A Text on functionality

THIS TEXT TO BE MOVED TO AUTOMATA PAPER

Definition A.1 An A-valuation $V : A \to \wp D$ is said to *separate* a set $B \subseteq A$ if $|V^\flat(d) \cap B| \leq 1$, for every $d \in D$. A sentence $\varphi \in \mathtt{ME}^\infty(A)$ is B*-separating* if for every valuation $V : A \to \wp D$ such that $(D, V) \models \varphi$ there is a B-separating valuation $U \leq_B V$ such that $(D, U) \models \varphi$. $\lhd$

Intuitively, a formula φ is B-separating if its truth in a monadic model never requires an element of the domain to satisfy two distinct predicates in B at the same time, in the sense that any valuation violating this constraint can be reduced to a valuation satisfying it, without sacrificing the truth of φ.

Lemma A.2 *With $B \subseteq A$, let $\varphi \in (\mathtt{ME}^\infty)^+(A)$ be a disjunction of formulas of the shape $\nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Psi \cup \Sigma, \Sigma)$, where $\Sigma \subseteq \wp A$, $\Psi, \overline{\mathbf{T}} \subseteq \wp B$ are such that (1) $B \cap \bigcup \Sigma = \varnothing$ and (2) each $S \in \overline{\mathbf{T}} \cup \Psi$ is either empty or a singleton from B. Then φ is both separating and continuous in B.*

As a consequence, under the same conditions as in the lemma, for every valuation $V : A \to \wp D$ such that $(D, V) \models \varphi$ we may find a B-separating valuation U such that $(D, U) \models \varphi$ and (1) $U(b)$ is a finite subset of $V(b)$, for all $b \in B$, and (2) $U(a) = V(a)$, for all $a \in A \setminus B$.

THE FOLLOWING IS LEFT OVER FROM THE SECTION ON MONOTONICITY

We isolate the case when basic forms enforce a semantic property called *functionality*. This will be useful when defining non-deterministic parity automata.

Definition A.3 We say that $\varphi \in \mathtt{ME}(A)$ is *functional* in $B \subseteq A$ if whenever $(D, V\ A \to \wp(D)) \models \varphi$ then there is a restriction V' of V such that $(D, V'\ A \to \wp(D)) \models \varphi$ and $s \in V'(b)$ for $b \in B$ implies $s \notin V'(a)$ for all $a \in A \setminus \{b\}$. $\lhd$

The syntactic shape guaranteeing functionality is the following.

Proposition A.4 *Suppose that $\varphi \in \mathtt{ME}(A)$ is equivalent to a sentence in the basic form $\bigvee \nabla^{+}_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ where $T_1, \ldots, T_k$ and each $S \in \Pi$ are either $\emptyset$ of singletons $\{b\}$ for some $b \in B$. Then φ is functional in B.* •

Proof. Let (D, V) be a model where φ is true. Thus one disjunct $\nabla^{+}_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ is true, that means, there are elements $s_1, \ldots, s_k$ of D witnessing types $T_1, \ldots, T_k$ respectively and all the other elements witness some $S \in \Pi$. By pruning from V any other assignment of predicates to elements of D but for the types they witness according to this description, we obtain a restriction V' of V such that $(D, V) \models \varphi$. Because $T_1, \ldots, T_k$ are either $\emptyset$ of singletons $\{b\}$ for some $b \in B$, such V' assigns at most one $b \in B$ to each element of D. Therefore φ is functional in B. QED

As for ME, we record a syntactic criterion for functionality, which is defined as in Definition A.3 with $\mathtt{ME}^{\infty}$ replacing ME.

Definition A.5 We say that $\varphi \in \mathtt{ME}(A)$ is *functionally continuous* in $B \subseteq A$ if whenever $(D, V\ A \to \wp(D)) \models \varphi$ then there is a restriction V' of V such that $(D, V'\ A \to \wp(D)) \models \varphi$ and for all $b \in B$

functionality $s \in V'(b)$ implies $s \notin V'(a)$ for all $a \in A \setminus \{b\}$

continuity $V'(b)$ is finite.

$\lhd$

Proposition A.6 *If $\varphi \in \mathtt{ME}(A)$ is in the basic form $\bigvee \nabla^{+}_{\mathtt{ME}^{\infty}}(\overline{\mathbf{T}}, \Pi, \Sigma)$ with all $T_1, \ldots, T_k, \Pi$ either empty or singletons, then it is functional in A.*

THE FOLLOWING IS LEFT OVER FROM THE SECTION ON CONTINUITY

Putting together the above lemmas we obtain Theorem 5.9. Moreover, a careful analysis of the translation gives us the following corollary, providing normal forms for the continuous fragment of $\mathtt{ME}^\infty$. Point ii below is tailored for applications to parity automata, see Theorem **??**. We call a formula $\varphi \in \mathtt{ME}^\infty(A)$ *functionally continuous* in $B \subseteq A$ when, given a model $(D, V), g$ where φ is true, a restriction V' of V can be found that both witnesses continuity in each $b \in B$ and also witnesses functionality in each $b \in B$, in the sense of Definition A.3.

Corollary A.7 *Let $\varphi \in \mathtt{ME}^\infty(A)$, the following hold:*

(i) The formula φ is continuous in $a \in A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla^a_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ for some types $\Sigma \subseteq \Pi \subseteq \wp A$ and $T_i \subseteq A$ such that $a \notin \bigcup \Sigma$.

(ii) The formula φ is functionally continuous in $B \subseteq A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Psi \cup \Sigma, \Sigma)$ for some types $\Sigma \subseteq \wp A$, $\Psi \subseteq \wp B$ and $T_i \subseteq B$ such that (1) for all $b \in B$, $b \notin \bigcup \Sigma$ and (2) $T_1, \ldots, T_k$ and each $S \in \Psi$ are either empty or singletons.

(iii) If φ is monotone in every element of A (i.e., $\varphi \in \mathtt{ME}^{\infty+}(A)$) then φ is continuous in $a \in A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ for some types $\Sigma \subseteq \Pi \subseteq A$ and $T_i \subseteq A$ such that $a \notin \bigcup \Sigma$.

Proof. For i and iii, the observation is that in order to obtain $\Sigma \subseteq \Pi$ in the above normal forms it is enough to use Proposition 3.16 before applying the translation. For ii, fix a model (D, V) where $\bigvee \nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Psi \cup \Sigma, \Sigma)$ is true. This means that a certain disjunct $\nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Psi \cup \Sigma, \Sigma)$, by definition unfolding as

$$\exists \overline{\mathbf{x}}.\Big(\mathrm{diff}(\overline{\mathbf{x}}) \wedge \bigwedge_i \tau^+_{T_i}(x_i) \ \wedge \forall z.(\neg \mathrm{diff}(\overline{\mathbf{x}}, z) \vee \bigvee_{S \in \Psi \cup \Sigma} \tau^+_S(z) \wedge \forall^\infty y. \bigvee_{S \in \Sigma} \tau^+_S(y)\Big) \wedge \bigwedge_{S \in \Sigma} \exists^\infty y. \tau^+_S(y),$$

is true. Because by assumption $B \cap \Sigma = \emptyset$, point i yields a restriction V' of V such that $(D, V') \models \nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Psi \cup \Sigma, \Sigma)$ and V' witnesses continuity of φ in B. functionality, observe that $\nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Psi \cup \Sigma)$ is the conjunction of sub-formulas $\nabla^+_{\mathtt{ME}}(\overline{\mathbf{T}}, \Psi \cup \Sigma)$ (first line of (**??**)) and ψ (second line).

For functionality, the syntactic shape of $\nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Psi \cup \Sigma, \Sigma)$ implies that (D, V') can be partitioned in three sets D_1, D_2 and D_3 as follows: D_1 contains elements $s_1, \ldots, s_k$ witnessing types $\tau^+_{T_1}, \ldots, \tau^+_{T_k}$, respectively; among the remaining elements, there are infinitely many witnessing τ^+_S for each $S \in \Sigma$ (these form D_2), and finitely many not witnessing any such τ^+_S but witnessing τ^+_R for some $R \in \Psi$ (these form D_3). If we now prune any other assignment of predicates to elements of D but for the types they witness according to this description, we obtain a restriction V'' of V' which still makes $\nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Psi \cup \Sigma, \Sigma)$ true. Moreover because $T_1, \ldots, T_k$ and each $S \in \Psi$ are empty or singleton subsets of B and $\Sigma \cap B = \emptyset$, it shows that $\nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Psi \cup \Sigma, \Sigma)$ is functional in B, and still witnesses its continuity because V' does and

V'' is a restriction of V'. Proposition A.6 yields a restriction V'' of V' such that $(D, V'') \models \nabla^{+}_{\mathtt{ME}}(\overline{\mathbf{T}}, \Psi \cup \Sigma)$ and V'' witnesses functionality in B of $\nabla^{+}_{\mathtt{ME}}(\overline{\mathbf{T}}, \Psi \cup \Sigma)$. In order for such V'' to witness continuity in B and make $\nabla^{+}_{\mathtt{ME}^{\infty}}(\overline{\mathbf{T}}, \Psi \cup \Sigma, \Sigma)$ true, it must preserve the condition imposed by ψ, i.e. that infinitely many nodes are marked with each $S \in \Sigma$ and only finitely many are not marked with any $S \in \Sigma$. It can be checked by inspection of $\nabla^{\Psi \cup \Sigma}_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)+$ that such a V'' is definable by pruning the valuation V' only for finitely many elements of D. Therefore, ψ remains true, $(D, V'') \models \nabla^{+}_{\mathtt{ME}^{\infty}}(\overline{\mathbf{T}}, \Psi \cup \Sigma, \Sigma)$ and thus also $(D, V'') \models \varphi$ as required. QED
