3. PARITY AUTOMATA AND MODAL μ-CALCULI

This section introduces and studies the parity automata that will be used in the characterisation of WMSO and NMSO. The transition function of these automata will be defined by formulas of a so-called one-step language $\mathcal{L}_1$. Subsection 3.4 illustrates one-step languages and give normal forms for relevant semantic properties such as monotonicity and continuity. We then have all the ingredients to formally introduce $\mathcal{L}_1$-parity automata, in Subsection 3.2. The remaining part of the section is devoted to showing that classes of parity automata are equivalently described as modal fixpoint logics. After introducing the concept of a modal fixpoint logic parametric in $\mathcal{L}_1$, in Subsection 3.3, we build an effective translation from $\mathcal{L}_1$-parity automata, in Subsection 3.4. 3.1

3.1. One-step languages and normal forms

Definition 3.1. Given a finite set A of *monadic predicates*, a *one-step model* is a tuple $(D, V) = (D, V)$ consisting of a *domain* set D and a *valuation* $V : A \to \wp D$. We say that $V' : A \to \wp D$ is a *restriction* of V if $V'(a) \subseteq V(a)$ for all $a \in A$. is a map assigning

A *one-step language* is a set $\mathcal{L}_1(A)$ of objects called *one-step formulas* over A. We assume that one-step languages come with a *truth* relation between one-step formulas and models, written $(D, V) \models \varphi$, and a *dual* $\varphi^\delta \in \mathcal{L}_1(A)$ for each one-step formula φ, with the property that $(D, V) \models \varphi$ iff $(D, V^c) \not\models \varphi^\delta$, where V^c is given by $V^c(a) := D \setminus V(a)$, for all a. a set $\mathcal{L}_1(A)$ ...

Also mention $L^+(A)$. ?

The following semantic properties will be useful when studying parity automata.

Definition 3.2. Given a one-step language $\mathcal{L}_1(A)$ and $\varphi \in \mathcal{L}_1(A)$,

— φ is *monotone* in $B \subseteq A$ if for every one step model (D, V) and $V' : A \to \wp(D)$ such that $V(b) \subseteq V'(b)$ for all $b \in B$, $(D, V) \models \varphi$ implies $(D, V') \models \varphi$. x modify def (see other paper).

— φ is *functional* in $B \subseteq A$ if it is monotone in $B \subseteq A$ and, whenever $(D, V) \models \varphi$, then there exists a restriction $V' : A \to \wp(D)$ of V such that $(D, V') \models \varphi$ and $s \in V'(b)$ for some $b \in B$ implies $s \notin V'(a)$ for all $a \in A \setminus \{b\}$. $B \setminus \{b\}$

— φ is *continuous* in $B \subseteq A$ if φ is monotone in B and, whenever $(D, V) \models \varphi$, then there exists a restriction $V' : A \to \wp(D)$ of V such that $(D, V') \models \varphi$ and $V'(b)$ is finite for all $b \in B$.

— φ is *co-continuous* in $B \subseteq A$ if its dual φ^δ is continuous in B.

Our chief examples of one-step languages will be variants of first-order logic.

Definition 3.3. The one-step language $\mathrm{FOE}_1(A)$ of *first-order logic with equality* on a set of predicates A and individual variables iVar is given by the sentences (variable-free formulas) generated by the following grammar, where $a \in A$ and $x, y \in$ iVar.: w.o. free variables

$$\varphi ::= a(x) \mid \neg a(x) \mid x \approx y \mid x \not\approx y \mid \exists x.\varphi \mid \forall x.\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \tag{8}$$

We use FO_1 for the equality-free fragment, where we omit clauses $x \approx y$ and $x \not\approx y$.

Formulas of FOE_1 will be interpreted over one-step models $(D, V : A \to \wp(D))$ with a variable assignment $g :$ iVar $\to D$. The semantics of FOE_1 (and also FO_1) is standard: