Student ID:	
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Signature:	

Primary Examination, Semester X, 202X

APP MTH 4121	**Modelling with ODEs - Hons**
APP MTH 7035	**Modelling with ODEs - PG**

Course Coordinator: Dr A.N. Other

Total writing time: 180 minutes

Question	Marks	
1	/20	
2	/17	
3	/12	
4	/15	
5	/16	
Total	/80	

Instructions for candidates

- This examination is **closed book**.
- Answer all questions in the spaces provided.
- Only work written in this question and answer booklet will be marked.
- Your solutions must show all relevant working.
- Examination materials must not be removed from the examination room.
- Write your name on the front cover of this booklet.

Materials

- Calculators are permitted, provided they **do not** have computer algebraic system (CAS) functionality or remote communication capabilities.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO
STOP WRITING IMMEDIATELY WHEN INSTRUCTED.

20 Total

Question 1

/5 marks

1(a) For the initial value problem

$$\frac{dx}{dt} = f(x,t) \quad \text{with} \quad x(0) = x_0,$$

the forward or explicit Euler method is given by

$$x_{j+1} = x_j + h\, f(x_j, t_j),$$

where h is the timestep. Show that this method is convergent to $O(h)$, clearly stating any theorems or results which you rely on.

Solution By Dahlquist's equivalence theorem, a method is convergent if it is consistent and zero stable, and the order of convergence is the same as the order to which it is consistent.

We begin by showing consistency. We need to consider:

$$\frac{\ell(h)}{h} = \frac{|x(t_{j+1}) - x_{j+1}|}{h}.$$

Using Taylor series, we have

$$x(t_{j+1}) = x(t_j) + hx'(t_j) + \frac{h^2}{2}x''(t_j) + \ldots$$

Thus, noting that $x'(t_j) = f(x_j, t_j)$ we have

$$\begin{aligned}\frac{\ell(h)}{h} &= \frac{1}{h}\left|x(t_j) + hx'(t_j) + \frac{h^2}{2}x''(t_j) + \ldots - (x_j + hx'(t_j))\right| \text{[1 mark]} \\ &= \frac{h}{2}|x''(t_j)| + \ldots = O(h). \quad \text{[1 mark]}\end{aligned}$$

Thus, we have shown that the method is consistent to $O(h)$. It remains to show the method is zero-stable. We consider the difference equation for the method with $f(x,t) = 0$, which is

$$x_{j+1} = x_j.$$

To solve this equation, we make the ansatz $x_j = p^j$ for some number p. This yields

$$p^j(p-1) = 0.\text{[1 mark]}.$$

The roots are $p = 0$ (repeated) and $p = 1$. Since the non-repeated root satisfies $|p| \leq 1$ and the repeated root satisfies $|p| < 1$, the method is indeed zero stable. [1 mark] Thus, by Dahlquist's equivalence theorem, the method is convergent to $O(h)$. [1 mark] .

/3 marks

1(b) Recall that the standard test problem used to define the region of absolute stability of a numerical method is

$$\frac{dx}{dt} = \lambda x, \qquad x(0) = 1,$$

where λ is a (complex) constant. Show that the region of absolute stability for the forward or explicit Euler method is

$$|1 + z| < 1.$$

Solution Using the explicit Euler method for our test problem, we have

$$x_{i+1} = x_i + \lambda h x_i = (1 + \lambda h)x_i \quad \Rightarrow \quad x_i = (1 + \lambda h)^i. \quad \text{[1 mark]}$$

Then $x_i \to 0$ as $i \to \infty$ provided $|1 + \lambda h| < 1$, [1 mark] and thus, setting $\lambda h = z$, the region of stability is

$$|1 + \lambda h| = |1 + z| < 1, \quad \text{[1 mark]}$$

as required. (This is the interior of the unit circle in the complex plane with centre $z = -1$.)

/1 mark

1(c) Is the explicit Euler method A-stable? Give a reason for your answer.

Solution The method is not A-stable, as the region of absolute stability does not include the entire left half-plane ($\Re(z) < 0$). [1 mark]

/6 marks

1(d) The central difference formula for the first derivative is

$$x'(t_n) \approx \frac{x_{n+1} - x_{n-1}}{2h} - \frac{h^2}{6}x'''(\xi), \qquad \text{where} \qquad \xi \in (t_n - h, t_n + h).$$

Let the total error be $e = x'(t_n) - \frac{x_{n+1}-x_{n-1}}{2h}$, which can also be written as the sum $e = e_T + e_R$, where e_T is the truncation error and e_R is the round-off error. Find the step size h that minimises the total error, in terms of $K = \max|x'''|$ and ϵ, where $|\epsilon_n| \le \epsilon$ for all n, and $\epsilon_n = x(t_n) - x_n$ is the round-off error in the solution at time $t_n = nh$.

Solution The total error is

$$\begin{aligned} e &= x'(t_n) - \frac{x_{n+1} - x_{n-1}}{2\,h} \\ &= x'(t_n) - \frac{x(t_{n+1}) - x(t_{n-1})}{2\,h} + \frac{\epsilon_{n+1} - \epsilon_{n-1}}{2\,h} \\ &= e_T + e_R \quad \text{[1 mark]} \end{aligned}$$

Now

$$|e| = |e_T + e_R| \le |e_T| + |e_R|,$$

Please turn over for page 4.

$$|e_T| = |\frac{h^2}{6} x'''(\xi)| \le \frac{K\, h^2}{6} \quad \text{[1 mark]}$$

and

$$|e_R| = \frac{|\epsilon_{n+1} - \epsilon_{n-1}|}{2\, h} \le \frac{\epsilon}{h}. \quad \text{[1 mark]}$$

Thus

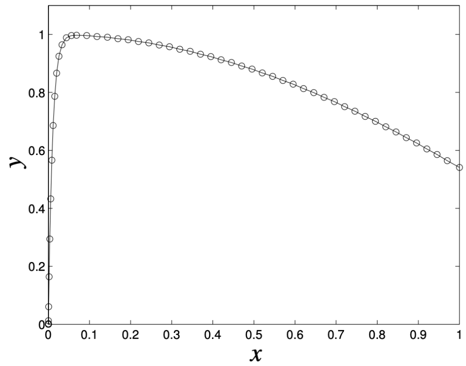
$$|e| \le \frac{K\, h^2}{6} + \frac{\epsilon}{h}, \quad \text{[1 mark]}$$

has minimum at $h = h_{\min}$, where

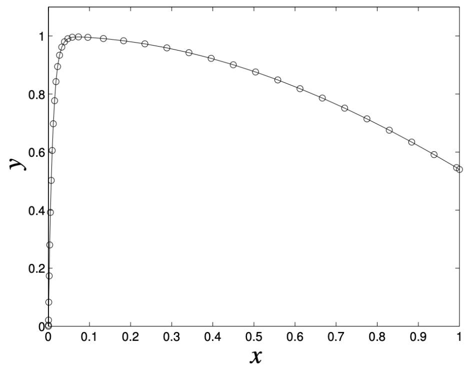
$$\frac{K\, h_{\min}}{3} - \frac{\epsilon}{h_{\min}^2} = 0 \quad \text{[1 mark]} \quad \Rightarrow \quad h_{\min} = \left(\frac{3\, \epsilon}{K}\right)^{1/3}. \quad \text{[1 mark]}$$

Please turn over for page 5.

1(e) Figures A and B below show numerical solutions to a stiff initial value problem. One figure was obtained with a Matlab ODE solver for stiff problems and the other with a Matlab solver for non-stiff problems. The circles show the gridpoints used by each solver.



(a) Figure A



(b) Figure B

/1 mark

(i) What feature of the solution indicates that the problem is stiff?

Solution The solution has a boundary layer in the region $0 < x < 0.05$ over which it changes very rapidly, and then changes more slowly over the region $x > 0.2$. [1 mark]

/4 marks

(ii) Which solution is given by the stiff solver and which is given by the non-stiff solver. Give reasons for your answer.

Solution Figure A is from the non-stiff solver; the step size is uniform over the entire range of x, even though the solution changes fairly slowly for x greater than around 0.2. [2 marks]

Figure B is from the stiff solver; a small step size is used for small values of x where the solution is changing rapidly and the step is increased for larger time when the solution changes more slowly. [2 marks]

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17 Total

Question 2

The population at time T of a certain kind of small rodent which is preyed upon by birds is denoted by $N(T)$. In the absence of the predators, the population undergoes logistic growth. A simple model for the dynamics of the population is

$$\frac{dN}{dT} = \alpha_1 N - \alpha_2 N^2 - \frac{\alpha_3 N^2}{\alpha_4^2 + N^2},$$

where the α_i $(i = 1, 2, 3, 4)$ are non-negative parameters.

/4 marks

2(a) Give a biological interpretation of the parameters α_1, α_2, α_3, α_4.

Solution α_1 - Birth rate of rodents [1 mark]
α_2 - Death rate of rodents in the absence of predators [1 mark]
α_3 - Maximum death rate of rodents due to predation [1 mark]
α_4 - Population of rodents when the rate of death due to predation is half the maximum [1 mark]

/4 marks

2(b) Show that this model can be nondimensionalised to give

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{k}\right) - \frac{n^2}{1+n^2} = f(n),$$

where n and t are the dimensionless versions of N and T, and r and k are dimensionless constants. You should clearly specify your scalings, and give the dimensionless constants in terms of the original, dimensional constants.

Solution We introduce the scalings $N = N_c n$ and and $T = T_c t$, and our ODE then becomes

$$\frac{N_c}{T_c}\frac{dn}{dt} = \alpha_1\, N_c\, n - \alpha_2\, N_c^2\, n^2 - \frac{\alpha_3\, N_c^2\, n^2}{\alpha_4^2 + N_c^2\, n^2}$$

$$\Rightarrow \quad \frac{dn}{dt} = \frac{T_c\,\alpha_1\, N_c}{N_c}\, n - \frac{T_c\,\alpha_2\, N_c^2}{N_c}\, n^2 - \frac{T_c\,\alpha_3\, N_c^2\, n^2}{N_c\,(\alpha_4^2 + N_c^2\, n^2)}$$

$$= T_c\,\alpha_1\, n - T_c\,\alpha_2\, N_c\, n^2 - \frac{T_c\,\alpha_3\, N_c\, n^2}{\alpha_4^2 + N_c^2\, n^2}$$

$$= T_c\,\alpha_1\, n\left(1 - \frac{\alpha_2\, N_c\, n}{\alpha_1}\right) - \frac{n^2}{\frac{\alpha_4^2}{T_c\,\alpha_3\, N_c} + \frac{N_c\, n^2}{T_c\,\alpha_3}}.$$

We choose

$$\frac{\alpha_4^2}{T_c\,\alpha_3\, N_c} = 1 \quad \text{and} \quad \frac{N_c}{T_c\,\alpha_3} = 1 \quad \text{i.e.} \quad N_c = \alpha_3\, T_c$$

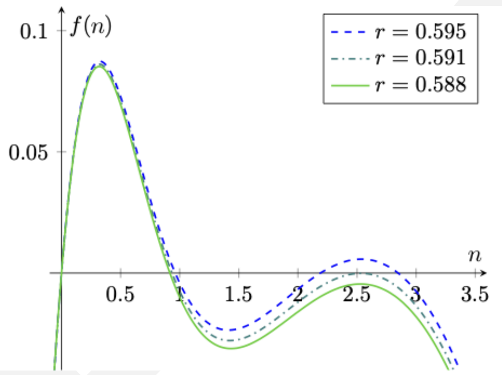
$$\Rightarrow \quad T_c = \frac{\alpha_4}{\alpha_3} \quad \text{and} \quad N_c = \alpha_4. \qquad \text{[2 marks]}$$

The parameters are

$$r = T_c\,\alpha_1 = \frac{\alpha_1\,\alpha_4}{\alpha_3} \quad \text{and} \quad k = \frac{\alpha_1}{\alpha_2\,N_c} = \frac{\alpha_1}{\alpha_2\,\alpha_4}. \qquad \text{[2 marks]}$$

/3 marks

2(c) The figure below shows $f(n)$ for $k = 6$ and three different values of r



State the number of fixed points for each value of r, and their stabilities, clearly explaining your reasoning.

Solution

- For $r = 0.588$ there are two fixed points: the smallest (at $x = 0$) is unstable because the slope of f is positive there; the largest (near $x = 1$) is stable because the slope is negative. [1 mark]
- For $r = 0.591$ there are three fixed points: the smallest (at $x = 0$) is unstable because the slope of f is positive there; the next (near $x = 1$) is stable because the slope is negative; and the largest (near $x = 2.5$) is semi-stable because f is the same sign (negative) on either side of the fixed point. [1 mark]
- For $r = 0.595$ there are four fixed points: the smallest (at $x = 0$) is unstable because the slope of f is positive there; the next (near $x = 1$) is stable because the slope is negative; the next (near $x = 2$) is unstable because the slope is positive; the largest (near $x = 3$) is stable because the slope is negative. [1 mark]

/2 marks

2(d) State with reason the type of bifurcation that occurs when r increases from 0.588 to 0.595.

Solution A saddle-node bifurcation [1 mark] , as two fixed points are created [1 mark] .

/2 marks

2(e) The figure below shows the bifurcation diagram for $k = 9$.

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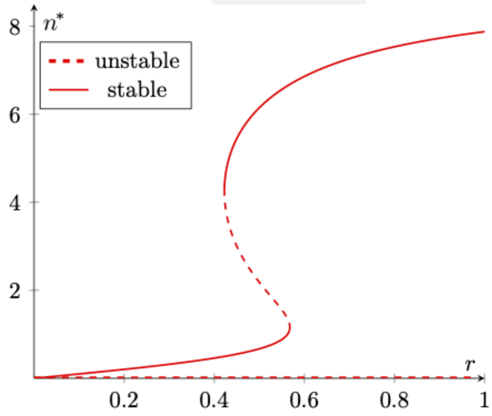

Suppose that initially n and r are both very small, and that r is slowly increased up to $r = 1$. Explain how the solution n evolves.

Solution The solution initially evolves along the only stable branch (emanating from the origin). It remains on there through the bistable interval after the first saddle-node bifurcation (around $r = 0.45$). [1 mark] When the stable branch disappears at the second saddle-node bifurcation (around $r = 0.6$), the solution jumps to the only stable branch available, and continues along it as r increases up to $r = 1$. [1 mark]

/2 marks

2(f) Does the system display hysteresis if r is then decreased? Explain your answer.

Solution Yes it does, since the the evolution is not simply the reverse of what happens when r is increased (described above). As r is decreased here, the solution follows the **upper stable** branch through the interval where there are two stable branches (not the lower one as in the previous part). [2 marks]

12 Total

Question 3

Consider the following linear systems of ODEs:

$$(A) \quad \dot{\boldsymbol{x}} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \boldsymbol{x},$$

$$(B) \quad \dot{\boldsymbol{y}} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \boldsymbol{y}.$$

/8 marks

3(a) For each system, classify the unique steady state at the origin. If the steady state is a centre or spiral, determine if trajectories travel clockwise or anticlockwise. If the steady state is a saddle, find the stable and unstable directions.

Solution For system (A): we let

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$$

then

$$\det(A) = 1 \times -1 - 3 \times 1 = -4$$

so that the steady state is a saddle. [1 mark] The eigenvalues are

$$\begin{aligned} \lambda_\pm &= \frac{1}{2}\left\{\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}\right\} \\ &= \pm\frac{1}{2}\sqrt{4^2} \\ &= \pm 2 \quad \text{[1 mark]} \end{aligned}$$

The eigenvectors are

$$\mathbf{v}_- = (1, -1)^T \quad \text{and} \quad \mathbf{v}_+ = (3, 1)^T \quad \text{[1 mark]}$$

Therefore, the stable direction is along the line

$$x = -y,$$

and the unstable direction is along

$$x = 3y. \quad \text{[1 mark]}$$

For system (B), we let

$$B = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$$

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then the eigenvalues $\lambda = \lambda_{\pm}$satisfy the characteristic equation

$$\lambda^2 + 4 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 2\mathrm{i} \in \mathrm{i}\mathbb{R} \quad \text{[1 mark]}$$

Therefore, the steady state is a centre. [1 mark] Suppose the initial conditions are $x(0) = x_0 > 0$ and $y(0) = 0$, then

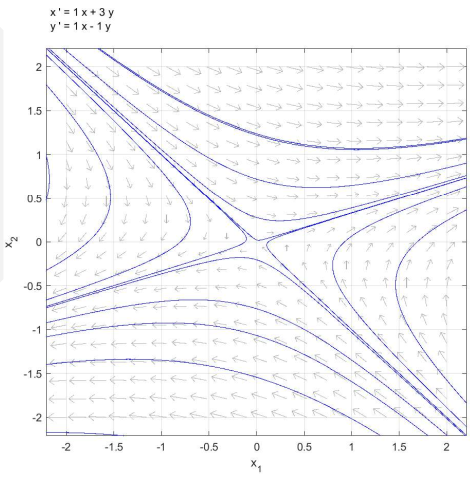
$$f(x, y) = y = 0 \quad g(x, y) = -4x = -4x_0 < 0 \quad \text{[1 mark]}$$

Therefore, trajectories travel clockwise. [1 mark]

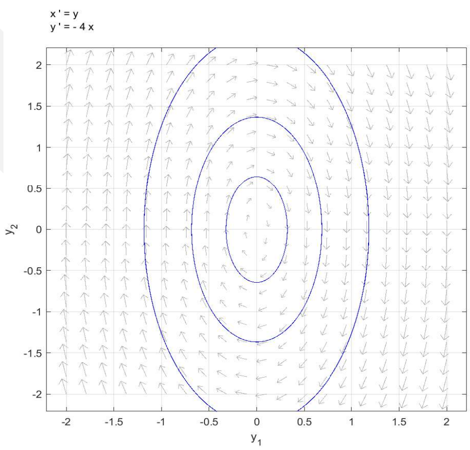
/4 marks

3(b) Sketch the phase portrait for each system.

Solution The phase plane sketch for (A) should resemble the one below. [2 marks]



The phase plane sketch for (B) should resemble the one below. [2 marks]



15 Total

Question 4

Let $x(t)$ and $y(t)$ be the populations at time t of two species which interact with each other. Assuming that each of the two populations grow logistically when the other is absent, a dimensionless model for this situation is:

$$\frac{dx}{dt} = x(1 - x + ay)$$
$$\frac{dy}{dt} = ry(1 - y + bx)$$

where a, b and r are all positive parameters and $ab \neq 1$.

/1 mark

4(a) What type of interaction occurs between the two populations in this model?

Solution This is a model of symbiosis or mutually beneficial interactions, since the interactions benefit both populations. [1 mark]

/6 marks

4(b) Find the four steady states of the model, giving any conditions on the parameters required for them to be biologically realistic.

Solution The steady states are the solutions $(\bar{x}, \bar{y})$ of

$$x(1 - x + ay) = 0.$$
$$ry(1 - y + bx) = 0. \quad \text{[1 mark]}$$

The steady states are: $(\bar{x}_1, \bar{y}_1) = (0, 0)$ [1 mark] , $(\bar{x}_2, \bar{y}_2) = (1, 0)$ [1 mark] , $(\bar{x}_3, \bar{y}_3) = (0, 1)$ [1 mark] and $(\bar{x}_4, \bar{y}_4) = \left(\frac{a+1}{1-ab}, \frac{b+1}{1-ab}\right)$[1 mark]

We require $ab < 1$ for the last steady state to be positive (which is required, since it represents a population). [1 mark]

/6 marks

4(c) Write down the Jacobian matrix for this system. Determine whether each of the steady states found in part (i) is stable or unstable. (You do not need to determine the type of steady state e.g., node, spiral, etc.)

Solution The stability of the steady states is determined by the eigenvalues of the Jacobian (or, equivalently, by its trace and determinant).

$$J = \begin{pmatrix} 1 - 2x + ay & ax \\ rby & r(1 - 2y + bx) \end{pmatrix} \quad \text{[1 mark]}$$

At $\bar{x}_1 = (0, 0)$, $J = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$ so $\bar{x}_1$ is unstable. [1 mark]

Please turn over for page 12.

At $\bar{x}_2 = (1, 0)$, $J = \begin{pmatrix} -1 & a \\ 0 & r(1+b) \end{pmatrix}$ hence $\det J = -r(1+b) < 0$ so $\bar{x}_2$ is a saddle, and therefore unstable.[1 mark]

At $\bar{x}_3 = (0, 1)$ $J = \begin{pmatrix} 1+a & 0 \\ rb & -r \end{pmatrix}$ so $\det\ J = -r(1+a) < 0$. Thus $\bar{x}_3$ is also a saddle, and so unstable.[1 mark]

At $(\bar{x}_4, \bar{y}_4)$ we have

$$J = \begin{pmatrix} 1 - \frac{2(a+1)}{1-ab} + \frac{a(b+1)}{1-ab} & \frac{a(a+1)}{1-ab} \\ \frac{rb(b+1)}{1-ab} & r\left(1 - \frac{2(b+1)}{1-ab} + \frac{b(a+1)}{1-ab}\right) \end{pmatrix}$$

Note that the diagonal elements of J can be simplified to:

$$\begin{aligned} J_{11} &= \frac{1}{1-ab}[1 - ab - 2a - 2 + ab + a] = \frac{-(a+1)}{1-ab} \\ J_{22} &= \frac{r}{1-ab}[1 - ab - 2b - 2 + ab + b] = \frac{-(b+1)r}{1-ab} \\ \text{so} \quad \operatorname{tr}(J) &= \frac{-(a+1) - r(b+1)}{1-ab} < 0 \quad \text{if } 1 - ab > 0 \\ \det J &= \frac{r(a+1)(b+1)}{(1-ab)^2} - \frac{rab(a+1)(b+1)}{(1-ab)^2} = \frac{r(a+1)(b+1)}{(1-ab)}. \end{aligned}$$

Thus we note $\det J > 0$ provided $1 - ab > 0$. Hence, $\bar{x}_4$ is stable [1 mark] when it exists. [1 mark]

/2 marks

4(d) Find a condition on the parameter values such that a possible long-term behaviour of the solution is a steady state where both species co-exist. What would happen to the two populations if this condition did not hold?

Solution For the steady state with both species coexisting to be a possible long-time solution of the model, we require $(\bar{x}_4, \bar{y}_4)$ to be a stable steady state, which is the case provided $ab < 1$ [1 mark] . Biologically, this means that that benefit of the interaction to both species must not be too great. If the condition does not hold, then there are no stable steady states, and the populations can grow without bound [1 mark] ($\dot{x}$ and $\dot{y}$ can be positive for large x and y).

16 Total

Question 5

Functions $y(x)$ which extremise functionals of the form

$$F\{y\} = \int_{x_0}^{x_1} f\left(x, y, y'\right) dx,$$

with the endpoint conditions $y\left(x_0\right) = y_0$, and $y\left(x_1\right) = y_1$, satisfy the Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0.$$

/5 marks

5(a) Suppose that, in the functional above, f has no explicit dependence on x, so we wish to extremise a functional of the form:

$$F\{y\} = \int_{x_0}^{x_1} f(y, y')\, dx,$$

subject to the same endpoint conditions as above. Show that in this case f must satisfy the equation:

$$f - y'\frac{\partial f}{\partial y'} = \text{const.}$$

Solution If $f(y, y')$ does not explicitly depend on x then by the chain rule

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \\ &= y' f_y + y'' f_{y'} \\ \Rightarrow \quad y' f_y &= \frac{df}{dx} - y'' f_{y'}. \quad \text{[1 mark]} \end{aligned} \tag{I}$$

From the Euler-Lagrange equation

$$\begin{aligned} f_y - \frac{d}{dx} f_{y'} &= 0 \\ y' f_y - y'\frac{d}{dx} f_{y'} &= 0. \quad \text{[1 mark]} \end{aligned}$$

and substitution from (I) yields

$$\frac{df}{dx} - y'' f_{y'} - y'\frac{d}{dx} f_{y'} = 0. \quad \text{[1 mark]}$$

But the last two terms are a derivative of a product and therefore

$$\frac{df}{dx} - \frac{d}{dx}\left(y' f_{y'}\right) = 0, \quad \text{[1 mark]}$$

and so integrating

$$f - y' f_{y'} = \text{const}, \quad \text{[1 mark]}$$

as required.

/5 marks

5(b) Find the curve $y(x)$ maximises or minimises the functional

$$F\{y\} = \int_0^1 \left(\frac{1}{2} y'^2 + yy' + y' + y \right) dx,$$

subject to the end point conditions

$$y(0) = 0, \quad y(1) = \frac{3}{2}.$$

Note: you do not need to determine if your solution is a minimum or a maximum.

Solution The integrand $f = \frac{1}{2}y'^2 + yy' + y' + y$ has no explicit x-dependence, so we can use the Beltrami identity. [1 mark] Denoting the constant by α we have:

$$\begin{aligned} y' \frac{\partial f}{\partial y'} - f &= \alpha \\ y'\left(y' + y + 1\right) - \left(\frac{1}{2} y'^2 + yy' + y' + y \right) &= \alpha \\ \frac{1}{2} y'^2 - y &= \alpha. \quad \text{[1 mark]} \end{aligned}$$

Therefore

$$y' = \sqrt{2}\sqrt{y + \alpha}. \quad \Rightarrow \quad \int \frac{1}{\sqrt{2}\sqrt{y+\alpha}}\, dy = \int dx. \quad \text{[1 mark]}$$

On integrating (and letting β be the constant of integration) we find

$$x + \beta = \sqrt{2}\sqrt{y + \alpha} \quad \Rightarrow \quad y = \frac{(x+\beta)^2}{2} - \alpha. \quad \text{[1 mark]}$$

Now applying endpoint conditions

$$y(0) = 0 \quad \Rightarrow \quad 0 = \frac{\beta^2}{2} - \alpha \quad \Rightarrow \quad \alpha = \frac{\beta^2}{2}.$$

So

$$y(x) = \frac{(x+1)^2}{2} - \frac{1}{2} = \frac{x(x+2)}{2}. \quad \text{[1 mark]}$$

5(c) Consider the problem of finding the curve $y(x)$ that extremises

$$F\{y\} = \frac{1}{2}\int_0^1 y'^2\,dx$$

subject to the constraint

$$G\{y\} = \int_0^1 y\,dx = \frac{1}{6},$$

and the end point conditions $y(0) = 0$ and $y(1) = 0$.

/3 marks

(i) Using the Euler-Lagrange equation, show that $y(x)$ is a quadratic in x.

Solution Note that this is an isoperimetric problem, where we need to extremise $F\{y\}$ subject to the integral constraint $G\{y\} = \frac{1}{6}$ Including the constraint with a Lagrange multiplier λ, we form the new functional that we must extremise:

$$\hat{F}\{y\} = \int_0^1 \frac{1}{2}y'^2 + \lambda y\,dx \quad \text{[1 mark]}$$

where the integrand is given by

$$f = \frac{1}{2}y'^2 + \lambda y.$$

This is independent of x but as it turns out the easiest solution is found via the Euler-Lagrange equation (hence the wording of the question), which is

$$\begin{aligned}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} &= 0\\ \frac{d}{dx}(y') - \lambda &= 0.\end{aligned}$$

Therefore

$$y'' = \lambda. \quad \text{[1 mark]}$$

Integrating twice we have

$$y = \frac{\lambda}{2}x^2 + \alpha x + \beta,$$

where α and β are constants of integration. Hence, we find that y is a quadratic, as stated. [1 mark]

/3 marks

(ii) Find the values of the constants in your solution for $y(x)$ so that it satisfies the end point conditions and the constraint.

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Solution Now applying endpoint conditions

$$y(0) = 0 \quad \Rightarrow \quad 0 = \beta$$
$$y(1) = 0 \quad \Rightarrow \quad 0 = \frac{\lambda}{2} + \alpha \quad \Rightarrow \quad \alpha = -\frac{\lambda}{2}.$$

So

$$y(x) = \frac{\lambda}{2}(x^2 - x). \quad \text{[2 marks]}$$

Now using the constraint

$$G\{y\} = \frac{\lambda}{2}\int_0^1 (x^2 - x)\,dx = \frac{\lambda}{2}\left(\frac{1}{3} - \frac{1}{2}\right) = -\frac{\lambda}{12}.$$

and since $K\{y\} = 1/6$ we find $\lambda = -2$ and so

$$y(x) = x(1 - x). \quad \text{[1 mark]}$$

End of examination questions.