CHAPTER 4. Finite Difference Method

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Abstract

Provide the reasoning of finite difference method (FDM) from discretize the derivative of the flux. A continued method for discretization of unequal distance meshing is provided based on FDM from the meshing transformation as to be introduced in Chap. 7.

We first give the Euler equation as in 1D situation:

$$\frac{\partial U}{\partial t}+\frac{\partial E}{\partial x}=0 \tag{1}$$

For the 1D situation, the boundary conditions and the governing equation can be written in combined as in Eq. 2, as visualized in 0:

$$\frac{\partial U}{\partial t}+a\frac{\partial U}{\partial x}=0 \tag{2}$$

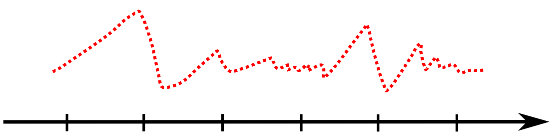

Fig. 1 Schematic for a 1D shock wave equation.

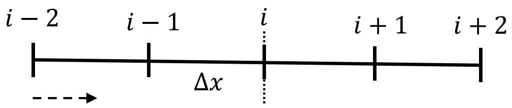

Fig. 2 Schematic for spatial discretization in a row.

As shown in Fig. 2, the spatial term in x direction can be expanded as Taylor series:

$$U_{i-1}=U_i-\frac{\partial U}{\partial x_i}\Delta x+O(\Delta^2)+\cdots \tag{3}$$

$$U_{i+1}=U_i+\frac{\partial U}{\partial x_i}\Delta x+O(\Delta^2)+\cdots \tag{4}$$

Subtracting Eq. 4 to Eq. 3, one obtains:

$$U_{i+1}-U_{i-1}=2\frac{\partial U}{\partial x_i}\Delta x \tag{5}$$

Hence, Eq. 5 can be reduced to:

$$\left.\frac{\partial U}{\partial x}\right|_i=\frac{U_{i+1}-U_{i-1}}{2\Delta x} \tag{6}$$

Which the discretization method is nominated as the central difference method (CDS).

Based on Eq. 3, we can write:

$$U_{i+1} \approx U_i + \frac{\partial U}{\partial x_i}\Delta x \tag{7}$$

Subtracting Eq. 7 to U_i one obtains:

$$U_{i+1} - U_i = \frac{\partial U}{\partial x_i}\Delta x \tag{8}$$

Hence, Eq. 7 can be reduced to:

$$\left.\frac{\partial U}{\partial x}\right|_i = \frac{U_{i+1} - U_i}{\Delta x} \tag{9}$$

Which the discretization is called the front difference scheme (FDS).

Referred from what is shown form Eq. 7 to 8, we obtains:

$$U_{i-1} = U_i - \frac{\partial U}{\partial x_i}\Delta x \tag{10}$$

Subtracting Eq. 10 to U_i one obtains:

$$U_i - U_{i-1} = \frac{U_i - U_{i-1}}{\Delta x} \tag{11}$$

Therefore, the derivative takes the form

$$\left.\frac{\partial U}{\partial x}\right|_i = \frac{U_i - U_{i-1}}{\Delta x} \tag{12}$$

Which is called the back-difference scheme (BDS).

For spatial discretization, we have

$$\left.\frac{\partial U}{\partial x}\right|_i = \begin{cases} \dfrac{U_{i+1} - U_i}{\Delta x} \to \text{FDS} \\ \dfrac{U_i - U_{i-1}}{\Delta x} \to \text{BDS} \\ \dfrac{U_{i+1} - U_{i-1}}{2\Delta x} \to \text{CDS} \end{cases} \tag{13}$$

For discretization of time derivatives, we apply similar strategies:

$$\left.\frac{dU}{dt}\right|_i = \begin{cases} \dfrac{U^{n+1} - U^n}{\Delta x} \to \text{FDS} \\ \dfrac{U^n - U^{n-1}}{\Delta x} \to \text{BDS} \\ \dfrac{U^{n+1} - U^{n-1}}{2\Delta x} \to \text{CDS} \end{cases} \tag{14}$$

Here we give the artificial dissipative term as to be discussed in Chap. 6 in spatial discretization based on CDS:

$$\frac{U_{i+1} - U_{i-1}}{2\Delta x} + \kappa_1 \frac{\partial^2 u}{\partial x^2} - \kappa_2 \frac{\partial^4 u}{\partial t^4} = 0 \tag{15}$$

Also, for unequal distance meshing as to be introduced in Chap. 7 as visualized in Fig. 3, we can discretize the term $\frac{\partial U}{\partial x}$ in the forms:

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial \zeta}\frac{\partial \zeta}{\partial x} \tag{16}$$

$$= \frac{U^{\zeta+1} - U^{\zeta}}{\Delta \zeta}\frac{\partial \zeta}{\partial x} \tag{17}$$

$$= \frac{U^{i+1} - U^{i}}{\Delta x} \tag{18}$$

Where $\zeta = \frac{1}{\Delta x}$.

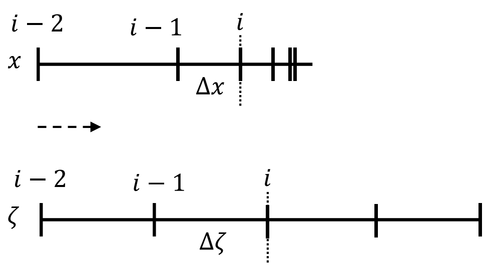

Fig. 3 Schematic for the unequal distance meshing in a row.

Here, we expand the Euler equation as in 2D situation:

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0 \tag{19}$$

Where

$$\begin{cases} \dfrac{\partial E}{\partial x} = \dfrac{\partial E}{\partial \zeta}\zeta_x + \dfrac{\partial E}{\partial \eta}\eta_x \\ \dfrac{\partial F}{\partial y} = \dfrac{\partial F}{\partial \zeta}\zeta_y + \dfrac{\partial F}{\partial \eta}\eta_y \end{cases} \tag{20}$$

Hence, the 2D Euler equation takes the form:

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial \zeta}\zeta_x + \frac{\partial E}{\partial \eta}\eta_x + \frac{\partial F}{\partial \zeta}\zeta_y + \frac{\partial F}{\partial \eta}\eta_y = 0 \tag{21}$$

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