CHAPTER 2. Nondimensionalization for Variables

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Abstract

Introduce the nondimensionalization methods for the equations involved in computational fluid dynamics. Provide an example with different reference variables on Euler equation to show basic strategies of nondimensionalization.

To solve the governing equations as given in Chap. 1, magnitude of variables needed to be considered. In storing the variables for calculations, the last few decimal points are neglected when magnitude doesn't match. Hence, nondimensionalization of variables in equations is important for computations for accurate results.

For nondimensionalizations, here we adopt the Navier-Stokes equation in 1D situation neglecting the spatial term as an example:

$$\frac{\partial U}{\partial t} - \frac{\partial E_v}{\partial x} = 0 \tag{1}$$

Where

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}, E = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u H \end{pmatrix} \tag{2}$$

In coordinate system, Eq. 1 can be expanded to the form:

$$\frac{\partial \rho u}{\partial t} - \mu \frac{\partial^2 u}{\partial x^2} = 0 \tag{3}$$

Recalled from Chap. 1, The pressure p, temperature T, and terms H, E follows:

$$\begin{cases} p = \rho R T \\ \rho H = \rho E + p \\ E = \dfrac{R}{\gamma - 1} \end{cases} \tag{4}$$

Hence, we obtains:

$$\rho E = \frac{pR}{\gamma - 1} + \frac{\rho}{2}(u^2 + v^2) \tag{5}$$

$$\Rightarrow \rho = (\gamma - 1)\left[\rho E - \frac{1}{2}\rho(u^2 + v^2)\right] \tag{6}$$

Here, we chose the reference variables for nondimensionalization:

$$\begin{cases}[\rho] = \rho_\infty = 1 \\ [p] = p_\infty = 1 \\ [T] = T_\infty = 1 \\ [l] = l_\infty = 1\end{cases} \tag{7}$$

Therefore, the time and velocity can be obtained through calculation as:

$$\begin{cases}[t] = \dfrac{[x]}{[u]} = \dfrac{l}{\sqrt{\dfrac{p_\infty}{\rho_\infty}}} \\ u_\infty = c = \sqrt{\dfrac{\gamma p}{\rho}}\end{cases} \tag{8}$$

Hence, all the terms involved in the equation can be nondimensionalized to the following forms:

$$\begin{cases}\rho = \rho'[\rho] = \rho'\rho_\infty \\ p = p'[p] = p'p_\infty \\ T = T'[T] = T'T_\infty \\ x = x'[l] = x'l_\infty \\ \mu = \mu'[\mu] = \mu'\mu_\infty \\ u = u'[u] = u'[c] = u'\sqrt{\dfrac{p_\infty}{\rho_\infty}} \\ t = t'[t] = t'\dfrac{l_\infty}{u} = t'\dfrac{l_\infty}{\sqrt{\dfrac{p_\infty}{\rho_\infty}}}\end{cases} \tag{9}$$

Substituting Eq. 9 into Eq. 3, we obtains:

$$\frac{\partial \rho' u'}{\partial t'} + \frac{\mu_\infty}{\rho_\infty l_\infty \sqrt{\dfrac{p_\infty}{\rho_\infty}}} \frac{\partial^2 u'}{\partial x'^2} = 0 \tag{10}$$

Due to $Re = \frac{\rho_\infty u_\infty l_\infty}{\mu_\infty}$ and $M = \frac{u_\infty}{c} = u_\infty \sqrt{\frac{\rho_\infty}{\gamma p_\infty}}$, the nondimensionalized equation can be written in the form:

$$\frac{\partial \rho' u'}{\partial t'} + \frac{M\sqrt{\gamma}}{Re} \frac{\partial^2 u'}{\partial x'^2} = 0 \tag{11}$$

References

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