CHAPTER 1. Introduction to Computational Fluid Dynamics

© Hanfeng Zhai

School of Mechanics and Engineering Science, Shanghai University

Shanghai 200444, China

Abstract

Introduce the concept of computational fluid dynamics (CFD) continued from fluid mechanics. Provide the governing equations, conceptions, and related parameters in CFD. Explained the basic ideology and goal for CFD.

Computational fluid dynamics (CFD), is a method to solve fluid mechanics problems through numerical methods. Navier-Stokes equation is the governing equation for fluid phenomena and physics. Due to the nonlinear nature of the Navier-Stokes equation, analytical solution for fluidic problems are limited to specific situations. Hence, to obtain solutions for more generalized fluidic problems, we introduce numerical methods like CFD to discretize the equations for results.

In fluid mechanics (or fluid dynamics), the governing equation for fluid motion and physics is the Navier-Stokes equation, which takes the form

$$\rho \frac{d\boldsymbol{V}}{dt} = \rho g - \nabla P + \mu \nabla^2 \boldsymbol{V} \tag{1}$$

Which can be written in the coordinate system for 3D situation:

$$\begin{aligned}
\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) &= f_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\
\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) &= f_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \\
\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) &= f_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)
\end{aligned} \tag{2}$$

Where ρ is the fluidic density, $\boldsymbol{V}$ is the speed vector; P is the pressure on fluid, u, v, w are the velocity component in the x, y, z directions, respectively. $\boldsymbol{f}$ is the force acting on the unit volume. The constant μ is the dynamics viscosity.

The Navier-Stokes equation taking the form in Eq. 2 elucidate the physics of fluidic nature. Yet for computation, we write the Navier-Stokes equation in the following terms for discretization.

$$\frac{\partial U}{\partial t} + \frac{\partial E_i}{\partial x_i} - \frac{\partial E_{vi}}{\partial x_i} - P_i = 0 \tag{3}$$

Where P_i, E_{vi} marks the external forces and viscosity terms, respectively.

In 3D situation, consider the coordinate system as compared with Eq. 2, Eq. 3 takes the form:

$$\frac{\partial U}{\partial t}+\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}+\frac{\partial G}{\partial z}-\frac{\partial E_v}{\partial x}-\frac{\partial F_v}{\partial y}-\frac{\partial G_v}{\partial z}-P=0 \tag{4}$$

Where

$$U=\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \rho E\end{pmatrix}, E=\begin{pmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uw\\ \rho uH\end{pmatrix}, F=\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ \rho vw\\ \rho vH\end{pmatrix}, G=\begin{pmatrix}\rho w\\ \rho uw\\ \rho vw\\ \rho w^2+p\\ \rho wH\end{pmatrix} \tag{5}$$

$$P=\begin{pmatrix}0\\ \rho f_x\\ \rho f_y\\ \rho f_z\\ \rho\left(uf_x+vf_y+wf_z\right)\end{pmatrix}, E_v=\begin{pmatrix}0\\ \sigma_{xx}\\ \sigma_{xy}\\ \sigma_{xz}\\ u\sigma_{xx}+v\sigma_{xy}+w\sigma_{xz}+\kappa\frac{\partial T}{\partial x}\end{pmatrix}, \tag{6}$$

$$F_v=\begin{pmatrix}0\\ \sigma_{yx}\\ \sigma_{yy}\\ \sigma_{yz}\\ u\sigma_{yx}+v\sigma_{yy}+w\sigma_{yz}+\kappa\frac{\partial T}{\partial x}\end{pmatrix}, G_v=\begin{pmatrix}0\\ \sigma_{zx}\\ \sigma_{zy}\\ \sigma_{zz}\\ u\sigma_{zx}+v\sigma_{zy}+w\sigma_{zz}+\kappa\frac{\partial T}{\partial x}\end{pmatrix}$$

Where $p=\rho RT=\rho(1-\gamma)\left(\rho E-\frac{1}{2}(u^2+v^2+w^2)\right)$.

Here, for classic aerodynamics problems, we neglect the external force as in specific flow fields (Lagrangian viewpoint). we take the equation in 2D situation for example:

$$\frac{\partial U}{\partial t}+\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}-\frac{\partial E_v}{\partial x}-\frac{\partial F_v}{\partial y}=0 \tag{7}$$

Where

$$U=\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho E\end{pmatrix}, E=\begin{pmatrix}\rho\\ \rho u^2+p\\ \rho uv\\ \rho uH\end{pmatrix}, F=\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ \rho vH\end{pmatrix} \tag{8}$$

$$E_v=\begin{pmatrix}0\\ \sigma_{xx}\\ \sigma_{xy}\\ u\sigma_{xx}+v\sigma_{xy}-\kappa\frac{\partial T}{\partial x}\end{pmatrix}, F_v=\begin{pmatrix}0\\ \sigma_{xy}\\ \sigma_{yy}\\ u\sigma_{xy}+v\sigma_{yy}-\kappa\frac{\partial T}{\partial y}\end{pmatrix} \tag{9}$$

In the given terms, p is the pressure, $\sigma_{x_ix_i}$ are the viscous terms. The given variables have the following relations:

The pressure p, temperature T, and terms H, E follows:

$$p=\rho RT \tag{10}$$

$$\rho H=\rho E+p \tag{11}$$

$$E=\frac{R}{\gamma-1} \tag{12}$$

Where $\gamma = \rho g$ and R is the thermal constant.

The viscous term $\sigma_{x_i x_i}$ takes the form:

$$\sigma_{x_i x_i} = \mu\left(\frac{\partial u_i}{\partial x_i}\right) \tag{13}$$

Where

$$\begin{cases} \sigma_{xx} = \mu\left(\frac{4}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial u}{\partial y}\right) \\ \sigma_{yy} = \mu\left(\frac{4}{3}\frac{\partial v}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y}\right) \\ \sigma_{xy} = \sigma_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \end{cases} \tag{13}$$

In fluid mechanics, the constants have the following relations:

The sound velocity:

$$c = \sqrt{\frac{\gamma P}{\rho}} \tag{14}$$

The Reynold's number:

$$Re = \frac{\rho u l}{\mu} \tag{15}$$

The thermal constant:

$$\kappa = \frac{\gamma R}{Pr(\gamma - 1)}\mu \tag{16}$$

Where

$$\mu = \frac{M}{Re}\sqrt{\gamma} \tag{17}$$

In which Re, Pr, M is the Reynold's number, Prandtl number and Mach number, respectively.

The constants relations showed from Eq. 14 to 17 will be further applied on the nondimensionalization for the equations to be given in Chap. 2.

References

[1] Yuxin Ren. Computational Fluid Dynamics (Basic). Department of Engineering Mechanics, Tsinghua University. 2003.

[2] Eleuterio F. Toro. Riemann Solvers and Numerical Methods for Fluid Dynamics. Springer-Verlag Berlin Heidelberg 2009. ISBN 978-3-540-25202-3. DOI 10.1007/978-3-540-49834-6.

[3] Antony Jameson. Advanced Computational Fluid Dynamics AA215A Lecture 4. Winter Quarter, 2012, Stanford, CA.

X. -Q. Yang. **Computational Fluid Dynamics** (2020).

Full notes series could be found at http://hanfengzhai.net/categories/note