Derivation of Navier-Stokes Equation

© Hanfeng Zhai

School of Mechanics and Engineering Science, Shanghai Univeristy
Shanghai 200444, PRC

Abstract

Navior-Stokes Equation is the governing equation in fluid mechanics. Here, we first introduce the mathematical basis of the derivation. Thence we present how to reason out the Navior-Stokes Equation from the introduction of Eulerian and Lagrangian viewpoints, and further deduce the final form based on the Newton's second law.

Preface

Before beginning the reasoning, we first need to introduce the mathematical basis for the equations.

What is the nabla operator ∇?

The nabla operator is a mathematical calculation for vectors. Which has the following forms. We consider ∇ as gradient when it multiples a variable.

Gradient $\nabla\phi$ is a vector, when ϕ is a scalar.

For example, if $\phi = 2xy + z$, we have

$$\nabla\phi = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} (2xy + z) \tag{1}$$

Which can be written as

$$\nabla\phi = \begin{pmatrix} \frac{\partial(2xy+z)}{\partial x} \\ \frac{\partial(2xy+z)}{\partial y} \\ \frac{\partial(2xy+z)}{\partial z} \end{pmatrix} \tag{2}$$

$$\nabla\phi = \begin{pmatrix} 2y \\ 2x \\ 1 \end{pmatrix} \tag{3}$$

Also, we define the Divergence as $\nabla \bullet \mathbf{v}$ is a scalar.

And we define the Curl as $\nabla \times \mathbf{v}$ is a vector.

Another question is hence elicited: does $\nabla \bullet \mathbf{v}$ equals $\mathbf{v} \bullet \nabla$?

The left-hand side

$$\nabla \bullet \mathbf{v} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \bullet \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{4}$$

equals $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$.

The right-hand side is

$$\mathbf{v} \bullet \nabla = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \tag{5}$$

equals $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$.

Hence, we can deduce that the answer is YES. Now, we know how the nabla operator works, which will be used for the next part.

Eulerian and Lagrangian viewpoints

Eulerian viewpoints mainly focuses on the fluid particles' motion in a stablized space; howbeit Lagrangian mainly focuses on the motion of a single fluid particle. Hence, when we refer the acceleration of fluid, we usually consider the Lagrangian viewpoints. When we are investigating the fluid distribution on a specific space area, we usually refer the Eulerian viewpoints.

Here, the schematic for the Eulerian and Lagrangian viewpoints are shown as in figure 1.

Here, we need to find the relationship between the Eulerian and Lagrangian viewpoints in mathematical form. We try to form the Lagrangian acceleration with Eulerian velocity.

$$\mathbf{v_L} = \mathbf{v_E}(t, x_p, y_p, \delta_p). \tag{6}$$

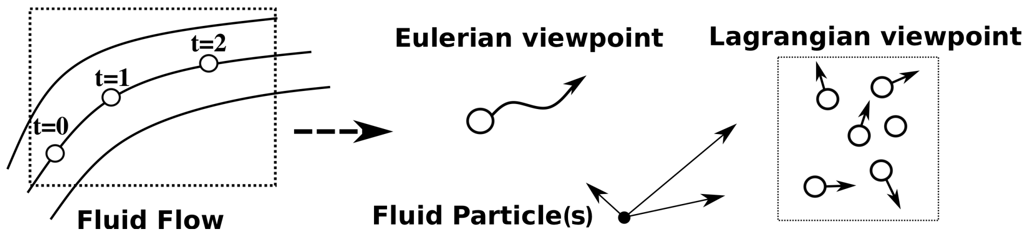

Figure 1: The schematic for the Eulerian and Lagrangian viewpoints.

Considering the time-dependent situation:

$$\mathbf{v_L} = \mathbf{v_E}[t, x_p(t), y_p(t), \delta_p(t)] \tag{7}$$

Hence, the acceleration in Lagrangian viewpoints is written as

$$\mathbf{a_L} = \frac{\mathrm{d}\mathbf{v_L}}{\mathrm{dt}} = \frac{\partial \mathbf{v_E}}{\partial t} + \frac{\partial \mathbf{v_E}}{\partial \mathbf{x}} \bullet \frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} \tag{8}$$

$$\mathbf{a_L} = \frac{\partial \mathbf{v_E}}{\partial t} + \mathbf{v} \bullet \frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} \tag{9}$$

In which $\mathbf{a_L}$ is the Lagrangian acceleration, equals to the Eulerian velocities. $\mathbf{x}$ equals (x, y, z), and $\mathbf{v}$ equals (u, v, w). Based on the defination of acceleration, we hence define the material derivative as

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \bullet \nabla) \bullet \mathbf{v} = \frac{D\mathbf{v}}{Dt} \tag{10}$$

Now, we consider the Newton's second law:

$$m\mathbf{a} = \mathbf{F} \tag{11}$$

Which can further written as

$$\rho V \mathbf{a} = \mathbf{F} \tag{12}$$

We substitute the material's derivative into the acceleration, and consider the external force of the fluid:

$$\rho V \frac{D\mathbf{v}}{Dt} = V(-\nabla P - \rho g \mathbf{k}) \tag{13}$$

In which P is the pressure on fluid, $\mathbf{k}$ is the direction vector.

The equation can be further written as:

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla P - \rho g\mathbf{k} \tag{14}$$

$$\rho[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v} \bullet \nabla) \bullet \mathbf{v}] = -\nabla P - \rho g\mathbf{k} \tag{15}$$

In the equations shown above, there is 4 unknown variables $\mathbf{v} = \ (u, v, w)$ and pressure P.

Based on the conservation of mass:

$$\nabla \bullet \mathbf{v} = 0 \tag{16}$$

Therefore, combining the equations shown above, we obtain the Euler's Equation, which is considered as the governing equation of perfect fluid.

$$\rho[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \cdot \mathbf{v}] = -\nabla P - \rho g\mathbf{k} \tag{17}$$

$$\nabla \bullet \mathbf{v} = 0 \tag{18}$$

Now, if we consider the viscosity of the fluid, we obtain the final form of the Navior-Stokes Equation:

$$\rho[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \cdot \mathbf{v}] = -\nabla P - \rho g\mathbf{k} + \mu\nabla^2\mathbf{v} \tag{19}$$

$$\nabla \bullet \mathbf{v} = 0 \tag{20}$$

Then, we will elicit the conservation of mass for a compressible fluid flow, as for the further discussion.

$$\frac{D\rho}{Dt} + \rho(\nabla\mathbf{v}) = 0 \tag{21}$$

References

[1] WY. Wu. Fluid Mechanics. *PKU Press*, **2014**.12. ISBN 978-7-301-00198-8/O-0030

[2] R. Alam. PAST 2501: Fluid Mechanics. *UC Berkeley Online Research Program*, **2020**.