DECISION PROBLEMS FOR GENERALIZED QUANTIFIERS - A SURVEY

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§ 1. Introduction.

In this survey by a generalized quantifier is meant a cardinality quantifier of the kind first introduced by Mostowski [1957]. We adopt the following notation. If φ is a formula with one free variable, then $\varphi^{\underline{\underline{A}}}$ denotes the subset of the universe of $\underline{\underline{A}}$ which φ defines in $\underline{\underline{A}}$. If φ has two free variables, then $\varphi^{\underline{\underline{A}}}$ denotes the subset of $A \times A$ defined by φ in $\underline{\underline{A}}$, and so on. Using this notation the existential quantifier $\exists$ can be defined by the satisfaction clause

$$\underline{\underline{A}} \models (\exists v)\, \varphi \iff \mathrm{card}(\varphi^{\underline{\underline{A}}}) \geq 1 .$$

For each ordinal α there is a quantifier Q_α which behaves syntactically like the existential quantifier and whose meaning is given by the satisfaction clause

$$\underline{\underline{A}} \models (Q_\alpha v)\, \varphi \iff \mathrm{card}(\varphi^{\underline{\underline{A}}}) \geq \aleph_\alpha .$$

If L is a (countable) first order language we denote by L_α the language obtained when we add the new quantifier symbol Q_α. In addition to these quantifiers there are three other cardinality quantifiers which merit attention in the present context. First there is the <u>Chang</u> or <u>equi</u>-<u>cardinal</u> quantifier, Q_C. This also binds one variable and is defined by

$$\underline{\underline{A}} \models (Q_C v)\ \varphi \iff \mathrm{card}(\varphi^{\underline{\underline{A}}}) = \mathrm{card}(\underline{\underline{A}})\ .$$

For each positive integer n and each ordinal α, the Malitz quantifier Q^n_α is one which binds n variables and whose interpretation is given by

$$\underline{\underline{A}} \models (Q^n_\alpha v_1 \ldots v_n)\varphi \iff \text{for some } X \subseteq A, \text{ with } \mathrm{card}(X) \geq \aleph_\alpha\ ,$$
$$(X^n)' \subseteq \varphi^{\underline{\underline{A}}}\ ,$$

where $(X^n)'$ denotes the set of all n-tuples of distinct elements of X. Thus when $n = 1$, Q^1_α is just the quantifier Q_α already mentioned. Finally there is the Hartig quantifier H. This binds one variable in each of a pair of formulas and its interpretation is given by

$$\underline{\underline{A}} \models (Hv)(\varphi;\psi) \iff \mathrm{card}(\varphi^{\underline{\underline{A}}}) = \mathrm{card}(\psi^{\underline{\underline{A}}})\ .$$

In terms of the Hartig quantifier we can define both the Chang quantifier and the Q_0 quantifier (which says "there exist infinitely many). Thus $(Q_C v)\varphi$ is equivalent to $(Hv)(\varphi;\ v = v)$ and $(Q_0 v)\ \varphi(v)$ is equivalent to $(\exists w)(\varphi(w) \wedge (Hv)(\varphi(v);\ \varphi(v) \wedge v \neq w))$.

In a structure of cardinality $< \aleph_\alpha$ the Q_α quantifier acts vacuously and therefore when we are dealing with this quantifier it is technically convenient to assume that all the structures we consider have cardinality at least $\aleph_\alpha$. Similarly when dealing with the Chang quantifier it is convenient to assume that all structures are infinite.

For each class K of structures we let $Th(K)$ be the first order theory of K and $Th_\alpha(K)$ be the theory of K in the language L_α. Thus

$$Th_\alpha(K) = \{\sigma : \sigma \text{ is a sentence of } L_\alpha \text{ and for all } \underline{\underline{A}} \in K\ ,$$
$$\text{with } \mathrm{card}(\underline{\underline{A}}) \geq \aleph_\alpha\ ,\ \underline{\underline{A}} \models \sigma\ \}\ .$$

Similarly $Th_C(K)$ denotes the theory of K in the language L_C with the Chang quantifier.

We shall be concerned with problems about the decidability of $Th_\alpha(K)$ for various classes K and ordinals α. Clearly $Th_\alpha(K)$ can only be decidable if $Th(K)$ is decidable. It is easy to provide artificial examples of classes K such that $Th(K)$ is decidable while $Th_\alpha(K)$ is not decidable for certain (or all) α. So we only want to consider classes K which are "natural" in some sense. Especially we consider the case where K is a first order elementary class. If K is the class of all models of the set Δ of first order sentences, we denote $Th_\alpha(K)$ by $Th_\alpha(\Delta)$ and $Th(K)$ by $Th(\Delta)$. Herre and Wolter [1975] have given an example of a theory Δ such that $Th_0(\Delta)$ is decidable while $Th_1(\Delta)$ is undecidable. It is not yet known whether an example of the converse situation exists. Herre and Wolter's example exploits the fact that while $Th(\Delta)$ is decidable, if a new unary predicate is added to the language in this extended language in this language the theory of Δ is undecidable.

§ 2. Basic results and methods

In this section we list some basic results and techniques which underlie the decidability results mentioned below. In connection with decision problems, and for other reasons, it is interesting to know, for fixed K, how $Th_\alpha(K)$ varies as α varies. The key method here is the reduction technique due to Fuhrken [1964,1965] (see also chapter 13 of Bell and Slomson [1971]). This shows how problems about the existence of models of sentences of L_α can be reduced to the existence of models of first order sentences with special properties, i.e. cardinal-like and two-cardinal models. Known results about these models then enable us to obtain the following comparison theorems. (Note that the assertion $Th_\alpha(\Delta) \subseteq Th_\beta(\Delta)$ must be interpreted as meaning that if in $Th_\alpha(\Delta)$ each occurrence of Q_α is replaced by an occurrence of Q_β, then we obtain a subset of $Th_\beta(\Delta)$. We adopt this convention throughout.)

<u>Comparison Theorems</u>

(1) For all α, $Th_\alpha(\Delta) \subseteq Th_0(\Delta)$.

(2) For all $\alpha > 0$ with $\aleph_\alpha$ regular, $Th_1(\Delta) \subseteq Th_\alpha(\Delta)$.

(3) (G.C.H.) For all α with $\aleph_\alpha$ regular $Th_{\alpha+1}(\Delta) \subseteq Th_1(\Delta)$.

(4) (V = L) For all α, $Th_{\alpha+1}(\Delta) \subseteq Th_1(\Delta)$.

(5) For all α,β with $\aleph_\alpha$ a strong limit cardinal and $\aleph_\beta$ singular, $Th_\alpha(\Delta) \subseteq Th_\beta(\Delta)$.

These results depend on theorems of MacDowell and Specker [1961], Morley and Vaught [1962], Chang [1965], Jensen (unpublished, see Chang and Keisler [1973]), and Keisler [1968], respectively. It follows from them that if we make some strong assumption such as $V = L$ + "there are no inaccessible cardinals" then for a given first order theory Δ there are only three distinct theories $Th_\alpha(\Delta)$ at most, namely $Th_0(\Delta)$, $Th_1(\Delta)$ and $Th_\omega(\Delta)$. Although in general these three theories are distinct in some special cases it is known these are equal. Some of these cases are noted below.

Of the three Q_α quantifiers Q_0, Q_1 and Q_ω that give rise to these three theories Q_0 is somewhat different in character from the other two. With this quantifier we can express a categorical recursive set of exioms for the standard model of arithmetic. It therefore follows from Gödel's Incompleteness theorem that L_0 is not axiomatizable. On the other hand the powerful theorem of Rabin [1969] on the decidability of the second order theory of two successor functions enables us to obtain the decidability of many theries in the language L_0 . Examples are given below. In contrast to the non-axiomatizability of L_0 we have the following:

<u>Axiomatizability results for</u> L_1 <u>and</u> L_ω .

(1) If Δ is recursively enumerable then so is $Th_1(\Delta)$.

(2) If Δ is recursively enumerable then so is $Th_\omega(\Delta)$.

(1) is an observation due to Vaught [1964]. An explicit axiomatization for L_1 has been given by Keisler [1970], and (2) is also due to Keisler [1968].

It follows that in cases where Δ is recursively enumerable to prove that $Th_1(\Delta)$ (or $Th_\omega(\Delta)$) is decidable it is sufficient to show that the sentences of L_1 (or L_ω) consistent with Δ form a recursively enumerable set.

<u>Ehrenfeucht's Game</u>

Ehrenfeucht [1961], extending the work of Fraissé [1954], showed that elementary equivalence of structures with respect to a first order language can be characterized in terms of a game played with these structures. Lipner [1970] and Brown [1971] independently showed how this game could be extended to cope with elementary equivalence in the languages L_α . Vinner [1972], also independently, gave a similar characterization but expressed in terms of partial isomorphisms and observed that it could be used to compare the L_α theory of one structure with the L_β theory of another. For an account of the game see Slomson [1972].

Badger [1975] has shown that this game can be generalized further to deal with the Malitz quantifiers.

Apart from the use of Ehrenfeucht's game the chief technique used in proving the results listed below is that of elimination of quantifiers.

§ 3. Decidability results.

(a) Monadic predicates

The theory of monadic predicates without equality was shown by Mostowski [1957] to be decidable in each of the languages L_α , and to be the same for each α . In Slomson [1968] this is extended to a language with equality. The argument here is given in terms of the Chang quantifier but is easily seen to work also for each language L_α . Vinner [1972] gives a more direct proof. Slomson [1968] also shows, using a theorem of Löb [1967] that the theory of monadic predicates, without equality, but with one unary function, is decidable in the language with the Chang quantifier.

(b) One equivalence relation

Rabin [1969] proved that 2S2 , the second order theory of two successor functions, is decidable. Vinner [1972] showed that the L_0 theory of one equivalence relation is interpretable in 2S2 and hence is decidable. He also proved that for all α the L_α theory of one equivalence relation is the same as the L_0 theory, and hence is decidable. Since the theory of two equivalence relations is undecidable these results cannot be improved.

(c) Trees and one unary function

A tree is a relational structure with a single symmetric binary relation and in which there are no circuits. The theory of trees in interpretable in the theory 2S2 and so it follows that the L_0 theory of trees is decidable. The same applies to the L_0 theory of one unary function. Vinner [1972] observed that the L_1 theory of one unary function is not the same as the L_0 theory and he proved that for all $\alpha > 0$, with $\aleph_\alpha$ regular, the L_α theory is the same as the L_1 theory. Herre [1975] proved that for $\alpha > 0$ and $\aleph_\alpha$ regular the L_α theories of trees and one unary function are decidable.

(d) Abelian groups

Baudisch [1975] has proved that the L_α theory of Abelian groups is the same for all α and is decidable. His method is to extend the basis for the theory of Abelian groups given by Szmielew [1955] in her proof of the decidability of the first order theory of Abelian groups.

(e) Arithmetic with + and <

In Wolter [1973] it is proved that the L_0 theory of the natural numbers with addition and the usual ordering is decidable, and in Wolter [1975] this is extended to the same theory in the language $L_{\alpha,\beta}$ which comes from L by adding the two quantifiers Q_α and Q_β, with $0 < \alpha,\beta$.

(f) p-adic numbers

Weese [1975] showed that a certain theory of p-adic number fields is decidable in the language L_α , for all α . The class of structures he considers is not "natural" in the sense mentioned above since the theory he works with includes some non-first-order axioms, for example, the axiom $(\forall x)[x \neq 0 \rightarrow ((Q_\alpha y)(x < y) \wedge (Q_\alpha y)(y < x))]$.

(g) Well-ordered sets

We identify each ordinal ξ with the well-ordered structure $(\xi,\in)$, and we denote the class of all ordinals by **On** . Of course **On** is not a first-order elementary class, but in a good sense it is a "natural" class of structures.

Lipner [1970] proved that for each ordinal ξ , and for all α with $\aleph_\alpha$ regular, $Th_\alpha(\xi)$ is decidable. From the decidability of the theory 2S2, Rabin [1969], it follows that $Th_0(On)$ is decidable. In Slomson [1972] it is proved that $Th_1(On)$ is decidable and a proof is also given of the result due to Vinner that for all $\alpha > 0$, $Th_\alpha(On) = Th_1(On)$, and hence is also decidable. These proofs use Ehrenfeucht's game. In his thesis Badger [1975] raises the question as to whether these results can be extended to the theory of ordinals in the language $L_\alpha^{<\omega}$, which contains all the Malitz quantifiers Q_α^n , for $n < \omega$ .It is not too difficult to see that the techniques of Slomson [1972] can be extended to give a positive answer to this question, and indeed to prove that if two ordinals are elementarily equivalent with respect to the language L_α , then they are also elementarily equivalent with respect to the language $L_\alpha^{<\omega}$.

Herre and Wolter [1975], using quantifier elimination arguments, show that the theory of well-ordered sets in the language with the two quantifiers Q_0 and Q_α is decidable. In contrast to these decidability results Weese [1975i] has proved that the theory of well-ordered sets in the language with the Hartig quantifier is undecidable.

In most of the examples above we also have the decidability of the corresponding theory in the language L_C with the Chang quantifier. This is because for any set of first order sentences Δ,

$$\mathrm{Th}_C(\Delta) = \bigcap_{\alpha \in \mathrm{On}} \mathrm{Th}_\alpha(\Delta) \quad .$$

§ 4. A Remark on Dense Linear Orderings

Perhaps the most notable omission from the list of theories given above is that of linear orderings. We let LO denote this theory. Again it follows from Rabin [1969] that $\mathrm{Th}_0(\mathrm{LO})$ is decidable, but the question as to the decidability of, for example, $\mathrm{Th}_1(\mathrm{LO})$ remains open. Rabin's method applies essentially to countable sets and so is not capable of immediate generalization to the language L_1. Similarly, the original proof of the decidability of the first order theory of linear orderings, due to Läuchli and Leonard [1966] makes essential use of Ramsey's theorem and so cannot be easily extended from the countable case to the uncountable case.

The difficulty of settling the decidability of $\mathrm{Th}_1(\mathrm{LO})$ is also seen if we look at the theory DLO of dense linear orderings without endpoints. As is well known, a famous theorem due to Cantor says that DLO is $\aleph_0$-categorical, hence by Vaught's test DLO is complete, and so being recursively axiomatizable it is decidable. In contrast the L_1 theory of dense linear orderings, i.e. $\mathrm{Th}_1(\mathrm{DLO})$ is not $\aleph_1$-categorical, but has $2^{\aleph_1}$ isomorphism types among its models of cardinal $\aleph_1$, and is far from complete, but has $2^{\aleph_0}$ complete extensions. Furthermore $\mathrm{Th}_1(\mathrm{LO})$ can be interpreted in $\mathrm{Th}_1(\mathrm{DLO})$. This can be seen as follows.

Let η be the order type of the rationals and let ϑ be the order type of a dense linear ordering without endpoints of cardinal $\aleph_1$ and with $\aleph_1$ points between any two distinct points (i.e. in case the G.C.H. holds ϑ is the order type of $(R,<)$ and otherwise of an L_1 elementary substructure of this structure of cardinal $\aleph_1$). Let λ be the order type $\eta + 1 + \vartheta$, and let $\varphi(x)$ be the formula

$$(\exists y)(y < x \wedge "(y,x) \models \sigma_1") \wedge (\exists y)(x < y \wedge "(x,y) \models \sigma_2") ,$$

where σ_1, σ_2 are finite axiomatizations of $Th_1(\eta)$ and $Th_1(\vartheta)$ respectively, and "$(y,x) \models \sigma_1$" , "$(x,y) \models \sigma_2$" denote their relativizations to the formulas $(y < v \wedge v < x)$ and $(x < v \wedge v < y)$ respectively. Then in an ordered set of order type λ the formula $\varphi(x)$ defines a unique element ("the 1 in the middle"). Further if γ is any order type, then in a structure of order type $\lambda\cdot\gamma$ $\varphi(x)$ defines a subset of order type γ . Also $\lambda\cdot\gamma$ is a dense linear order type without endpoints. It therefore follows that for any sentence σ of the language L_1 for linear orderings with the Q_1 quantifier,

$$LO \models \sigma \iff DLO \models \sigma^{(\varphi)} \wedge (\exists x)\, \varphi(x) ,$$

where $\sigma^{(\varphi)}$ denotes the relativization of σ to the formula $\varphi(x)$. Thus, in striking contrast to the first order case, in the language L_1 the decidability of the theory of dense linear orderings is no easier than of the theory of all linear orderings.

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