Diagrammatic algebra: from linear to concurrent systems

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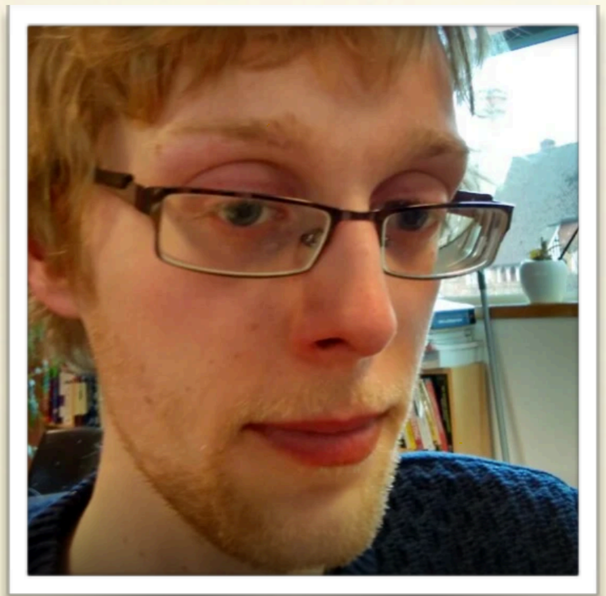

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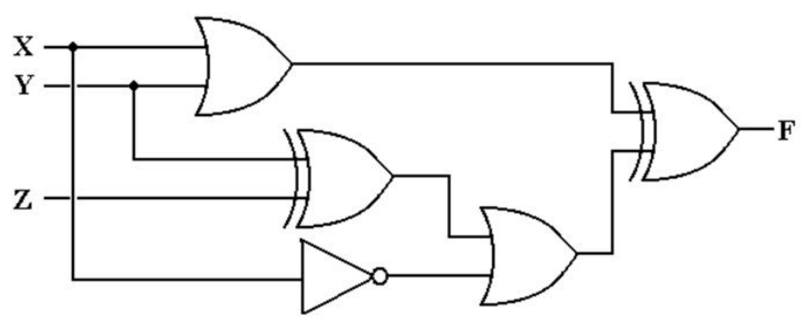
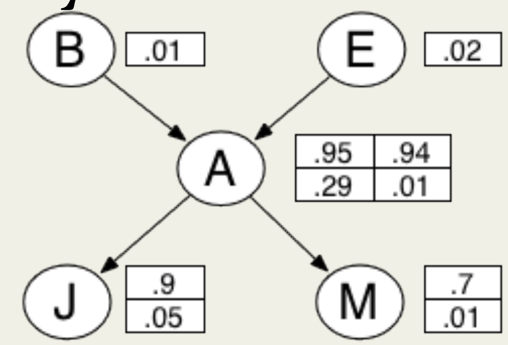
Introduction

Component-Based Systems

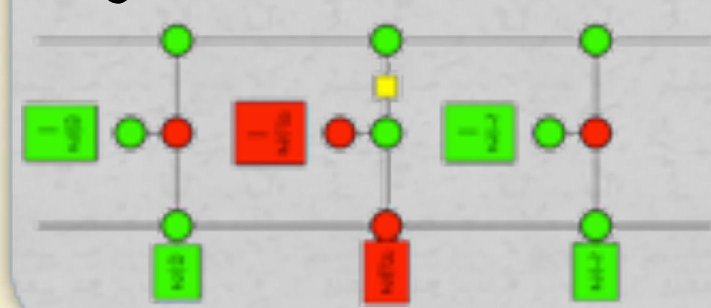
Digital Circuits



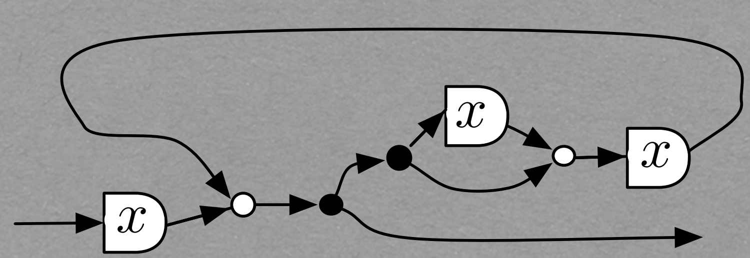
Bayesian Networks



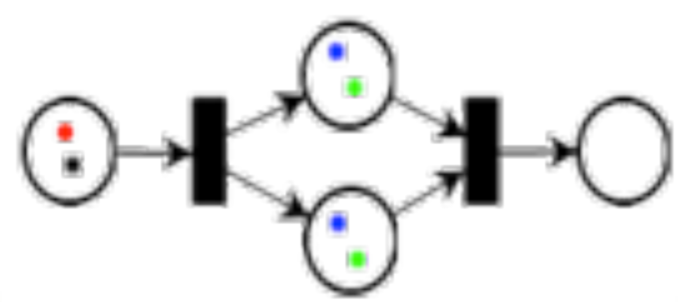
Quantum Processes



Signal Flow Graphs



Petri Nets



Compositional Modelling

There is an emerging, multi-disciplinary field aiming at studying graphical models of computation **compositionally**, inspired by the **algebraic methods** of programming language semantics.

Diagrams are first-class citizens of the theory. The appropriate algebraic setting is **monoidal** (and not **cartesian**) categories.

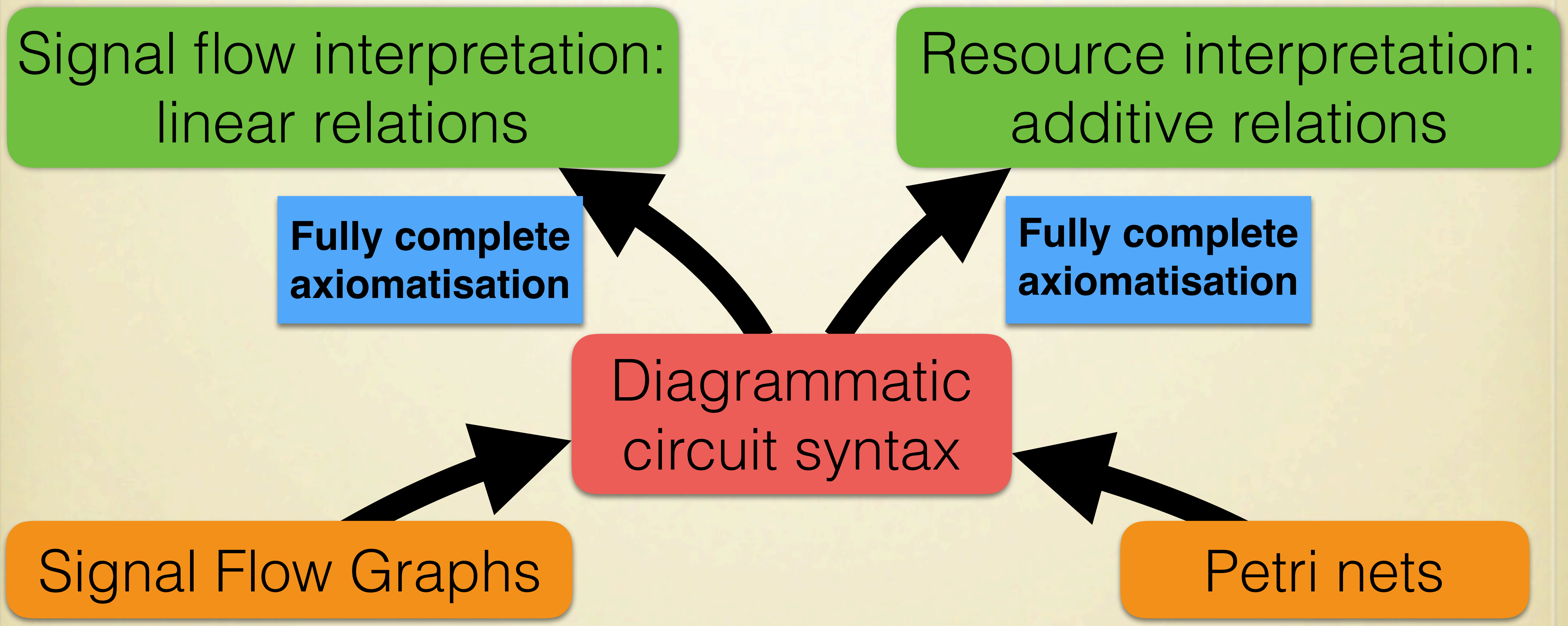
In this talk

- We start from a simple diagrammatic language.

- We **axiomatise** two different interpretations of the connectors appearing in the language.
  - With the *signal-flow* interpretation, we model **linear dynamical systems.**
  - With the *resource* interpretation, we model **concurrent systems**.

- Our analysis shows that seemingly diverse computational models can be studied within **the same** algebraic framework.

Overview



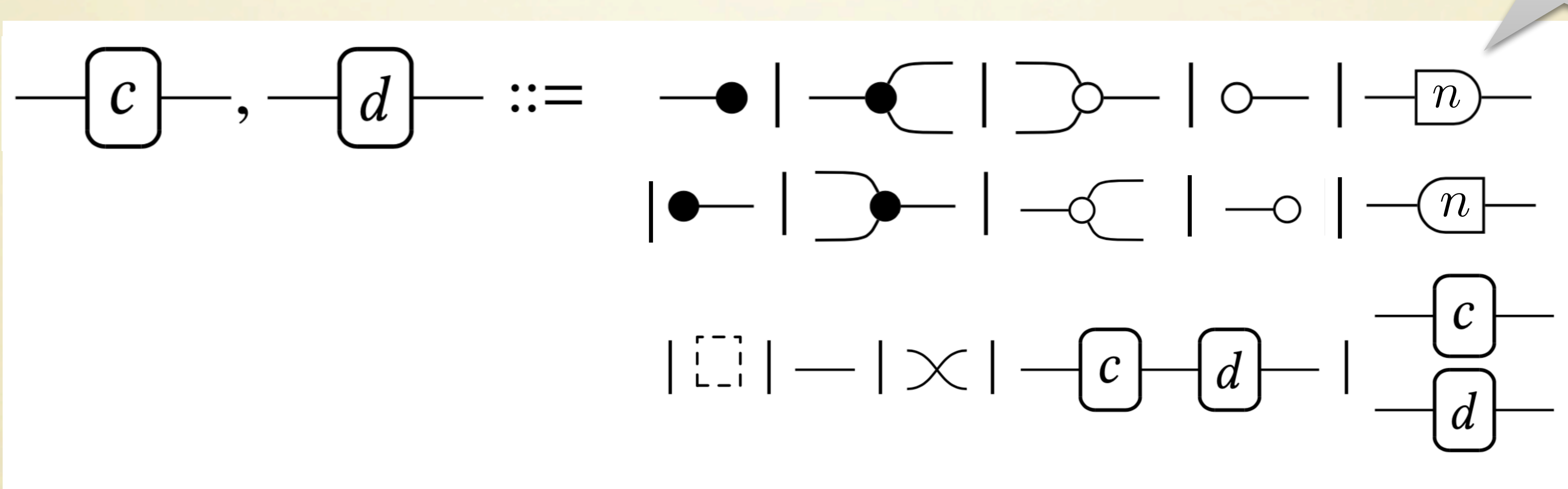
Signal flow interpretation:
linear relations
Resource interpretation:
additive relations
Fully complete
axiomatisation
Fully complete
axiomatisation
Diagrammatic
circuit syntax
Signal Flow Graphs
Petri nets

The core language

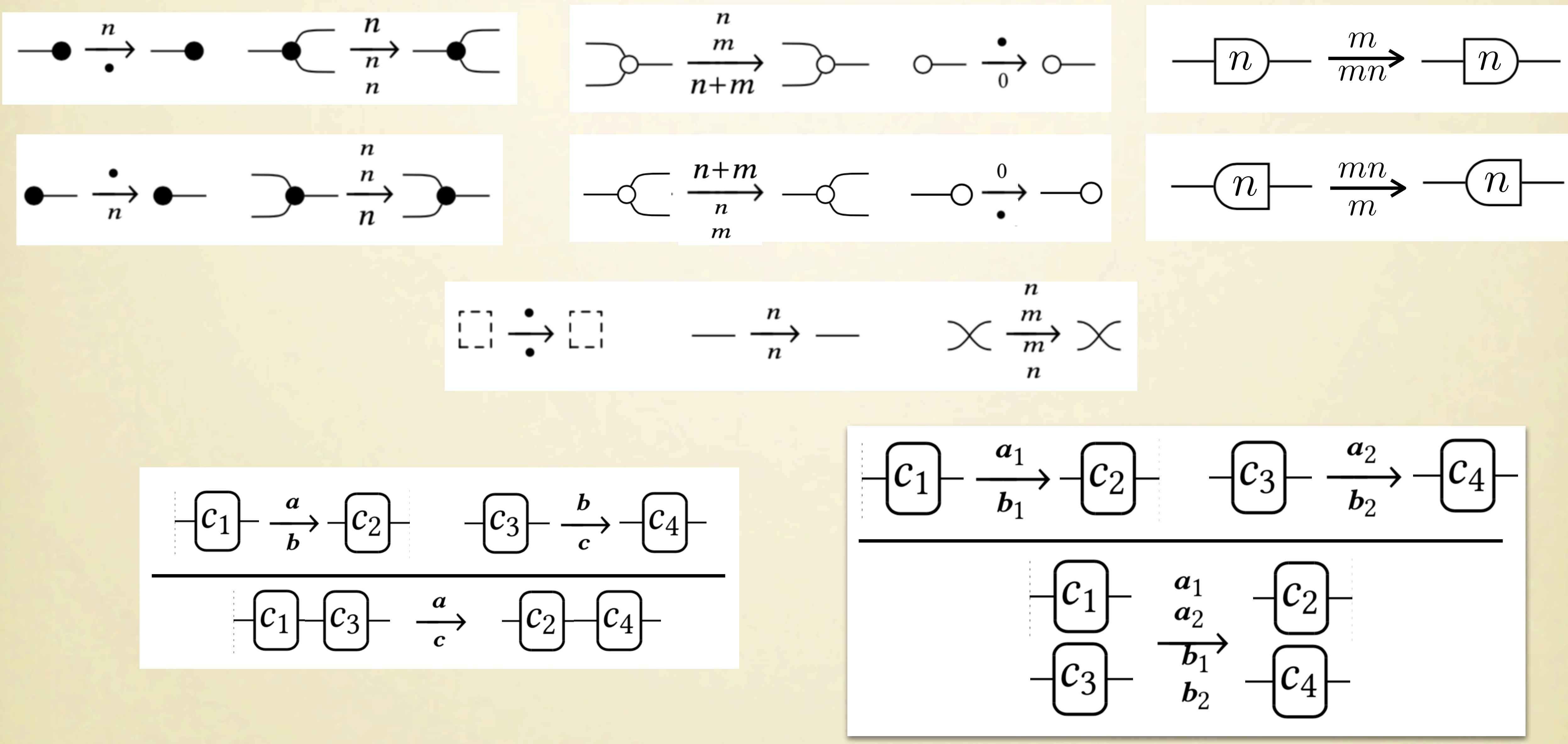
A simple circuit syntax

Circ

elements of a semiring



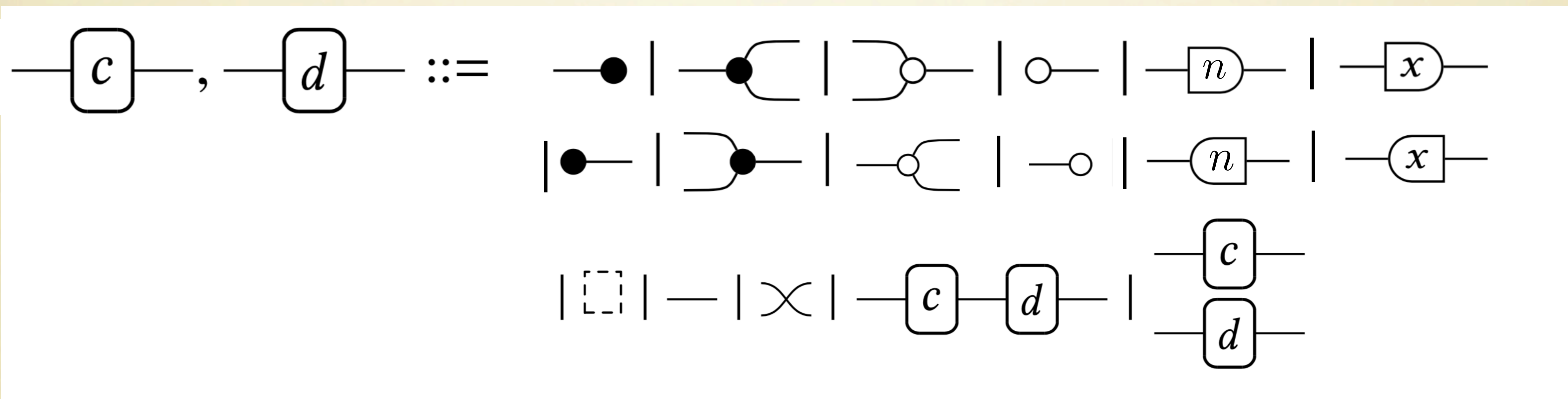
Circuit Behaviour



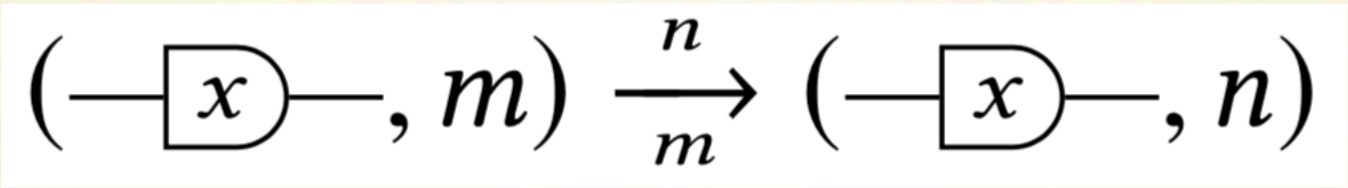
$$[\![c]\!] := \{(\boldsymbol{a},\boldsymbol{b}) \mid c \xrightarrow[\boldsymbol{b}]{\boldsymbol{a}} c\}$$

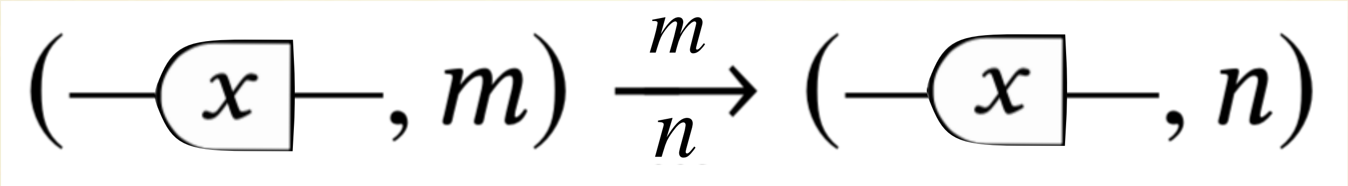
Stateful extension: syntax

Circ$_s$



Stateful extension: behaviour





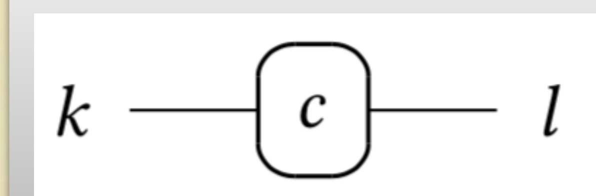
Stateless semantics can be inductively extended to a stateful one:

$$[\![c]\!] := \{(s_1, \boldsymbol{a}, s_2, \boldsymbol{b}) \mid (c, s_1) \xrightarrow[\boldsymbol{b}]{\boldsymbol{a}} (c, s_2)\}$$

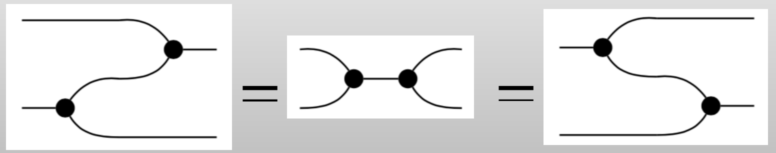
The register is canonical

Registers and their semantics are justified by the iso

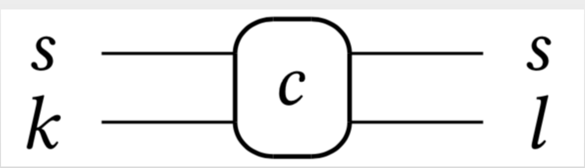
Arrows $k \to l$ are



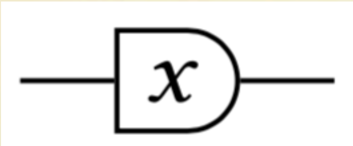
$$(\mathrm{Circ}_s)_{/Fr} \cong \mathrm{St}(\mathrm{Circ}_{/Fr})$$



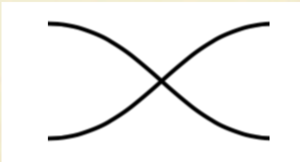
Arrows $k \to l$ are

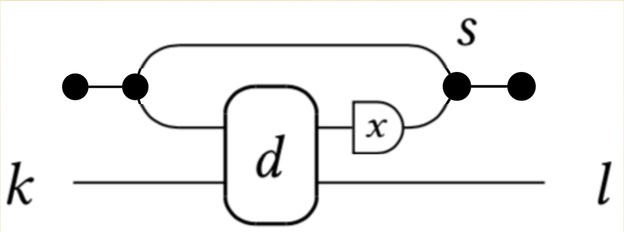


Cf. Katis, Sabadini, Walters. Bicategories of Processes, *JPAA* 1997

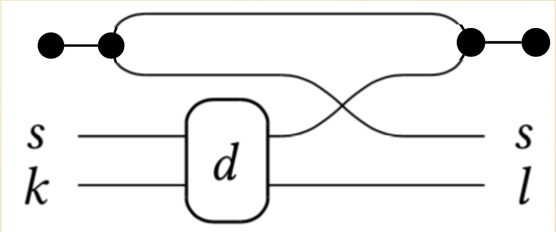


$\longmapsto$

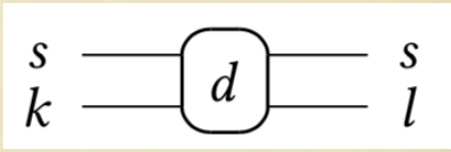




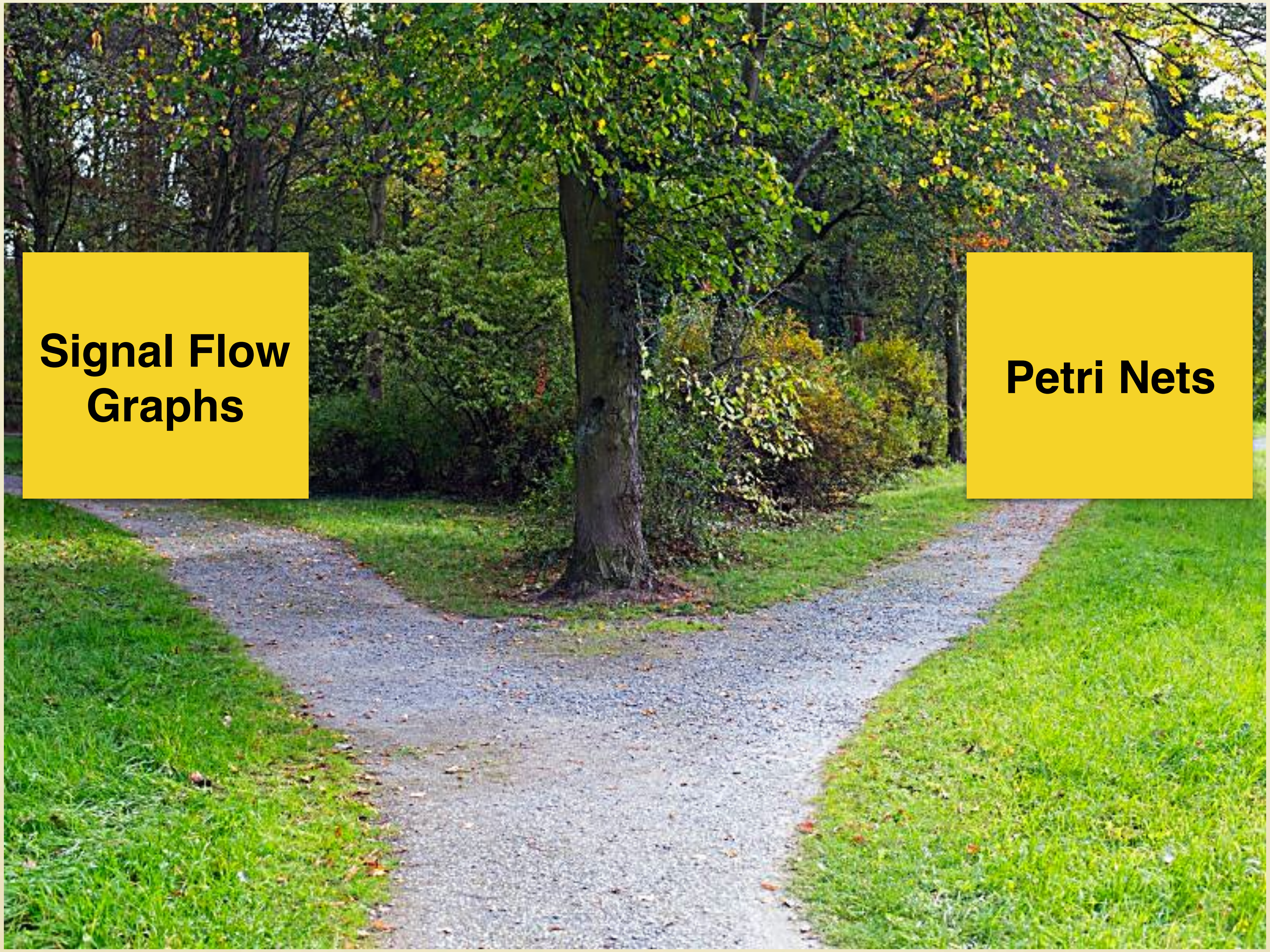
$\longmapsto$



$\overset{Fr}{=}$



What's coming up



The signal flow perspective

(CONCUR'14, PoPL'15)

Linear interpretation

Circ -diagrams are interpreted on vectors over $\mathbb{R}$.

$$[\![c]\!] := \{(\boldsymbol{a}, \boldsymbol{b}) \mid c \xrightarrow[\boldsymbol{b}]{\boldsymbol{a}} c\}$$

For c of type $k \to l$, the relation $[\![c]\!]$ is a finite-dimensional *subspace* of $\mathbb{R}^k \times \mathbb{R}^l$, a.k.a. a *linear relation* between $\mathbb{R}^k$ and $\mathbb{R}^l$.

Proposition

Finite-dimensional linear relations form a category $\mathbf{LinRel}_{\mathbb{R}}$

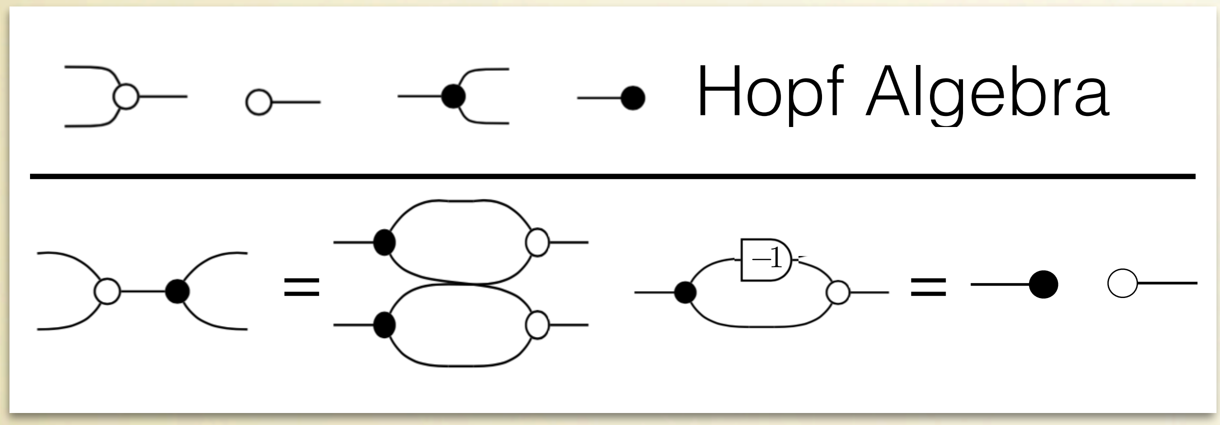
Examples

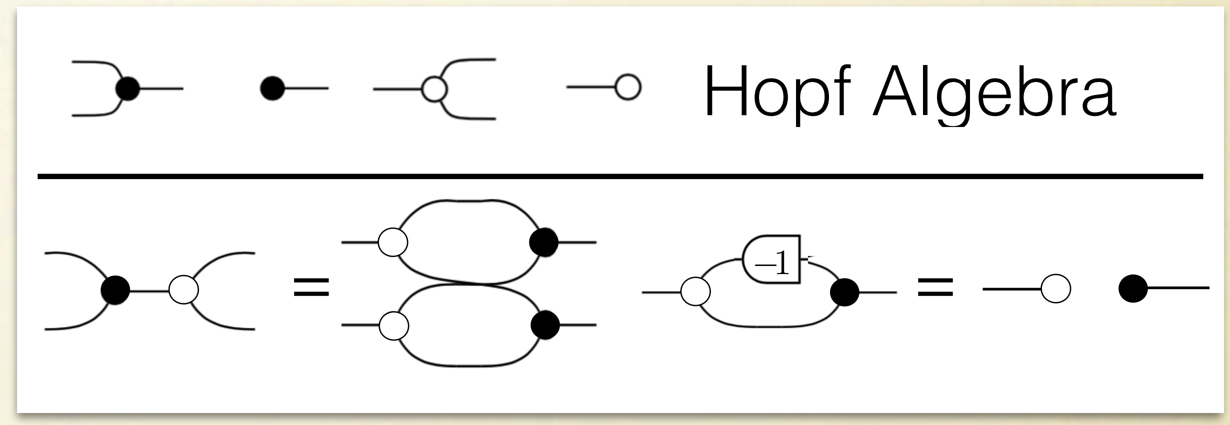
Graphical Linear Algebra

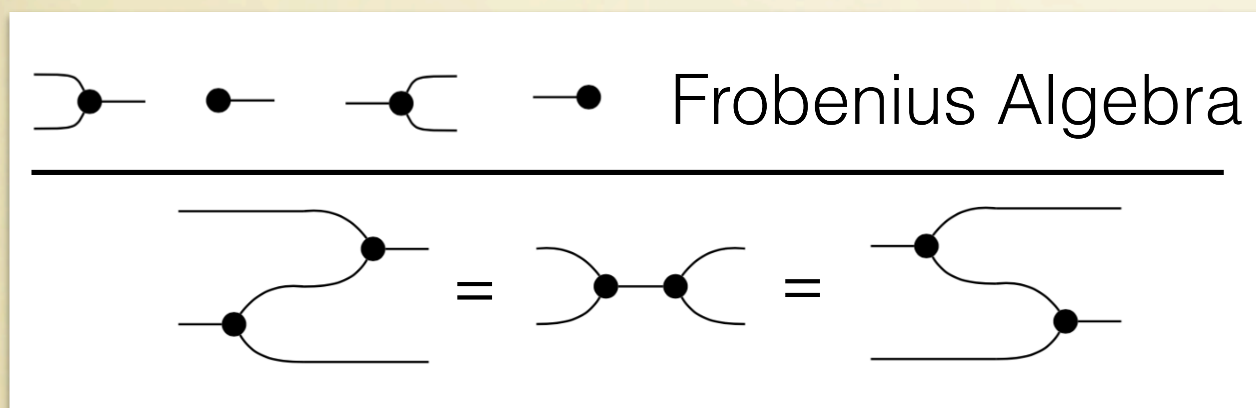
On the board

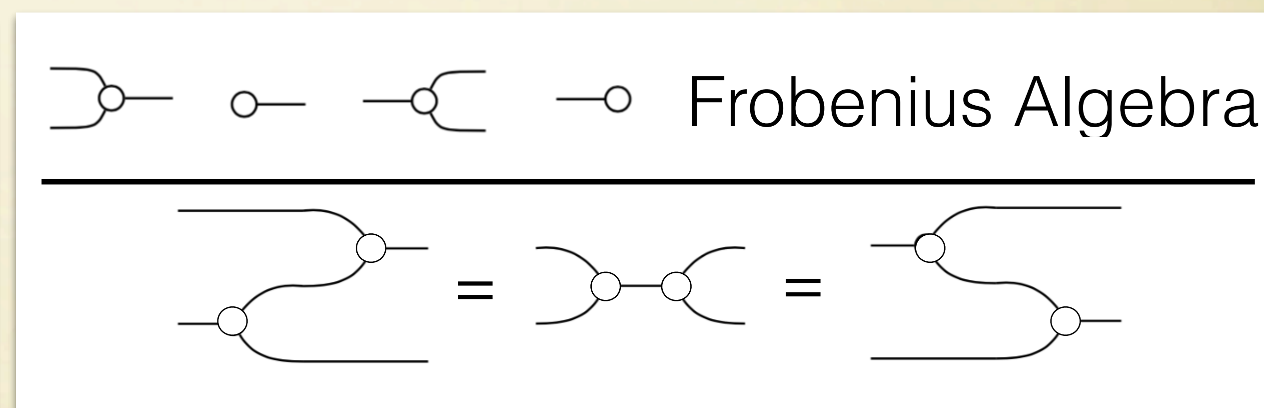
Equational Theory

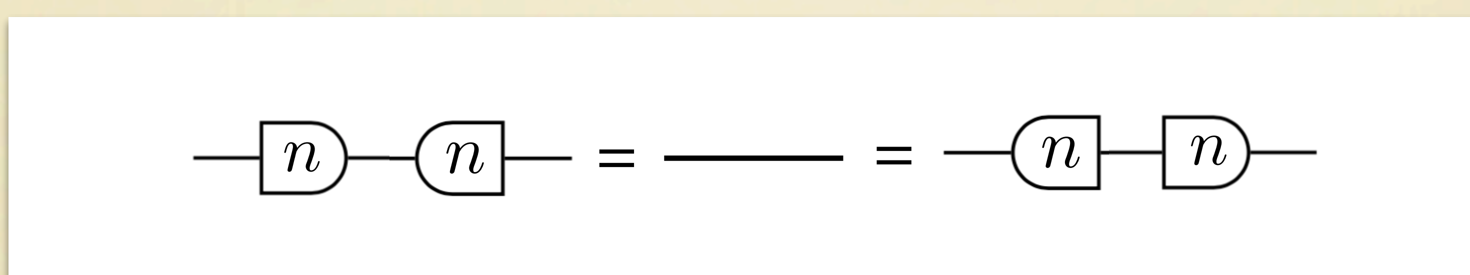
$\mathbb{IH}$: Interacting Hopf Algebras











$n \neq 0$

Completeness

Theorem $\mathsf{Circ}_{/\mathbb{IH}} \cong \mathsf{LinRel}_{\mathbb{R}}$

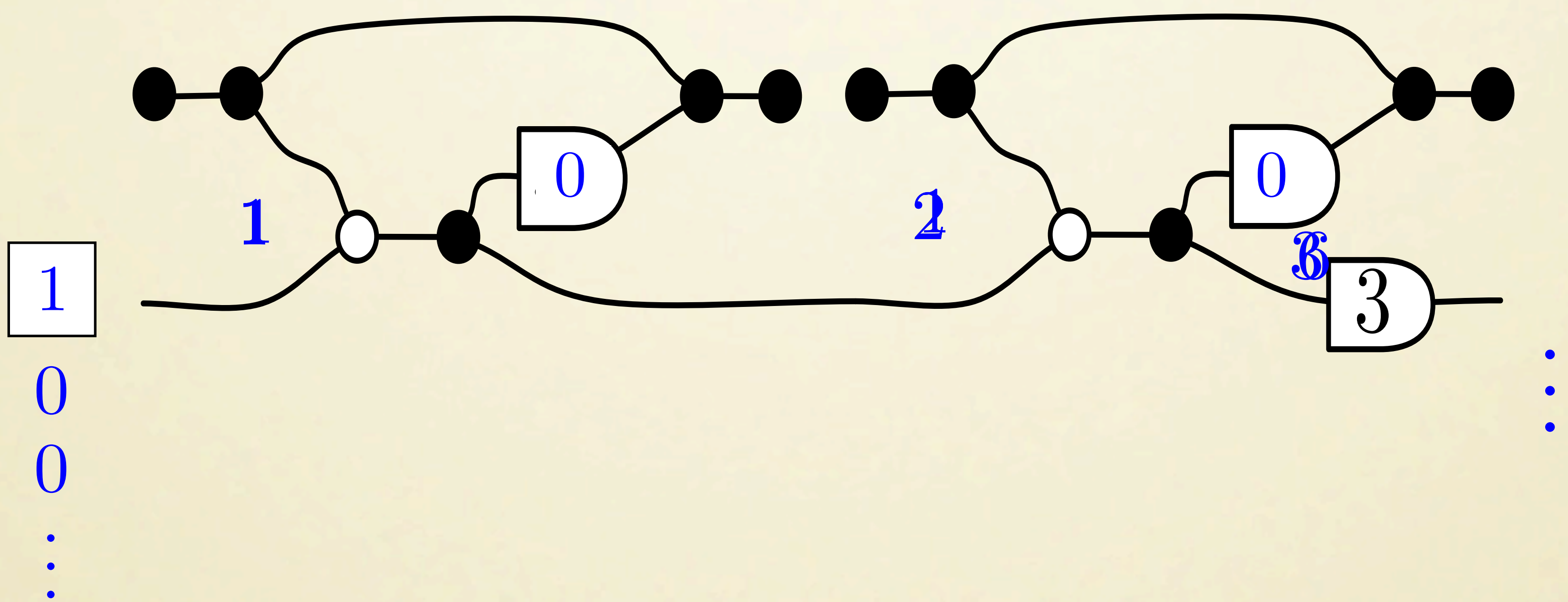
Corollary

$$[\![c]\!] = [\![d]\!] \iff c \overset{\mathbb{IH}}{=} d$$

Corollary

$$(\mathsf{Circ}_s)_{/\mathbb{IH}} \cong \mathsf{St}(\mathsf{LinRel}_{\mathbb{R}})$$

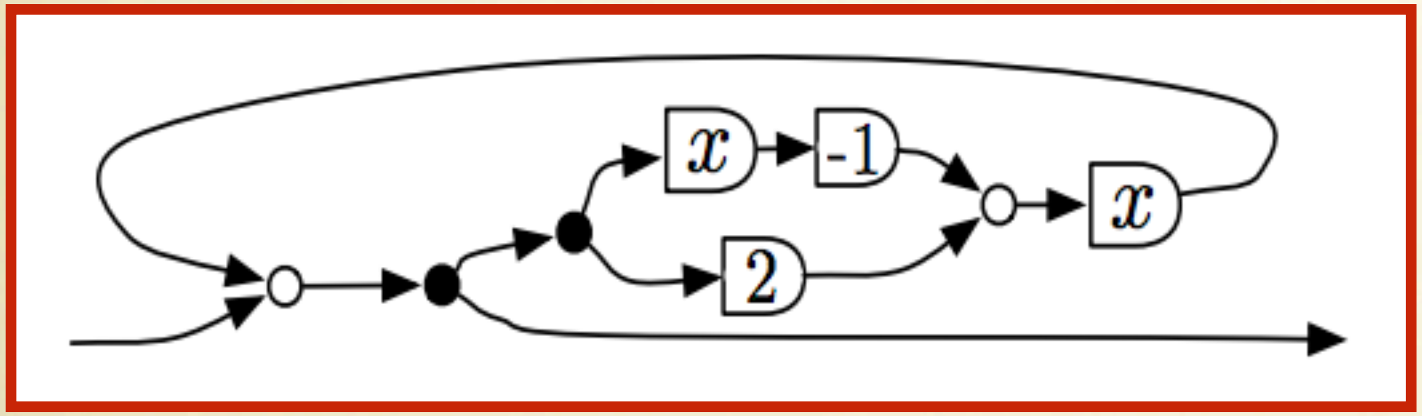
Stateful Example



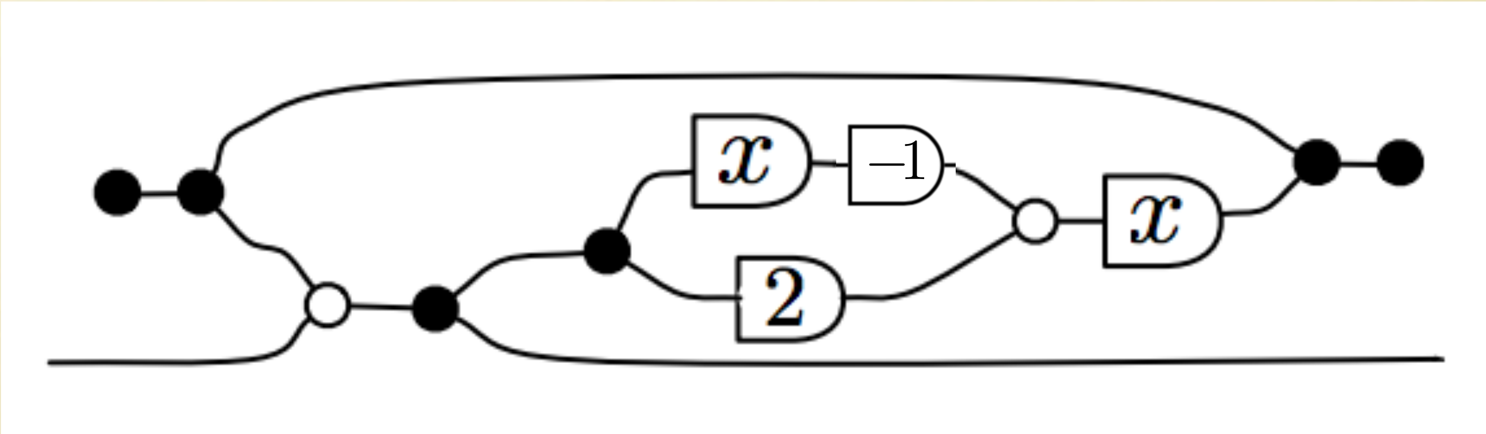
$$(c,0,0) \xrightarrow[3]{1} (c,1,1) \xrightarrow[6]{0} (c,1,2) \xrightarrow[9]{0} (c,1,3) \xrightarrow[12]{0} \ldots$$

Signal Flow Graphs

In fact, the class of **signal flow graphs** embeds in Circ_s



$\mapsto$

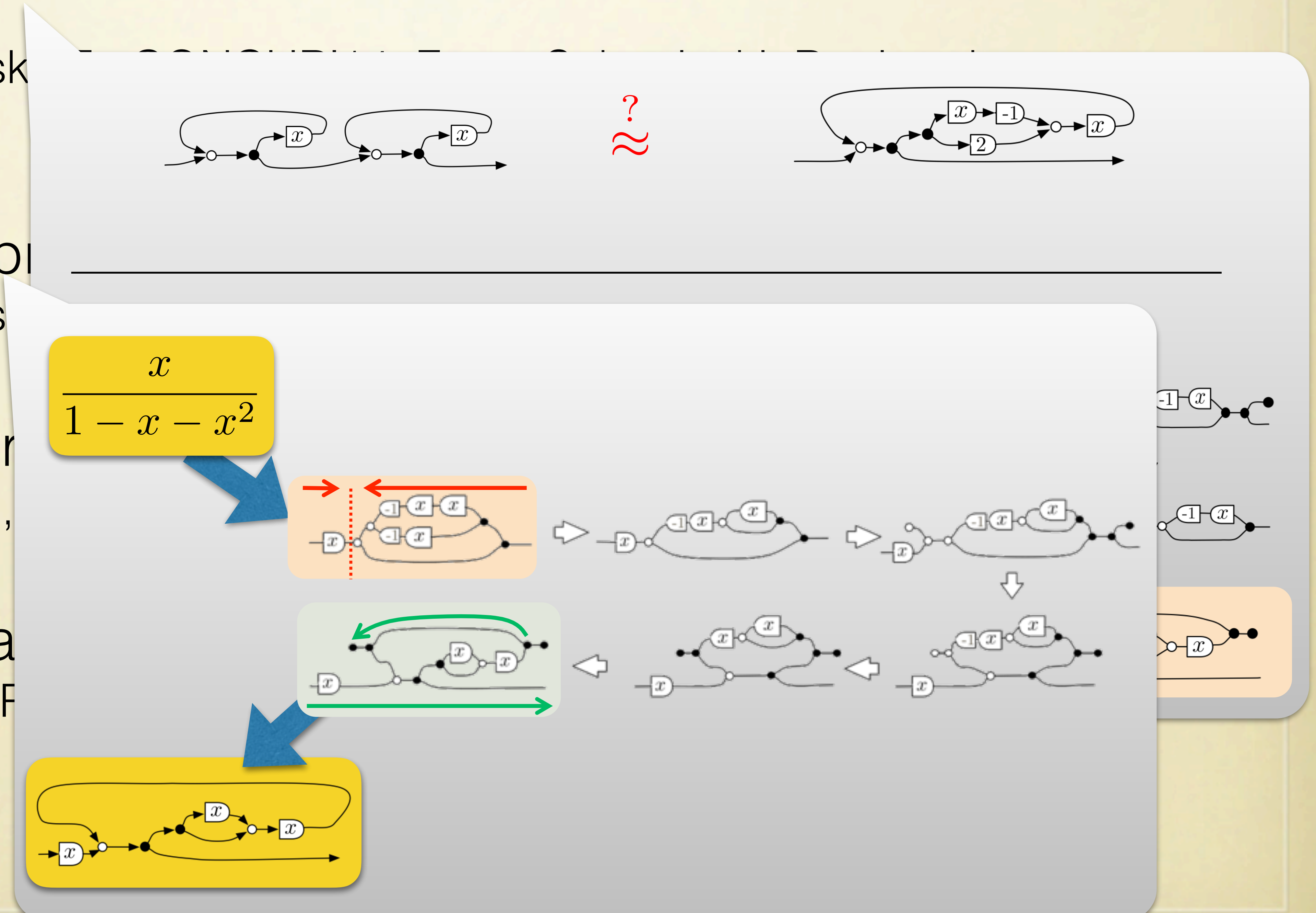


Further Developments

- **Stream semantics** of signal flow graphs: move from $\mathbb{R}$ to the field $\mathbb{L}$ of *Laurent series* (generalised streams).
- This semantics can be also **axiomatised**, by an extension $\mathbb{IH}'$ of $\mathbb{IH}$ (only the semiring axioms change: from $\mathbb{R}$ to $\mathbb{R}[x]$).
- ``**Kleene's Theorem**'': signal flow graphs modulo $\mathbb{IH}'$ identify precisely the *rational* behaviours in the semantic domain.

Further Developments

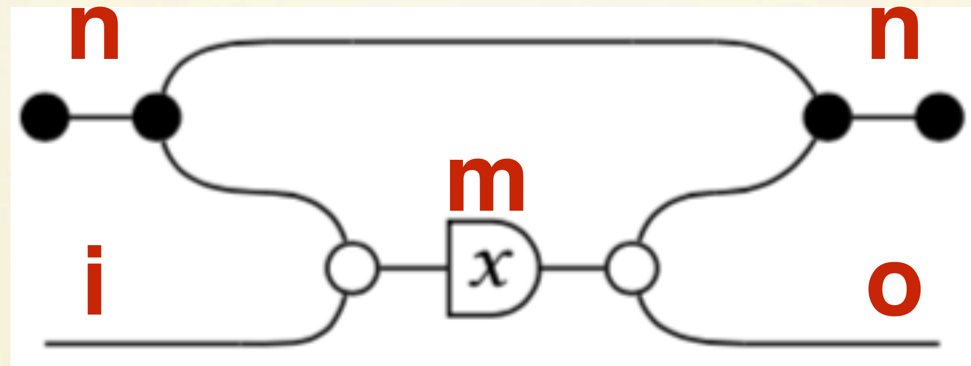
- **Graphical Equational Reasoning** for Linear Dynamical Systems.
  (Bonchi, Sobocinsk
  LICS'16)
- **Realisability** pr
  (Bonchi, Sobocins
- Syntactic criter
  (Fong, Sobocinski,
- Signal flow gra
  (Bonchi, Holland, P



The resource perspective

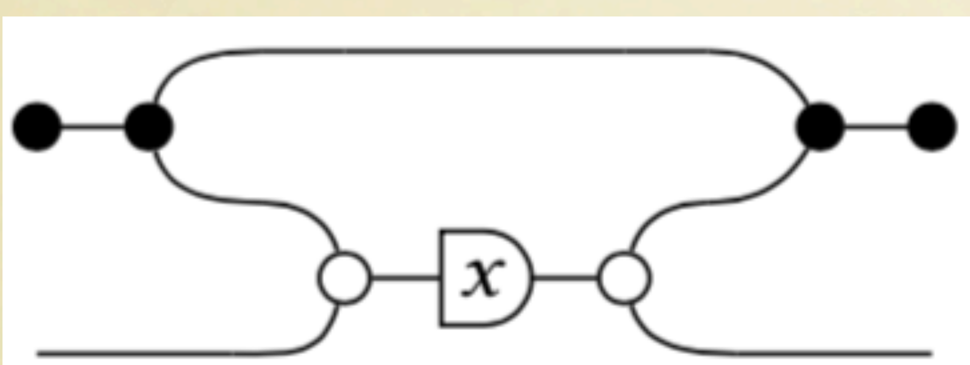
(PoPL'19)

Motivating Example



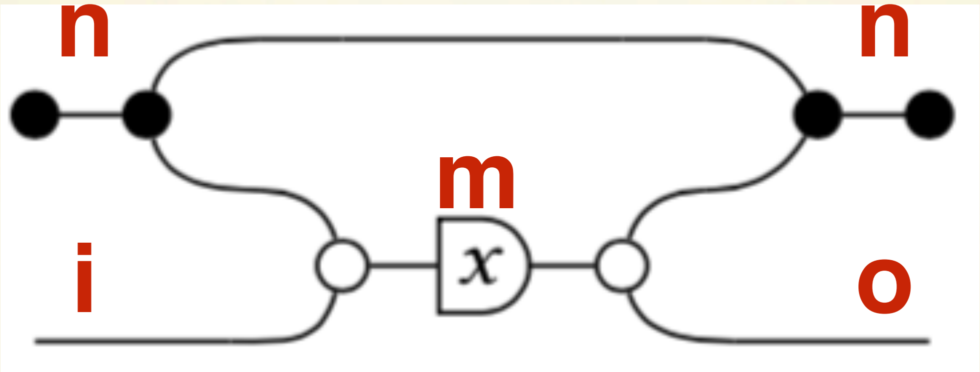
$\exists$ n. m = o+n

Over $\mathbb{R}$, this is always true.



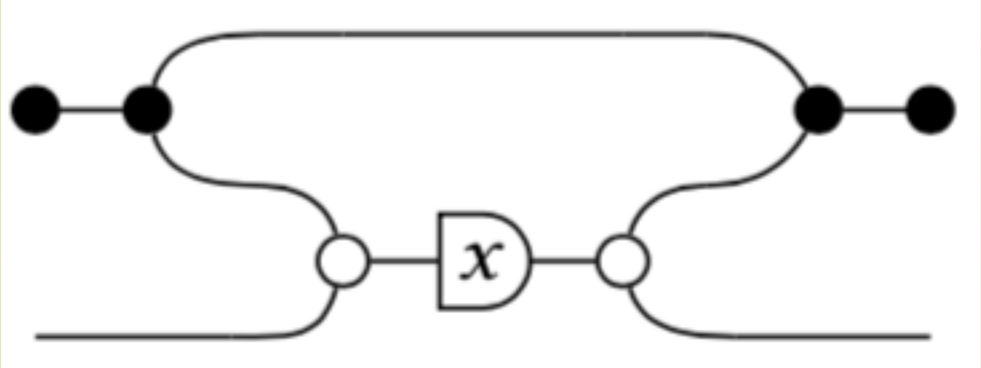
$\stackrel{\mathbb{IH}}{=}$

Motivating Example

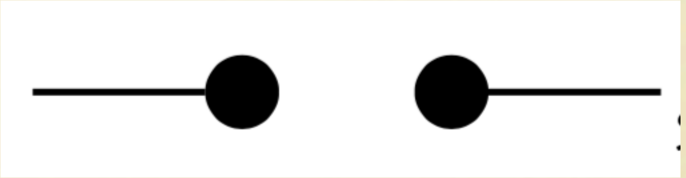


∃ n. m = o+n

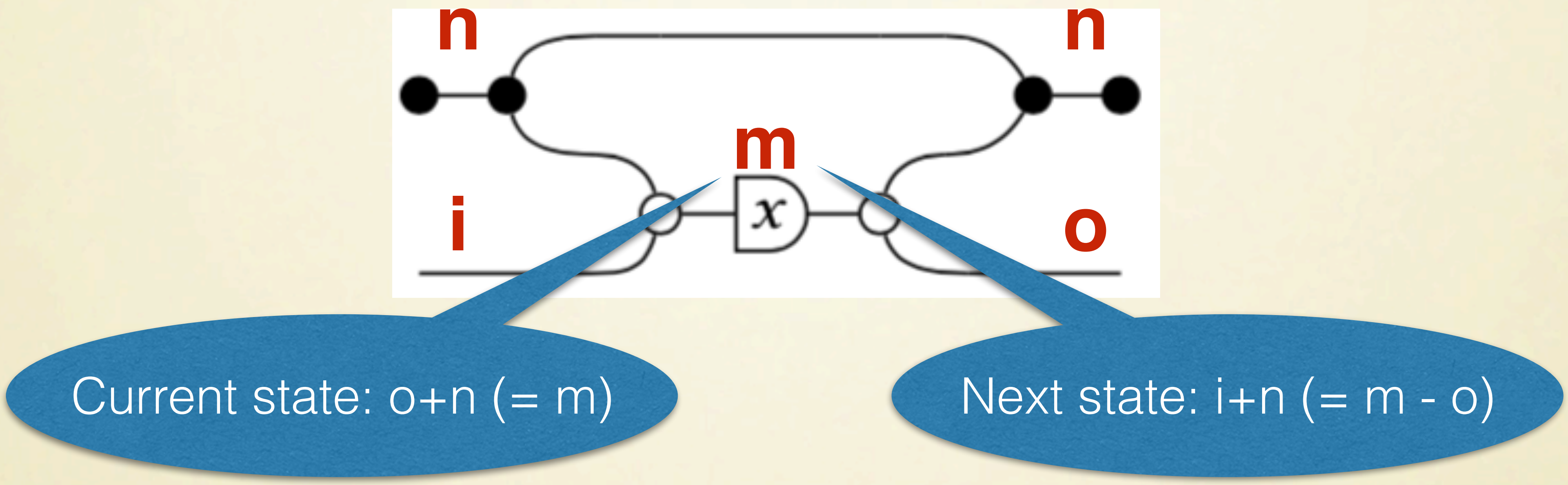
Over $\mathbb{N}$, only true if m ≥ o.



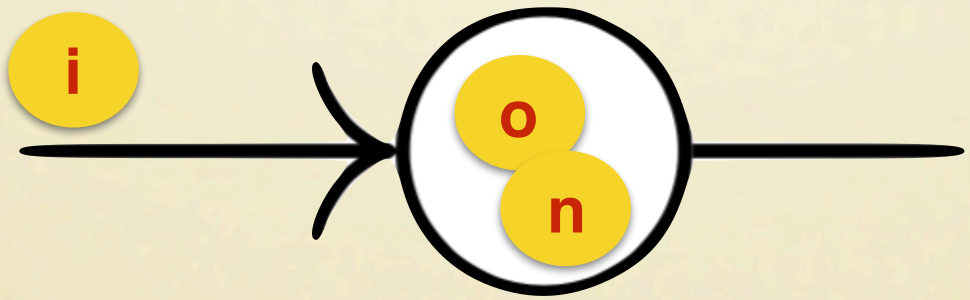
$\neq$



Motivating Example



In fact, over $\mathbb{N}$, the circuit behaves as the **place of a Petri net.**



Additive Relations

Circ -diagrams are interpreted on vectors over $\mathbb{N}$.

$$[\![c]\!] := \{(\boldsymbol{a}, \boldsymbol{b}) \mid c \xrightarrow[\boldsymbol{b}]{\boldsymbol{a}} c\}$$

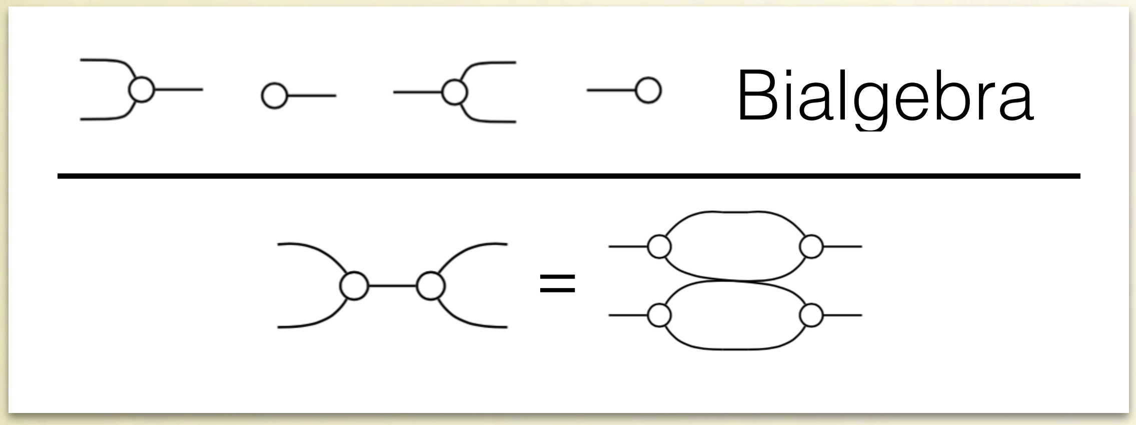
For *c* of type $k \to l$, $[\![c]\!]$ is an *additive relation*, that is, a subset of $\mathbb{N}^k \times \mathbb{N}^l$ that contains $(\mathbf{0},\mathbf{0})$ and is closed under addition.

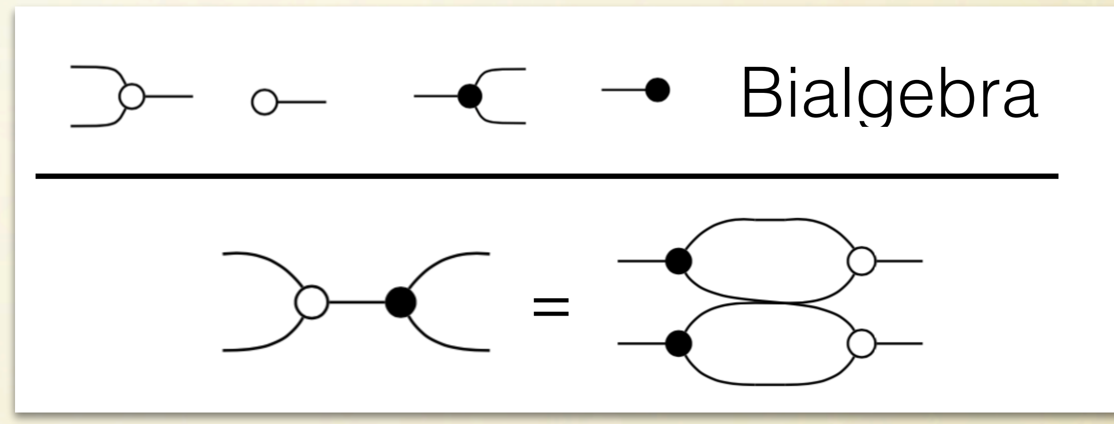
Proposition (non-trivial!)

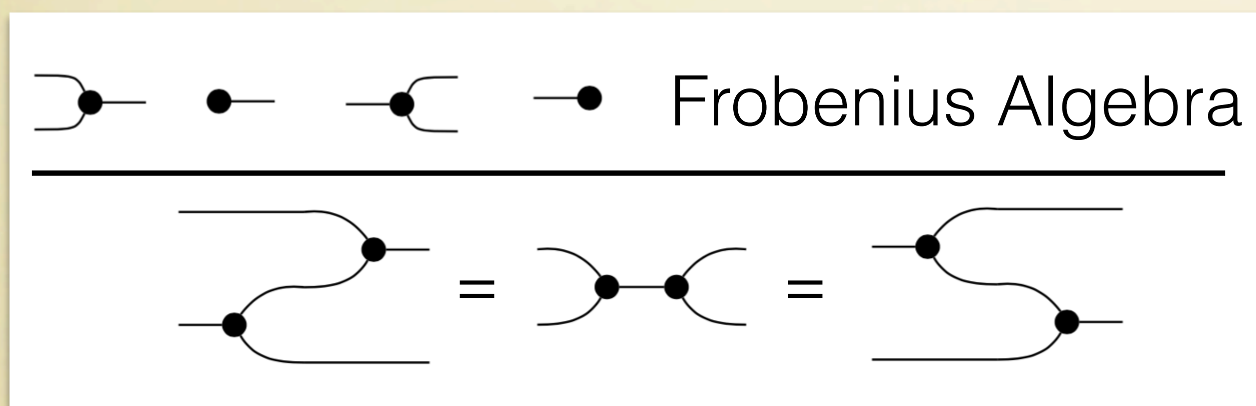
Finitely-generated additive relations form a category $\mathbf{AddRel}_{\mathbb{N}}$

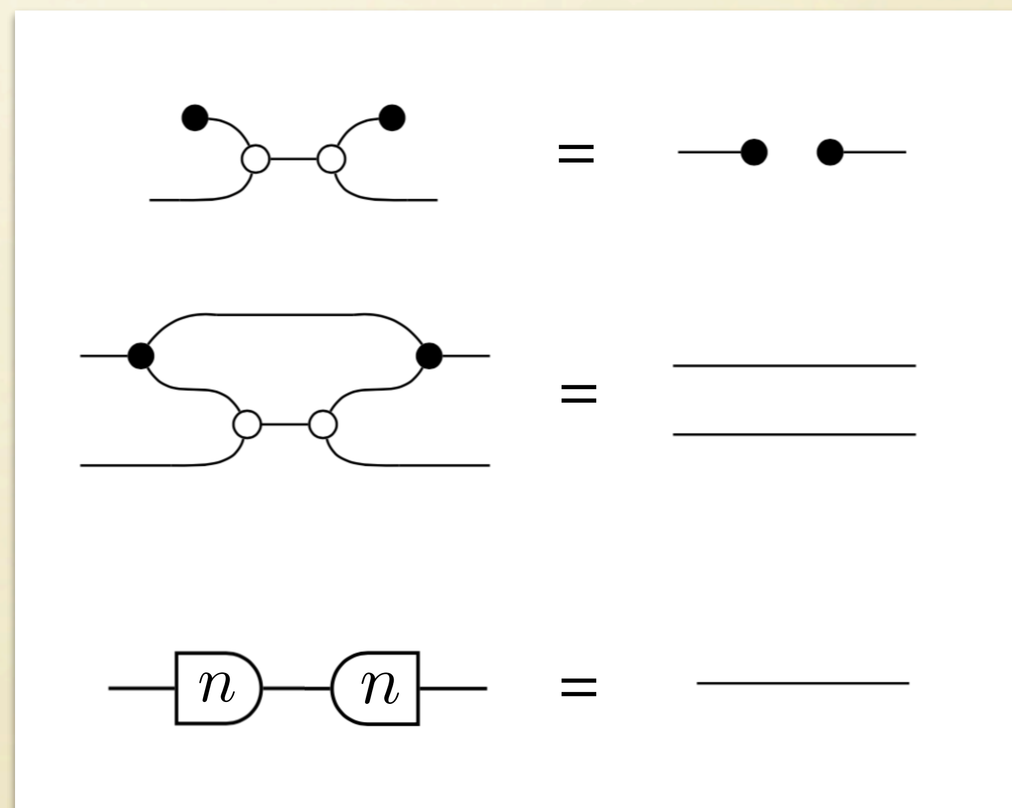
Equational Theory

$\mathbb{AR}$: Algebra of Resources



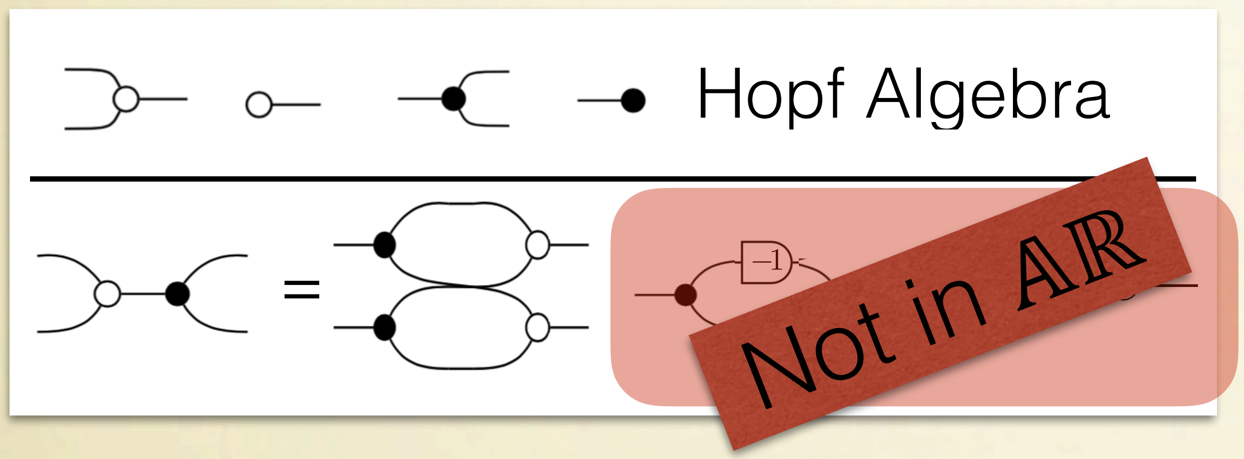


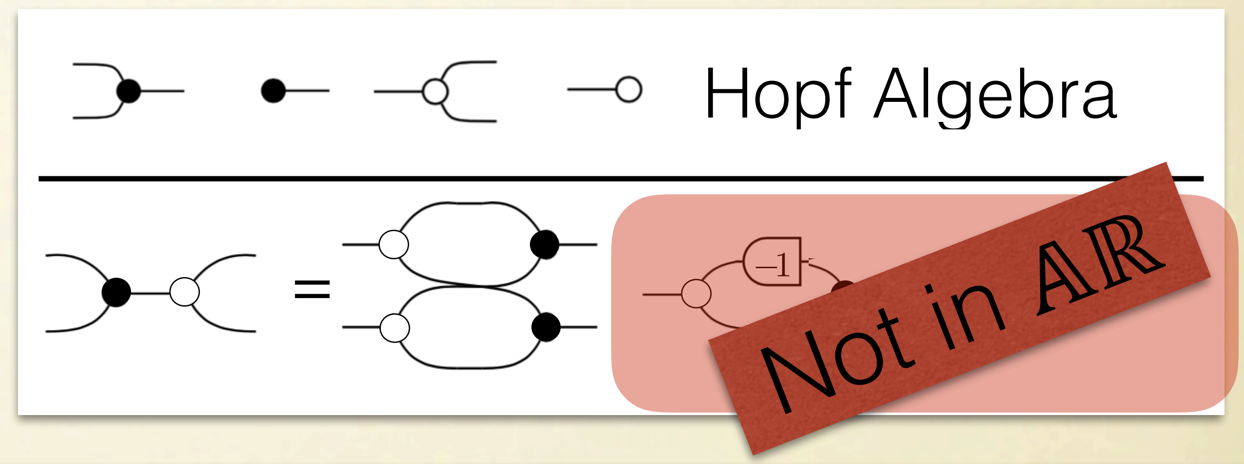


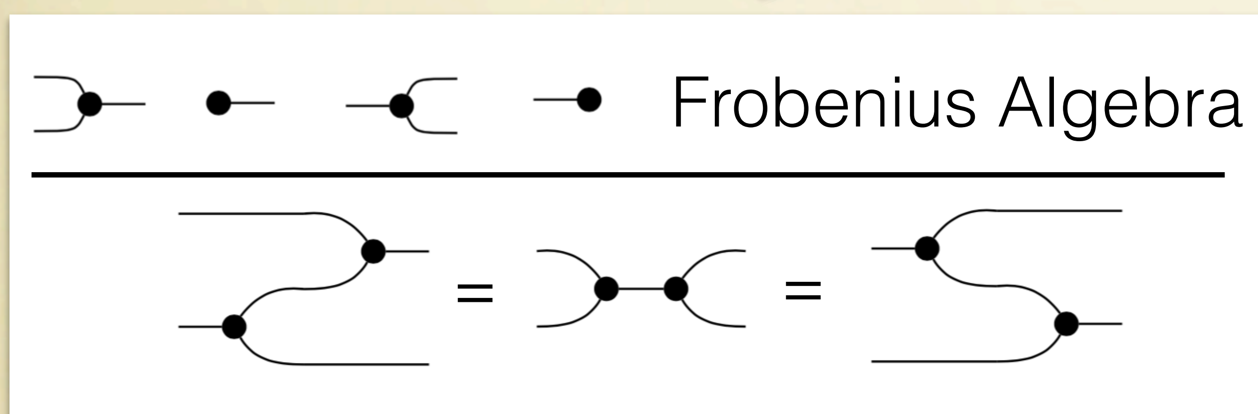


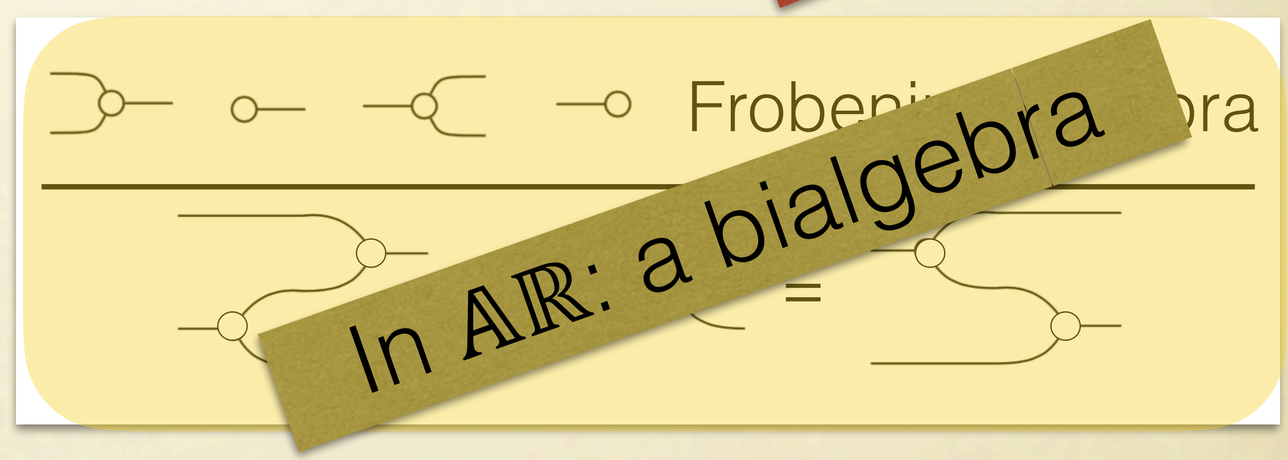
$n \neq 0$

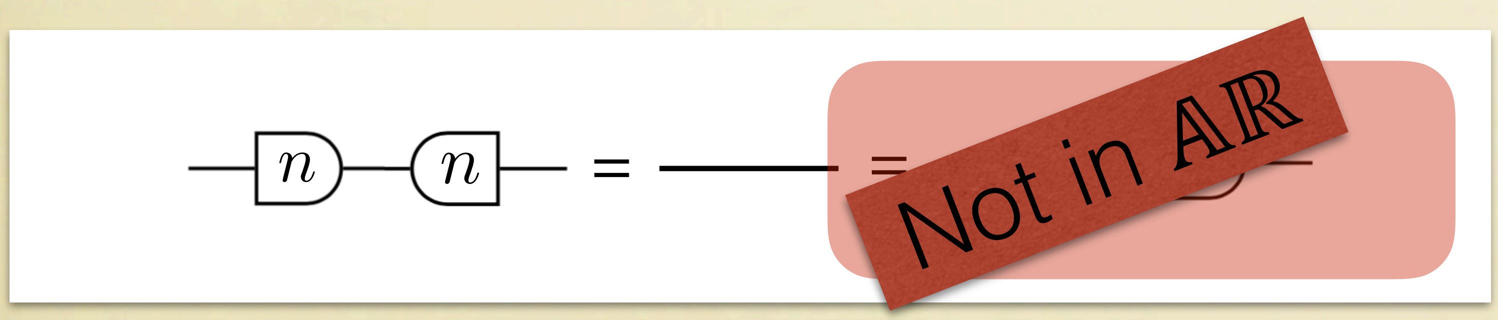
$\mathbb{AR}$ vs $\mathbb{IH}$

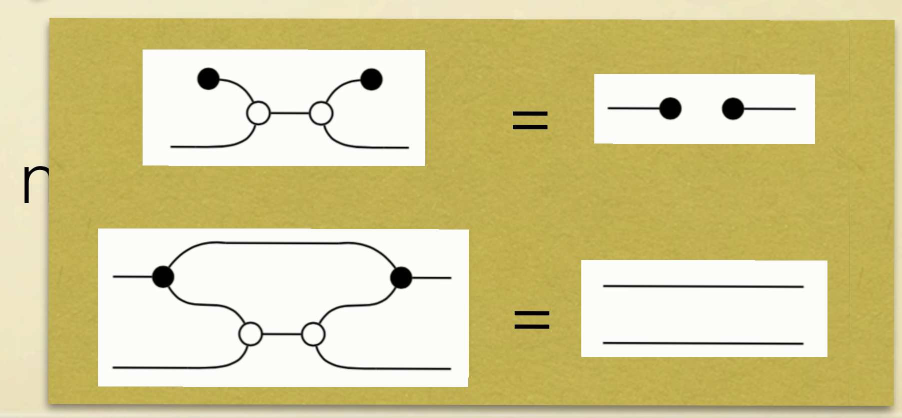












Completeness

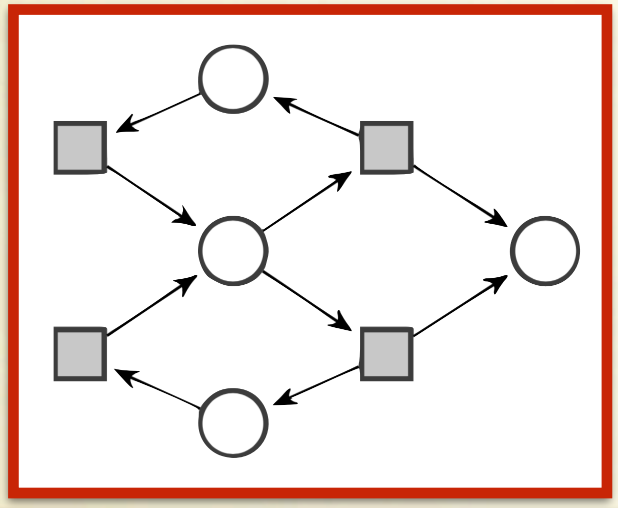
Theorem $\mathsf{Circ}_{/\mathbb{AR}} \cong \mathsf{AddRel}_{\mathbb{N}}$

$$[\![c]\!] = [\![d]\!] \iff c \stackrel{\mathbb{AR}}{=} d$$

Corollary

$$(\mathsf{Circ}_s)_{/\mathbb{AR}} \cong \mathsf{St}(\mathsf{AddRel}_{\mathbb{N}})$$

Embedding Petri Nets



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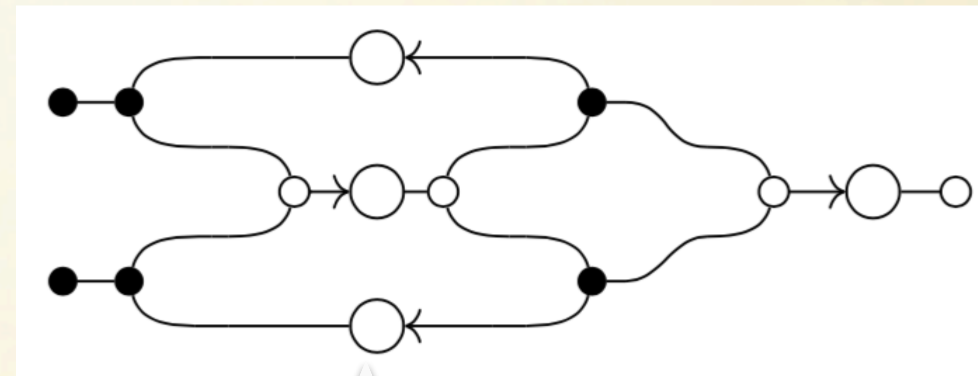
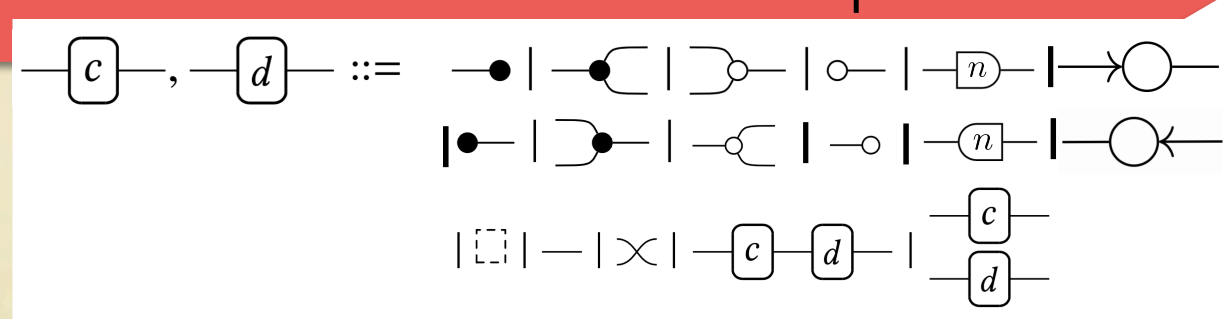
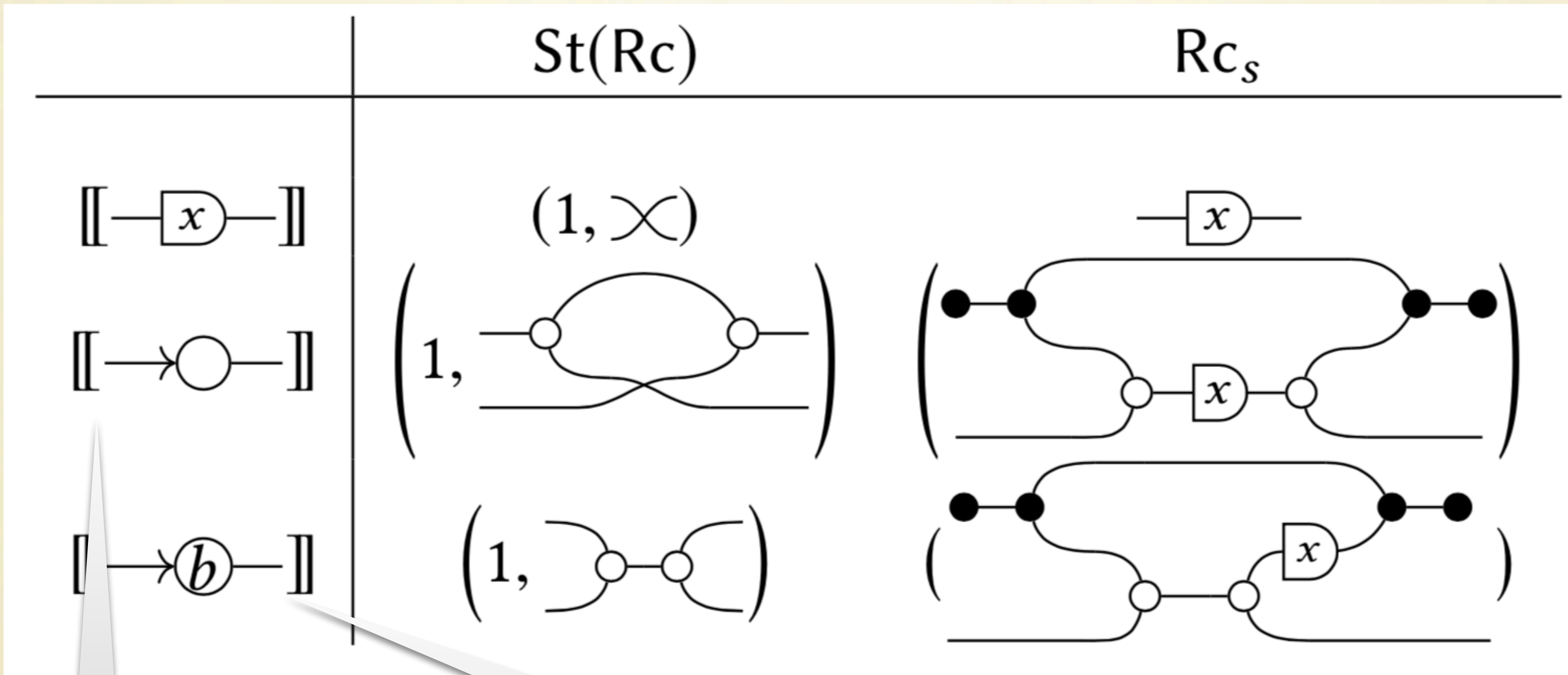


Theorem Petri Nets and 0 → 0 string diagrams in **Circ**$_p$ are in 1-1 correspondence modulo semantic equivalence.



We can thus use $\mathbb{AR}$ for equational reasoning about Petri Nets.

Classifying Place Semantics



Firing semantics

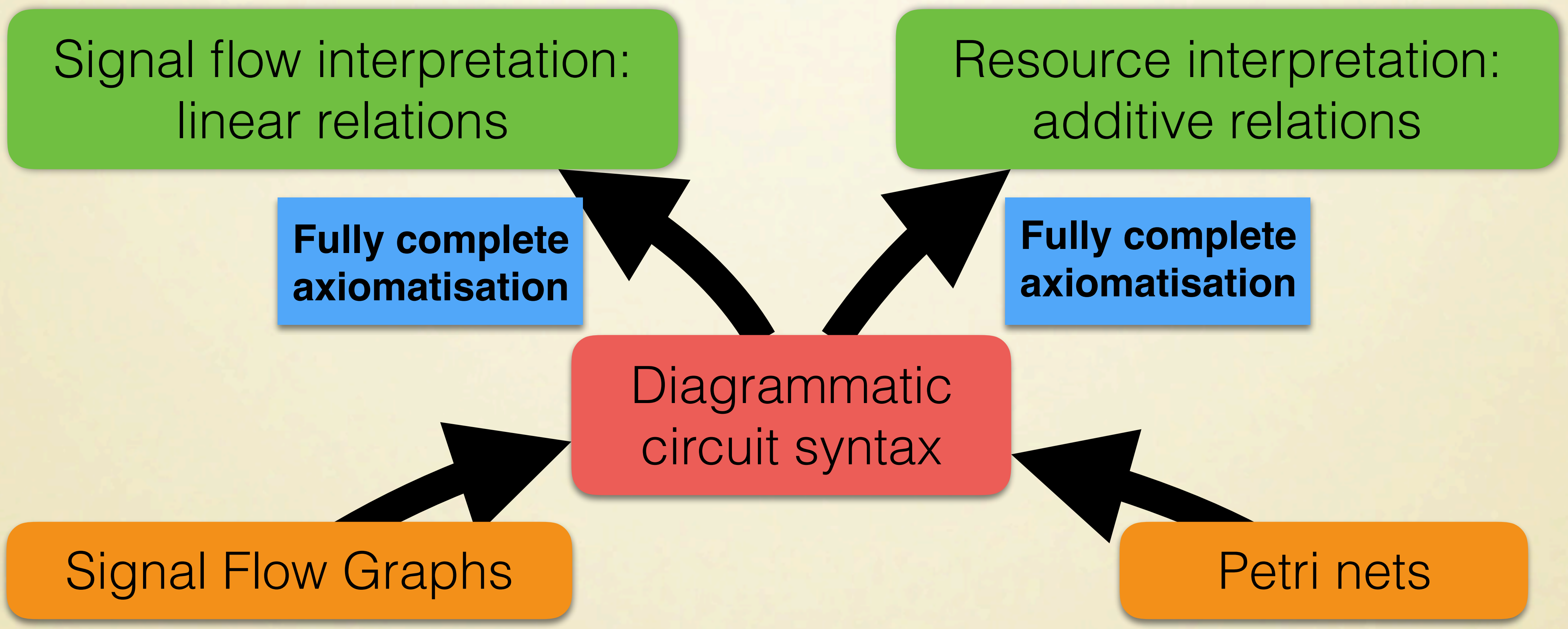
$$\frac{o \le m}{(\rightarrow\!\bigcirc\!-, m) \xrightarrow[o]{i} (\rightarrow\!\bigcirc\!-, m - o + i)}$$

Banking semantics

$$\frac{m + i = m' + o}{(\rightarrow\!\textcircled{b}\!-, m) \xrightarrow[o]{i} (\rightarrow\!\textcircled{b}\!-, m')}$$

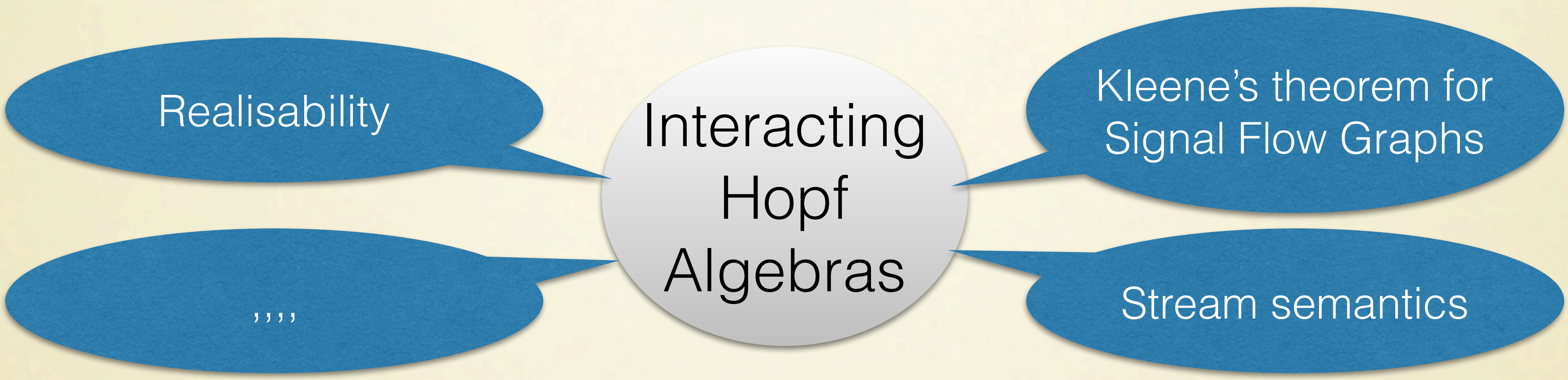
Conclusions

Summary

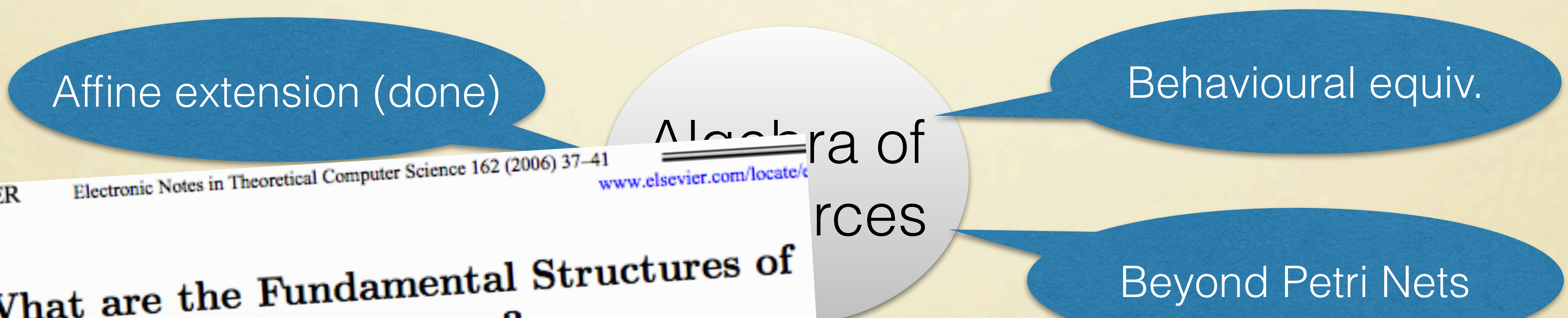

Signal flow interpretation: linear relations
Resource interpretation: additive relations
Fully complete axiomatisation
Fully complete axiomatisation
Diagrammatic circuit syntax
Signal Flow Graphs
Petri nets

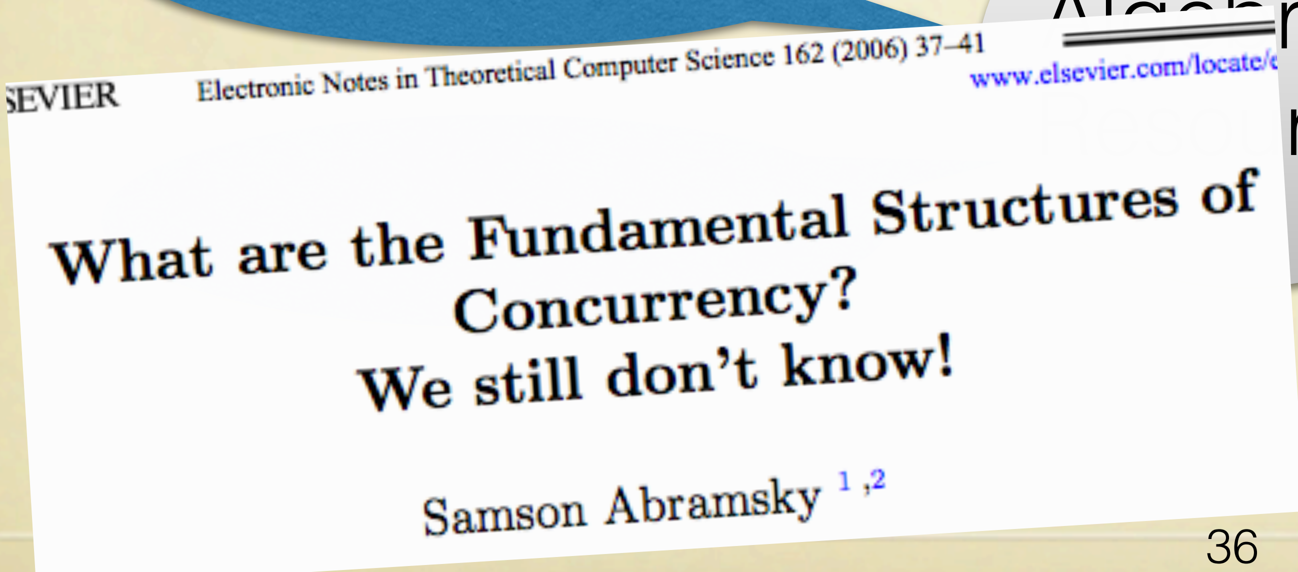
Ramifications

- The signal flow perspective is more developed (2014-'18).



- In the resource perspective, much remains to be explored.

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www.elsevier.com/locate/

What are the Fundamental Structures of Concurrency?
We still don't know!

Samson Abramsky [1,2]

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