Interacting Hopf Algebras

The Theory of Linear Systems

Fabio Zanasi

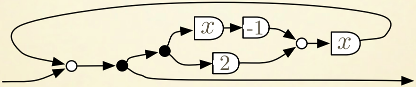
ENS DE LYON

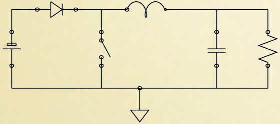
5 October 2015

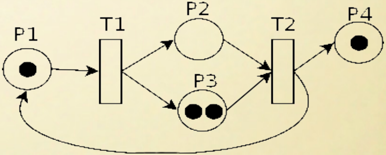
Introduction

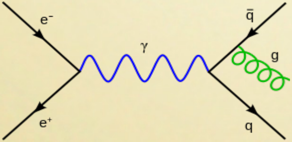
Graphical formalisms

Diagrammatic notations play an important role in various branches of science.









Formal approaches to network diagrams

monolithic approach

influenced by physics

compositional approach

influenced by computer science

In this thesis

A compositional theory of linear systems

In this thesis

A compositional theory of linear systems

- (diagrammatic) syntax
- denotational semantics
- sound and complete axiomatisation
  - the theory of *interacting Hopf algebras*

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A compositional theory of linear systems

- (diagrammatic) syntax
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leading example:
signal flow diagrams

In this thesis

A compositional theory of linear systems

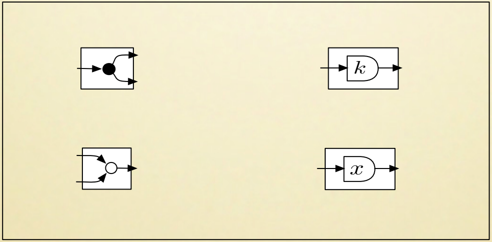
- (diagrammatic) syntax
- structural operational semantics
- denotational semantics
- sound and complete axiomatisation
  - the theory of *interacting Hopf algebras*
- full abstraction
- realisability

leading example:
signal flow diagrams

Syntax and Semantics

Signal Flow Graphs

- Signal Flow Graphs are **stream processing circuits** studied in Control Theory since the 1950s.
- Constructed combining four kinds of gate



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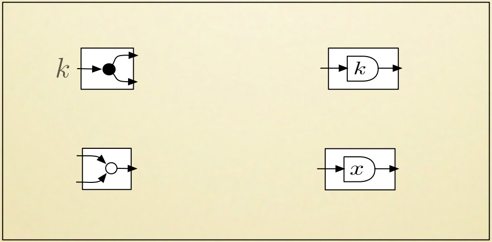

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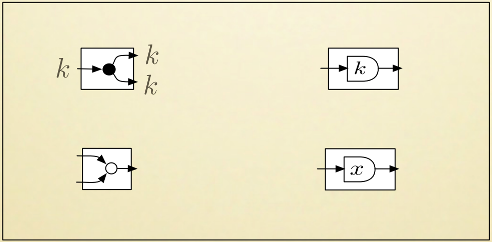

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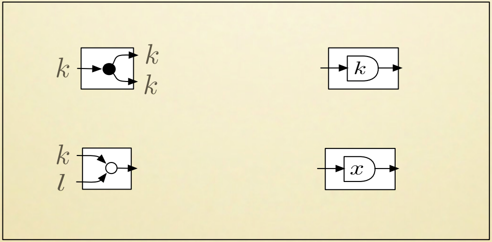

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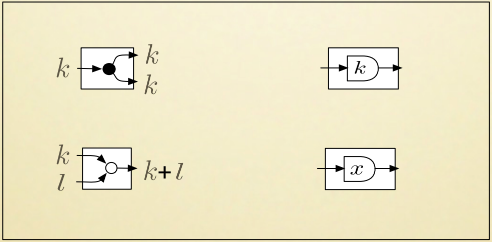

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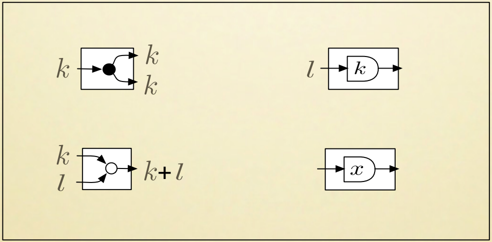

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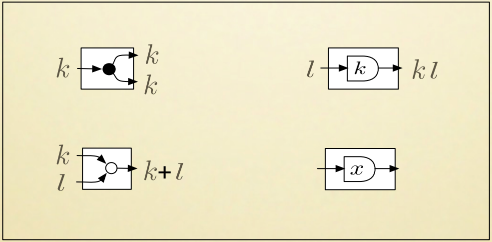

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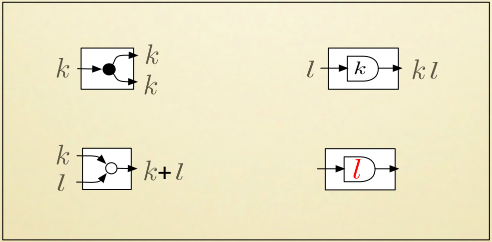

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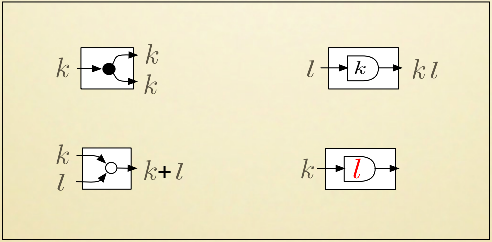

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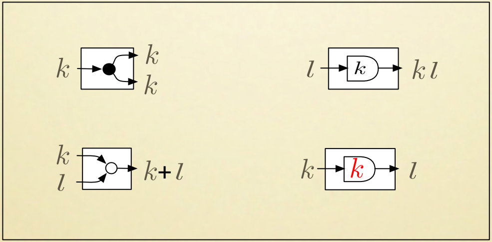

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Signal flow graphs

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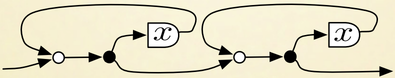

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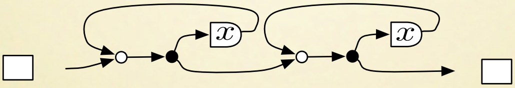

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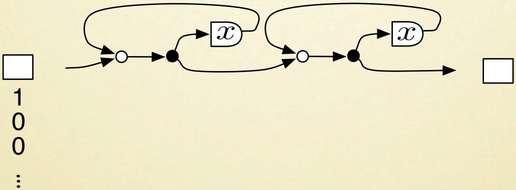

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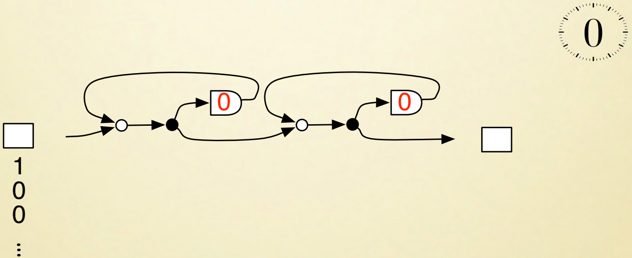

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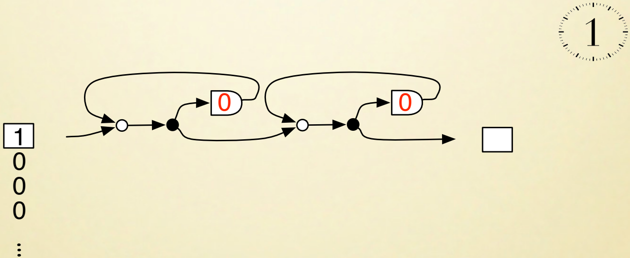

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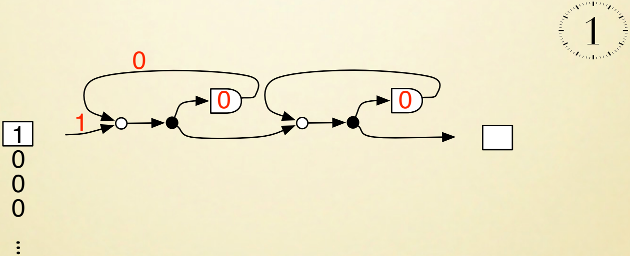

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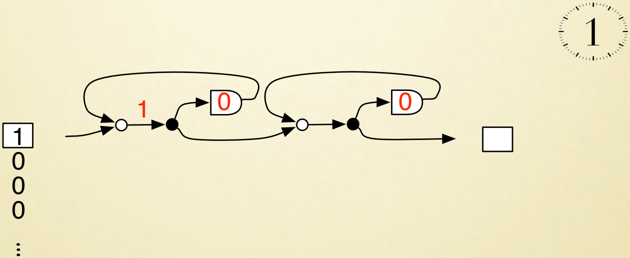

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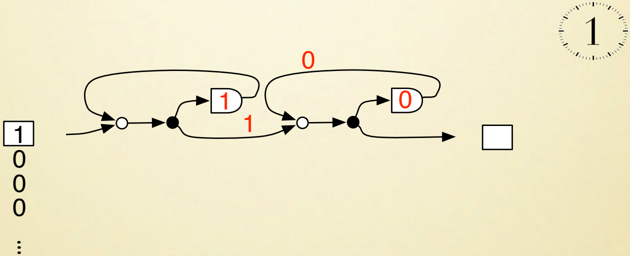

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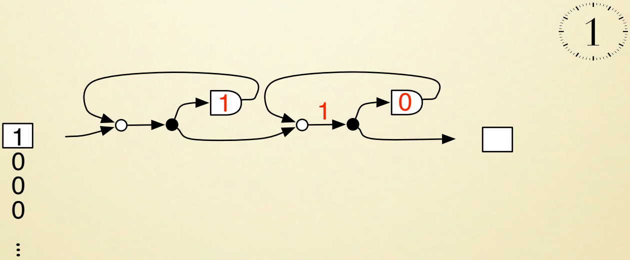

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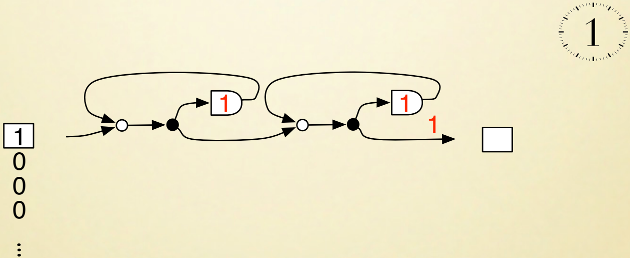

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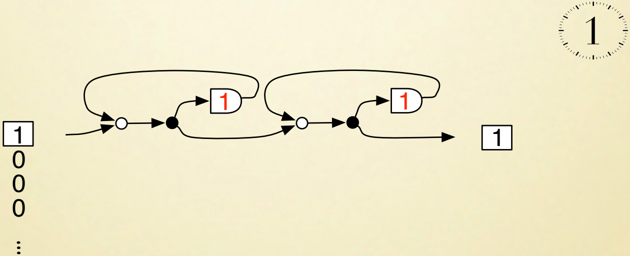

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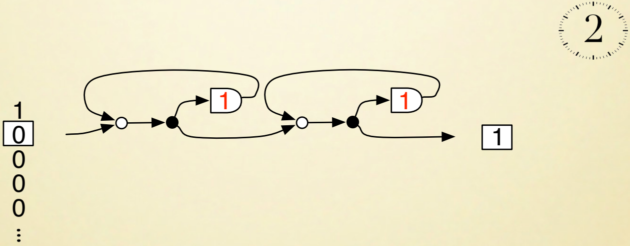

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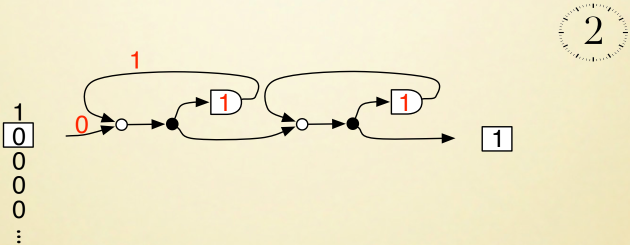

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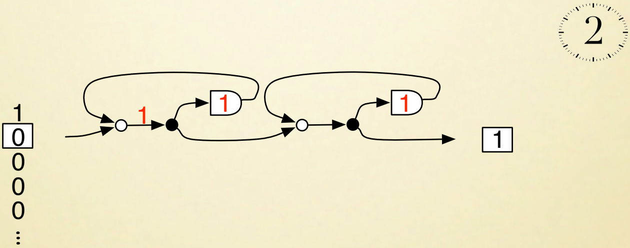

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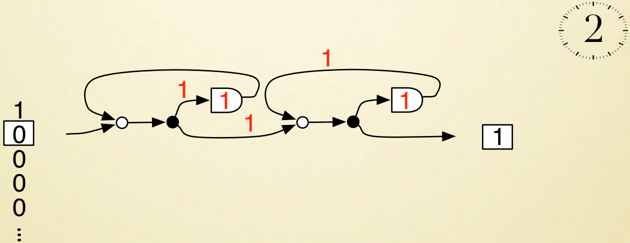

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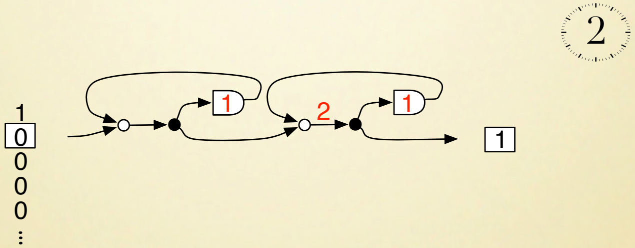

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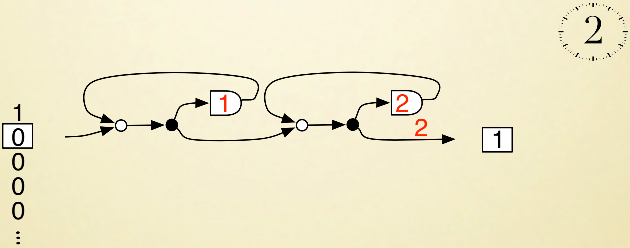

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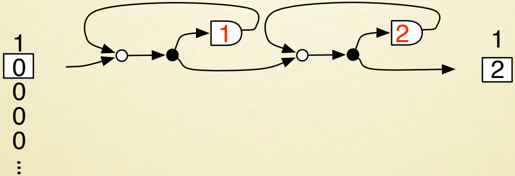

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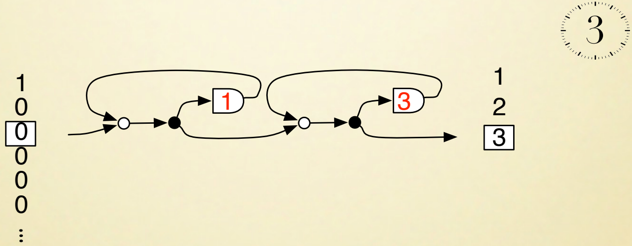

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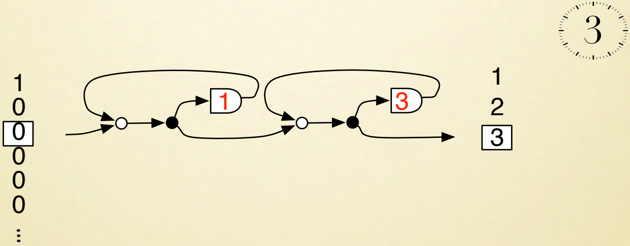
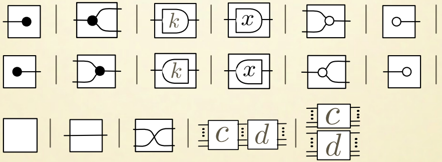


Input $1000\ldots$ produces $1234\ldots$.

Syntax

diagrams are generated by the grammar

$c, d ::=$



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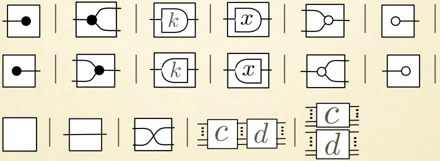
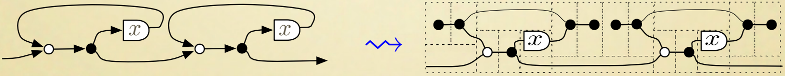


We can represent (orthodox) signal flow graphs as diagrams:



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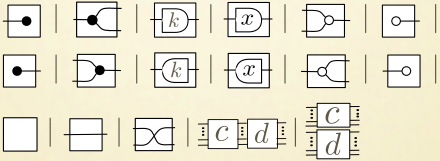

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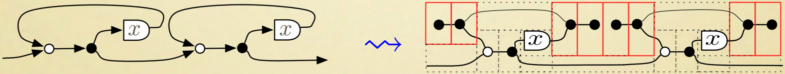
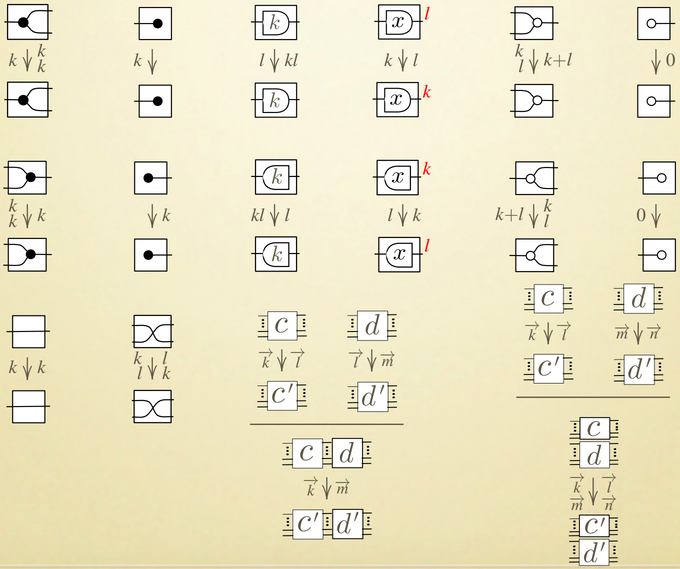


Structural Operational Semantics



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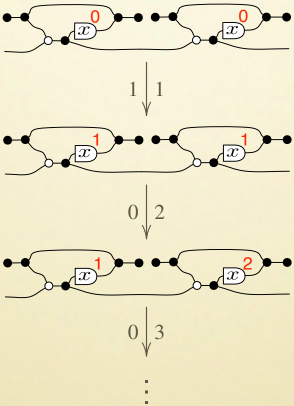
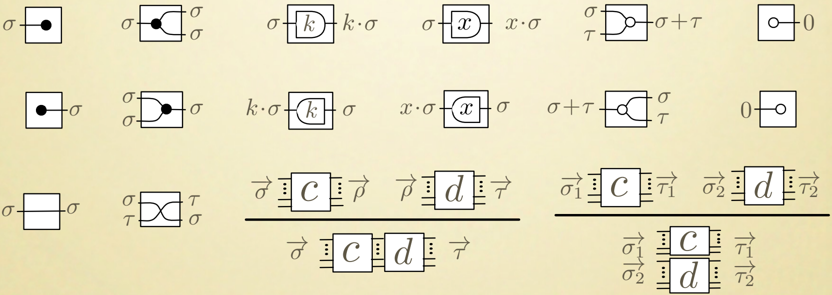


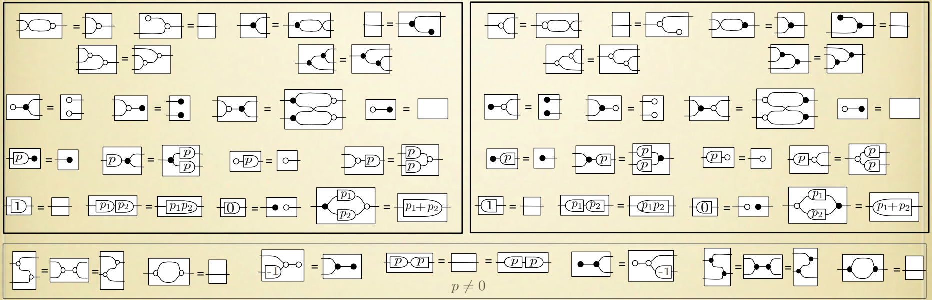
Denotational Semantics

The semantics $\langle\!\langle\cdot\rangle\!\rangle$ maps a diagram to a linear relation between stream vectors (i.e. a subspace over streams).



Interacting Hopf Algebras

The equational theory $\mathbb{IH}$

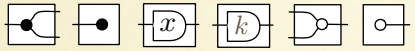


Theorem (soundness and completeness)

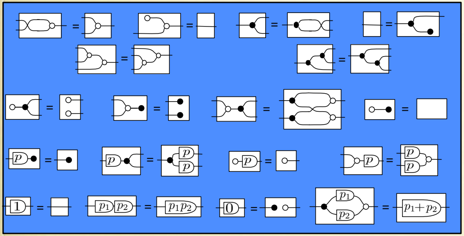
For any diagrams c, d,

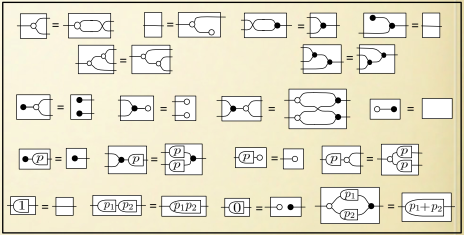
$$c \stackrel{\mathbb{IH}}{=} d \quad \Leftrightarrow \quad \langle\langle c \rangle\rangle = \langle\langle d \rangle\rangle.$$

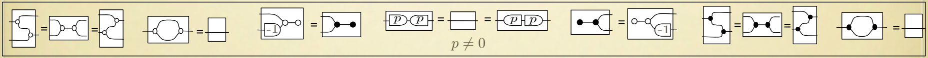
The equational theory $\mathbb{IH}$

interaction of 

the theory $\mathbb{HA}_{\mathsf{k}[x]}$ of $\mathsf{k}[x]$-Hopf algebras







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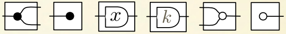
Modular construction of $\mathbb{IH}$

interaction of 

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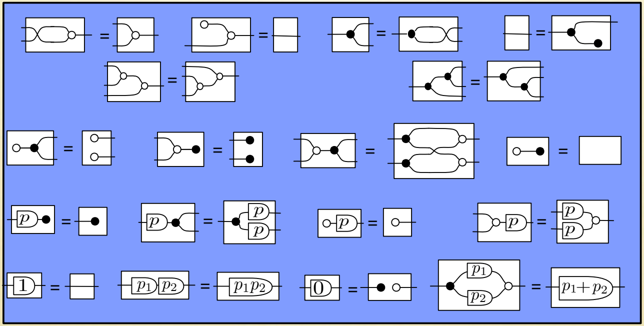

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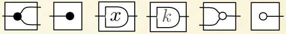
interaction of 

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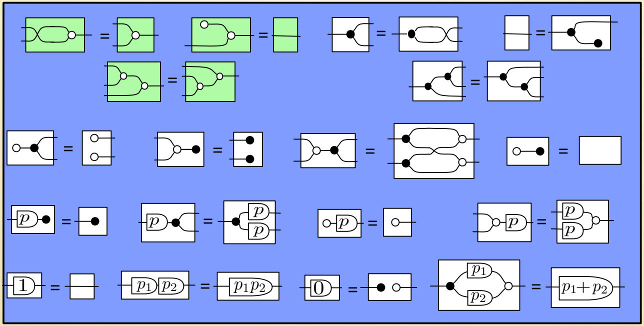

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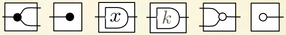
interaction of 

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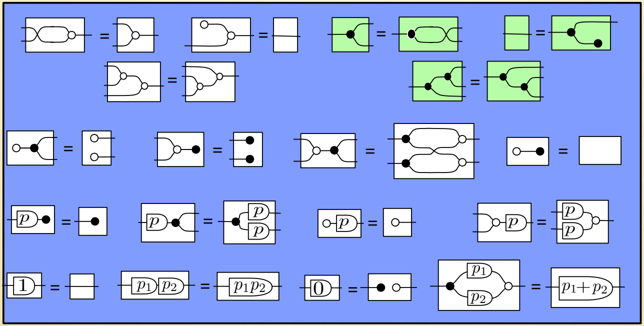

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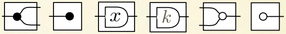
interaction of 

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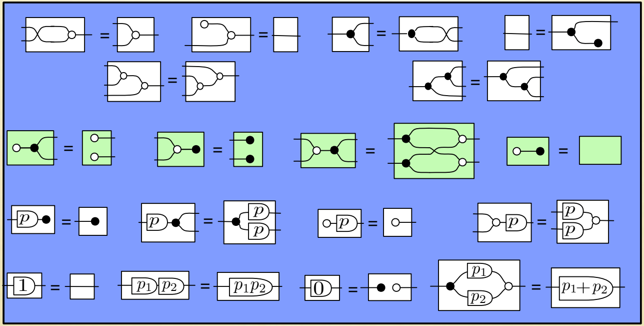

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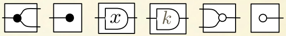
interaction of 

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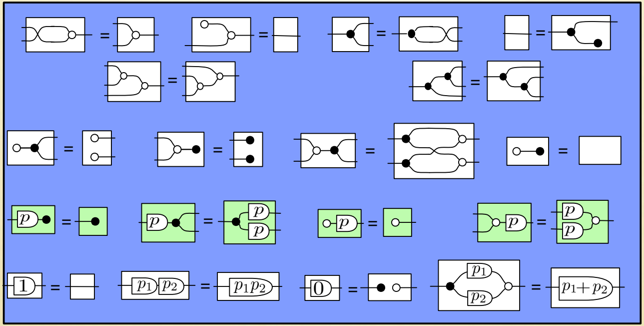
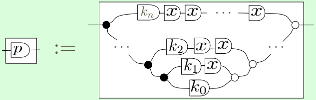


the diagram  expresses a polynomial
$p = k_0 + k_1 x + \cdots + k_n x^n$



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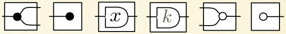
interaction of 

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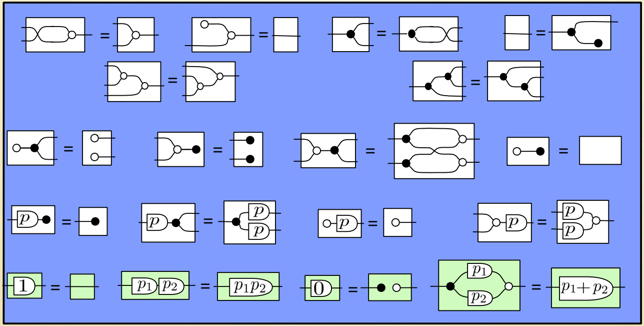

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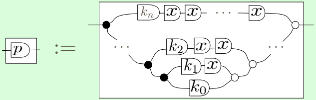

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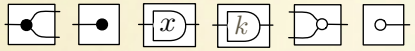
interaction of 

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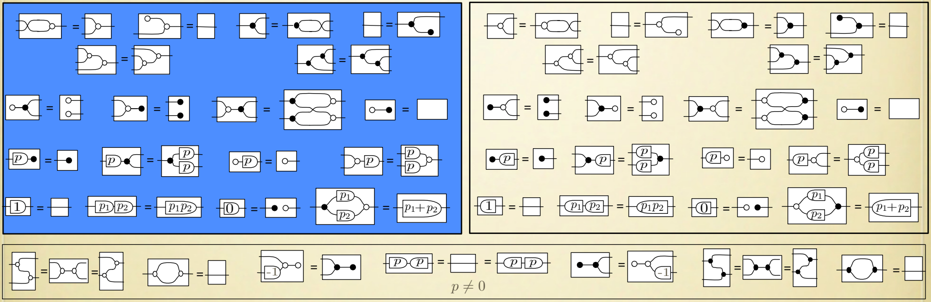

Theorem (soundness and completeness)

For any diagrams c, d,

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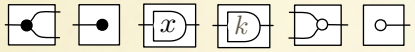
The equational theory $\mathbb{IH}$

interaction of 

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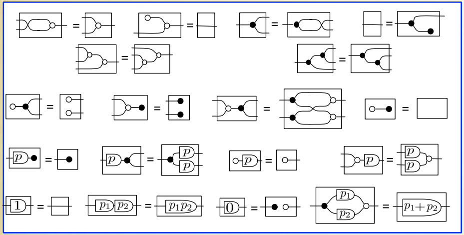
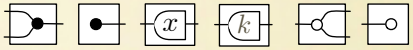
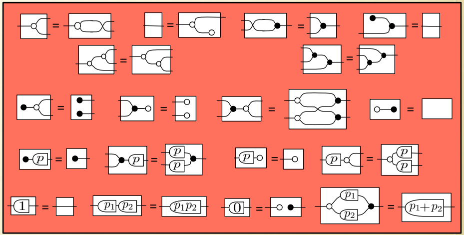
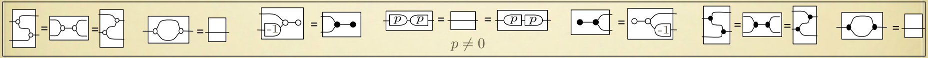


interaction of 

the theory $\mathbb{HA}^{op}_{\mathsf{k}[x]}$ of “co”$\mathsf{k}[x]$-Hopf algebras





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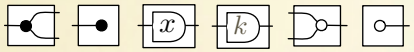
interaction of 

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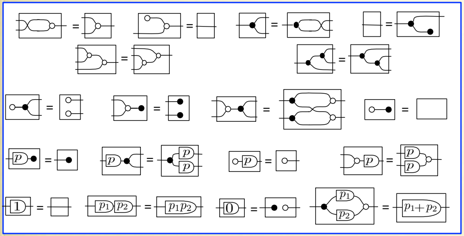
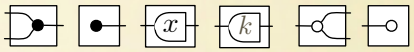


interaction of 

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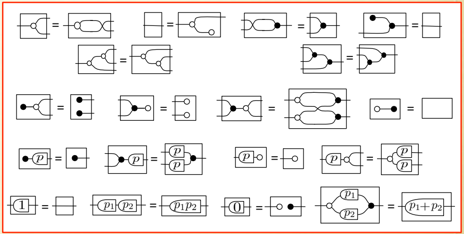
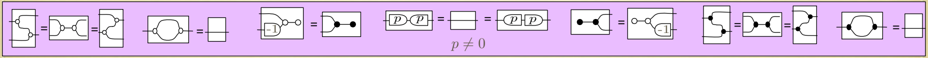
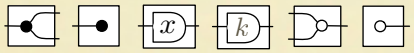
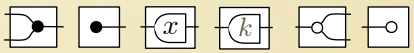




interaction of  with 

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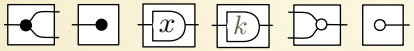
interaction of 

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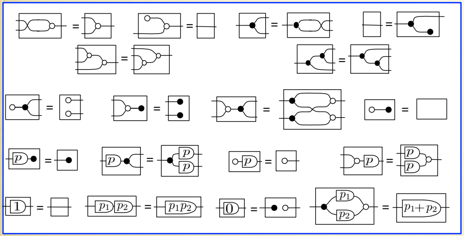
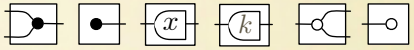


interaction of 

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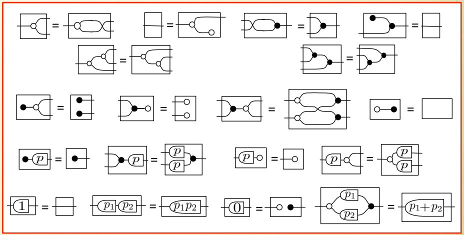
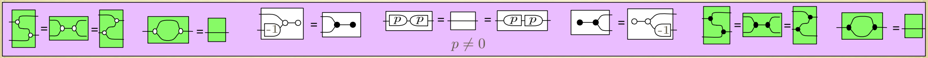
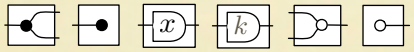
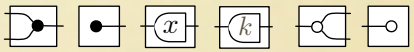




interaction of  with 

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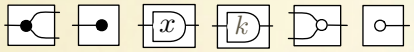
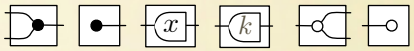
interaction of 

the theory $\mathbb{HA}_{\mathsf{k}[x]}$ of $\mathsf{k}[x]$-Hopf algebras

interaction of 

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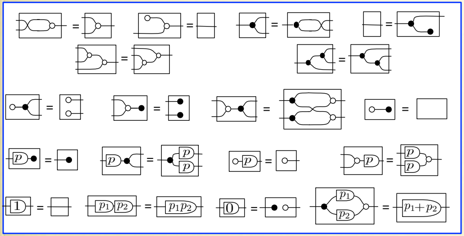
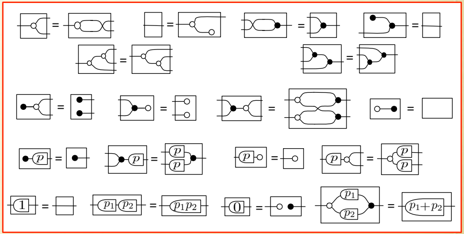
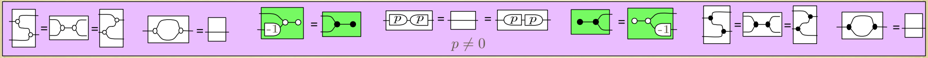
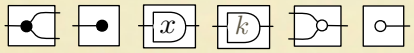
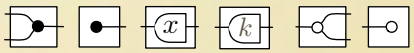






interaction of  with 

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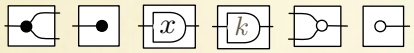
interaction of 

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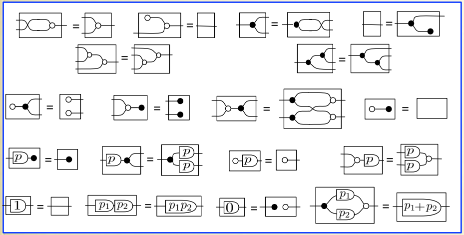
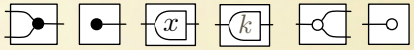


interaction of 

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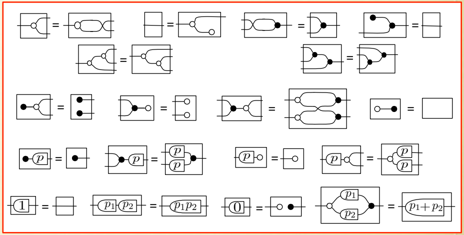
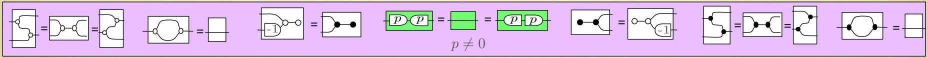
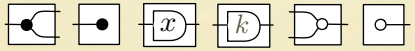
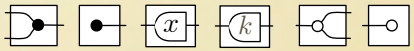




interaction of  with 

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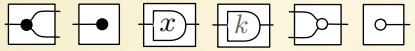
interaction of 

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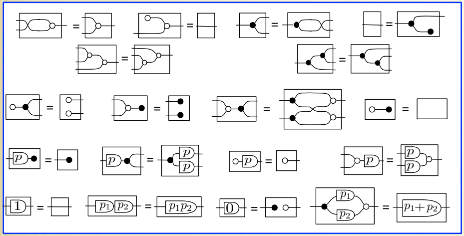
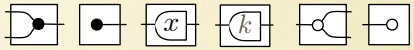


interaction of 

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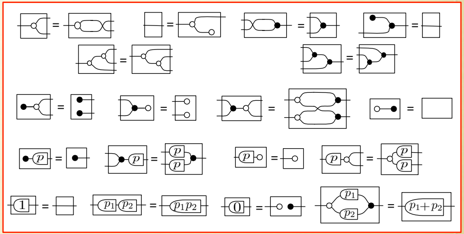
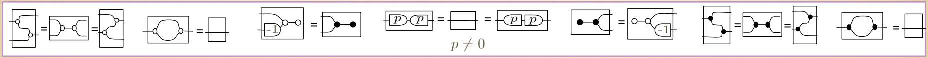
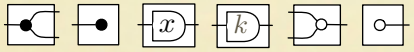
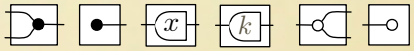




interaction of  with 

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Modular construction of $\mathbb{II}$

$\boxed{\mathbb{HA}_{\mathsf{k}[x]}}$ $\boxed{\mathbb{HA}^{op}_{\mathsf{k}[x]}}$

Modular construction of $\mathbb{IH}$

$$\begin{tikzcd} \boxed{\mathbb{HA}_{\mathsf{k}[x]}} \arrow[d, "\cong"'] & \boxed{\mathbb{HA}^{op}_{\mathsf{k}[x]}} \arrow[d, "\cong"'] \\ \mathsf{Mat}_{\mathsf{k}[x]} & \mathsf{Mat}^{op}_{\mathsf{k}[x]} \end{tikzcd}$$

Modular construction of $\mathbb{IH}$

$$
\begin{tikzcd}
\boxed{\mathbb{HA}_{\mathsf{k}[x]}} + \boxed{\mathbb{HA}^{op}_{\mathsf{k}[x]}} \arrow[d, "\cong"'] \\
\mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]}
\end{tikzcd}
$$

Modular construction of $\mathbb{IH}$

$$\begin{tikzcd}
\boxed{\mathbb{HA}_{\mathsf{k}[x]}} + \boxed{\mathbb{HA}^{op}_{\mathsf{k}[x]}} \arrow[d, "\cong"'] & & \boxed{\mathbb{IH}_{\mathsf{k}[x]}} \\
\mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} & &
\end{tikzcd}$$

Modular construction of $\mathbb{IH}$

$$
\begin{tikzcd}
 & \boxed{\mathbb{HA}_{\mathsf{k}[x]}} + \boxed{\mathbb{HA}^{op}_{\mathsf{k}[x]}} \arrow[r] \arrow[dl] \arrow[dd, "\cong"'] & \mathbb{HA}^{op}_{\mathsf{k}[x]} \mathbin{;} \mathbb{HA}_{\mathsf{k}[x]} \\
\mathbb{HA}_{\mathsf{k}[x]} \mathbin{;} \mathbb{HA}^{op}_{\mathsf{k}[x]} & & \boxed{\mathbb{IH}_{\mathsf{k}[x]}} \\
 & \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} &
\end{tikzcd}
$$

Modular construction of $\mathbb{IH}$

distributive law

distributive law

$$
\begin{tikzcd}
& \boxed{\mathbb{HA}_{\mathsf{k}[x]}} + \boxed{\mathbb{HA}^{op}_{\mathsf{k}[x]}} \arrow[dl] \arrow[r] \arrow[dd, "\cong"'] & \boxed{\mathbb{HA}^{op}_{\mathsf{k}[x]}} ; \boxed{\mathbb{HA}_{\mathsf{k}[x]}} \\
\boxed{\mathbb{HA}_{\mathsf{k}[x]}} ; \boxed{\mathbb{HA}^{op}_{\mathsf{k}[x]}} & & \boxed{\mathbb{IH}_{\mathsf{k}[x]}} \\
& \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} &
\end{tikzcd}
$$

Modular construction of $\mathbb{IH}$

distributive law

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$$
\begin{tikzcd}
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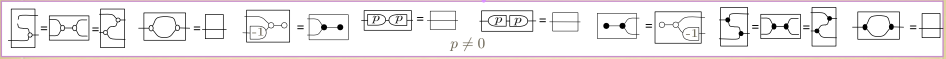
\mathbb{HA}_{\mathsf{k}[x]} ; \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] & & \mathbb{IH}_{\mathsf{k}[x]} & \\
 & \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} & &
\end{tikzcd}
$$



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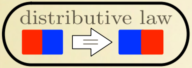
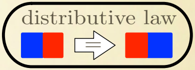
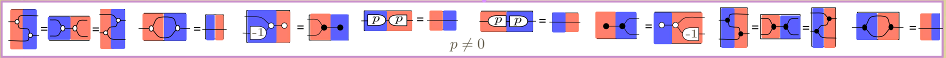




$$
\begin{tikzcd}
 & \mathbb{HA}_{\mathsf{k}[x]} + \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{k}[x]} ; \mathbb{HA}_{\mathsf{k}[x]} \arrow[dl] \\
\mathbb{HA}_{\mathsf{k}[x]} ; \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] & & \mathbb{IH}_{\mathsf{k}[x]} & \\
 & \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} & &
\end{tikzcd}
$$



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$$
\begin{tikzcd}
 & \mathbb{HA}_{\mathsf{k}[x]} + \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{k}[x]} \,;\, \mathbb{HA}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{k}[x]} \,;\, \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{k}[x]} & \\
 & \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{k}[x]} \,;\, \mathsf{Mat}_{\mathsf{k}[x]} \\
\mathsf{Mat}_{\mathsf{k}[x]} \,;\, \mathsf{Mat}^{op}_{\mathsf{k}[x]} & & &
\end{tikzcd}
$$

Modular construction of $\mathbb{IH}$

$$\begin{tikzcd}
 & \mathbb{HA}_{\mathsf{k}[x]} + \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{k}[x]} \,;\mathbb{HA}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{k}[x]} \,;\mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{k}[x]} & \\
 & \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{k}[x]} \,;\mathsf{Mat}_{\mathsf{k}[x]} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{k}[x]} \,;\mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] & & \mathsf{SV}_{\mathsf{k}(x)} &
\end{tikzcd}$$

Modular construction of $\mathbb{IH}$

$$
\begin{tikzcd}
 & \mathbb{HA}_{\mathsf{k}[x]} + \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{k}[x]} \mathbin{;} \mathbb{HA}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{k}[x]} \mathbin{;} \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{k}[x]} & \\
 & \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{k}[x]} \mathbin{;} \mathsf{Mat}_{\mathsf{k}[x]} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{k}[x]} \mathbin{;} \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] & & \mathsf{SV}_{\boxed{\mathsf{k}(x)}} &
\end{tikzcd}
$$

the field of fractions
of polynomials

Modular construction of $\mathbb{IH}$

$$
\begin{tikzcd}
& \mathbb{HA}_{\mathsf{k}[x]} + \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{k}[x]} \,;\, \mathbb{HA}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{k}[x]} \,;\, \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{k}[x]} \arrow[dd, dotted, "\cong"] & \\
& \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{k}[x]} \,;\, \mathsf{Mat}_{\mathsf{k}[x]} \arrow[dl] \\
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\end{tikzcd}
$$

the field of fractions
of polynomials

Modular construction of $\mathbb{IH}$

$$
\begin{tikzcd}
 & \mathbb{HA}_{\mathsf{k}[x]} + \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{k}[x]} \,;\mathbb{HA}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{k}[x]} \,;\mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{k}[x]} \arrow[dd, dotted, "\cong"] & \\
 & \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathsf{Mat}^{op}_{\mathsf{k}[x]} \,;\mathsf{Mat}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd] \\
\mathsf{Mat}_{\mathsf{k}[x]} \,;\mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd] & & \mathsf{SV}_{\mathsf{k}(x)} \arrow[dd, dotted] & \\
 & \mathsf{Mat}_{\mathsf{k}[[x]]} + \mathsf{Mat}^{op}_{\mathsf{k}[[x]]} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{k}[[x]]} \,;\mathsf{Mat}_{\mathsf{k}[[x]]} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{k}[[x]]} \,;\mathsf{Mat}^{op}_{\mathsf{k}[[x]]} \arrow[rr] & & \mathsf{SV}_{\mathsf{k}((x))} &
\end{tikzcd}
$$

Modular construction of $\mathbb{IH}$

$$
\begin{tikzcd}
 & \mathbb{HA}_{\mathsf{k}[x]} + \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{k}[x]} \,;\, \mathbb{HA}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{k}[x]} \,;\, \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{k}[x]} \arrow[dd, dotted, "\cong"] & \\
 & \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathsf{Mat}^{op}_{\mathsf{k}[x]} \,;\, \mathsf{Mat}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd] \\
\mathsf{Mat}_{\mathsf{k}[x]} \,;\, \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd] & & \mathsf{SV}_{\mathsf{k}(x)} \arrow[dd, dotted] & \\
 & \mathsf{Mat}_{\mathsf{k}[[x]]} + \mathsf{Mat}^{op}_{\mathsf{k}[[x]]} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{k}[[x]]} \,;\, \mathsf{Mat}_{\mathsf{k}[[x]]} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{k}[[x]]} \,;\, \mathsf{Mat}^{op}_{\mathsf{k}[[x]]} \arrow[rr] & & \mathsf{SV}_{\boxed{\mathsf{k}((x))}} &
\end{tikzcd}
$$

the field of formal
Laurent series (streams)

Modular construction of $\mathbb{IH}$

$$
\begin{tikzcd}[column sep=small]
 & \mathbb{HA}_{\mathsf{k}[x]} + \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{k}[x]} \,;\, \mathbb{HA}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{k}[x]} \,;\, \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{k}[x]} \arrow[dd, dotted, "\cong"] \arrow[dddd, bend left, red, "\langle\langle\cdot\rangle\rangle"] & \\
 & \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathsf{Mat}^{op}_{\mathsf{k}[x]} \,;\, \mathsf{Mat}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd] \\
\mathsf{Mat}_{\mathsf{k}[x]} \,;\, \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd] & & \mathsf{SV}_{\mathsf{k}(x)} \arrow[dd, dotted] & \\
 & \mathsf{Mat}_{\mathsf{k}[[x]]} + \mathsf{Mat}^{op}_{\mathsf{k}[[x]]} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{k}[[x]]} \,;\, \mathsf{Mat}_{\mathsf{k}[[x]]} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{k}[[x]]} \,;\, \mathsf{Mat}^{op}_{\mathsf{k}[[x]]} \arrow[rr] & & \mathsf{SV}_{\boxed{\mathsf{k}((x))}} &
\end{tikzcd}
$$

the field of formal
Laurent series (streams)

Modular construction of $\mathbb{IH}$

$$
\begin{tikzcd}[column sep=small]
 & \mathbb{HA}_{\mathsf{k}[x]} + \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{k}[x]} \,;\, \mathbb{HA}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{k}[x]} \,;\, \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{k}[x]} \arrow[dd, dotted, "\cong"] \arrow[dddd, bend left, red, "\langle\!\langle \cdot \rangle\!\rangle"] & \\
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\mathsf{Mat}_{\mathsf{k}[x]} \,;\, \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd] & & \mathsf{SV}_{\mathsf{k}(x)} \arrow[dd, dotted] & \\
 & \mathsf{Mat}_{\mathsf{k}[[x]]} + \mathsf{Mat}^{op}_{\mathsf{k}[[x]]} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{k}[[x]]} \,;\, \mathsf{Mat}_{\mathsf{k}[[x]]} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{k}[[x]]} \,;\, \mathsf{Mat}^{op}_{\mathsf{k}[[x]]} \arrow[rr] & & \mathsf{SV}_{\boxed{\mathsf{k}((x))}} &
\end{tikzcd}
$$

the field of formal
Laurent series (streams)

Theorem (soundness and completeness)

For any diagrams c, d,

$$c \stackrel{\mathbb{IH}}{=} d \quad \Leftrightarrow \quad \langle\!\langle c \rangle\!\rangle = \langle\!\langle d \rangle\!\rangle.$$

One cube to rule them all

$$
\begin{tikzcd}
& \mathbb{HA}_{\mathsf{k}[x]} + \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{k}[x]} \mathbin{;} \mathbb{HA}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{k}[x]} \mathbin{;} \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{k}[x]} \arrow[dd, dotted, "\cong"] & \\
& \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{k}[x]} \mathbin{;} \mathsf{Mat}_{\mathsf{k}[x]} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{k}[x]} \mathbin{;} \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] & & \mathsf{SV}_{\mathsf{k}(x)} &
\end{tikzcd}
$$

One cube to rule them all

$$
\begin{tikzcd}[column sep=small]
& \mathbb{HA}_{\mathsf{k}[x]} + \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{k}[x]} \,;\mathbb{HA}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{k}[x]} \,;\mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr, crossing over] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{k}[x]} & \\
& \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{k}[x]} \,;\mathsf{Mat}_{\mathsf{k}[x]} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{k}[x]} \,;\mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] & & \mathsf{SV}_{\mathsf{k}(x)} \arrow[from=uu, dotted, crossing over, "\cong"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
& \mathbb{HA}_{\mathsf{R}} + \mathbb{HA}^{op}_{\mathsf{R}} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{R}} \,;\mathbb{HA}_{\mathsf{R}} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{R}} \,;\mathbb{HA}^{op}_{\mathsf{R}} \arrow[rr, crossing over] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}} & \\
& \mathsf{Mat}_{\mathsf{R}} + \mathsf{Mat}^{op}_{\mathsf{R}} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{R}} \,;\mathsf{Mat}_{\mathsf{R}} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{R}} \,;\mathsf{Mat}^{op}_{\mathsf{R}} \arrow[rr] & & \mathsf{SV}_{\mathsf{k}} \arrow[from=uu, dotted, crossing over, "\cong"] &
\end{tikzcd}
$$

One cube to rule them all

$$\begin{tikzcd}
 & \mathbb{HA}_{\mathsf{k}[x]} + \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{k}[x]} \,;\mathbb{HA}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{k}[x]} \,;\mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{k}[x]} \arrow[dd, dotted, "\cong"] & \\
 & \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{k}[x]} \,;\mathsf{Mat}_{\mathsf{k}[x]} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{k}[x]} \,;\mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] & & \mathsf{SV}_{\mathsf{k}(x)} &
\end{tikzcd}$$

principal
ideal domain

$$\begin{tikzcd}
 & \mathbb{HA}_{\boxed{\mathsf{R}}} + \mathbb{HA}^{op}_{\mathsf{R}} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{R}} \,;\mathbb{HA}_{\mathsf{R}} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{R}} \,;\mathbb{HA}^{op}_{\mathsf{R}} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}} \arrow[dd, dotted, "\cong"] & \\
 & \mathsf{Mat}_{\mathsf{R}} + \mathsf{Mat}^{op}_{\mathsf{R}} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{R}} \,;\mathsf{Mat}_{\mathsf{R}} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{R}} \,;\mathsf{Mat}^{op}_{\mathsf{R}} \arrow[rr] & & \mathsf{SV}_{\boxed{\mathsf{k}}} &
\end{tikzcd}$$

the field of
fractions of R

One cube to rule them all

case $\mathsf{k}[x]$: calculus of signal flow diagrams (control theory)

$$
\begin{tikzcd}
& \mathbb{HA}_{\mathsf{k}[x]} + \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{k}[x]} \,;\mathbb{HA}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{k}[x]} \,;\mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr, crossing over] \arrow[dd, "\cong"'] & & \boxed{\mathbb{IH}_{\mathsf{k}[x]}} & \\
& \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{k}[x]} \,;\mathsf{Mat}_{\mathsf{k}[x]} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{k}[x]} \,;\mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] & & \mathsf{SV}_{\mathsf{k}(x)} \arrow[from=uu, dotted, "\cong", crossing over] &
\end{tikzcd}
$$

principal ideal domain

$$
\begin{tikzcd}
& \mathbb{HA}_{\boxed{\mathsf{R}}} + \mathbb{HA}^{op}_{\mathsf{R}} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{R}} \,;\mathbb{HA}_{\mathsf{R}} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{R}} \,;\mathbb{HA}^{op}_{\mathsf{R}} \arrow[rr, crossing over] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}} & \\
& \mathsf{Mat}_{\mathsf{R}} + \mathsf{Mat}^{op}_{\mathsf{R}} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{R}} \,;\mathsf{Mat}_{\mathsf{R}} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{R}} \,;\mathsf{Mat}^{op}_{\mathsf{R}} \arrow[rr] & & \mathsf{SV}_{\boxed{\mathsf{k}}} \arrow[from=uu, dotted, "\cong", crossing over] &
\end{tikzcd}
$$

the field of fractions of R

One cube to rule them all

case $\mathsf{k}[x]$: calculus of signal flow diagrams (control theory)

$$
\begin{tikzcd}
& \mathbb{HA}_{\mathsf{k}[x]} + \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{k}[x]} \,;\mathbb{HA}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{k}[x]} \,;\mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd, "\cong"'] & & \boxed{\mathbb{IH}_{\mathsf{k}[x]}} \arrow[dd, dotted, "\cong"] & \\
& \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr, dashed] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{k}[x]} \,;\mathsf{Mat}_{\mathsf{k}[x]} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{k}[x]} \,;\mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] & & \mathsf{SV}_{\mathsf{k}(x)} &
\end{tikzcd}
$$

principal ideal domain

$$
\begin{tikzcd}
& \mathbb{HA}_{\boxed{\mathsf{R}}} + \mathbb{HA}^{op}_{\mathsf{R}} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{R}} \,;\mathbb{HA}_{\mathsf{R}} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{R}} \,;\mathbb{HA}^{op}_{\mathsf{R}} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}} \arrow[dd, dotted, "\cong"] & \\
& \mathsf{Mat}_{\mathsf{R}} + \mathsf{Mat}^{op}_{\mathsf{R}} \arrow[rr, dashed] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{R}} \,;\mathsf{Mat}_{\mathsf{R}} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{R}} \,;\mathsf{Mat}^{op}_{\mathsf{R}} \arrow[rr] & & \mathsf{SV}_{\boxed{\mathsf{k}}} &
\end{tikzcd}
$$

the field of fractions of R

$$
\begin{tikzcd}
& \mathbb{HA}_{\mathsf{Z}} + \mathbb{HA}^{op}_{\mathsf{Z}} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{Z}} \,;\mathbb{HA}_{\mathsf{Z}} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{Z}} \,;\mathbb{HA}^{op}_{\mathsf{Z}} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{Z}} \arrow[dd, dotted, "\cong"] & \\
& \mathsf{Mat}_{\mathsf{Z}} + \mathsf{Mat}^{op}_{\mathsf{Z}} \arrow[rr, dashed] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{Z}} \,;\mathsf{Mat}_{\mathsf{Z}} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{Z}} \,;\mathsf{Mat}^{op}_{\mathsf{Z}} \arrow[rr] & & \mathsf{SV}_{\mathsf{Q}} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathbb{HA}_{\mathsf{Z}_2} + \mathbb{HA}^{op}_{\mathsf{Z}_2} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{Z}_2} \,;\mathbb{HA}_{\mathsf{Z}_2} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{Z}_2} \,;\mathbb{HA}^{op}_{\mathsf{Z}_2} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{Z}_2} \arrow[dd, dotted, "\cong"] & \\
& \mathsf{Mat}_{\mathsf{Z}_2} + \mathsf{Mat}^{op}_{\mathsf{Z}_2} \arrow[rr, dashed] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{Z}_2} \,;\mathsf{Mat}_{\mathsf{Z}_2} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{Z}_2} \,;\mathsf{Mat}^{op}_{\mathsf{Z}_2} \arrow[rr] & & \mathsf{SV}_{\mathsf{Z}_2} &
\end{tikzcd}
$$

One cube to rule them all

case $\mathsf{k}[x]$: calculus of signal flow diagrams (control theory)

$$
\begin{tikzcd}
& \mathbb{HA}_{\mathsf{k}[x]} + \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{k}[x]} \,;\mathbb{HA}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{k}[x]} \,;\mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd, "\cong"'] & & \boxed{\mathbb{IH}_{\mathsf{k}[x]}} \arrow[dd, dotted, "\cong"] & \\
& \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{k}[x]} \,;\mathsf{Mat}_{\mathsf{k}[x]} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{k}[x]} \,;\mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] & & \mathsf{SV}_{\mathsf{k}(x)} &
\end{tikzcd}
$$

principal ideal domain

$$
\begin{tikzcd}
& \mathbb{HA}_{\boxed{\mathsf{R}}} + \mathbb{HA}^{op}_{\mathsf{R}} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{R}} \,;\mathbb{HA}_{\mathsf{R}} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{R}} \,;\mathbb{HA}^{op}_{\mathsf{R}} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}} \arrow[dd, dotted, "\cong"] & \\
& \mathsf{Mat}_{\mathsf{R}} + \mathsf{Mat}^{op}_{\mathsf{R}} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{R}} \,;\mathsf{Mat}_{\mathsf{R}} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{R}} \,;\mathsf{Mat}^{op}_{\mathsf{R}} \arrow[rr] & & \mathsf{SV}_{\boxed{\mathsf{k}}} &
\end{tikzcd}
$$

the field of fractions of R

$$
\begin{tikzcd}
& \mathbb{HA}_{\mathsf{Z}} + \mathbb{HA}^{op}_{\mathsf{Z}} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{Z}} \,;\mathbb{HA}_{\mathsf{Z}} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{Z}} \,;\mathbb{HA}^{op}_{\mathsf{Z}} \arrow[rr] \arrow[dd, "\cong"'] & & \boxed{\mathbb{IH}_{\mathsf{Z}}} \arrow[dd, dotted, "\cong"] & \\
& \mathsf{Mat}_{\mathsf{Z}} + \mathsf{Mat}^{op}_{\mathsf{Z}} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{Z}} \,;\mathsf{Mat}_{\mathsf{Z}} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{Z}} \,;\mathsf{Mat}^{op}_{\mathsf{Z}} \arrow[rr] & & \mathsf{SV}_{\mathsf{Q}} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathbb{HA}_{\mathsf{Z}_2} + \mathbb{HA}^{op}_{\mathsf{Z}_2} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{Z}_2} \,;\mathbb{HA}_{\mathsf{Z}_2} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{Z}_2} \,;\mathbb{HA}^{op}_{\mathsf{Z}_2} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{Z}_2} \arrow[dd, dotted, "\cong"] & \\
& \mathsf{Mat}_{\mathsf{Z}_2} + \mathsf{Mat}^{op}_{\mathsf{Z}_2} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{Z}_2} \,;\mathsf{Mat}_{\mathsf{Z}_2} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{Z}_2} \,;\mathsf{Mat}^{op}_{\mathsf{Z}_2} \arrow[rr] & & \mathsf{SV}_{\mathsf{Z}_2} &
\end{tikzcd}
$$

case Z: a graphical syntax for rational subspaces

One cube to rule them all

case $\mathsf{k}[x]$: calculus of signal flow diagrams (control theory)

$$
\begin{tikzcd}
& \mathbb{HA}_{\mathsf{k}[x]} + \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{k}[x]} ; \mathbb{HA}_{\mathsf{k}[x]} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{k}[x]} ; \mathbb{HA}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dd, "\cong"'] & & \boxed{\mathbb{IH}_{\mathsf{k}[x]}} \arrow[dd, dotted, "\cong"] & \\
& \mathsf{Mat}_{\mathsf{k}[x]} + \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{k}[x]} ; \mathsf{Mat}_{\mathsf{k}[x]} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{k}[x]} ; \mathsf{Mat}^{op}_{\mathsf{k}[x]} \arrow[rr] & & \mathsf{SV}_{\mathsf{k}(x)} &
\end{tikzcd}
$$

principal ideal domain

$$
\begin{tikzcd}
& \mathbb{HA}_{\boxed{\mathsf{R}}} + \mathbb{HA}^{op}_{\mathsf{R}} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{R}} ; \mathbb{HA}_{\mathsf{R}} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{R}} ; \mathbb{HA}^{op}_{\mathsf{R}} \arrow[rr] \arrow[dd, "\cong"'] & & \mathbb{IH}_{\mathsf{R}} \arrow[dd, dotted, "\cong"] & \\
& \mathsf{Mat}_{\mathsf{R}} + \mathsf{Mat}^{op}_{\mathsf{R}} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{R}} ; \mathsf{Mat}_{\mathsf{R}} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{R}} ; \mathsf{Mat}^{op}_{\mathsf{R}} \arrow[rr] & & \mathsf{SV}_{\boxed{\mathsf{k}}} &
\end{tikzcd}
$$

the field of fractions of R

$$
\begin{tikzcd}
& \mathbb{HA}_{\mathsf{Z}} + \mathbb{HA}^{op}_{\mathsf{Z}} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{Z}} ; \mathbb{HA}_{\mathsf{Z}} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{Z}} ; \mathbb{HA}^{op}_{\mathsf{Z}} \arrow[rr] \arrow[dd, "\cong"'] & & \boxed{\mathbb{IH}_{\mathsf{Z}}} \arrow[dd, dotted, "\cong"] & \\
& \mathsf{Mat}_{\mathsf{Z}} + \mathsf{Mat}^{op}_{\mathsf{Z}} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{Z}} ; \mathsf{Mat}_{\mathsf{Z}} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{Z}} ; \mathsf{Mat}^{op}_{\mathsf{Z}} \arrow[rr] & & \mathsf{SV}_{\mathsf{Q}} &
\end{tikzcd}
$$

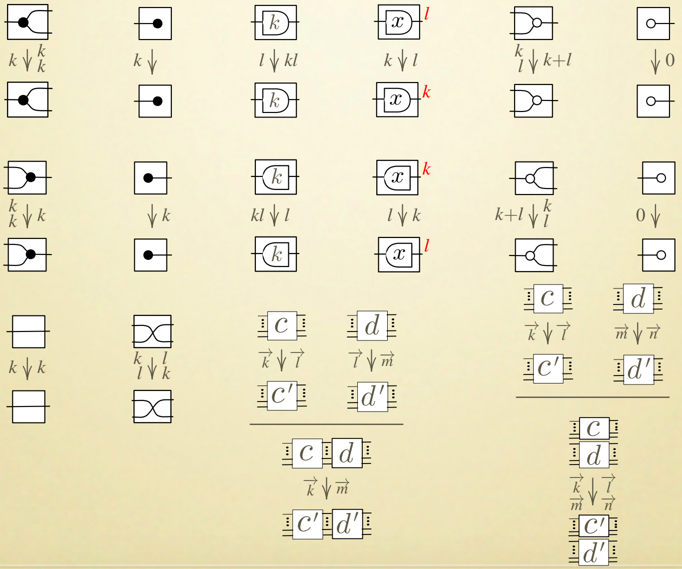
$$
\begin{tikzcd}
& \mathbb{HA}_{\mathsf{Z}_2} + \mathbb{HA}^{op}_{\mathsf{Z}_2} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbb{HA}^{op}_{\mathsf{Z}_2} ; \mathbb{HA}_{\mathsf{Z}_2} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{HA}_{\mathsf{Z}_2} ; \mathbb{HA}^{op}_{\mathsf{Z}_2} \arrow[rr] \arrow[dd, "\cong"'] & & \boxed{\mathbb{IH}_{\mathsf{Z}_2}} \arrow[dd, dotted, "\cong"] & \\
& \mathsf{Mat}_{\mathsf{Z}_2} + \mathsf{Mat}^{op}_{\mathsf{Z}_2} \arrow[rr] \arrow[dl] & & \mathsf{Mat}^{op}_{\mathsf{Z}_2} ; \mathsf{Mat}_{\mathsf{Z}_2} \arrow[dl] \\
\mathsf{Mat}_{\mathsf{Z}_2} ; \mathsf{Mat}^{op}_{\mathsf{Z}_2} \arrow[rr] & & \mathsf{SV}_{\mathsf{Z}_2} &
\end{tikzcd}
$$

case Z: a graphical syntax for rational subspaces

case Z_2
phase-free ZX-calculus
(categorical quantum mechanics)
tweak of the calculus of stateless connectors (concurrency theory)

Full Abstraction and Realisability

Structural Operational Semantics



Full Abstraction

The *observable behaviour* $\langle c \rangle$ of a diagram c is the set of all traces starting from an initial state for c (i.e. one where all the registers are labeled with 0).

Full Abstraction

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Theorem (?)

For any diagrams c and d

$$\langle\langle c \rangle\rangle = \langle\langle d \rangle\rangle \iff \langle c \rangle = \langle d \rangle$$

Full Abstraction

The *observable behaviour* $\langle c \rangle$ of a diagram c is the set of all traces starting from an initial state for c (i.e. one where all the registers are labeled with 0).

Theorem (?)

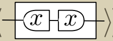
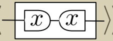
For any diagrams c and d

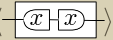
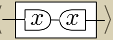
$$\langle\langle c \rangle\rangle = \langle\langle d \rangle\rangle \iff \langle c \rangle = \langle d \rangle$$

Not true in general.
The denotational semantics is *coarser* than the operational semantics.

Full Abstraction

A counterexample

$\langle\!\langle$  $\rangle\!\rangle = \langle\!\langle$  $\rangle\!\rangle = \langle\!\langle$  $\rangle\!\rangle$

$\langle$  $\rangle \subsetneq \langle$  $\rangle \subsetneq \langle$  $\rangle$

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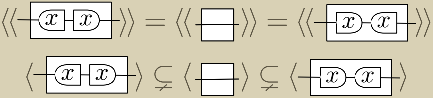

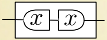

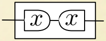

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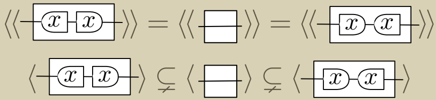

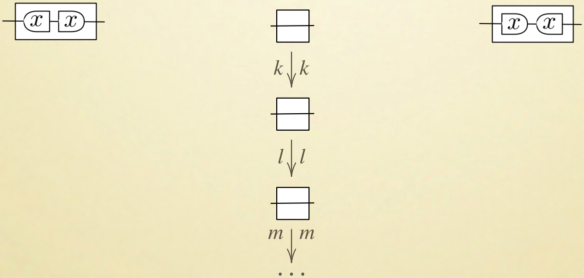

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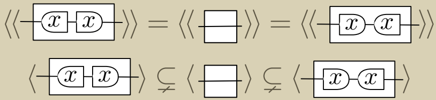

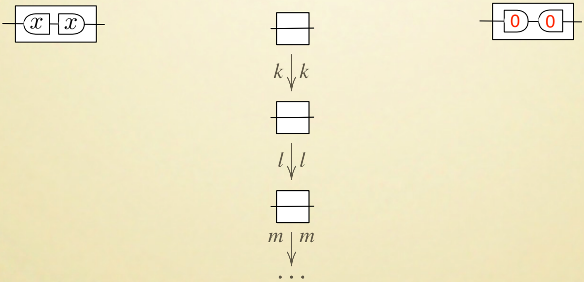

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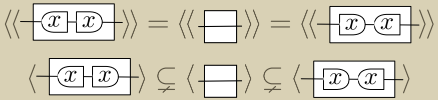

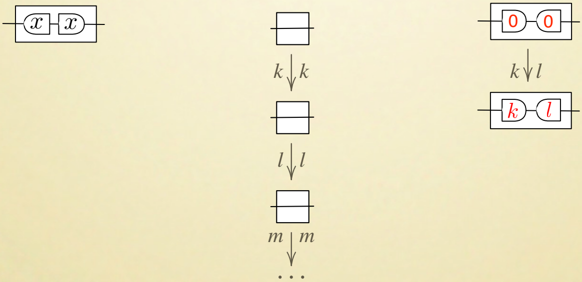

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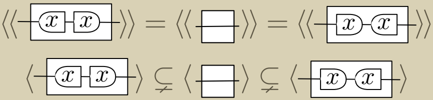

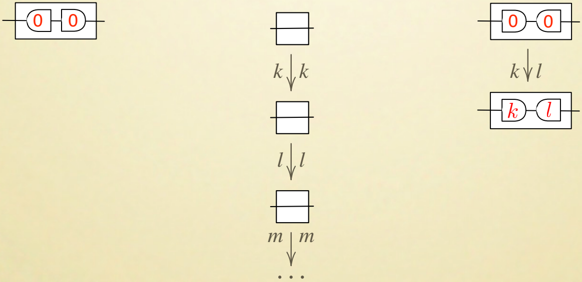

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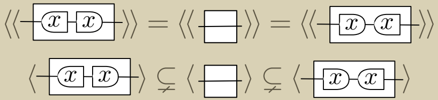

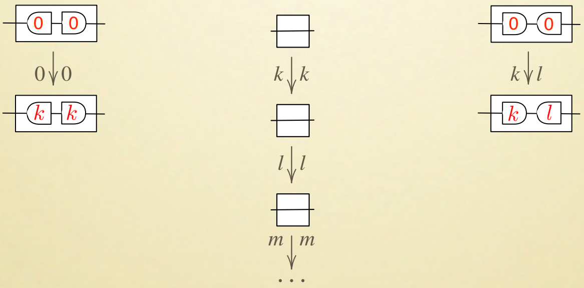

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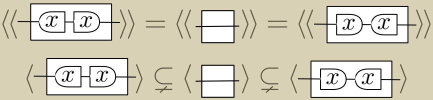

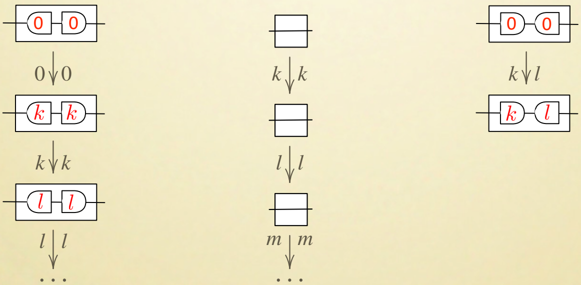

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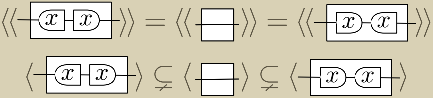
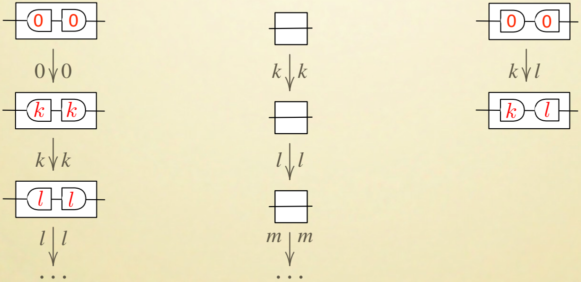
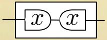
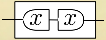




We say that  has *deadlocks* and  needs *initialisation*.

Full Abstraction

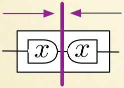
Theorem

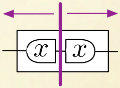
For any diagrams c, d **deadlock and initialisation free**

$$\langle\langle c \rangle\rangle = \langle\langle d \rangle\rangle \iff \langle c \rangle = \langle d \rangle$$

Realisability

In presence of deadlocks or initialisation, we cannot determine directionality of the flow.





A trace for these diagrams cannot be thought as the execution of a state-machine.

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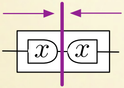
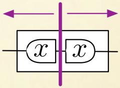




A trace for these diagrams cannot be thought as the execution of a state-machine.

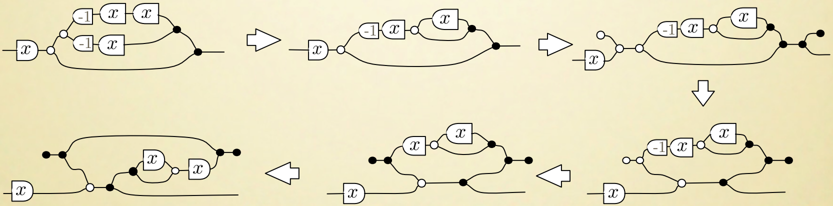
However, all the diagrams can be put into an executable form using the equational theory $\stackrel{\mathbb{IH}}{=}$.

Realisability Theorem

For any diagram c there exists
d deadlock and initialisation free such that $c \stackrel{\mathbb{IH}}{=} d$.

Realisation via $\mathbb{IH}$-rewriting

Implementing the Fibonacci circuit



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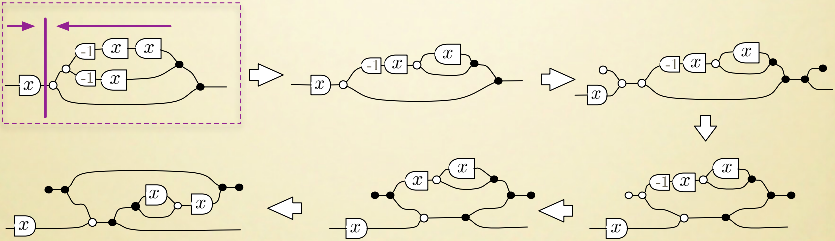

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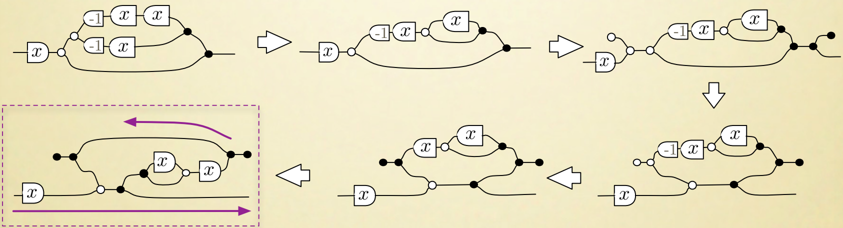

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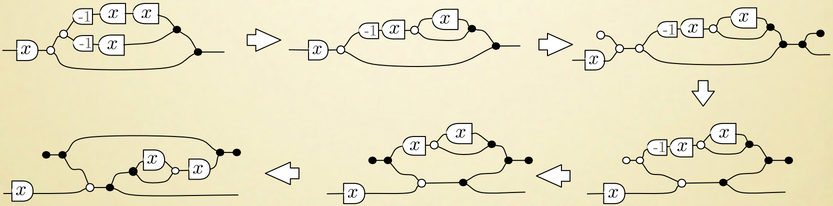

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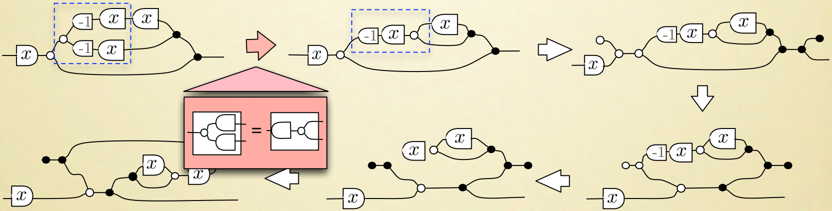

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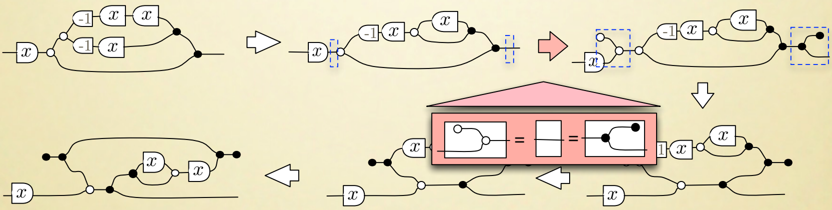

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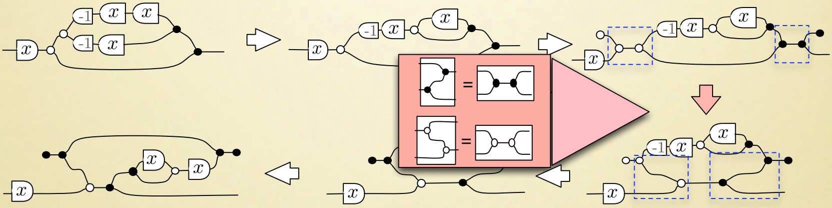

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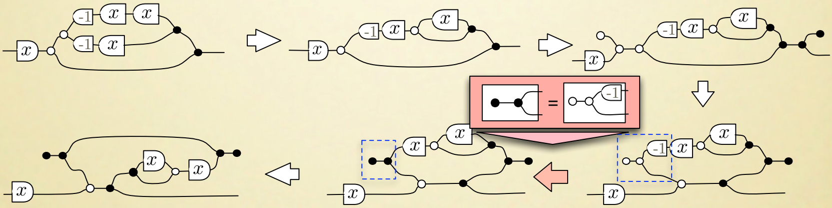
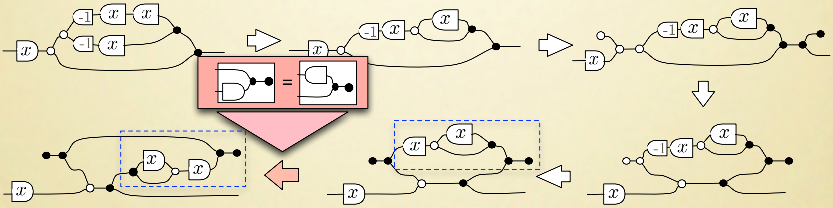
Implementing the Fibonacci circuit



Realisation via IH-rewriting

Implementing the Fibonacci circuit



Realisation via $\mathbb{IH}$-rewriting

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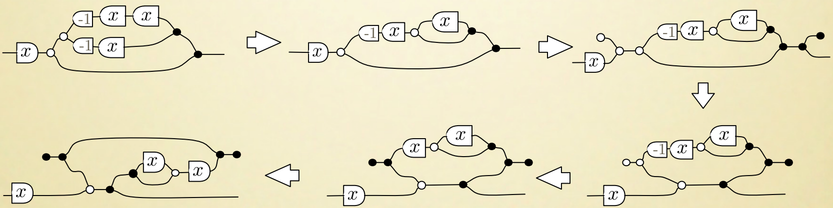

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- The construction of $\mathbb{IH}$ is based on **modular techniques** for composing algebraic theories via distributive laws, which are developed in the thesis extending recent work by S. Lack and E. Cheng.
- Interesting instances of $\mathbb{IH}$ are
  - $\mathbb{IH}_{\mathsf{Z}}$ — graphical linear algebra over rational subspaces
  - $\mathbb{IH}_{\mathsf{Z}_2}$ — relevant for categorical quantum mechanics and concurrency theory
  - $\mathbb{IH}_{\mathsf{k}[x]}$ — calculus of signal flow diagrams, presented in this talk

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  Adding a signal flow direction is often a figment of one's imagination, [which] needlessly complicates matters, mathematically and conceptually. A good theory of systems takes the behavior as the basic notion [...] and switches back and forth between a wide variety of convenient representations. (J. C. Willems - 2009)