B):

$$\varphi ::= \psi \mid b(x) \mid \exists x.\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x.\psi.$$

(c) For $B \subseteq A$, the *B-cocontinuous* fragment of $\mathrm{FOE}_1^{\mid}(A)$, written $\mathrm{FOE}_1\overline{\mathsf{CON}}_B(A)$, is the set $\{\varphi \mid \varphi^\delta \in \mathrm{FOE}_1^\infty\mathsf{CON}_B(A)\}$.

It will be convenient to introduce some abbreviations.

Definition 3.7.

(i) A *type* T is just a subset of A. It defines a FOE_1-formula $\tau_T^+(x) := \bigwedge_{a\in T} a(x)$. Given a one-step model (D, V), $s \in D$ *witnesses* a type T if $(D, V), g[x \mapsto s] \models \tau_T^+(x)$ for any g.
(ii) We say that $\varphi \in \mathrm{FOE}_1(A)$ is in *basic form* if $\varphi = \bigvee \nabla^+_{\mathrm{FOE}}(\overline{\mathbf{T}}, \Pi)$ where each disjunct is of the form

$$\nabla^+_{\mathrm{FOE}}(\overline{\mathbf{T}}, \Pi) = \exists \overline{\mathbf{x}}.\big(\mathrm{diff}(\overline{\mathbf{x}}) \wedge \bigwedge_i \tau^+_{T_i}(x_i) \wedge \forall z.(\mathrm{diff}(\overline{\mathbf{x}}, z) \to \bigvee_{S\in\Pi} \tau^+_S(z))\big)$$

such that $\overline{\mathbf{T}} \in \wp(A)^k$ for some k and $\Pi \subseteq \overline{\mathbf{T}}$. The predicate $\mathrm{diff}(\overline{\mathbf{y}})$, stating that the elements $\overline{\mathbf{y}}$ are distinct, is defined as $\mathrm{diff}(y_1, \ldots, y_n) := \bigwedge_{1\le m<m'\le n}(y_m \not\approx y_{m'})$.

Intuitively, the formula $\nabla^+_{\mathrm{FOE}}(\overline{\mathbf{T}}, \Pi)$ says that each one-step model satisfying it admits a partition of its domain in two parts: distinct elements $t_1, \ldots, t_n$ witnessing types $T_1, \ldots, T_n$, and all the remaining elements witnessing some type S of Π. The following result syntactically characterises the monotone fragment $\mathrm{FOE}_1^{\mid}(A)$ of $\mathrm{FOE}_1(A)$.

THEOREM 3.8 ([?]).

(i) $\varphi \in \mathrm{FOE}_1(A)$ *is monotone if and only if it belongs to* $\mathrm{FOE}_1^{\mid}(A)$. X ...iff there is an
(ii) $\varphi \in \mathrm{FOE}_1^+(A)$ *is continuous in* $B \subseteq A$ *if and only if it belongs to* $\mathrm{FOE}_1\mathsf{CON}_B(A)$. X equivalent fma φ'
(iii) $\varphi \in \mathrm{FOE}_1^+(A)$ *is cocontinuous in* $B \subseteq A$ *if and only if it belongs to* $\mathrm{FOE}_1\overline{\mathsf{CON}}_B(A)$. X ...
(iv) There is an effective translation $(\cdot)^\bullet\colon \mathrm{FOE}_1^{\mid}(A) \to \mathrm{FOE}_1^+(A)$ *mapping* φ *into an equivalent sentence in basic form* $\varphi^\bullet = \bigvee \nabla^+_{\mathrm{FOE}}(\overline{\mathbf{T}}, \Pi)$ *such that:*
  (a) φ *is functional in* B *if and only if each* $T_1, \ldots, T_k$ *and* $S \in \Pi$ *in* $\varphi^\bullet$ *are either* $\emptyset$ *of singletons* $\{b\}$ *for some* $b \in B$.
  (b) φ *is continuous in* B *if and only if* $b \notin \bigcup \Sigma$ *for all* $b \in B$.

We now provide analogous characterisations for $\mathrm{FOE}_1^\infty(A)$. The capacity of $\mathrm{FOE}_1^\infty(A)$ to tear apart finite and infinite sets of elements requires extra care.

Definition 3.9.

(a) The *positive* fragment of $\mathrm{FOE}_1^\infty(A)$, written $\mathrm{FOE}_1^{\infty+}(A)$, is the set of sentences generated by the grammar (8) without clauses $\neg a(x)$.
(b) For $B \subseteq A$, the *B-continuous* fragment of $\mathrm{FOE}_1^{\infty+}(A)$, written $\mathrm{FOE}_1^\infty\mathsf{CON}_B(A)$, is the set of sentences generated by the following grammar, where $b \in B$, $\psi \in \mathrm{FOE}_1^{\infty+}(A \setminus B)$ and and $\mathbf{W}x.(\varphi, \psi) := \forall x.(\varphi(x) \vee \psi(x)) \wedge \forall^\infty x.\psi(x)$. explain/discuss W

$$\varphi ::= \psi \mid b(x) \mid \exists x.\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathbf{W}x.(\varphi, \psi)$$

(c) For $B \subseteq A$, the *B-cocontinuous* fragment of $\mathrm{FOE}_1^{\infty+}(A)$, written $\mathrm{FOE}_1^\infty\overline{\mathsf{CON}}_B(A)$, is the set $\{\varphi \mid \varphi^\delta \in \mathrm{FOE}_1^\infty\mathsf{CON}_B(A)\}$.