Monadic Predicate Logic with the infinity quantifier: some model theory

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Abstract

We study some model theory of a predicate logic $\mathtt{ME}^{\infty}$ that allows only monadic predicate symbols and no function symbols, but that goes beyond standard first-order logic with equality in that it features the generalized quantifier 'there are infinitely many'.

Extending known results on monadic first-order logic, as our first contribution we provide normal forms for the sentences of $\mathtt{ME}^{\infty}$. We then use these normal forms to provide syntactic characterizations for a number of semantic properties pertaining to sentences of the language. The properties include that of being monotone, (Scott) continuous and completely additive in a given set of monadic predicates; we also consider the sentences of which the truth is preserved under taking submodels or invariant under taking quotients.

In each case, our proof method is based on an effectively defined map that translates an arbitrary sentence of the language to a sentence belonging to the characterizing syntactic fragment. The key observation on these translations is that in each case, the original sentence has the semantic property under discussion if and only if is equivalent to its translation.

Since the logic $\mathtt{ME}^{\infty}$ is known to have a decidable satisfiability problem, as a corollary we obtain a number of decidability results. In particular, we show that it is decidable whether the truth of an $\mathtt{ME}^{\infty}$-sentence is preserved under submodels or invariant under quotients, and whether such a sentence is monotone, continuous or completely additive in a fixed set of predicate symbols.

While we believe these observations to be of interest in their own right, they are of crucial importance in a companion paper, where results on the logic $\mathtt{ME}^{\infty}$ and its first-order fragments are used to obtain automata-theoretic characterizations for and bisimulation-invariance theorems on variants of monadic second-order logic.

STILL TO DO: MANY THINGS, BUT DON'T FORGET THE FOLLOWING:

- `check & refer to Jouko's book on Models and Games`
- `add results/proofs about decidability`
- $\bot, \top$ `needed?`
- `improve style, in particular avoid repetitive sentences`
- `uniform definition of translations`
- `historical notes (normal forms, model-theoretic results, etc.)`
- `lemma vs proposition`

1 Introduction

- `Monadic predicate logic is very nice`

2 Basics

In this section we provide the basic definitions of the monadic predicate liftings that we study in this paper. Throughout this paper we fix a finite set A of objects that we shall refer to as *(monadic) predicate symbols* or *names*. We shall also assume an infinite set iVar of *individual variables*.

Definition 2.1 Given a finite set A we define a *(monadic) model* to be a tuple $\mathbb{D} = (D, V)$ consisting of a set D, which we call the *domain* of $\mathbb{D}$, and an interpretation or *valuation* $V : A \to \wp D$. The class of all models will be denoted by $\mathfrak{M}$. $\lhd$

Remark 2.2 Observe that a valuation $V : A \to \wp(D)$ can equivalently be presented via its associated *coloring* $V^\flat : D \to \wp(A)$ given by

$$V^\flat(d) := \{a \in A \mid d \in V(a)\}.$$

We will use these perspectives interchangeably, calling the set $V^\flat(d) \subseteq A$ the *color* or *type* of d. $\lhd$

In this paper we study three languages of monadic predicate logic: the languages $\mathtt{ME}$ and $\mathtt{M}$ of monadic first-order logic with and without equality, respectively, and the extension $\mathtt{ME}^\infty$ of the first language with the generalized quantifiers $\exists^\infty$ abd $\forall^\infty$. Probably the most concise definition of the full language of monadic predicate logic would be given by the following grammar:

$$\varphi ::= a(x) \mid x \approx y \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x.\varphi \mid \exists^\infty x.\varphi,$$

where $a \in A$ and x and y belong to the set iVar of individual variables. In this set-up we would need to introduce the quantifiers $\forall$ and $\forall^\infty$ as abbreviations of $\neg\exists\neg$ and $\neg\exists^\infty\neg$, respectively. However, for our purposes it will be more convenient to work with a variant of this language where all formulas are in negation normal form; that is, we only permit the occurrence of the negation symbol $\neg$ in front of an atomic formula. In addition, for technical reasons we will add $\bot$ and $\top$ as constants, and we will write $\neg x \approx y$ as $x \not\approx y$. Thus we arrive at the • following definition of our syntax.

Definition 2.3 The set $\mathtt{ME}^\infty(A)$ of *monadic formulas* is given by the following grammar:

$$\varphi ::= \top \mid \bot \mid a(x) \mid \neg a(x) \mid x \approx y \mid x \not\approx y \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x.\varphi \mid \forall x.\varphi \mid \exists^\infty x.\varphi \mid \forall^\infty x.\varphi$$

where $a \in A$ and $x, y \in \mathsf{iVar}$. The language $\mathtt{ME}(A)$ of *first-order logic with equality* is defined as the fragment of $\mathtt{ME}^\infty(A)$ where occurrences of the generalized quantifiers $\exists^\infty$ and $\forall^\infty$ are not allowed:

$$\varphi ::= \top \mid \bot \mid a(x) \mid \neg a(x) \mid x \approx y \mid x \not\approx y \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x.\varphi \mid \forall x.\varphi$$

Finally, the language $\mathtt{M}(A)$ of *first-order logic* is the equality-free fragment of $\mathtt{ME}(A)$; that is, atomic formulas of the form $x \approx y$ and $x \not\approx y$ are not permitted either:

$$\varphi ::= \top \mid \bot \mid a(x) \mid \neg a(x) \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x.\varphi \mid \forall x.\varphi$$

In all three languages we use the standard definition of free and bound variables, and we call a formula a *sentence* if it has no free variables. In the sequel we will often use the symbol $\mathtt{L}$ to denote either of the languages $\mathtt{M}$, $\mathtt{ME}$ or $\mathtt{ME}^{\infty}$.

For each of the languages $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^{\infty}\}$, we define the *positive fragment* $\mathtt{L}^{+}$ of $\mathtt{L}$ as the language obtained by almost the same grammar as for $\mathtt{L}$, but with the difference that we do not allow negative formulas of the form $\neg a(x)$. $\lhd$

The semantics of these languages is given as follows.

Definition 2.4 Let $\varphi \in \mathtt{ME}^{\infty}(A)$ be a formula, let $\mathbb{D} = (D, V)$ be a model, and let $g : \mathsf{iVar} \to \wp(D)$ be an assignment. We define the *truth* relation $\models$ by a straightforward induction on the complexity of $\mathtt{ME}^{\infty}$-formulas; we explicitly provide the clauses of the quantifiers:

$$
\begin{array}{lll}
\mathbb{D}, g \models \exists x.\varphi & \text{iff} & \mathbb{D}, g[x \mapsto d] \models \varphi \text{ for some } d \in D, \\
\mathbb{D}, g \models \forall x.\varphi & \text{iff} & \mathbb{D}, g[x \mapsto d] \models \varphi \text{ for all } d \in D, \\
\mathbb{D}, g \models \exists^{\infty} x.\varphi & \text{iff} & \mathbb{D}, g[x \mapsto d] \models \varphi \text{ for infinitely many } d \in D, \\
\mathbb{D}, g \models \forall^{\infty} x.\varphi & \text{iff} & \mathbb{D}, g[x \mapsto d] \models \varphi \text{ for all but at most finitely many } d \in D.
\end{array}
$$

The clauses for the atomic formulas and for the Boolean connectives are standard. $\lhd$

In the remainder of the section we fix some further definitions and notations, starting with some useful syntactic abbreviations.

Definition 2.5 Given a list $\overline{\mathbf{y}} = y_1 \cdots y_n$ of individual variables, we use the formula

$$
\mathrm{diff}(\overline{\mathbf{y}}) := \bigwedge_{1 \leq m < m' \leq n} (y_m \not\approx y_{m'})
$$

to state that the elements $\overline{\mathbf{y}}$ are all distinct. An *A-type* is a formula of the form

$$
\tau_S(x) := \bigwedge_{a \in S} a(x) \wedge \bigwedge_{a \in A \setminus S} \neg a(x).
$$

where $S \subseteq A$. Here and elsewhere we use the convention that $\bigwedge \varnothing = \top$ (and $\bigvee \varnothing = \bot$). The *positive* A-type $\tau^{+}_S(x)$ only bears positive information, and is defined as

$$
\tau^{+}_S(x) := \bigwedge_{a \in S} a(x).
$$

Given a one-step model $\mathbb{D} = (D, V)$ we define

$$
|S|_{\mathbb{D}} := |\{d \in D \mid \mathbb{D} \models \tau_S(d)\}|
$$

as the number of elements of $\mathbb{D}$ that realize the type τ_S. $\lhd$

We often blur the distinction between the formula $\tau_S(x)$ and the subset $S \subseteq A$, calling S an A-type as well. Note that we have $\mathbb{D} \models \tau_S(d)$ iff $V^{\flat}(d) = S$, so that we may refer to $V^{\flat}(d)$ as the *type of* $d \in D$ indeed.

Definition 2.6 The quantifier rank $\mathtt{qr}(\varphi)$ of a formula $\varphi \in \mathtt{ME}^\infty$ (hence also for $\mathtt{M}$ and $\mathtt{ME}$) is defined as follows:

$$\begin{array}{lcll} \mathtt{qr}(\varphi) & := & 0 & \text{if } \varphi \text{ is atomic,} \\ \mathtt{qr}(\neg\psi) & := & \mathtt{qr}(\psi) & \\ \mathtt{qr}(\psi_1 \heartsuit \psi_2) & := & \max\{\mathtt{qr}(\psi_1), \mathtt{qr}(\psi_2)\} & \text{where } \heartsuit \in \{\wedge, \vee\} \\ \mathtt{qr}(Qx.\psi) & := & 1 + \mathtt{qr}(\psi), & \text{where } Q \in \{\exists, \forall, \exists^\infty, \forall^\infty\} \end{array}$$

Given a monadic logic $\mathtt{L}$ we write $\mathbb{D} \equiv_k^{\mathtt{L}} \mathbb{D}'$ to indicate that the models $\mathbb{D}$ and $\mathbb{D}'$ satisfy exactly the same sentences $\varphi \in \mathtt{L}$ with $\mathtt{qr}(\varphi) \leq k$. We write $\mathbb{D} \equiv^{\mathtt{L}} \mathbb{D}'$ if $\mathbb{D} \equiv_k^{\mathtt{L}} \mathbb{D}'$ for all k. When clear from context, we may omit explicit reference to $\mathtt{L}$. $\triangleleft$

Definition 2.7 A *partial isomorphism* between two models (D, V) and (D', V') is a partial function $f : D \rightharpoonup D'$ which is injective and satisfies that $d \in V(a) \Leftrightarrow f(d) \in V'(a)$ for all $a \in A$ and $d \in \mathsf{Dom}(f)$. Given two sequences $\overline{\mathbf{d}} \in D^k$ and $\overline{\mathbf{d'}} \in {D'}^k$ we use $f : \overline{\mathbf{d}} \mapsto \overline{\mathbf{d'}}$ to denote the partial function $f : D \rightharpoonup D'$ defined as $f(d_i) := d'_i$. We will take care to avoid cases where there exist d_i, d_j such that $d_i = d_j$ but $d'_i \neq d'_j$. $\triangleleft$

In the paper [?] where Mostowski introduced general quantifiers such as $\exists^\infty$ and $\forall^\infty$, he proved the following decidability result for their monadic fragment.

Fact 2.8 *For each logic* $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^\infty\}$*, the problem whether a given* $\mathtt{L}$*-sentence* φ *is satisfiable, is decidable.*

Finally, for future reference we briefly discuss the notion of *Boolean duals*. We first give a concrete definition of a dualisation operator on the set of monadic formulas.

Definition 2.9 The *(Boolean) dual* $\varphi^\delta \in \mathtt{ME}^\infty(A)$ of $\varphi \in \mathtt{ME}^\infty(A)$ is the formula given by:

$$\begin{array}{rlrl} (a(x))^\delta & := a(x) & (\neg a(x))^\delta & := \neg a(x) \\ (\top)^\delta & := \bot & (\bot)^\delta & := \top \\ (x \approx y)^\delta & := x \not\approx y & (x \not\approx y)^\delta & := x \approx y \\ (\varphi \wedge \psi)^\delta & := \varphi^\delta \vee \psi^\delta & (\varphi \vee \psi)^\delta & := \varphi^\delta \wedge \psi^\delta \\ (\exists x.\psi)^\delta & := \forall x.\psi^\delta & (\forall x.\psi)^\delta & := \exists x.\psi^\delta \\ (\exists^\infty x.\psi)^\delta & := \forall^\infty x.\psi^\delta & (\forall^\infty x.\psi)^\delta & := \exists^\infty x.\psi^\delta \end{array}$$

$\triangleleft$

Remark 2.10 Where $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^\infty\}$, observe that if $\varphi \in \mathtt{L}(A)$ then $\varphi^\delta \in \mathtt{L}(A)$. Moreover, the operator preserves positivity of the predicates, that is, if $\varphi \in \mathtt{L}^+(A)$ then $\varphi^\delta \in \mathtt{L}^+(A)$. $\triangleleft$

The following proposition states that the formulas φ and φ^δ are Boolean duals. We omit its proof, which is a routine check.

Proposition 2.11 *Let* $\varphi \in \mathtt{ME}^\infty(A)$ *be a monadic formula. Then* φ *and* φ^δ *are Boolean duals, in the sense that for every monadic model* (D, V) *we have that*

$$(D, V) \models \varphi \textit{ iff } (D, V^c) \not\models \varphi^\delta,$$

where $V^c : A \to \wp D$ *is the valuation given by* $V^c(a) := D \setminus V(a)$*.*

3 Normal forms

In this section we provide normal forms for the logics $\mathtt{M}$, $\mathtt{ME}$ and $\mathtt{ME}^\infty$. These normal forms will be pivotal for characterizing the different fragments of these logics in later sections.

3.1 Normal form for $\mathtt{M}$

We start by introducing a normal form for monadic first-order logic without equality.

Definition 3.1 Given sets of types $\Sigma, \Pi \subseteq \wp A$, we define the following formulas:

$$\begin{array}{lll} \nabla_{\mathtt{M}}(\Sigma,\Pi) & := & \bigwedge_{S\in\Sigma} \exists x.\tau_S(x) \wedge \forall x. \bigvee_{S\in\Pi} \tau_S(x) \\ \nabla_{\mathtt{M}}(\Sigma) & := & \nabla_{\mathtt{M}}(\Sigma,\Sigma) \end{array}$$

A sentence of $\mathtt{M}(A)$ is in *basic form* if it is a disjunction of formulas of the form $\nabla_{\mathtt{M}}(\Sigma)$. $\lhd$

Clearly the meaning of the formula $\nabla_{\mathtt{M}}(\Sigma)$ is that Σ is a complete description of the collection of types that are realized in a monadic model.

Every $\mathtt{M}$-formula is effectively equivalent to a formula in basic form.

Fact 3.2 *There is an effective procedure that transforms an arbitrary $\mathtt{M}$-sentence φ into an equivalent formula φ^* in basic form.*

This observation is easy to prove using Ehrenfeucht-Fraïssé games — proof sketches can be found in [**?**, Lemma 16.23] and [**?**, Proposition 4.14]. We omit a full proof because it is very similar to the following more complex cases.

3.2 Normal form for $\mathtt{ME}$

When considering a normal form for $\mathtt{ME}$, the fact that we can 'count types' using equality yields a more involved basic form.

Convention 3.3 Here and in the sequel it will often be convenient to blur the distinction between lists and sets. For instance, identifying the list $\overline{\mathbf{T}} = T_1 \cdots T_n$ with the set $\{T_1, \ldots, T_n\}$, we may write statements like $S \in \overline{\mathbf{T}}$ or $\Pi \subseteq \overline{\mathbf{T}}$.

Definition 3.4 We say that a formula $\varphi \in \mathtt{ME}(A)$ is in *basic form* if $\varphi = \bigvee \nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ where each disjunct is of the form

$$\nabla_{\mathtt{ME}}(\overline{\mathbf{T}},\Pi) = \exists \overline{\mathbf{x}}.\big(\mathrm{diff}(\overline{\mathbf{x}}) \wedge \bigwedge_i \tau_{T_i}(x_i) \wedge \forall z.(\mathrm{diff}(\overline{\mathbf{x}}, z) \to \bigvee_{S\in\Pi} \tau_S(z))\big)$$

such that $\overline{\mathbf{T}} \in \wp(A)^k$ for some k and $\Pi \subseteq \overline{\mathbf{T}}$. $\lhd$

We prove that every sentence of monadic first-order logic with equality is equivalent to a formula in basic form. Although this result seems to be folklore, we provide a detailed proof because some of its ingredients will be used later, when we give a normal form for $\mathtt{ME}^\infty$. We start by defining the following relation between monadic models.

Definition 3.5 For every $k \in \mathbb{N}$ we define the relation $\sim_k^=$ on the class $\mathfrak{M}$ of monadic models by putting

$$\mathbb{D} \sim_k^= \mathbb{D}' \Longleftrightarrow \forall S \subseteq A \ \big(|S|_{\mathbb{D}} = |S|_{\mathbb{D}'} < k \text{ or } |S|_{\mathbb{D}}, |S|_{\mathbb{D}'} \geq k\big),$$

where $\mathbb{D}$ and $\mathbb{D}'$ are arbitrary monadic models. $\lhd$

Intuitively, two models are related by $\sim_k^=$ when their type information coincides 'modulo k'. Later we will prove that this is the same as saying that they cannot be distinguished by a formula of $\mathtt{ME}$ with quantifier rank lower or equal to k. For the moment, we record the following properties of $\sim_k^=$.

Proposition 3.6 *The following hold:*

1. *The relation $\sim_k^=$ is an equivalence relation of finite index.*
2. *Every $E \in \mathfrak{M}/{\sim_k^=}$ is characterized by a formula $\varphi_E^= \in \mathtt{ME}(A)$ with $\mathtt{qr}(\varphi_E^=) = k$.*

Proof. We only prove the second statement. Let $E \in \mathfrak{M}/{\sim_k^=}$ and let $\mathbb{D} \in E$ be a representative. Call $S_1, \ldots, S_n \subseteq A$ to the types such that $|S_i|_{\mathbb{D}} = n_i < k$ and $S'_1, \ldots, S'_m \subseteq A$ to those satisfying $|S'_i|_{\mathbb{D}} \geq k$. Now define

$$\begin{aligned}
\varphi_E^= \quad := \quad & \bigwedge_{i \leq n} \Big(\exists x_1, \ldots, x_{n_i}. \text{diff}(x_1, \ldots, x_{n_i}) \ \wedge \bigwedge_{j \leq n_i} \tau_{S_i}(x_j) \\
& \qquad\qquad \wedge \forall z. \text{diff}(x_1, \ldots, x_{n_i}, z) \to \neg \tau_{S_i}(z) \Big) \\
& \wedge \bigwedge_{i \leq m} \big(\exists x_1, \ldots, x_k. \text{diff}(x_1, \ldots, x_k) \wedge \bigwedge_{j \leq k} \tau_{S'_i}(x_j) \big)
\end{aligned}$$

First note that the union of all the S_i and S'_i yields all the possible A-types, and that if a type S_j is not realized at all, we take $n_j = 0$. It is easy to see that $\mathtt{qr}(\varphi_E^=) = k$ and that $\mathbb{D}' \models \varphi_E^=$ iff $\mathbb{D}' \in E$. Observe that $\varphi_E^=$ gives a specification of E "type by type". QED

Next we recall a (standard) notion of Ehrenfeucht-Fraïssé game for $\mathtt{ME}$ which will be used to establish the connection between $\sim_k^=$ and $\equiv_k^{\mathtt{ME}}$.

Definition 3.7 Let $\mathbb{D}_0 = (D_0, V_0)$ and $\mathbb{D}_1 = (D_1, V_1)$ be monadic models. We define the game $\text{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)$ between $\forall$ and $\exists$. If $\mathbb{D}_i$ is one of the models we use $\mathbb{D}_{-i}$ to denote the other model. A position in this game is a pair of sequences $\overline{\mathbf{s_0}} \in D_0^n$ and $\overline{\mathbf{s_1}} \in D_1^n$ with $n \leq k$. The game consists of k rounds where in round $n+1$ the following steps are made:

1. $\forall$ chooses an element d_i in one of the $\mathbb{D}_i$;
2. $\exists$ responds with an element d_{-i} in the model $\mathbb{D}_{-i}$.

In this way, the sequences $\overline{\mathbf{s_i}} \in D_i^n$ of elements chosen up to round n are extended to $\overline{\mathbf{s_i}}' := \overline{\mathbf{s_i}} \cdot d_i$. Player $\exists$ survives the round iff she does not get stuck and the function f_{n+1} :

$\overline{\mathbf{s_0}}' \mapsto \overline{\mathbf{s_1}}'$ is a partial isomorphism of monadic models. Finally, player $\exists$ wins the match iff she survives all k rounds.

Given $n \leq k$ and $\overline{\mathbf{s_i}} \in D_i^n$ such that $f_n : \overline{\mathbf{s_0}} \mapsto \overline{\mathbf{s_1}}$ is a partial isomorphism, we write $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$ to denote the (initialized) game where n moves have been played and $k - n$ moves are left to be played. $\lhd$

Lemma 3.8 *The following are equivalent:*

1. $\mathbb{D}_0 \equiv_k^{\mathtt{ME}} \mathbb{D}_1$,
2. $\mathbb{D}_0 \sim_k^= \mathbb{D}_1$,
3. *$\exists$ has a winning strategy in* $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)$.

Proof. Step (1) to (2) is direct by Proposition 3.6. For (2) to (3) we give a winning strategy for $\exists$ in $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)$ by showing the following claim.

Claim 1 Let $\mathbb{D}_0 \sim_k^= \mathbb{D}_1$ and $\overline{\mathbf{s_i}} \in D_i^n$ be such that $n < k$ and $f_n : \overline{\mathbf{s_0}} \mapsto \overline{\mathbf{s_1}}$ is a partial isomorphism; then $\exists$ can survive one more round in $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$.

Proof of Claim Let $\forall$ pick $d_i \in D_i$ such that the type of d_i is $T \subseteq A$. If d_i had already been played then $\exists$ picks the same element as before and $f_{n+1} = f_n$. If d_i is new and $|T|_{\mathbb{D}_i} \geq k$ then, as at most $n < k$ elements have been played, there is always some new $d_{-i} \in D_{-i}$ that $\exists$ can choose to match d_i. If $|T|_{\mathbb{D}_i} = m < k$ then we know that $|T|_{\mathbb{D}_{-i}} = m$. Therefore, as d_i is new and f_n is injective, there must be a $d_{-i} \in D_{-i}$ that $\exists$ can choose. ◀

Step (3) to (1) is a standard result [1, Corollary 2.2.9] which we prove anyway because we will need to extend it later. We prove the following loaded statement.

Claim 2 Let $\overline{\mathbf{s_i}} \in D_i^n$ and $\varphi(z_1, \ldots, z_n) \in \mathtt{ME}(A)$ be such that $\mathtt{qr}(\varphi) \leq k - n$. If $\exists$ has a winning strategy in the game $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$ then $\mathbb{D}_0 \models \varphi(\overline{\mathbf{s_0}})$ iff $\mathbb{D}_1 \models \varphi(\overline{\mathbf{s_1}})$.

Proof of Claim If φ is atomic the claim holds because of $f_n : \overline{\mathbf{s_0}} \mapsto \overline{\mathbf{s_1}}$ being a partial isomorphism. The Boolean cases are straightforward. Let $\varphi(z_1, \ldots, z_n) = \exists x.\psi(z_1, \ldots, z_n, x)$ and suppose $\mathbb{D}_0 \models \varphi(\overline{\mathbf{s_0}})$. Hence, there exists $d_0 \in D_0$ such that $\mathbb{D}_0 \models \psi(\overline{\mathbf{s_0}}, d_0)$. By hypothesis we know that $\exists$ has a winning strategy for $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$. Therefore, if $\forall$ picks $d_0 \in D_0$ she can respond with some $d_1 \in D_1$ and have a winning strategy for $\mathrm{EF}_k^=(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}{\cdot}d_0, \overline{\mathbf{s_1}}{\cdot}d_1)$. By induction hypothesis, because $\mathtt{qr}(\psi) \leq k - (n+1)$, we have that $\mathbb{D}_0 \models \psi(\overline{\mathbf{s_0}}, d_0)$ iff $\mathbb{D}_1 \models \psi(\overline{\mathbf{s_1}}, d_1)$ and hence $\mathbb{D}_1 \models \exists x.\psi(\overline{\mathbf{s_1}}, x)$. The opposite direction is proved by a symmetric argument. ◀

We finish the proof of the lemma by combining these two claims. QED

Theorem 3.9 *There is an effective procedure that transforms an arbitrary* ME*-sentence φ into an equivalent formula φ^* in basic form.*

Proof. Let $\mathtt{qr}(\psi) = k$ and let $[\![\psi]\!]$ be the class of models satisfying ψ. As $\mathfrak{M}/{\equiv_k^{\mathtt{ME}}}$ is the same as $\mathfrak{M}/{\sim_k^{=}}$ by Lemma 3.8, it is easy to see that ψ is equivalent to $\bigvee\{\varphi_E^{=} \mid E \in [\![\psi]\!]/{\sim_k^{=}}\}$. Now it only remains to see that each $\varphi_E^{=}$ is equivalent to the formula $\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ for some $\overline{\mathbf{T}}, \Pi \subseteq \wp A$ with $\Pi \subseteq \overline{\mathbf{T}}$.

The crucial observation is that we will use $\overline{\mathbf{T}}$ and Π to give a specification of the types "element by element". Let $\mathbb{D}$ be a representative of the equivalence class E. Call $S_1, \ldots, S_n \subseteq A$ to the types such that $|S_i|_{\mathbb{D}} = n_i < k$ and $S'_1, \ldots, S'_m \subseteq A$ to those satisfying $|S'_j|_{\mathbb{D}} \geq k$. The size of the sequence $\overline{\mathbf{T}}$ is defined to be $(\sum_{i=1}^n n_i) + k \times m$ where $\overline{\mathbf{T}}$ contains exactly n_i occurrences of type S_i and at least k occurrences of each S'_j. On the other hand we set $\Pi := \{S'_1, \ldots, S'_m\}$. It is straightforward to check that $\Pi \subseteq \overline{\mathbf{T}}$ and $\varphi_E^{=}$ is equivalent to $\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$. (Observe however, that the quantifier rank of the latter is only bounded by $k \times 2^{|A|} + 1$.) QED

3.3 Normal form for $\mathtt{ME}^\infty$

The logic $\mathtt{ME}^\infty$ extends $\mathtt{ME}$ with the capacity to tear apart finite and infinite sets of elements. This is reflected in the normal form for $\mathtt{ME}^\infty$ by adding extra information to the normal form of $\mathtt{ME}$.

Definition 3.10 We say that a formula $\varphi \in \mathtt{ME}^\infty(A)$ is in *basic form* if $\varphi = \bigvee \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ where each disjunct is of the form

$$\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) := \nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi \cup \Sigma) \wedge \nabla_\infty(\Sigma)$$

where

$$\nabla_\infty(\Sigma) := \bigwedge_{S \in \Sigma} \exists^\infty y.\tau_S(y) \wedge \forall^\infty y. \bigvee_{S \in \Sigma} \tau_S(y).$$

Here $\overline{\mathbf{T}} \in \wp(A)^k$ for some k, and $\Pi, \Sigma \subseteq \wp A$ are such that $\Sigma \cup \Pi \subseteq \overline{\mathbf{T}}$. $\lhd$

Intuitively, the formula $\nabla_\infty(\Sigma)$ says that (1) for every type $S \in \Sigma$, there are infinitely many elements satisfying S and (2) only finitely many elements do not satisfy any type in Σ. As a special case, the formula $\nabla_\infty(\varnothing)$ expresses that the model is finite. A short argument reveals that, intuitively, every disjunct of the form $\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ expresses that any monadic model satisfying it admits a partition of its domain in three parts:

(i) distinct elements $t_1, \ldots, t_n$ with respective types $T_1, \ldots, T_n$,

(ii) finitely many elements whose types belong to Π, and

(iii) for each $S \in \Sigma$, infinitely many elements with type S.

Observe that basic formulas of $\mathtt{ME}$ are *not* basic formulas of $\mathtt{ME}^\infty$.

In the same way as before, we define an equivalence relation $\sim_k^\infty$ on monadic models which refines $\sim_k^{=}$ by adding information about the (in-)finiteness of the types.

Definition 3.11 For every $k \in \mathbb{N}$ we define the relation $\sim_k^\infty$ on the class $\mathfrak{M}$ of monadic models by putting

$$
\begin{aligned}
\mathbb{D} \sim_0^\infty \mathbb{D}' &\iff \text{always} \\
\mathbb{D} \sim_{k+1}^\infty \mathbb{D}' &\iff \forall S \subseteq A \ \big(|S|_{\mathbb{D}} = |S|_{\mathbb{D}'} < k \text{ or } k \leq |S|_{\mathbb{D}}, |S|_{\mathbb{D}'} < \omega \text{ or } |S|_{\mathbb{D}}, |S|_{\mathbb{D}'} \geq \omega \ \big),
\end{aligned}
$$

where $\mathbb{D}$ and $\mathbb{D}'$ are arbitrary monadic models. $\lhd$

Proposition 3.12 *The following hold:*

1. *The relation $\sim_k^\infty$ is an equivalence relation of finite index.*
2. *The relation $\sim_k^\infty$ is a refinement of $\sim_k^=$.*
3. *Every $E \in \mathfrak{M}/{\sim_k^\infty}$ is characterized by a formula $\varphi_E^\infty \in \mathtt{ME}^\infty(A)$ with $\mathtt{qr}(\varphi) = k$.*

Proof. We only prove the last point, for $k > 0$. Let $E \in \mathfrak{M}/{\sim_k^\infty}$ and let $\mathbb{D} \in E$ be a representative of the class. Let $E' \in \mathfrak{M}/{\sim_k^=}$ be the equivalence class of $\mathbb{D}$ with respect to $\sim_k^=$. Let $S_1, \dots, S_n \subseteq A$ be all the types such that $|S_i|_{\mathbb{D}} \geq \omega$, and define

$$\varphi_E^\infty := \varphi_{E'}^= \wedge \nabla_\infty(\{S_1, \dots, S_n\}).$$

It is not difficult to see that $\mathtt{qr}(\varphi_E^\infty) = k$ and that $\mathbb{D}' \models \varphi_E^\infty$ iff $\mathbb{D}' \in E$. QED

Now we give a version of the Ehrenfeucht-Fraïssé game for $\mathtt{ME}^\infty$. This game, which extends $\mathrm{EF}_k^=$ with moves for $\exists^\infty$, can be seen as an adaptation of the Ehrenfeucht-Fraïssé game for monotone generalized quantifiers found in [2] to the case of full monadic first-order logic. •

Definition 3.13 Let $\mathbb{D}_0 = (D_0, V_0)$ and $\mathbb{D}_1 = (D_1, V_1)$ be monadic models. We define the game $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1)$ between $\forall$ and $\exists$. A position in this game is a pair of sequences $\overline{\mathbf{s_0}} \in D_0^n$ and $\overline{\mathbf{s_1}} \in D_1^n$ with $n \leq k$. The game consists of k rounds, where in round $n+1$ the following steps are made. First $\forall$ chooses to perform one of the following types of moves:

(a) second-order move:

1. $\forall$ chooses an infinite set $X_i \subseteq D_i$;
2. $\exists$ responds with an infinite set $X_{-i} \subseteq D_{-i}$;
3. $\forall$ chooses an element $d_{-i} \in X_{-i}$;
4. $\exists$ responds with an element $d_i \in X_i$.

(b) first-order move:

1. $\forall$ chooses an element $d_i \in D_i$;
2. $\exists$ responds with an element $d_{-i} \in D_{-i}$.

The sequences $\overline{\mathbf{s_i}} \in D_i^n$ of elements chosen up to round n are then extended to $\overline{\mathbf{s_i}}' := \overline{\mathbf{s_i}} \cdot d_i$. $\exists$ survives the round iff she does not get stuck and the function $f_{n+1} : \overline{\mathbf{s_0}}' \mapsto \overline{\mathbf{s_1}}'$ is a partial isomorphism of monadic models. $\lhd$

Lemma 3.14 *The following are equivalent:*

1. $\mathbb{D}_0 \equiv_k^{\mathtt{ME}^\infty} \mathbb{D}_1$*,*

2. $\mathbb{D}_0 \sim_k^\infty \mathbb{D}_1$*,*

3. $\exists$ *has a winning strategy in* $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1)$*.*

Proof. Step (1) to (2) is direct by Proposition 3.12. For (2) to (3) we show the following.

Claim 1 Let $\mathbb{D}_0 \sim_k^\infty \mathbb{D}_1$ and $\overline{\mathbf{s_i}} \in D_i^n$ be such that $n < k$ and $f_n : \overline{\mathbf{s_0}} \mapsto \overline{\mathbf{s_1}}$ is a partial isomorphism. Then $\exists$ can survive one more round in $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$.

Proof of Claim We focus on the second-order moves because the first-order moves are the same as in the corresponding Claim of Lemma 3.8. Let $\forall$ choose an infinite set $X_i \subseteq D_i$, we would like $\exists$ to choose an infinite set $X_{-i} \subseteq D_{-i}$ such that the following conditions hold:

(a) The map f_n is a well-defined partial isomorphism between the restricted monadic models $\mathbb{D}_0{\restriction}X_0$ and $\mathbb{D}_1{\restriction}X_1$,

(b) For every type S there is an element $d \in X_i$ of type S which is *not* connected by f_n iff there is such an element in X_{-i}.

First we prove that such a set X_{-i} exists. To satisfy item (a) $\exists$ just needs to add to X_{-i} the elements connected to X_i by f_n; this is not a problem.

For item (b) we proceed as follows: for every type S such that there is an element $d \in X_i$ of type S, we add a new element $d' \in D_{-i}$ of type S to X_{-i}. To see that this is always possible, observe first that $\mathbb{D}_0 \sim_k^\infty \mathbb{D}_1$ implies $\mathbb{D}_0 \sim_k^= \mathbb{D}_1$. Using the properties of this relation, we divide in two cases:

- If $|S|_{D_i} \geq k$ we know that $|S|_{D_{-i}} \geq k$ as well. From the elements of D_{-i} of type S, at most $n < k$ are used by f_n. Hence, there is at least one $d' \in D_{-i}$ of type S to choose from.

- If $|S|_{D_i} < k$ we know that $|S|_{D_i} = |S|_{D_{-i}}$. From the elements of D_i of type S, at most $|S|_{D_i} - 1$ are used by f_n. (The reason for the '-1' is that we are assuming that we have just chosen a $d \in X_i$ which is not in f_n.) Using that $|S|_{D_i} = |S|_{D_{-i}}$ and that f_n is a partial isomorphism we can again conclude that there is at least one $d' \in D_{-i}$ of type S to choose from.

Finally, we need to show that $\exists$ can choose X_{-i} to be infinite. To see this, observe that X_i is infinite, while there are only finitely many types. Hence there must be some S such that $|S|_{X_i} \geq \omega$. It is then safe to add infinitely many elements for S in X_{-i} while considering point (b). Moreover, the existence of infinitely many elements satisfying S in D_{-i} is guaranteed by $\mathbb{D}_0 \sim_k^\infty \mathbb{D}_1$.

Having shown that $\exists$ can choose a set X_{-i} satisfying the above conditions, it is now clear that using point (b) $\exists$ can survive the "first-order part" of the second-order move we were considering. This finishes the proof of the claim. ◀

Returning to the proof of Lemma 3.14, for step (3) to (1) we prove the following.

Claim 2 Let $\overline{\mathbf{s_i}} \in D_i^n$ and $\varphi(z_1, \dots, z_n) \in \mathtt{ME}^\infty(A)$ be such that $\mathtt{qr}(\varphi) \leq k - n$. If $\exists$ has a winning strategy in $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$ then $\mathbb{D}_0 \models \varphi(\overline{\mathbf{s_0}})$ iff $\mathbb{D}_1 \models \varphi(\overline{\mathbf{s_1}})$.

Proof of Claim All the cases involving operators of $\mathtt{ME}$ are the same as in Lemma 3.8. We prove the inductive case for the generalized quantifier. Let $\varphi(z_1, \dots, z_n)$ be of the form $\exists^\infty x.\psi(z_1, \dots, z_n, x)$ and let $\mathbb{D}_0 \models \varphi(\overline{\mathbf{s_0}})$. Hence, the set $X_0 := \{d_0 \in D_0 \mid \mathbb{D}_0 \models \psi(\overline{\mathbf{s_0}}, d_0)\}$ is infinite.

By assumption $\exists$ has a winning strategy in $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}, \overline{\mathbf{s_1}})$. Therefore, if $\forall$ plays a second-order move by picking $X_0 \subseteq D_0$ she can respond with some infinite set $X_1 \subseteq D_1$. We claim that $\mathbb{D}_1 \models \psi(\overline{\mathbf{s_1}}, d_1)$ for every $d_1 \in X_1$. First observe that if this holds then the set $X_1' := \{d_1 \in D_1 \mid \mathbb{D}_1 \models \psi(\overline{\mathbf{s_1}}, d_1)\}$ must be infinite, and hence $\mathbb{D}_1 \models \exists^\infty x.\psi(\overline{\mathbf{s_1}}, x)$.

Assume, for a contradiction, that $\mathbb{D}_1 \not\models \psi(\overline{\mathbf{s_1}}, d_1')$ for some $d_1' \in X_1$. Let $\forall$ play this d_1' as the second part of his move. Then, as $\exists$ has a winning strategy, she will respond with some $d_0' \in X_0$ for which she has a winning strategy in $\mathrm{EF}_k^\infty(\mathbb{D}_0, \mathbb{D}_1)@(\overline{\mathbf{s_0}}{\cdot}d_0', \overline{\mathbf{s_1}}{\cdot}d_1')$. But then by our induction hypothesis, which applies since $\mathtt{qr}(\psi) \leq k - (n+1)$, we may infer from $\mathbb{D}_1 \not\models \psi(\overline{\mathbf{s_1}}, d_1')$ that $\mathbb{D}_0 \not\models \psi(\overline{\mathbf{s_0}}, d_0')$. This clearly contradicts the fact that $d_0' \in X_0$. ◀

Combining the claims finishes the proof of the lemma. QED

Theorem 3.15 *There is an effective procedure that transforms an arbitrary $\mathtt{ME}^\infty$-sentence φ into an equivalent formula φ^* in basic form.*

Proof. This can be proved using the same argument as in Theorem 3.9. Hence we only focus on showing that $\varphi_E^\infty \equiv \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ for some $\overline{\mathbf{T}}, \Pi, \Sigma \subseteq \wp A$ such that $\Sigma \cup \Pi \subseteq \overline{\mathbf{T}}$. Recall that

$$\varphi_E^\infty = \varphi_{E'}^{\equiv} \wedge \nabla_\infty(\Sigma)$$

where Σ is the collection of types that are realized by infinitely many elements. Using Theorem 3.9 on $\varphi_{E'}^{\equiv}$ we know that this is equivalent to

$$\varphi_E^\infty = \nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi') \wedge \nabla_\infty(\Sigma)$$

where $\Pi' \subseteq \overline{\mathbf{T}}$. Observe that we may assume that $\Sigma \subseteq \Pi$, otherwise the formula would be inconsistent. Now separate Π' as $\Pi' = \Pi \uplus \Sigma$ where $\Pi := \Pi' \setminus \Sigma$ consists of the types that are satisfied by finitely many elements. Then we find

$$\varphi_E^\infty \equiv \nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi \cup \Sigma) \wedge \nabla_\infty(\Sigma).$$

Therefore, we can conclude that $\varphi_E^\infty \equiv \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. QED

► `The following stronger normal form will be useful in later chapters --- is it?`

Proposition 3.16 *For every formula in the basic form $\bigvee \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ it is possible to assume, without loss of generality, that $\Sigma \subseteq \Pi$.*

Proof. This is direct from observing that $\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ is equivalent to $\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi \cup \Sigma, \Sigma)$. To check it we just unravel the definitions and observe that $\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi \cup \Sigma) \wedge \nabla_\infty(\Sigma)$ is equivalent to $\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi \cup \Sigma \cup \Sigma) \wedge \nabla_\infty(\Sigma)$. QED

4 Monotonicity

▶ `write section intro`

Definition 4.1 Let V and V' be two valuations on the same domain D, then we say that V' is a B-*extension of* V, notation: $V \leq_B V'$, if $V(b) \subseteq V'(b)$ for every $b \in B$, and $V(a) = V'(a)$ for every $a \in A \setminus B$.

Given a monadic logic $\mathtt{L}$ and a formula $\varphi \in \mathtt{L}(A)$ we say that φ is *monotone in* $B \subseteq A$ if

$$(D, V), g \models \varphi \text{ and } V \leq_B V' \text{ imply } (D, V'), g \models \varphi, \tag{1}$$

for every pair of monadic models (D, V) and (D, V') and every assignment $g : \mathsf{iVar} \to D$. $\lhd$

Remark 4.2 It is easy to prove that a formula is monotone in $B \subseteq A$ if and only if it is monotone in every $b \in B$. $\lhd$

The semantic property of monotonicity can usually be linked to the syntactic notion of positivity. That is, for many logics, a formula φ is monotone in $a \in A$ iff φ is equivalent to a formula where all occurrences of a have a positive polarity, that is, they are situated in the scope of an even number of negations. This is the case for all three logics considered here.

Definition 4.3 For $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}\}$ we define the fragment of A-formulas that are *positive* in all predicates in B, in short: the B-*positive formulas* by the following grammar:

$$\varphi ::= \psi \mid b(x) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x.\varphi \mid \forall x.\varphi,$$

where $b \in B$ and $\psi \in \mathtt{L}(A \setminus B)$ (that is, there are no occurrences of any $b \in B$ in ψ). Similarly, the B-positive fragment of $\mathtt{ME}^\infty$ is given by

$$\varphi ::= \psi \mid b(x) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x.\varphi \mid \forall x.\varphi \mid \exists^\infty x.\varphi \mid \forall^\infty x.\varphi,$$

where $b \in B$ and $\psi \in \mathtt{ME}^\infty(A \setminus B)$.

In all three cases, we let $\mathtt{Pos}_B(\mathtt{L}(A))$ denote the set of B-positive sentences. $\lhd$

Note that the difference between the fragments $\mathtt{Pos}_B(\mathtt{M}(A))$ and $\mathtt{Pos}_B(\mathtt{ME}(A))$ lies in the fact that in the latter case, the 'B-free' formulas ψ may contain the equality symbol.

Theorem 4.4 *Let φ be a sentence of the monadic logic $\mathtt{L}(A)$, where $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^\infty\}$. Then φ is monotone in a set $B \subseteq A$ if and only if there is an equivalent formula $\varphi^\oslash \in \mathtt{Pos}_B(\mathtt{L}(A))$. Furthermore, it is decidable whether a formula $\varphi \in \mathtt{L}(A)$ has this property or not.*

The 'easy' direction of the theorem is taken care of by the following proposition.

Lemma 4.5 *Every formula $\varphi \in \mathtt{Pos}_B(\mathtt{L}(A))$ is monotone in B, where $\mathtt{L}$ is one of the logics $\{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^\infty\}$.*

Proof. The proof is a routine argument by induction on the complexity of φ. That is, we show by induction, that any formula φ in the B-positive fragment (which may not be a sentence) satisfies (1), for every monadic model (D, V), valuation $V' \geq_B V$ and assignment $g : \mathsf{iVar} \to D$. We focus on the generalized quantifiers. Let $(D, V), g \models \varphi$ and $V \leq_B V'$.

- Case $\varphi = \exists^\infty x.\varphi'(x)$. By definition there exists an infinite set $I \subseteq D$ such that for all $d \in I$ we have $(D, V), g[x \mapsto d] \models \varphi'(x)$. By induction hypothesis $(D, V'), g[x \mapsto d] \models \varphi'(x)$ for all $d \in I$. Therefore $(D, V'), g \models \exists^\infty x.\varphi'(x)$.

- Case $\varphi = \forall^\infty x.\varphi'(x)$. Hence there exists $C \subseteq D$ such that for all $d \in C$ we have $(D, V), g[x \mapsto d] \models \varphi'(x)$ and $D \setminus C$ is *finite*. By induction hypothesis $(D, V'), g[x \mapsto d] \models \varphi'(x)$ for all $d \in C$. Therefore $(D, V'), g \models \forall^\infty x.\varphi'(x)$.

This finishes the proof. QED

We will prove the three manifestations of the 'hard' direction of the theorem separately, in three respective subsections. The following definition will be used throughout.

Definition 4.6 Given $S \subseteq A$ and $B \subseteq A$ we use the following notation

$$\tau_S^B(x) := \bigwedge_{b \in S} b(x) \wedge \bigwedge_{b \in A \setminus (S \cup B)} \neg b(x),$$

for what we call the *B-positive* A-type τ_S^B. $\lhd$

Intuitively, τ_S^B works almost like the A-type τ_S, the difference being that τ_S^B discards the negative information for the names in B. If $B = \{a\}$ we write τ_S^a instead of $\tau_S^{\{a\}}$. Observe that with this notation, τ_S^+ is equivalent to τ_S^A.

4.1 Monotone fragment of $\mathtt{M}$

To prove that the fragment $\mathtt{Pos}_B(\mathtt{M})$ is complete for monotonicity in B, we need to show that every formula which is monotone in B is equivalent to some formula in $\mathtt{Pos}_B(\mathtt{M})$. We prove a stronger result: we give a translation that constructively maps arbitrary formulas into $\mathtt{Pos}_B(\mathtt{M})$. The interesting observation is that the translation will preserve truth iff the given formula is monotone in B. To formulate the translation we need to introduce some new notation.

Definition 4.7 Let $B \subseteq A$ be a finite set of names. The B-positive variant of $\nabla_{\mathtt{M}}(\Sigma)$ is given as follows:

$$\nabla_{\mathtt{M}}^B(\Sigma) := \bigwedge_{S \in \Sigma} \exists x.\tau_S^B(x) \wedge \forall x. \bigvee_{S \in \Sigma} \tau_S^B(x).$$

We also introduce the following generalized forms of the above notation:

$$\nabla_{\mathtt{M}}^B(\Sigma, \Pi) := \bigwedge_{S \in \Sigma} \exists x.\tau_S^B(x) \wedge \forall x. \bigvee_{S \in \Pi} \tau_S^B(x).$$

The *positive* variants of the above notations are defined as $\nabla_{\mathtt{M}}^+(\Sigma) := \nabla_{\mathtt{M}}^A(\Sigma)$ and $\nabla_{\mathtt{M}}^+(\Sigma, \Pi) := \nabla_{\mathtt{M}}^A(\Sigma, \Pi)$. $\lhd$

Lemma 4.8 *There exists a translation* $(-)^{\oslash} : \mathtt{M}(A) \to \mathtt{Pos}_B(\mathtt{M}(A))$ *such that a sentence* $\varphi \in \mathtt{M}(A)$ *is monotone in* $a \in A$ *if and only if* $\varphi \equiv \varphi^{\oslash}$.

Proof. To define the translation we assume, without loss of generality, that φ is in the normal form $\bigvee \nabla_{\mathtt{M}}(\Sigma)$ given in Definition 3.1, where $\nabla_{\mathtt{M}}(\Sigma) = \bigwedge_{S\in\Sigma} \exists x.\tau_S(x) \wedge \forall x. \bigvee_{S\in\Sigma} \tau_S(x)$. We define the translation as

$$(\bigvee \nabla_{\mathtt{M}}(\Sigma))^{\oslash} := \bigvee \nabla^B_{\mathtt{M}}(\Sigma).$$

From the construction it is clear that $\varphi^{\oslash} \in \mathtt{Pos}_B(\mathtt{M}(A))$ and therefore the right-to-left direction of the lemma is immediate by Lemma 4.5. For the left-to-right direction assume that φ is monotone in B, we have to prove that $(D, V) \models \varphi$ if and only if $(D, V) \models \varphi^{\oslash}$.

$\boxed{\Rightarrow}$ This direction is trivial.

$\boxed{\Leftarrow}$ Assume $(D, V) \models \varphi^{\oslash}$ and let Σ be such that $(D, V) \models \nabla^B_{\mathtt{M}}(\Sigma)$.

Because of the existential part of $\nabla^B_{\mathtt{M}}(\Sigma)$, every type $S \in \Sigma$ has a 'B-witness' in $\mathbb{D}$, that is, an element $d_S \in D$ such that $(D, V) \models \tau^B_S(d_S)$. It is in fact safe to assume that all these witnesses are *distinct* (this is because (D, V) can be proved to be $\mathtt{M}$-equivalent to such a model, cf. Proposition 6.10). But because of the universal part of $\nabla^B_{\mathtt{M}}(\Sigma)$, we may assume that for all states d in D there is a type S_d in Σ such that $(D, V) \models \tau^B_{S_d}(d)$. Putting these observations together we may assume that the map $d \mapsto S_d$ is surjective.

Note however, that where we have $(D, V) \models \tau^B_S(d)$, this does not necessarily imply that $(D, V) \models \tau_S(d)$: it might well be the case that $d \in V(b)$ but $b \notin S_d$, for some $b \in B$. What we want to do now is to shrink V in such a way that the witnessed type (S_d) and the actually satisfied type coincide. That is, we consider the valuation U defined as $U^{\flat}(d) := S_d$.[1] It is then immediate by the surjectivity of the map $d \mapsto S_d$ that $(D, U) \models \nabla_{\mathtt{M}}(\Sigma)$, which implies that $(D, U) \models \varphi$.

We now claim that

$$U \leq_B V. \tag{2}$$

To see this, observe that for $a \in A \setminus B$ we have the following equivalences:

$$d \in U(a) \iff a \in S_d \iff (D, V) \models a(d) \iff d \in V(a),$$

while for $b \in B$ we can prove

$$d \in U(b) \iff b \in S_d \Longrightarrow (D, V) \models b(d) \iff d \in V(b).$$

This suffices to prove (2).

But from (2) and the earlier observation that $(D, U) \models \varphi$ it is immediate by the monotonity of φ in B that $(D, V) \models \varphi$. QED

Putting together the Lemmas 4.5 and 4.8 we obtain the instantiation of Theorem 4.4 for $\mathtt{L} = \mathtt{M}$. Moreover, a careful analysis of the translation gives us the following corollary, providing normal forms for the monotone fragment of $\mathtt{M}$.

[1] Recall that a valuation $U : A \to \wp D$ can also be represented as a marking $U^{\flat} : D \to \wp A$ given by $U^{\flat}(d) := \{a \in A \mid d \in V(a)\}$.

Corollary 4.9 *For any sentence $\varphi \in \mathtt{M}(A)$, the following hold.*

1. *The formula φ is monotone in $B \subseteq A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla^B_{\mathtt{M}}(\Sigma)$ for some types $\Sigma \subseteq \wp A$.*

2. *The formula φ is monotone in every $a \in A$ iff φ is equivalent to a formula $\bigvee \nabla^+_{\mathtt{M}}(\Sigma)$ for some types $\Sigma \subseteq \wp A$.*

4.2 Monotone fragment of ME

In order to prove the ME-variant of Lemma 4.5, we need to introduce some new notation.

Definition 4.10 Let $B \subseteq A$ be a finite set of names. The B-monotone variant of $\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ is given as follows:

$$\nabla^B_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi) := \exists \overline{\mathbf{x}}.\big(\mathrm{diff}(\overline{\mathbf{x}}) \wedge \bigwedge_i \tau^B_{T_i}(x_i) \wedge \forall z.(\mathrm{diff}(\overline{\mathbf{x}}, z) \to \bigvee_{S \in \Pi} \tau^B_S(z))\big).$$

When the set B is a singleton $\{a\}$ we will write a instead of B. The positive variant $\nabla^+_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ of $\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ is defined as above but with $+$ in place of B. $\lhd$

Lemma 4.11 *There exists a translation $(-)^{\oslash} : \mathtt{ME}(A) \to \mathtt{Pos}_B(\mathtt{ME}(A))$ such that a sentence $\varphi \in \mathtt{ME}(A)$ is monotone in B if and only if $\varphi \equiv \varphi^{\oslash}$.*

Proof. In Lemma 4.14 this result is proved for $\mathtt{ME}^\infty$ (i.e., ME extended with generalized quantifiers). It is not difficult to adapt the proof for ME. QED

The variant of Theorem 4.4 for $\mathtt{L} = \mathtt{ME}$ follows immediately from the Lemmas 4.11 and 4.8. Moreover, combining the normal form for ME and the above lemma, we obtain the following corollary providing a normal form for the monotone fragment of ME.

Corollary 4.12 *For any sentence $\varphi \in \mathtt{M}(A)$, the following hold.*

1. *The formula φ is monotone in $B \subseteq A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla^B_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ where for each disjunct we have $\overline{\mathbf{T}} \in \wp(A)^k$ for some k and $\Pi \subseteq \overline{\mathbf{T}}$.*

2. *The formula φ is monotone in all $a \in A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla^+_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)$ where for each disjunct we have $\overline{\mathbf{T}} \in \wp(A)^k$ for some k and $\Pi \subseteq \overline{\mathbf{T}}$.*

4.3 Monotone fragment of $\mathtt{ME}^\infty$

Before going on, we introduce some notation.

Definition 4.13 Let $B \subseteq A$ be a finite set of names. The B-positive variant of $\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ is given as follows:

$$\begin{aligned}
\nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) &:= \nabla^B_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi \cup \Sigma) \wedge \nabla^B_\infty(\Sigma) \\
\nabla^B_{\mathtt{ME}}(\overline{\mathbf{T}}, \Lambda) &:= \exists \overline{\mathbf{x}}.\big(\mathrm{diff}(\overline{\mathbf{x}}) \wedge \bigwedge_i \tau^B_{T_i}(x_i) \wedge \forall z.(\mathrm{diff}(\overline{\mathbf{x}}, z) \to \bigvee_{S \in \Lambda} \tau^B_S(z))\big) \\
\nabla^B_\infty(\Sigma) &:= \bigwedge_{S \in \Sigma} \exists^\infty y.\tau^B_S(y) \wedge \forall^\infty y. \bigvee_{S \in \Sigma} \tau^B_S(y).
\end{aligned}$$

When the set B is a singleton $\{a\}$ we will write a instead of B. The positive variant of $\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ is defined as $\nabla^{+}_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) := \nabla^{A}_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. $\lhd$

We are now ready to give the translation.

Lemma 4.14 *There is a translation* $(-)^{\oslash} : \mathtt{ME}^\infty(A) \to \mathtt{Pos}_B(\mathtt{ME}^\infty(A))$ *such that a formula* $\varphi \in \mathtt{ME}^\infty(A)$ *is monotone in* B *if and only if* $\varphi \equiv \varphi^{\oslash}$.

Proof. We assume that φ is in the normal form $\bigvee \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) = \nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi \cup \Sigma) \wedge \nabla_\infty(\Sigma)$ for some sets of types $\Pi, \Sigma \subseteq \wp A$ and each $T_i \subseteq A$. For the translation we define

$$\Big(\bigvee \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)\Big)^{\oslash} := \bigvee \nabla^{B}_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma).$$

From the construction it is clear that $\varphi^{\oslash} \in \mathtt{Pos}_B(\mathtt{ME}^\infty(A))$ and therefore the right-to-left direction of the lemma is immediate by Lemma 4.5. For the left-to-right direction assume that φ is monotone in B, we have to prove that $(D, V) \models \varphi$ if and only if $(D, V) \models \varphi^{\oslash}$.

$\boxed{\Rightarrow}$ This direction is trivial.

$\boxed{\Leftarrow}$ Assume $(D, V) \models \varphi^{\oslash}$, and in particular that $(D, V) \models \nabla^{B}_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. Observe that the elements of D can be partitioned in the following way:

(a) distinct elements $t_i \in D$ such that each t_i satisfies $\tau^{B}_{T_i}(x)$,

(b) for every $S \in \Sigma$ an infinite set D_S, such that every $d \in D_S$ satisfies τ^{B}_{S},

(c) a *finite* set D_Π of elements, each satisfying one of the B-positive types τ^{B}_{S} with $S \in \Pi$.

Following this partition, with every element $d \in D$ we may associate a type S_d in, respectively, (a) $\overline{\mathbf{T}}$, (b) Σ, or (c) Π, such that d satisfies $\tau^{B}_{S_d}$. As in the proof of Lemma 4.8, we now consider the valuation U defined as $U^\flat(d) := S_d$, and as before we can show that $U \leq_B V$. Finally, it easily from the definitions that $(D, U) \models \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, implying that $(D, U) \models \varphi$. But then by the assumed B-monotonicity of φ it is immediate that $(D, V) \models \varphi$, as required. QED

Putting together the Lemmas 4.14 and 4.8 we obtain Theorem 4.4 for the case $\mathtt{L} = \mathtt{ME}^\infty$. Moreover, a careful analysis of the translation gives us the following corollary, providing normal forms for the monotone fragment of $\mathtt{ME}^\infty$.

Corollary 4.15 *Let* $\varphi \in \mathtt{ME}^\infty(A)$, *the following hold:*

1. *The formula* φ *is monotone in* $B \subseteq A$ *iff it is equivalent to a formula* $\bigvee \nabla^{B}_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ *for* $\Sigma \subseteq \Pi \subseteq \wp A$ *and* $\overline{\mathbf{T}} \in \wp(A)^k$ *for some* k.

2. *The formula* φ *is monotone in every* $a \in A$ *iff it is equivalent to a formula in the basic form* $\bigvee \nabla^{+}_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ *for types* $\Sigma \subseteq \Pi \subseteq \wp A$ *and* $\overline{\mathbf{T}} \in \wp(A)^k$ *for some* k.

Proof. We only remark that to obtain $\Sigma \subseteq \Pi$ in the above normal forms it is enough to use Proposition 3.16 before applying the translation. QED

5 Continuity

In this section we study the formulas that are *continuous* in some set B of monadic predicate symbols.

Definition 5.1 Let U and V be two A-valuations on the same domain D. For a set $B \subseteq A$, we write $U \leq_B^\omega V$ if $U \leq_B V$ and $U(b)$ is finite, for every $b \in B$.

Given a monadic logic $\mathtt{L}$ and a formula $\varphi \in \mathtt{L}(A)$ we say that φ is *continuous in* $B \subseteq A$ if φ is monotone in B and satisfies the following:

$$\text{if } (D, V), g \models \varphi \text{ then } (D, U), g \models \varphi \text{ for some } U \leq_B^\omega V. \tag{3}$$

for every monadic model (D, V) and every assignment $g : \mathsf{iVar} \to D$. ◁

Remark 5.2 As for monotonicity, but with slightly more effort, one may show that a formula φ is continuous in a set B iff it is continuous in every $b \in B$. ◁

- ▶ `explain the name 'continuity' and discuss/motivate the notion`
- ▶ `mention van Benthem's characterization and other known results`

In this section we will characterise the continuous fragments of $\mathtt{M}$ and $\mathtt{ME}^\infty$

- ▶ `What about ME?`

5.1 Continuous fragment of M

The fragment of $\mathtt{M}$ that characterizes the property of continuity can be defined as follows.

Definition 5.3 The fragment $\mathtt{Con}_B(\mathtt{M}(A))$ of $\mathtt{M}(A)$ of formulas that are *syntactically continuous* in a subset $B \subseteq A$ is defined by the following grammar:

$$\varphi ::= \psi \mid b(x) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x.\varphi,$$

where $b \in B$ and $\psi \in \mathtt{M}(A \setminus B)$. ◁

Theorem 5.4 *Let φ be a sentence of the logic $\mathtt{M}(A)$. Then φ is continuous in a set $B \subseteq A$ if and only if there is a equivalent formula $\varphi^\ominus \in \mathtt{Pos}_B(\mathtt{M}(A))$. Furthermore, it is decidable whether a formula $\varphi \in \mathtt{M}(A)$ has this property or not.*

The theorem will follow from the next two lemmas and Remark 5.2.

Lemma 5.5 *Every $\varphi \in \mathtt{Con}_B(\mathtt{M}(A))$ is continuous in B.*

Proof. First observe that φ is monotone in B by Lemma 4.5. We show, by induction, that any one-step formula φ in the fragment (which may not be a sentence) satisfies (3), for every one-step model (D, V) and assignment $g : \mathsf{iVar} \to D$.

- If $\varphi = \psi \in \mathtt{M}(A \setminus B)$, changes in the B part of the valuation will not affect the truth value of φ and hence the condition is trivial.

- Case $\varphi = b(x)$ for some $b \in B$: if $(D, V), g \models b(x)$ then $g(x) \in V(b)$. Clearly, $g(x) \in V[b \mapsto \{g(x)\}](b)$ and hence $(D, V[b \mapsto \{g(x)\}]), g \models b(x)$, while it is obvious from the definitions that $V[b \mapsto \{g(x)\}] \leq^{\omega}_{B} V$.

- Case $\varphi = \varphi_1 \vee \varphi_2$: assume $(D, V), g \models \varphi$. Without loss of generality we can assume that $(D, V), g \models \varphi_1$ and hence by induction hypothesis there is $U \leq^{\omega}_{B} V$ such that $(D, U), g \models \varphi_1$ which clearly implies $(D, U), g \models \varphi$.

- Case $\varphi = \varphi_1 \wedge \varphi_2$: assume $(D, V), g \models \varphi$. By induction hypothesis we have $U_1, U_2 \leq^{\omega}_{B} V$ such that $(D, U_1), g \models \varphi_1$ and $(D, U_2), g \models \varphi_2$. Let U be the valuation defined by putting $U(a) := U_1(a) \cup U_2$; then clearly we have $U \leq^{\omega}_{B} V$, while it follows by monotonicity that $(D, U), g \models \varphi_1$ and $(D, U), g \models \varphi_2$. Clearly then $(D, U), g \models \varphi$.

- Case $\varphi = \exists x.\varphi'(x)$ and $(D, V), g \models \varphi$. By definition there exists $d \in D$ such that $(D, V), g[x \mapsto d] \models \varphi'(x)$. By induction hypothesis there is a valuation $U \leq^{\omega}_{B} V$ such that $(D, U), g[x \mapsto d] \models \varphi'(x)$ and hence $(D, U), g \models \exists x.\varphi'(x)$.

This finishes the proof. QED

The main result underlying the proof of Theorem 5.4 is the following.

Lemma 5.6 *There is a translation* $(-)^{\ominus} : \mathtt{Pos}_B(\mathtt{M}(A)) \to \mathtt{Con}_B(\mathtt{M}(A))$ *such that a formula* $\varphi \in \mathtt{Pos}_B(\mathtt{M}(A))$ *is continuous in* $B \subseteq A$ *if and only if* $\varphi \equiv \varphi^{\ominus}$.

Proof. To define the translation we assume, without loss of generality, that φ is in the basic form $\bigvee \nabla^{B}_{\mathtt{M}}(\Sigma)$. For the translation, let

$$(\bigvee \nabla^{B}_{\mathtt{M}}(\Sigma))^{\ominus} := \bigvee \nabla^{B}_{\mathtt{M}}(\Sigma, \Sigma^{-}_{B})$$

where $\Sigma^{-}_{B} := \{S \in \Sigma \mid B \cap S = \varnothing\}$. From the construction it is clear that $\varphi^{\ominus} \in \mathtt{Con}_B(\mathtt{M}(A))$ and therefore the right-to-left direction of the lemma is immediate by Lemma 5.5.

For the left-to-right direction assume that φ is continuous in B, we have to prove that $(D, V) \models \varphi$ iff $(D, V) \models \varphi^{\ominus}$, for every one-step model (D, V). Our proof strategy consists of proving the same equivalence for the model $(D \times \omega, V_{\pi})$, where $D \times \omega$ consists of ω many copies of each element in D and V_{π} is the valuation given by $V_{\pi}(a) := \{(d, k) \mid d \in V(a), k \in \omega\}$. It is easy to see that $(D, V) \equiv^{\mathtt{M}} (D \times \omega, V_{\pi})$ (see Proposition 6.10) and so it suffices indeed to prove that

$$(D \times \omega, V_{\pi}) \models \varphi \text{ iff } (D \times \omega, V_{\pi}) \models \varphi^{\ominus}.$$

$\boxed{\Rightarrow}$ Let $(D \times \omega, V_{\pi}) \models \varphi$. As φ is continuous in B there is a valuation $U \leq^{\omega}_{B} V_{\pi}$ satisfying $(D \times \omega, U) \models \varphi$. This means that $(D \times \omega, U) \models \nabla^{B}_{\mathtt{M}}(\Sigma)$ for some disjunct $\nabla^{B}_{\mathtt{M}}(\Sigma)$ of φ.

Our claim is now that $(D \times \omega, U) \models \nabla^{B}_{\mathtt{M}}(\Sigma, \Sigma^{-}_{B})$. The existential part of $\nabla^{B}_{\mathtt{M}}(\Sigma, \Sigma^{-}_{B})$ is trivially true. To cover the universal part, it remains to show that every element of $(D \times \omega, U)$

realizes a B-positive type in Σ^-_B. Take an arbitrary pair $(d,k) \in D \times \omega$ and let T be the (full) type of (d,k), that is, let $T := U^\flat(d,k)$. If $B \cap T = \varnothing$ then trivially $T \in \Sigma^-_B$ and we are done. So suppose $B \cap T \neq \varnothing$. In this case we must have $T = V^\flat_\pi(d,k)$. Observe that in $D \times \omega$ we have infinitely many copies of $d \in D$. Hence, as $U(b)$ is finite for every $b \in B$, there must be some (d,k') with type $U^\flat(d,k') = V^\flat_\pi(d,k') \setminus B = V^\flat_\pi(d,k) \setminus B = T \setminus B$. Abbreviate $T' := T \setminus B$. But if $U^\flat(d,k') = T'$ then it follows from $(D \times \omega, U) \models \nabla^B_{\mathtt{M}}(\Sigma)$ and $\tau_{T'} = \tau^b_{T'}$ that $T' \in \Sigma$ and hence $T' \in \Sigma^-_B$. It is then easy to see that (d,k) realises the B-positive type T'.

Finally, by monotonicity it directly follows from $(D \times \omega, U) \models \nabla^B_{\mathtt{M}}(\Sigma, \Sigma^-_B)$ that $(D \times \omega, V_\pi) \models \nabla^B_{\mathtt{M}}(\Sigma, \Sigma^-_B)$, and from this it is immedate that $(D \times \omega, V_\pi) \models \varphi^\ominus$.

$\boxed{\Leftarrow}$ Let $(D \times \omega, V_\pi) \models \nabla^B_{\mathtt{M}}(\Sigma, \Sigma^-_B)$. To show that $(D \times \omega, V_\pi) \models \nabla^B_{\mathtt{M}}(\Sigma)$, the existential part is trivial. For the universal part just observe that $\Sigma^-_B \subseteq \Sigma$. QED

Putting together the above lemmas we obtain Theorem 5.4. Moreover, a careful analysis of the translation gives us the following corollary, providing normal forms for the continuous fragment of M.

Corollary 5.7 *For any sentence $\varphi \in \mathtt{M}(A)$, the following hold.*

1. *The formula φ is continuous in $B \subseteq A$ iff it is equivalent to a formula $\bigvee \nabla^B_{\mathtt{M}}(\Sigma, \Sigma^-_B)$ for some types $\Sigma \subseteq \wp A$, where $\Sigma^-_B := \{S \in \Sigma \mid B \cap S = \varnothing\}$.*
2. *If φ is monotone in A then φ is continuous in $B \subseteq A$ iff it is equivalent to a formula in the basic form $\bigvee \nabla^+_{\mathtt{M}}(\Sigma, \Sigma^-_B)$ for some types $\Sigma \subseteq \wp A$, where $\Sigma^-_B := \{S \in \Sigma \mid B \cap S = \varnothing\}$.*

5.2 Continuous fragment of $\mathtt{ME}^\infty$

To define the syntactically continuous fragment of $\mathtt{ME}^\infty$, we first introduce the following binary generalized quantifier $\mathbf{W}$: given to formulas $\varphi(x)$ and ψ, we set

$$\mathbf{W}x.(\varphi, \psi) := \forall x.(\varphi(x) \vee \psi(x)) \wedge \forall^\infty x.\psi(x).$$

The intuition behind $\mathbf{W}$ is the following. If $(D,V), g \models \mathbf{W}x.(\varphi, \psi)$, then because of the second conjunct there are only finitely many $d \in D$ refuting ψ. The point is that this weakens the universal quantification of the first conjunct to the effect that only the finitely many mentioned elements refuting ψ need to satisfy φ.

Definition 5.8 The fragment $\mathtt{Con}_B(\mathtt{ME}^\infty(A))$ of $\mathtt{ME}^\infty(A)$-formulas that are *syntactically continuous* in a subset $B \subseteq A$ is given by the following grammar:

$$\varphi ::= \psi \mid a(x) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x.\varphi \mid \mathbf{W}x.(\varphi, \psi),$$

where $b \in B$ and $\psi \in \mathtt{ME}^\infty(A \setminus B)$. $\lhd$

Theorem 5.9 *Let φ be a sentence of the logic $\mathtt{ME}^\infty(A)$. Then φ is continuous in a set $B \subseteq A$ if and only if there is an equivalent formula $\varphi^\ominus \in \mathtt{Con}_B(\mathtt{ME}^\infty(A))$. Furthermore, it is decidable whether a formula $\varphi \in \mathtt{M}(A)$ has this property or not.*

The theorem follows from the next two lemmas.

Lemma 5.10 *Every $\varphi \in \mathtt{Con}_B(\mathtt{ME}^\infty(A))$ is semantically continuous in B.*

Proof. Observe that monotonicity of φ is guaranteed by Lemma 4.5.

We show, by induction, that any formula of the fragment (which need not be a sentence) satisfies (3), for every one-step model (D, V) and assignment $g : \mathsf{iVar} \to D$. We focus on the inductive case of the new quantifier. Let $\varphi' = \mathbf{W}x.(\varphi, \psi)$, and define the formulas $\alpha(x)$ and β as follows:

$$\varphi' = \forall x. \underbrace{(\varphi(x) \vee \psi(x))}_{\alpha(x)} \wedge \underbrace{\forall^\infty x.\psi(x)}_{\beta}.$$

Suppose that $(D, V), g \models \varphi'$. By the induction hypothesis, for every $d \in D$ which satisfies $(D, V), g_d \models \alpha(x)$ (where we write $g_d := g[x \mapsto d]$) there is a valuation $U_d \leq^\omega_B V$ such that $(D, U_d), g_d \models \alpha(x)$. The crucial observation is that because of β, only finitely many elements of d refute $\psi(x)$. Let U be the valuation defined by putting $U(a) := \bigcup\{U_d(a) \mid (D, V), g_d \not\models \psi(x)\}$. Note that for each $b \in B$, the set $U(b)$ is a finite union of finite sets, and hence finite itself; it follows that $U \leq^\omega_B V$. We claim that

$$(D, U), g \models \varphi'. \tag{4}$$

It is clear that $(D, U), g \models \beta$ because ψ (and hence β) is B-free. To prove that $(D, U), g \models \forall x\, \alpha(x)$, take an arbitrary $d \in D$, then we have to show that $(D, U), g_d \models \varphi(x) \vee \psi(x)$. We consider two cases: If $(D, V), g_d \models \psi(x)$ we are done, again because ψ is B-free. On the other hand, if $(D, V), g_d \not\models \psi(x)$, then $(D, U_d), g_d \models \alpha(x)$ by assumption on U_d, while it is obvious that $U_d \leq_B U$; but then it follows by monotonicity of α that $(D, V_U), g_d \models \alpha(x)$. This finishes the proof of (4), and hence, that of the lemma. QED

Lemma 5.11 *There is a translation $(-)^\ominus : \mathtt{Pos}_B(\mathtt{ME}^\infty(A)) \to \mathtt{Con}_B(\mathtt{ME}^\infty(A))$ such that a formula $\varphi \in \mathtt{Pos}_B(\mathtt{ME}^\infty(A))$ is continuous in B if and only if $\varphi \equiv \varphi^\ominus$.*

Proof. We assume that φ is in basic normal form, i.e., $\varphi = \bigvee \nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. For the translation let $\big(\bigvee \nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)\big)^\ominus := \bigvee \nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)^\ominus$ where

$$\nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)^\ominus := \begin{cases} \bot & \text{if } B \cap \bigcup \Sigma \neq \varnothing \\ \nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma) & \text{otherwise.} \end{cases}$$

First we prove the right-to-left direction of the lemma. By Lemma 5.10 it is enough to show that $\varphi^\ominus \in \mathtt{Con}_B(\mathtt{ME}^\infty(A))$. We focus on the disjuncts of $\varphi^\ominus$. The interesting case is where $B \cap \bigcup \Sigma = \varnothing$. If we rearrange $\nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ somewhat and define the formulas φ', ψ as follows:

$$\exists \overline{\mathbf{x}}.\Big(\mathrm{diff}(\overline{\mathbf{x}}) \wedge \bigwedge_i \tau^B_{T_i}(x_i) \ \wedge \forall z.(\underbrace{\neg\mathrm{diff}(\overline{\mathbf{x}}, z) \vee \bigvee_{S \in \Pi} \tau^B_S(z)}_{\varphi'(\overline{\mathbf{x}}, z)} \vee \underbrace{\bigvee_{S \in \Sigma} \tau^B_S(z)}_{\psi(z)}) \ \wedge \forall^\infty y. \underbrace{\bigvee_{S \in \Sigma} \tau^B_S(y)}_{\psi(y)}\Big)$$

$$\wedge \bigwedge_{S \in \Sigma} \exists^\infty y. \tau^B_S(y),$$

then we find that

$$\nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}},\Pi,\Sigma) \equiv \exists \overline{\mathbf{x}}.\Big(\mathrm{diff}(\overline{\mathbf{x}}) \wedge \bigwedge_i \tau^B_{T_i}(x_i) \wedge \mathbf{W}z.(\varphi'(\overline{\mathbf{x}},z),\psi(z))\Big) \wedge \bigwedge_{S\in\Sigma} \exists^\infty y.\tau^B_S(y),$$

which belongs to the required fragment because $B \cap \bigcup \Sigma = \varnothing$.

For the left-to-right direction of the lemma we have to prove that $\varphi \equiv \varphi^\ominus$.

$\boxed{\Rightarrow}$ Let $(D,V) \models \varphi$. Because φ is continuous in B we may assume that $V(b)$ is finite, for all $b \in B$. Let $\nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}},\Pi,\Sigma)$ be a disjunct of φ such that $(D,V) \models \nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}},\Pi,\Sigma)$. Suppose for contradiction that $B \cap \bigcup \Sigma \neq \varnothing$, then there must be some $S \in \Sigma$ with $B \cap S \neq \varnothing$. Because $(D,V) \models \nabla^B_{\mathtt{ME}^\infty}(\overline{\mathbf{T}},\Pi,\Sigma)$ we have, in particular, that $(D,V) \models \exists^\infty y.\tau^B_S(x)$ and hence $V(b)$ must be infinite, for any $b \in B \cap S$, which is absurd. It follows that $B \cap \bigcup \Sigma = \varnothing$, but then we trivially conclude that $(D,V) \models \varphi^\ominus$ because the disjunct remains unchanged.

$\boxed{\Leftarrow}$ Let $(D,V) \models \varphi^\ominus$. The only difference between φ and $\varphi^\ominus$ is that some disjuncts may have been replaced by $\perp$. Therefore this direction is trivial. QED

Putting together the above lemmas we obtain Theorem 5.9. Moreover, a careful analysis of the translation gives us the following corollary, providing normal forms for the continuous fragment of $\mathtt{ME}^\infty$.

Corollary 5.12 *For any sentence $\varphi \in \mathtt{ME}^\infty(A)$, the following hold.*

1. *The formula φ is continuous in $B \subseteq A$ iff φ is equivalent to a formula, effectively obtainable from φ, which is a disjunction of formulas $\nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}},\Pi,\Sigma)$ where $\overline{\mathbf{T}},\Sigma$ and Π are such that $\Sigma \subseteq \Pi \subseteq \overline{\mathbf{T}}$ and $B \cap \bigcup \Sigma = \varnothing$.*
2. *If φ is positive (i.e., $\varphi \in \mathtt{ME}^{\infty+}(A)$) then φ is continuous in $B \subseteq A$ iff it is equivalent to a formula, effectively obtainable from φ, which is a disjunction of formulas $\bigvee \nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}},\Pi,\Sigma)$, where $\overline{\mathbf{T}},\Sigma$ and Π are such that $\Sigma \subseteq \Pi \subseteq \overline{\mathbf{T}}$ and $B \cap \bigcup \Sigma = \varnothing$.*

Proof.

► `give some proof details?`

QED

5.3 Dual fragments

▶ `Shouldn't this part move to the other paper?`

In this subsection we give syntactic characterizations of the co-continuous fragment of several one-step logics. This notion is dual to continuity.

We are now ready to give the syntactic definition of the dual fragments for the one-step logics into consideration.

Definition 5.13 The fragment $\mathtt{Cocon}_B(\mathtt{ME}^\infty(A))$ is given by the sentences generated by: YV: incorrect

$$\varphi ::= \psi \mid b(x) \mid \forall x.\varphi \mid \forall^\infty x.\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi$$

where $b \in B$ and $\psi \in \mathtt{ME}^\infty(A \setminus B)$. Observe that the equality is included in ψ. The fragment $\mathtt{Cocon}_B(\mathtt{M}(A))$ is defined as $\mathtt{Cocon}_B(\mathtt{ME}^\infty(A))$ but without the clause for $\forall^\infty$ and with $\psi \in \mathtt{M}(A \setminus B)$. $\lhd$

The following proposition states that the above fragments are actually the duals of the fragments defined earlier in this chapter.

Proposition 5.14 *The following hold:*

$$\mathtt{Cocon}_B(\mathtt{ME}^\infty(A)) = \{\varphi \mid \varphi^\delta \in \mathtt{Con}_B(\mathtt{ME}^\infty(A))\}$$
$$\mathtt{Cocon}_B(\mathtt{M}(A)) = \{\varphi \mid \varphi^\delta \in \mathtt{Con}_B(\mathtt{M}(A))\}.$$

Proof. Easily proved by induction. QED

As a corollary, we get a characterisation for co-continuity.

Corollary 5.15 *Let* $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}^\infty\}$. *A formula* $\varphi \in \mathtt{L}(A)$ *is co-continuous in* $a \in A$ *if and only if it is equivalent to some* $\varphi' \in \mathtt{Cocon}_a(\mathtt{L})(A)$.

Proof. This is a onsequence of Proposition 5.14 and 2.11. QED

6 Submodels and quotients

There are various natural notions of morphism between monadic models; the one that we will be interested here is that of a (strong) homomorphism.

Definition 6.1 Let $\mathbb{D} = (D, V)$ and $\mathbb{D}' = (D', V')$ be two monadic models. A map $f : D \to D'$ is a *homomorphism* from $\mathbb{D}$ to $\mathbb{D}'$, notation: $f : \mathbb{D} \to \mathbb{D}'$, if we have $d \in V(a)$ iff $f(d) \in V'(a)$, for all $a \in A$ and $d \in D$. $\lhd$

In this section we will be interested in the sentences of $\mathtt{M}, \mathtt{ME}$ and $\mathtt{ME}^{\infty}$ that are preserved under taking submodels and the ones that are invariant under quotients.

Definition 6.2 Let $\mathbb{D} = (D, V)$ and $\mathbb{D}' = (D', V')$ be two monadic models. We call $\mathbb{D}$ a *submodel* of $\mathbb{D}'$ if $D \subseteq D'$ and the inclusion map $\iota_{DD'} : D \hookrightarrow D'$ is a homomorphism, and we say that $\mathbb{D}'$ is a *quotient* of $\mathbb{D}$ if there is a surjective homomorphism $f : \mathbb{D} \to \mathbb{D}'$.

Now let φ be an $\mathtt{L}$-sentence, where $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^{\infty}\}$. We say that φ is *preserved under taking submodels* if $\mathbb{D} \models \varphi$ implies $\mathbb{D}' \models \varphi$, whenever $\mathbb{D}'$ is a submodel of $\mathbb{D}$. Similarly, φ is *invariant under taking quotients* if we have $\mathbb{D} \models \varphi$ iff $\mathbb{D}' \models \varphi$, whenever $\mathbb{D}'$ is a quotient of $\mathbb{D}$. $\lhd$

The first of these properties (preservation under taking submodels) is well known from classical model theory — it is for instance the topic of the Łos-Tarski Theorem. When it comes to quotients, in model theory one is usually more interested in the formulas that are *preserved* under surjective homomorphisms (and the definition of homomorphism may also differ from ours): for instance, this is the property that is characterized by Lyndon's Theorem. Our preference for the notion of *invariance* under quotients stems from the fact that the property of invariance under quotients plays a key role in characterizing the *bisimulation-invariant fragments* of various monadic second-order logics, as is explained in our companion paper [**?**].

6.1 Preservation under submodels

In this subsection we characterize the fragments of our predicate logics consisting of the sentences that are preserved under taking submodels. That is, the main result of this subsection is a Łos-Tarksi Theorem for $\mathtt{ME}^{\infty}$.

Definition 6.3 The *universal fragment* of the set $\mathtt{ME}^{\infty}(A)$ is the collection $\mathtt{Univ}(\mathtt{ME}^{\infty}(A))$ of formulas given by the following grammar:

$$\varphi ::= \top \mid \bot \mid a(x) \mid \neg a(x) \mid x \approx y \mid x \not\approx y \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \forall x.\varphi \mid \forall^{\infty} x.\varphi$$

where $x, y \in \mathsf{iVar}$ and $a \in A$. The universal fragment $\mathtt{Univ}(\mathtt{ME}(A))$ is obtained by deleting the clause for $\forall^{\infty}$ from this grammar, and we obtain the universal fragment $\mathtt{Univ}(\mathtt{M}(A))$ by further deleting both clauses involving the equality symbol. $\lhd$

Theorem 6.4 *Let φ be a sentence of the monadic logic $\mathtt{L}(A)$, where $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^{\infty}\}$. Then φ is preserved under taking submodels if and only if there is an equivalent formula $\varphi^{\otimes} \in \mathtt{Univ}(\mathtt{L}(A))$. Furthermore, it is decidable whether a formula $\varphi \in \mathtt{L}(A)$ has this property or not.*

We first show the 'easy' side of the theorem.

Lemma 6.5 *Let $\varphi \in \mathtt{ME}^\infty(A)$ be a universal sentence. Then φ is preserved under taking submodels.*

Proof. Let (D', V') be a submodel of the monadic model (D, V). By induction on the complexity of a formula $\varphi \in \mathtt{Univ}(\mathtt{ME}^\infty(A))$ we will show that for any assignment $g : \mathsf{iVar} \to D'$ we have

$$(D, V), g \models \varphi \text{ implies } (D', V') \models \varphi.$$

We will only consider the inductive step of the proof where φ is of the form $\forall^\infty x.\psi$. Define $X_{D,V} := \{d \in D \mid (D, V), g[x \mapsto d] \models \psi\}$, and similarly, $X_{D',V'} := \{d \in D' \mid (D', V'), g[x \mapsto d] \models \psi\}$. By the inductive hypothesis we have that $X_{D,V} \cap D' \subseteq X_{D',V'}$, implying that $D' \setminus X_{D',V'} \subseteq D \setminus X_{D,V}$. But from this we immediately obtain that

$$|D \setminus X_{D,V}| < \omega \text{ implies } |D' \setminus X_{D',V'}| < \omega,$$

which means that $(D, V), g \models \varphi$ implies $(D', V'), g \models \varphi$, as required. QED

For the 'hard' side of the theorem we need the following lemma.

Lemma 6.6 *For any monadic logic $\mathtt{L} \in \{\mathtt{M}, \mathtt{ME}, \mathtt{ME}^\infty\}$ there is a translation $(-)^\otimes : \mathtt{L}(A) \to \mathtt{Univ}(\mathtt{L}(A))$ such that a formula $\varphi \in \mathtt{L}(A)$ is preserved under taking submodels if and only if $\varphi \equiv \varphi^\otimes$.*

Proof. We only consider the case where $\mathtt{L} = \mathtt{ME}^\infty$, leaving the other cases to the reader. For simple basic formulas, the translation $(-)^\otimes$ is given as follows:

$$(\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma))^\otimes := \forall z \bigvee_{S \in \overline{\mathbf{T}} \cup \Pi \cup \Sigma} \tau_S(z) \wedge \forall^\infty z \bigvee_{S \in \Sigma} \tau_S(z).$$

This translation is then extended to the full language $\mathtt{ME}$ in the usual way.

It is easy to see that $\varphi^\otimes \in \mathtt{Univ}(\mathtt{ME}^\infty(A))$, for every sentence $\varphi \in \mathtt{ME}^\infty(A)$; but then it is immediate by Lemma 6.5 that φ is preserved under taking submodels if $\varphi \equiv \varphi^\otimes$.

For the left-to-right direction of the lemma, assume that φ is preserved under taking submodels. It is easy to see that φ implies $\varphi^\otimes$, so we focus on proving the opposite. That is, we suppose that $(D, V) \models \varphi^\otimes$, and aim to show that $(D, V) \models \varphi$.

By Corollary 5.12 we may assume without loss of generality that φ is a disjunction of formulas of the form $\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, where $\Sigma \subseteq \Pi \subseteq \overline{\mathbf{T}}$. It follows that (D, V) satisfies some such disjunct $\forall z \bigvee_{S \in \overline{\mathbf{T}} \cup \Pi \cup \Sigma} \tau_S(z) \wedge \forall^\infty z \bigvee_{S \in \Sigma} \tau_S(z)$ of $\left(\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)\right)^\otimes$. Expand D with finitely many elements $\overline{\mathbf{d}}$, in one-one correspondence with $\overline{\mathbf{T}}$, and ensure that the type of each d_i is T_i. In addition, add, for each $S \in \Sigma$, infinitely many elements $\{e_n^S \mid n \in \omega\}$, each of type S. Call the resulting monadic model $\mathbb{D}' = (D', V')$.

This construction is tailored to ensure that $(D', V') \models \nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, and so we obtain $(D', V') \models \varphi$. But obviously, $\mathbb{D}$ is a submodel of $\mathbb{D}'$, whence $(D, V) \models \varphi$ by our assumption on φ. QED

The proof of Theorem 6.4 is an immediate consequence of the Lemmas 6.5 and 6.6.

6.2 Invariance under quotients

The following theorem states that monadic first-order logic *without* equality (M) provides the quotient-invariant fragment of both monadic first-order logic with equality (ME), and of infinite-monadic predicate logic ($\mathtt{ME}^\infty$).

Theorem 6.7 *Let φ be a sentence of the monadic logic $\mathtt{L}(A)$, where $\mathtt{L} \in \{\mathtt{ME}, \mathtt{ME}^\infty\}$. Then φ is preserved under taking submodels if and only if there is a equivalent sentence in* M. *Furthermore, it is decidable whether a formula $\varphi \in \mathtt{L}(A)$ has this property or not.*

We first state the 'easy' part of the Theorem. Note that in fact, we have already been using this observation in earlier parts of the paper.

Proposition 6.8 *Every sentence in* M *is invariant under taking quotients.*

Proof. Let $F : D \to D'$ provide a surjective homomorphism between the models (D, V) and (D', V'), and observe that for any assignment $g : \mathsf{iVar} \to D$ on D, the composition $f \circ g : \mathsf{iVar} \to D'$ is an assignment on D'.

In order to prove the proposition one may show that, for an arbitrary M-formula φ and an arbitrary assignment $g : \mathsf{iVar} \to D$, we have

$$(D, V), g \models \varphi \text{ iff } (D', V', f \circ g \models \varphi. \tag{5}$$

We leave the proof of (5), which proceeds by a straightforward induction on the complexity of φ, as an exercise to the reader. QED

To prove the 'hard' part of Theorem 6.7, by the Theorems 3.9 and 3.15 it suffices to provide a translation for sentences in basic form.

Definition 6.9 For ME-sentences in basic form we first define

$$\Big(\nabla_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)\Big)^\circ := \bigwedge_i \exists x_i.\tau_{T_i}(x_i) \land \forall x. \bigvee_{S\in\Pi} \tau_S(x),$$

and then set $(\bigvee_i \alpha_i)^\circ := \bigvee \alpha_i^\circ$. This translation is extended to the collection of all ME-sentences by defining $\varphi^\circ := (\varphi^*)^\circ$.

For $\mathtt{ME}^\infty$-sentences in basic form we start with defining

$$\Big(\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)\Big)^\bullet := \bigwedge_i \exists x_i.\tau_{T_i}(x_i) \land \forall x. \bigvee_{S\in\Sigma} \tau_S(x),$$

and then extend this definition to the set of all $\mathtt{ME}^\infty$-sentences in the analogous way. $\lhd$

Note that the two translations may give *different* translations for ME-sentences. Also observe that the Π 'disappears' in the translation of the formula $\nabla_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$.

The key property of these translations is the following.

Proposition 6.10 *1. For every one-step model (D, V) and every $\varphi \in \texttt{ME}(A)$ we have*

$$(D, V) \models \varphi^\circ \textit{ iff } (D \times \omega, V_\pi) \models \varphi. \tag{6}$$

2. For every one-step model (D, V) and every $\varphi \in \texttt{ME}^\infty(A)$ we have

$$(D, V) \models \varphi^\bullet \textit{ iff } (D \times \omega, V_\pi) \models \varphi. \tag{7}$$

Here V_π is the induced valuation given by $V_\pi(a) := \{(d, k) \mid d \in V(a), k \in \omega\}$.

Proof. We only prove the claim for $\texttt{ME}^\infty$ (i.e., the second part of the proposition), the case for $\texttt{ME}$ being similar. Clearly it suffices to prove (7) for formulas of the form $\alpha = \nabla_{\texttt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$.

$\boxed{\Rightarrow}$ Assume $(D, V) \models \alpha^\bullet$, we will show that $(D \times \omega, V_\pi) \models \nabla_{\texttt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. Let d_i be such that $V^\flat(d_i) = T_i$ in (D, V). It is clear that the (d_i, i) provide *distinct* elements, with each (d_i, i) satisfying τ_{T_i} in $(D \times \omega, V_\pi)$ and therefore the first-order existential part of α is satisfied. With a similar but easier argument it is straightforward that the $\exists^\infty$-part of α is also satisfied. For the universal parts of $\nabla_{\texttt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$ it is enough to observe that, because of the universal part of $\alpha^\bullet$, *every* $d \in D$ realizes a type in Σ. By construction, the same applies to $(D \times \omega, V_\pi)$, therefore this takes care of both universal quantifiers.

$\boxed{\Leftarrow}$ Assuming that $(D \times \omega, V_\pi) \models \nabla_{\texttt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, we will show that $(D, V) \models \alpha^\bullet$. The existential part of $\alpha^\bullet$ is trivial. For the universal part we have to show that every element of D realizes a type in Σ. Suppose not, and let $d \in D$ be such that $\neg\tau_S(d)$ for all $S \in \Sigma$. Then we have $(D \times \omega, V_\pi) \not\models \tau_S(d, k)$ for all k. That is, there are infinitely many elements not realizing any type in Σ. Hence we have $(D \times \omega, V_\pi) \not\models \forall^\infty y. \bigvee_{S \in \Sigma} \tau_S(y)$. Absurd, because this formula is a conjunct of $\nabla_{\texttt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$. QED

We will now show how the theorem follows from this.

Proof of Theorem 6.7. Let φ be a sentence of $\texttt{ME}^\infty$ (we only cover the case of $\texttt{L} = \texttt{ME}^\infty$, the case for $\texttt{L} = \texttt{ME}$ is similar). We will show that

$$\varphi \equiv \varphi^\bullet \text{ iff } \varphi \text{ is invariant under taking quotients.} \tag{8}$$

The direction from right to left is immediate by Proposition 6.8. For the other direction it suffices to observe that any model (D, V) is a quotient of its 'ω-product' $(D \times \omega, V_\pi)$, and to reason as follows:

$$\begin{aligned} (D, V) \models \varphi \text{ iff } & (D \times \omega, V_\pi) \models \varphi && \text{(assumption on } \varphi) \\ \text{iff } & (D, V) \models \varphi^\bullet && \text{(Proposition 6.10)} \end{aligned}$$

Finally, it is immediate by the decidability of $\texttt{ME}^\infty$ and (8) that it is decidable whether a given $\texttt{ME}^\infty$-sentence φ is invariant under taking quotients or not. QED

In our companion paper on automata, we need versions of these results for the monotone and the continuous fragment. For this purpose we define some slight modifications of the translations $(\cdot)^\circ$ and $(\cdot)^\bullet$ which map positive and syntactically continuous formulas to respectively positive and syntactically continuous formulas.

Theorem 6.11 *There are effective translations* $(\cdot)^\circ : \mathtt{ME}^+(A) \to \mathtt{M}^+(A)$ *and* $(\cdot)^\bullet : \mathtt{ME}^{\infty+}(A) \to \mathtt{M}^+(A)$ *such that* $\varphi \equiv \varphi^\circ$ *(respectively,* $\varphi \equiv \varphi^\bullet$*) iff* φ *is invariant under quotients. Moreover, we may assume that* $(\cdot)^\bullet : \mathtt{Con}_B(\mathtt{ME}^{\infty+}(A)) \to \mathtt{Con}_B(\mathtt{M}^+(A))$*, for any* $B \subseteq A$*.*

Definition 6.12 We define translations $(\cdot)^\circ : \mathtt{ME}^+(A) \to \mathtt{M}^+(A)$ and $(\cdot)^\bullet : \mathtt{ME}^{\infty+}(A) \to \mathtt{M}^+(A)$ as follows. For $\mathtt{ME}^+, \mathtt{ME}^{\infty+}$-sentences in simple basic form we define

$$\begin{array}{lcl} \left(\nabla^+_{\mathtt{ME}}(\overline{\mathbf{T}}, \Pi)\right)^\circ & := & \bigwedge_i \exists x_i.\tau^+_{T_i}(x_i) \wedge \forall x. \bigvee_{S \in \Pi} \tau^+_S(x), \\ \left(\nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)\right)^\bullet & := & \bigwedge_i \exists x_i.\tau^+_{T_i}(x_i) \wedge \forall x. \bigvee_{S \in \Sigma} \tau^+_S(x), \end{array}$$

and then we use, respectively, the Corollaries 4.12 and 4.15 to extend these translations to the full positive fragments $\mathtt{ME}^+$ and $\mathtt{ME}^{\infty+}$, as we did in Definition 6.9 for the full language. $\lhd$

Proof of Theorem 6.11. We leave it as an exercise for the reader to prove the analogon of Proposition 6.10 for these translations, and to show how the first statements of Theorem 6.11 follows from this.

Finally, to see why we may assume that $(\cdot)^\bullet$ restricts to a map from the syntactically B-continuous fragment of $\mathtt{ME}^\infty(A)$ to the syntactically B-continuous fragment of $\mathtt{M}(A)$, assume that $\varphi \in \mathtt{ME}^\infty(A)$ is continuous in $B \subseteq A$. By Corollary 5.12 we may assume that φ is a disjunction of formulas of the form $\nabla^+_{\mathtt{ME}^\infty}(\overline{\mathbf{T}}, \Pi, \Sigma)$, where $B \cap \bigcup \Sigma = \varnothing$. This implies that in the formula $\varphi^\bullet$ no predicate symbol $b \in B$ occurs in the scope of a universal quantifier, and so $\varphi^\bullet$ is syntactically continuous in B indeed. QED

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