A Universal Construction for (Co)relations

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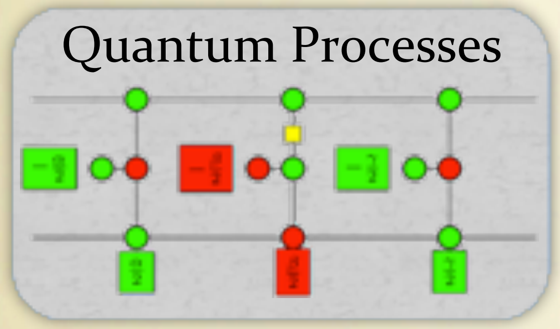
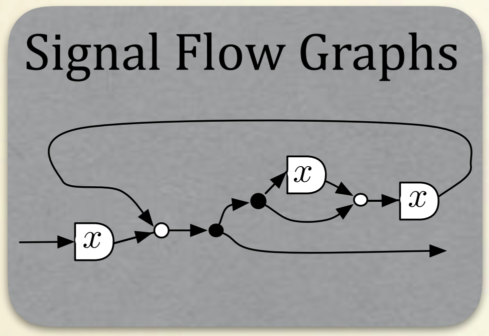
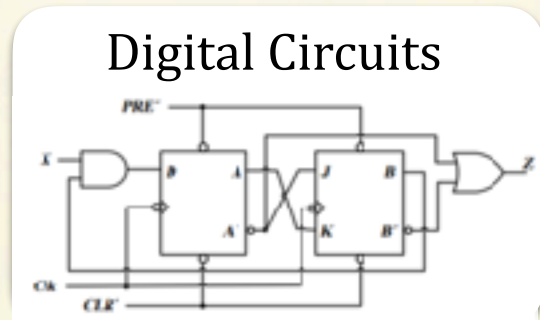
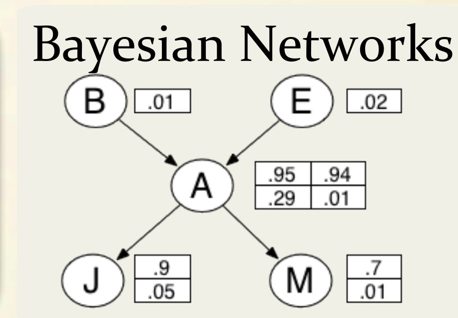
Context and Motivation

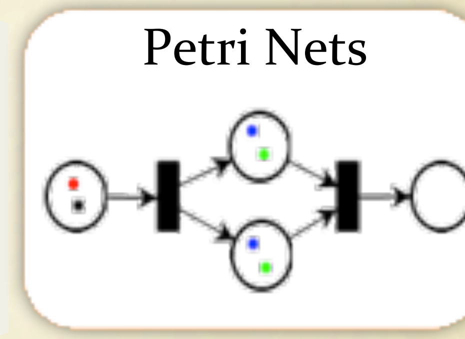
The algebraic approach to network diagrams







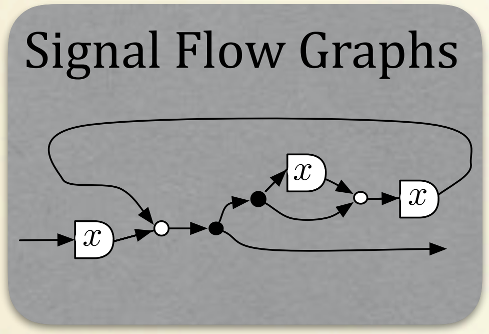




Diagrammatic languages studied using the compositional methods of programming language theory.

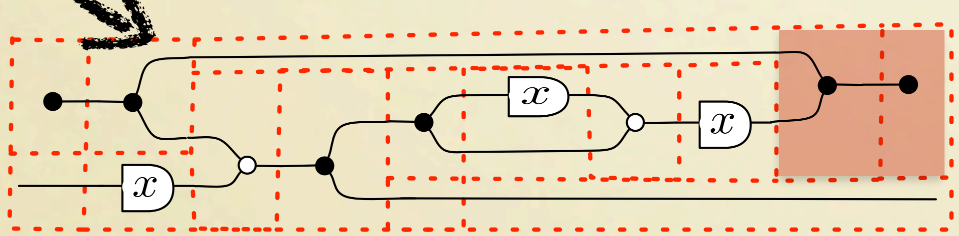
Syntax → **Semantics**

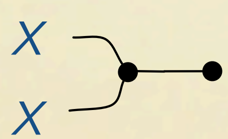
Relational Semantics



Props
(SMCs with set of objects isomorphic to $\mathbb{N}$)

SFG





LRel_k

Linear Relation
(subspace of $k \times k$)

field of streams

Bonchi, Sobocinski, Zanasi CONCUR'14

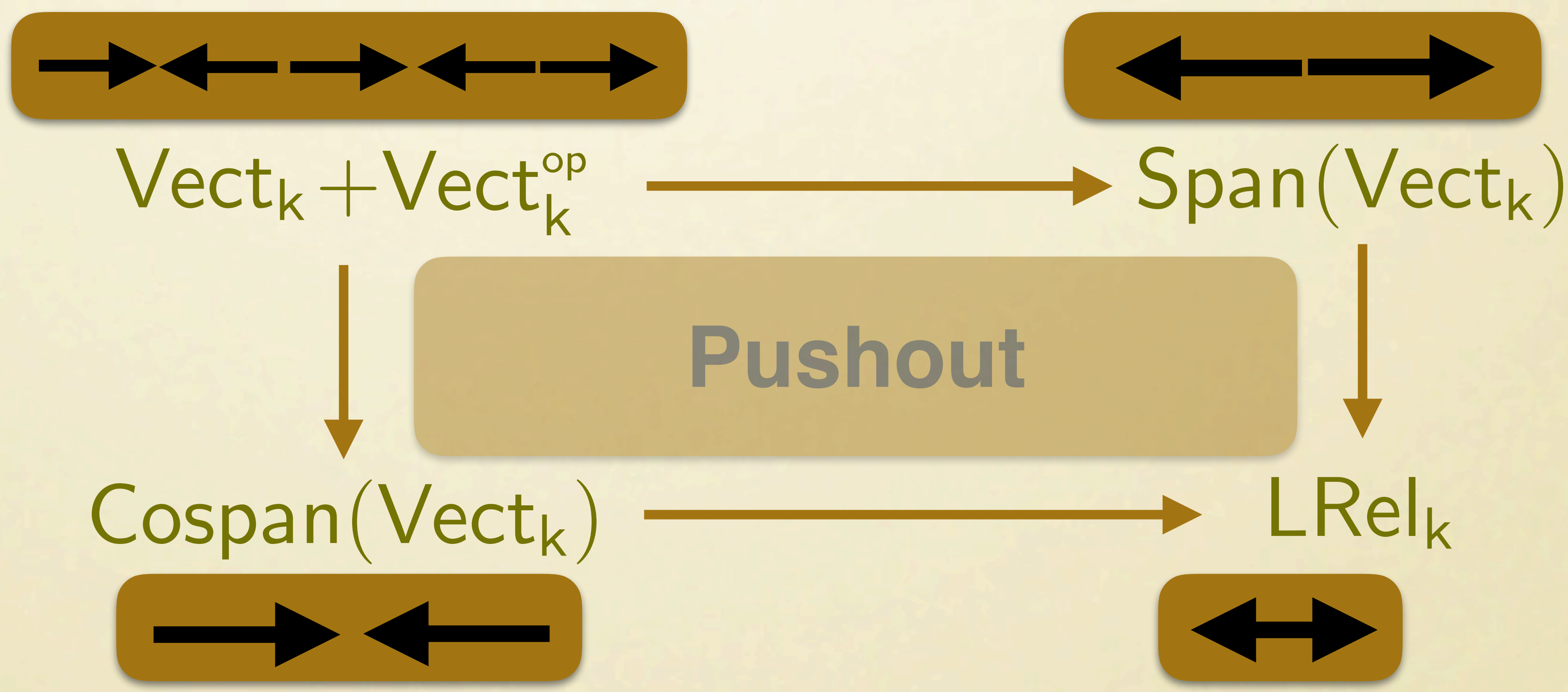
A Modular Recipe for Axioms

$$
\begin{tikzcd}
\mathsf{Hopf} + \mathsf{Hopf}^{\mathrm{op}} \arrow[rr] \arrow[d] \arrow[ddr, hook] \arrow[drr, phantom, "\textbf{Pushout}"] & & \mathsf{Hopf}^{\mathrm{op}} ; \mathsf{Hopf} \arrow[d] \arrow[ddr, hook] & \\
\mathsf{Hopf} ; \mathsf{Hopf}^{\mathrm{op}} \arrow[rr] \arrow[ddr, hook] & & \mathsf{SFG}/_{\equiv} \arrow[ddr, hook] & \\
& \mathsf{Vect}_{\mathsf{k}} + \mathsf{Vect}_{\mathsf{k}}^{\mathrm{op}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\textbf{Pushout}"] & & \mathsf{Span}(\mathsf{Vect}_{\mathsf{k}}) \arrow[d] \\
& \mathsf{Cospan}(\mathsf{Vect}_{\mathsf{k}}) \arrow[rr] & & \mathsf{LRel}_{\mathsf{k}}
\end{tikzcd}
$$

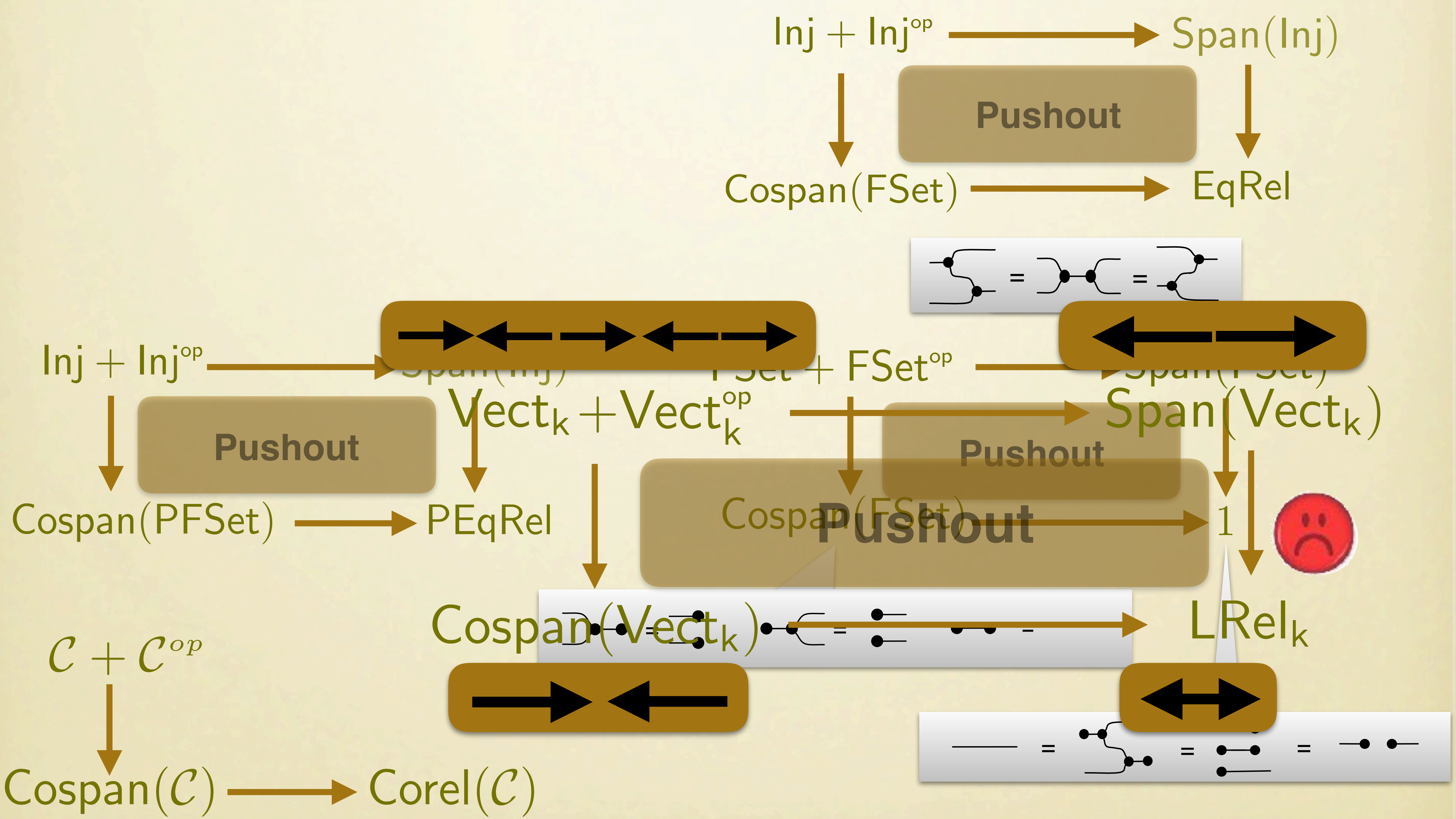
Soundness and Completeness

$$c \equiv d \Leftrightarrow [\![c]\!] = [\![d]\!]$$

An Ubiquitous Construction



An Ubiquitous Construction



The General Pattern

(Co)relations, categorically

Factorisation system $(\mathscr{E},\mathscr{M})$

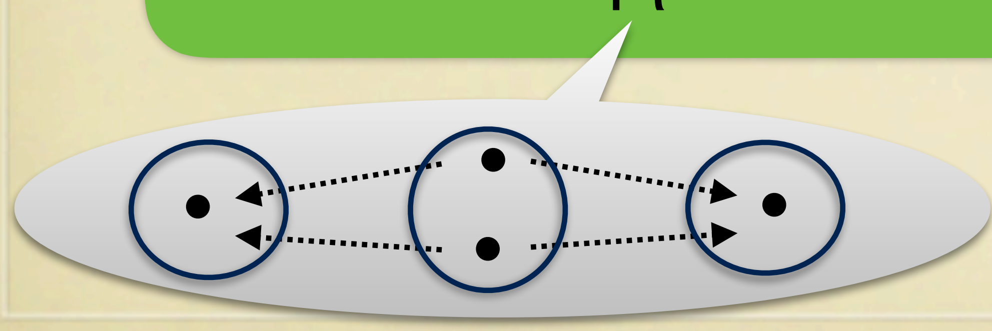
relation on X and Y

jointly-in-$\mathscr{M}$ span

$$X \longleftarrow R \longrightarrow Y$$

eq. class of spans

$$\begin{tikzcd} X & R \arrow[l] \arrow[r] & Y \\ & R' \arrow[ul] \arrow[u, "\in \mathscr{E}"'] \arrow[ur] & \end{tikzcd}$$



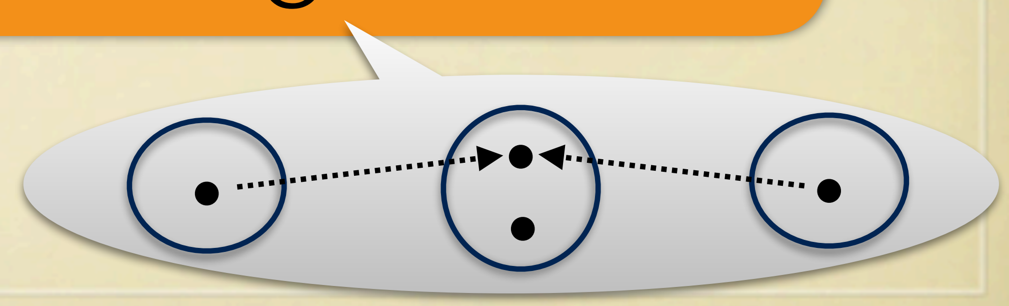
corelation on X and Y

jointly-in-$\mathscr{E}$ cospan

$$X \longrightarrow C \longleftarrow Y$$

eq. class of cospans

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & C \arrow[d, "\in \mathscr{M}"] & Y \arrow[l] \arrow[dl] \\ & C' & \end{tikzcd}$$



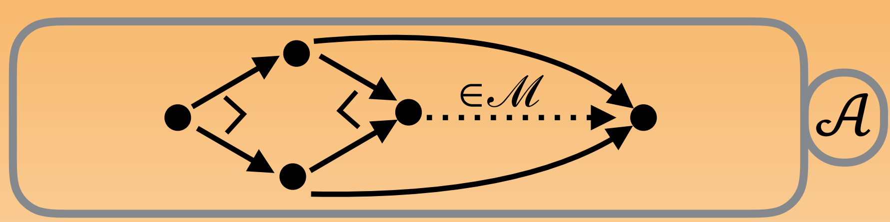
The Universal Characterisation

Assumptions

A prop $\mathcal{C}$ with pullbacks and pushouts

A fact. system $(\mathcal{E},\mathcal{M})$ s.t. $\mathcal{M} \subseteq Monos(\mathcal{C})$

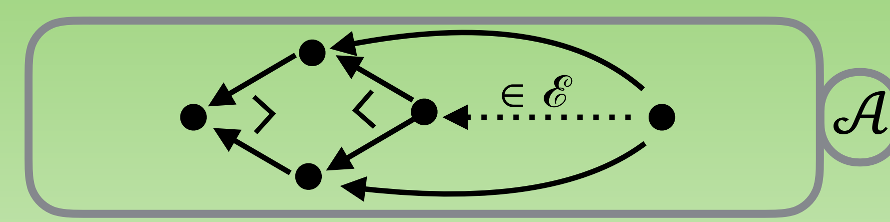
A sub-prop $\mathcal{A}$ such that $\mathcal{M} \subseteq \mathcal{A}$



Then

$$\begin{tikzcd} \mathcal{A} + \mathcal{A}^{op} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{Pushout}"] & \mathsf{Span}(\mathcal{A}) \arrow[d] \\ \mathsf{Cospan}(\mathcal{C}) \arrow[r] & \mathsf{Corel}(\mathcal{C}) \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{Inj} + \mathsf{Inj}^{op} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{Pushout}"] & \mathsf{Span}(\mathsf{Inj}) \arrow[d] \\ \mathsf{Cospan}(\mathsf{FSet}) \arrow[r] & \mathsf{EqRel} \end{tikzcd}$$



Then

$$\begin{tikzcd} \mathcal{A} + \mathcal{A}^{op} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{Pushout}"] & \mathsf{Cospan}(\mathcal{A}) \arrow[d] \\ \mathsf{Span}(\mathcal{C}) \arrow[r] & \mathsf{Rel}(\mathcal{C}) \end{tikzcd}$$

The Abelian Case

Assumptions

An abelian prop $\mathcal{C}$

(pick $\mathcal{C}$ itself as subcategory $\mathcal{A}$)

Then

$$\begin{tikzcd}
\mathcal{C}+\mathcal{C}^{op} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{Pushout}"] & \mathrm{Span}(\mathcal{C}) \arrow[d] \\
\mathrm{Cospan}(\mathcal{C}) \arrow[r] & \mathrm{Rel}(\mathcal{C}) \cong \mathrm{Corel}(\mathcal{C})
\end{tikzcd}$$

Then

$$\begin{tikzcd}
\mathcal{A}+\mathcal{A}^{op} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{Pushout}"] & \mathrm{Span}(\mathcal{A}) \arrow[d] \\
\mathrm{Cospan}(\mathcal{C}) \arrow[r] & \mathrm{Corel}(\mathcal{C})
\end{tikzcd}$$

Then

$$\begin{tikzcd}
\mathcal{A}+\mathcal{A}^{op} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{Pushout}"] & \mathrm{Cospan}(\mathcal{A}) \arrow[d] \\
\mathrm{Span}(\mathcal{C}) \arrow[r] & \mathrm{Rel}(\mathcal{C})
\end{tikzcd}$$

A Modular Recipe for Axioms

Corel($\mathcal{C}$) is presented by generators $\xrightarrow{\in\mathcal{C}}$ $\xleftarrow{\in\mathcal{C}}$

and equations

$$\xrightarrow{p\in\mathcal{A}}\xleftarrow{q\in\mathcal{A}} \;=\; \xleftarrow{r\in\mathcal{A}}\xrightarrow{s\in\mathcal{A}} \qquad \begin{tikzcd} \bullet \arrow[r, "r"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "q"'] & \bullet \end{tikzcd}$$

$$\xleftarrow{f\in\mathcal{C}}\xrightarrow{g\in\mathcal{C}} \;=\; \xrightarrow{h\in\mathcal{C}}\xleftarrow{i\in\mathcal{C}} \qquad \begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & \bullet \arrow[d, "h"] \\ \bullet \arrow[r, "i"'] & \bullet \end{tikzcd}$$

Rel($\mathcal{C}$) is presented by generators $\xrightarrow{\in\mathcal{C}}$ $\xleftarrow{\in\mathcal{C}}$

and equations

$$\xleftarrow{p\in\mathcal{A}}\xrightarrow{q\in\mathcal{A}} \;=\; \xrightarrow{r\in\mathcal{A}}\xleftarrow{s\in\mathcal{A}} \qquad \begin{tikzcd} \bullet \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner"] & \bullet \arrow[d, "r"] \\ \bullet \arrow[r, "s"'] & \bullet \end{tikzcd}$$

$$\xrightarrow{f\in\mathcal{C}}\xleftarrow{g\in\mathcal{C}} \;=\; \xleftarrow{h\in\mathcal{C}}\xrightarrow{i\in\mathcal{C}} \qquad \begin{tikzcd} \bullet \arrow[r, "h"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "g"'] & \bullet \end{tikzcd}$$

Examples

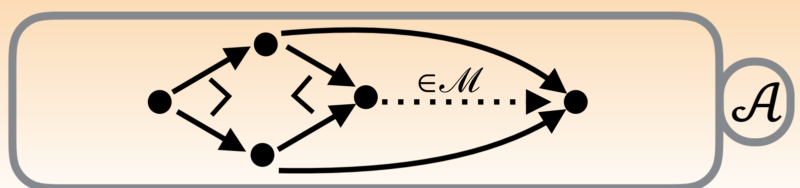
Equivalence Relations

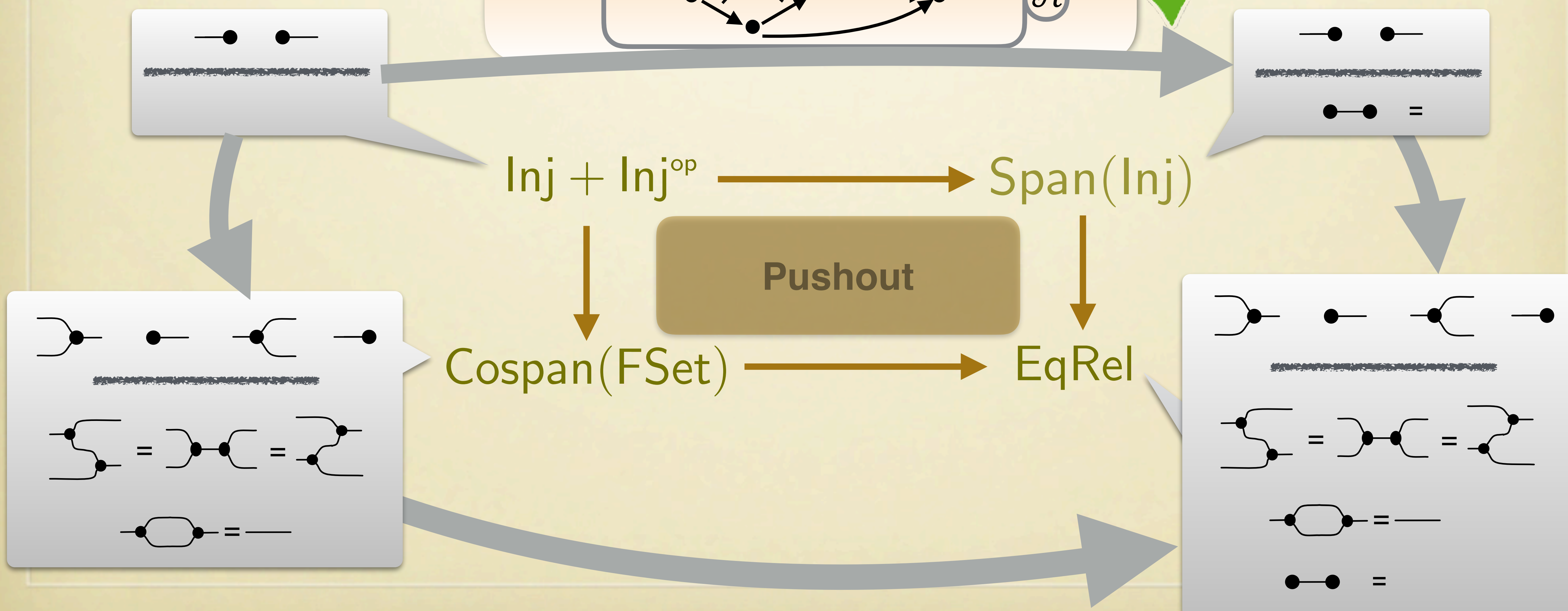
Assumptions

A prop $\mathcal{C}$ with pullbacks and pushouts — **FSet**

A fact. system $(\mathcal{E},\mathcal{M})$ s.t. $\mathcal{M}\subseteq Monos(\mathcal{C})$ — **(Surj,Inj)**

A sub-prop $\mathcal{A}$ such that $\mathcal{M}\subseteq\mathcal{A}$ — **Inj**





A non-example

Assumptions

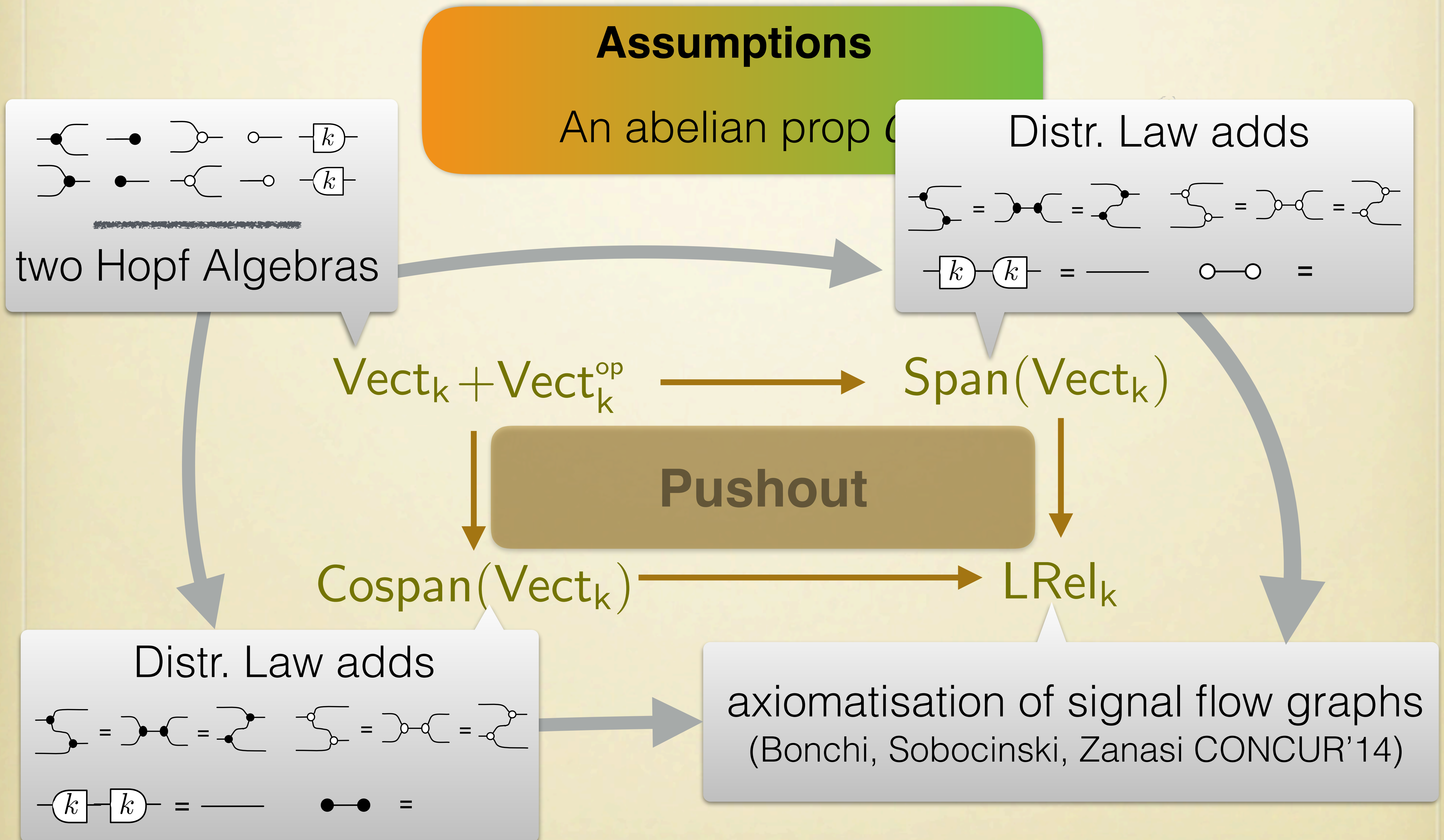
A prop $\mathcal{C}$ with pullbacks and pushouts — **FSet**

A fact. system $(\mathscr{E},\mathscr{M})$ s.t. $\mathscr{M} \subseteq Monos(\mathcal{C})$ — **(Surj,Inj)**

A sub-prop $\mathcal{A}$ such that $\mathscr{M} \subseteq \mathcal{A}$ — **FSet**

$$
\begin{tikzcd}
\mathsf{FSet} + \mathsf{FSet}^{op} \arrow[r] \arrow[d] \arrow[dr, phantom, "\textbf{Pushout}"] & \mathsf{Span}(\mathsf{FSet}) \arrow[d] \\
\mathsf{Cospan}(\mathsf{FSet}) \arrow[r] & 1
\end{tikzcd}
$$

Linear Subspaces



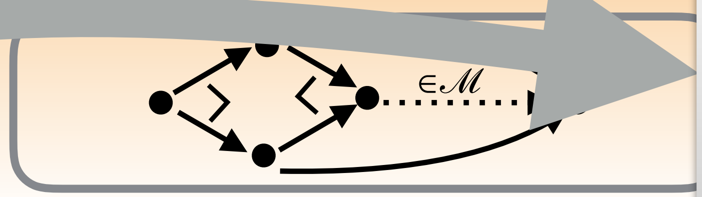
Corelations of Free Ring Modules

Assumptions

A prop $\mathcal{C}$ with pullbacks and pushouts

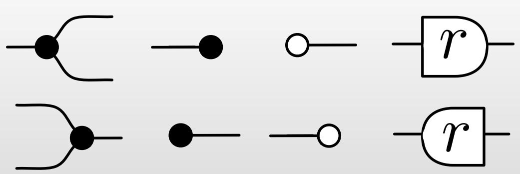
A fact. system $(\mathcal{E},\mathcal{M})$ s.t. $\mathcal{M} \subseteq Monos(\mathcal{C})$

sub-prop $\mathcal{A}$ such that $\mathcal{M} \subseteq \mathcal{A}$

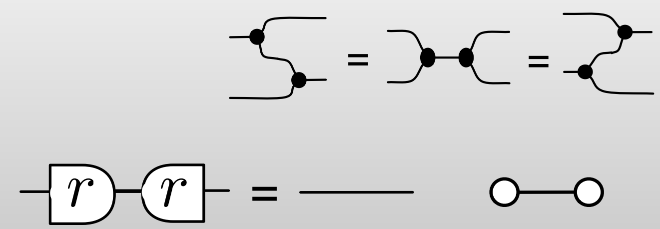


Free R-modules

(Epi,Split monos)

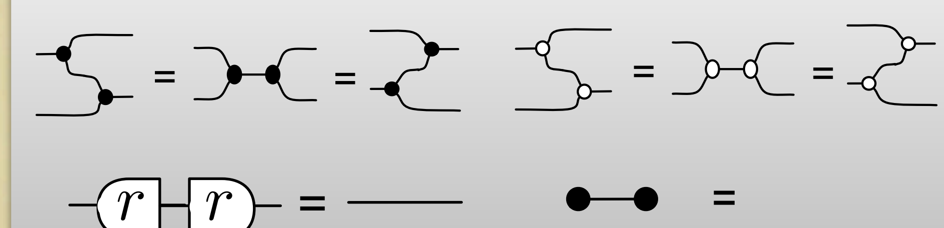


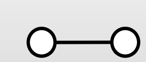
Distr. Law adds



$$\begin{array}{ccc} MFMod_R + MFMod_R^{op} & \longrightarrow & Span(MFMod_R) \\ \downarrow & \textbf{Pushout} & \downarrow \\ Cospan(FMod_R) & \longrightarrow & Corel(FMod_R) \end{array}$$

Distr. Law adds



Quotient Cospan(FMod$_R$) by  =

axiomatisation of LTI dynamical systems

(Fong, Rapisarda, Sobocinski LICS'16)

Concluding Remarks

- Universal characterisation of **Rel(C)** and **Corel(C)**
  - main insight: a certain subcategory $\mathcal{A}$ is the crux of the matter
  - modular recipe for axiomatisation
  - relevant for program semantics, control theory and quantum theory
- Results hold in **Cat** and **SymMonCat**, not just **Prop**
- More examples are to be found in the paper…
  - Partial equivalence relations
  - Relations of Eilenberg-Moore algebras
- ….but others aren't.
  - Hypergraph categories?
  - More categories of relations?
  - Higher-categorical extension? (Bicategories of relations)