(y).$$

for some set of types $\Pi, \Sigma \subseteq \wp A$ and $T_1, \ldots, T_k \subseteq A$.

Intuitively, the formula $\nabla^+_\infty(\Sigma)$ extends the information given by $\nabla^+_{\mathrm{FOE}}(\overline{\mathbf{T}}, \Pi \cup \Sigma)$ by saying that (1) for every type $S \in \Sigma$, there are infinitely many elements satisfying S and (2) only finitely many elements do not satisfy any type in Σ. In short, any one-step model satisfying $\nabla^+_{\mathrm{FOE}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ admits a partition of its domain in three parts: (1) distinct elements $t_1, \ldots, t_n$ witnessing types $T_1, \ldots, T_n$ respectively; (2) finitely many elements whose types belong to Π, and (3) for each $S \in \Sigma$, infinitely many elements witnessing type S.

THEOREM 3.11 ([?]).

(i) $\varphi \in \mathrm{FOE}_1^\infty(A)$ *is monotone if and only if it belongs to* $\mathrm{FOE}_1^{\infty+}(A)$.
(ii) $\varphi \in \mathrm{FOE}_1^{\infty+}(A)$ *is continuous in* $B \subseteq A$ *if and only if it belongs to* $\mathrm{FOE}_1^\infty\mathsf{CON}_B(A)$.
(iii) $\varphi \in \mathrm{FOE}_1^{\infty+}(A)$ *is cocontinuous in* $B \subseteq A$ *if and only if it belongs to* $\mathrm{FOE}_1^\infty\overline{\mathsf{CON}}_B(A)$.
(iv) There is an effective translation $(\cdot)^\bullet : \mathrm{FOE}_1^{\infty+}(A) \to \mathrm{FOE}_1^{\infty+}(A)$ *mapping* φ *into an equivalent sentence in basic form* $\varphi^\bullet = \bigvee \nabla^+_{\mathrm{FOE}^\infty}(\overline{\mathbf{T}}, \Psi \cup \Sigma, \Sigma)$ *such that:*
   (a) φ *is functional in* B *if and only if each* $T_1, \ldots, T_k$ *and* $S \in \Psi$ *in* $\varphi^\bullet$ *are either* $\emptyset$ *of singletons* $\{b\}$ *for some* $b \in B$.
   (b) φ *is continuous in* B *if and only if* $b \notin \bigcup \Sigma$ *for all* $b \in B$.

Remark 3.12. We focussed on normal form results for monotone and (co)continuous sentences, as these are the ones relevant to our study of parity automata. However, generic sentences both of FO_1, FOE_1 and FOE_1^∞ also enjoy normal form results, with the syntactic formats given by variations of the "basic form" above. The interested reader may find in [?] a detailed overview of these results.

3.2. Parity automata

Throughout the rest of the section we fix, next to a set P of proposition letters, a one-step language $\mathcal{L}_1$, as defined in Subsection 3.1. In light of the results therein, we assume that we have isolated fragments $\mathcal{L}_1^+(A)$, $\mathcal{L}_1\mathsf{CON}_B(A)$ and $\mathcal{L}_1\overline{\mathsf{CON}}_B(A)$ consisting of one-step formulas in $\mathcal{L}_1(A)$ that are respectively monotone, B-continuous and B-cocontinuous, for $B \subseteq A$.

We first recall the definition of a general parity automaton, adapted to this setting.

Definition 3.13 (*Parity Automata*). A ***parity automaton*** based on the one-step language $\mathcal{L}_1$ and the set P of proposition letters, or briefly: an $\mathcal{L}_1$*-automaton*, is a tuple $\mathbb{A} = \langle A, \Delta, \Omega, a_I \rangle$ such that A is a finite set of states, $a_I \in A$ is the initial state, $\Delta : A \times \wp(\mathsf{P}) \to \mathcal{L}_1^+(A)$ is the transition map, and $\Omega : A \to \mathbb{N}$ is the parity map. The class of such automata will be denoted by $Aut(\mathcal{L}_1)$.

Acceptance of a P-transition system $\mathbb{S} = \langle S, R, \kappa, s_I \rangle$ by $\mathbb{A}$ is determined by the *acceptance game* $\mathcal{A}(\mathbb{A}, \mathbb{S})$ of $\mathbb{A}$ on $\mathbb{S}$. This is the parity game defined according to the rules of the following table.

Position	Player	Admissible moves	Parity
$(a, s) \in A \times S$	$\exists$	$\{V : A \to \wp(R[s]) \mid (R[s], V) \models \Delta(a, \kappa(s))\}$	$\Omega(a)$
$V : A \to \wp(S)$	$\forall$	$\{(b, t) \mid t \in V(b)\}$	$\max(\Omega[A])$