CHAPTER 3. Finite Volume Method

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Abstract

Provide the reasoning of finite volume method based on the integration to discretize the Euler equation. We first integrate the Euler equation and hence simplify each term of U and H separately. We therefore give the full term to discretize the Euler equation.

We first give the Euler equation as the controlling equation:

$$\frac{\partial U}{\partial t}+\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}=0 \tag{1}$$

Where

$$U=\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho E\end{pmatrix}, E=\begin{pmatrix}\rho u\\ \rho u^2+P\\ \rho uv\\ \rho vu\end{pmatrix}, F=\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+P\\ \rho vu\end{pmatrix} \tag{2}$$

We hence integral the Euler equation, as visualized in Fig. 1:

$$\int\left(\frac{\partial U}{\partial t}+\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}\right)dS=0 \tag{3}$$

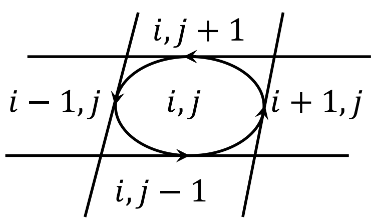

Fig. 1 The integral area of the finite volume method based on single mesh element.

Eq. 3 can be written as:

$$\int\left(\frac{\partial U}{\partial t}\right)dS+\int\left(\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}\right)dS=0 \tag{4}$$

Now we define the term *H*:

$$\vec{H}=E\vec{\imath}+F\vec{\jmath} \tag{5}$$

With the Green-Gauss integration, the right term in Eq. 4 can be written as:

$$\int\vec{v}\cdot\vec{H}d\Omega=\oint\vec{H}d\vec{S} \tag{6}$$

$$=\sum_{i=1}^{\Delta}H_i n_i\,\Delta S_i \tag{7}$$

Based on the Taylor expansion, the term U can be written as:

$$U = U_c + \frac{\partial U}{\partial x}(x - x_c) + \frac{\partial U}{\partial y}(y - y_c) + O(\Delta^2) \tag{8}$$

Hence, the mean of the term U can be obtained through integration:

$$\int \frac{U d\Omega}{d\Omega} = \int U_c d\Omega \tag{9}$$

$$= \frac{U d\Omega}{d\Omega} + \int \frac{\partial U}{\partial x}(x - x_c) d\Omega \tag{10}$$

Here we define the mean of U as:

$$\int \frac{U_c d\Omega}{d\Omega} = \overline{U} \tag{11}$$

The term $U = U(x, t)$ can be decomposed as the separation of variation:

$$U = \sum_{i=1}^{N} U_i(t) b(x_i) \tag{12}$$

Thence, the term U can be considered:

$$U(x, t) \cong \overline{U}(t) \tag{13}$$

The integration of the first term is written as:

$$\int \frac{\partial U}{\partial t} d\Omega = \int \frac{dU}{dt} d\Omega = \frac{dU}{dt}\Omega \tag{14}$$

Hence, Euler equation can be written as:

$$\frac{dU}{dt}\Omega + \oint \vec{H} \cdot \vec{n} \cdot dS = 0 \tag{15}$$

The right term can be discretized as the following form.

$$\oint \vec{H} \cdot \vec{n} \cdot dS = \overline{H}$$

$$= H_{i+\frac{1}{2},j} - H_{i-\frac{1}{2},j} + H_{i,j+\frac{1}{2}} - H_{i,j-\frac{1}{2}} \tag{16}$$

We therefore define the term in Eq. 16 as *RHS*.

$$RHS = H_{i+\frac{1}{2},j} - H_{i-\frac{1}{2},j} + H_{i,j+\frac{1}{2}} - H_{i,j-\frac{1}{2}} \tag{17}$$

The discretized form of the Euler equation is given as the form.

$$\frac{dU}{dt}\Omega + RHS = 0 \tag{18}$$

APPENDIX. Jacobian Matrix

The Jacobian matrix A as formerly introduced in Chap. 1 can be further diagonalized for obtaining the eigenvalue λ. Here we show how the A matrix and the eigen value is derived.

We first give the term H based on Eq. 5:

$$H = \vec{H} \cdot \vec{n}$$

$$= \begin{pmatrix} \rho g \\ \rho u g + P n_x \\ \rho v g + P n_y \\ \rho H g \end{pmatrix} \tag{19}$$

Where

$$\rho H = \rho E + P \tag{20}$$

Hence, we deduce that the Jacobian matrix A can be written as:

$$A = \frac{\partial H}{\partial U}$$

$$= \begin{pmatrix} g & 0 & 0 & 0 \\ \frac{P n_x + \rho g u}{r} & 0 & 0 & 0 \\ \frac{P n_y + \rho g v}{r} & 0 & 0 & 0 \\ \frac{P g + \rho E g}{r} & 0 & 0 & 0 \end{pmatrix} \tag{21}$$

Therefore, matrix A is written as:

$$A = \frac{\partial E}{\partial u} n_x + \frac{\partial E}{\partial v} n_y \tag{22}$$

The eigenvalues of matrix A are

$$\begin{cases} \lambda_1 = q = u n_x + v n_y \\ \lambda_2 = q = u n_x + v n_y \\ \lambda_3 = q + c = u n_x + v n_y + c \\ \lambda_4 = q - c = u n_x + v n_y + c \end{cases} \tag{23}$$

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