CHAPTER 8. Implicit Difference Scheme

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Abstract

Provide an implicit scheme method with a given half differential equation. We first discretize the equation with finite difference; hence elicit the terms D, L, U to linearize the given equation and update the solution w.

Let us consider a simple differential equation:

$$\frac{dw}{dt} = RHS \tag{1}$$

The equation can be discretized with the finite difference method; in the *n+1* moment:

$$\frac{w^{n+1} - w^n}{\Delta t} = RHS^{n+1} \tag{2}$$

In which

$$RHS^{n+1} = RHS^n + \frac{\partial RHS}{\partial w}(w^{n+1} - w^n) + O(\Delta w^2) \tag{3}$$

Let us consider $w^{n+1} - w^n = \Delta w$. Hence, Eq. 3 can be reduced to

$$RHS^{n+1} \approx RHS^n + \left(\frac{\partial RHS}{\partial w}\Delta w\right)^n \tag{4}$$

Eq. 2 can be further written as

$$\frac{\Delta w}{\Delta t} - \frac{\partial RHS}{\partial w}\Delta w = RHS^n \tag{5}$$

Which can be reduced to

$$\left(\frac{I}{\Delta t} - \frac{\partial RHS}{\partial w}\right)\Delta w = RHS^n \tag{6}$$

Now, we can consider Eq. 6 as a simple linear equation form, which is written as:

$$A\Delta w = RHS^n \tag{7}$$

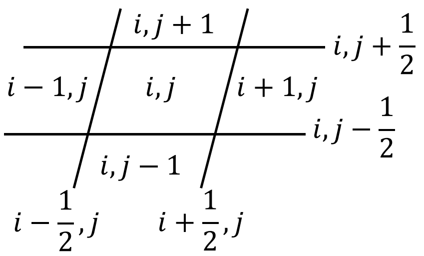

Fig. 1 Schematic for the single mesh element.

From Fig. 1, the term *RHS* can be decomposed as:

$$(RHS)_{ij} = H_{i+\frac{1}{2},j} - H_{i-\frac{1}{2},j} + H_{i,j+\frac{1}{2}} - H_{i,j-\frac{1}{2}} \tag{8}$$

In which

$$H_{i+\frac{1}{2},j} = H\big(w_{i,j}, w_{i+1,j}\big) \tag{9}$$

$$H_{i-\frac{1}{2},j} = H\big(w_{i,j}, w_{i-1,j}\big) \tag{10}$$

$$H_{i,j+\frac{1}{2}} = H\big(w_{i,j}, w_{i,j+1}\big) \tag{11}$$

$$H_{i,j-\frac{1}{2}} = H\big(w_{i,j}, w_{i,j-1}\big) \tag{12}$$

Here we define three terms from term *RHS* and *w*:

$$D = \frac{\partial RHS_{ij}}{\partial w_{ij}} \tag{13}$$

$$L = \frac{\partial RHS_{ij}}{\partial w_{i-1,j}} + \frac{\partial RHS_{ij}}{\partial w_{i,j-1}} \tag{14}$$

$$U = \frac{\partial RHS_{ij}}{\partial w_{i+1,j}} + \frac{\partial RHS_{ij}}{\partial w_{i,j+1}} \tag{15}$$

Hence the linear equation Eq. 7 could be written as:

$$(L + D + U)\Delta w = \frac{\partial RHS}{\partial w} \tag{16}$$

From Eq. 14 to 15, we obtain:

$$L\Delta w = \frac{\partial RHS_{ij}}{\partial w_{i-1,j}}\Delta w_{i-1,j} + \frac{\partial RHS_{ij}}{\partial w_{i,j-1}}\Delta w_{i,j-1} \tag{17}$$

$$U\Delta w = \frac{\partial RHS_{ij}}{\partial w_{i+1,j}}\Delta w_{i+1,j} + \frac{\partial RHS_{ij}}{\partial w_{i,j+1}}\ \Delta w_{i,j+1} \tag{18}$$

Therefore, we could simplify Eq. 16 as

$$(L + D + U)\Delta w \approx (L + D)D^{-1}(U + D)\ \Delta w \tag{19}$$

The right term in Eq. 19 could be further simplified as:

$$\begin{aligned}(L + D)D^{-1}(U + D)\Delta w &= (L \cdot D^{-1} + I)(U + D)\ \Delta w \\ &= L \cdot D^{-1}U + (L + U + D)\ \Delta w^{-1}\end{aligned} \tag{20}$$

Where $L \cdot D^{-1}U$ could be considered as an infinitesimal of high order.

Substituting Eq. 19 into Eq. 16:

$$(L + D)D^{-1}(U + D)\Delta w = RHS \tag{21}$$

Eq. 21 can be further linearized as:

$$(L + D)\Delta Q = RHS \tag{22}$$

$$D\Delta Q = RHS - L\Delta Q \tag{23}$$

$$\Delta Q = D^{-1}(RHS - L\Delta Q) \tag{24}$$

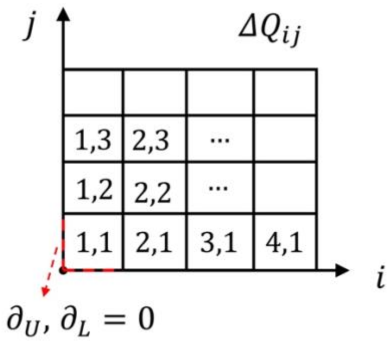

Fig. 2 Schematic for the meshing.

From Fig. 2 we could write the term ΔQ_{ij} as the following terms:

$$\Delta Q_{11} = {D_{11}}^{-1} RHS_{11} \tag{25}$$

$$\Delta Q_{21} = {D_{11}}^{-1}(RHS_{21} - L_{21}\Delta Q_{11}) \tag{26}$$

$$\Delta Q_{12} = {D_{11}}^{-1}(RHS_{12} - L_{12}\Delta Q_{11}) \tag{27}$$

$$\cdots$$

$$\Delta Q_{nm} = \cdots \tag{28}$$

Hence, the full sets of the implicit scheme algorithm could be summarized as:

Step 1: $\Delta Q = D^{-1}(RHS - L\Delta Q)$

Step 2: $\Delta Q = (D^{-1}U + I)\Delta w$

$$\Delta w = \Delta Q - D^{-1}U\,\Delta w$$

The solution can be updated as

$$\downarrow$$

Step 3: $w^{n+1} = w^n + \Delta w$

References

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