Model Theory of Monadic Predicate Logic with the Infinity Quantifier

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Abstract

This paper establishes model-theoretic properties of $\mathtt{ME}^{\infty}$, a variation of monadic first-order logic that features the generalised quantifier $\exists^{\infty}$ ('there are infinitely many').

We provide syntactically defined fragments of $\mathtt{ME}^{\infty}$ characterising four different semantic properties of $\mathtt{ME}^{\infty}$-sentences: (1) being monotone and (2) (Scott) continuous in a given set of monadic predicates; (3) having truth preserved under taking submodels or (4) invariant under taking quotients. In each case, we produce an effectively defined map that translates an arbitrary sentence φ to a sentence φ^{p} belonging to the corresponding syntactic fragment, with the property that φ is equivalent to φ^{p} precisely when it has the associated semantic property.

Our methodology is first to provide these results in the simpler setting of monadic first-order logic with ($\mathtt{ME}$) and without ($\mathtt{M}$) equality, and then move to $\mathtt{ME}^{\infty}$ by including the generalised quantifier $\exists^{\infty}$ into the picture.

As a corollary of our developments, we obtain that the four semantic properties above are decidable for $\mathtt{ME}^{\infty}$-sentences. Moreover, our results are directly relevant to the characterisation of automata and expressiveness modulo bisimilirity for variants of monadic second-order logic. This application is developed in a companion paper.

1 Introduction

Model theory investigates the relationship between formal languages and semantics. From this perspective, among the most important results are the so called *preservation theorems*. Such results typically characterise a certain language as the fragment of another, richer language satisfying a certain model-theoretic property. In doing so, they therefore link the syntactic shape of a formula with the semantic properties of the class of models it defines. In the case of classical first-order logic, notable examples are the Łoś-Tarski theorem, stating that a first-order formula is equivalent to a universal one if and only if the class of its models is closed under taking submodels, and Lyndon's theorem, stating that a first-order formula is equivalent to one for which each occurrence of a relation symbol R is positive if and only if it is monotone with respect to the interpretation of R (see e.g. [13]).

The aim of this paper is to show that similar results also hold when considering the predicate logic $\mathtt{ME}^{\infty}$ that allows only monadic predicate symbols and no function symbols, but that goes beyond

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standard first-order logic with equality in that it features the generalised quantifier 'there are infinitely many'.

Generalised quantifiers were introduced by Mostowski in [19], and in a more general sense by Lindström in [17], the main motivation being the observation that standard first-order quantifiers 'there are some' and 'for all' are not sufficient for expressing some basic mathematical concepts. Since then, they have attracted a lot of interests, insomuch that their study constitutes nowadays a well-established field of logic with important ramifications in disciplines such as linguistics and computer science.[1].

Despite the fact that the absence of polyadic predicates clearly restricts its expressing power, monadic first-order logic (with identity) displays nice properties, both from a computational and a model-theoretic point of view. Indeed, the satisfiability problem becomes decidable [3, 18], and, in addition of an immediate application of Łoś-Tarski and Lyndon's theorems, one can also obtain a Lindström like characterisation result [21]. Moreover, adding the possibility of quantifying over predicates does not increase the expressiveness of the language [2], meaning that when restricted to monadic predicates, monadic second order logic collapses into first-order logic.

For what concerns monadic first-order logic extended with an infinity quantifier, in [19] Mostowski, already proved its decidability, whereas from work of Väänänen [22] we know that its expressive power coincides with that of weak monadic second-order logic restricted to monadic predicates, that is monadic first-order logic extended with a second order quantifier ranging over finite sets[2].

Preservation results and proof outline.

A preservation result involves some fragment $\mathtt{L}_{\mathfrak{P}}$ of a given yardstick logic $\mathtt{L}$, related to a certain semantic property $\mathfrak{P}$. It is usually formulated as

$$\varphi \in \mathtt{L} \text{ has the property } \mathfrak{P} \text{ iff } \varphi \text{ is equivalent to some } \varphi' \in \mathtt{L}_{\mathfrak{P}}. \tag{1}$$

In this work, our main yardstick logic will be $\mathtt{ME}^\infty$. Table 1 summarises the semantic properties ($\mathfrak{P}$) we are going to consider, the corresponding expressively complete fragment ($\mathtt{L}_{\mathfrak{P}}$) and preservation theorem.

$\mathfrak{P}$	$\mathtt{L}_{\mathfrak{P}}$	Preservation Theorem
Monotonicity (Definition 4.1)	Positive fragment $\mathtt{Pos}(\mathtt{ME}^\infty)$	Theorem 4.4
Continuity (Definition 5.1)	Continuous fragment $\mathtt{Con}(\mathtt{ME}^\infty)$	Theorem 5.6
Preservation under submodels (Definition 6.2(6.2))	Universal fragment $\mathtt{Univ}(\mathtt{ME}^\infty)$	Theorem 6.4
Invariance under quotients (Definition 6.2(6.2))	Monadic first-order logic $\mathtt{M}$	Theorem 6.9

Table 1: A summary of our preservation theorems

The proof of each preservation theorem is composed of two parts. The first, simpler one concerns the claim that each sentence in the fragment satisfies the concerned property. It is usually proved by induction on the structure of the sentence. The other direction is the *expressive completeness*

[1] For an overview see e.g. [26, 23, 30]. For an introduction to the model theory of generalised quantifiers, the interested reader can consult for instance [24, Chapter 10].

[2] Extensions of monadic first-order logic with other generalised quantifiers have also been studied (see e.g. [20, 4]).

statement, stating that within the considered logic, the fragment is expressively complete for the property. Its verification generally requires more effort. In this paper, we will actually verify a stronger expressive completeness statement. Namely, for each semantic property $\mathfrak{P}$ and corresponding fragment $\mathtt{L}_{\mathfrak{P}}$ from Table 1, we are going to provide an effective translation operation $(\cdot)^{\mathsf{p}} : \mathtt{ME}^{\infty} \to \mathtt{L}_{\mathfrak{P}}$ such that

$$\text{if } \varphi \in \mathtt{ME}^{\infty} \text{ has the property } \mathfrak{P} \text{ then } \varphi \text{ is equivalent to } \varphi^{\mathsf{p}}. \tag{2}$$

Since the satisfiability problem for $\mathtt{ME}^{\infty}$ is decidable and the translation $(\cdot)^{\mathsf{p}}$ is effectively computable, we obtain, as an immediate corollary of (2), that for each property $\mathfrak{P}$ listed in Table 1

$$\text{the problem whether a } \mathtt{ME}^{\infty}\text{-sentence satisfies property } \mathfrak{P} \text{ or not is decidable.} \tag{3}$$

The proof of each instance of (2) will follow an uniform pattern, analogous to the one employed in the aim of obtaining similar results in the context of the modal μ-calculus [15, 8, 11]. The crux of the adopted proof method is that, extending known results on monadic first-order logic, for each sentence φ in $\mathtt{ME}^{\infty}$ it is possible to compute a logically equivalent sentence in *basic normal norm.* Such normal forms will take the shape of a disjunction $\bigvee \nabla_{\mathtt{ME}^{\infty}}$, where each disjunct $\nabla_{\mathtt{ME}^{\infty}}$ characterises a class of models of φ satisfying the same set of $\mathtt{ME}^{\infty}$-sentences of equal quantifier rank as φ. Based on this, it will therefore be enough to define an effective translation $(\cdot)^{\mathsf{p}}$ for sentences in normal form, point-wise in each disjunct $\nabla_{\mathtt{ME}^{\infty}}$, and then verify that it indeed satisfies (2).

As a corollary of the employed proof method, we thus obtain effective normal forms for sentences satisfying the considered property.

In addition to $\mathtt{ME}^{\infty}$, in this paper we are also going to consider monadic first-order logic with and without equality, denoted respectively by $\mathtt{ME}$ and $\mathtt{M}$. Table 2 shows a summary of the expressive completeness and normal form results presented in this paper.

		Language		
		$\mathtt{M}$	$\mathtt{ME}$	$\mathtt{ME}^{\infty}$
	Normal forms	Fact 3.3	Thm. 3.9	Thm. 3.15
Monotonicity	Completeness	Prop. 4.9	Prop. 4.12	Prop. 4.15
	Normal forms	Cor. 4.10	Cor. 4.13	Cor. 4.16
Continuity	Completeness	Prop. 5.9	Fact 5.5	Prop. 5.11
	Normal forms	Cor. 5.10	–	Cor. 5.12
Preservation under submodels	Completeness	Prop. 6.7		
	Normal forms	Cor. 6.8(1)	Cor. 6.8(2)	Cor. 6.8(3)
Invariance under quotients	Completeness	Prop. 6.10	Prop. 6.13	
	Normal forms	Fact 3.3	Cor. 6.14	

Table 2: An overview of our expressive completeness and normal form results.

Application of obtained results: the companion paper

Parity automata are finite-state systems playing a crucial role in obtaining decidability and expressiveness results in fixpoint logic (see e.g. [27]). They are specified by a finite set of states A, a distinguished, initial state $a \in A$, a function Ω assigning to each states a priority (a natural number), and a transition function Δ whose co-domain is usually given by a monadic logic in which the set of (monadic) predicates coincides with A. Hence, each monadic logic $\mathtt{L}$ induces its own class of automata $\mathtt{Aut(L)}$.

A landmark result in this area is Janin and Walukiewicz's theorem stating that the bisimulation-invariant fragment of monadic second order logic coincides with the modal μ-calculus [15], and the proof of this result is an interesting mix of the theory of parity automata and the model theory of monadic predicate logic. First, preservation and normal forms results are used to verify that (on tree models) $\mathtt{Aut}(\mathtt{Pos}(\mathtt{ME}))$ is the class of automata characterising the expressive power of monadic second order logic [29], whereas $\mathtt{Aut}(\mathtt{Pos}(\mathtt{M}))$ corresponds to the modal μ-calculus [14], where $\mathtt{Pos}(\mathtt{L})$ denote the positive fragment of the monadic logic $\mathtt{L}$. Then, Janin-Walukieiwcz's expressiveness theorem is a consequence of these automata characterisations and the fact that positive monadic first-order logic without equality provides the quotient-invariant fragment of positive monadic first-order logic with equality (see Theorem 6.15).

In our companion paper [7], we provide a Janin-Walukiewicz type characterisation result for *weak* monadic second order logic. Our proof, analogously to the case of full monadic second order logic discussed previously, crucially employs preservation and normal form results for $\mathtt{ME}^\infty$ listed in Tables 1 and 2.

Other versions

Results in this paper first appeared in the first author's PhD thesis ([5, Chapter 5]); this journal version largely expands material first published as part of the conference papers [10, 6]. In particular, the whole of Section 6 below contains new results.

2 Basics

In this section we provide the basic definitions of the monadic predicate liftings that we study in this paper. Throughout this paper we fix a finite set A of objects that we shall refer to as *(monadic) predicate symbols* or *names*.

We shall also assume an infinite set iVar of *individual variables*.

Definition 2.1 Given a finite set A we define a *(monadic) model* to be a pair $\mathbb{D} = (D, V)$ consisting of a set D, which we call the *domain* of $\mathbb{D}$, and an interpretation or *valuation* $V : A \to \wp(D)$. The class of all models will be denoted by $\mathfrak{M}$. $\lhd$

Remark 2.2 Note that we make the somewhat unusual choice of allowing the domain of a monadic model to be *empty*. In view of the applications of our results to automata theory (see Section 1) this choice is very natural, even if it means that some of our proofs here become more laborious in requiring an extra check. Observe that there is exactly one monadic model based on the empty domain; we shall denote this model as $\mathbb{D}_\varnothing := (\varnothing, \varnothing)$. $\lhd$

Definition 2.3 Observe that a valuation $V : A \to \wp(D)$ can equivalently be presented via its associated *colouring* $V^\flat : D \to \wp(A)$ given by

$$V^\flat(d) := \{a \in A \mid d \in V(a)\}.$$

We will use these perspectives interchangeably, calling the set $V^\flat(d) \subseteq A$ the *colour* or *type* of d. In case $D = \varnothing$, $V^\flat$ is simply the empty map. $\lhd$

In this paper we study three languages of monadic predicate logic: the languages $\mathtt{ME}$ and $\mathtt{M}$ of monadic first-order logic with and without equality, respectively, and the extension $\mathtt{ME}^\infty$ of $\mathtt{ME}$ with the generalised quantifiers $\exists^\infty$ and $\forall^\infty$. Probably the most concise definition of the full language of monadic predicate logic would be given by the following grammar:

$$\varphi ::= a(x) \mid x \approx y \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x.\varphi \mid \exists^\infty x.\varphi,$$

where $a \in A$ and x and y belong to the set iVar of individual variables. In this set-up we would need to introduce the quantifiers $\forall$ and $\forall^\infty$ as abbreviations of $\neg\exists\neg$ and $\neg\exists^\infty\neg$, respectively. However, for our purposes it will be more convenient to work with a variant of this language where all formulas are in negation normal form; that is, we only permit the occurrence of the negation symbol $\neg$ in front of an atomic formula. In addition, for technical reasons we will add $\bot$ and $\top$ as constants, and we will write $\neg(x \approx y)$ as $x \not\approx y$. Thus we arrive at the following definition of our syntax.

Definition 2.4 The set $\mathtt{ME}^\infty(A)$ of *monadic formulas* is given by the following grammar:

$$\varphi ::= \top \mid \bot \mid a(x) \mid \neg a(x) \mid x \approx y \mid x \not\approx y \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x.\varphi \mid \forall x.\varphi \mid \exists^\infty x.\varphi \mid \forall^\infty x.\varphi$$

where $a \in A$ and $x, y \in \mathsf{iVar}$. The language $\mathtt{ME}(A)$ of *first-order logic with equality* is defined as the fragment of $\mathtt{ME}^\infty(A)$ where occurrences of the generalised quantifiers $\exists^\infty$ and $\forall^\infty$ are not allowed:

$$\varphi ::= \top \mid \bot \mid a(x) \mid \neg a(x) \mid x \approx y \mid x \not\approx y \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x.\varphi \mid \forall x.\varphi$$

Finally, the language $\mathtt{M}(A)$ of *first-order logic* is the equality-free fragment of $\mathtt{ME}(A)$; that is, atomic formulas of the form $x \approx y$ and $x \not\approx y$ are not permitted either:

$$\varphi ::= \top \mid \bot \mid a(x) \mid \neg a(x) \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x.\varphi \mid \forall x.\varphi$$

In all three languages we use the standard definition of free and bound variables, and we call a formula a *sentence* if it has no free variables. In the sequel we will often use the symbol $\mathtt{L}$ to denote either of the languages $\mathtt{M}$, $\mathtt{ME}$ or $\mathtt{ME}^\infty$.

For each of the languages $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^\infty\}$, we define the *positive fragment* $\mathtt{L}^+$ of $\mathtt{L}$ as the language obtained by almost the same grammar as for $\mathtt{L}$, but with the difference that we do not allow negative formulas of the form $\neg a(x)$. $\lhd$

The semantics of these languages is given as follows.

Definition 2.5 The semantics of the languages $\mathtt{M}, \mathtt{ME}$ and $\mathtt{ME}^\infty$ is given in the form of a truth relation $\models$ between models and sentences of the language. To define the truth relation on a model $\mathbb{D} = (D, V)$, we distinguish cases.

Case $D = \varnothing$**:** We define the *truth relation* $\models$ on the empty model $\mathbb{D}_\varnothing$ for all formulas that are Boolean combinations of sentences of the form $Qx.\varphi$, where $Q \in \{\exists, \exists^\infty, \forall, \forall^\infty\}$ is a quantifier. The definition is by induction on the complexity of such sentences; the "atomic" clauses, where the sentence is of the form $Qx.\varphi$, is as follows:

$$\begin{array}{lll} \mathbb{D}_\varnothing \not\models Qx.\varphi & \text{if} & Q \in \{\exists, \exists^\infty\}, \\ \mathbb{D}_\varnothing \models Qx.\varphi & \text{if} & Q \in \{\forall, \forall^\infty\}. \end{array}$$

The clauses for the Boolean connectives are as expected.

Case $D \neq \varnothing$**:** In the (standard) case of a non-empty model $\mathbb{D}$, we extend the truth relation to arbitrary formulas, involving assignments of individual variables to elements of the domain. That is, given a model $\mathbb{D} = (D, V)$, an assignment $g : \mathsf{iVar} \to D$ and a formula $\varphi \in \mathtt{ME}^\infty(A)$ we define the *truth* relation $\models$ by a straightforward induction on the complexity of φ. Below we explicitly provide the clauses of the quantifiers:

$$\begin{array}{lll} \mathbb{D}, g \models \exists x.\varphi & \text{iff} & \mathbb{D}, g[x \mapsto d] \models \varphi \text{ for some } d \in D, \\ \mathbb{D}, g \models \forall x.\varphi & \text{iff} & \mathbb{D}, g[x \mapsto d] \models \varphi \text{ for all } d \in D, \\ \mathbb{D}, g \models \exists^\infty x.\varphi & \text{iff} & \mathbb{D}, g[x \mapsto d] \models \varphi \text{ for infinitely many } d \in D, \\ \mathbb{D}, g \models \forall^\infty x.\varphi & \text{iff} & \mathbb{D}, g[x \mapsto d] \models \varphi \text{ for all but at most finitely many } d \in D. \end{array}$$

The clauses for the atomic formulas and for the Boolean connectives are standard.

In what follows, when discussing the truth of φ on the empty model, we always implicitly assume that φ is a sentence. $\lhd$

As mentioned in the introduction, general quantifiers such as $\exists^\infty$ and $\forall^\infty$ were introduced by Mostowski [19], who proved the decidability for the language obtained by extending $\mathtt{M}$ with such quantifiers. The decidability of the full language $\mathtt{ME}^\infty$ was then proved by Slomson in [20].[3] The case for $\mathtt{M}$ and $\mathtt{ME}$ goes back already to [3, 18].

Fact 2.6 *For each logic* $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^\infty\}$*, the problem of whether a given* $\mathtt{L}$*-sentence* φ *is satisfiable, is decidable.*

In the remainder of the section we fix some further definitions and notations, starting with some useful syntactic abbreviations.

Definition 2.7 Given a list $\overline{\mathbf{y}} = y_1 \cdots y_n$ of individual variables, we use the formula

$$\mathrm{diff}(\overline{\mathbf{y}}) := \bigwedge_{1 \le m < m' \le n} (y_m \not\approx y_{m'})$$

to state that the elements $\overline{\mathbf{y}}$ are all distinct. An *A-type* is a formula of the form

$$\tau_S(x) := \bigwedge_{a \in S} a(x) \wedge \bigwedge_{a \in A \setminus S} \neg a(x).$$

where $S \subseteq A$. Here and elsewhere we use the convention that $\bigwedge \varnothing = \top$ (and $\bigvee \varnothing = \bot$). The *positive* A-type $\tau^+_S(x)$ only bears positive information, and is defined as

$$\tau^+_S(x) := \bigwedge_{a \in S} a(x).$$

Given a one-step model $\mathbb{D} = (D, V)$ we define

$$|S|_{\mathbb{D}} := |\{d \in D \mid \mathbb{D} \models \tau_S(d)\}|$$

as the number of elements of $\mathbb{D}$ that realise the type τ_S. $\lhd$

We often blur the distinction between the formula $\tau_S(x)$ and the subset $S \subseteq A$, calling S an A-type as well. Note that we have $\mathbb{D} \models \tau_S(d)$ iff $V^\flat(d) = S$, so that we may refer to $V^\flat(d)$ as the *type of* $d \in D$ indeed.

Definition 2.8 The quantifier rank $\mathtt{qr}(\varphi)$ of a formula $\varphi \in \mathtt{ME}^\infty$ (hence also for $\mathtt{M}$ and $\mathtt{ME}$) is defined as follows:

$$\begin{array}{lcll}
\mathtt{qr}(\varphi) & := & 0 & \text{if } \varphi \text{ is atomic,} \\
\mathtt{qr}(\neg\psi) & := & \mathtt{qr}(\psi) & \\
\mathtt{qr}(\psi_1 \heartsuit \psi_2) & := & \max\{\mathtt{qr}(\psi_1), \mathtt{qr}(\psi_2)\} & \text{where } \heartsuit \in \{\wedge, \vee\} \\
\mathtt{qr}(Qx.\psi) & := & 1 + \mathtt{qr}(\psi), & \text{where } Q \in \{\exists, \forall, \exists^\infty, \forall^\infty\}
\end{array}$$

Given a monadic logic $\mathtt{L}$ we write $\mathbb{D} \equiv^{\mathtt{L}}_k \mathbb{D}'$ to indicate that the models $\mathbb{D}$ and $\mathbb{D}'$ satisfy exactly the same sentences $\varphi \in \mathtt{L}$ with $\mathtt{qr}(\varphi) \le k$. We write $\mathbb{D} \equiv^{\mathtt{L}} \mathbb{D}'$ if $\mathbb{D} \equiv^{\mathtt{L}}_k \mathbb{D}'$ for all k. When clear from context, we may omit explicit reference to $\mathtt{L}$. $\lhd$

[3] The argument in [20] is given in terms of the so called *Chang quantifier* but is easily seen to work also for $\exists^\infty$ and $\forall^\infty$. Both Mostowski's and Slomson's decidability results can be extended to the case of the empty domain.

Definition 2.9 A *partial isomorphism* between two models (D, V) and (D', V') is a partial function $f : D \rightharpoonup D'$ which is injective and satisfies that $d \in V(a) \Leftrightarrow f(d) \in V'(a)$ for all $a \in A$ and $d \in \mathsf{Dom}(f)$. Given two sequences $\overline{\mathbf{d}} \in D^k$ and $\overline{\mathbf{d}'} \in {D'}^k$ we use $f : \overline{\mathbf{d}} \mapsto \overline{\mathbf{d}'}$ to denote the partial function $f : D \rightharpoonup D'$ defined as $f(d_i) := d'_i$. We will take care to avoid cases where there exist d_i, d_j such that $d_i = d_j$ but $d'_i \neq d'_j$. $\triangleleft$

Finally, for future reference we briefly discuss the notion of *Boolean duals.* We first give a concrete definition of a dualisation operator on the set of monadic formulas.

Definition 2.10 The *(Boolean) dual* $\varphi^\delta \in \mathtt{ME}^\infty(A)$ of $\varphi \in \mathtt{ME}^\infty(A)$ is the formula given by:

$$
\begin{aligned}
(a(x))^\delta &:= a(x) & \qquad (\neg a(x))^\delta &:= \neg a(x)\\
(\top)^\delta &:= \bot & (\bot)^\delta &:= \top\\
(x \approx y)^\delta &:= x \not\approx y & (x \not\approx y)^\delta &:= x \approx y\\
(\varphi \wedge \psi)^\delta &:= \varphi^\delta \vee \psi^\delta & (\varphi \vee \psi)^\delta &:= \varphi^\delta \wedge \psi^\delta\\
(\exists x.\psi)^\delta &:= \forall x.\psi^\delta & (\forall x.\psi)^\delta &:= \exists x.\psi^\delta\\
(\exists^\infty x.\psi)^\delta &:= \forall^\infty x.\psi^\delta & (\forall^\infty x.\psi)^\delta &:= \exists^\infty x.\psi^\delta
\end{aligned}
$$

$\triangleleft$

Remark 2.11 Where $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^\infty\}$, observe that if $\varphi \in \mathtt{L}(A)$ then $\varphi^\delta \in \mathtt{L}(A)$. Moreover, the operator preserves positivity of the predicates, that is, if $\varphi \in \mathtt{L}^+(A)$ then $\varphi^\delta \in \mathtt{L}^+(A)$. $\triangleleft$

The following proposition states that the formulas φ and φ^δ are Boolean duals. We omit its proof, which is a routine check.

Proposition 2.12 *Let $\varphi \in \mathtt{ME}^\infty(A)$ be a monadic formula. Then φ and φ^δ are indeed Boolean duals, in the sense that for every monadic model (D, V) we have that*

$$(D, V) \models \varphi \textit{ iff } (D, V^c) \not\models \varphi^\delta,$$

where $V^c : A \to \wp(D)$ is the valuation given by $V^c(a) := D \setminus V(a)$.

3 Normal forms

In this section we provide normal forms for the logics $\mathtt{M}$, $\mathtt{ME}$ and $\mathtt{ME}^\infty$. These normal forms will be pivotal for characterising the different fragments of these logics in later sections.

Convention 3.1 Here and in the sequel it will often be convenient to blur the distinction between lists and sets. For instance, identifying the list $\overline{\mathbf{T}} = T_1 \cdots T_n$ with the set $\{T_1, \ldots, T_n\}$, we may write statements like $S \in \overline{\mathbf{T}}$ or $\Pi \subseteq \overline{\mathbf{T}}$. Moreover, given a finite set $\Phi = \{\varphi_1, \ldots, \varphi_n\}$, we write $\varphi_1 \wedge \cdots \wedge \varphi_n$ as $\bigwedge \Phi$, and $\varphi_1 \vee \cdots \vee \varphi_n$ as $\bigvee \Phi$. If Φ is empty, we set as usual $\bigwedge \Phi = \top$ and $\bigvee \Phi = \bot$. Finally, notice that we write $\bigvee_{1 \leq m < m' \leq n} (y_m \approx y_{m'}) \vee \psi$ as $\mathrm{diff}(\overline{\mathbf{y}}) \to \psi$.

3.1 Normal form for $\mathtt{M}$

We start by introducing a normal form for monadic first-order logic without equality.

Definition 3.2 Given sets of types $\Sigma, \Pi \subseteq \wp(A)$, we define the following formulas:

$$\begin{array}{lll} \nabla_{\mathtt{M}}(\Sigma, \Pi) & := & \bigwedge_{S\in\Sigma} \exists x.\tau_S(x) \wedge \forall x. \bigvee_{S\in\Pi} \tau_S(x) \\ \nabla_{\mathtt{M}}(\Sigma) & := & \nabla_{\mathtt{M}}(\Sigma, \Sigma) \end{array}$$

A sentence of $\mathtt{M}(A)$ is in *basic form* if it is a disjunction of formulas of the form $\nabla_{\mathtt{M}}(\Sigma)$. ◁

Clearly the meaning of the formula $\nabla_{\mathtt{M}}(\Sigma)$ is that Σ is a complete description of the collection of types that are realised in a monadic model. Notice that $\nabla_{\mathtt{M}}(\Sigma, \Pi) = \nabla_{\mathtt{M}}(\Sigma) = \forall x.\bot$, for $\Sigma = \Pi = \varnothing$.

Every $\mathtt{M}$-formula is effectively equivalent to a formula in basic form.

Fact 3.3 *There is an effective procedure that transforms an arbitrary* $\mathtt{M}$*-sentence* φ *into an equivalent formula* φ^* *in basic form.*

This observation is easy to prove using Ehrenfeucht-Fraïssé games – proof sketches can be found in [12, Lemma 16.23] and [28, Proposition 4.14] –, and the decidability of the satisfiability problem for $\mathtt{M}$ (Fact 2.6). We omit a full proof because it is very similar to the following more complex cases.

3.2 Normal form for ME

When considering a normal form for $\mathtt{ME}$, the fact that we can 'count types' using equality yields a more involved basic form.

Definition 3.4 We say that a formula $\varphi \in \mathtt{ME}(A)$ is in *basic form* if $\varphi = \bigvee \nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ where each disjunct is of the form

$$\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi) = \exists \overline{\mathbf{x}}.\big(\mathrm{diff}(\overline{\mathbf{x}}) \wedge \bigwedge_i \tau_{T_i}(x_i) \wedge \forall z.(\mathrm{diff}(\overline{\mathbf{x}}, z) \to \bigvee_{S\in\Pi} \tau_S(z))\big)$$

such that $\overline{\mathbf{T}} \in \wp(A)^k$ for some k and $\Pi \subseteq \overline{\mathbf{T}}$. ◁

We prove that every sentence of monadic first-order logic with equality is equivalent to a formula in basic form. Although this result seems to be folklore, we provide a detailed proof because some of its ingredients will be used later, when we give a normal form for $\mathtt{ME}^\infty$. We start by defining the following relation between monadic models.

Definition 3.5 For every $k \in \mathbb{N}$ we define the relation $\sim_k^=$ on the class $\mathfrak{M}$ of monadic models by putting

$$\mathbb{D} \sim_k^= \mathbb{D}' \iff \forall S \subseteq A\ \big(|S|_{\mathbb{D}} = |S|_{\mathbb{D}'} < k \text{ or } |S|_{\mathbb{D}}, |S|_{\mathbb{D}'} \geq k\big),$$

where $\mathbb{D}$ and $\mathbb{D}'$ are arbitrary monadic models. ◁

Intuitively, two models are related by $\sim_k^=$ when their type information coincides 'modulo k'. Later on we prove that this is the same as saying that they cannot be distinguished by a sentence of $\mathtt{ME}$ with quantifier rank at most k. As a special case, observe that any two monadic models are related by $\sim_0^=$.

For the moment, we record the following properties of these relations.

Proposition 3.6 *The following hold:*

1. *The relation* $\sim_k^=$ *is an equivalence relation of finite index.*
2. *Every* $E \in \mathfrak{M}/{\sim_k^=}$ *is characterised by a sentence* $\varphi_E^= \in \mathtt{ME}(A)$ *with* $\mathtt{qr}(\varphi_E^=) = k$.

Proof. We only prove the second statement, and first we consider the case where $k = 0$. The equivalence relation $\sim_0^=$ has the class $\mathfrak{M}$ of all monadic models as its unique equivalence class, so here we may define $\varphi_{\mathfrak{M}}^= := \top$.

From now on we assume that $k > 0$. Let $E \in \mathfrak{M}/{\sim_k^=}$ and let $\mathbb{D} \in E$ be a representative. Call $S_1, \ldots, S_n \subseteq A$ to the types such that $|S_i|_{\mathbb{D}} = n_i < k$ and $S'_1, \ldots, S'_m \subseteq A$ to those satisfying $|S'_i|_{\mathbb{D}} \geq k$. Note that the union of all the S_i and S'_i yields all the possible A-types, and that if a type S_j is not realised at all, we take $n_j = 0$. Now define

$$\begin{aligned}\varphi_E^= \quad := \quad & \bigwedge_{i \leq n} \Big(\exists x_1, \ldots, x_{n_i}.\mathrm{diff}(x_1, \ldots, x_{n_i}) \ \wedge \ \bigwedge_{j \leq n_i} \tau_{S_i}(x_j) \\ & \qquad \wedge \forall z.\mathrm{diff}(x_1, \ldots, x_{n_i}, z) \to \neg \tau_{S_i}(z) \Big) \\ & \wedge \bigwedge_{i \leq m} \big(\exists x_1, \ldots, x_k.\mathrm{diff}(x_1, \ldots, x_k) \wedge \bigwedge_{j \leq k} \tau_{S'_i}(x_j) \big),\end{aligned}$$

where we understand that any conjunct of the form $\exists x_1, \ldots, x_l.\psi$ with $l = 0$ is simply omitted (or, to the same effect, defined as $\top$). It is easy to see that $\mathtt{qr}(\varphi_E^=) = k$ and that $\mathbb{D}' \models \varphi_E^=$ iff $\mathbb{D}' \in E$. Intuitively, $\varphi_E^=$ gives a specification of E "type by type"; in particular observe that $\varphi_{\mathbb{D}_\varnothing}^= \equiv \forall x.\bot$. QED

Next we recall a (standard) notion of Ehrenfeucht-Fraïssé game for $\mathtt{ME}$ which will be used to establish the connection between $\sim_k^=$ and $\equiv_k^{\mathtt{ME}}$.

Definition 3.7 Let $\mathbb{D}_0 = (D_0, V_0)$ and $\mathbb{D}_1 = (D_1, V_1)$ be monadic models. We define the game $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)$ between $\forall$ and $\exists$. If $\mathbb{D}_i$ is one of the models we use $\mathbb{D}_{-i}$ to denote the other model. A position in this game is a pair of sequences $\overline{\mathbf{s_0}} \in D_0^n$ and $\overline{\mathbf{s_1}} \in D_1^n$ with $n \leq k$. The game consists of k rounds where in round $n+1$ the following steps are made:

1. $\forall$ chooses an element d_i in one of the $\mathbb{D}_i$;
2. $\exists$ responds with an element d_{-i} in the model $\mathbb{D}_{-i}$.

In this way, the sequences $\overline{\mathbf{s_i}} \in D_i^n$ of elements chosen up to round n are extended to $\overline{\mathbf{s_i}}' := \overline{\mathbf{s_i}} \cdot d_i$. Player $\exists$ survives the round iff she does not get stuck and the function $f_{n+1} : \overline{\mathbf{s_0}}' \mapsto \overline{\mathbf{s_1}}'$ is a partial isomorphism of monadic models. Finally, player $\exists$ wins the match iff she survives all k rounds.

Given $n \leq k$ and $\overline{\mathbf{s_i}} \in D_i^n$ such that $f_n : \overline{\mathbf{s_0}} \mapsto \overline{\mathbf{s_1}}$ is a partial isomorphism, we write $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$ to denote the (initialised) game where n moves have been played and $k-n$ moves are left to be played. $\lhd$

Proposition 3.8 *The following are equivalent:*

1. $\mathbb{D}_0 \equiv_k^{\mathtt{ME}} \mathbb{D}_1$,
2. $\mathbb{D}_0 \sim_k^= \mathbb{D}_1$,
3. $\exists$ *has a winning strategy in* $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)$.

Proof. Step (1) to (2) is direct by Proposition 3.6. For (2) to (3) we give a winning strategy for $\exists$ in $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)$ by showing the following claim.

CLAIM 1 Let $\mathbb{D}_0 \sim_k^= \mathbb{D}_1$ and $\overline{\mathbf{s_i}} \in D_i^n$ be such that $n < k$ and $f_n : \overline{\mathbf{s_0}} \mapsto \overline{\mathbf{s_1}}$ is a partial isomorphism; then $\exists$ can survive one more round in $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$.

PROOF OF CLAIM Let $\forall$ pick $d_i \in D_i$ such that the type of d_i is $T \subseteq A$. If d_i had already been played then $\exists$ picks the same element as before and $f_{n+1} = f_n$. If d_i is new and $|T|_{\mathbb{D}_i} \geq k$ then, as at most $n < k$ elements have been played, there is always some new $d_{-i} \in D_{-i}$ that $\exists$ can choose to match d_i. If $|T|_{\mathbb{D}_i} = m < k$ then we know that $|T|_{\mathbb{D}_{-i}} = m$. Therefore, as d_i is new and f_n is injective, there must be a $d_{-i} \in D_{-i}$ that $\exists$ can choose. ◀

Step (3) to (1) is a standard result [9, Corollary 2.2.9] which we prove anyway because we will need to extend it later. We prove the following loaded statement.

Claim 2 Let $\overline{\mathbf{s_i}} \in D_i^n$ and $\varphi(z_1, \ldots, z_n) \in \mathtt{ME}(A)$ be such that $\mathtt{qr}(\varphi) \leq k - n$. If $\exists$ has a winning strategy in the game $\mathrm{EF}_k^{=}(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$ then $\mathbb{D}_0 \models \varphi(\overline{\mathbf{s_0}})$ iff $\mathbb{D}_1 \models \varphi(\overline{\mathbf{s_1}})$.

Proof of Claim If φ is atomic the claim holds because of $f_n : \overline{\mathbf{s_0}} \mapsto \overline{\mathbf{s_1}}$ being a partial isomorphism. The Boolean cases are straightforward. Let $\varphi(z_1, \ldots, z_n) = \exists x.\psi(z_1, \ldots, z_n, x)$ and suppose $\mathbb{D}_0 \models \varphi(\overline{\mathbf{s_0}})$. Hence, there exists $d_0 \in D_0$ such that $\mathbb{D}_0 \models \psi(\overline{\mathbf{s_0}}, d_0)$. By hypothesis we know that $\exists$ has a winning strategy for $\mathrm{EF}_k^{=}(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$. Therefore, if $\forall$ picks $d_0 \in D_0$ she can respond with some $d_1 \in D_1$ and have a winning strategy for $\mathrm{EF}_k^{=}(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}{\cdot}d_0, \overline{\mathbf{s_1}}{\cdot}d_1)$. By induction hypothesis, because $\mathtt{qr}(\psi) \leq k - (n+1)$, we have that $\mathbb{D}_0 \models \psi(\overline{\mathbf{s_0}}, d_0)$ iff $\mathbb{D}_1 \models \psi(\overline{\mathbf{s_1}}, d_1)$ and hence $\mathbb{D}_1 \models \exists x.\psi(\overline{\mathbf{s_1}}, x)$. The opposite direction is proved by a symmetric argument. ◀

We finish the proof of the proposition by combining these two claims. QED

Theorem 3.9 *There is an effective procedure that transforms an arbitrary* ME*-sentence φ into an equivalent formula φ^* in basic form.*

Proof. Let $\mathtt{qr}(\psi) = k$ and let $[\![\psi]\!]$ be the class of models satisfying ψ. As $\mathfrak{M}/{\equiv_k^{\mathtt{ME}}}$ is the same as $\mathfrak{M}/{\sim_k^{=}}$ by Proposition 3.8, it is easy to see that ψ is equivalent to $\bigvee\{\varphi_E^{=} \mid E \in [\![\psi]\!]/{\sim_k^{=}}\}$. Now it only remains to see that each $\varphi_E^{=}$ is equivalent to the sentence $\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ for some $\overline{\mathbf{T}}, \Pi \subseteq \wp(A)$ with $\Pi \subseteq \overline{\mathbf{T}}$.

The crucial observation is that we will use $\overline{\mathbf{T}}$ and Π to give a specification of the types "element by element". Let $\mathbb{D}$ be a representative of the equivalence class E. Call $S_1, \ldots, S_n \subseteq A$ to the types such that $|S_i|_{\mathbb{D}} = n_i < k$ and $S'_1, \ldots, S'_m \subseteq A$ to those satisfying $|S'_j|_{\mathbb{D}} \geq k$. The size of the sequence $\overline{\mathbf{T}}$ is defined to be $(\sum_{i=1}^{n} n_i) + k \times m$ where $\overline{\mathbf{T}}$ contains exactly n_i occurrences of type S_i and at least k occurrences of each S'_j. On the other hand we set $\Pi := \{S'_1, \ldots, S'_m\}$. It is straightforward to check that $\Pi \subseteq \overline{\mathbf{T}}$ and $\varphi_E^{=}$ is equivalent to $\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$. (Observe however, that the quantifier rank of the latter is only bounded by $k \times 2^{|A|} + 1$.) In particular $\varphi_{\mathbb{D}_\varnothing}^{=} \equiv \nabla_{\mathtt{ME}}(\varnothing, \varnothing) = \forall x.\bot$.

The effectiveness of the procedure hence follows from the fact that, given the previous bound on the size of a normal form, it is possible to non-deterministically guess the number of disjuncts, types and associated parameters for each conjunct and repeatedly check whether the formulas φ and $\bigvee \nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ are equivalent, this latter problem being decidable by Fact 2.6. QED

3.3 Normal form for $\mathtt{ME}^\infty$

The logic $\mathtt{ME}^\infty$ extends ME with the capacity to tear apart finite and infinite sets of elements. This is reflected in the normal form for $\mathtt{ME}^\infty$ by adding extra information to the normal form of ME.

Definition 3.10 We say that a formula $\varphi \in \mathtt{ME}^\infty(A)$ is in *basic form* if $\varphi = \bigvee \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ where each disjunct is of the form

$$\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) := \nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi \cup \Sigma) \wedge \nabla_\infty(\Sigma)$$

where

$$\nabla_\infty(\Sigma) := \bigwedge_{S \in \Sigma} \exists^\infty y.\tau_S(y) \wedge \forall^\infty y. \bigvee_{S \in \Sigma} \tau_S(y).$$

Here $\overline{\mathbf{T}} \in \wp(A)^k$ for some k, and $\Pi, \Sigma \subseteq \wp(A)$ are such that $\Sigma \cup \Pi \subseteq \overline{\mathbf{T}}$. ◁

Intuitively, the formula $\nabla_\infty(\Sigma)$ says that (1) for every type $S \in \Sigma$, there are infinitely many elements satisfying S and (2) only finitely many elements do not satisfy any type in Σ. As a special case, the formula $\nabla_\infty(\varnothing)$ expresses that the model is finite. A short argument reveals that, intuitively, every disjunct of the form $\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ expresses that any monadic model satisfying it admits a partition of its domain in three parts:

- (i) distinct elements $t_1, \dots, t_n$ with respective types $T_1, \dots, T_n$,
- (ii) finitely many elements whose types belong to Π, and
- (iii) for each $S \in \Sigma$, infinitely many elements with type S.

Observe that basic formulas of $\mathtt{ME}$ are *not* basic formulas of $\mathtt{ME}^\infty$.

In the same way as before, we define an equivalence relation $\sim_k^\infty$ on monadic models which refines $\sim_k^=$ by adding information about the (in-)finiteness of the types.

Definition 3.11 For every $k \in \mathbb{N}$ we define the relation $\sim_k^\infty$ on the class $\mathfrak{M}$ of monadic models by putting

$$\begin{aligned} \mathbb{D} \sim_0^\infty \mathbb{D}' \quad &\iff \quad \text{always} \\ \mathbb{D} \sim_{k+1}^\infty \mathbb{D}' \quad &\iff \quad \forall S \subseteq A \ \big(|S|_{\mathbb{D}} = |S|_{\mathbb{D}'} < k \text{ or } k \leq |S|_{\mathbb{D}}, |S|_{\mathbb{D}'} < \omega \text{ or } |S|_{\mathbb{D}}, |S|_{\mathbb{D}'} \geq \omega\big), \end{aligned}$$

where $\mathbb{D}$ and $\mathbb{D}'$ are arbitrary monadic models. $\lhd$

Proposition 3.12 *The following hold:*

1. *The relation $\sim_k^\infty$ is an equivalence relation of finite index.*
2. *The relation $\sim_k^\infty$ is a refinement of $\sim_k^=$.*
3. *Every $E \in \mathfrak{M}/{\sim_k^\infty}$ is characterised by a sentence $\varphi_E^\infty \in \mathtt{ME}^\infty(A)$ with $\mathtt{qr}(\varphi) = k$.*

Proof. We only prove the last point, for $k > 0$. Let $E \in \mathfrak{M}/{\sim_k^\infty}$ and let $\mathbb{D} \in E$ be a representative of the class. Let $E' \in \mathfrak{M}/{\sim_k^=}$ be the equivalence class of $\mathbb{D}$ with respect to $\sim_k^=$. Let $S_1, \dots, S_n \subseteq A$ be all the types such that $|S_i|_{\mathbb{D}} \geq \omega$, and define

$$\varphi_E^\infty := \varphi_{E'}^= \wedge \nabla_\infty(\{S_1, \dots, S_n\}).$$

It is not difficult to see that $\mathtt{qr}(\varphi_E^\infty) = k$ and that $\mathbb{D}' \models \varphi_E^\infty$ iff $\mathbb{D}' \in E$. In particular $\varphi_{\mathbb{D}_\varnothing}^\infty \equiv \nabla_{\mathtt{ME}^\infty}(\varnothing, \varnothing, \varnothing) = \forall x.\bot \wedge \forall^\infty y.\bot$. QED

Now we give a version of the Ehrenfeucht-Fraïssé game for $\mathtt{ME}^\infty$. This game, which extends $\mathrm{EF}_k^=$ with moves for $\exists^\infty$, is the adaptation of the Ehrenfeucht-Fraïssé game for monotone generalised quantifiers found in [16] to the case of full monadic first-order logic.

Definition 3.13 Let $\mathbb{D}_0 = (D_0, V_0)$ and $\mathbb{D}_1 = (D_1, V_1)$ be monadic models. We define the game $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1)$ between $\forall$ and $\exists$. A position in this game is a pair of sequences $\overline{\mathbf{s_0}} \in D_0^n$ and $\overline{\mathbf{s_1}} \in D_1^n$ with $n \leq k$. The game consists of k rounds, where in round $n+1$ the following steps are made. First $\forall$ chooses to perform one of the following types of moves:

(a) second-order move:

1. $\forall$ chooses an infinite set $X_i \subseteq D_i$;
2. $\exists$ responds with an infinite set $X_{-i} \subseteq D_{-i}$;
3. $\forall$ chooses an element $d_{-i} \in X_{-i}$;
4. $\exists$ responds with an element $d_i \in X_i$.

(b) first-order move:

1. $\forall$ chooses an element $d_i \in D_i$;
2. $\exists$ responds with an element $d_{-i} \in D_{-i}$.

The sequences $\overline{\mathbf{s_i}} \in D_i^n$ of elements chosen up to round n are then extended to $\overline{\mathbf{s_i}}' := \overline{\mathbf{s_i}} \cdot d_i$. $\exists$ survives the round iff she does not get stuck and the function $f_{n+1} : \overline{\mathbf{s_0}}' \mapsto \overline{\mathbf{s_1}}'$ is a partial isomorphism of monadic models. $\lhd$

Proposition 3.14 *The following are equivalent:*

1. $\mathbb{D}_0 \equiv_k^{\mathtt{ME}^\infty} \mathbb{D}_1$,
2. $\mathbb{D}_0 \sim_k^\infty \mathbb{D}_1$,
3. *$\exists$ has a winning strategy in* $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1)$.

Proof. Step (1) to (2) is direct by Proposition 3.12. For (2) to (3) we show the following.

Claim 1 Let $\mathbb{D}_0 \sim_k^\infty \mathbb{D}_1$ and $\overline{\mathbf{s_i}} \in D_i^n$ be such that $n < k$ and $f_n : \overline{\mathbf{s_0}} \mapsto \overline{\mathbf{s_1}}$ is a partial isomorphism. Then $\exists$ can survive one more round in $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$.

Proof of Claim We focus on the second-order moves because the first-order moves are the same as in the corresponding Claim of Proposition 3.8. Let $\forall$ choose an infinite set $X_i \subseteq D_i$, we would like $\exists$ to choose an infinite set $X_{-i} \subseteq D_{-i}$ such that the following conditions hold:

(a) The map f_n is a well-defined partial isomorphism between the restricted monadic models $\mathbb{D}_0{\restriction}X_0$ and $\mathbb{D}_1{\restriction}X_1$,

(b) For every type S there is an element $d \in X_i$ of type S which is *not* connected by f_n iff there is such an element in X_{-i}.

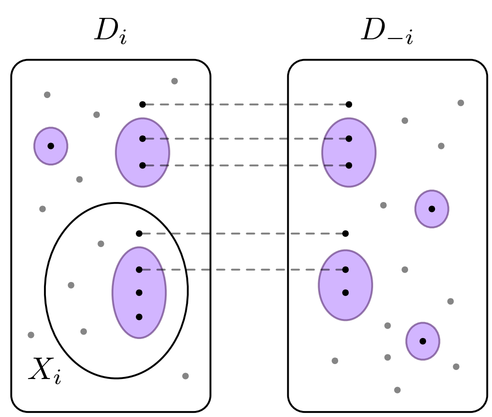

Figure 1: Elements of type S have coloured background.

First we prove that such a set X_{-i} exists. To satisfy item (a) $\exists$ just needs to add to X_{-i} the elements connected to X_i by f_n; this is not a problem.

For item (b) we proceed as follows: for every type S such that there is an element $d \in X_i$ of type S, we add a new element $d' \in D_{-i}$ of type S to X_{-i}. To see that this is always possible, observe first that $\mathbb{D}_0 \sim_k^\infty \mathbb{D}_1$ implies $\mathbb{D}_0 \sim_k^= \mathbb{D}_1$. Using the properties of this relation, we divide in two cases:

- If $|S|_{D_i} \geq k$ we know that $|S|_{D_{-i}} \geq k$ as well. From the elements of D_{-i} of type S, at most $n < k$ are used by f_n. Hence, there is at least one $d' \in D_{-i}$ of type S to choose from.
- If $|S|_{D_i} < k$ we know that $|S|_{D_i} = |S|_{D_{-i}}$. From the elements of D_i of type S, at most $|S|_{D_i} - 1$ are used by f_n. (The reason for the '-1' is that we are assuming that we have just chosen a $d \in X_i$ which is not in f_n.) Using that $|S|_{D_i} = |S|_{D_{-i}}$ and that f_n is a partial isomorphism we can again conclude that there is at least one $d' \in D_{-i}$ of type S to choose from.

Finally, we need to show that $\exists$ can choose X_{-i} to be infinite. To see this, observe that X_i is infinite, while there are only finitely many types. Hence there must be some S such that $|S|_{X_i} \geq \omega$. It is then safe to add infinitely many elements for S in X_{-i} while considering point (b). Moreover, the existence of infinitely many elements satisfying S in D_{-i} is guaranteed by $\mathbb{D}_0 \sim_k^\infty \mathbb{D}_1$.

Having shown that $\exists$ can choose a set X_{-i} satisfying the above conditions, it is now clear that using point (b) $\exists$ can survive the "first-order part" of the second-order move we were considering. This finishes the proof of the claim. ◀

Returning to the proof of Proposition 3.14, for step (3) to (1) we prove the following.

Claim 2 Let $\overline{\mathbf{s_i}} \in D_i^n$ and $\varphi(z_1, \ldots, z_n) \in \mathtt{ME}^\infty(A)$ be such that $\mathtt{qr}(\varphi) \leq k - n$. If $\exists$ has a winning strategy in $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$ then $\mathbb{D}_0 \models \varphi(\overline{\mathbf{s_0}})$ iff $\mathbb{D}_1 \models \varphi(\overline{\mathbf{s_1}})$.

Proof of Claim All the cases involving operators of $\mathtt{ME}$ are the same as in Proposition 3.8. We prove the inductive case for the generalised quantifier. Let $\varphi(z_1, \ldots, z_n)$ be of the form $\exists^\infty x.\psi(z_1, \ldots, z_n, x)$ and let $\mathbb{D}_0 \models \varphi(\overline{\mathbf{s_0}})$. Hence, the set $X_0 := \{d_0 \in D_0 \mid \mathbb{D}_0 \models \psi(\overline{\mathbf{s_0}}, d_0)\}$ is infinite.

By assumption $\exists$ has a winning strategy in $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$. Therefore, if $\forall$ plays a second-order move by picking $X_0 \subseteq D_0$ she can respond with some infinite set $X_1 \subseteq D_1$. We claim that $\mathbb{D}_1 \models \psi(\overline{\mathbf{s_1}}, d_1)$ for every $d_1 \in X_1$. First observe that if this holds then the set $X_1' := \{d_1 \in D_1 \mid \mathbb{D}_1 \models \psi(\overline{\mathbf{s_1}}, d_1)\}$ must be infinite, and hence $\mathbb{D}_1 \models \exists^\infty x.\psi(\overline{\mathbf{s_1}}, x)$.

Assume, for a contradiction, that $\mathbb{D}_1 \not\models \psi(\overline{\mathbf{s_1}}, d_1')$ for some $d_1' \in X_1$. Let $\forall$ play this d_1' as the second part of his move. Then, as $\exists$ has a winning strategy, she will respond with some $d_0' \in X_0$ for which she has a winning strategy in $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}\cdot d_0', \overline{\mathbf{s_1}}\cdot d_1')$. But then by our induction hypothesis, which applies since $\mathtt{qr}(\psi) \leq k - (n+1)$, we may infer from $\mathbb{D}_1 \not\models \psi(\overline{\mathbf{s_1}}, d_1')$ that $\mathbb{D}_0 \not\models \psi(\overline{\mathbf{s_0}}, d_0')$. This clearly contradicts the fact that $d_0' \in X_0$. ◀

Combining the claims finishes the proof of the proposition. QED

Theorem 3.15 *There is an effective procedure that transforms an arbitrary $\mathtt{ME}^\infty$-sentence φ into an equivalent formula φ^* in basic form.*

Proof. This can be proved using the same argument as in Theorem 3.9 but based on Proposition 3.14. Hence we only focus on showing that $\varphi_E^\infty \equiv \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ for some $\overline{\mathbf{T}}, \Pi, \Sigma \subseteq \wp(A)$ such that $\Sigma \cup \Pi \subseteq \overline{\mathbf{T}}$, where φ_E^∞ is the sentence characterising $E \in \mathfrak{M}/{\sim_k^\infty}$ from Proposition 3.12(2). Recall that

$$\varphi_E^\infty = \varphi_{E'}^{\equiv} \wedge \nabla_\infty(\Sigma)$$

where Σ is the collection of types that are realised by infinitely many elements. Using Theorem 3.9 on $\varphi_{E'}^{\equiv}$ we know that this is equivalent to

$$\varphi_E^\infty = \nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi') \wedge \nabla_\infty(\Sigma)$$

where $\Pi' \subseteq \overline{\mathbf{T}}$. Observe that we may assume that $\Sigma \subseteq \Pi$, otherwise the formula would be inconsistent. Now separate Π' as $\Pi' = \Pi \uplus \Sigma$ where $\Pi := \Pi' \setminus \Sigma$ consists of the types that are satisfied by finitely many elements. Then we find

$$\varphi_E^\infty \equiv \nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi \cup \Sigma) \wedge \nabla_\infty(\Sigma).$$

Therefore, we can conclude that $\varphi_E^\infty \equiv \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. QED

The following slightly stronger normal form will be useful in later chapters.

Proposition 3.16 *For every sentence in the basic form* $\bigvee \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ *it is possible to assume, without loss of generality, that* $\Sigma \subseteq \Pi \subseteq \overline{\mathbf{T}}$.

Proof. This is direct from observing that $\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ is equivalent to $\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi \cup \Sigma, \Sigma)$. To check it we just unravel the definitions and observe that $\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi \cup \Sigma) \wedge \nabla_\infty(\Sigma)$ is equivalent to $\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi \cup \Sigma \cup \Sigma) \wedge \nabla_\infty(\Sigma)$. QED

4 Monotonicity

In this section we provide our first characterisation result, which concerns the notion of monotonicity.

Definition 4.1 Let V and V' be two valuations on the same domain D, then we say that V' is a *B-extension of V*, notation: $V \leq_B V'$, if $V(b) \subseteq V'(b)$ for every $b \in B$, and $V(a) = V'(a)$ for every $a \in A \setminus B$.

Given a monadic logic $\mathtt{L}$ and a formula $\varphi \in \mathtt{L}(A)$ we say that φ is *monotone in* $B \subseteq A$ if

$$(D, V), g \models \varphi \text{ and } V \leq_B V' \text{ imply } (D, V'), g \models \varphi, \tag{4}$$

for every pair of monadic models (D, V) and (D, V') and every assignment $g : \mathsf{iVar} \to D$. $\lhd$

Remark 4.2 It is easy to prove that a formula is monotone in $B \subseteq A$ if and only if it is monotone in every $b \in B$. $\lhd$

The semantic property of monotonicity can usually be linked to the syntactic notion of positivity. Indeed, for many logics, a formula φ is monotone in $a \in A$ iff φ is equivalent to a formula where all occurrences of a have a positive polarity, that is, they are situated in the scope of an even number of negations.

Definition 4.3 For $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}\}$ we define the fragment of A-formulas that are *positive* in all predicates in B, in short: the *B-positive formulas* by the following grammar:

$$\varphi ::= \psi \mid b(x) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x.\varphi \mid \forall x.\varphi,$$

where $b \in B$ and $\psi \in \mathtt{L}(A \setminus B)$ (that is, there are no occurrences of any $b \in B$ in ψ). Similarly, the B-positive fragment of $\mathtt{ME}^\infty$ is given by

$$\varphi ::= \psi \mid b(x) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x.\varphi \mid \forall x.\varphi \mid \exists^\infty x.\varphi \mid \forall^\infty x.\varphi,$$

where $b \in B$ and $\psi \in \mathtt{ME}^\infty(A \setminus B)$.

In all three cases, we let $\mathtt{Pos}_B(\mathtt{L}(A))$ denote the set of B-positive sentences. $\lhd$

Note that the difference between the fragments $\mathtt{Pos}_B(\mathtt{M}(A))$ and $\mathtt{Pos}_B(\mathtt{ME}(A))$ lies in the fact that in the latter case, the 'B-free' formulas ψ may contain the equality symbol. Clearly $\mathtt{Pos}_A(\mathtt{L}(A)) = \mathtt{L}^+$.

Theorem 4.4 *Let φ be a sentence of the monadic logic $\mathtt{L}(A)$, where $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^\infty\}$. Then φ is monotone in a set $B \subseteq A$ if and only if there is an equivalent formula $\varphi^\oslash \in \mathtt{Pos}_B(\mathtt{L}(A))$. Furthermore, it is decidable whether a sentence $\varphi \in \mathtt{L}(A)$ has this property or not.*

The 'easy' direction of the first claim of the theorem is taken care of by the following proposition.

Proposition 4.5 *Every formula* $\varphi \in \texttt{Pos}_B(\texttt{L}(A))$ *is monotone in* B*, where* $\texttt{L}$ *is one of the logics* $\{\texttt{M}, \texttt{ME}, \texttt{ME}^\infty\}$.

Proof. The case for $D = \varnothing$ being immediate, we assume $D \neq \varnothing$. The proof is a routine argument by induction on the complexity of φ. That is, we show by induction, that any formula φ in the B-positive fragment (which may not be a sentence) satisfies (4), for every monadic model (D, V), valuation $V' \geq_B V$ and assignment $g : \mathsf{iVar} \to D$. We focus on the generalised quantifiers. Let $(D, V), g \models \varphi$ and $V \leq_B V'$.

- Case $\varphi = \exists^\infty x.\varphi'(x)$. By definition there exists an infinite set $I \subseteq D$ such that for all $d \in I$ we have $(D, V), g[x \mapsto d] \models \varphi'(x)$. By induction hypothesis $(D, V'), g[x \mapsto d] \models \varphi'(x)$ for all $d \in I$. Therefore $(D, V'), g \models \exists^\infty x.\varphi'(x)$.
- Case $\varphi = \forall^\infty x.\varphi'(x)$. Hence there exists $C \subseteq D$ such that for all $d \in C$ we have $(D, V), g[x \mapsto d] \models \varphi'(x)$ and $D \setminus C$ is *finite*. By induction hypothesis $(D, V'), g[x \mapsto d] \models \varphi'(x)$ for all $d \in C$. Therefore $(D, V'), g \models \forall^\infty x.\varphi'(x)$.

This finishes the proof. QED

The 'hard' direction of the first claim of the theorem states that the fragment $\texttt{Pos}_B(\texttt{M})$ is complete for monotonicity in B. In order to prove it, we need to show that every sentence which is monotone in B is equivalent to some formula in $\texttt{Pos}_B(\texttt{M})$. We actually are going to prove a stronger result.

Proposition 4.6 *Let* $\texttt{L}$ *be one of the logics* $\{\texttt{M}, \texttt{ME}, \texttt{ME}^\infty\}$. *There exists an effective translation* $(-)^\oslash : \texttt{L}(A) \to \texttt{Pos}_B(\texttt{L}(A))$ *such that a sentence* $\varphi \in \texttt{L}(A)$ *is monotone in* $B \subseteq A$ *only if* $\varphi \equiv \varphi^\oslash$.

We prove the three manifestations of Proposition 4.6 separately, in three respective subsections.

Proof of Theorem 4.4. The first claim of the Theorem is an immediate consequence of Proposition 4.6. By effectiveness of the translation and Fact 2.6, it is therefore decidable whether a sentence $\varphi \in \texttt{L}(A)$ is monotone in $B \subseteq A$ or not. QED

The following definition will be used throughout in the remaining of the section.

Definition 4.7 Given $S \subseteq A$ and $B \subseteq A$ we use the following notation

$$\tau_S^B(x) := \bigwedge_{b \in S} b(x) \wedge \bigwedge_{b \in A \setminus (S \cup B)} \neg b(x),$$

for what we call the *B-positive* A-type τ_S^B. ◁

Intuitively, τ_S^B works almost like the A-type τ_S, the difference being that τ_S^B discards the negative information for the names in B. If $B = \{a\}$ we write τ_S^a instead of $\tau_S^{\{a\}}$. Observe that with this notation, τ_S^+ is equivalent to τ_S^A.

4.1 Monotone fragment of $\texttt{M}$

In this subsection we prove the $\texttt{M}$-variant of Proposition 4.6. That is, we give a translation that constructively maps arbitrary sentences into $\texttt{Pos}_B(\texttt{M})$ and that moreover it preserves truth iff the given sentence is monotone in B. To formulate the translation we need to introduce some new notation.

Definition 4.8 Let $B \subseteq A$ be a finite set of names. The B-positive variant of $\nabla_{\mathtt{M}}(\Sigma)$ is given as follows:

$$\nabla^B_{\mathtt{M}}(\Sigma) := \bigwedge_{S\in\Sigma} \exists x.\tau^B_S(x) \wedge \forall x. \bigvee_{S\in\Sigma} \tau^B_S(x).$$

We also introduce the following generalised forms of the above notation:

$$\nabla^B_{\mathtt{M}}(\Sigma,\Pi) := \bigwedge_{S\in\Sigma} \exists x.\tau^B_S(x) \wedge \forall x. \bigvee_{S\in\Pi} \tau^B_S(x).$$

The *positive* variants of the above notations are defined as $\nabla^+_{\mathtt{M}}(\Sigma) := \nabla^A_{\mathtt{M}}(\Sigma)$ and $\nabla^+_{\mathtt{M}}(\Sigma,\Pi) := \nabla^A_{\mathtt{M}}(\Sigma,\Pi)$. ◁

Proposition 4.9 *There exists an effective translation* $(-)^{\oslash} : \mathtt{M}(A) \to \mathtt{Pos}_B(\mathtt{M}(A))$ *such that a sentence* $\varphi \in \mathtt{M}(A)$ *is monotone in* $B \subseteq A$ *if and only if* $\varphi \equiv \varphi^{\oslash}$.

Proof. To define the translation, by Fact 3.3, we assume, without loss of generality, that φ is in the normal form $\bigvee \nabla_{\mathtt{M}}(\Sigma)$ given in Definition 3.2, where $\nabla_{\mathtt{M}}(\Sigma) = \bigwedge_{S\in\Sigma} \exists x.\tau_S(x) \wedge \forall x. \bigvee_{S\in\Sigma} \tau_S(x)$. We define the translation as

$$(\bigvee \nabla_{\mathtt{M}}(\Sigma))^{\oslash} := \bigvee \nabla^B_{\mathtt{M}}(\Sigma).$$

From the construction it is clear that $\varphi^{\oslash} \in \mathtt{Pos}_B(\mathtt{M}(A))$ and therefore the right-to-left direction of the proposition is immediate by Proposition 4.5. For the left-to-right direction assume that φ is monotone in B, we have to prove that $(D,V) \models \varphi$ if and only if $(D,V) \models \varphi^{\oslash}$.

$\boxed{\Rightarrow}$ This direction is trivial.

$\boxed{\Leftarrow}$ Assume $(D,V) \models \varphi^{\oslash}$ and let Σ be such that $(D,V) \models \nabla^B_{\mathtt{M}}(\Sigma)$. If $D = \varnothing$, then $\Sigma = \varnothing$ and $\nabla^B_{\mathtt{M}}(\Sigma) = \nabla_{\mathtt{M}}(\Sigma)$. Hence, assume $D \neq \varnothing$, and clearly $\Sigma \neq \varnothing$.

Because of the existential part of $\nabla^B_{\mathtt{M}}(\Sigma)$, every type $S \in \Sigma$ has a 'B-witness' in $\mathbb{D}$, that is, an element $d_S \in D$ such that $(D,V) \models \tau^B_S(d_S)$. It is in fact safe to assume that all these witnesses are *distinct* (this is because (D,V) can be proved to be $\mathtt{M}$-equivalent to such a model, cf. Proposition 6.12). But because of the universal part of $\nabla^B_{\mathtt{M}}(\Sigma)$, we may assume that for all states d in D there is a type S_d in Σ such that $(D,V) \models \tau^B_{S_d}(d)$. Putting these observations together we may assume that the map $d \mapsto S_d$ is surjective.

Note however, that where we have $(D,V) \models \tau^B_S(d)$, this does not necessarily imply that $(D,V) \models \tau_S(d)$: it might well be the case that $d \in V(b)$ but $b \notin S_d$, for some $b \in B$. What we want to do now is to shrink V in such a way that the witnessed type (S_d) and the actually satisfied type coincide. That is, we consider the valuation U defined as $U^\flat(d) := S_d$.[4] It is then immediate by the surjectivity of the map $d \mapsto S_d$ that $(D,U) \models \nabla_{\mathtt{M}}(\Sigma)$, which implies that $(D,U) \models \varphi$.

We now claim that

$$U \leq_B V. \tag{5}$$

To see this, observe that for $a \in A \setminus B$ we have the following equivalences:

$$d \in U(a) \iff a \in S_d \iff (D,V) \models a(d) \iff d \in V(a),$$

while for $b \in B$ we can prove

$$d \in U(b) \iff b \in S_d \Longrightarrow (D,V) \models b(d) \iff d \in V(b).$$

This suffices to prove (5).

But from (5) and the earlier observation that $(D,U) \models \varphi$ it is immediate by the monotonicity of φ in B that $(D,V) \models \varphi$. QED

[4] Recall that a valuation $U : A \to \wp(D)$ can also be represented as a colouring $U^\flat : D \to \wp(A)$ given by $U^\flat(d) := \{a \in A \mid d \in V(a)\}$.

A careful analysis of the translation gives us the following corollary, providing normal forms for the monotone fragment of $\mathtt{M}$.

Corollary 4.10 *For any sentence $\varphi \in \mathtt{M}(A)$, the following hold.*

1. *The formula φ is monotone in $B \subseteq A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla^B_{\mathtt{M}}(\Sigma)$ for some types $\Sigma \subseteq \wp(A)$.*
2. *The formula φ is monotone in every $a \in A$ iff φ is equivalent to a formula $\bigvee \nabla^+_{\mathtt{M}}(\Sigma)$ for some types $\Sigma \subseteq \wp(A)$.*

In both cases the norma forms are effective.

4.2 Monotone fragment of ME

In order to prove the $\mathtt{ME}$-variant of Proposition 4.6, we need to introduce some new notation.

Definition 4.11 Let $B \subseteq A$ be a finite set of names. The B-monotone variant of $\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ is given as follows:

$$\nabla^B_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi) := \exists \overline{\mathbf{x}}.\big(\mathrm{diff}(\overline{\mathbf{x}}) \wedge \bigwedge_i \tau^B_{T_i}(x_i) \wedge \forall z.(\mathrm{diff}(\overline{\mathbf{x}}, z) \to \bigvee_{S \in \Pi} \tau^B_S(z))\big).$$

When the set B is a singleton $\{a\}$ we will write a instead of B. The positive variant $\nabla^+_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ of $\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ is defined as above but with $+$ in place of B. $\lhd$

Proposition 4.12 *There exists an effective translation $(-)^{\oslash} : \mathtt{ME}(A) \to \mathtt{Pos}_B(\mathtt{ME}(A))$ such that a sentence $\varphi \in \mathtt{ME}(A)$ is monotone in B if and only if $\varphi \equiv \varphi^{\oslash}$.*

Proof. In proposition 4.15 this result is proved for $\mathtt{ME}^\infty$ (i.e., $\mathtt{ME}$ extended with generalised quantifiers). It is not difficult to adapt the proof for $\mathtt{ME}$. The translation is defined as follows. By Theorem 3.9 ,without loss of generality, assume that φ is in basic normal form $\bigvee \nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$. Then $\varphi^{\oslash} := \bigvee \nabla^B_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$. QED

Combining the normal form for $\mathtt{ME}$ and the proof of the above proposition, we therefore obtain a normal form for the monotone fragment of $\mathtt{ME}$.

Corollary 4.13 *For any sentence $\varphi \in \mathtt{M}(A)$, the following hold.*

1. *The formula φ is monotone in $B \subseteq A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla^B_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ where for each disjunct we have $\overline{\mathbf{T}} \in \wp(A)^k$ for some k and $\Pi \subseteq \overline{\mathbf{T}}$.*
2. *The formula φ is monotone in all $a \in A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla^+_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ where for each disjunct we have $\overline{\mathbf{T}} \in \wp(A)^k$ for some k and $\Pi \subseteq \overline{\mathbf{T}}$.*

In both cases, normal forms are effective.

4.3 Monotone fragment of ME$^\infty$

First, in this case too we introduce some notation for the positive variant of a sentence in normal form.

Definition 4.14 Let $B \subseteq A$ be a finite set of names. The B-positive variant of $\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ is given as follows:

$$\begin{aligned}
\nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) &:= \nabla^B_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi \cup \Sigma) \wedge \nabla^B_\infty(\Sigma) \\
\nabla^B_{\mathtt{ME}}(\overline{\mathbf{T}}, \Lambda) &:= \exists \overline{\mathbf{x}}.\big(\mathrm{diff}(\overline{\mathbf{x}}) \wedge \bigwedge_i \tau^B_{T_i}(x_i) \wedge \forall z.(\mathrm{diff}(\overline{\mathbf{x}}, z) \to \bigvee_{S \in \Lambda} \tau^B_S(z))\big) \\
\nabla^B_\infty(\Sigma) &:= \bigwedge_{S \in \Sigma} \exists^\infty y.\tau^B_S(y) \wedge \forall^\infty y. \bigvee_{S \in \Sigma} \tau^B_S(y).
\end{aligned}$$

When the set B is a singleton $\{a\}$ we will write a instead of B. The positive variant of $\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ is defined as $\nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) := \nabla^A_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. ◁

We are now ready to proceed with the proof of the $\mathtt{ME}^\infty$-variant of Proposition 4.6 and thus to give the translation.

Proposition 4.15 *There is an effective translation $(-)^\oslash : \mathtt{ME}^\infty(A) \to \mathtt{Pos}_B(\mathtt{ME}^\infty(A))$ such that a sentence $\varphi \in \mathtt{ME}^\infty(A)$ is monotone in B if and only if $\varphi \equiv \varphi^\oslash$.*

Proof. By Theorem 3.15, we assume that φ is in the normal form $\bigvee \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) = \nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi \cup \Sigma) \wedge \nabla_\infty(\Sigma)$ for some sets of types $\Pi, \Sigma \subseteq \wp(A)$ and each $T_i \subseteq A$. For the translation we define

$$\Big(\bigvee \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)\Big)^\oslash := \bigvee \nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma).$$

From the construction it is clear that $\varphi^\oslash \in \mathtt{Pos}_B(\mathtt{ME}^\infty(A))$ and therefore the right-to-left direction of the proposition is immediate by Proposition 4.5. For the left-to-right direction assume that φ is monotone in B, we have to prove that $(D, V) \models \varphi$ if and only if $(D, V) \models \varphi^\oslash$.

$\boxed{\Rightarrow}$ This direction is trivial.

$\boxed{\Leftarrow}$ Assume $(D, V) \models \varphi^\oslash$, and in particular that $(D, V) \models \nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. If $D = \varnothing$, then $\Sigma = \Pi = \overline{\mathbf{T}} = \varnothing$ and $\nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) = \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. Hence, assume $D \neq \varnothing$. Observe that the elements of D can be partitioned in the following way:

(a) distinct elements $t_i \in D$ such that each t_i satisfies $\tau^B_{T_i}(x)$,

(b) for every $S \in \Sigma$ an infinite set D_S, such that every $d \in D_S$ satisfies τ^B_S,

(c) a *finite* set D_Π of elements, each satisfying one of the B-positive types τ^B_S with $S \in \Pi$.

Following this partition, with every element $d \in D$ we may associate a type S_d in, respectively, (a) $\overline{\mathbf{T}}$, (b) Σ, or (c) Π, such that d satisfies $\tau^B_{S_d}$. As in the proof of proposition 4.9, we now consider the valuation U defined as $U^\flat(d) := S_d$, and as before we can show that $U \leq_B V$. Finally, it easily from the definitions that $(D, U) \models \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, implying that $(D, U) \models \varphi$. But then by the assumed B-monotonicity of φ it is immediate that $(D, V) \models \varphi$, as required. QED

As with the previous two cases, the translation provides normal forms for the monotone fragment of $\mathtt{ME}^\infty$.

Corollary 4.16 *For any sentence $\varphi \in \mathtt{ME}^\infty(A)$, the following hold:*

1. *The formula φ is monotone in $B \subseteq A$ iff it is equivalent to a formula $\bigvee \nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ for $\Sigma \subseteq \Pi \subseteq \wp(A)$ and $\overline{\mathbf{T}} \in \wp(A)^k$ for some k.*

2. *The formula φ is monotone in every $a \in A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla^{+}_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ for types $\Sigma \subseteq \Pi \subseteq \wp(A)$ and $\overline{\mathbf{T}} \in \wp(A)^k$ for some k.*

In both cases, normal forms are effective.

Proof. We only remark that to obtain $\Sigma \subseteq \Pi$ in the above normal forms it is enough to use Proposition 3.16 before applying the translation. QED

5 Continuity

In this section we study the sentences that are *continuous* in some set B of monadic predicate symbols.

Definition 5.1 Let U and V be two A-valuations on the same domain D. For a set $B \subseteq A$, we write $U \leq^\omega_B V$ if $U \leq_B V$ and $U(b)$ is finite, for every $b \in B$.

Given a monadic logic $\mathtt{L}$ and a formula $\varphi \in \mathtt{L}(A)$ we say that φ is *continuous in* $B \subseteq A$ if φ is monotone in B and satisfies the following:

$$\text{if } (D, V), g \models \varphi \text{ then } (D, U), g \models \varphi \text{ for some } U \leq^\omega_B V. \tag{6}$$

for every monadic model (D, V) and every assignment $g : \mathsf{iVar} \to D$. $\lhd$

Remark 5.2 As for monotonicity, but with slightly more effort, one may show that a formula φ is continuous in a set B iff it is continuous in every $b \in B$. $\lhd$

What explains both the name and the importance of this property is its equivalence to so called *Scott continuity*. To understand it, we may formalise the dependence of the meaning of a monadic sentence φ with m-free variables $\overline{\mathbf{x}}$ in a one-step model $\mathbb{D} = (D, V)$ on a fixed name $b \in A$ as a map $\varphi^{\mathbb{D}}_b : \wp(D) \to \wp(D^m)$ defined by

$$X \subseteq D \mapsto \{\overline{\mathbf{d}} \in D^m \mid (D, V[b \mapsto X]) \models \varphi(\overline{\mathbf{d}})\}.$$

One can then verify that a sentence φ is continuous in b if and only if the operation $\varphi^{\mathbb{D}}_b$ is continuous with respect to the Scott topology on the powerset algebras[5]. Scott continuity is of key importance in many areas of theoretical computer sciences where ordered structures play a role, such as domain theory (see e.g. [1]).

Similarly as for monotonicity, the semantic property of continuity can also be provided with a corresponding syntactical formulation.

Definition 5.3 Let $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}\}$ The fragment of $\mathtt{M}(A)$ of formulas that are *syntactically continuous* in a subset $B \subseteq A$ is defined by the following grammar:

$$\varphi ::= \psi \mid b(x) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x.\varphi,$$

where $b \in B$ and $\psi \in \mathtt{L}(A \setminus B)$. In both cases, we let $\mathtt{Con}_B(\mathtt{L}(A))$ denote the set of B-continuous sentences. $\lhd$

To define the syntactically continuous fragment of $\mathtt{ME}^\infty$, we first introduce the following binary generalised quantifier $\mathbf{W}$: given two formulas $\varphi(x)$ and ψ, we set

$$\mathbf{W}x.(\varphi, \psi) := \forall x.(\varphi(x) \vee \psi(x)) \wedge \forall^\infty x.\psi(x).$$

The intuition behind $\mathbf{W}$ is the following. If $(D, V), g \models \mathbf{W}x.(\varphi, \psi)$, then because of the second conjunct there are only finitely many $d \in D$ refuting ψ. The point is that this weakens the universal quantification of the first conjunct to the effect that only the finitely many mentioned elements refuting ψ need to satisfy φ.

[5] The interested reader is referred to [11, Sec. 8] for a more precise discussion of the connection.

Definition 5.4 The fragment of $\mathtt{ME}^\infty(A)$-formulas that are *syntactically continuous* in a subset $B \subseteq A$ is given by the following grammar:

$$\varphi ::= \psi \mid a(x) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x.\varphi \mid \mathbf{W}x.(\varphi, \psi),$$

where $b \in B$ and $\psi \in \mathtt{ME}^\infty(A \backslash B)$. We let $\mathtt{Con}_B(\mathtt{ME}^\infty(A))$ denote the set of B-continuous $\mathtt{ME}^\infty$-sentences. $\lhd$

For $\mathtt{M}$ and $\mathtt{ME}$, the equivalence between the semantical and syntactical properties of continuity was established by van Benthem in [25].

Proposition 5.5 *Let φ be a sentence of the monadic logic $\mathtt{L}(A)$, where $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}\}$. Then φ is continuous in a set $B \subseteq A$ if and only if there is a equivalent sentence $\varphi^\ominus \in \mathtt{Con}_B(\mathtt{L}(A))$.*

Proof. The direction from right to left is covered by Proposition 5.7 below, so we immediately turn to the completeness part of the statement. The case of $\mathtt{M}$ being treated in Subsection 5.1, we only discuss the statement for $\mathtt{ME}$. Hence, let $\varphi \in \mathtt{ME}(A)$ be continuous in B. For simplicity in the exposition, we assume $B = \{b\}$, the case of an arbitrary B being easily generalisable from what follows. Let $\overline{\mathbf{y}} := y_0 \dots y_{k-1}$ be a list of k variables not occurring in φ. Consider the formula $\varphi_k(\overline{\mathbf{y}})$ obtained from φ by substituting each occurrence of an atomic formula of the form $b(x)$ with the formula $\bigvee_{\ell<k} x = y_\ell$. Define $\Phi := \{\exists \overline{\mathbf{y}}.\varphi_k(\overline{\mathbf{y}}) \mid k \in \omega\} \cup \{\varphi_{\mathbb{D}_\varnothing}\}$, where $\varphi_{\mathbb{D}_\varnothing} := \forall x.\bot$ if $\mathbb{D}_\varnothing \models \varphi$ and $\varphi_{\mathbb{D}_\varnothing} := \exists x.\bot$ otherwise. By construction $\Phi \subset \mathtt{Con}_B(\mathtt{ME}(A))$. Now, notice that $\neg\Phi \cup \{\varphi\}$ is inconsistent. Hence, by compactness of first-order logic, there is a $k \in \omega$ such that $\varphi \models \bigvee_{\ell<k} \exists \overline{\mathbf{y}}.\varphi_k(\overline{\mathbf{y}}) \vee \varphi_{\mathbb{D}_\varnothing}$. By monotonicity, $\exists \overline{\mathbf{y}}.\varphi_k(\overline{\mathbf{y}}) \models \varphi$, for every $k \in \omega$, and by definition $\varphi_{\mathbb{D}_\varnothing} \models \varphi$. We therefore conclude that $\varphi \equiv \bigvee_{\ell<k} \exists \overline{\mathbf{y}}.\varphi_k(\overline{\mathbf{y}}) \vee \varphi_{\mathbb{D}_\varnothing}$. As $\mathtt{Con}_B(\mathtt{ME}(A))$ is closed under disjunctions, this ends the proof of the statement. QED

In this paper, we extend such a characterisation to $\mathtt{ME}^\infty$. Moreover, analogously to what we did in the previous section, for $\mathtt{M}$ and $\mathtt{ME}^\infty$ we provide both an explicit translation and a decidability result. From this latter perspective, the case of $\mathtt{ME}$ remains however open.

Theorem 5.6 *Let φ be a sentence of the monadic logic $\mathtt{L}(A)$, where $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}^\infty\}$. Then φ is continuous in a set $B \subseteq A$ if and only if there is a equivalent sentence $\varphi^\ominus \in \mathtt{Con}_B(\mathtt{L}(A))$. Furthermore, it is decidable whether a sentence $\varphi \in \mathtt{L}(A)$ has this property or not.*

Analogously to the previous case of monotonicity, the proof of the theorem is composed of two parts. We start with the right-left implication of the first claim (the preservation statement), which also holds for $\mathtt{ME}$.

Proposition 5.7 *Every sentence $\varphi \in \mathtt{Con}_B(\mathtt{L}(A))$ is continuous in B, where $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^\infty\}$.*

Proof. First observe that φ is monotone in B by Proposition 4.5. The case for $D = \varnothing$ being clear, we assume $D \neq \varnothing$. We show, by induction, that any one-step formula φ in the fragment (which may not be a sentence) satisfies (6), for every non-empty one-step model (D, V) and assignment $g : \mathsf{iVar} \to D$.

- If $\varphi = \psi \in \mathtt{L}(A \setminus B)$, changes in the B part of the valuation will not affect the truth value of φ and hence the condition is trivial.
- Case $\varphi = b(x)$ for some $b \in B$: if $(D, V), g \models b(x)$ then $g(x) \in V(b)$. Let U be the valuation given by $U(b) := \{g(x)\}$, $U(a) := \varnothing$ for $a \in B \setminus \{b\}$ and $U(a) := V(a)$ for $a \in A \setminus B$. Then it is obvious that $(D, U), g \models b(x)$, while it is immediate by the definitions that $U \leq^\omega_B V$.

- Case $\varphi = \varphi_1 \vee \varphi_2$: assume $(D,V), g \models \varphi$. Without loss of generality we can assume that $(D,V), g \models \varphi_1$ and hence by induction hypothesis there is $U \leq_B^\omega V$ such that $(D,U), g \models \varphi_1$ which clearly implies $(D,U), g \models \varphi$.

- Case $\varphi = \varphi_1 \wedge \varphi_2$: assume $(D,V), g \models \varphi$. By induction hypothesis we have $U_1, U_2 \leq_B^\omega V$ such that $(D,U_1), g \models \varphi_1$ and $(D,U_2), g \models \varphi_2$. Let U be the valuation defined by putting $U(a) := U_1(a) \cup U_2$; then clearly we have $U \leq_B^\omega V$, while it follows by monotonicity that $(D,U), g \models \varphi_1$ and $(D,U), g \models \varphi_2$. Clearly then $(D,U), g \models \varphi$.

- Case $\varphi = \exists x.\varphi'(x)$ and $(D,V), g \models \varphi$. By definition there exists $d \in D$ such that $(D,V), g[x \mapsto d] \models \varphi'(x)$. By induction hypothesis there is a valuation $U \leq_B^\omega V$ such that $(D,U), g[x \mapsto d] \models \varphi'(x)$ and hence $(D,U), g \models \exists x.\varphi'(x)$.

- Case $\varphi = \mathbf{W}x.(\varphi', \psi) \in \mathtt{Con}_B(\mathtt{ME}^\infty(A))$ and $(D,V), g \models \varphi$. Define the formulas $\alpha(x)$ and β as follows:
$$\varphi = \forall x. \underbrace{(\varphi'(x) \vee \psi(x))}_{\alpha(x)} \wedge \underbrace{\forall^\infty x.\psi(x)}_{\beta}.$$
Suppose that $(D,V), g \models \varphi$. By the induction hypothesis, for every $d \in D$ which satisfies $(D,V), g_d \models \alpha(x)$ (where we write $g_d := g[x \mapsto d]$) there is a valuation $U_d \leq_B^\omega V$ such that $(D,U_d), g_d \models \alpha(x)$. The crucial observation is that because of β, only finitely many elements of d refute $\psi(x)$. Let U be the valuation defined by putting $U(a) := \bigcup\{U_d(a) \mid (D,V), g_d \not\models \psi(x)\}$. Note that for each $b \in B$, the set $U(b)$ is a finite union of finite sets, and hence finite itself; it follows that $U \leq_B^\omega V$. We claim that
$$(D,U), g \models \varphi. \tag{7}$$
It is clear that $(D,U), g \models \beta$ because ψ (and hence β) is B-free. To prove that $(D,U), g \models \forall x\, \alpha(x)$, take an arbitrary $d \in D$, then we have to show that $(D,U), g_d \models \varphi'(x) \vee \psi(x)$. We consider two cases: If $(D,V), g_d \models \psi(x)$ we are done, again because ψ is B-free. On the other hand, if $(D,V), g_d \not\models \psi(x)$, then $(D,U_d), g_d \models \alpha(x)$ by assumption on U_d, while it is obvious that $U_d \leq_B U$; but then it follows by monotonicity of α that $(D,U), g_d \models \alpha(x)$.

This finishes the proof. QED

The second part of the proof of the theorem, is thus constituted by the following stronger version of the expressive completeness result that provides as a corollary normal forms for the syntactically continuous fragments.

Proposition 5.8 *Let* $\mathtt{L}$ *be one of the logics* $\{\mathtt{M}, \mathtt{ME}^\infty\}$. *There exists an effective translation* $(-)^\ominus : \mathtt{L}(A) \to \mathtt{Con}_B(\mathtt{L}(A))$ *such that a sentence* $\varphi \in \mathtt{L}(A)$ *is continuous in* $B \subseteq A$ *if and only if* $\varphi \equiv \varphi^\ominus$.

We prove the two manifestations of Proposition 5.8 separately, in two respective subsections.
By putting together the two propositions above, we are thence able to conclude.

Proof of Theorem 5.6. The first claim follows from Proposition 5.8. Hence, by applying Fact 2.6 to Proposition 5.8, the problem of checking whether a sentence $\varphi \in \mathtt{L}(A)$ is continuous in $B \subseteq A$ or not, is decidable. QED

We conjecture that Proposition 5.8, and therefore Theorem 5.6, holds also for $\mathtt{L} = \mathtt{ME}$.

5.1 Continuous fragment of $\mathtt{M}$

Since continuity implies monotonicity, by Theorem 4.4, in order to verify the $\mathtt{M}$-variant of Proposition 5.8, it is enough to proof the following result.

Proposition 5.9 *There is an effective translation* $(-)^{\ominus} : \mathtt{Pos}_B(\mathtt{M}(A)) \to \mathtt{Con}_B(\mathtt{M}(A))$ *such that a sentence* $\varphi \in \mathtt{Pos}_B(\mathtt{M}(A))$ *is continuous in* $B \subseteq A$ *if and only if* $\varphi \equiv \varphi^{\ominus}$.

Proof. By Corollary 4.10, to define the translation we assume, without loss of generality, that φ is in the basic form $\bigvee \nabla^B_{\mathtt{M}}(\Sigma)$. For the translation, let

$$(\bigvee \nabla^B_{\mathtt{M}}(\Sigma))^{\ominus} := \bigvee \nabla^B_{\mathtt{M}}(\Sigma, \Sigma^-_B)$$

where $\Sigma^-_B := \{S \in \Sigma \mid B \cap S = \varnothing\}$. From the construction it is clear that $\varphi^{\ominus} \in \mathtt{Con}_B(\mathtt{M}(A))$ and therefore the right-to-left direction of the proposition is immediate by Proposition 5.7.

For the left-to-right direction assume that φ is continuous in B, we have to prove that $(D, V) \models \varphi$ iff $(D, V) \models \varphi^{\ominus}$, for every one-step model (D, V). Our proof strategy consists of proving the same equivalence for the model $(D \times \omega, V_\pi)$, where $D \times \omega$ consists of ω many copies of each element in D and V_π is the valuation given by $V_\pi(a) := \{(d, k) \mid d \in V(a), k \in \omega\}$. It is easy to see that $(D, V) \equiv^{\mathtt{M}} (D \times \omega, V_\pi)$ (see Proposition 6.12) and so it suffices indeed to prove that

$$(D \times \omega, V_\pi) \models \varphi \text{ iff } (D \times \omega, V_\pi) \models \varphi^{\ominus}.$$

Consider first $D = \varnothing$. Then $(D \times \omega, V_\pi) = \mathbb{D}_\varnothing$, and therefore the claim is true since $\nabla^B_{\mathtt{M}}(\varnothing) = \nabla^B_{\mathtt{M}}(\varnothing, \varnothing^-_B)$ and $\mathbb{D}_\varnothing \models \nabla^B_{\mathtt{M}}(\Sigma)$ iff $\Sigma = \varnothing$. Hence, assume $D \neq \varnothing$.

$\boxed{\Rightarrow}$ Let $(D \times \omega, V_\pi) \models \varphi$. As φ is continuous in B there is a valuation $U \leq^\omega_B V_\pi$ satisfying $(D \times \omega, U) \models \varphi$. This means that $(D \times \omega, U) \models \nabla^B_{\mathtt{M}}(\Sigma)$ for some disjunct $\nabla^B_{\mathtt{M}}(\Sigma)$ of φ. Below we will use the following fact (which can easily be verified):

$$(D \times \omega), U \models \tau^B_S(d, k) \text{ iff } S \setminus B = U^\flat(d, k) \setminus B \text{ and } S \cap B \subseteq U^\flat(d, k). \tag{8}$$

Our claim is now that $(D \times \omega, U) \models \nabla^B_{\mathtt{M}}(\Sigma, \Sigma^-_B)$.

The existential part of $\nabla^B_{\mathtt{M}}(\Sigma, \Sigma^-_B)$ is trivially true. To cover the universal part, it remains to show that every element of $(D \times \omega, U)$ realizes a B-positive type in Σ^-_B. Take an arbitrary pair $(d, k) \in D \times \omega$ and let T be the (full) type of (d, k), that is, let $T := U^\flat(d, k)$. If $B \cap T = \varnothing$ then trivially $T \in \Sigma^-_B$ and we are done. So suppose $B \cap T \neq \varnothing$. Observe that in $D \times \omega$ we have infinitely many copies of $d \in D$. Hence, as $U(b)$ is finite for every $b \in B$, there must be some (d, k') with type $U^\flat(d, k') = V^\flat_\pi(d, k') \setminus B = V^\flat_\pi(d, k) \setminus B = T \setminus B$. It follows from $(D \times \omega, U) \models \nabla^B_{\mathtt{M}}(\Sigma)$ and (8) that there is some $S \in \Sigma$ such that $S \setminus B = U^\flat(d, k') \setminus B = U^\flat(d, k')$ and $S \cap B \subseteq U^\flat(d, k) \cap B = \varnothing$. From this we easily derive that $S = U^\flat(d, k')$ and $S \in \Sigma^-_B$. Finally, we observe that $S \setminus B = U^\flat(d, k') \setminus B = U^\flat(d, k) \setminus B$ and $S \cap B = \varnothing \subseteq U^\flat(d, k)$, so that by (8) we find that $D \times \omega, U) \models \tau^B_S(d, k)$ indeed.

Finally, by monotonicity it directly follows from $(D \times \omega, U) \models \nabla^B_{\mathtt{M}}(\Sigma, \Sigma^-_B)$ that $(D \times \omega, V_\pi) \models \nabla^B_{\mathtt{M}}(\Sigma, \Sigma^-_B)$, and from this it is immediate that $(D \times \omega, V_\pi) \models \varphi^{\ominus}$.

$\boxed{\Leftarrow}$ Let $(D \times \omega, V_\pi) \models \nabla^B_{\mathtt{M}}(\Sigma, \Sigma^-_B)$. To show that $(D \times \omega, V_\pi) \models \nabla^B_{\mathtt{M}}(\Sigma)$, the existential part is trivial. For the universal part just observe that $\Sigma^-_B \subseteq \Sigma$. QED

A careful analysis of the translation gives us the following corollary, providing normal forms for the continuous fragment of $\mathtt{M}$.

Corollary 5.10 *For any sentence* $\varphi \in \mathtt{M}(A)$*, the following hold.*

1. *The formula φ is continuous in $B \subseteq A$ iff it is equivalent to a formula $\bigvee \nabla^B_{\mathtt{M}}(\Sigma, \Sigma^-_B)$ for some types $\Sigma \subseteq \wp(A)$, where $\Sigma^-_B := \{S \in \Sigma \mid B \cap S = \varnothing\}$.*

2. *If φ is monotone in A then φ is continuous in $B \subseteq A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla^+_{\mathtt{M}}(\Sigma, \Sigma^-_B)$ for some types $\Sigma \subseteq \wp(A)$, where $\Sigma^-_B := \{S \in \Sigma \mid B \cap S = \varnothing\}$.*

5.2 Continuous fragment of $\mathtt{ME}^\infty$

As for the previous case, the $\mathtt{ME}^\infty$-variant of Proposition 5.8 is an immediate consequence of Theorem 4.4 and the following proposition.

Proposition 5.11 *There is an effective translation $(-)^\ominus : \mathtt{Pos}_B(\mathtt{ME}^\infty(A)) \to \mathtt{Con}_B(\mathtt{ME}^\infty(A))$ such that a sentence $\varphi \in \mathtt{Pos}_B(\mathtt{ME}^\infty(A))$ is continuous in B if and only if $\varphi \equiv \varphi^\ominus$.*

Proof. By Corollary 4.16, we assume that φ is in basic normal form, i.e., $\varphi = \bigvee \nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. For the translation let $\big(\bigvee \nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)\big)^\ominus := \bigvee \nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)^\ominus$ where

$$\nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)^\ominus := \begin{cases} \bot & \text{if } B \cap \bigcup \Sigma \neq \varnothing \\ \nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) & \text{otherwise.} \end{cases}$$

First we prove the right-to-left direction of the proposition. By Proposition 5.7 it is enough to show that $\varphi^\ominus \in \mathtt{Con}_B(\mathtt{ME}^\infty(A))$. We focus on the disjuncts of $\varphi^\ominus$. The interesting case is where $B \cap \bigcup \Sigma = \varnothing$. If we rearrange $\nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ somewhat and define the formulas φ', ψ as follows:

$$\exists \overline{\mathbf{x}}.\Big(\mathrm{diff}(\overline{\mathbf{x}}) \wedge \bigwedge_i \tau^B_{T_i}(x_i) \ \wedge \forall z.(\underbrace{\neg \mathrm{diff}(\overline{\mathbf{x}}, z) \vee \bigvee_{S \in \Pi} \tau^B_S(z)}_{\varphi'(\overline{\mathbf{x}}, z)} \vee \underbrace{\bigvee_{S \in \Sigma} \tau^B_S(z)}_{\psi(z)}) \ \wedge \forall^\infty y. \underbrace{\bigvee_{S \in \Sigma} \tau^B_S(y)}_{\psi(y)}\Big)$$
$$\wedge \bigwedge_{S \in \Sigma} \exists^\infty y. \tau^B_S(y).$$

Then we find that

$$\nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) \equiv \exists \overline{\mathbf{x}}.\Big(\mathrm{diff}(\overline{\mathbf{x}}) \wedge \bigwedge_i \tau^B_{T_i}(x_i) \wedge \mathbf{W} z.(\varphi'(\overline{\mathbf{x}}, z), \psi(z))\Big) \wedge \bigwedge_{S \in \Sigma} \exists^\infty y. \tau^B_S(y),$$

which belongs to the required fragment because $B \cap \bigcup \Sigma = \varnothing$.

For the left-to-right direction of the proposition we have to prove that $\varphi \equiv \varphi^\ominus$.

$\boxed{\Rightarrow}$ Let $(D, V) \models \varphi$. Because φ is continuous in B we may assume that $V(b)$ is finite, for all $b \in B$. Let $\nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ be a disjunct of φ such that $(D, V) \models \nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. If $D = \varnothing$, then $\overline{\mathbf{T}} = \Pi = \Sigma = \varnothing$, and $\nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) = (\nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma))^\ominus$. Hence, let $D \neq \varnothing$. Suppose for contradiction that $B \cap \bigcup \Sigma \neq \varnothing$, then there must be some $S \in \Sigma$ with $B \cap S \neq \varnothing$. Because $(D, V) \models \nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ we have, in particular, that $(D, V) \models \exists^\infty y. \tau^B_S(x)$ and hence $V(b)$ must be infinite, for any $b \in B \cap S$, which is absurd. It follows that $B \cap \bigcup \Sigma = \varnothing$, but then we trivially conclude that $(D, V) \models \varphi^\ominus$ because the disjunct remains unchanged.

$\boxed{\Leftarrow}$ Let $(D, V) \models \varphi^\ominus$. The only difference between φ and $\varphi^\ominus$ is that some disjuncts may have been replaced by $\bot$. Therefore this direction is trivial. QED

We conclude the section by stating the following corollary, providing normal forms for the continuous fragment of $\mathtt{ME}^\infty$.

Corollary 5.12 *For any sentence $\varphi \in \mathtt{ME}^{\infty}(A)$, the following hold.*

1. *The formula φ is continuous in $B \subseteq A$ iff φ is equivalent to a formula, effectively obtainable from φ, which is a disjunction of formulas $\nabla^{B}_{\mathtt{ME}^{\infty}}(\overline{\mathbf{T}}, \Pi, \Sigma)$ where $\overline{\mathbf{T}}, \Sigma$ and Π are such that $\Sigma \subseteq \Pi \subseteq \overline{\mathbf{T}}$ and $B \cap \bigcup \Sigma = \varnothing$.*

2. *If φ is monotone (i.e., $\varphi \in \mathtt{ME}^{\infty +}(A)$) then φ is continuous in $B \subseteq A$ iff it is equivalent to a formula, effectively obtainable from φ, which is a disjunction of formulas $\bigvee \nabla^{+}_{\mathtt{ME}^{\infty}}(\overline{\mathbf{T}}, \Pi, \Sigma)$, where $\overline{\mathbf{T}}, \Sigma$ and Π are such that $\Sigma \subseteq \Pi \subseteq \overline{\mathbf{T}}$ and $B \cap \bigcup \Sigma = \varnothing$.*

Proof. Notice that, from Proposition 3.16, every sentence in the basic form $\bigvee \nabla_{\mathtt{ME}^{\infty}}(\overline{\mathbf{T}}, \Pi, \Sigma)$ can be assumed such that $\Sigma \subseteq \Pi \subseteq \overline{\mathbf{T}}$. The claims hence follow by construction of the translation. QED

6 Submodels and quotients

There are various natural notions of morphism between monadic models; the one that we will be interested here is that of a (strong) homomorphism.

Definition 6.1 Let $\mathbb{D} = (D, V)$ and $\mathbb{D}' = (D', V')$ be two monadic models. A map $f : D \to D'$ is a *homomorphism* from $\mathbb{D}$ to $\mathbb{D}'$, notation: $f : \mathbb{D} \to \mathbb{D}'$, if we have $d \in V(a)$ iff $f(d) \in V'(a)$, for all $a \in A$ and $d \in D$. $\triangleleft$

In this section we will be interested in the sentences of $\mathtt{M}, \mathtt{ME}$ and $\mathtt{ME}^{\infty}$ that are preserved under taking submodels and the ones that are invariant under quotients.

Definition 6.2 Let $\mathbb{D} = (D, V)$ and $\mathbb{D}' = (D', V')$ be two monadic models. We call $\mathbb{D}$ a *submodel* of $\mathbb{D}'$ if $D \subseteq D'$ and the inclusion map $\iota_{DD'} : D \hookrightarrow D'$ is a homomorphism, and we say that $\mathbb{D}'$ is a *quotient* of $\mathbb{D}$ if there is a surjective homomorphism $f : \mathbb{D} \to \mathbb{D}'$.

Now let φ be an $\mathtt{L}$-sentence, where $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^{\infty}\}$. We say that φ is *preserved under taking submodels* if $\mathbb{D} \models \varphi$ implies $\mathbb{D}' \models \varphi$, whenever $\mathbb{D}'$ is a submodel of $\mathbb{D}$. Similarly, φ is *invariant under taking quotients* if we have $\mathbb{D} \models \varphi$ iff $\mathbb{D}' \models \varphi$, whenever $\mathbb{D}'$ is a quotient of $\mathbb{D}$. $\triangleleft$

The first of these properties (preservation under taking submodels) is well known from classical model theory — it is for instance the topic of the Łos-Tarski Theorem. When it comes to quotients, in model theory one is usually more interested in the formulas that are *preserved* under surjective homomorphisms (and the definition of homomorphism may also differ from ours): for instance, this is the property that is characterised by Lyndon's Theorem. Our preference for the notion of *invariance* under quotients stems from the fact that the property of invariance under quotients plays a key role in characterising the *bisimulation-invariant fragments* of various monadic second-order logics, as is explained in our companion paper [7].

6.1 Preservation under submodels

In this subsection we characterise the fragments of our predicate logics consisting of the sentences that are preserved under taking submodels. That is, the main result of this subsection is a Łos-Tarksi Theorem for $\mathtt{ME}^{\infty}$.

Definition 6.3 The *universal fragment* of the set $\mathtt{ME}^{\infty}(A)$ is the collection $\mathtt{Univ}(\mathtt{ME}^{\infty}(A))$ of formulas given by the following grammar:

$$\varphi ::= \top \mid \bot \mid a(x) \mid \neg a(x) \mid x \approx y \mid x \not\approx y \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \forall x.\varphi \mid \forall^{\infty} x.\varphi$$

where $x, y \in \mathsf{iVar}$ and $a \in A$. The universal fragment $\mathtt{Univ}(\mathtt{ME}(A))$ is obtained by deleting the clause for $\forall^\infty$ from this grammar, and we obtain the universal fragment $\mathtt{Univ}(\mathtt{M}(A))$ by further deleting both clauses involving the equality symbol. $\lhd$

Theorem 6.4 *Let φ be a sentence of the monadic logic $\mathtt{L}(A)$, where $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^\infty\}$. Then φ is preserved under taking submodels if and only if there is a equivalent formula $\varphi^\otimes \in \mathtt{Univ}(\mathtt{L}(A))$. Furthermore, it is decidable whether a sentence $\varphi \in \mathtt{L}(A)$ has this property or not.*

We start by verifying that universal formulas satisfy the property.

Proposition 6.5 *Let $\varphi \in \mathtt{Univ}(\mathtt{L}(A))$ be a universal sentence of the monadic logic $\mathtt{L}(A)$, where $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^\infty\}$. Then φ is preserved under taking submodels.*

Proof. It is enough to directly consider the case $\mathtt{L} = \mathtt{ME}^\infty$. Let (D', V') be a submodel of the monadic model (D, V). The case for $D = \varnothing$ being immediate, let us assume $D \neq \varnothing$. By induction on the complexity of a formula $\varphi \in \mathtt{Univ}(\mathtt{ME}^\infty(A))$ we will show that for any assignment $g : \mathsf{iVar} \to D'$ we have

$$(D, V), g' \models \varphi \text{ implies } (D', V'), g \models \varphi,$$

where $g' := g \circ \iota_{D'D}$. We will only consider the inductive step of the proof where φ is of the form $\forall^\infty x.\psi$. Define $X_{D,V} := \{d \in D \mid (D, V), g'[x \mapsto d] \models \psi\}$, and similarly, $X_{D',V'} := \{d \in D' \mid (D', V'), g[x \mapsto d] \models \psi\}$. By the inductive hypothesis we have that $X_{D,V} \cap D' \subseteq X_{D',V'}$, implying that $D' \setminus X_{D',V'} \subseteq D \setminus X_{D,V}$. But from this we immediately obtain that

$$|D \setminus X_{D,V}| < \omega \text{ implies } |D' \setminus X_{D',V'}| < \omega,$$

which means that $(D, V), g' \models \varphi$ implies $(D', V'), g \models \varphi$, as required. QED

Before verifying the 'hard' side of the theorem, we define the appropriate translations from each monadic logic into its universal fragment.

Definition 6.6 We start by defining the translations for sentences in basic normal forms.

For $\mathtt{M}$-sentences in basic form we first set

$$\Big(\nabla_{\mathtt{M}}(\Sigma)\Big)^\otimes := \forall z \bigvee_{S \in \Sigma} \tau_S(z)$$

Second, we define $(\bigvee_i \alpha_i)^\otimes := \bigvee \alpha_i^\otimes$. Finally, we extend the translation $(-)^\otimes$ to the collection of all $\mathtt{M}$-sentences by defining $\varphi^\circ := (\varphi^*)^\otimes$, where φ^* is the basic normal form of φ as given by Fact 3.3.

Similarly, for $\mathtt{ME}$-sentences we first define

$$\Big(\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)\Big)^\otimes := \forall z \bigvee_{S \in \overline{\mathbf{T}} \cup \Pi} \tau_S(z)$$

and then we extend it to the full language by distributing over disjunction and applying Theorem 3.9 to convert an arbitrary $\mathtt{ME}$ sentence into an equivalent sentence in basic normal form.

Finally, for simple basic formulas of $\mathtt{ME}^\infty$, the translation $(-)^\otimes$ is given as follows:

$$(\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma))^\otimes := \forall z \bigvee_{S \in \overline{\mathbf{T}} \cup \Pi \cup \Sigma} \tau_S(z) \wedge \forall^\infty z \bigvee_{S \in \Sigma} \tau_S(z).$$

The definition is thus extended to the full language $\mathtt{ME}^\infty$ as expected: given a $\mathtt{ME}^\infty$-sentence φ, by Theorem 3.15 and Proposition 3.16 we compute an equivalent basic form $\bigvee \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, with $\Sigma \subseteq \Pi \subseteq \overline{\mathbf{T}}$, and therefore we set $\varphi^\otimes := \bigvee (\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma))^\otimes$. $\lhd$

The missing parts in the proof of the theorem is thence covered by the following result.

Proposition 6.7 *For any monadic logic* $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^\infty\}$ *there is an effective translation* $(-)^\otimes : \mathtt{L}(A) \to \mathtt{Univ}(\mathtt{L}(A))$ *such that a sentence* $\varphi \in \mathtt{L}(A)$ *is preserved under taking submodels if and only if* $\varphi \equiv \varphi^\otimes$.

Proof. We only consider the case where $\mathtt{L} = \mathtt{ME}^\infty$, leaving the other cases to the reader.

It is easy to see that $\varphi^\otimes \in \mathtt{Univ}(\mathtt{ME}^\infty(A))$, for every sentence $\varphi \in \mathtt{ME}^\infty(A)$; but then it is immediate by Proposition 6.5 that φ is preserved under taking submodels if $\varphi \equiv \varphi^\otimes$.

For the left-to-right direction, assume that φ is preserved under taking submodels. It is easy to see that φ implies $\varphi^\otimes$, so we focus on proving the opposite. That is, we suppose that $(D, V) \models \varphi^\otimes$, and aim to show that $(D, V) \models \varphi$.

By Theorem 3.15 and Proposition 3.16 we may assume without loss of generality that φ is a disjunction of setences of the form $\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, where $\Sigma \subseteq \Pi \subseteq \overline{\mathbf{T}}$. It follows that (D, V) satisfies some disjunct $\forall z \bigvee_{S \in \overline{\mathbf{T}} \cup \Pi \cup \Sigma} \tau_S(z) \wedge \forall^\infty z \bigvee_{S \in \Sigma} \tau_S(z)$ of $\Big(\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)\Big)^\otimes$. Expand D with finitely many elements $\overline{\mathbf{d}}$, in one-one correspondence with $\overline{\mathbf{T}}$, and ensure that the type of each d_i is T_i. In addition, add, for each $S \in \Sigma$, infinitely many elements $\{e^S_n \mid n \in \omega\}$, each of type S. Call the resulting monadic model $\mathbb{D}' = (D', V')$.

This construction is tailored to ensure that $(D', V') \models \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, and so we obtain $(D', V') \models \varphi$. But obviously, $\mathbb{D}$ is a submodel of $\mathbb{D}'$, whence $(D, V) \models \varphi$ by our assumption on φ. QED

Proof of Theorem 6.4. The first part of the theorem is an immediate consequence of Proposition 6.7. By applying Fact 2.6 to Proposition 6.7 we finally obtain that for the three concerned formalisms the problem of deciding whether a sentence is preserved under taking submodels is decidable. QED

As an immediate consequence of the proof of the previous Proposition 6.7, we get effective normal forms for the universal fragments.

Corollary 6.8 *The following hold:*

1. *A sentence* $\varphi \in \mathtt{ME}(A)$ *is preserved under taking submodels iff it is equivalent to a formula* $\bigvee \big(\forall z \bigvee_{S \in \Sigma} \tau_S(z)\big)$, *for types* $\Sigma \subseteq \wp(A)$.
2. *A sentence* $\varphi \in \mathtt{ME}(A)$ *is preserved under taking submodels iff it is equivalent to a formula* $\bigvee \big(\forall z \bigvee_{S \in \overline{\mathbf{T}} \cup \Pi} \tau_S(z)\big)$, *for types* $\Pi \subseteq \wp(A)$ *and* $\overline{\mathbf{T}} \in \wp(A)^k$ *for some* k.
3. *A sentence* $\varphi \in \mathtt{ME}^\infty(A)$ *is preserved under taking submodels iff it is equivalent to a formula* $\bigvee \big(\forall z \bigvee_{S \in \overline{\mathbf{T}} \cup \Pi \cup \Sigma} \tau_S(z) \wedge \forall^\infty z \bigvee_{S \in \Sigma} \tau_S(z)\big)$ *for types* $\Sigma \subseteq \Pi \subseteq \wp(A)$ *and* $\overline{\mathbf{T}} \in \wp(A)^k$ *for some* k.

In all three cases, normal forms are effective.

6.2 Invariance under quotients

The following theorem states that monadic first-order logic *without* equality ($\mathtt{M}$) provides the quotient-invariant fragment of both monadic first-order logic with equality ($\mathtt{ME}$), and of infinite-monadic predicate logic ($\mathtt{ME}^\infty$).

Theorem 6.9 *Let* φ *be a sentence of the monadic logic* $\mathtt{L}(A)$, *where* $\mathtt{L} \in \{\mathtt{ME}, \mathtt{ME}^\infty\}$. *Then* φ *is invariant under taking quotients if and only if there is an equivalent sentence in* $\mathtt{M}$. *Furthermore, it is decidable whether a sentence* $\varphi \in \mathtt{L}(A)$ *has this property or not.*

We first state the 'easy' part of the first claim of the theorem. Note that in fact, we have already been using this observation in earlier parts of the paper.

Proposition 6.10 *Every sentence in* $\mathtt{M}$ *is invariant under taking quotients.*

Proof. Let $f : D \to D'$ provide a surjective homomorphism between the models (D, V) and (D', V'), and observe that for any assignment $g : \mathsf{iVar} \to D$ on D, the composition $f \circ g : \mathsf{iVar} \to D'$ is an assignment on D'.

In order to prove the proposition one may show that, for an arbitrary $\mathtt{M}$-formula φ and an arbitrary assignment $g : \mathsf{iVar} \to D$, we have

$$(D, V), g \models \varphi \text{ iff } (D', V'), f \circ g \models \varphi. \tag{9}$$

We leave the proof of (9), which proceeds by a straightforward induction on the complexity of φ, as an exercise to the reader. QED

To prove the remaining part of Theorem 6.9, we start with providing translations from respectively $\mathtt{ME}$ and $\mathtt{ME}^{\infty}$ to $\mathtt{M}$.

Definition 6.11 For $\mathtt{ME}$-sentences in basic form we first define

$$\left(\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)\right)^{\circ} := \bigwedge_{i} \exists x_i.\tau_{T_i}(x_i) \wedge \forall x. \bigvee_{S \in \Pi} \tau_S(x),$$

whereas for $\mathtt{ME}^{\infty}$-sentences in basic form we start with defining

$$\left(\nabla_{\mathtt{ME}^{\infty}}(\overline{\mathbf{T}}, \Pi, \Sigma)\right)^{\bullet} := \bigwedge_{i} \exists x_i.\tau_{T_i}(x_i) \wedge \forall x. \bigvee_{S \in \Sigma} \tau_S(x).$$

In both cases, the translations is then extended to the full language as in Definition 6.6. $\lhd$

Note that the two translations may give *different* translations for $\mathtt{ME}$-sentences. Also observe that the Π 'disappears' in the translation of the formula $\nabla_{\mathtt{ME}^{\infty}}(\overline{\mathbf{T}}, \Pi, \Sigma)$.

The key property of these translations is the following.

Proposition 6.12 *1. For every one-step model* (D, V) *and every* $\varphi \in \mathtt{ME}(A)$ *we have*

$$(D, V) \models \varphi^{\circ} \textit{ iff } (D \times \omega, V_{\pi}) \models \varphi. \tag{10}$$

2. For every one-step model (D, V) *and every* $\varphi \in \mathtt{ME}^{\infty}(A)$ *we have*

$$(D, V) \models \varphi^{\bullet} \textit{ iff } (D \times \omega, V_{\pi}) \models \varphi. \tag{11}$$

Here V_{π} *is the induced valuation given by* $V_{\pi}(a) := \{(d, k) \mid d \in V(a), k \in \omega\}$.

Proof. We only prove the claim for $\mathtt{ME}^{\infty}$ (i.e., the second part of the proposition), the case for $\mathtt{ME}$ being similar. Clearly it suffices to prove (11) for formulas of the form $\alpha = \nabla_{\mathtt{ME}^{\infty}}(\overline{\mathbf{T}}, \Pi, \Sigma)$.

First of all, if $\mathbb{D}$ is the empty model, we find $\overline{\mathbf{T}} = \Pi = \Sigma = \varnothing$, $(D, V) = (D \times \omega, V_{\pi})$, and $\nabla_{\mathtt{ME}^{\infty}}(\overline{\mathbf{T}}, \Pi, \Sigma) = (\nabla_{\mathtt{ME}^{\infty}}(\overline{\mathbf{T}}, \Pi, \Sigma))^{\bullet}$. In other words, in this case there is nothing to prove.

In the sequel we assume that $D \neq \varnothing$.

$\boxed{\Rightarrow}$ Assume $(D, V) \models \varphi^{\bullet}$, we will show that $(D \times \omega, V_{\pi}) \models \nabla_{\mathtt{ME}^{\infty}}(\overline{\mathbf{T}}, \Pi, \Sigma)$. Let d_i be such that $V^{\flat}(d_i) = T_i$ in (D, V). It is clear that the (d_i, i) provide *distinct* elements, with each (d_i, i) satisfying τ_{T_i} in $(D \times \omega, V_{\pi})$ and therefore the first-order existential part of α is satisfied. With a similar argument it is straightforward to verify that the $\exists^\infty$-part of α is also satisfied — here we critically use the observation that $\Sigma \subseteq \overline{\mathbf{T}}$, so that every type in Σ is witnessed in the model (D, V), and hence witnessed infinitely many times in $(D \times \omega, V_\pi)$.

For the universal parts of $\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ it is enough to observe that, because of the universal part of $\alpha^\bullet$, *every* $d \in D$ realizes a type in Σ. By construction, the same applies to $(D \times \omega, V_\pi)$, therefore this takes care of both universal quantifiers.

$\boxed{\Leftarrow}$ Assuming that $(D \times \omega, V_\pi) \models \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, we will show that $(D, V) \models \varphi^\bullet$. The existential part of $\alpha^\bullet$ is trivial. For the universal part we have to show that every element of D realizes a type in Σ. Suppose not, and let $d \in D$ be such that $\neg\tau_S(d)$ for all $S \in \Sigma$. Then we have $(D \times \omega, V_\pi) \not\models \tau_S(d, k)$ for all k. That is, there are infinitely many elements not realising any type in Σ. Hence we have $(D \times \omega, V_\pi) \not\models \forall^\infty y. \bigvee_{S \in \Sigma} \tau_S(y)$. Absurd, because this formula is a conjunct of $\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. QED

We will now show how the theorem follows from this. First of all we verify that in both cases $\mathtt{M}$ is expressively complete for the property of being invariant under taking quotients.

Proposition 6.13 *For any monadic logic* $\mathtt{L} \in \{\mathtt{ME}, \mathtt{ME}^\infty\}$ *there is an effective translation* $(-)^\circledast : \mathtt{L}(A) \to \mathtt{M}$ *such that a sentence* $\varphi \in \mathtt{L}(A)$ *is invariant under taking quotients if and only if* $\varphi \equiv \varphi^\circledast$.

Proof. Let φ be a sentence of $\mathtt{ME}^\infty$, and let $\varphi^\circledast := \varphi^\bullet$ (we only cover the case of $\mathtt{L} = \mathtt{ME}^\infty$, the case for $\mathtt{L} = \mathtt{ME}$ is similar, just take $\varphi^\circledast := \varphi^\circ$) We will show that

$$\varphi \equiv \varphi^\circledast \text{ iff } \varphi \text{ is invariant under taking quotients.} \tag{12}$$

The direction from right to left is immediate by Proposition 6.10. For the other direction it suffices to observe that any model (D, V) is a quotient of its 'ω-product' $(D \times \omega, V_\pi)$, and to reason as follows:

$$\begin{aligned} (D, V) \models \varphi &\text{ iff } (D \times \omega, V_\pi) \models \varphi && \text{(assumption on } \varphi) \\ &\text{ iff } (D, V) \models \varphi^\bullet && \text{(Proposition 6.12)} \end{aligned}$$

QED

Hence we can conclude.

Proof of Theorem 6.9. The theorem is an immediate consequence of Proposition 6.13. Finally, the effectiveness of translation $(\cdot)^\bullet$, decidability of $\mathtt{ME}^\infty$ (Fact 2.6) and (12) yield that it is decidable whether a given $\mathtt{ME}^\infty$-sentence φ is invariant under taking quotients or not. QED

As a corollary, we therefore obtain:

Corollary 6.14 *Let* φ *be a sentence of the monadic logic* $\mathtt{L}(A)$, *where* $\mathtt{L} \in \{\mathtt{ME}, \mathtt{ME}^\infty\}$. *Then* φ *is invariant under taking quotients if and only if there is an equivalent sentence* $\bigvee\big(\bigwedge_{S\in\Sigma} \exists x.\tau_S(x) \wedge \forall x. \bigvee_{S\in\Sigma} \tau_S(x)\big)$, *for types* $\Sigma \subseteq \wp(A)$. *Moreover, such a normal form is effective.*

In our companion paper [7] on automata, we need versions of these results for the monotone and the continuous fragment. For this purpose we define some slight modifications of the translations $(\cdot)^\circ$ and $(\cdot)^\bullet$ which map positive and syntactically continuous sentences to respectively positive and syntactically continuous formulas.

Theorem 6.15 *There are effective translations* $(\cdot)^\circ : \mathtt{ME}^+ \to \mathtt{M}^+$ *and* $(\cdot)^\bullet : \mathtt{ME}^{\infty+} \to \mathtt{M}^+$ *such that* $\varphi \equiv \varphi^\circ$ *(respectively,* $\varphi \equiv \varphi^\bullet$*) iff* φ *is invariant under quotients. Moreover, we may assume that* $(\cdot)^\bullet : \mathtt{Con}_B(\mathtt{ME}^\infty(A)) \cap \mathtt{ME}^{\infty+} \to \mathtt{Con}_B(\mathtt{M}(A)) \cap \mathtt{M}^+$, *for any* $B \subseteq A$.

Proof.

We define translations $(\cdot)^\circ : \mathtt{ME}^+ \to \mathtt{M}^+$ and $(\cdot)^\bullet : \mathtt{ME}^{\infty+} \to \mathtt{M}^+$ as follows. For $\mathtt{ME}^+, \mathtt{ME}^{\infty+}$-sentences in simple basic form we define

$$\begin{array}{lll} \left(\nabla^+_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)\right)^\circ & := & \bigwedge_i \exists x_i.\tau^+_{T_i}(x_i) \wedge \forall x. \bigvee_{S\in\Pi} \tau^+_S(x), \\ \left(\nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)\right)^\bullet & := & \bigwedge_i \exists x_i.\tau^+_{T_i}(x_i) \wedge \forall x. \bigvee_{S\in\Sigma} \tau^+_S(x), \end{array}$$

and then we use, respectively, the Corollaries 4.13 and 4.16 to extend these translations to the full positive fragments $\mathtt{ME}^+$ and $\mathtt{ME}^{\infty+}$, as we did in Definition 6.11 for the full language.

We leave it as an exercise for the reader to prove the analogue of Proposition 6.12 for these translations, and to show how the first statements of the theorem follows from this.

Finally, to see why we may assume that $(\cdot)^\bullet$ restricts to a map from the syntactically B-continuous fragment of $\mathtt{ME}^{\infty+}(A)$ to the syntactically B-continuous fragment of $\mathtt{M}^+(A)$, assume that $\varphi \in \mathtt{ME}^\infty(A)$ is continuous in $B \subseteq A$. By Corollary 5.12 we may assume that φ is a disjunction of formulas of the form $\nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, where $B \cap \bigcup\Sigma = \varnothing$. This implies that in the formula $\varphi^\bullet$ no predicate symbol $b \in B$ occurs in the scope of a universal quantifier, and so $\varphi^\bullet$ is syntactically continuous in B indeed. QED

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