$$
\begin{array}{lll}
(D,V),g \models a(x) & \text{iff} & x \in V(a) \\
(D,V),g \models \neg a(x) & \text{iff} & x \notin V(a) \\
(D,V),g \models\mid x \approx y & \text{iff} & g(x) = g(y) \\
(D,V),g \models\mid x \not\approx y & \text{iff} & g(x) \neq g(y) \\
(D,V),g \models \varphi \vee \psi & \text{iff} & (D,V),g \models \varphi \text{ or } \mathbb{M},g \models \psi \\
(D,V),g \models \varphi \wedge \psi & \text{iff} & (D,V),g \models \varphi \text{ and } \mathbb{M},g \models \psi \\
(D,V),g \models \exists x.\varphi & \text{iff} & \text{there is } s \in D \text{ such that } (D,V),g[x \mapsto s] \models \varphi \\
(D,V),g \models \forall x.\varphi & \text{iff} & \text{for all } s \in D,\ (D,V),g[x \mapsto s] \models \varphi.
\end{array}
$$

desired

As wished, this definition induces a truth relation $(D,V) \models \varphi$ between one-step models and one-step formulas, as the one-step formulas of $\mathrm{FOE}_1(A)$ are defined to be the *sentences*, thus there is no need of an explicit free variables assignment g in determining their semantics. rewrite (too long)

In order for the semantic notion of co-continuity to be well-defined (Definition 3.2), we also need to define what is the dual of a formula. This just coincides with the familiar notion of boolean dual.

Definition 3.4. The *dual* $\varphi^\delta \in \mathrm{FOE}_1^\infty(A)$ of $\varphi \in \mathrm{FOE}_1^\infty(A)$ is given by:

$$
\begin{array}{rlrl}
(a(x))^\delta := a(x) & & (\neg a(x))^\delta := \neg a(x) \\
(\top)^\delta := \bot & & (\bot)^\delta := \top \\
(x \approx y)^\delta := x \not\approx y & & (x \not\approx y)^\delta := x \approx y \\
(\varphi \wedge \psi)^\delta := \varphi^\delta \vee \psi^\delta & & (\varphi \vee \psi)^\delta := \varphi^\delta \wedge \psi^\delta \\
(\exists x.\psi)^\delta := \forall x.\psi^\delta & & (\forall x.\psi)^\delta := \exists x.\psi^\delta
\end{array}
$$

We now introduce an extension of first-order logic with two additional quantifiers, which first appeared in the context of Mostowski's study [Mostowski 1957] of generalised quantifiers. The first, written $\exists^\infty x.\varphi$, expresses that there exist infinitely many elements satisfying a formula φ. Its dual, written $\forall^\infty x.\varphi$, expresses that there are *at most finitely many* elements *falsifying* the formula φ. The formal definition goes as follows, where Q ranges over $\{\exists^\infty, \forall^\infty\}$.

why not use simpler def as in other paper?

$$
\begin{array}{c}
\exists^\infty := \{(J,X) \mid |X| \geq \aleph_0\} \qquad \forall^\infty = \{(J,X) \mid |J \setminus X| < \aleph_0\} \\
(D,V),g \models Qx.\varphi(x) \quad \text{iff} \quad (D,\{s \in D \mid (D,V),g[x \mapsto s] \models \varphi(x)\}) \in Q
\end{array} \tag{9}
$$

(8)

Definition 3.5. The one-step language $\mathrm{FOE}_1^\infty(A)$ is defined by adding to the grammar of $\mathrm{FOE}_1(A)$ the cases $\exists^\infty x.\varphi$ and $\forall^\infty x.\varphi$. The truth relation $(D,V) \models \varphi$ is defined by extending the one for $\mathrm{FOE}_1(A)$ with clauses (9).

In the rest of the subsection we recall from [?] syntactic characterisations for semantic properties of the first-order logics FOE_1 and FOE_1^∞. We first discuss FOE_1. The properties of monotonicity and continuity will be characterised both with a grammar and a normal form.

Definition 3.6.

(a) The *positive* fragment of $\mathrm{FOE}_1(A)$, written $\mathrm{FOE}_1^+(A)$, is the set of sentences generated by the grammar (8) without clauses $\neg a(x)$.
(b) For $B \subseteq A$, the *B-continuous* fragment of $\mathrm{FOE}_1^+(A)$, written $\mathrm{FOE}_1\mathsf{CON}_B(A)$, is the set of sentences generated by the following grammar, where $b \in B$ and $\psi \in \mathrm{FOE}_1^+(A\setminus$