CHAPTER 5. Boundary Conditions

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Abstract

Introduce the basic strategy to analyze and give the boundary conditions in different scenarios based on an 1D non-viscous situation (Euler equation). Introduce the boundary conditions in three different scenarios.

In solving mechanics problems, whether involved in fluid or solid, boundary conditions (BCs) are always considered one of the most important factors. In computational fluid dynamics (CFD), slightly difference in BCs may strongly variate the calculation results. Therefore, choosing the right BCs for whether specific engineering problems or research works are critical and must be scrutinized. Here, we introduce three basic BCs model commonly encountered in CFD.

We first give the Euler equation as in 1D situation:

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} = 0 \tag{1}$$

For the 1D situation, the boundary conditions and the governing equation can be written in combined as in Eq. 2, as recalled from Chap. 4, can be visualized in Fig. 1:

$$\begin{cases} \dfrac{\partial U}{\partial t} + a\dfrac{\partial U}{\partial x} = 0 \\ U = U_0 \\ U_{BC} = \mathrm{C} \end{cases} \tag{2}$$

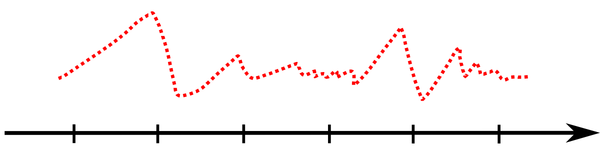

Fig. 1 Schematic for a 1D shock wave equation.

Recall the flux H as referred form Eq. 7 and Eq. 11 in Chap. 1, we have:

$$H = \begin{pmatrix} \rho q \\ \rho u q + p n_x \\ \rho v q + p n_y \\ \rho H q \end{pmatrix} \tag{3}$$

Where $q = u n_x + v n_y$.

Here, we introduce the three basic scenarios for different boundary conditions:

I. *No-slip wall*

The no-slip wall BCs is the most commonly encountered BCs when the fluid-solid interactions are involved. Here, Fig. 2 visualize a typical model with meshing on of the no-slip BCs, which shows how the continuum pressure values is discretized nearing the wall boundary when the normal vector is parallel to the pressure variation.

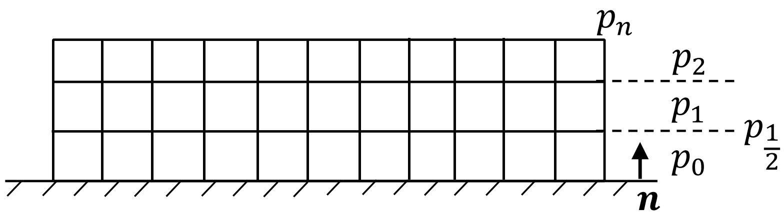

Fig. 2 Schematic for the no-slip boundary conditions.

For no-slip BCs, we have $q = 0$, thence the flux takes the form:

$$H = \begin{pmatrix} 0 \\ pn_x \\ pn_y \\ 0 \end{pmatrix} \tag{4}$$

The velocities on the wall obeys:

$$\begin{cases} u_{wall} = 0 \\ v_{wall} = 0 \end{cases} \tag{5}$$

The pressure on the wall obeys:

$$P_{wall} = P_L \tag{6}$$

Here we provide a coding example (FORTRAN) on giving the no-slip wall boundary conditions. Where the mesh generation of a 2D situation applied on a no-slip BCs is shown.

```
FOR J = JL, i = 1, IL
    FOR I = 1, j = 1, JL
        FOR = 1L, j = 1, JL, J = JL
              CUT J = 1, I = 1, IOI
              CUT J = 1, I = IOI, LL
        END
    END
END
```

II. *Symmetry wall*

Symmetry wall BCs are widely encountered for symmetric geometries including the NACA "00 series" airfoils, bullets, ships, etc. Here in Fig. 3 we show a schematic of the symmetry BCs with $\boldsymbol{V} = \binom{u_1}{v_1}$ as the velocity vector and $\boldsymbol{n}$ as the position vector.

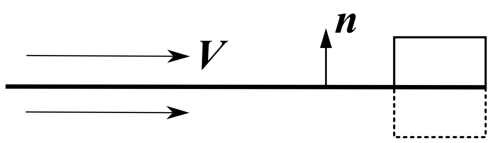

Fig. 3 Schematic for the symmetry boundary conditions.

For symmetry situation, the velocities follows:

$$u_{wall}n_x + v_{wall}n_y = 0 \tag{7}$$

Where the velocities in the two directions obeys

$$u_{wall} = u_1 - (u_1 n_x + v_1 n_y) \cdot n_x \tag{8}$$

$$v_{wall} = v_1 - (u_1 n_x + v_1 n_y) \cdot n_y \tag{9}$$

The flux on the boundary takes the form

$$H = \begin{pmatrix} 0 \\ pn_x \\ pn_y \\ 0 \end{pmatrix} \tag{10}$$

The pressure obeys:

$$P_{wall} = P_L \tag{11}$$

III. *Far field*

Far field BCs are widely applied on noise calculation, thermal estimation and other problems common in aerodynamics. Here in Fig. 4 we show how a typical far field BCs is adopted based on an airfoil meshing, where mostly applied on noise calculations. Usually, for aerodynamic and fluid mechanics, the flow field is chosen to be much larger than the targeted structure to obey the far field BCs.

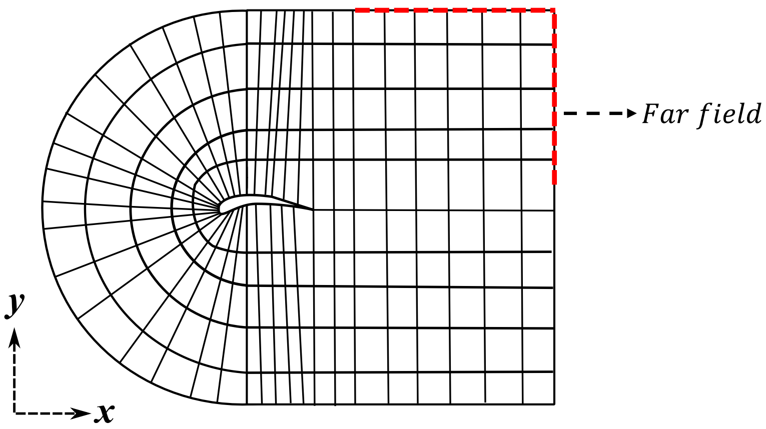

Fig. 4 Schematic for the far field boundary conditions, with a structural meshing of an airfoil.

The far field BCs involves the following situations:

i. For $Ma \le -1$:

When the Mach number obeys $Ma \le -1$, we refer the situation as "supersonic outlet", where the flux U can be written as:

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_1 = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_2 = \begin{pmatrix} \rho_\infty \\ \rho u_\infty \\ \rho v_\infty \\ \rho E_\infty \end{pmatrix} \tag{12}$$

Hence the flux H can be reasoned:

$$\Rightarrow H = \begin{pmatrix} \rho_\infty q_\infty \\ \rho u_\infty q_\infty + p_\infty n_x \\ \rho v_\infty q_\infty + p_\infty n_y \\ \rho H_\infty q_\infty \end{pmatrix} \tag{13}$$

ii. For $-1 < Ma < 0$:

The BCs obeys $-1 < Ma < 0$ are considered as "subsonic inlet", where the flux H can also be reasoned through the flux U:

$$\begin{pmatrix} \rho \\ \rho u_1 \\ \rho v_1 \\ \rho E_1 \end{pmatrix} = \begin{pmatrix} \rho_\infty \\ \rho u_\infty \\ \rho v_\infty \\ \rho E_\infty \end{pmatrix} \Rightarrow H = F(u_1) + F(u_2) + D_1 + \cdots \tag{14}$$

iii. For $0 < Ma < 1$:

The BCs is considered "subsonic outlet", where the flux takes the form:

$$\begin{pmatrix} \rho \\ \rho u_1 \\ \rho v_1 \end{pmatrix} = \begin{pmatrix} \rho_\infty \\ \rho u_\infty \\ \rho v_\infty \end{pmatrix} \Rightarrow \rho E_1 = \frac{p_\infty}{\gamma - 1} + \frac{\rho_2}{2}({u_2}^2 + {v_2}^2) \tag{15}$$

Hence the flux H is reasoned:

$$\Rightarrow H = \begin{pmatrix} \rho_2 q_2 \\ \rho u_2 q_2 + p_2 n_x \\ \rho v_2 q_2 + p_2 n_y \\ \rho E_2 + p_\infty \end{pmatrix} \tag{16}$$

iv. For $Ma > 1$:

The BCs $Ma > 1$ is the "supersonic outlet", where the flux H can be reasoned:

$$\begin{pmatrix} \rho \\ \rho u_1 \\ \rho v_1 \\ \rho E_1 \end{pmatrix} = \begin{pmatrix} \rho \\ \rho u_2 \\ \rho v_2 \\ \rho E_2 \end{pmatrix} \Rightarrow H = \begin{pmatrix} \rho_2 q_2 \\ \rho u_2 q_2 + p_2 n_x \\ \rho v_2 q_2 + p_2 n_y \\ \rho E_2 \end{pmatrix} \tag{17}$$

Where

$$\rho E = \frac{p_\infty}{\gamma - 1} + \frac{\rho}{2} V^2 \tag{18}$$

and

$$\rho H = \rho E + p \tag{19}$$

References

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